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# Valuation of Real Options

*Construction of a Model to Evaluate Real Investment Projects*

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NORGES HANDELSHØYSKOLE

# Summary

In this thesis I use two existing models to construct a model that can evaluate the exploration phase and the production phase of a real investment project simultaneously. I assume that the value of the production phase is the value of the outcome of an exploration project and can therefore make a model that combines the two phases. I assume that the exploration phase has on-going investment costs until completion and that the time to completion is uncertain. I allow the exploration project to shift between an active and a passive state and the production can be shut down and restarted whenever this is optimal. The model is applicable for R&D projects and natural resource exploration projects such as mine or oil exploration projects.

# Foreword

This thesis is written as the final part of the Master of Science in Economics and Business Administration program at Norges Handelshøyskole.  My major is in Financial Economics.

I have always found options to be an interesting field in finance, and I particularly wanted to learn more about the valuation of real options. The appliance of options to real investment projects is very fascinating and I think it will be more and more used in the future. I got help from my advisor, Kristian R. Miltersen to specify the topic for this thesis. He has, together with Eduardo S. Schwartz, developed a model for evaluating real option problems with uncertain maturity which is highly applicable to analyze various real investment projects. I found this topic very intriguing, and I decided to use their model as a basis for further development in my thesis.

Writing this thesis has been a great experience, although it has been a process of both up and downturns. The work has required thoroughness and patience. I have learned a lot about the technical issues and also about working independently and working with the same project over a longer time period.

I would like to thank my advisor, Kristian R. Miltersen, for suggestion to the topic of this thesis and for all the help and support he has given during the process.

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Mette Rørvik Rutgerson

# Contents

# 1. Introduction

The purpose of this thesis is to construct a model that can be used to evaluate the exploration phase and the production phase of an investment project simultaneously to find out if it is profitable to invest in the project. The model should give closed form solutions to the value of the investment project. The outcome of this thesis can be useful for investors in the starting phase of a real investment project, for example an R&D project or a natural resource exploration project such mine or oil exploration.

**Problem:**

Can real option theory be used to make a model that simultaneously evaluates the exploration phase and production phase of an investment project?

I will construct a model that uses the value of the production phase as the value of the outcome of an exploration project. The model computes the present value of future cash flows that can be generated from selling the product/commodity/mineral that is discovered. The output price of the production phase is treated as a stochastic variable. I assume that the exploration phase has on-going investment costs until completion and that the time to completion is uncertain. I will consider the possibility of shutting down the exploration project at any time if the future prospects are not good enough, and restarting the project if this is profitable. I will also consider the possibility of closing down and restarting the production/extraction of the product/commodity/mineral after what is optimal for the value of the investment project. The model should give closed form solutions to the value of the investment project and to the optimal switching points between an active and a passive investment project and between a closed and an open production.

The assumptions make my model highly applicable for R&D projects and mine or oil exploration project. These industries can experience great price swings, and it is therefore essential that the output price is treated as stochastic. Such projects also require high on-going investment costs until completion. The time to completion is uncertain, and hence, the total on-going investment costs are uncertain. The ability to shut down a money losing project and restart it again when this is profitable is important.

As a basis for the exploration phase in my model I will use a model developed by Miltersen and Schwartz in their article “Real Options with Uncertain Maturity and Competition”.

Miltersen and Schwartz analyze two general types of models; monopoly models where the owner of the investment project has exclusive rights to outcome, and duopoly models where there are two (or more) owners with similar investment projects who compete to get the value of the outcome. They analyze the models with different options; model with abandonment option, model with switching option and model with both abandonment and switching options. To limit the extent of this thesis I choose only to consider the monopoly model with switching option.

Schwartz has also, together with Brennan, developed a model for evaluating natural resource investments. They treat output prices as stochastic and allow the project to be closed down and reopened when output prices fall/rise far enough. The model is from 1985 but it is still applicable and I will use this as a basis for the production phase in my model. I think it might give a more correct model when I combine two models that are developed (to a certain degree) by the same person. The models may have more similar characteristics and can more easily be compared and combined.

I will in the next chapter present introductory theory about options and real options. Further I will present Miltersen and Schwartz’s model as the theory about the valuation of the exploration phase and Brennan and Schwartz’s model as the theory about the valuation of the exploration phase. In chapter 3 I will combine the two models to construct a model that evaluates the two phases simultaneously. A numerical example is created in chapter 4 to illustrate the model and to see how the model behaves when central parameter values are changed. I show the complex deriving of new equations in the appendixes.

# 2. Theory

In this part I will give a brief introduction to option theory and real options. After that I will present Miltersen and Schwartz's model for valuation of real option with uncertain time to completion, and lastly I will present Brennan and Schwartz's model for evaluating a natural resource investment.

## 2.1 An Introduction to Options and Real Options

An option is a derivative, which means a financial instrument that has a value determined by the price of something else (McDonald 2006). Derivatives are used for in for example risk management, as insurance to reduce the risk and in speculation to secure an investment.

**Call Options and Put Options**

An option gives the holder the right to do something. The holder does not have to exercise this right (Hull 1997). A call option gives the holder the right but not the obligation to buy the underlying asset by a certain date for a certain price. A put option gives the holder the right but not the obligation to sell the underlying asset by a certain date for a certain price. This certain price is called the strike price or exercise price, and the date is the expiration date or maturity. They are both prearranged and written in the contract. If the option is not exercised the holder of the option would get zero.

I will consider two option styles: American option and European option. The American option can be exercised at any time up to maturity, while the European option can only be exercised at maturity. European options are generally easier to analyze than American options. Even though most of the options that are traded on exchanges are American options (Hull 1997) I will only show the payoffs of a European option.

When you take a long position in the option it is the same as buying the call or the put option. You enter a call option contract to get the right to buy the underlying asset at maturity for the exercise price. If the exercise price $X$ is lower than the spot price at maturity $\tau$, $S_\tau$ , you will exercise the call option. The option is then in-the-money because it gives a positive payoff. The opposite will be if the option was out-of-the-money. This is the case when it is not profitable to exercise the option, when the payoff is negative. The option holder will not exercise the option and get a payoff of zero. If the option is at-the-money the exercise price is equal to the spot price. The payoff from a long position in a European call option is

$$\max(S_\tau - X\,, 0)$$

The same principles apply for a put option; you enter a put option contract to get the right to sell the underlying asset at maturity for the exercise price. If the exercise price is greater than the spot price at maturity the option is in-the-money and you will exercise the put option. The payoff from a long position in a European put option will be

$$\max(X - S_\tau\,, 0)$$

If you expect the price of the underlying asset to become higher you should buy a call option, and if you expect the price to be lower you should buy a put option. You can also sell or write the option. You then sell something you don't have, and this is called taking a short position in the option. You will sell a call option if you expect the price of the underlying asset to fall. At maturity the buyer of the call option has the right to exercise the option. If the spot price at maturity is greater than the exercise price the buyer of the contract would exercise the option. The writer of the call option would then lose the difference between the spot price at maturity and the exercise price. The payoff from a short position in a European call option is the opposite of the payoff from a long position and will be

$$-\max(S_\tau - X\,, 0)$$

You will sell a put option if you expect the price of the underlying asset to rise. If the spot price at maturity is lower than the exercise price the buyer of the contract would exercise the option and the writer of the put option would lose the difference between the exercise price and the spot price. The payoff from a short position in a European put option is the opposite of the payoff from a long position and will be

$$-\max(X - S_\tau\,, 0)$$

Short selling is more risky than taking a long position because you have to pay the difference between the exercise price and the spot price if the option holder exercises the option. There can also be an option premium added to the scenarios above. The option buyer pay a premium at the contract date to enter the contract, and this premium is deducted from the payoff from the long positions (when you are a buyer) and added to the payoff of the short position (when you are a seller).

**Real options**

Real options can be defined as the application of derivatives theory to the operation and valuation of real investment projects (McDonald 2006). Any real investment can be viewed as a call option with the investment costs equal to the strike price and the present value of future cash flows equal to the price of the underlying asset. The present value of future cash flows is then compared to the investment costs, and if the present value is greater it is profitable to exercise the option, or in real option terms: it is profitable to invest in the project. The payoff from the investment project is:

$$\max(V_\tau - K\ , 0)$$

$V_\tau$ is the present value of future cash flows at maturity $\tau$ and $K$ is the investment costs.

In real investment decisions, as well as with financial options, you have to make a decision about whether and when to invest in the project and consider the ability to shut down, restart, and abandon projects. The decision about whether to invest has, as I said before, the properties as a standard call option. If the net present value is negative, it might be profitable to wait to invest. Waiting to invest can make the investment project profitable if the net present value was originally negative or more profitable if the net present value was already positive. The ability to temporarily shut down or abandon a money-losing project is important to investors because it is an insurance against greater losses. This can be viewed as having the investment project plus a put option; if the value drops under a certain threshold level it is profitable to shut down or abandon the investment project. There are often costs attached to shutting down or abandoning the project. When there are such costs the threshold level for shutting down or abandoning is lower and the insurance provided by the option is therefore less. Having the option to restart the project once it is shut down makes it easier to shut down because you can then keep the project in a “passive” state. The project will be restarted when it has reached a threshold level where it is profitable to pay the restarting costs. The option to restart can be viewed as a call option. When you decide to shut down a project you exercise the put option and at the same time you acquire a call option to restart (McDonald 2006). This increases the value of the investment project and makes investors more willing to invest.

Real options can be used in the valuation of research and development (R&D) projects (McDonald 2006). These projects involve paying R&D costs today to receive future cash flows. If the R&D is successful a project can be undertaken if the net present value is positive. This is a call option; exercise the option if the present value of the future cash flows exceeds the final investment necessary. The R&D costs leading up to the completion date can be viewed as an option premium. The uncertainty of the results of an R&D project requires that there is a possibility to temporarily shut down, restart or abandon the project.

Real options can also be used as a valuation tool and to make investment decisions in natural resources investment projects. The extraction of a natural resource has great resemblance to the exercise of a financial option; by paying the extraction costs you can receive the present value of the future cash flows the extracted resource will generate. It is important to have the option to temporarily shut down, restart or abandon the extraction if the investment project becomes unprofitable.

## 2.2 Valuation of the Exploration Phase

In the article "Real Options with Uncertain Maturity and Competition" Miltersen and Schwartz (2006) develop a new approach to dealing with real option problems with uncertain maturity. The approach is highly applicable to analyze R&D investments and mine or oil exploration projects. There was some literature on this subject before[1], but they involve complex numerical solution techniques, like elliptical partial differential equations or the Monte Carlo simulation. Miltersen and Schwartz simplify the framework to get closed form solutions to the values of the investment project without losing the important elements for the valuation.

Their main simplification is that completion of the project is governed by an independent exponential random variable, which means that the conditional probability of completion per unit of time is constant. This simplification implies that the value of the project will be a solution to an ordinary differential equation, instead of a partial differential equation.

[1] Articles on this subject include Pindyck (1993), Schwartz and Moon (2000), Schwartz (2004), Miltersen and Schwartz (2004), and Hsu and Schwartz (2006). Reference to articles is taken from Miltersen and Schwartz (2006).

