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Asset Pricing Theory and the LIBOR Market Model
av
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Asset Pricing Theory and the LIBOR Market Model

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The first part of this thesis is a general presentation of no-arbitrage asset pricing theory in continuous time. The standard mathematical formulations of models with Brownian motion as random variables is presented, as well as the two approaches of partial differential equations and martingale methods. The second part narrows in on a particular application of this theory: The market models of interest rates. The LIBOR and swap market model are presented together with limitations on extension to multiple currencies.

Contents

Chapter 1

Fundamentals of Asset Pricing

Finance is about pricing cashflows. The cashflows we are interested in pricing will appear over a future period and they will be uncertain — so a fundamental challenge is how to take time and uncertainty into account.

To find principles for the pricing of uncertain future cashflows, the theory relies on the economic reason of no arbitrage. Arbitrage is about consistent prices. Informally, absence of arbitrage means that similar claims must have similar price. We sketch the idea behind absence of arbitrage as the following: if two claims are available paying equal cash flows but at unequal cost, we have an arbitrage opportunity. If offered such opportunities, we could adopt a strategy of buying one claim and selling the other in such a way that we would get something and pay nothing. We state that any claim, if possible to replicate by some combination of other available claims, must have the value of the replicating claims. A formal definition of this will be given later in the paper. Here we hope the above is sufficient and note that as maximizing investors getting something for nothing, we could no longer solve our optimizing problem. No equilibrium could result, so we state that arbitrage must be absent from the financial markets we study.[1]

To apply the pricing principles, we need to describe the distribution of cashflows over time and states. The description of our claim will be provided by a *model*. As we want to model indeterminate future values, our model will have to be a *stochastic* model. The modelling framework we use rests on *the efficient market hypothesis* (EMH), stating that asset prices at any point in time take full account of the information known at that time. Then, if the EMH is correct, it also means that we are not able to (and we will not try to) *predict* future asset prices. All we can do, assuming that the EMH is correct, is to model the *dynamics* of price processes.

[1]Absence of arbitrage and equilibrium are not the same. The absence of arbitrage is a necessary, but not sufficient condition for the existence of an equilibrium.

Much of the available asset pricing theory is devoted to *derivatives*. This is not solely based on the needs of financial markets, as both academic interest and convenience is seen to favor the development of pricing models for derivative contracts. The value at maturity of a derivative contract is derived from one or possibly several underlying securities, and we shall see that with the help of results presented below, the price today of a derivative can be computed as a function of (the price of) the underlying security and some other information observable today. This is based on the argument that the derivative is *replicable* by a combination of a limited number of other assets, as opposed to the underlying or basic securities that might depend on an endless number of factors. A derivative is a *contingent claim* conditioned on the underlying security, and often we need not model all the factors driving the underlying security. A model of its dynamics is sufficient.

1.1 The model of the underlying securities

[2]The standard choice of stochastic process to model asset prices in continuous time is the *Brownian motion* or *Wiener process*. Originating from botanist Robert Brown, this process was first used to describe the motion of pollen grains or other small particles in fluid (1827). Einstein explained the physics of the phenomena, and Norbert Wiener established the mathematical foundation of the process (1923). In the literature, the process is denoted both as a Brownian and Wiener process. We will use the name Brownian motion. The brownian motion B is defined by the following properties:

Definition (Brownian motion)

1. The initial value of the process is zero: $B_0 = 0$.
2. Increments are Gaussian or normally distributed with zero mean and variance equal to the length of the time interval: $(B_t - B_s) \sim N(0, t-s)$, where $s \leq t$.[3]
3. Increments over disjoint time intervals are independently distributed: for $s < t \leq u < v$, $(B_t - B_s)$ and $(B_v - B_u)$ are independent random variables.
4. The process is a continuous function of time: $t \rightarrow B_t(\omega)$ is continuous for all $\omega \in \Omega$.

[2]The contents of this section is based on [1], [4] and [12].

[3]B_t may also be a multidimensional process, in which case increments are multinormally distributed.

The first use of the Brownian motion in a model of asset prices was Louis Bacheliers model of the French bond market in 1900. Though innovative, the Bachelier model suffered from a flaw common to the purely Gaussian models: models of the form $X_t = (\cdot)B_t$ assign positive probability to the occurrence of negative asset prices. For assets with limited liability, such as stocks, the price cannot be negative. We therefore want to modify to get a model where asset prices are positive with probability one. We also want to incorporate another element into the asset price model: drift or pure time-dependency. A way of achieving both is by using the exponential form, as in the following and preliminary model for the price of asset X at time t:

$$X_t = X_0 exp(\mu t + \sigma B_t) \tag{1.1}$$

where μ and σ are parameters denoting the rates of *drift* and *diffusion*. Note that the exponent $(\mu t + \sigma B_t)$ is a normally distributed variable by property 2 of Brownian motion, so X_t is log-normal. X_t is called a *geometric Brownian motion*. Note also that apart from the diffusion term σB_t, X_t is an asset with a continuously compounded growth rate of μ.

We also want to express the dynamics of our asset X_t, so we need to find an expression in differential form. We then encounter the problem that the Brownian motion B_t is *nowhere differentiable* with respect to time. Trajectories of a Brownian motion are nowhere sufficiently smooth to be differentiable. An expression of the form

$$X_t = X_0 + \int_0^t (\cdot)ds + \int_0^t (\cdot)dB_s \tag{1.2}$$

is not straightforward calculus as we know it from the rule

$$\int_0^t g(s)df(s) = \int_0^t g(s)f'(s)ds$$

applicable to a function $f(s)$ of *limited variation*. The integral involving dB_s in (1.2) is a *stochastic integral*, and a term dB_s/ds is not defined. We need another approach. Consider the following:

$$\int_0^t Y_s(\omega)dB_s(\omega) = \lim_{n\to\infty} \sum_{i=1}^{n-1} Y_{s_i}(\omega)(B_{s_{i+1}}(\omega) - B_{s_i}(\omega)) \tag{1.3}$$

where we call Y_s the *integrand process*. For the limit on the right-hand side of (1.3) to exist, we need the sum to be of limited variation. The variance of the single increment of the Brownian motion from time s_i to s_{i+1} is

$$E\left[(\Delta B_{s_i})^2\right] - (E\left[\Delta B_{s_i}\right])^2 = \Delta s_i \tag{1.4}$$

by property 2 of Brownian motion. Note that in (1.3) we have a *forward increment* of B, and assume that the integrand Y_{s_i} is independent from the forward increment ΔB_{s_i}. If so, we can compute the variance of each increment on the right-hand side of (1.3) as

$$\begin{aligned}
\text{variance of increment} &= E\left[((Y_{s_i}(\Delta B_{s_i}))^2\right] \\
&= E\left[(Y_{s_i})^2\right] E\left[(\Delta B_{s_i})^2\right] \\
&= E\left[(Y_{s_i})^2\right] \Delta s_i
\end{aligned}$$

To compute the variance of the whole sum in (1.3), we first use that the expectation of increments of the Brownian motion is zero, so that the square of the expectation is also zero. This property cancels out the square expectation term of the variance. Next, we use property 3 of Brownian motion telling us that increments are independent. The expectation of cross terms is then the product of the expectation of each term - which again is zero and cross terms are canceled. The variance of the sum of increments is then reduced to

$$\begin{aligned}
\text{variance of sum} \quad &= \quad E\left[\left(\sum_{i=1}^{n-1} Y_{s_i}\Delta B_{s_i}\right)^2\right] \\
&= \quad \sum_{i=1}^{n-1} E\left[(Y_{s_i})^2\right] \Delta s_i \\
\overrightarrow{\overset{=}{n \to \infty}} \quad & \quad \int_0^t E\left[(Y_{s_i})^2\right] ds
\end{aligned}$$

Remember that the stochastic integral is defined if the variance of the sum is finite. We have found this variance assuming that the integrand process Y is functionally independent of the forward increment of the Brownian motion. This is a desired property of the price process model. We do not want, however, the value of the price process Y at any time t to be independent of increments prior to t. On the contrary, we want the asset price Y_t to be completely determined when the path of the Brownian motion up to time t is known.

This is formalized in the following probabilistic set-up. Let the state space be denoted Ω. Investors exposed to the not yet known future states of Ω have available a restricted set of information on which they base their choice of exposure. The set of information available in the financial market will be denoted $\mathcal{F}$. $\mathcal{F}$ is a σ-*algebra* (also called *tribe* or *field*) on Ω if the following properties are satisfied:

1. $\emptyset \in \mathcal{F}$. (The empty set is in $\mathcal{F}$.)

2. $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}, \quad \forall A \in \Omega$. (For any set in $\mathcal{F}$, the compliment of that set in Ω is also in $\mathcal{F}$.)

3. $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$. (For any combination of sets that are in $\mathcal{F}$, the union of those sets is also in $\mathcal{F}$.)

$(\Omega, \mathcal{F})$ is then called a *measurable space*. Further, if we have a function $P : \mathcal{F} \rightarrow [0, 1]$ on a measurable space $(\Omega, \mathcal{F})$ such that

$$P(\emptyset) = 0, P(\Omega) = 1$$

and if for $A_1, A_2, \ldots \in \mathcal{F}$ and $A_i \bigcap A_j = \emptyset, \quad \forall \{i, j : i \neq j\}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P\left(A_i\right)$$

P is called a *probability measure*. The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space*.

Given a probability space $(\Omega, \mathcal{F}, P)$, we say that the random variable $Y : \Omega \rightarrow \mathrm{R}$ is *$\mathcal{F}$-measurable* or *adapted* if for any set of values U of the random variable Y, the set of arguments (states) giving U are in the σ-algebra $\mathcal{F}$:

$$Y^{-1}(U) := \{\omega \in \Omega : Y(\omega) \in U\} \in \mathcal{F}$$

Note that a *σ-algebra* is a collection of subsets of the state space, so the information set is an ordering of the possible states. To accommodate for time in the model, we define the *filtration* $\mathbf{F} = \{\mathcal{F}_t : t \in [0, T^*]\}$. In our case, $\mathbf{F}$ will be generated by the Brownian motion B such that an increasing degree of detail is given as time passes and B evolves. That is, $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s \leq t$. If the asset price process Y is $\mathcal{F}_t$-adapted the value of Y_s is determined for $s < t$, while Y_s is a random variable for $s > t$.

We then return to our search for a differential form of the asset price process X_t modeled in equation (1.1). The usual way to obtain this is by Taylor expansion of X_t. Remember that we want to model X_t as a function of both time and the Brownian motion B, so we have $X_t = f(t, B_t)$. We assume that f is differentiable in time and twice differentiable in B.

$$f(t, B_t) \quad = \quad f(0, B_0) + \sum_i \frac{\partial f}{\partial t} \Delta t_i + \sum_i \frac{\partial f}{\partial B_t} \Delta B_{t_i}$$

$$+ \quad \frac{1}{2}\sum_i \frac{\partial^2 f}{\partial t^2}(\Delta t_i)^2 + \frac{1}{2}\sum_i \frac{\partial^2 f}{\partial B_t^2}(\Delta B_{t_i})^2$$
$$+ \quad \sum_i \frac{\partial^2 f}{\partial t \partial B_t}(\Delta t_i)(\Delta B_{t_i}) + \sum_i O((\Delta t_i)^2 + (\Delta B_{t_i})^2)$$

If $\Delta t_i \to 0$, the term involving the first order derivative in time tends to a conventional integral. From the discussion above, it follows that the term with the first order derivative in B_t tends to a stochastic integral like that in equation (3). Terms involving $(\Delta t_i)^2$, $(\Delta t_i)(\Delta B_{t_i})$ and $(\Delta B_{t_i})^2)$ are all of higher order than Δt_i. If $\Delta t_i \to 0$, these terms will then tend to zero with one important exception: Recall from the discussion of the stochastic integral above that

$$\begin{aligned} E\left[\sum_i \frac{\partial^2 f}{\partial B_{t_i}^2}(\Delta B_{t_i})^2\right] &= \sum_i E\left[\frac{\partial^2 f}{\partial B_{t_i}^2}\right]\Delta t_i \\ &\xrightarrow[\Delta t_i \to 0]{=} \int_0^t \frac{\partial^2 f}{\partial B_s^2} ds \end{aligned}$$

where we use that the asset price process is adapted so that $\partial^2 f/\partial B_t^2$ is independent from $(\Delta B_{t_i})^2$. Property 2 of Brownian motion is used to obtain Δt_i as the expectation of the squared forward increment of B_{t_i}. As $(\Delta B_{t_i})^2$-terms are shown to be of order Δt_i they cannot be overlooked, and this result distinguishes stochastic integration from ordinary integration.