I will present Miltersen and Schwartz's monopoly model with a switching option. The owner can at any time switch between an active and a passive investment project. This is equivalent to an American option; it can be exercised at any time up till maturity. At completion the owner has the right to the value of the outcome of the investment project. This is equivalent to a European option; it can only be exercised at maturity. It is assumed that the investment project will be completed at a random date $\tau$. Up until this date the owner has to pay the on-going investment costs at the rate of $k$ per unit of time. Since the time to maturity is uncertain the total on-going investment costs is also uncertain. At maturity the owner of the investment project has to compare the final investment cost $K$ to the present value of future cash flows to decide whether it is profitable to make the final investment necessary to make use of the resource. This present value is referred to as the value of the outcome $V$. The value of the investment project at completion date would be

$$\max\left(V_\tau - K, 0\right)$$

It is assumed that the value of the outcome evolves stochastically through time and that it can be observed or estimated by the owner of the investment project at any point in date $t$. This estimated value is denoted $V_t$. The dynamics of $V$ is given by the geometric Brownian motion[2]

$$dV_t = V_t \mu \, dt + V_t \sigma \, dW_t$$

where $\sigma$ is the instantaneous volatility of the value process, $\mu$ is the instantaneous drift and $W$ is the increment of a Brownian motion.

It is further assumed that the random time to completion, $\tau$, is exponentially distributed with intensity $\lambda$ and that the time to completion is independent of the value process. The expected time to completion is therefore $T = {}^{1}\!/_{\lambda}$ . $\lambda$ is also interpreted as probability of completion per unit of time. The riskless rate $r$ is constant and strictly greater than $\mu$. This is to avoid the possibility of infinite values of the investment project. The expected time to completion $T$

[2] A Brownian motion (also called a Wiener Process) is a continuous time stochastic process with three important properties; (i) it is a Markov Process which implies that only current value is useful for forecasting the future path of the process, (ii) it has independent increments, and (iii) changes in the process over any finite time interval are normally distributed (Dixit and Pindyck 1994). The increment of the Brownian motion represents the randomness of the change in the value of the outcome. For a geometric Brownian motion the percentage change in the value of the outcome is normally distributed.

does not depend on calendar time. It has the same distribution as $\tau$, and it is therefore no need to distinguish between these two terms. The value of the investment project at any date $t$ depends only on $V_t$ and not at date $t$ itself. This simplifies the analysis and makes it possible to obtain closed form solutions.

When you have an investment project with a switching option the owner has the option to temporarily suspend investing in the project by switching to a passive state. He can at any time switch back to an active state. The active state incurs on-going investment costs and has a positive probability of completion, while the passive state has no on-going investment cost and no chance of completion; $k = 0$ and $\lambda = 0$. Miltersen and Schwartz assume that it is costless to switch between the two states. There is a threshold level $S_N$ at where it is optimal to switch between the two states. If the value of the outcome is above this threshold level it is optimal to keep the investment project active, and if the value is below this level it is optimal to switch to the passive state. The optimal switching point will be above the final investment costs, $K$, because by keeping the investment passive when the value is less than $K$ the owner can avoid completing the project when it is out of the money. The investment project is therefore in the money whenever it is active. It will never be optimal to abandon the project because there are no costs linked to a passive investment project.

$N(V)$ is the value of the investment project and must satisfy the following set of ordinary differential equations

$$1/2\,\sigma^2 V^2 N''(V) + \mu V N'(V) - rN(V) = 0 \qquad \text{when } V < S_N$$

$$1/2\,\sigma^2 V^2 N''(V) + \mu V N'(V) - (r+\lambda)N(V) - k + \lambda(V - K) = 0 \qquad \text{when } S_N < V$$

The first equation describes the value of the investment project when the value of the outcome is less than the optimal switching point; when the project is passive. Here $k$ and $\lambda$ are equal to zero. The value of the investment project is zero in the passive state. The second equation describes the value when the investment project is active. It reflects that with intensity $\lambda$ the value of the investment project will jump to the completion value $(V - K)$. This corresponds to a change in value of $V - K - N(V)$. In addition the owner has to pay the on-going investment costs $k$ per unit of time to keep the investment project active. Remember that $\lambda = 1/T$ . The general solutions to the ordinary differential equations are

$N_1(V) = n_{11}V^{x_1} + n_{12}V^{x_2}$ when $V \leq S_N$

$N_2(V) = n_{21}V^{y_1} + n_{22}V^{y_2} + \frac{V}{1+(r-\mu)T} - \frac{kT+K}{1+rT}$ when $S_N \leq V$

The powers are given by

$$x_1 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}}{\sigma^2} > 1$$

$$x_2 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) - \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}}{\sigma^2} < 0$$

$$y_1 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2(r+\lambda)\sigma^2}}{\sigma^2} > 1$$

$$y_2 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) - \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2(r+\lambda)\sigma^2}}{\sigma^2} < 0$$

Since $x_2 < 0$ the value of the investment project $N(V)$ is increasing when the value of the outcome $V$ converge to zero. $N(V)$ must be zero when this happens and the $V^{x_2}$ term therefore has to be eliminated. This is done by giving $n_{12}$ the value zero. $N(V)$ can also never exceed the value of the outcome, and since $y_1 > 1$ the $V^{y_1}$ term has to be eliminated as well. This gives the simplified solutions

$N_1(V) = n_{11}V^{x_1}$ when $V \leq S_N$

$N_2(V) = n_{22}V^{y_2} + \frac{V}{1+(r-\mu)T} - \frac{kT+K}{1+rT}$ when $S_N \leq V$

The value of the investment project has the following boundary conditions, which reflect that the value function should be continuous and differentiable at the point where the two ordinary differential equations meet at the switching point

$$N_1(S_N) = N_2(S_N)$$

$$N_1'(S_N) = N_2'(S_N)$$

The optimal switching point $S_N$ is found by weighing the instantaneous cost and benefits from switching between an active and a passive investment project. The increased instantaneous benefit from switching from a passive to an active state is the increased intensity of completion which has a value flow $\lambda(V-K)$per unit of time. The increased instantaneous costs of switching are the increased intensity of losing the investment project which has a value flow $\lambda N_2(V)$ per unit of time. In addition there are the increased on-going investment costs $k$ per unit of time. This gives the following equilibrium equation for the optimal switching point

$$\lambda(S_N - K) = \lambda N_2(S_N) + k$$

It is also correct to use $N_1(S_N)$ instead of $N_2(S_N)$ because of the first boundary condition.

The unknowns $n_{11}$ and $n_{22}$ are found by solving the boundary conditions

$$n_{11} = \frac{y_2(1+(r-\mu)T)(kT+K) + (1-y_2)(1+rT)S_N}{(x_1-y_2)(1+(r-\mu)T)(1+rT)S_N^{x_1}}$$

$$n_{22} = \frac{x_1(1+(r-\mu)T)(kT+K) - (x_1-1)(1+rT)S_N}{(x_1-y_2)(1+(r-\mu)T)(1+rT)S_N^{y_2}}$$

These are together with the equilibrium equation used to find the equation for the optimal switching point

$$S_N = \frac{(x_1 + (x_1-y_2)rT)(1+(r-\mu)T)(kT+K)}{(x_1 - 1 + (x_1-y_2)(r-\mu)T)(1+rT)}$$

| Parameters values – Base case | | | |
|---|---|---|---|
| Instantaneous drift of the value process | $\mu$ | 3 | % per year |
| Instantaneous volatility of the value process | $\sigma$ | 40 | % per year |
| Expected time to completion | $T$ | 5 | years |
| On-going investment costs rate | $k$ | 1 | million $ per year |
| Final (fixed) investment costs | $K$ | 5 | million $ |
| Interest rate | $r$ | 5 | % per year |

*TABLE 1: Parameter values in Miltersen and Schwartz's model - Base case*

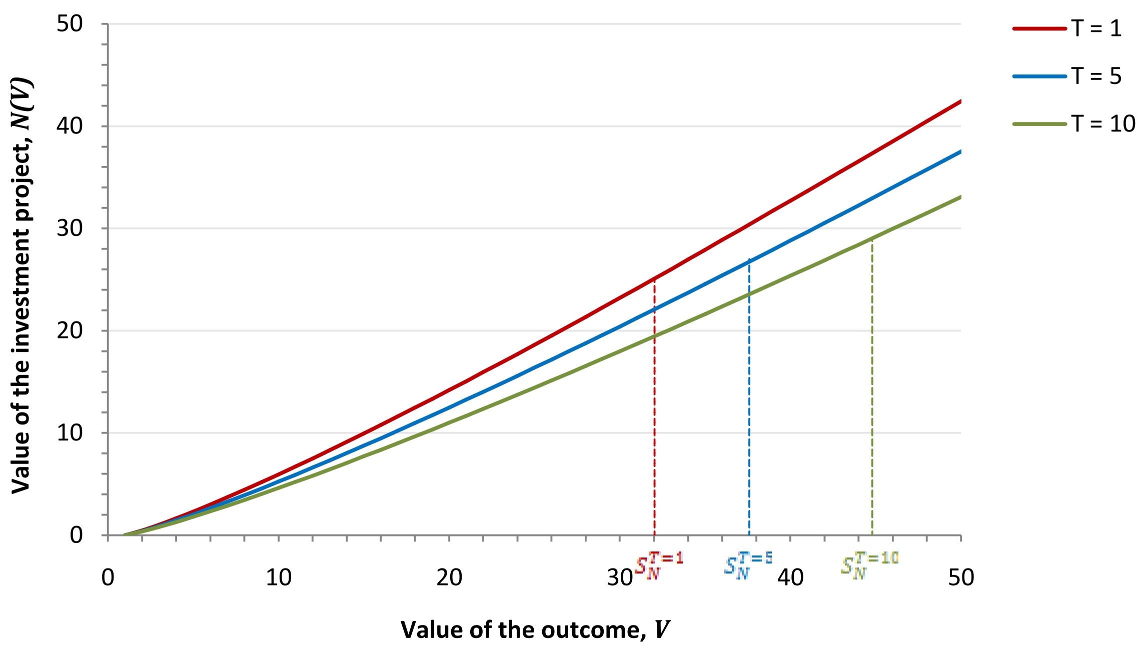


*FIGURE 1: Values of investment projects as a function of the value of the outcome for different expected time to completion. The value of the investment project increases when the expected time to completion decreases, and vice versa. The base case has an optimal switching point at* $S_N^{T=5}$ *= 37,58. The threshold levels for* $T = 1$ *and* $T = 10$ *are* $S_N^{T=1}$ *= 32,04 and* $S_N^{T=10}$ *= 44,79.*

Miltersen and Schwartz use the parameter values shown in table 1 as a base case for numerical illustration. The base case has an expected time to completion equal to 5 years, and they change this value to $T = 1$ and $T = 10$ years to see how the value of the investment project and the optimal switching points are affected. Figure 1 shows the solutions for the value of the investment project as a function of the value of the outcome. They find that the value of the investment project is higher when the expected time to completion is shorter and that the values of the investment project are strictly positive. This is because there are no costs related to keeping the investment project passive. The switching levels are higher when the expected time to completion is higher.

## 2.3 Valuation of the Production Phase

Brennan and Schwartz published in 1985 a new model for evaluation of investment projects in their article "Evaluating Natural Resource Investments". The standard technique before Brennan and Schwartz discounts expected cash flows from an investment project at a rate appropriate to the risk, and the present value is compared to the cost of the project. This does not take the stochastic characteristic of output prices into account. Brennan and Schwartz made a model that treats output prices as stochastic. This is of great importance in the natural resource industries where there may be large price swings. They also consider the possibility that a project may be closed down or abandoned if the output prices fall under a certain level. The model is useful to corporations considering when, whether, and how to develop a given resource, and to financial analysts concerned with the valuation of such corporations.

Brennan and Schwartz begin their paper with developing a general model for valuing the cash flow from a natural resource investment, which they later present in a more specialized or simplified version. It is only possible to get closed form solutions from the simplified model. The assumptions are that the convenience yield can be written as a function of the output price, the interest rate is constant, the resource is of a known amount and the costs are known. The convenience yield is the flow of services that accrues to an owner of the physical commodity and not to the owner of a contract for future delivery of the commodity.

They use an example of a hypothetical mine that produces a single homogenous commodity. The spot price of the commodity $S$ is determined competitively and follows an exogenously given stochastic process

$$dS = \mu S\,dt + \sigma S\,dz$$

where $\sigma$ is the instantaneous standard deviation of the spot price, $\mu$ is the instantaneous drift and $dz$ is the increment to a standard Gauss-Wiener process[3].

The value of the mine $H$ depends on whether the mine is currently open, $j = 1$, or closed, $j = 0$, the current commodity price $S$, the physical inventory in the mine $Q$, calendar time $t$ and the mine operating policy $\phi$. Under the value maximizing operating policy $\phi^*$ the values of the open mine $V$ and the closed mine $W$ are given by

$$V(S, Q, t) \equiv \max_{\phi} H(S, Q, t; j = 1, \phi)$$

$$W(S, Q, t) \equiv \max_{\phi} H(S, Q, t; j = 0, \phi)$$

[3] A Wiener Process is equivalent to a Brownian motion (see footnote 2).

The after tax cash flow is

$$q(S-A)-M(1-j)-\lambda_j H-T$$

where $A$ is the average cash cost rate, $M$ is the after-tax fixed-cost rate of maintaining the mine when it is closed, $T$ is the total income tax and royalties imposed on the mine when it is operating, and $\lambda_j$ is the proportional rate of tax on the value of the mine. $\lambda_j$ can also be interpreted as the intensities of Poisson processes governing the event of uncompensated expropriation of the owners of the mine. When using this interpretation of $\lambda_j$ the expression above represents the cash flow of the net expected cost of expropriation.

There is a cost to close and open the mine, represent by $K_1$ and $K_2$ respectively. The value of the mine depends on calendar time because the costs $A$, $M$, $K_1$ and $K_2$ and the convenience yield $C$ depends on time. This can be changed. Suppose the convenience yield can be written as $\kappa S$. If there is a constant rate of inflation $\pi$ in all of the variables, they can be deflated by multiplying each variable with $e^{-\pi t}$. The deflated values are written in small letters; $a$, $f$ (deflated value of $M$), $k_1$, $k_2$, $s$, $v$ and $w$. The real interest rate is $r=\rho-\pi$.

This leads to a set of partial differential equation that has to be solved numerically. To get closed form solutions to this model it is assumed that the physical inventory of the mine $Q$ is infinite. $Q$ was previously of a known amount. When $Q$ is infinite it means that $Q$ is no longer a state variable and the partial differential equations for the value of the mine can be replaced with ordinary differential equations. It is further assumed that the tax system allows for full loss offset and finally that the mine only has two operating rates, $q^*$ when it is open, and zero when it is closed. The (deflated) value of the mine when it is open satisfies the ordinary differential equation

$$1/2\,\sigma^2 s^2 v''(s)+(r-\kappa)s\,v'(s)+ms-n-(r+\lambda)v=0$$

where $m=q^*(1-t_1)(1-t_2)$ and $n=q^*a(1-t_2)$

Assuming that the periodic maintenance cost for a closed mine $f$ is equal to zero, the value of the closed mine satisfies this differential equation

$$1/2\,\sigma^2 s^2 w''(s)+(r-\kappa)s\,w'(s)-(r+\lambda)w=0$$

The boundary conditions are as follows

$$w(0) = 0$$

$$v(s_1^*) = \max[w(s_1^*) - k_1, 0]$$

$$w(s_2^*) = v(s_2^*) - k_2$$

$$v'(s_1^*) = \begin{cases} w'(s_1^*) \text{ if } w(s_1^*) - k_1 \geq 0 \\ 0 \text{ if } w(s_1^*) - k_1 < 0 \end{cases}$$

$$w'(s_2^*) = v'(s_2^*)$$

$s_1^*$, and $s_2^*$ are the critical commodity prices: $s_1^*$ is the threshold level to close the mine if it was already open, and $s_2^*$ is the threshold level to open the mine if it was already closed. The complete solutions to the differential equations are

$$w(s) = \beta_1 s^{\gamma_1} + \beta_2 s^{\gamma_2}$$

$$v(s) = \beta_3 s^{\gamma_1} + \beta_4 s^{\gamma_2} + \frac{ms}{(\lambda + \kappa)} - \frac{n}{(r + \lambda)}$$

$$\text{where } \gamma_1 = \alpha_1 + \alpha_2 \text{ and } \gamma_2 = \alpha_1 - \alpha_2$$

$$\alpha_1 = 1/2 - \frac{(r - \kappa)}{\sigma^2} \quad \text{and} \quad \alpha_2 = \sqrt{\alpha_1^2 + \frac{2(r + \lambda)}{\sigma^2}}$$

It is necessary that $(r + \lambda) > 0$ for the present value of the future cost to be finite. $\gamma_1 > 1$ and $\gamma_2 < 0$. The value of a closed mine $w(s)$ must remain finite as $s$ approaches zero and since $\gamma_2$ is negative $\beta_2$ has to be zero. The value of an open mine $v(s)$ must remain finite as $s$ goes to infinity and because $\gamma_1$ is greater than 1 $\beta_3$ also has to be zero. This leaves the shortened solutions

$$w(s) = \beta_1 s^{\gamma_1}$$

$$v(s) = \beta_4 s^{\gamma_2} + \frac{ms}{(\lambda + \kappa)} - \frac{n}{(r + \lambda)}$$

The term $\beta_1 s^{\gamma_1}$ represents the value of the option to open the mine and the term $\beta_4 s^{\gamma_2}$ represents the value of the closure option. If there was no such option the value of the mine would be given by $\frac{ms}{(\lambda+\kappa)} - \frac{n}{(r+\lambda)}$. $\beta_1$, $\beta_4$ and the optimal prices for when to close and open the mine, $s_1^*$ and $s_2^*$, are determined by the boundary conditions which gives these solutions

$$\beta_1 = \frac{ds_2^*(\gamma_2 - 1) + b\gamma_2}{(\gamma_2 - \gamma_1)s_2^{*\gamma_1}}$$

$$\beta_4 = \frac{ds_2^*(\gamma_1 - 1) + b\gamma_1}{(\gamma_2 - \gamma_1)s_1^{*\gamma_2}}$$

$$s_2^* = \frac{\gamma_2(e - bx^{\gamma_1})}{(x^{\gamma_1} - x)d(\gamma_2 - 1)}$$

$$\frac{s_1^*}{s_2^*} = x$$

$$\frac{(x^{\gamma_2} - x)(\gamma_1 - 1)}{\gamma_1(e - bx^{\gamma_2})} = \frac{(x^{\gamma_1} - x)(\gamma_2 - 1)}{\gamma_2(e - bx^{\gamma_1})}$$

Where $e = k_1 - \frac{n}{r+\lambda}$ , $b = -k_2 - \frac{n}{r+\lambda}$ and $d = \frac{m}{\lambda+\kappa}$

$x$ is the ratio of commodity prices at which the mine is closed and opened. $x$ is found by solving the non-linear equation above.

Figure 2 shows the values of the mine when it is open and closed as functions of the commodity price $s$. If the price is below $s_1^*$ the value of the mine is sufficiently greater when it is closed for it to be profitable to pay the cost $k_1$ to close the mine. Because of the cost of opening the mine it is profitable to open the mine again when the price reaches $s_2^*$. If the cost of opening and closing the mine was larger the gap between $w(s)$ and $v(s)$ in $s_1^*$ and $s_2^*$ would be greater and the closure option will eventually become worthless. On the other hand, if the cost of opening and closing the mine was lower $s_1^*$ and $s_2^*$ would move closer, and if $k_1$ and $k_2$ were zero the value of the mine would be one single curve.

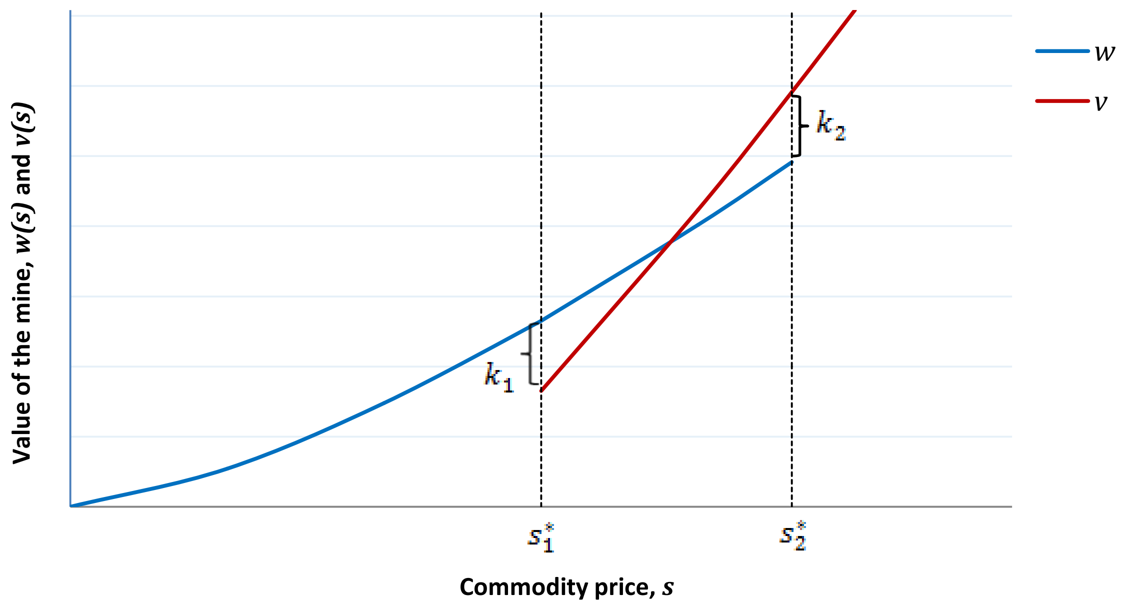


*FIGURE 2: Value of the mine when it is closed and open as a function of the commodity price. The optimal switching points $s_1^*$ and $s_2^*$ are shown on the horizontal axis.*

Brennan and Schwartz use a hypothetical mine to illustrate the general model. They do not have an example of the simplified model but I will use the parameter values from the general model in my analysis. The values, assuming $Q$ is infinite and $f = 0$, are shown in table 2.

| Parameter values for the mine | | | |
|---|---|---|---|
| Output rate of the mine | $q^*$ | 10 | million pounds per year |
| Mine inventory | $Q$ | infinite | million pounds |
| Initial average cost of production | $a$ | 0,5 | \$ per pound |
| Initial cost of opening and closing | $k_1, k_2$ | 0,2 | million \$ |
| Convenience yield | $\kappa$ | 1 | % per year |
| Price variance | $\sigma^2$ | 8 | % per year |
| Intensity of expropriation of the mine | $\lambda_1, \lambda_2$ | 2 | % per year |
| Income tax | $t_2$ | 50 | % |
| Royalty | $t_1$ | 0 | % |
| Inflation | $\pi$ | 8 | % per year |
| Interest rate | $\rho$ | 10 | % per year |

*TABLE 2: Parameter values for a hypothetical mine in Brennan and Schwartz's model.*

# 3. Results

I will combine the two models I just described to come up with a model that can evaluate both the exploration phase and the production phase simultaneously. I will refer to Miltersen and Schwartz's model as the exploration phase and Brennan and Schwartz's model as the production phase.