Collecting these last results, we have a formula for the integration of stochastic processes based on Brownian motion, and this formula is called the *Itô formula* after K. Itô.[4] The processes are called *Itô processes*. The integral form of our asset price process is then

$$X_t = X_0 + \int_0^t \left(\frac{\partial X}{\partial s} + \frac{1}{2}\frac{\partial^2 X}{\partial B_s^2}\right) ds + \int_0^t \frac{\partial X}{\partial B_s} dB_s$$

and the more common differential form is

$$dX_t = \left(\frac{\partial X}{\partial t} + \frac{1}{2}\frac{\partial^2 X}{\partial B_t^2}\right) dt + \frac{\partial X}{\partial B_t} dB_t, \quad X_0 = x$$

which is an initial value problem involving a *stochastic differential equation* (SDE).

In equation (1.1) we suggested the model

[4]The Itô formula was published in 1951 in the paper *Multiple Wiener Integral*, J. Math. Soc. Japan 3, 157-169.

$$X_t = X_0 exp(\mu t + \sigma B_t)$$

from which we calculate the derivatives

$$\frac{\partial X}{\partial t} = \mu X_t, \quad \frac{\partial X}{\partial B_t} = \sigma X_t, \quad \frac{\partial^2 X}{\partial B_t^2} = \sigma^2 X_t \tag{1.5}$$

resulting in the SDE

$$dX_t = \left(\mu + \frac{1}{2}\sigma^2\right) X_t dt + \sigma X_t dB_t$$

To simplify the form of the SDE, X_t is often expressed as

$$X_t = X_0 exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t)\right) \tag{1.6}$$

giving us the new derivatives

$$\frac{\partial X}{\partial t} = (\mu - \frac{1}{2}\sigma^2)X_t, \quad \frac{\partial X}{\partial B_t} = \sigma X_t, \quad \frac{\partial^2 X}{\partial B_t^2} = \sigma^2 X_t$$

and the new SDE

$$dX_t = \mu X_t dt + \sigma X_t dB_t \tag{1.7}$$

This SDE together with an initial value constitutes an initial value problem to which (1.6) is a solution. Notice that from the discussion following the Taylor-expansion above, both the drift rate μ and the diffusion rate σ may be functions. μ and σ may even be stochastic, as long as they are adapted processes. X_t may also be a vector stochastic process, thus representing a portfolio or even the whole market of securities. This basic model then offers enough flexibility to be the one set-up we will use throughout the paper.

1.2 Arbitrage pricing of derivative securities

Having developed the basic model of asset prices we are now ready to investigate the pricing of claims to these assets. We will present two approaches to arbitrage derivatives pricing, of which one is regarded as traditional and the other as modern.

The traditional approach goes back to the paper of Fischer Black and Myron Scholes of 1973 [2], where a formula is calculated for the price of a European call option on an underlying asset modeled as the geometric Brownian motion (1.6). Black and Scholes construct a portfolio replicating the dynamics of the option, and derive a partial differential equation (PDE) describing the dynamics of this portfolio — hence the dynamics of the option. The option price is then calculated as the solution to this PDE with the boundary condition that at the time of maturity, the value of the replicating portfolio must equal the payoff of the option if arbitrage is absent. We will call this traditional approach the PDE approach.

The modern approach follows from the work of Michael Harrison together with David Kreps and Stanley Pliska. This is a probabilistic approach using the techniques of *equivalent martingale measures*. A martingale is a stochastic process with constant mean, and a martingale measure is a probability measure so that the processes we consider are martingales under this measure.[5] If all assets in the market are martingales, we cannot expect to consistently profit (or lose) in this market. In the words of [14], a "martingale is the mathematical formalisation of the concept of a fair game."[6] Expressing price processes so that the market becomes a fair game is what links the martingale method to the absence of arbitrage.

In a market where all assets have constant mean, the investors must be risk neutral and there can be no time value of money. Both risk aversion and time value of money are, however, present in the model. The time value of money is included in the model by normalizing the price processes. We say that price processes are *deflated* or that they are divided by a *price deflator* or *numeraire*. The so deflated price processes are what we model as martingales. Investor risk aversion is included in the way the true probability measure is *altered* to become the martingale measure.

Let the price process S be a geometric Brownian motion as in equation (1.7). Let there also be a bank account β with continuously compounded and constant interest rate r. These two securities together with the derivative Y is our *market*:

$$S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\right) \tag{1.8}$$

$$\beta_t = \beta_0 \exp(rt) \tag{1.9}$$

We assume that we are free to trade in the security S and the bank account β. Let there also be a derivative contract on Y on S, and assume that the price of the derivative can be written as a function of time and the price process of the underlying security, such that

[5] As the mean of asset prices is obviously not constant, we should note that a martingale measure can be a constructed measure different from the true probability measure.

[6] See [14], p. 2.

$$Y_t = f(S_t, t)$$

The method to find the price of Y, which is the function f, will be to use the securities S and β to replicate f. Assume f is twice continuously differentiable in S and continuously differentiable in t, and apply Itô's formula to get the dynamics of the option.

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial S} dS_t + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS_t)^2 && (1.10) \\ &= \frac{\partial f}{\partial S}(\mu S_t dt + \sigma S_t dB_t) + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2 dt && (1.11) \\ &= \left(\frac{\partial f}{\partial S}\mu S_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2\right) dt + \frac{\partial f}{\partial S}\sigma S_t dB_t && (1.12) \end{aligned}$$

Note that if our assumptions are correct, the price process Y of the derivative has a form similar to that of the underlying security S: there is a diffusion term in both processes involving the very same Brownian motion, and there is a drift term in each process. Note also that our definition of the market (1.8) gives a set of linear equations in dt and dB_t. Replication of Y should not be that far away if we assume that the positions (a, b) of the portfolio can be any real number, so that short positions are permitted and any fraction of the securities can be bought and sold. A market such as this, where a linear combination of marketed securities can replicate another security, is called *(dynamically) complete.*[7]

We should also note that a linear combination of securities will in general not replicate the dynamics of another security for more than a moment. We therefore need to assume there are no taxes, transaction costs or any other friction — so that the portfolio, in order to perfectly replicate the derivative dynamics, can be continuously rebalanced without any cost.

The trading strategy of positions $\theta = (a, b)$ in (S, β) that replicates Y can be constructed as a *self-financing trading strategy.* This means that we put money into the portfolio only when the trading strategy is initiated, and that we take nothing out of the portfolio until the strategy is terminated.

Definition (self-financing) The portfolio V of positions $\theta = (a, b)$ in (S, β), where a and S can be vectors, is a self-financing trading strategy if

$$V_t = a_t S_t + b_t \beta_t = a_0 S_0 + b_0 \beta_0 + \int_0^t a_u dS_u + \int_0^t b_u d\beta_u$$

for any time t in the life-time of the strategy.

[7] Definition follows.

To avoid arbitrage, we also need to imply constraints ruling out so-called *doubling strategies*. An example of an arbitrage strategy is given by [4]: Choose an arbitrary amount α to be earned before some date. Assume that an infinite number of bets can be made in the period, for example by betting half-way into the period, at 3/4 of the period, at 7/8 of the period and so on. (Already assuming it is possible to constantly rebalance a portfolio, assuming an infinite number of bets does not extend our assumptions.) By betting α on the result of coin tosses, a strategy of quitting when succesful and doubling when not will lead to an infinitesimal probability of loosing and consequently a riskless gain of α.

As only finite amounts of goods are available, the existence of doubling strategies does not make economic sense. Doubling strategies are therefore ruled out by technical (integrability) constraints and by credit constraints. We say that a strategy is *admissible* if it satisfies such constraints.

Definition (admissible) The trading strategy θ is admissible if both the following two conditions are satisfied:

1. (Integrability constraints)

$$P(\int_0^T \theta_t^2 dt < \infty) = 1 \text{ and } E[\int_0^T \theta_t^2 dt] < \infty$$

   The space of trading strategies θ satisfying these constraints is called $\mathcal{H}^2$.

2. (credit constraint)

   There is some constant k such that $P(V_t \geq k) = 1 \ \forall\, t$, that is the portfolio V_t is bounded from below. The space of strategies θ satisfying this credit constraints is called $\underline{\Theta}$.

Common to both approaches is the basic assumption that there is no arbitrage in the financial market. This is a fundamental assumption. To formally define arbitrage, let X represent the price processes of the available investments and let θ denote a trading strategy in these assets.

Definition (Arbitrage) The admissible and self-financing trading strategy θ is an arbitrage if $\theta_0 \cdot X_0 < 0$ and $\theta_T \cdot X_T \geq 0$ or if $\theta_0 \cdot X_0 \leq 0$ and $\theta_T \cdot X_T > 0$.[8]

[8] This definition is from [4], chapter 6 A. See also the preface of [4] for more on notation: $x \geq 0$ means that x is not negative in any coordinate. $x > 0$ means that x is not negative in any coordinate and strictly positive in some, but not necessarily strictly positive in all coordinates.

So there is arbitrage if a gain is possible but not certain, and there is no risk of loss. If arbitrage is absent, any claim to a certainly non-negative and possibly strictly positive pay-off should have a strictly positive price.

Then we have our pricing method: If there is no arbitrage, if the price process of the derivative Y is equal to the function $f(S_t, t)$ and if there is a self-financing trading strategy V such that $V_T = Y_T$, we state that at any time t prior to maturity T we must have $V_t = Y_t$. The price of the derivative contract at time t is then simply the value of the replicating portfolio $a_t S_t + b_t \beta_t$.

1.2.1 The PDE approach

We want to find the price of a European derivative on a security. The pricing method we shall apply here rests on assumptions regarding arbitrage, the differentiability of the derivative and the existence of a portfolio replicating the price process of the derivative contract.

The above is the rationale of the pricing method, but of course we do not yet know enough to compute the numerical value of the derivative. We need to find expressions for the positions (a, b) of the self-financing replicating portfolio. To this end, we use the uniqueness of stochastic differential equations to demand that both drift and diffusion are equal.[9]

Consider the complete market (S, β) specified in the previous section. The value process of the derivative security $Y = f(S, t)$ can be replicated by an admissible self-financing portfolio V of positions (a, b) in (S, β) so that $V_T = Y_T$. Assuming that this market does not permit arbitrage, the price of the derivative must be equal to the price of the replicating portfolio. The approach here is to derive a characterization in the form of a PDE of the replicating portfolio V. Observe that V evolves as $dV_t = a_t dS_t + b_t d\beta_t$. Equate the diffusion of V with the diffusion of the derivative dynamics given by (1.10)

$$a_t \sigma S_t dB_t = \frac{\partial f}{\partial S} \sigma S_t dB_t$$

or

$$a_t = \frac{\partial f}{\partial S}(S_t, t)$$

Substitute for a_t into the portfolio V_t to get an expression for b_t

$$\frac{\partial f}{\partial S} S_t + b_t \beta_t = Y_t = f$$

[9]See [12], 5.2.

and solve to get

$$b_t = \frac{f - \frac{\partial f}{\partial S} S_t}{\beta_t}$$

Substitute then for a_t and b_t into the drift term of the portfolio in V and equate to the drift term of the derivative in (1.10)

$$\frac{\partial f}{\partial S}\mu S_t + \frac{f - \frac{\partial f}{\partial S} S_t}{\beta_t} r\beta_t = \frac{\partial f}{\partial S}\mu S_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S_t^2$$

Rearranging this gives the following partial differential equation (PDE)

$$f(S_t, t)r - \frac{\partial f}{\partial S}(S_t, t)S_t r - \frac{\partial f}{\partial t}(S_t, t) - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(S_t, t)\sigma^2 S_t^2 = 0 \tag{1.13}$$

which must hold for all $(S_t, t) \in [0, \infty) \times [0, T]$. Note that the PDE does not include the drift rate μ of the security S, but the drift rate r of the risk-free asset.