## 3.1 Combination of the Exploration Phase and the Production Phase

To be able to combine the two models I have to make some assumptions. Firstly, the two models differ in one fundamental condition; in the exploration phase the value of the outcome follows a geometric Brownian motion while it is the commodity price that follows a geometric Brownian motion in the production phase. To combine the two models I therefore have to assume that the commodity price at any date $t$ is the value of the outcome at any date $t$, $S_t = V_t$. I will from now on refer to the commodity price as the value of the outcome $V$, and it follows the same geometric Brownian motion as before

$$dV_t = V_t \mu \, dt + V_t \sigma \, dW_t$$

where $\sigma$ is the instantaneous volatility of the value process, $\mu$ is the instantaneous drift and $W$ is the increment of a Brownian motion[4].

To be able to use the simplified model for the production phase I have to assume that the mine inventory $Q$ is infinite and the maintenance cost of a closed mine $f$ is zero. I think of $\lambda_j$ as the intensity of uncompensated expropriation of the owners of the mine. I also change the symbol for this to $\delta$ because the exploration phase has a different $\lambda$ which symbolizes the probability of completion of the investment project. For simplification reasons I assume that there is no final investment cost $K$ needed to exploit the value of the mine. Further, I assume that it is costless to switch between a closed and an open mine, $k_1 = k_2 = 0$.

[4] Brennan and Schwartz write the increment of the Brownian motion as dz while Miltersen and Schwartz use dW. These are equivalent; the increment is just given different symbols.

The last assumption simplifies the combination of the models to a great extent. Recall that the switching point $s_1^*$ and $s_2^*$ becomes on combined switching point when $k_1 = k_2 = 0$. The value of an open and a closed mine is therefore equal in this point. Look at the ordinary differential equations for the production phase

$$1/_2\, \sigma^2 s^2 v''(s) + (r - \kappa)s\, v'(s) + ms - n - (r + \lambda)v = 0$$

$$1/_2\, \sigma^2 s^2 w''(s) + (r - \kappa)s\, w'(s) - (r + \lambda)w = 0$$

In the switching point the value of an open and a closed mine is the same, $v(s) = w(s)$. The optimal switching point is found by weighing thee instantaneous benefits and costs of switching between the two states. The instantaneous benefit of switching to an open mine has the value flow of $ms$ and the instantaneous cost has a value flow equal to $n$. The trade-off between benefits and costs results in the equilibrium equation $ms = n$. The optimal switching point is therefore $s = \frac{n}{m}$.

When I combine the two models I start by finding out what the completion value of the exploration phase will be. At completion, that is when a mineral or a product is found, the owner of the investment project has the option to the value of the outcome. This will not be $(V - K)$ like in Miltersen and Schwartz's model, but it will be an option to the value of the mine in the production phase. I call this option value $P(V)$. The value of the option at completion will be

$$\max(P(V_\tau), 0)$$

The investment project will jump to the completion value $P(V)$ with intensity $\lambda$. The solutions to my model must satisfy these ordinary differential equations:

$$1/_2\, \sigma^2 V^2 L''(V) + \mu V L'(V) - rL(V) = 0 \qquad \text{when } V < S$$

$$1/_2\, \sigma^2 V^2 L''(V) + \mu V L'(V) - (r + \lambda)L(V) - k + \lambda P(V) = 0 \qquad \text{when } V \geq S$$

Where $P(V) = \begin{cases} w(V) & \text{if } V < \frac{n}{m} \\ v(V) & \text{if } V \geq \frac{n}{m} \end{cases}$

$w(V)$ and $v(V)$ are the values of a closed and an open mine. If the value of the outcome is less that the switching point between an open and a closed mine, the owner of the investment project has the option to get the value of a closed mine at maturity. The owner will get the option to the value of an open mine if the value of the outcome is greater than the same switching point. In the equations below I have switched the symbols $\gamma_1$ and $\gamma_2$ to $z_1$ and $z_2$ respectively to make it less confusing since I also use $y_1$ and $y_2$ in the exploration phase. The equations for $\gamma_1$ and $\gamma_2$ and the other unknowns are given by Brennan and Schwartz's model in chapter 2.3.

$$w(V) = \beta_1 V^{z_1}$$

$$v(V) = \beta_4 V^{z_2} + \frac{mV}{\delta + \kappa} - \frac{n}{r + \delta}$$

To find the complete set of ordinary equations that have to be satisfied I have to separate between $S_H > \frac{n}{m}$ and $S_L < \frac{n}{m}$. $S_H > \frac{n}{m}$ describes the situation when the switching point between an active and a passive investment project is above the switching point for an open and a closed the mine in the production phase. $S_L < \frac{n}{m}$ describes the situation when the switching point between an active and a passive investment project is above the switching point for an open and a closed the mine in the production phase. The switching point $S_L$ will apply if the switching point $S_H$ turns out to be less than $\frac{n}{m}$. Whenever the value of the outcome $V$ is below $S_H$ and $S_L$ the investment project will be passive, and vice versa. A passive state has no chance of completion, $\lambda = 0$, and no on-going investment costs until completion, $k = 0$. The investment project will never be completed in this state, and since it is costless to stay here and the value of a passive investment project will therefore always be zero. An active investment project has two outcomes; if $V$ is below $\frac{n}{m}$ the completion value will be equivalent to the value of a closed mine $w(V)$, and if $V$ is greater than $\frac{n}{m}$ the completion value will jump to the value of an open mine $v(V)$. This is illustrated in figure 3.

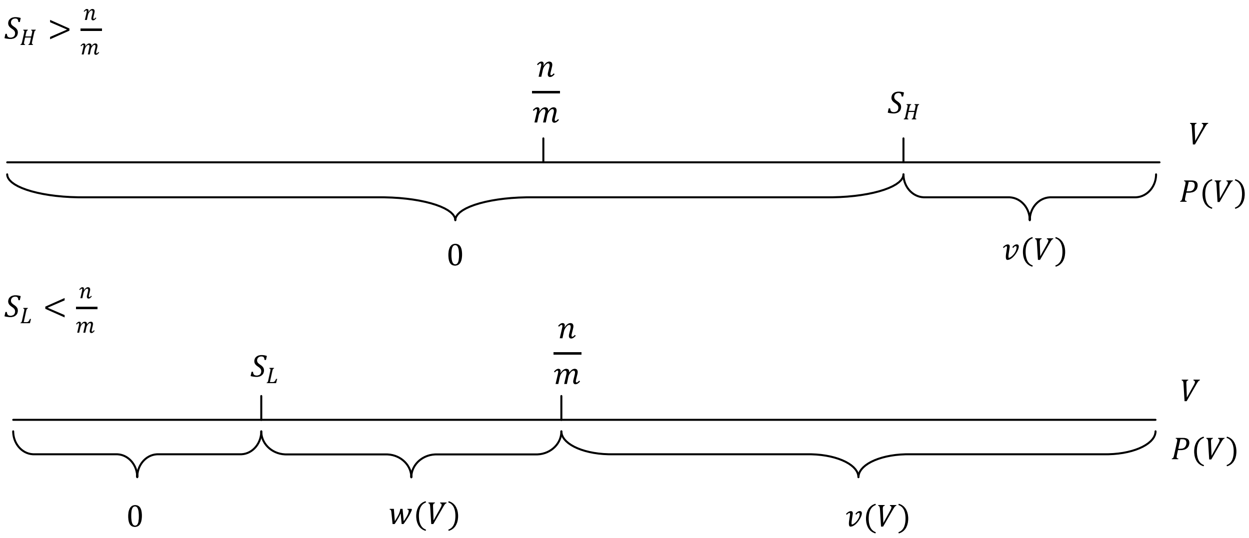


*FIGURE 3: The value of the investment project at completion depends on the value of the switching points. The value of a passive investment project will always be zero. The completion value of an active investment project will be equal to the value of the mine; if* $V < \frac{n}{m}$ *the completion value will be equal to the value of a closed mine, and if* $V > \frac{n}{m}$ *it will be equal to the value of an open mine.*

This set of ordinary equations has to be satisfied

- $S_H > \frac{n}{m}$

$$\frac{1}{2}\sigma^2 V^2 L_1''(V) + \mu V L_1'(V) - rL_1(V) = 0 \qquad \text{when } V < S_H$$

$$\frac{1}{2}\sigma^2 V^2 L_2''(V) + \mu V L_2'(V) - (r+\lambda)L_2(V) - k + \lambda(\beta_4 V^{z_2} + mV - n) = 0$$

$$\text{when } V \geq S_H$$

- $S_L < \frac{n}{m}$

$$\frac{1}{2}\sigma^2 V^2 L_3''(V) + \mu V L_3'(V) - rL_3(V) = 0 \qquad \text{when } V < S_L$$

$$\frac{1}{2}\sigma^2 V^2 L_4''(V) + \mu V L_4'(V) - (r+\lambda)L_4(V) - k + \lambda(\beta_1 V^{z_1}) = 0 \qquad \text{when } S_L \leq V < \frac{n}{m}$$

$$\frac{1}{2}\sigma^2 V^2 L_5''(V) + \mu V L_5'(V) - (r+\lambda)L_5(V) - k + \lambda(\beta_4 V^{z_2} + mV - n) = 0$$

$$\text{when } V \geq \frac{n}{m}$$

For the first two equations the switching point for an open and a closed mine in the production phase is below the switching point for an active and a passive investment project, $\frac{n}{m} < S_H$. The equation for $L_1(V)$ describes the value of the investment project when the value of the outcome is below the switching point for the investment project $V < S_H$. This corresponds to a passive investment project. It does not matter if $V$ is greater of less than $\frac{n}{m}$, the value of the passive investment project will always be zero. The equation for $L_2(V)$ describes an active investment project. The value of the outcome is above both the switching points, $\frac{n}{m} < S_H \leq V$, the investment project will therefore always be in the money in this state. The completion value of will jump to $\beta_4 V^{z_2} + mV - n$ with intensity $\lambda$, deducted the value of the increased probability of losing the investment project $\lambda L_2(V)$ and the on-going investment costs $k$ per unit of time. For the following three equations the switching point for an open and a closed mine in the production phase is above the switching point for an active and a passive investment project, $\frac{n}{m} > S_L$. For $L_3(V)$ the value of the outcome is below both of the switching points, $V < S_L < \frac{n}{m}$. The investment project is passive and the value is zero. The equation for $L_4(V)$ describes the situation when the value of the outcome is between the two switching points, $S_L \leq V < \frac{n}{m}$. The investment project is active, but it is not optimal to open the mine at completion. The completion value is $\beta_1 V^{z_1}$ with intensity $\lambda$, deducted the increased value of the probability of losing the investment project $\lambda L_4(V)$ the on-going investment cost $k$ per unit of time. For the last equation, $L_5(V)$, the value of the outcome is greater than both switching points, and the investment project is also here always in the money when it is active. The value terms are the same as for $L_2(V)$; The completion value of will jump to $\beta_4 V^{z_2} + mV - n$ with intensity $\lambda$, deducted the value of the increased probability of losing the investment project $\lambda L_5(V)$ and the on-going investment costs $k$ per unit of time.

The general solutions to the ordinary differential equations are

- $S_H > \frac{n}{m}$

$$L_1(V) = l_{11} V^{x_1} + l_{12} V^{x_2} \qquad \text{when } V < S_H$$

$$L_2(V) = l_{21} V^{y_1} + l_{22} V^{y_2} + \frac{\lambda m V}{r+\lambda-\mu} - \frac{\lambda n + k}{r+\lambda} + \frac{\lambda \beta_4 V^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} \qquad \text{when } V \geq S_H$$

- $S_L < \frac{n}{m}$

$$L_3(V) = l_{31}V^{x_1} + l_{32}V^{x_2} \qquad \text{when } V < S_L$$

$$L_4(V) = l_{41}V^{y_1} + l_{42}V^{y_2} - \frac{k}{r+\lambda} + \frac{\lambda\beta_1 V^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2} \qquad \text{when } S_L \leq V < \frac{n}{m}$$

$$L_5(V) = l_{51}V^{y_1} + l_{52}V^{y_2} + \frac{\lambda mV}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4 V^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} \qquad \text{when } V \geq \frac{n}{m}$$

Equations for all the unknowns are given in the theory about the exploration phase, chapter 2.3. Since $x_2 < 0$, the value of the investment project will increase when the value of the outcome decreases. This has to be prevented and the $V^{x_2}$ terms must therefore be eliminated and consequently $l_{12}$ and $l_{32}$ has to be zero. Furthermore, the value of the investment project never can exceed the value of the outcome. Since $y_1 > 1$ the value of the investment project will increase more than $V$. $l_{21}$ and $l_{51}$ has to be zero to eliminate the $V^{y_1}$ terms. $l_{41}$ does not have the value zero because the equation for $L_4(V)$ is two-sided, $S_L \leq V < \frac{n}{m}$. This leaves the simplified solutions

- $S_H > \frac{n}{m}$

$$L_1(V) = l_{11}V^{x_1} \qquad \text{when } V < S_H$$

$$L_2(V) = l_{22}V^{y_2} + \frac{\lambda mV}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4 V^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} \qquad \text{when } V \geq S_H$$

- $S_L < \frac{n}{m}$

$$L_3(V) = l_{31}V^{x_1} \qquad \text{when } V < S_L$$

$$L_4(V) = l_{41}V^{y_1} + l_{42}V^{y_2} - \frac{k}{r+\lambda} + \frac{\lambda\beta_1 V^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2} \qquad \text{when } S_L \leq V < \frac{n}{m}$$

$$L_5(V) = l_{52}V^{y_2} + \frac{\lambda mV}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4 V^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} \qquad \text{when } V \geq \frac{n}{m}$$

The following boundary conditions apply for the situation when the optimal switching point between an active and a passive state is greater than the switching point between an open and a closed mine in the production phase, $S_H > \frac{n}{m}$

$$L_1(S_H) = L_2(S_H)$$

$$L_1{}'(S_H) = L_2{}'(S_H)$$

$L_1(V)$ should be equal to $L_2(V)$ in the switching point between an active and a passive investment project. The optimal switching point is found by weighing the instantaneous costs and benefits for switching between an active and a passive state. The increased benefit from switching to an active state is the value of the increased intensity of completion. This has a value flow $\lambda(\beta_4 {S_H}^{z_2} + mS_H - n)$ per unit of time. The increased costs of switching to an active state are the increased intensity of losing the investment project, and also the increased on-going investment costs $k$ per unit of time. The costs has a value flow $\lambda L_2(V) + k$. This trade-off gives an equilibrium equation which is used to find the optimal switching point

$$\lambda(\beta_4 {S_H}^{z_2} + mS_H - n) = \lambda L_2(S_H) + k$$

The equation can be rearranged, and because of the first boundary condition $L_2(S_H)$ can be replaced by $L_1(S_H)$

$$L_1(S_H) = L_2(S_H) = \beta_4 {S_H}^{z_2} + mS_H - n - \frac{k}{\lambda}$$

The boundary conditions are used to find $l_{11}$ and $l_{22}$. I derive the equations in appendix A.

$$l_{11} = \frac{1}{(y_2 - x_1){S_H}^{x_1}} \left( \frac{(y_2 - 1)\lambda m S_H}{r + \lambda - \mu} + \frac{(y_2 - z_2)\lambda \beta_4 {S_H}^{z_2}}{r + \lambda - z_2\mu - \frac{1}{2} z_2(z_2 - 1)\sigma^2} - \frac{y_2(\lambda n + k)}{r + \lambda} \right)$$

$$l_{22} = -\frac{1}{(y_2 - x_1){S_H}^{y_2}} \left( \frac{(1 - x_1)\lambda m S_H}{r + \lambda - \mu} + \frac{(z_2 - x_1)\lambda \beta_4 {S_H}^{z_2}}{r + \lambda - z_2\mu - \frac{1}{2} z_2(z_2 - 1)\sigma^2} + \frac{x_1(\lambda n + k)}{r + \lambda} \right)$$

The optimal switching point is found by inserting for $l_{11}$or $l_{22}$ in $L_1(V)$ or $L_2(V)$ and using the equilibrium equation. $L(S_H)$ in the equilibrium equation can be replaced by $L_1(V)$ or $L_2(V)$ when $V = S_H$. This is shown in appendix B. The optimal switching point is found by solving a non-linear equation

$$\left(\frac{(y_2-1)\lambda}{r+\lambda-\mu}-(y_2-x_1)\right)mS_H+\left(\frac{(y_2-z_2)\lambda}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}-(y_2-x_1)\right)\beta_4{S_H}^{z_2}$$
$$-\frac{y_2(\lambda n+k)}{r+\lambda}+\frac{(y_2-x_1)(\lambda n+k)}{\lambda}=0$$

The following boundary conditions apply for the situation when the optimal switching point between an active and a passive state is below the switching point between an open and a closed mine in the production phase, $S_L < \frac{n}{m}$

$$L_3(S_L) = L_4(S_L)$$

$$L_3{}'(S_L) = L_4{}'(S_L)$$

$$L_4\left(\frac{n}{m}\right) = L_5\left(\frac{n}{m}\right)$$

$$L_4{}'\left(\frac{n}{m}\right) = L_5{}'\left(\frac{n}{m}\right)$$

$L_3(V)$ should be equal to $L_4(V)$ in the switching point between an active and a passive investment project $S_L$ , and $L_4(V)$ should be equal to $L_5(V)$ in the switching point between an open and a closed mine in the production phase $\frac{n}{m}$. The optimal switching point for the investment project is found the same way as above, by weighing the instantaneous costs and benefits from switching between an active and a passive state. The increased benefits of switching to an active state have a value flow $\lambda(\beta_1 V^{z_1})$, and the increased costs have the value flow $\lambda L_4(V)+k$. The trade-off between the benefits and costs gives an equilibrium equation for the optimal switching point

$$\lambda(\beta_1 {S_L}^{z_1}) = \lambda L_4(S_L)+k$$

Since $L_3(S_L)$ is equal to $L_4(S_L)$, it can replace $L_4(S_L)$ in the equation. Rearranging the equation gives this equilibrium equation

$$L_3(S_L) = L_4(S_L) = \beta_1 {S_L}^{z_1} - \frac{k}{\lambda}$$

The boundary conditions are used to find $l_{31}$, $l_{41}$, $l_{42}$ and $l_{52}$. The equations are derived in appendix A.

$$l_{31} = \frac{1}{(y_2 - x_1){S_L}^{x_1}}\left((y_2 - y_1)l_{41}{S_L}^{y_1} - \frac{y_2 k}{r+\lambda} + \frac{(y_2 - z_1)\lambda\beta_1 {S_L}^{z_1}}{r+\lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2}\right)$$

$$l_{41} = \frac{1}{(y_2 - y_1)\left(\frac{n}{m}\right)^{y_1}}\left(\frac{(y_2 - 1)\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu} - \frac{y_2\lambda n}{r+\lambda} - \frac{(y_2 - z_1)\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2} + \frac{(y_2 - z_2)\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda - z_2\mu - \frac{1}{2}z_2(z_2 - 1)\sigma^2}\right)$$

$$l_{42} = -\frac{1}{(y_2 - x_1){S_L}^{y_2}}\left((y_1 - x_1)l_{41}{S_L}^{y_1} + \frac{(z_1 - x_1)\lambda\beta_1 {S_L}^{z_1}}{r+\lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2} + \frac{x_1 k}{r+\lambda}\right)$$

$$l_{52} = l_{42} + \frac{1}{(y_2 - y_1)\left(\frac{n}{m}\right)^{y_2}}\left(-\frac{(z_2 - y_1)\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda - z_2\mu - \frac{1}{2}z_2(z_2 - 1)\sigma^2} - \frac{(y_1 - z_1)\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2} + \frac{(y_1 - 1)\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu} - \frac{y_1\lambda n}{r+\lambda}\right)$$

The optimal switching point is found by inserting for $l_{31}$ in $L_3(V)$ or for $l_{41}$ and $l_{42}$ in $L_4(V)$ and using the equilibrium equation. $L(S_H)$ in the equilibrium equation can be replaced by $L_3(V)$ or $L_4(V)$ when $V = S_H$. This is shown in appendix B. The equation for the optimal switching point is also here a non-linear equation

$$\left((y_2 - x_1) - \frac{(y_2 - z_1)\lambda}{r+\lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2}\right)\beta_1 {S_L}^{z_1} - (y_2 - y_1)l_{41}{S_L}^{y_1} - \frac{(y_2 - x_1)k}{\lambda} + \frac{y_2 k}{r+\lambda} = 0$$

I now have all the equations I need to evaluate the exploration phase and productions phase simultaneously. In the next section I will construct an example to see how this works in practice and how the value investment project and the switching points are affected by changes in the parameter values.

# 4. Analysis

In this part I construct an example to illustrate my model numerically. I examine how the value investment project and the switching points are affected by changes in the parameter values. I mostly use the parameter values from Miltersen and Schwartz's and Brennan and Schwartz's examples but I make some simplifying assumptions. To be able to use the ordinary differential equation from Brennan and Schwartz's model I have already assumed that the mine inventory $Q$ is infinite and that there is no maintenance costs for a closed mine, $f = 0$. In my model I have also assumed that the final investment cost $K$ is equal to zero and that there are no switching costs between an open and a closed mine, $k_1 = k_2 = 0$. To simplify the computation of the model I further assume that there is no inflation $\pi = 0$, no taxes $t_1 = t_2 = 0$, and that the interest rates in the two phases are constant and the same, 5 %. After combining the models the commodity price of the resource extracted from the mine is the same as value of the outcome in the exploration phase. The value of the outcome of the two phases is therefore the same; hence, the volatility is the same.