If the form $f(S_T, T) = g(S_T)$ of the derivative at maturity is specified for any positive real value of S_T, this specification gives a boundary condition for the PDE (1.13). We also have the initial condition $S_t = s$. The solution $f(S_t, t)$ to this boundary value problem then give the value of the derivative at time t as a function of time and the known variable S_t (as S_t is $\mathcal{F}_t$-adapted).

We may note that so far we have made some assumptions on the dynamics of the underlying security and on the differentiability of the derivative, but we have not specified the functional form of the derivative contract. The PDE (1.13) applies generally to all derivatives satisfying this particular market set-up, and it is therefore denoted the *fundamental PDE* or the Black-Scholes PDE after the paper where it was introduced to finance.

The boundary value problem can be solved directly and approximatively by numerical methods, or analytically by the probabilistic *Feynman-Kač formula.*[10] This formula applies to more general equations, and the solution is in our case given as the probabilistic representation

$$f(S_t, t) = e^{-r(T-t)} E_{s,t}\left[g(Z_T)\right]$$

where $E_{s,t}[\cdot]$ denotes expectation conditioned on $\mathcal{F}_t$ where the initial condition is $Z_t = S_t = s$. From this starting point the process Z evolves as

$$dZ_u = rZ_u du + \sigma Z_u dB_u \tag{1.14}$$

[10]See [4] appendix E for details and a more general version of the Feynman-Kač formula.

The process Z is similar to S, except that the drift rate of Z is not μ but the risk-free rate r. The two processes also start at the same point at the initial time t. The Feynman-Kač formula then changes the drift rate of the underlying security to the risk-free interest rate, and the resulting pay-off at maturity is discounted by that rate.

1.2.2 Example: Traditional Black-Scholes

The above model of a financial market is similar to for example the Black-Scholes model. Let Y be a European call option on the underlying security S specified as in (1.8). The maturity is T and strike price K, so that $g(Z_T) = (Z_T - K)^+$ and $Z_t = S_t = s$. We can derive the Black-Scholes formula as the solution to this pricing problem.

Denote the value of the European call by C. According to the Feynman-Kač formula the time t price of the call in a Black-Scholes market is:

$$C_t = e^{-r(T-t)} E_{s,t}\left[(Z_T - K)^+\right] \tag{1.15}$$

where the underlying security Z will follow a path given by he dynamics of equation (1.14) from the value s at time t. To evaluate this expectation we introduce an indicator function. Let A denote the set of states where the option is exercised, i.e. the set of states where $Z_T \geq K$:

$$A = \{\omega : Z_T(\omega) \geq K\}$$

The indicator function $\mathbf{1}_A$ is then defined by

$$\mathbf{1}_A = \begin{cases} 1 & \text{if } Z_T \geq K \\ 0 & \text{if } Z_T < K \end{cases}$$

i.e. $\mathbf{1}_A$ takes the value 1 if the state of the world is in the set A, otherwise it will be zero. By using the indicator function, the expectation (1.15) is rewritten to

$$C_t = e^{-r(T-t)} \left(E_t\left[Z_T \mathbf{1}_A\right] - E_t\left[K \mathbf{1}_A\right]\right) \tag{1.16}$$

We evaluate the last expectation first, noting that the expectation of the indicator variable is the probability that a state in the set A occur.

$$E_t\left[K \mathbf{1}_A\right] = KP(A|\mathcal{F}_t) = KP\left(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T - B_t)} \geq K|\mathcal{F}_t\right)$$

Take logarithms on both sides of the inequality and rearrange:

$$= KP\left(-\sigma(B_T - B_t) \leq \ln\frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)|\mathcal{F}_t\right)$$

Use that $(B_T - B_t)$ is normally distributed according to property 2 of Brownian motion, and normalize to get a standard normal probability:

$$= KP\left(\frac{\ln\frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \geq \frac{-\sigma(B_T - B_t)}{\sigma\sqrt{T-t}}|\mathcal{F}_t\right) \tag{1.17}$$

Let this familiar expression from the Black-Scholes formula be denoted d_2:

$$= KN(d_2), \quad d_2 = \frac{\ln\frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

where N denotes the cumulative standard normal distribution.

Then go on to evaluate the first expectation in (1.16). Note that, as follows from (1.17), the option will only be exercised when the standard normally distributed variable $(B_T - B_t)/\sqrt{T-t})$ is smaller than or equal to d_2. We can rewrite the expression for Z_T by changing $(B_T - B_t)$ with the equally distributed variable $N\sqrt{T-t}$, where N is a standard normal variable. To evaluate the first expectation in (1.16) it then suffices to use the standard normal density function and integrate over the interval $(-\infty, d_2]$. Remember that the density of the standard normally distributed variable N is given by

$$f(n) = \frac{1}{\sqrt{2\pi}}e^{\frac{-n^2}{2}}$$

Remember also that the expectation of a function $g(x)$ of the random variable X is given by

$$E\left[g(x)\right] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

where $f(x)$ is the density of the random variable X. Apply this to obtain

$$E_t\left[Z_T\mathbf{1}_A\right] = \int_{-\infty}^{d_2} Z_t e^{\left((r-\frac{1}{2}\sigma^2)(T-t)+\sigma n\sqrt{T-t}\right)}\frac{1}{\sqrt{2\pi}}e^{\frac{-n^2}{2}}dn$$

Rearrange to form a familiar complete square in the exponents

$$\begin{aligned} &= \frac{Z_t e^{r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{1}{2}\sigma^2(T-t)+\sigma n\sqrt{T-t}}e^{-\frac{n^2}{2}}dn \\ &= \frac{Z_t e^{r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{1}{2}(n-\sigma\sqrt{T-t})^2}dn \end{aligned}$$

where the random variable $(n - \sigma\sqrt{T-t})$ is normally distributed with variance 1 and mean $-\sigma\sqrt{T-t}$. Making a correction for the non-zero mean we get:

$$
\begin{aligned}
&= Z_t e^{r(T-t)} N(d_2 + \sigma\sqrt{T-t}) \\
&= Z_t e^{r(T-t)} N(d_1), \quad d_1 = \frac{\ln \frac{Z_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

Collecting terms into (1.16) and using $Z_t = S_t$ the value of the European call is then given as the Black-Scholes formula:

$$C_t = S_t N(d_1) - e^{-r(T-t)} K N(d_2)$$

1.2.3 The martingale method

The uncertainty of future cash flows must be taken into account. Risk averse investors demand compensation to carry risk, so the pricing principles we apply must incorporate the risk premium demanded in the market. This is often obtained by discounting uncertain future cash flows with a discount rate higher than the risk free rate. The martingale method is a different approach. Here uncertainty is compensated by *altering the probability measure*. We shall see that altering the probability measure will often make it easier to compute prices — analytically as well as numerically. And we will see that also the martingale method rests on the arguments of replication to construct arbitrage free prices.

We have already encountered probability in the section on the PDE approach. There we applied the Feynman-Kač formula which expressed the call price as an expectation. We should bear in mind that in practical reality we do not know very precisely the probabilities of possible events. And even when we assume that assets behave as we model them (so that probabilities and expectations are implicitly given), we do not know from this alone what is the correct present value of a future expectation — probability and expectation does not provide any principle for the time value. Both the Feynman-Kač formula and the methods in this section result in pricing formulae involving expectations, but we shall see that these expressions are the result of probability measures mimicked in a certain way so that the price of the replicating portfolio can be given in the form of an expectation.

To demonstrate that expectation is useless for pricing when there is a replication strategy available, the standard example is the derivation of the arbitrage-free price of the forward contract. A forward contract is a contract where the holder of the long position will pay on maturity T the forward price $F(0,T)$ to the counter-party in exchange for the underlying asset S. The forward price is the price the parties agree on at the contract date $t=0$ so that the value of the contract at that date is zero. The cash flow resulting from the contract is then zero up to the maturity date, when the long position pays $F(0,T)$ and receives S_T. Why should the parties of the contract not use probability or expectation when they set the forward price $F(0,T)$? The cash flow from the contract on the time of maturity can also be obtained by initiating the following strategy at the contract date: buy the underlying asset S to obtain S_T, and borrow $F(0,T)$ T-bonds (each paying one unit of account at time T) to pay $F(0,T)$ at maturity. As this strategy replicates the cash flow of the forward contract at maturity and through the life time of the contract, it must have the same value as the contract also at the contract date

$$S_0 - F(0,T)P(0,T) = 0$$

so the forward price $F(0,T)$ with maturity T on the asset S is $S_0/P(0,T)$. Any other price would be an arbitrage.

Why does the martingale method use expectation then? Generally expectation does not take time value into account and equals the replication cost or arbitrage free price only by accident. The martingale method however, applies a pseudo probability measure that rests on replication and takes full account of the risk premium. Under a martingale measure all cash flows are then discounted by the same asset (for example a risk free interest rate investment). This means that for any claim discounted by this chosen asset, the expected future value under the pseudo martingale probability measure will equal the value the claim trades for today.

Consider an introductory example. Let us say we want to pin all securities to the risk free interest rate, so that under the pseudo measure the expectation of all securities discounted by the risk free rate will equal the same price the securities trade for today. This means that we want a measure Q that takes care of all risk compensation and leaves the compensation for time value to the risk free rate r. If we denote the bank account β, where $\beta_t = \exp(\int_o^t r_u du)$, the price of the security X can then be expressed as

$$X_t = E_t^Q \left[\exp\left(-\int_t^T r_u du \right) X_T \right] = E_t^Q \left[\frac{\beta_t}{\beta_T} X_T \right]$$

or, equivalently

$$\frac{X_t}{\beta_t} = E_t^Q \left[\frac{X_T}{\beta_T} \right] \tag{1.18}$$

Two points should be noted before we go on. First, the latter form (1.18) of the pricing equation is general not only for discounting with the bank account β — any security can be used to normalize the securities we want to price. Second, there is a one-to-one correspondence between the measure and the security used for normalization. β is called a *numeraire*, and Q is called the *martingale measure associated with the numeraire* β.

We need to sort out the technicalities. First we define what is meant by a *martingale*:

Definition (Martingale) A stochastic process Y is a martingale if it is adapted and satisfies $E_t[Y_u] = Y_t$ for $t \leq u$ and also satisfies the regularity condition $E_t[|Y_u|] < \infty$ for $t \leq u$.[11]

Processes that satisfy both conditions are martingales. Processes that satisfy the first part but not the regularity condition are called *local martingales.*

Note that the process used to model the random (diffusion) part of our price processes, the Brownian motion B, is a martingale: B is adapted to the filtration generated by itself, and by property 2 of the Brownian motion the mean of any forward increment of B is zero.

$$E_t[B_u] = E_t[B_t + \int_t^u dB_s] = B_t, \qquad t \leq u$$

This is under the true probability measure P. P assigns a certain probability $P(\omega)$ to any state ω in the state space Ω. When expectation of B is calculated using the probability distribution assigned by P, B is a martingale. Assume now that we can change the probability measure, by altering the probability distribution over Ω so that B is no longer a martingale. Let us call this new probability measure Q. Also assume that the process B can be changed into a process B^Q that is a Q-martingale. If such a change of probability measure is possible, we can change back and forth between measures obtaining martingales as we like. For the process Y on $(\Omega, \mathcal{F}, P)$ we want to find $F(\omega)$ that satisfies

$$E^Q[Y] = \int_\Omega y dQ(\omega) = \int_\Omega y F(\omega) dP(\omega) = E^P\left[F(\omega)Y\right] \tag{1.19}$$

Such a change of probability measure is sometimes possible and is then performed according to the *Radon-Nikodym theorem* and *Girsanov's theorem.*

$F(\omega)$ is, if it exists, called the *Radon-Nikodym derivative.* From (1.19) we see that it can also be written $\frac{dQ}{dP}$. For the Radon-Nikodym derivative to be well-defined, we can not have one measure assigning positive probability

[11] $E_t[\cdot]$ denotes expectation conditional on the information set $\mathcal{F}_t$ available at time t.

to states the other measure assigns zero probability. We say that the two measures must be *equivalent*.