## 4.1 Base Case

The parameter values for the base case scenario are shown in table 3. The different variable values that follows from the base case parameters are shown in table 4.

| Parameter values – Base case | | | |
|---|---|---|---|
| Instantaneous drift of the value process | $\mu$ | 3 | % per year |
| Instantaneous volatility of the value process | $\sigma$ | 20 | % per year |
| Expected time to completion of the project | $T$ | 5 | years |
| On-going investment costs rate until completion | $k$ | 1 | million $ per year |
| Riskless interest rate | $r$ | 5 | % per year |
| Mine output rate | $q*$ | 10 | million pounds per year |
| Mine inventory | $Q$ | infinite | million pounds |
| Initial average cost of production | $a$ | 0,5 | $ per pound |
| Convenience yield of the commodity | $\kappa$ | 1 | % per year |
| Intensity of expropriation of the mine | $\delta$ | 2 | % per year |

*TABLE 3: Parameter values in the combined model - Base case*

| Variable values – Base case | |
|---|---|
| $\lambda$ | 0,20 |
| $x_1$ | 1,35 |
| $x_2$ | -1,85 |
| $y_1$ | 3,29 |
| $y_2$ | -3,79 |
| $m$ | 10,00 |
| $n$ | 5,00 |
| $\frac{n}{m}$ | 0,50 |
| $\alpha_1$ | -0,50 |
| $\alpha_2$ | 1,94 |
| $z_1$ | 1,44 |
| $z_2$ | -2,44 |
| $b$ | -71,43 |
| $d$ | 333,33 |
| $\beta_1$ | 278,64 |
| $\beta_4$ | 1,42 |

*TABLE 4: Variable values derived from the base case parameters. These are calculated using Excel.*

From table 4 you see that the switching point between an open and a closed mine $\frac{n}{m}$ is equal to \$0,5 million. The switching point depends on the average cost of production and taxes. This follows from the formulas for *m* and *n*

$$m = q^*(1 - t_1)(1 - t_2)$$

$$n = q^* a(1 - t_2)$$

$$\frac{n}{m} = \frac{a}{(1 - t_1)}$$

Since there are no taxes the switching point between an open and a closed mine is equal to the average cost of production and it is constant even when other parameters changes. To find the optimal switching point between an active and a passive investment project I have to solve the non-linear equations I found in the previous chapter

$$\left(\frac{(y_2-1)\lambda}{r+\lambda-\mu}-(y_2-x_1)\right)mS_H+\left(\frac{(y_2-z_2)\lambda}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}-(y_2-x_1)\right)\beta_4 S_H{}^{z_2}$$
$$-\frac{y_2(\lambda n+k)}{r+\lambda}+\frac{(y_2-x_1)(\lambda n+k)}{\lambda}=0$$

and

$$\left((y_2{-}x_1)-\frac{(y_2-z_1)\lambda}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}\right)\beta_1 S_L{}^{z_1}-(y_2-y_1)l_{41}S_L{}^{y_1}-\frac{(y_2{-}x_1)k}{\lambda}$$
$$+\frac{y_2 k}{r+\lambda}=0$$

I will use the equation for $S_H$ first. If I find that the optimal switching point for the investment project is less than the switching point for an open and a closed mine (less than \$0,5 million) I will have to use the equation for $S_L$. Using the equation for $S_H$ I find two optimal switching points: $S_H$ = \$0,61 million and $S_H$ = \$2,62 million. The values of the non-linear equation for different values of the switching point are shown in figure 4. The optimal switching points are found when the equation is equal to zero. This is in the intersection between the graph and the horizontal axis.

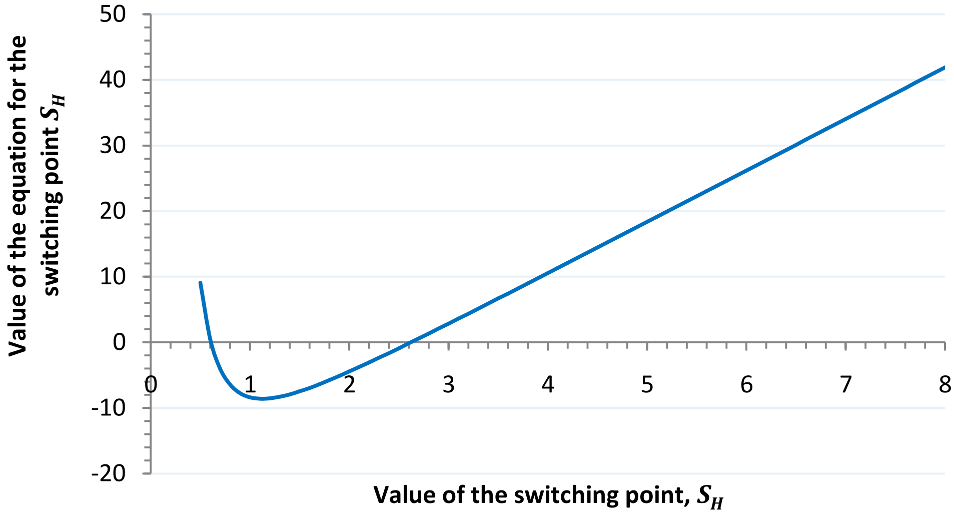


*FIGURE 4: Optimal switching point for an active and a passive investment project using the base case parameters. The figure shows the values of the non-linear equation for $S_H$ as a result of different values of $S_H$. The optimal switching points are found in the intersection with the horizontal axis, when $S_H$ = 0,61 and $S_H$ = 2,62.*

Why are there two switching points? This is because it is a non-linear equation and the chosen parameter values give this result. The equation will give two switching points for all my scenarios. I choose always to use the highest switching point. The results from using the lowest switching points are inconsistent with theory and I find that these switching points are not valid. The analysis of the switching points are shown in appendix C.

The investment project is active when the value of the outcome is greater than $2,62 million, and passive when the value of the outcome is below this threshold level. The values of the investment project for different values of the outcome are shown in table 5 and graphically in figure 5.

| Value of the outcome $V$ | $S_H > \frac{n}{m}$ | | |
|---|---|---|---|
| | $V < S_H$ $L_1(V)$ | $V \geq S_H$ $L_2(V)$ | Value of the investment project $L(V)$ |
| 0 | 0,00 | 0,00 | 0,00 |
| 1 | 4,45 | 16,35 | 4,45 |
| 2 | 11,35 | 11,49 | 11,35 |
| 2,62 | 16,34 | 16,34 | 16,34 |
| 3 | 19,62 | 19,61 | 19,61 |
| 4 | 28,94 | 28,50 | 28,50 |
| 5 | 39,12 | 37,52 | 37,52 |
| 6 | 50,04 | 46,58 | 46,58 |
| 7 | 61,63 | 55,66 | 55,66 |
| 8 | 73,81 | 64,74 | 64,74 |
| 9 | 86,54 | 73,83 | 73,83 |
| 10 | 99,77 | 82,92 | 82,92 |

*TABLE 5: Value of the investment project for different values of the outcome using the base case parameters. The values of a passive and an active investment project are shown separately, and they are combined in the last column to get the value of the investment project. In the switching point $S_H$ = 2,62 the value of an active and a passive investment project are equal.*

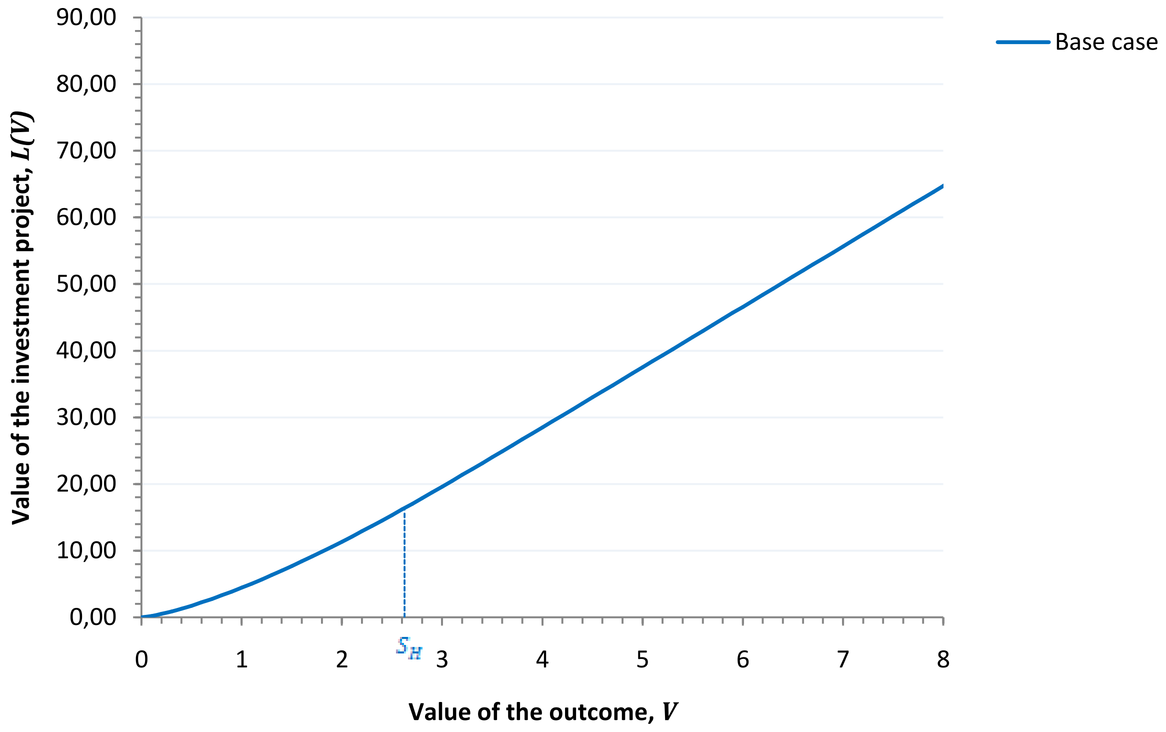


*FIGURE 5: Value of the investment project for different values of the outcome using the base case parameters. The switching point $S_H$ is 2,62. The value of the investment project is equivalent to the value of a passive investment project when $V$ is below this threshold level and equivalent to value of an active investment project when $V$ is above this threshold level.*

To see how the value of the investment project is affected by changes in the parameter values I have tried with different values for expected time to completion $T$ and on-going investment costs until completion $k$. This affects the value of the exploration phase. My model behaves in the same way as Miltersen and Schwartz's model from equivalent changes in parameter values.

## 4.2 Change in Expected Time to Completion

Recall that the expected time to completion is one divided by the probability of completion per unit of time, $T = {}^{1}/_{\lambda}$. From this simple relationship I see that the expected time to completion goes towards infinity when the probability of completion goes towards zero and towards one when the probability of completion goes towards one. Hence, the shorter the expected time to completion, the greater the probability of completion. I tried with different expected time to completion to see what happens to the value of the investment project. When $T$ = 1 the switching point between an active and a passive investment project, $S_H^{T=1}$, equal to \$1,72 million. This threshold level is lower than the base case scenario. On the other hand, when $T$ = 10 the switching points raises to $S_H^{T=10}$ = \$3,44 million.

Figure 6 shows how the value of the investment project changes when expected time to completion changes. Shorter expected time to completion gives a higher probability of completion and therefore a higher value of the investment project. The on-going investment costs until completion are smaller when the expected time to completion is shorter, which also increases the value of the investment project. The rise in value of the investment project is shown as a shift to the left in the figure. Conversely, when the expected time to completion is higher there is a smaller probability of completion and the on-going investment costs are greater. The value of the investment project is therefore lower.