Definition (Equivalent) The probability measures Q and P on $(\Omega, \mathcal{F})$ are equivalent if $P(A) = 0 \Leftrightarrow Q(A) = 0, \ \forall A \in \mathcal{F}$.

Q being equivalent to P is often denoted $Q \sim P$.[12]

We have seen that martingales can not have drift. The martingale measure must therefore cancel out the drift of a process to turn it into a martingale. This is done by the Girsanov theorem, which introduces another Brownian motion. The process B is a standard Brownian motion under the measure P. According to the Girsanov theorem, B can be transformed into the Q-Brownian motion B^Q by the rule

$$B_t^Q = \int_0^t \eta(s, \omega) ds + B_t$$

for the right choice of η. The dynamics of the process Y is given by

$$\begin{aligned} dY_t &= \mu(t, \omega)dt + \sigma(t, \omega)dB_t \\ &= \mu(t, \omega)dt + \sigma(t, \omega)(dB_t^Q - \eta(t, \omega))dt \\ &= (\mu(t, \omega) - \sigma(t, \omega)\eta(t, \omega))\, dt + \sigma(t, \omega)dB_t^Q \end{aligned}$$

Here η cancels out the drift term and turns Y into a Q-martingale if we have

$$\sigma(t, \omega)\eta(t, \omega) = \mu(t, \omega) \tag{1.20}$$

If η satisfies the necessary conditions to be well-defined, the Radon-Nikodym derivative is defined as follows

$$\frac{dQ(\omega)}{dP(\omega)} = \exp\left(- \int_0^T \eta(s, \omega) dB_s - \frac{1}{2} \int_0^T \eta(s, \omega) \cdot \eta(s, \omega) ds \right) \tag{1.21}$$

η is fundamental in the construction of the martingale measure, and we shall see that we can interpret it to be an important economic variable. But before we go on with this, we sort out the regularity conditions we have to impose on η to be able to construct martingale measures and pricing formulae. The first condition is the *Novikov condition*.

[12] If $(P(A) = 0) \Rightarrow (Q(A) = 0), \ \forall A \in \mathcal{F}$, the measure Q is also said to be *absolutely continuous* w.r.t. P (denoted $Q << P$). Equivalent measures are absolutely continuous with respect to each other, which is necessary for the Radon-Nikodym derivative not to explode.

$$E\left[\exp\left(\frac{1}{2}\int_0^T \eta\cdot\eta\,dt\right)\right] < \infty \tag{1.22}$$

The second condition is that the variance of the Radon-Nikodym derivative (1.21) is finite. If both these conditions are satisfied, we say that η is L^2-*reducible.*[13]

Definition (L^2-reducible) If there is a solution η to the linear equation (1.20) that satisfies the Novikov condition (1.22) and the Radon-Nikodym derivative (1.21) has finite variance, we say that η is L^2-reducible.

The Girsanov theorem states that if a process Y has a solution η to the equation (1.20) and this η is L^2-reducible, then there is an equivalent martingale measure for Y. We are then ready to define what is an equivalent martingale measure.

Definition (Equivalent martingale measure) The measure Q on $(\Omega, \mathcal{F})$ and equivalent to P, is an equivalent martingale measure for the process Y if Y is a martingale under Q and the solution η to the linear equation (1.20) is L^2-reducible.

We have now obtained the necessary tools to obtain a very fundamental result, which is at the core of applying the martingale method to asset pricing:

Theorem 1 *If the price process Y admits an equivalent martingale measure, no admissible self-financing trading strategy in Y is an arbitrage.*

To prove the theorem, let θ be an admissible self-financing trading strategy in Y, which implies that $\theta_0 \cdot Y_0 = \theta_T \cdot Y_T - \int_0^T \theta_t dY_t$. Let Q be an equivalent martingale measure for Y, so that $E^Q\left[\int_0^T \theta_t dY_t\right] = 0$. It follows that

$$\theta_0 \cdot Y_0 = E^Q\left[\theta_T \cdot Y_T - \int_0^T \theta_t dY_t\right] = E^Q\left[\theta_T \cdot Y_T\right]$$

For $\theta_T \cdot Y_T \geq 0$, we must have $\theta_0 \cdot X_0 \geq 0$. Likewise, for $\theta_T \cdot X_T > 0$ we must have $\theta_0 \cdot X_0 > 0$. θ is therefore not an arbitrage.[14]

[13] See [12] on necessary and sufficient regularity conditions and the Novikov condition. The combination of Novikov's condition and finite variance of the Radon-Nikodym derivative (L^2-reducibility) is from [4].

[14] See also [4] (particularly chapter 6 E and G) for proofs and a more rigorous treatment of the existence of an equivalent martingale measure and its relation to the absence of arbitrage.

The contrary implication to that in the theorem above also applies: if there is no arbitrage, there exists an equivalent martingale measure. This is not as extensively treated in the literature, and we shall not review it further here. Proof is given in [4] chapter 6 K and [12].

For the purpose of arbitrage pricing, a benchmark price process may be convenient. A chosen *numeraire* can be used to normalize or deflate the price processes in the market. If Y is the chosen numeraire process, the inverse $\frac{1}{Y}$ is called a *numeraire deflator*. The price process deflated by itself will always be 1, and all other price processes will be related to the numeraire. If there is no arbitrage, there will be a martingale measure under which all deflated price processes are martingales. Thus there is a correspondence between the equivalent martingale measure and the numeraire. We shall see later that we can change martingale measure or numeraire, and that a change of numeraire corresponds to a change of measure as we already have done from P to Q. We shall also see that calculations can be made easier by a change of numeraire (choice of which process to be constantly equal to 1).

A much used numeraire is the bank account $\beta_t = \beta_0 \exp(\int_o^t r_u du)$, normally with initial investment $\beta_0 = 1$. The short rate r_t may have the form $r(t, \omega)$, but it is assumed to be bounded and in some cases modeled as a constant. The dynamics of the bank account will be

$$d\beta_t = r_t \beta_t dt$$

and we say that the bank account is *locally risk free* as it is independent of the Brownian motion B. From now on we say that the market consists of the $n+1$ price processes expressed as (X, β).

The equivalent martingale measure corresponding to the numeraire β is the measure Q which satisfies, as we have already seen:

$$\frac{X_t}{\beta_t} = E_t^Q \left[\frac{X_T}{\beta_T} \right]$$

or, equivalently

$$X_t = E_t^Q \left[\frac{\beta_t}{\beta_T} X_T \right]$$

for all $t \leq T$. From now on we let Q denote the equivalent martingale measure corresponding to using this numeraire. The market deflated by β is denoted X^β. Assume for the simplicity of the following example that X is one-dimensional.

$$X^\beta = \frac{X_t}{\beta_t} = \frac{X_0}{\beta_0} \exp\left((\mu - \frac{1}{2}\sigma^2 - r)t + \sigma B_t \right)$$

and its P-dynamics are by the Itô formula

$$\begin{aligned} dX_t^\beta &= \left(\frac{\partial X^\beta}{\partial t} + \frac{1}{2}\frac{\partial^2 X^\beta}{\partial B_t^2}\right) dt + \frac{\partial X^\beta}{\partial B_t} dB_t \\ &= (\mu - r) X^\beta dt + \sigma X^\beta dB_t \end{aligned}$$

Using Girsanov's theorem, we look for a market price of risk η to make the following apply:

$$\begin{aligned} dX_t^\beta &= (\mu - r) X^\beta dt + \sigma X^\beta \left(dB_t^Q - \eta dt\right) \\ &= (\mu - r - \eta) X^\beta dt + \sigma X^\beta dB_t^Q \end{aligned}$$

For X^β to be a martingale, η must satisfy

$$\sigma\eta = \mu - r$$

so we have

$$\eta = \frac{\mu - r}{\sigma}$$

Assuming that this η is L^2-reducible (for constant μ, σ and r this is trivially the case), the Q-dynamics of the non-deflated process X is

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t \left(dB_t^Q - \frac{\mu - r}{\sigma}\right) dt \\ &= r X_t dt + \sigma X_t dB_t^Q \end{aligned}$$

so under the martingale measure Q, the non-deflated price process X has the same drift rate as the numeraire. This intuitively corresponds to the deflated price process being a martingale. This result, however, is particular to the numeraire β. For a numeraire with diffusion dynamics, and under the corresponding martingale measure, the second order derivatives will not vanish but be included in the drift term and make the drift of the deflated price processes different from that of the numeraire process. The connection between drift and diffusion in η achieves an *adjustment for risk*: The martingale method uses the probability measure (and nothing else) to adjust for investor risk aversion. Martingale measures are sometimes called risk-neutral measures: In a market where all price processes are martingales, variations in risk or diffusion is not compensated. Note that under Q the diffusion is not changed. We say that diffusion is preserved under a change of measure. The drift adjustment η is called *the market price of risk*. It is also interesting to notice that in this example the market price of risk equals the Sharpe ratio, which is a well-known expression for risk premium.

For technical reasons, we sometimes want to change from one numeraire to another. This is because the price process deflated by itself will be constant, so a clever choice of numeraire can simplify price expressions and make calculations easier. The numeraire is used to fix a reference for the other assets in the market. This means that there must be a correspondence the numeraire and the martingale measure for the market deflated by this numeraire. We shall see more on this correspondence in the following. For now let Y and U be to assets we will use as numeraires, and let Q^Y and Q^U be equivalent martingale measures corresponding to deflating with Y and U, respectively.[15]

To derive the change of numeraire/change of measure formula, start with a look at the following two ways of expressing the deflated price process X as a martingale expectation:

$$X_t = E_t^{Q^Y}\left[\frac{Y_t}{Y_T}X_T\right] = \int_\Omega \frac{Y_t}{Y_T}X_T dQ^Y$$

$$X_t = E_t^{Q^U}\left[\frac{U_t}{U_T}X_T\right] = \int_\Omega \frac{U_t}{U_T}X_T dQ^U$$

The following equality must be true

$$\frac{Y_t}{Y_T}dQ^Y = \frac{U_t}{U_T}dQ^U$$

or after rearranging

$$\frac{dQ^Y}{dQ^U} = \frac{U_t/U_T}{Y_t/Y_T} \tag{1.23}$$

Accordingly, a change of measure is then performed like this:

$$X_t = E_t^{Q^Y}\left[\frac{Y_t}{Y_T}X_T\right] = E_t^{Q^U}\left[\frac{dQ^Y}{dQ^U}\frac{Y_t}{Y_T}X_T\right] = E_t^{Q^U}\left[\frac{U_t}{U_T}X_T\right]$$

This change of measure technique is a result general to any two available equivalent martingale measures and corresponding numeraire processes. An early review of this technique and proof of it was given in [5].

[15] Note that if $P \sim Q^Y$ and $P \sim Q^U$, it must also be true that $Q^Y \sim Q^U$.

The market (X, β) is a set of linear equations in dt (drift) and dB_t (diffusion). This linearity is exploited in the construction of the martingale measure, and it is also essential for the replication of price processes. Using the linearity, we may construct a *hedge portfolio* to cancel out the diffusion part of our investments or we may construct a martingale measure to cancel drift. We have seen that (1.20) gives a set of linear equations for η to solve, and that if there is a solution (or several) there is no arbitrage. The hedging possibilities also depend on properties of the linear equations constructing the market. To replicate any particular claim, we need the available assets to span a space of the same dimension as the random process B. This is equal to having available as many linearly independent assets as there are dimensions of B. This property is called *completeness*. We say that a *T-claim* is a claim of the form $f(B_T)$, and say that the market is *complete* if the payoff from any T-claim can be replicated by a linear combination of the marketed assets.

Definition (Complete) The deflated market $Y_t = \mu(t,\omega)dt + \sigma(t,\omega)dB_t$, where B is of dimension d, is complete if and only if the diffusion matrix σ is of rank d.

There is not a straightforward connection between completeness and (absence of) arbitrage. In an incomplete market, i.e. where rank $(\sigma) < d$, the possibility of arbitrage depends on the availability of solutions η to the equation (1.20). There may be no solution and arbitrage, or there may be multiple solutions and no arbitrage. In a complete market there must be a solution η and there can not be arbitrage. Moreover, the solution in a complete deflated market must be unique.

To see that the martingale measure must be unique for a deflated and complete market, remember that the rank of the diffusion matrix (or the number of linearly independent assets) equals the dimension of the market price of risk vector η. Then there can not be any free variables in η, and it follows that the martingale measure is uniquely determined. To see that there must be an equivalent martingale measure at all, remember that in a complete market any claim can be replicated. Such a market will not permit arbitrage, and if there is no arbitrage there must be an equivalent martingale measure. The uniqueness of the martingale measure in a complete market is a general result: If a market is complete, the deflated market will have only one equivalent martingale measure for each numeraire deflator. The opposite is also true: If the martingale measure is uniquely determined for a given numeraire, the market is complete. In complete markets there is then a one-to-one correspondence between numeraire and martingale measure.

The theoretical foundation for replication still remains. We have already stated that in a complete market, we can use the linear form of available asset price dynamics to span the entire set of T-claims. The tool we use is the *Martingale representation theorem*: If the process M is a martingale with respect to the filtration generated by the Brownian motion B, there is a unique Itô-integrable process v such that

$$M_t = M_0 + \int_0^t v_u dB_u$$

Let there be a T-claim F and a self-financing trading strategy θ with value process V in the complete market (X, β). By completeness θ may be a replicating portfolio, meaning that

$$V_T = F(T, \omega)$$

We know that all deflated price processes are martingales under the equivalent martingale measure. For example are V^β and F^β Q-martingales. We then have

$$V_T^\beta = V_0^\beta + \int_0^T \beta^{-1}\theta\sigma dB_t^Q$$

where θ is unique by the martingale representation theorem. As θ is a replicating portfolio and there is no arbitrage, V^β must satisfy

$$V_t^\beta = E_t^Q\left[V_T^\beta\right] = E_t^Q\left[\beta_T^{-1}F(T,\omega)\right]$$

and V_t^β is the β-deflated price of F at any time $t \in [0, T]$.

The general pricing formula of the martingale method is then, expressed under the martingale measure Q^Y corresponding to the numeraire Y:

$$F_t = E_t^{Q^Y}\left[\frac{Y_t}{Y_T}F(T,\omega)\right] \tag{1.24}$$

There are certain similarities between this pricing formula (1.24) and the theory of *state prices* (state price deflator) or *Arrow-Debreu prices* and *stochastic discount factors*. Use the Radon-Nikodym derivative $\frac{dQ^Y}{dP}$ to rewrite the above expression to an ordinary P-expectation

$$F_t = E_t\left[\frac{dQ^Y}{dP}\frac{Y_t}{Y_T}F(T,\omega)\right] \tag{1.25}$$

where the term $\frac{dQ^Y}{dQ}\frac{Y_t}{Y_T}$ will be a random variable that can be interpreted as a state price deflator or a stochastic discount factor. Note also that in the general pricing formulae (1.24) and (1.25) utility is not mentioned. This will appear more clearly in the following section where we derive a fully specified pricing formula for a derivative. Pricing formulae include asset prices (this follows from the replication argument), and we assume that asset prices are the result of investor preferences and optimization.

1.2.4 Example: Modern Black-Scholes

As a demonstration of the martingale method, we will again derive the Black-Scholes formula. The formula was previously derived by the use of the Feynman-Kač formula and the traditional approach. Here we obtain the same result by applying the change of numeraire technique.

In the Black-Scholes market there are two marketed assets, the stock S and the bank account β. Price processes are given on the probability space $(\Omega, \mathcal{F}, Q)$, where the filtration $\mathbf{F} = \{\mathcal{F}_t : t \in [0, T^*]\}$ is generated by the standard Q-Brownian motion B:

$$\begin{aligned} S_t &= S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\right) \\ \beta_t &= \beta_0 \exp(rt) \end{aligned}$$

Let the bank account with initial investment $\beta_0 = 1$ be the numeraire. The deflated price process S^β is:

$$S_t^\beta = \frac{S_t}{\beta_t} = S_0 \exp\left((\mu - r - \frac{1}{2}\sigma^2)t + \sigma B_t\right) \tag{1.26}$$

Use the Itô formula to find the dynamics of the deflated stock price process:

$$\begin{aligned} dS_t^\beta &= \left(\mu - r - \frac{1}{2}\sigma^2\right) S_t^\beta dt + \sigma S_t^\beta dB_t + \frac{1}{2}\sigma^2 S_t^\beta (dB_t)^2 \\ &= (\mu - r)\, S_t^\beta dt + \sigma S_t^\beta dB_t \end{aligned}$$

As β is now the numeraire, β^β is always 1 and trivially a martingale. To see if the Black-Scholes market permits an equivalent martingale measure, i.e. excludes arbitrage, we must solve for η in the following:

$$\begin{aligned} (\mu - r)\, S_t^\beta dt + \sigma S_t^\beta dB_t &= \sigma S_t^\beta dB_t^Q \\ &= \sigma S_t^\beta (dB_t + \eta_t dt) \\ (\mu - r)\, S_t^\beta dt &= \sigma S_t^\beta \eta_t dt \\ \eta_t &= \frac{\mu - r}{\sigma} \end{aligned}$$

So there is a solution η to the market price of risk equation, and η satisfies the Novikov condition as μ, σ and r are constant. Then there is an equivalent martingale measure for the Black-Scholes market.

We shall then find the price of a T-claim: the European call option on S with strike price K. By the martingale pricing formula (1.24), the price of this claim C at time t is

$$C_t = E_t^Q \left[\frac{\beta_t}{\beta_T} (S_T - K)^+ \right] \tag{1.27}$$

Rewrite this expression by using the given dynamics of β and by introducing indicator functions. The value of the indicator function $\mathbf{1}_A$ will be 1 if the state of the world is in the set A, otherwise it will be 0. In our case, define:

$$A = \{\omega : S_T(\omega) \geq K\}$$

Equation (1.27) then becomes:

$$C_t = E_t^Q \left[e^{-r(T-t)} S_T \mathbf{1}_A \right] - E_t^Q \left[e^{-r(T-t)} K \mathbf{1}_A \right] \tag{1.28}$$

To evaluate the first term of (1.28), change numeraire from β to S by the use of (1.23):

$$\begin{aligned} E_t^Q \left[e^{-r(T-t)} S_T \mathbf{1}_A \right] &= E_t^{Q^S} \left[\frac{dQ}{dQ^S} e^{-r(T-t)} S_T \mathbf{1}_A \right] \\ &= E_t^{Q^S} \left[\frac{S_t/S_T}{e^{-r(T-t)}} e^{-r(T-t)} S_T \mathbf{1}_A \right] \\ &= S_t E_t^{Q^S} [\mathbf{1}_A] \end{aligned}$$

This expectation is equal to the probability of A occurring evaluated under the probability measure Q^S:

$$E_t^{Q^S} [\mathbf{1}_A] = Q^S(A \mid \mathcal{F}_t) \tag{1.29}$$

Before we evaluate this Q^S-probability, consider the term involving the strike price K in (1.28). This term can be rewritten to a Q-probability. This is a result of constant interest rate r, making β deterministic:[16]

[16]For stochastic interest rates, a change of numeraire to the zero-coupon bond paying 1 at maturity T (the T-bond) and the corresponding martingale measure Q^T (called the T-forward measure) will give a Q^T-probability. This is, of course, assuming that the T-bond is a traded asset in the market.

$$E_t^Q\left[e^{-r(T-t)}K\mathbf{1}_A\right] = e^{-r(T-t)}KQ(A \mid \mathcal{F}_t) \tag{1.30}$$

To evaluate this Q-probability, remember that β is the numeraire of the Q-measure so that $\frac{S_t}{\beta_t}$ is a Q-martingale. The process expressed under the measure Q in (1.26) can then also be expressed as:

$$S_T^\beta = \frac{S_T}{\beta_T} = \frac{S_t}{\beta_t}\exp\left(-\frac{1}{2}\sigma^2(T-t) + \sigma(B_T^Q - B_t^Q)\right) \tag{1.31}$$

We then go on to evaluate the Q-probability in (1.30). Use that β is numeraire of the Q-measure and deflate. Use also that the price processes are exponential and take logarithms:

$$Q\left(A \mid \mathcal{F}_t\right) = Q\left(S_T \geq K \mid \mathcal{F}_t\right) = Q\left(\ln\frac{S_T}{\beta_T} \geq \ln\frac{K}{\beta_T} \mid \mathcal{F}_t\right) \tag{1.32}$$

Using (1.31) we get:

$$= Q\left(\ln S_t - rt - \frac{1}{2}\sigma^2(T-t) + \sigma(B_T^Q - B_t^Q) \geq \ln K - rT\right)$$

Rewrite to:

$$= Q\left(\ln\frac{S_t}{K} + r(T-t) - \frac{1}{2}\sigma^2(T-t) \geq -\sigma(B_T^Q - B_t^Q)\right)$$

B^Q is a standard Brownian motion under Q. In the Black-Scholes market it is one-dimensional, so by property 2 of Brownian motion the random variable $-\sigma(B_T^Q - B_t^Q)$ is normally distributed with zero mean and variance $\sigma^2(T-t)$. Normalize to obtain the following cumulative probability of the standard normal distribution:

$$\begin{aligned}
&= Q\left(\frac{\ln\frac{S_t}{K} + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \geq \frac{-\sigma(B_T^Q - B_t^Q)}{\sigma\sqrt{T-t}}\right) \\
&= N(d_2), \quad d_2 = \frac{\ln\frac{S_t}{K} + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}$$

where N denotes the cumulative standard normal distribution.

The Q^S-probability in (1.29) can be evaluated similarly. Under Q^S the deflated price process β^S is a martingale. This process can be expressed as:

$$\beta_T^S = \frac{\beta_T}{S_T} = \frac{\beta_t}{S_t}\exp\left(-\frac{1}{2}\sigma^2(T-t) - \sigma(B_T^{Q^S} - B_t^{Q^S})\right) \tag{1.33}$$

Proceeding as in (1.32):

$$Q^S\left(A \mid \mathcal{F}_t\right) = Q^S\left(S_T \geq K \mid \mathcal{F}_t\right) = Q^S\left(\ln\frac{\beta_T}{K} \geq \ln\frac{\beta_T}{S_T} \mid \mathcal{F}_t\right)$$

Using (1.33):

$$= Q^S\left(rT - \ln K \geq rt - \ln S_t - \frac{1}{2}\sigma^2(T-t) - \sigma(B_T^{Q^S} - B_t^{Q^S})\right)$$

Rewriting and normalizing:

$$\begin{aligned} &= Q^S\left(\frac{\ln\frac{S_t}{K} + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \geq \frac{-\sigma(B_T^{Q^S} - B_t^{Q^S})}{\sigma\sqrt{T-t}}\right) \\ &= N(d_1), \quad d_1 = \frac{\ln\frac{S_t}{K} + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

Collecting terms, (1.28) is then solved to

$$\begin{aligned} C_t &= S_t Q^S\left(S_t \geq K\right) - e^{-r(T-t)}KQ\left(S_t \geq K\right) \\ &= S_t N(d_1) - e^{-r(T-t)}KN(d_2) \end{aligned}$$

where the random variables d_1 and d_2 are as given above. This is the Black-Scholes formula for the price of a European call.

Chapter 2

Market Models of the Interest Rate

Having reviewed the general theory of asset pricing, we will investigate more specific applications of the theory. We will look at a particular kind of interest rate models, the market models, and we will look into a limitation arising when a certain extension is performed in these models.

With the introduction of the LIBOR market model in the mid 1990-ies, a term structure model was given for interest rates as they are observed in the market. By modelling discretely compounded or simple rates, the LMM gives pricing formulae applicable to the real market interest rates. As these rates are log-normal in the model, the LMM supports market practice of applying closed-form Black or Black-Scholes type pricing formulae in the interest rate market.

Since the Black and Scholes paper was published in 1973, their formula has been applied by market practitioners also to price term structure derivatives. In the market for caps and swaptions, prices were quoted in terms of implied Black-Scholes volatilities. This practice was based on analogy and implicitly assuming log-normal interest rates. This implicit modelling of the instantaneous, continuously compounded interest rate as log-normal, has been shown to cause the problem of exploding rates, as explained in [16]. They proposed instead to model simple rates as log-normal. Furthermore, interest rate derivatives are written on rates that are simple (or discretely compounded), such as e.g. 3-month LIBOR. A log-normal no-arbitrage term structure of simple LIBOR-type interest rates, justifying the various closed-form solutions presented, was then given in [3], [7] and [10].