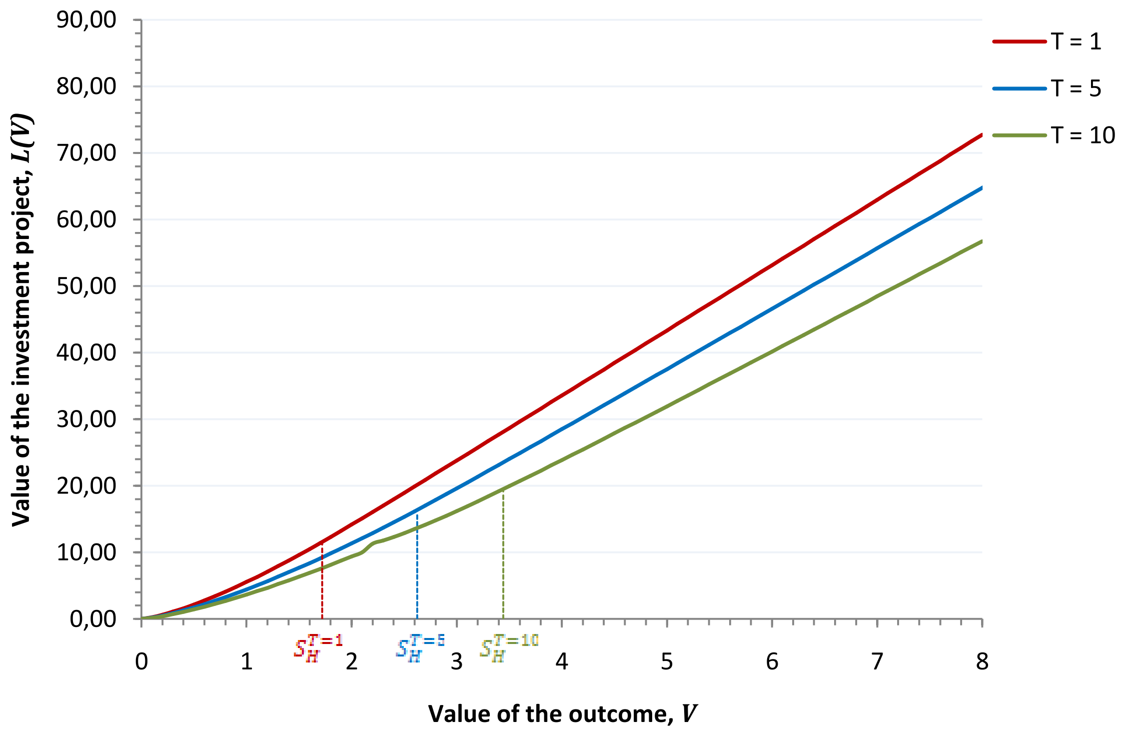


*FIGURE 6: Value of the investment project when the expected time to completion changes. $T$ = 5 represents the base case scenario with switching point $S_H^{T=5}$ = 2,62. When $T$ = 1 the value of the investment project rises, and the threshold level is $S_H^{T=1}$ = 1,72. When $T$ = 10 the value of the investment project falls and the threshold level is $S_H^{T=10}$ = 3,44.*

## 4.3 Change in On-Going Investment Costs until Completion

The investment project has on-going investment cost per unit of time until completion when it is active. A change in these costs will affect the value of the investment project. When the on-going investment costs are smaller the value of the investment project will obviously be higher. When $k$ = $0,2 million the optimal switching point between an active and a passive state decreases to $S_H^{k=0,2}$ = $1,26 million. When the on-going investment costs are higher the value of the investment project will decrease and the optimal switching point will be higher. The investment project will therefore switch to a passive state faster if the value of the outcome declines; it is less profitable to be in the active state. If $k$ = $2 million the optimal switching point between an active and a passive state is $S_H^{k=2}$ = $4 million. The solution is shown in figure 7.

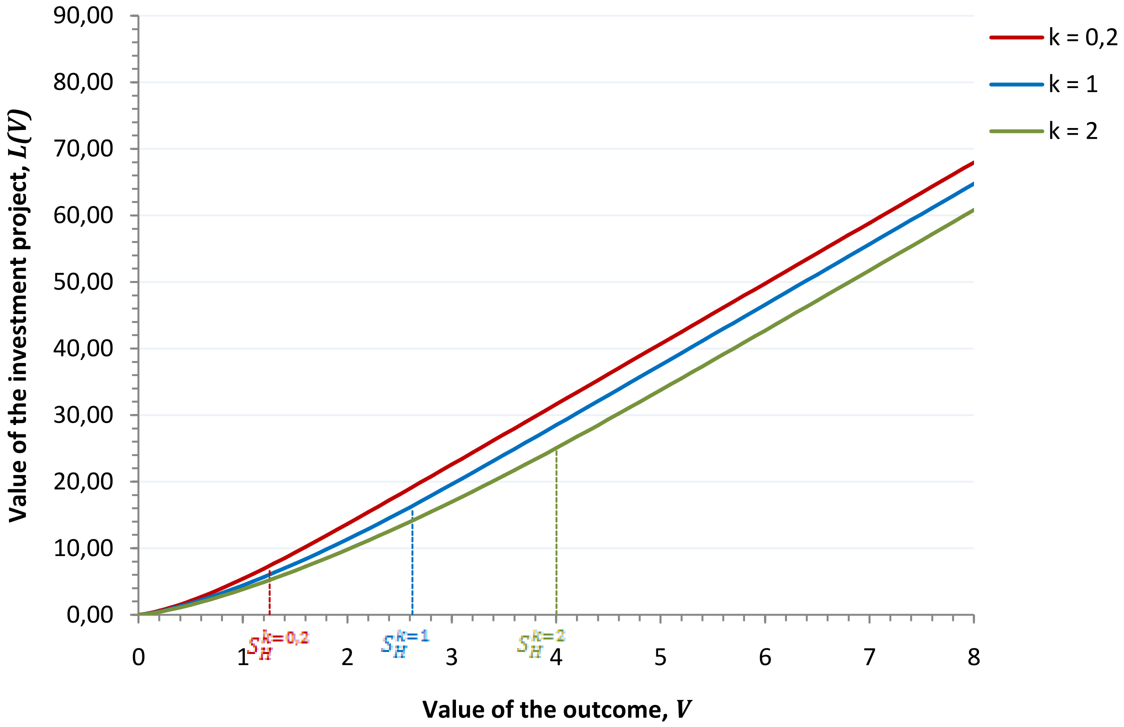


*FIGURE 7: Value of the investment project when the on-going investment costs until completion changes. When k is lower the value of the investment project will increase, and vice versa. k = 1 represents the base case scenario with switching point $S_H^{k=1}$ = 2,62. k = 0,2 has a threshold level at $S_H^{k=0,2}$ = 1,26 and k = 2 has a threshold level at $S_H^{k=2}$ = 4,00.*

## 4.4 Summary of the Analysis

For all the scenarios I have analysed the optimal switching point for an active and a passive investment project turned out to be above the switching point for an open and a closed mine. I therefore used the optimal switching point $S_H$ in all cases. Notice that the value of the investment project is always positive. This is because there are no costs incurred in the passive state. By switching to the passive state the owner of the investment project can protect the project against losses when the value of the outcome decreases under a certain threshold level. The same applies for the value of the mine; there is no costs incurred when the mine is closed and the production can be shut down when the value of the outcome decreases under the (different) threshold level for closing the mine.

The outcome of the analysis is consistent with theory. When the expected time to completion is higher the value of the investment project will decrease, and vice versa. The value of the investment project will also decrease if there are higher on-going investment costs until completion.

# 5. Conclusions

Can real option theory be used to make a model that simultaneously evaluates the exploration phase and production phase of an investment project?

I have managed construct a model that uses the value of the production phase to evaluate an exploration project. At completion of the exploration phase the owner of the investment project has an option to the value of the outcome. The value of the outcome is the present value of future cash flows that can be generated from producing/extracting the outcome of the exploration. The model includes switching options in both phases.

I am able to obtain closed form solutions to the value of the investment project and to the optimal switching points. I have found that the model's behavior is consistent with theory; longer expected time to completion and higher on-going investment costs until completion will reduce the value of the investment project, and vice versa.

The model is applicable for R&D projects and natural resource exploration projects such as mine or oil exploration project. The model has important qualities for the valuation of such investment projects; the output price is treated as stochastic, there are on-going investment costs in the exploration phase, the time to completion is uncertain, and it considers the ability to shut down a money losing project and restart it again when in both the exploration phase and the production phase.

To make the model even more applicable to real investment projects, some of the assumptions would have to be relaxed. The model would appear more realistic if I removed the assumption that there are no final investment costs in the exploration phase. This would not be very complicated and the model would not change drastically, but it would make the model a lot more realistic. Adding final investment costs would cause the value of the investment project to decrease. It would be more complex to relax the assumption that there are no switching costs to open and close the mine in the production phase, and no costs to switch between an active and a passive investment project. This could be interesting to take a look at in a further development of the model.

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# Appendix A – Equations for $\mathbf{l_{11}}$, $\mathbf{l_{22}}$, $\mathbf{l_{31}}$,$\mathbf{l_{41}}$, $\mathbf{l_{42}}$ and $\mathbf{l_{52}}$

Here I derive the equations for $l_{11}$, $l_{22}$, $l_{31}$,$l_{41}$, $l_{42}$ and $l_{52}$. I use the general solutions and the boundary condition from chapter 3.1 to find these.

- $S_H > \frac{n}{m}$

$$L_1(V) = l_{11}V^{x_1} \qquad \text{when } V < S_H$$

$$L_2(V) = l_{22}V^{y_2} + \frac{\lambda mV}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4 V^{z2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} \qquad \text{when } V \geq S_H$$

Boundary conditions

$$L_1(S_H) = L_2(S_H)$$

$$L_1{}'(S_H) = L_2{}'(S_H)$$

- $S_L < \frac{n}{m}$

$$L_3(V) = l_{31}V^{x_1} \qquad \text{when } V < S_L$$

$$L_4(V) = l_{41}V^{y_1} + l_{42}V^{y_2} - \frac{k}{r+\lambda} + \frac{\lambda\beta_1 V^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2} \qquad \text{when } S_L \leq V < \frac{n}{m}$$

$$L_5(V) = l_{52}V^{y_2} + \frac{\lambda mV}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4 V^{z2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} \qquad \text{when } V \geq \frac{n}{m}$$

Boundary conditions

$$L_3(S_L) = L_4(S_L)$$

$$L_3{}'(S_L) = L_4{}'(S_L)$$

$$L_4\left(\frac{n}{m}\right) = L_5\left(\frac{n}{m}\right)$$

$$L_4{}'\left(\frac{n}{m}\right) = L_5{}'\left(\frac{n}{m}\right)$$

**Deriving the equation for $l_{11}$**

$L_1(V) = l_{11}V^{x_1}$ when $V < S_H$

$L_2(V) = l_{22}V^{y_2} + \frac{\lambda mV}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4 V^{z2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$ when $V \geq S_H$

I set $L_1(S_H) = L_2(S_H)$

$$l_{11}{S_H}^{x_1} = l_{22}{S_H}^{y_2} + \frac{\lambda mS_H}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

I set the 1st derivatives equal to each other $L_1{}'(S_H) = L_2{}'(S_H)$ and solve for $l_{22}$

$$x_1 l_{11}{S_H}^{x_1-1} = y_2 l_{22}{S_H}^{y_2-1} + \frac{\lambda m}{r+\lambda-\mu} + \frac{z_2\lambda\beta_4 {S_H}^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

$$l_{22} = \frac{1}{y_2{S_H}^{y_2-1}}\left(x_1 l_{11}{S_H}^{x_1-1} - \frac{\lambda m}{r+\lambda-\mu} - \frac{z_2\lambda\beta_4 {S_H}^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}\right)$$

I insert for $l_{22}$ in $L_1(S_H) = L_2(S_H)$

$$l_{11}{S_H}^{x_1} = \frac{{S_H}^{y_2}}{y_2{S_H}^{y_2-1}}\left(x_1 l_{11}{S_H}^{x_1-1} - \frac{\lambda m}{r+\lambda-\mu} - \frac{z_2\lambda\beta_4 {S_H}^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}\right) + \frac{\lambda mS_H}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

$$l_{11}{S_H}^{x_1} = \frac{S_H}{y_2}\left(x_1 l_{11}{S_H}^{x_1-1} - \frac{\lambda m}{r+\lambda-\mu} - \frac{z_2\lambda\beta_4 {S_H}^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}\right) + \frac{\lambda mS_H}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

I multiply by $y_2$ to remove the brackets

$$y_2 l_{11}{S_H}^{x_1} = x_1 l_{11}{S_H}^{x_1} - \frac{\lambda mS_H}{r+\lambda-\mu} - \frac{z_2\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} + \frac{y_2\lambda mS_H}{r+\lambda-\mu} - \frac{y_2(\lambda n+k)}{r+\lambda} + \frac{y_2\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