Models previous to the LMM also offer closed-form solutions for interest rate derivatives. However, these models are all short rate models, so their pricing formulae may be mathematically elegant for short rate derivatives, but the underlying short rate is not observable in the market. The pricing of contracts written on the simple rates given in the market is then complicated. Many of the short rate models also imply negative interest rates with positive probability. See [10] and [4] chapter 7C for more on this. In the LMM, LIBOR-type interest rates are log-normally distributed. Log-normal rates will not be negative, so arbitrage cannot result from cost- and riskless storage of money.

2.1 The LIBOR

The LIBOR, or the London Interbank Offered Rate, is a rate offered for dollar loans to creditors with a credit rating of at least AA.[1] All LIBOR loans follow the same conventions: The basis is in years, for the α-LIBOR interest is compounded for α-fractions of one year and the α-LIBOR is fixed for the time interval from today (time t) to an α-fraction of one year out into the future (time $t+\alpha$). This means that if e.g. the 3-month LIBOR is 4%, a loan of 1.000.000 today will have accrued interest of $\alpha \times \text{LIBOR rate} \times \text{sum} = \frac{3}{12} \cdot 0,04 \cdot 1.000.000 = 10.000$ in 3 months from now. The LIBOR may also be a forward rate. To get a forward LIBOR we just move the interval $[t, t+\alpha]$ forward in time.

We then define the forward LIBOR $L(t, T, \alpha)$ as the rate given at time t for the time interval $[T, T+\alpha]$. Let $P(t, T)$ be the time t price of the default-free zero-coupon bond paying one unit of account at time T, $0 \leq t \leq T \leq T^* - \alpha$, where T^* is our time horizon. $P(t, T)$ may also be denoted the *discount bond of maturity* T, or simply the T*-bond*. We link the bonds to the forward interest rate by the following arbitrage argument. Let one investment strategy be to buy one $[T, T+\alpha]$-forward rate contract at time t. Another strategy is to sell one T-bond and buy a fraction $\frac{P(t,T)}{P(t,T+\alpha)}$ of the $(T+\alpha)$-bond. The cash flow from the two strategies will be as follows:

strategy\time	t	T	$T+\alpha$
forward rate	0	-1	$1+\alpha L(t,T,\alpha)$
bonds	$P(t,T) - \frac{P(t,T)}{P(t,T+\alpha)} P(t, T+\alpha)$	-1	$\frac{P(t,T)}{P(t,T+\alpha)}$

Table 1. Determining $L(t, T, \alpha)$.

We see that the the two strategies have identical cash flows at time t and T. As we do not have any cash flows at times other than t, T and $T+\alpha$, the

[1] LIBOR loans are then defaultable, but we will not consider credit risk in this paper. [10] give references to several papers incorporating credit risk into the model by adjusting the volatility function.

two strategies must by absence of arbitrage have identical cash flows also at time $T+\alpha$. We then solve for L to get:

$$L(t,T,\alpha) = \frac{1}{\alpha}\left(\frac{P(t,T)}{P(t,T+\alpha)} - 1\right) \tag{2.1}$$

It is also the case that the LIBOR itself can be replicated by trading a portfolio of bonds of the relevant maturities. The following replicating strategy is given by [9]. Consider the LIBOR $L(t,T,\alpha)$ and a portfolio consisting of the following positions:

At time t:

sell a number $\frac{1}{\alpha}$ of the forward contract with settlement date T on the $(T+\alpha)$-bond with time t forward price $\frac{P(t,T+\alpha)}{P(t,T)}$

buy a number $\frac{1}{\alpha}\left(1-\frac{P(t,T+\alpha)}{P(t,T)}\right)$ of the T-bond

As the forward contract by definition has value 0 at the contract date t, the only cash flow at this date is from the bond:

$$-\frac{1}{\alpha}\left(1-\frac{P(t,T+\alpha)}{P(t,T)}\right)P(t,T)$$

At time T:

receive $\frac{1}{\alpha}\frac{P(t,T+\alpha)}{P(t,T)}$ in exchange for the commitment to pay $\frac{1}{\alpha}$ units of account on the date $(T+\alpha)$

receive $\frac{1}{\alpha}\left(1-\frac{P(t,T+\alpha)}{P(t,T)}\right)$ from the T-bond

buy a number $\frac{1}{\alpha P(T,T+\alpha)}$ of the $(T+\alpha)$-bond

Cash flow in at date T is then equal to cash flow out, so there is only a rebalancing of the portfolio at this time:

$$\begin{aligned} \text{cash flow in} &= \text{cash flow out} \\ \frac{1}{\alpha}\frac{P(t,T+\alpha)}{P(t,T)} + \frac{1}{\alpha}\left(1-\frac{P(t,T+\alpha)}{P(t,T)}\right) &= \frac{1}{\alpha P(T,T+\alpha)}P(T,T+\alpha) \end{aligned}$$

At the final date $T+\alpha$ the following takes place:

pay $\frac{1}{\alpha}$ as was committed by the forward contract

receive $\frac{1}{\alpha P(T,T+\alpha)}$ from the position in the $(T+\alpha)$-bond

Net cash flow at date $T+\alpha$ is then equal to the rate $L(T,T,\alpha)$:

$$-\frac{1}{\alpha}+\frac{1}{\alpha P(T,T+\alpha)}=\frac{1}{\alpha}\left(\frac{1}{P(T,T+\alpha)}-1\right)$$

This portfolio is rebalanced at time T at zero cost and the final cash flow from the portfolio, at time $T+\alpha$, is identical to $L(T,T,\alpha)$ by our previous definition of the LIBOR in (2.1).

Absence of arbitrage implies that under an equivalent martingale measure, any self-financing trading strategy discounted with the respective numeraire must be a martingale. By the strategy presented above, we know that the LIBOR $L(t,T,\alpha)$ can be replicated. We also saw that net cash flow from the replicating portfolio will be zero at all other times than t and $T+\alpha$. This portfolio is then a self-financing trading strategy, and as such should be a martingale when properly discounted. Define the *forward measure* $Q^{T+\alpha}$ as the equivalent martingale measure having the $(T+\alpha)$-bond as numeraire. This numeraire has value $P(t,T+\alpha)$ at time t and value 1 at time $T+\alpha$. In general, if there is no arbitrage any traded asset Y should satisfy

$$\frac{Y_t}{P(t,T+\alpha)}=E_t^{Q^{T+\alpha}}\left[Y_{T+\alpha}\right] \tag{2.2}$$

And this is the case with the LIBOR interest rate! The self-financing and replicating strategy pays $L(T,T,\alpha)$ at time $T+\alpha$. At time t the value was $\frac{1}{\alpha}\left(1-P(t,T+\alpha)\right)$. Divide by the time t value of the numeraire as in the left-hand side of (2.2) to get $L(t,T,\alpha)$. So the LIBOR rate to be paid at date $T+\alpha$ is then itself a $(T+\alpha)$-martingale.

2.2 The LIBOR market model

As any LIBOR is a martingale under the appropriate forward measure, the dynamics of $L(t,T,\alpha)$ under $Q^{T+\alpha}$ can then be given by the *stochastic differential equation* (SDE)

$$dL(t,T,\alpha)_t=L(t,T,\alpha)\lambda(\cdot)dB_t^{T+\alpha}, \qquad t\in[0,T]$$

where $B^{T+\alpha}$ is standard Brownian motion under $Q^{T+\alpha}$ and $L(t,T,\alpha)$ satisfies the initial condition

$$L(0,T,\alpha) = \frac{1}{\alpha}\left(\frac{P(0,T)}{P(0,T+\alpha)} - 1\right)$$

To be able to arrive at pricing formulae of the Black and Scholes type, we need $L(\cdot)$ to be log-normal. We know that increments of $\{B_t^{T+\alpha}\}$ are normally distributed (with expectation zero) under $Q^{T+\alpha}$, so for $L(\cdot)$ to be log-normal it suffices that the *volatility function* $\lambda(\cdot)$ is deterministic. We also note that if this is satisfied, $L(\cdot)$ will be log-normal under any measure equivalent to Q and $Q^{T+\alpha}$. This follows from the Girsanov theorem, telling us that a change of measure will only change the drift of the process, and that the drift will change as a function of the (deterministic) volatility only. The model for $L(t,T,\alpha)$ is then set to be

$$dL(t,T,\alpha)_t = L(t,T,\alpha)\lambda(t,T,\alpha)dB_t^{T+\alpha}, \qquad t \in [0,T]$$

where the volatility function $\lambda(t,T,\alpha)$ is state independent and $L(t,T,\alpha)$ satisfies the initial condition (6) above The differential is with respect to the current time variable t. The solution to this SDE is the geometric Brownian motion

$$L(t,T,\alpha) = L(0,T,\alpha)\exp\left(-\frac{1}{2}\int_0^t \lambda(s,T,\alpha)^2 ds + \int_0^t \lambda(s,T,\alpha)dB_s^{T+\alpha}\right) \tag{2.3}$$

2.2.1 Example: caplet formula from LIBOR market model

To exemplify, we apply the LIBOR market model to price an interest rate derivative. This will demonstrate how closely linked the LIBOR market model for interest rates is to the Black-Scholes model. A *caplet* is the interest rate equivalent to the call option: If the interest rate at maturity is above an agreed level, the cap rate, the caplet pays the difference between the actual interest rate and the cap rate for an agreed notional amount and accrual period. In the case of LIBOR rates, the period is of length α. The caplet gives one single payment, but caplet contracts for subsequent maturities in the tenor structure can be put together to a *cap contract*.

The caplet or cap is then suitable for floating rate debtors wanting to hedge the risk of high interest rates. With a cap contract the cost of the debt can be capped as interest above the cap rate can be offset by gains from the cap contract. The caplet and cap contracts are therefore often referred to as debtor insurance or protection against high interest rate.

For a loan accruing floating LIBOR over the interval $[T, T+\alpha]$, the interest will be random until the LIBOR $L(T,T,\alpha)$ is fully determined (i.e. no longer stochastic) at time T. At any time t prior to T a debtor with such a loan can cap his interest due at time $T+\alpha$ by buying a caplet contract. The caplet on this forward LIBOR with cap rate $\overline{L}$ is a contract that pays the difference $\left(L(T,T,\alpha)-\overline{L}\right)^+$ times the length of the time interval times the face value. This cash flow is paid at time $T+\alpha$, the date when $L(T,T,\alpha)$-loans are due. We say that payment is made *in arrear*. As $L(T,T,\alpha)$ is determined at time T the amount to be paid from the caplet is also completely determined at time T — i.e. given $\mathcal{F}_T$ the amount $\left(L(T,T,\alpha)-\overline{L}\right)^+$ is no longer a random variable.

$$\text{caplet at time } T+\alpha = \alpha V(L(T,T,\alpha)-\overline{L})^+$$

As the forward LIBOR $L(T,T,\alpha)$ is a martingale under the $T+\alpha$ forward measure, the time t value of the caplet can be given as

$$\text{caplet at time } t = \alpha V P(t,T+\alpha) E_t^{Q^{T+\alpha}}\left[(L(T,T,\alpha)-\overline{L})^+\right]$$

This expectation can be evaluated using the same indicator function trick as we have already used to derive the Black-Scholes formula. Let D denote the exercise set of the caplet, so that the indicator function $\mathbf{1}_D$ takes the value 1 if the state of the world is in the exercise set D and zero otherwise. The caplet at time t is

$$= \alpha V P(t,T+\alpha)\left(E_t^{Q^{T+\alpha}}[L(T,T,\alpha)\mathbf{1}_D] - E_t^{Q^{T+\alpha}}[\overline{L}\mathbf{1}_D]\right) \tag{2.4}$$

We evaluate the last expectation first. Use the solution (2.3) of the LIBOR to compute this $Q^{T+\alpha}$-probability. Note also that the forward LIBOR is completely determined at time T, so the volatility function λ is only integrated up to this date.

$$\begin{aligned}
E_t^{Q^{T+\alpha}}[\overline{L}\mathbf{1}_D] &= \overline{L}Q^{T+\alpha}(L(T,T,\alpha) \geq \overline{L} \mid \mathcal{F}_t) \\
&= \overline{L}Q^{T+\alpha}\left(\ln L(t,T,\alpha) - \frac{1}{2}\int_t^T \lambda^2 ds + \int_t^T \lambda dB_s^{T+\alpha} \geq \ln \overline{L}\right) \\
&= \overline{L}Q^{T+\alpha}\left(\ln \frac{L(t,T,\alpha)}{\overline{L}} - \frac{1}{2}\int_t^T \lambda^2 ds \geq -\int_t^T \lambda dB_s^{T+\alpha}\right)
\end{aligned}$$

The random variable $\int_t^T \lambda(s,T,\alpha)dB_s^{T+\alpha}$ is normally distributed with zero mean and variance $\int_t^T \lambda(s,T,\alpha)^2 ds$. Use this to normalize:

$$
\begin{aligned}
&= \overline{L}Q^{T+\alpha}\left(\frac{\ln\frac{L(t,T,\alpha)}{\overline{L}} - \frac{1}{2}\int_t^T \lambda^2 ds}{\sqrt{\int_t^T \lambda^2 ds}} \geq -\frac{dB_T^{T+\alpha} - dB_t^{T+\alpha}}{\sqrt{\int_t^T \lambda^2 ds}}\right) \\
&= \overline{L}N(a_2)
\end{aligned}
$$

Where the value a_2 written in full notation is

$$
a_2 = \frac{\ln\frac{L(t,T,\alpha)}{\overline{L}} - \frac{1}{2}\int_t^T \lambda(s,T,\alpha)^2 ds}{\sqrt{\int_t^T \lambda(s,T,\alpha)^2 ds}} \tag{2.5}
$$

To evaluate the first expectation in (2.4), we proceed as we did when calculating the Black-Scholes formula from Feynman-Kač. Let Z denote a standard normally distributed random variable which will be used to replace $(B_T^{T+\alpha} - B_t^{T+\alpha})$. We can then rewrite the LIBOR to

$$
L(t,T,\alpha) = L(0,T,\alpha)\exp\left(-\frac{1}{2}\int_0^t \lambda(s,T,\alpha)^2 ds + z\int_0^t \lambda(s,T,\alpha)ds\right)
$$

The exercise set is the set of states where $z \leq a_2$. We then have

$$
\begin{aligned}
& E_t^{Q^{T+\alpha}}[L(T,T,\alpha)\mathbf{1}_D] \\
&= \int_{-\infty}^{\infty} L(T,T,\alpha)\mathbf{1}_D df(z) \\
&= \frac{L(t,T,\alpha)}{\sqrt{2\pi}}\int_{-\infty}^{a_2} \exp\left(-\frac{1}{2}\int_t^T \lambda^2 ds + z\sqrt{\int_t^T \lambda^2 ds}\right)\exp\left(\frac{-z^2}{2}\right)dz \\
&= \frac{L(t,T,\alpha)}{\sqrt{2\pi}}\int_{-\infty}^{a_2} \exp\left(-\frac{1}{2}\left(z - \sqrt{\int_t^T \lambda^2 ds}\right)^2\right)dz \\
&= L(t,T,\alpha)N\left(a_2 + \sqrt{\int_t^T \lambda^2 ds}\right) \\
&= L(t,T,\alpha)N(a_1)
\end{aligned}
$$

Where the value a_1 is given as

$$
a_1 = \frac{\ln\frac{L(t,T,\alpha)}{\overline{L}} + \frac{1}{2}\int_t^T \lambda(s,T,\alpha)^2 ds}{\sqrt{\int_t^T \lambda(s,T,\alpha)^2 ds}}
$$

The LIBOR market model caplet formula can then be summed up to

$$\begin{aligned}\text{caplet at time } t &= \alpha V P(t, T+\alpha) E_t^{Q^{T+\alpha}} \Big[(L(T,T,\alpha) - \overline{L})^+\Big] \\ &= \alpha V P(t, T+\alpha) \Big(L(t,T,\alpha) N(a_1) - \overline{L} N(a_2)\Big)\end{aligned}$$

2.3 The swap rate market model

We may also use the idea from Jamshidian [7] of modelling a term structure of swap rates. When we model the LIBOR, dynamics are given under the forward measure with the discount bond as numeraire. Here we model the swap rate dynamics under the *forward swap measure*, where the numeraire is what we will call the *annuity*. But first we will have a closer look at the contract of rate-swapping.

A swap contract has one party paying floating rate and receiving fixed, and vice versa for the other party. Payoff from the floating leg of the swap contract where rates are swapped over the time interval $[T_0, T_n]$ is a series of interest payments according to the forward LIBORs maturing in the interval. Each payment has the value $\alpha L(T_{i-1}, T_{i-1}, \alpha)$ at its payoff date T_i. The time t value of the floating leg is then, using that the LIBOR is a martingale under the forward measure of its payoff date:

$$\begin{aligned}\text{floating leg} &= \sum_{i=1}^{n} P(t,T_i) \alpha E_t^{Q^{T_i}} [L(T_{i-1}, T_{i-1}, \alpha)] \\ &= \sum_{i=1}^{n} P(t,T_i) \alpha L(t, T_{i-1}, \alpha)\end{aligned}$$

Substitute in for $L(\cdot)$ from equation (1) to get a telescoping series folding up into

$$= P(t,T_0) - P(t,T_n)$$

Each interest payment to the fixed leg with the fixed rate s is of the amount αs, so the time t value of the fixed leg is

$$\begin{aligned}\text{fixed leg} &= \sum_{i=1}^{n} P(t,T_i) \alpha s \\ &= s\alpha \sum_{i=1}^{n} P(t,T_i) \\ &= sB(t; T_o, \ldots, T_n)\end{aligned}$$

where the expression $B(t;T_o,\cdots,T_n)$ will be called the annuity. In the literature, $B(\cdot)$ is also called the *present value of a basis point (PVBP)*.[2]

Each party of the swap contract is short one leg and long the other, and the (par) swap rate is defined as the value of the fixed rate s that gives the contract zero value at the time of initiation, say time t:

$$\begin{aligned} s(t;T_o,\ldots,T_n)B(t;T_o,\ldots,T_n) &= P(t,T_0)-P(t,T_n) \\ s(t;T_o,\ldots,T_n) &= \frac{P(t,T_0)-P(t,T_n)}{B(t;T_o,\ldots,T_n)} \end{aligned}$$

Note that $P(t,T_0)-P(t,T_n)$ is the price process of a self-financing trading strategy up to time T_0, so by absence of arbitrage there must be a martingale measure equivalent to Q such that this price process is a martingale when properly discounted. As the annuity is a portfolio of α units of each of the zero-coupon bonds, it must be a strictly positive process. We can then use the annuity as a numeraire, and define the forward swap measure Q^{T_0,T_n} as the equivalent martingale measure with numeraire $B(t;T_o,\ldots,T_n)$. The swap rate

$$s(t;T_o,\ldots,T_n)=\frac{P(t,T_0)-P(t,T_n)}{B(t;T_o,\ldots,T_n)}$$

is then a Q^{T_0,T_n}-martingale.

Note also that the swap rate $s(\cdot)$ can be expressed in terms of the LIBORs:

$$s(t;T_o,\ldots,T_n)=\frac{\sum_{i=1}^{n}P(t,T_i)L(t,T_{i-1},\alpha)}{\sum_{i=1}^{n}P(t,T_i)}$$

so we cannot have deterministic volatility for both LIBOR and swap rates, meaning that we cannot consistently model both LIBOR and swap rates as log-normal. The LIBOR and the swap market model are then inconsistent with each other.[3]

Analogously to the LIBOR market model, we then specify the dynamics of the par swap rate over the time interval $[T_i,T_j]$ under the Q^{T_i,T_j}-measure as

$$ds(t;T_i,\ldots,T_j)_t = s(t;T_i,\ldots,T_j)\psi(t;T_i,\ldots,T_j)dB_t^{T_i,T_j}$$

so that the swap rate is a martingale under the appropriate forward swap measure, and is log-normal if the volatility function $\psi(\cdot)$ is deterministic.

[2]See [14] and [18].

[3]This was pointed out in the original paper of Jamshidian [7].

We have from [14] that we can model two possible sets of log-normal swap rates. For each set the rest of the swap rates will be given, but they will in general not be log-normally distributed. The reason follows from the no-arbitrage argument that a swap contract over some interval must have the same value if it is broken up into several contracts together covering the same interval as the original contract. This means that a model of the swap rates over e.g. $[T_0, T_n]$ and $[T_0, T_i]$ implicitly gives the swap rate for $[T_i, T_n]$. In our modelling of the swap rate we use a tenor of n time intervals (of length α), giving us n degrees of freedom.

Note first that the swap rate over a time interval of length α (so that interest payments are swapped for only one maturity) is simply the LIBOR rate for that interval. A model of these swap rates would be the LIBOR model once again. Note also that as the LIBOR rates will be given when swap rates are modelled, it follows again that the log-normal swap model is not consistent with log-normal LIBORs.

The first set of (proper) swap rates we can model is then the set of n rates $s(t; T_i, \ldots, T_n)$, $\; i = 0, \ldots, n-1$, the other possible set is the model of varying final dates $s(t; T_0, \ldots, T_i)$, $\; i = 1, \ldots, n$. For the first of these sets, [14] shows how to bring all swap rates under the single forward measure of their shared final date, Q^{T_n}. For the other set, a model is derived under the forward measure of the shared start date, namely Q^{T_0}. However, as drift terms are complicated in the swap market model, [14] uses Monte Carlo methods here.

2.4 Extension to dual-currency markets

A *quanto* is a security which is denominated in one currency but determined in another currency. Quanto contracts are traded on several interest rate products, for example caps. It is therefore desired that the models can price quantos. As financial markets are international, it is also desirable to have a consistent description in several currencies in the same model. Research on extending the LIBOR market model to a dual-currency economy is published in [17] and in the working papers of [13] and [8]. Before we go into detail on this work, we sketch some common features of the models.

Let the superscript d and f denote domestic and foreign, respectively. The dynamics of the domestic and foreign LIBORs are then

$$dL^d(t, T, \alpha)_t = L^d(t, T, \alpha)\lambda^d(t, T, \alpha)dB_t^{d, T+\alpha}$$

$$dL^f(t, T, \alpha)_t = L^f(t, T, \alpha)\lambda^f(t, T, \alpha)dB_t^{f, T+\alpha}$$

$B_t^{d,T+\alpha}$ denote standard Brownian motion under the domestic forward measure $Q^{d,T+\alpha}$. The foreign process and measure are correspondingly denoted by superscript f. We still assume the volatility functions $\lambda^d(\cdot)$ and $\lambda^f(\cdot)$ are deterministic, so that both rates are log-normal.

The *forward exchange rate* denoted $X(t,T)$ is the time t forward price in the domestic currency for one unit of the foreign currency when the actual exchange is taking place at time T. We also model the forward exchange rate as a geometric Brownian motion:

$$dX(t,T)_t = X(t,T)\sigma_X(t,T)dB_t^{d,T} \tag{2.6}$$

where we would like $\sigma_X(\cdot)$ to be deterministic in order to make the forward exchange rate log-normal.

Note that (whether the exchange rate volatility is deterministic or not) the above model (2.6) of the forward exchange rate assumes the process is a martingale under the domestic T-forward measure $Q^{d,T}$. To see that this must be true, consider two alternative ways to obtain one unit of the foreign currency with certainty at time T: We can exchange immediately to the foreign currency and buy the foreign T-bond, or we can buy the domestic bond and a T-forward contract on the exchange rate. As the two strategies pay equally in the future, the values at the current time t must also be equal:

$$X(t)P^f(t,T) = P^d(t,T)X(t,T) \tag{2.7}$$

Consequently,

$$X(t,T) = \frac{X(t)P^f(t,T)}{P^d(t,T)}$$

so the T-forward exchange rate is a $Q^{d,T}$-martingale.

Erik Schlögl [17] shows that the domestic and foreign forward rates and the forward exchange rate can all be log-normal only for certain combinations of maturities. Schlögl's result follows from evaluation of the dynamics of the exchange rate. This evaluation is based on the dynamics of the domestic and foreign interest rates and the interest rate parity relation (2.7). Remember also that we have postulated the dynamics of the forward exchange rate process in (2.6), a specification that is to be elaborated in the following.

First step in the review of Schlögl's work is to derive the necessary change of measure formulas. Change of numeraire and measure within the domestic (or the foreign) economy is equivalent to the change of numeraire presented under the single-currency model above. A change between domestic and foreign measure is slightly more complicated. Under the domestic T-forward measure $Q^{d,T}$, the numeraire is the domestic T-bond. This numeraire gives the same payoff at time T as can be obtained by exchanging to foreign currency at the current time t spot rate, buying the foreign T-bond and changing back to domestic currency at time T according to the T-forward exchange rate as it was given at time t. This follows from the interest rate parity (2.7). The spot exchange rate, the bond and the forward exchange rate are all strictly positive, so the process they form together can be used as a numeraire. As this gives two possible numeraire processes, and as these two numeraires must be equivalent (meaning that they have the same value both at time t and at time T), the measures corresponding to these numeraires are the same in a complete market. These measures can be applied in the domestic economy to price assets denoted in domestic currency: any asset denoted in domestic currency must be a $Q^{d,T}$-martingale when deflated by the domestic T-bond or by the above combination of the exchange rate and the foreign bond.

$$\frac{A_t}{P^d(t,T)} = E_t^{Q^{d,T}}[A_T]$$

We can also use the exchange rate to hold a foreign asset B in the domestic economy:

$$\frac{X(t)B_t}{P^d(t,T)} = E_t^{Q^{d,T}}[X(T)B_T]$$

We can even use a foreign asset as a domestic numeraire if we combine it with the exchange rate

$$\frac{X(t)B_t}{X(t)P^f(t,T)} = \frac{B_t}{P^f(t,T)} = E_t^{Q^{f,T}}[B_T]$$

Note that the expectation is given under the foreign measure. This follows from the uniqueness of the martingale measure for a given numeraire in a complete market

Here we have two valid numeraires for the domestic economy: the domestic bond and the combination of the foreign exchange rate and the foreign bond, where we have seen that the last numeraire must correspond to the foreign forward measure. Use Geman et al. and the two valid domestic numeraires from above

$$\begin{aligned}\frac{dQ^{f,T}}{dQ^{d,T}} &= \frac{P^d(t,T)/P^d(T,T)}{X(t)P^f(t,T)/X(T)P^f(T,T)} \\ &= \frac{X(T)P^f(T,T)/P^d(T,T)}{X(t)P^f(t,T)/P^d(t,T)}\end{aligned}$$

$$= \frac{X(T,T)}{X(t,T)}$$

which is similar to the change of measure formula given in Schlögl. This multi-currency change of measure formula is also given in [14] without the final rewriting to forward exchange rates, and in [11].

The change of measure can also be given by the Girsanov theorem. To find the vital variable η of this theorem, we relate a process dynamics given under the domestic measure to the same process given under the foreign measure. We have given the dynamics of the forward exchange rate process $X(t,T)$, which specifies the amount of domestic currency charged for one unit of the foreign currency, as a martingale under the domestic forward measure. The inverse of this process specifies the amount of foreign currency charged for one unit of domestic currency. By analogy, this inverted process must be a martingale under the foreign forward measure. The dynamics of $\left\{\frac{1}{X(t,T)}\right\}$ can be derived from inverting the solution to (2.6) and applying Itô's lemma:

$$d\left(\frac{1}{X(t,T)}\right) = \|\sigma_X(t,T)\|^2 \frac{1}{X(t,T)}dt - \sigma_X(t,T)\frac{1}{X(t,T)}dB_t^{d,T}$$

Keeping the diffusion term unchanged under a change of measure, we must have

$$\begin{aligned} d\left(\frac{1}{X(t,T)}\right) &= -\sigma_X(t,T)\frac{1}{X(t,T)}dB_t^{f,T} \\ &= -\sigma_X(t,T)\frac{1}{X(t,T)}\left(dB_t^{d,T} - \eta(t,T)dt\right) \end{aligned}$$

Matching this with the above dynamics under the domestic measure, we find that $\eta = \sigma_X$. The change of measure can then be performed by using the relation

$$dB_t^{f,T} = dB_t^{d,T} - \sigma_X(t,T)dt$$

Our next step in the review of Schlögl's result is rewriting the interest rate dynamics to expressions involving *forward processes*. This reformulation gives expressions for the volatility function of the forward processes that will reappear in the volatility of the forward exchange rate dynamics. It will be shown that this limits which forward exchange rates and interest rates that can have a deterministic volatility rate, i.e. be log-normal.

[11] defines the forward process $F(\cdot, T, \alpha)$ as $\frac{P(t,T)}{P(t,T+\alpha)}$, $t \leq T$. This process is a $Q^{T+\alpha}$-martingale, so its dynamics is modelled as

$$dF(t, T, \alpha) = F(t, T, \alpha)\gamma(t, T, \alpha)dB_t^{T+\alpha}$$

Recall the definition of the LIBOR and express it in terms of a forward process:

$$\begin{aligned} L(t, T, \alpha) &= \frac{1}{\alpha}\left(\frac{P(t, T)}{P(t, T + \alpha)} - 1\right) \\ &= \frac{1}{\alpha}\left(F(t, T, \alpha) - 1\right) \end{aligned}$$

The LIBOR must then evolve as a scaled forward process:

$$dL(t, T, \alpha)_t = \frac{1}{\alpha}F(t, T, \alpha)\gamma(t, T, \alpha)dB_t^{T+\alpha}$$

Let the forward process in this term be expressed in terms of the LIBOR

$$dL(t, T, \alpha)_t = \frac{1}{\alpha}(1 + \alpha L(t, T, \alpha))\gamma(t, T, \alpha)dB_t^{T+\alpha}$$

The LIBOR must also satisfy

$$dL(t, T, \alpha)_t = L(t, T, \alpha)\lambda(t, T, \alpha)dB_t^{T+\alpha}$$

where the volatility function $\lambda(\cdot)$ is deterministic according to the log-normal LIBOR market model we know from earlier sections. We can then express the volatility function of the forward process as

$$\gamma(t, T, \alpha) = \frac{\alpha L(t, T, \alpha)\lambda(t, T, \alpha)}{1 + \alpha L(t, T, \alpha)} \tag{2.8}$$

So the volatility function $\gamma(\cdot)$ of the forward process is LIBOR- and hence state-dependent.

By elaborating on the forward exchange rate as we have expressed it in (2.7), Schlögl shows that the forward process volatility γ of (2.8) appears in the forward exchange rate dynamics. This is what limits which of the interest rate processes and forward exchange rate processes that can be log-normal.

The forward exchange rate for an arbitrary date in the tenor, say T, can be expressed in terms of the forward exchange rate for the next date, $T + \alpha$, in the following manner.[4]

[4]This also gives a recursive relationship which will be applied later.

$$
\begin{aligned}
X(t,T) &= X(t)\frac{P^f(t,T)}{P^d(t,T)} \\
&= X(t)\frac{P^f(t,T+\alpha)P^d(t,T+\alpha)}{P^d(t,T+\alpha)P^f(t,T+\alpha)}\frac{P^f(t,T)}{P^d(t,T)} \\
&= X(t,T+\alpha)\frac{P^d(t,T+\alpha)}{P^d(t,T)}\frac{P^f(t,T)}{P^f(t,T+\alpha)} \\
&= X(t,T+\alpha)F^d(t,T,\alpha)^{-1}F^f(t,T,\alpha)
\end{aligned}
$$

Exchange rate dynamics are then related to a domestic and a foreign forward process. After a change to the domestic forward measure for the foreign forward process, Schlögl applies Itô's lemma and the given dynamics of the forward processes and the forward exchange rate to derive the following expression for the exchange rate dynamics:

$$
dX(t,T) = X(t,T)\left(\gamma^f(t,T,\alpha) - \gamma^d(t,T,\alpha) + \sigma_x(t,T+\alpha)\right)dB_t^{d,T}
$$

Matching the volatility function of this expression with the volatility given as in (2.6), we must have

$$
\sigma_x(t,T) = \gamma^f(t,T,\alpha) - \gamma^d(t,T,\alpha) + \sigma_x(t,T+\alpha) \tag{2.9}
$$

valid for arbitrary dates T in the tenor structure. All exchange rate volatilities are then linked together by this recursive relationship.

By substituting the LIBOR-dependent forward process volatilities into (2.9) it is demonstrated that if we model both the domestic and foreign LIBOR as log-normal, the exchange rate volatility $\sigma_X(t,T)$ can only be deterministic for a single maturity, e.g. τ. For any maturity $T \neq \tau$, the volatility function $\sigma_X(t,T)$ will be stochastic. Then the forward exchange rates $X(t,T)$, $T \neq \tau$, cannot be log-normal. From a chosen maturity τ where the two markets are linked by a log-normal exchange rate, all measures and exchange rate volatilities are given.

The basic quanto security is the single payment contract where payoff is made according to the foreign rate but paid in the domestic currency. This basic quanto contract paying the interest rate $L^f(T,T,\alpha)$ in domestic currency at time $T+\alpha$ (according to the time $T+\alpha$ spot exchange rate), has the time t value

$$
\alpha P^d(t,T+\alpha)E_t^{Q^{d,T+\alpha}}\left[L^f(T,T,\alpha)\right]
$$

from [13]. Assuming both domestic and foreign rates are log-normal for all maturities, and thereby having only one maturity for which the exchange rate can be log-normal, [13] give closed-form solutions for single-maturity quanto securities such as the caplet and exchange option. Pricing of (interest rate) quanto contracts over multiple dates or payments in a lognormal market model framework seems to be an area still open to research.

2.5 Concluding remarks

So far we have considered discretely compounded forward rates in a discrete tenor structure. The market models provide theoretical foundation for what is already applied in the market. The modeled rates are specified as are the market rates, and the formulae obtained are the same as market practitioners apply. These models also avoid negative as well as exploding interest rates. On the other hand, there are limitations to the market these models are capable of covering. As we have seen, extensions to several maturities or currencies can violate the no-arbitrage assumption. There is also the question of fitting the model to observed variables in the market. Empirical work suggests that the log-normal model (with deterministic volatility) does not entirely capture the dynamics observed in the market. A lot of work has been carried out on "volatility smiles" where the models are run backwards to read out volatilities implied by actual market data (and the data does not fit the model assumptions), as well as on stochastic volatility models.[5] Another feature missing in the market models as well as in most other interest rate models, is credit risk. Interest rate instruments are implicitly assumed to be default-free, so the models do not price the risk arising from potential default.

Most interest rate models have in common that they model the term structure from one single variable (the short rate). This one-factor specification contributes to simplify application as well as provide for analytical solutions to some common derivative contracts. On the other hand, restricting the model to one explanatory variable is not overly realistic and can make it harder to obtain a good fit to observed variables. Heath, Jarrow and Morton (HJM, [6]) provides a more general framework for forward rate modeling. The HJM framework allows a stochastic model of an entire continuous term structure, i.e. any one of infinitely many points on the forward curve can be a stochastic variable. At the same time this framework has the flexibility to allow specification of most other forward rate models.

[5]Volatility smiles and stochastic volatility in a LIBOR market model framework is treated in [15].

In its most general form HJM models the instantaneous continuously compounded forward rate (a point on the forward rate curve) for the whole term structure. HJM specifies a no-arbitrage drift restriction that is used to determine the model. From this restriction it follows that if the volatility is deterministic, the model is fully determined by a specification of the volatility and this class of models are called Gaussian HJM models. The Gaussian HJM can lead to analytical solutions for pricing formulae, as published by HJM themselves, but may also lead to negative interest rates. The non-Gaussian HJM forward rates need not be Markov processes. In that case they will not permit a PDE or Feynman-Kač approach, so prices can only be calculated by numerical methods. HJM may also be discretized to model LIBOR-type rates.[6]

In sum, the various specifications of interest rate models have a range of qualities making the practitioner having to compromise between properties such as ease of use, accuracy and scope. A wide repertoire is beneficial, and one should not finish after having learned only one model!

[6] [11] gives an overview of Gaussian models. [15] covers the LIBOR market model with empirical examples and several extensions, and [4] gives a good overview over how various common one-factor models relate to each other.

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