I collect and draw together similar terms

$$(y_2 - x_1)l_{11}{S_H}^{x_1} = \frac{(y_2 - 1)\lambda m S_H}{r + \lambda - \mu} + \frac{(y_2 - z_2)\lambda\beta_4 {S_H}^{z_2}}{r + \lambda - z_2\mu - \frac{1}{2}z_2(z_2 - 1)\sigma^2} - \frac{y_2(\lambda n + k)}{r + \lambda}$$

I solve for $l_{11}$ and get the solution

$$l_{11} = \frac{1}{(y_2 - x_1){S_H}^{x_1}}\left(\frac{(y_2 - 1)\lambda m S_H}{r + \lambda - \mu} + \frac{(y_2 - z_2)\lambda\beta_4 {S_H}^{z_2}}{r + \lambda - z_2\mu - \frac{1}{2}z_2(z_2 - 1)\sigma^2} - \frac{y_2(\lambda n + k)}{r + \lambda}\right)$$

**Deriving the equation for $l_{22}$**

$L_1(V) = l_{11}V^{x_1}$ when $V < S_H$

$L_2(V) = l_{22}V^{y_2} + \frac{\lambda m V}{r+\lambda-\mu} - \frac{\lambda n + k}{r+\lambda} + \frac{\lambda\beta_4 V^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$ when $V \geq S_H$

I set $L_1(S_H) = L_2(S_H)$

$$l_{11}{S_H}^{x_1} = l_{22}{S_H}^{y_2} + \frac{\lambda m S_H}{r + \lambda - \mu} - \frac{\lambda n + k}{r + \lambda} + \frac{\lambda\beta_4 {S_H}^{z_2}}{r + \lambda - z_2\mu - \frac{1}{2}z_2(z_2 - 1)\sigma^2}$$

I set the 1st derivatives equal to each other $L_1{}'(S_H) = L_2{}'(S_H)$ and solve for $l_{11}$

$$x_1 l_{11}{S_H}^{x_1 - 1} = y_2 l_{22}{S_H}^{y_2 - 1} + \frac{\lambda m}{r + \lambda - \mu} + \frac{z_2\lambda\beta_4 {S_H}^{z_2 - 1}}{r + \lambda - z_2\mu - \frac{1}{2}z_2(z_2 - 1)\sigma^2}$$

$$l_{11} = \frac{1}{x_1{S_H}^{x_1 - 1}}\left(y_2 l_{22}{S_H}^{y_2 - 1} + \frac{\lambda m}{r + \lambda - \mu} + \frac{z_2\lambda\beta_4 {S_H}^{z_2 - 1}}{r + \lambda - z_2\mu - \frac{1}{2}z_2(z_2 - 1)\sigma^2}\right)$$

I insert for $l_{11}$ in $L_1(S_H) = L_2(S_H)$

$$\frac{{S_H}^{x_1}}{x_1{S_H}^{x_1 - 1}}\left(y_2 l_{22}{S_H}^{y_2 - 1} + \frac{\lambda m}{r + \lambda - \mu} + \frac{z_2\lambda\beta_4 {S_H}^{z_2 - 1}}{r + \lambda - z_2\mu - \frac{1}{2}z_2(z_2 - 1)\sigma^2}\right)$$
$$= l_{22}{S_H}^{y_2} + \frac{\lambda m S_H}{r + \lambda - \mu} - \frac{\lambda n + k}{r + \lambda} + \frac{\lambda\beta_4 {S_H}^{z_2}}{r + \lambda - z_2\mu - \frac{1}{2}z_2(z_2 - 1)\sigma^2}$$

$$\frac{S_H}{x_1}\left(y_2 l_{22} {S_H}^{y_2-1} + \frac{\lambda m}{r+\lambda-\mu} + \frac{z_2\lambda\beta_4 {S_H}^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}\right)$$
$$= l_{22}{S_H}^{y_2} + \frac{\lambda m S_H}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

I multiply by $x_1$ to remove the brackets

$$y_2 l_{22}{S_H}^{y_2} + \frac{\lambda m S_H}{r+\lambda-\mu} + \frac{z_2\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$
$$= x_1 l_{22}{S_H}^{y_2} + \frac{x_1\lambda m S_H}{r+\lambda-\mu} - \frac{x_1(\lambda n+k)}{r+\lambda} + \frac{x_1\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

I collect and draw together similar terms

$$y_2 l_{22}{S_H}^{y_2} - x_1 l_{22}{S_H}^{y_2} + \frac{\lambda m S_H}{r+\lambda-\mu} - \frac{x_1\lambda m S_H}{r+\lambda-\mu} + \frac{z_2\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$
$$- \frac{x_1\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} = -\frac{x_1(\lambda n+k)}{r+\lambda}$$

$$(y_2-x_1)l_{22}{S_H}^{y_2} + \frac{(1-x_1)\lambda m S_H}{r+\lambda-\mu} + \frac{(z_2-x_1)\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} = -\frac{x_1(\lambda n+k)}{r+\lambda}$$

I solve for $l_{22}$ and get the solution

$$l_{22} = -\frac{1}{(y_2-x_1){S_H}^{y_2}}\left(\frac{(1-x_1)\lambda m S_H}{r+\lambda-\mu} + \frac{(z_2-x_1)\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} + \frac{x_1(\lambda n+k)}{r+\lambda}\right)$$

**Deriving the equation for $l_{31}$**

$L_3(V) = l_{31}V^{x_1}$ when $V < S_L$

$L_4(V) = l_{41}V^{y_1} + l_{42}V^{y_2} - \frac{k}{r+\lambda} + \frac{\lambda\beta_1 V^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$ when $S_L \le V < \frac{n}{m}$

I set $L_3(S_L) = L_4(S_L)$

$$l_{31}{S_L}^{x_1} = l_{41}{S_L}^{y_1} + l_{42}{S_L}^{y_2} - \frac{k}{r+\lambda} + \frac{\lambda\beta_1 {S_L}^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

I insert for $l_{42} = -\frac{1}{(y_2-x_1)S_L{}^{y_2}}\left((y_1-x_1)l_{41}S_L{}^{y_1} + \frac{(z_1-x_1)\lambda\beta_1 S_L{}^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2} + \frac{x_1 k}{r+\lambda}\right)$

$$l_{31}S_L{}^{x_1} = l_{41}S_L{}^{y_1} - \frac{S_L{}^{y_2}}{(y_2-x_1)S_L{}^{y_2}}\left((y_1-x_1)l_{41}S_L{}^{y_1} + \frac{(z_1-x_1)\lambda\beta_1 S_L{}^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2} + \frac{x_1 k}{r+\lambda}\right) - \frac{k}{r+\lambda} + \frac{\lambda\beta_1 S_L{}^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

I multiply all terms with $(y_2-x_1)$ to remove the bracket

$$(y_2-x_1)l_{31}S_L{}^{x_1} = (y_2-x_1)l_{41}S_L{}^{y_1} - (y_1-x_1)l_{41}S_L{}^{y_1} - \frac{(z_1-x_1)\lambda\beta_1 S_L{}^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2} - \frac{x_1 k}{r+\lambda} - \frac{(y_2-x_1)k}{r+\lambda} + \frac{(y_2-x_1)\lambda\beta_1 S_L{}^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

I collect and draw together similar terms

$$(y_2-x_1)l_{31}S_L{}^{x_1} = (y_2-y_1)l_{41}S_L{}^{y_1} - \frac{y_2 k}{r+\lambda} + \frac{(y_2-z_1)\lambda\beta_1 S_L{}^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

I solve for $l_{31}$ and get the solution

$$l_{31} = \frac{1}{(y_2-x_1)S_L{}^{x_1}}\left((y_2-y_1)l_{41}S_L{}^{y_1} - \frac{y_2 k}{r+\lambda} + \frac{(y_2-z_1)\lambda\beta_1 S_L{}^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}\right)$$

**Deriving the equation for $l_{41}$**

$L_4\left(\frac{n}{m}\right) = l_{41}\left(\frac{n}{m}\right)^{y_1} + l_{42}\left(\frac{n}{m}\right)^{y_2} - \frac{k}{r+\lambda} + \frac{\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$ when $S_L \leq V < \frac{n}{m}$

$L_5\left(\frac{n}{m}\right) = l_{52}\left(\frac{n}{m}\right)^{y_2} + \frac{\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$ when $V \geq \frac{n}{m}$

I set $L_4\left(\frac{n}{m}\right) = L_5\left(\frac{n}{m}\right)$

$$l_{41}\left(\frac{n}{m}\right)^{y_1} + l_{42}\left(\frac{n}{m}\right)^{y_2} - \frac{k}{r+\lambda} + \frac{\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$
$$= l_{52}\left(\frac{n}{m}\right)^{y_2} + \frac{\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

I set the 1st derivatives equal to each other $L_4'\left(\frac{n}{m}\right) = L_5'\left(\frac{n}{m}\right)$ and solve for $l_{52}$

$$y_1 l_{41}\left(\frac{n}{m}\right)^{y_1-1} + y_2 l_{42}\left(\frac{n}{m}\right)^{y_2-1} + \frac{z_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1-1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$
$$= y_2 l_{52}\left(\frac{n}{m}\right)^{y_2-1} + \frac{\lambda m}{r+\lambda-\mu} + \frac{z_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

$$l_{52} = \frac{1}{y_2\left(\frac{n}{m}\right)^{y_2-1}}\left(y_1 l_{41}\left(\frac{n}{m}\right)^{y_1-1} + y_2 l_{42}\left(\frac{n}{m}\right)^{y_2-1} - \frac{\lambda m}{r+\lambda-\mu}\right.$$
$$\left.+ \frac{z_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1-1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2} - \frac{z_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}\right)$$

I insert for $l_{52}$ in $L_4\left(\frac{n}{m}\right) = L_5\left(\frac{n}{m}\right)$

$$l_{41}\left(\frac{n}{m}\right)^{y_1} + l_{42}\left(\frac{n}{m}\right)^{y_2} - \frac{k}{r+\lambda} + \frac{\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$
$$= \frac{\left(\frac{n}{m}\right)^{y_2}}{y_2\left(\frac{n}{m}\right)^{y_2-1}}\left(y_1 l_{41}\left(\frac{n}{m}\right)^{y_1-1} + y_2 l_{42}\left(\frac{n}{m}\right)^{y_2-1} - \frac{\lambda m}{r+\lambda-\mu}\right.$$
$$\left.+ \frac{z_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1-1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2} - \frac{z_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}\right)$$
$$+ \frac{\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

I multiply all terms with $y_2$ to remove the bracket

$$y_2 l_{41}\left(\frac{n}{m}\right)^{y_1}+y_2 l_{42}\left(\frac{n}{m}\right)^{y_2}-\frac{y_2 k}{r+\lambda}+\frac{y_2\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$
$$=y_1 l_{41}\left(\frac{n}{m}\right)^{y_1}+y_2 l_{42}\left(\frac{n}{m}\right)^{y_2}-\frac{\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu}+\frac{z_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$
$$-\frac{z_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}+\frac{y_2\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu}-\frac{y_2(\lambda n+k)}{r+\lambda}$$
$$+\frac{y_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

I collect and draw together similar terms

$$y_2 l_{41}\left(\frac{n}{m}\right)^{y_1}-y_1 l_{41}\left(\frac{n}{m}\right)^{y_1}+y_2 l_{42}\left(\frac{n}{m}\right)^{y_2}-y_2 l_{42}\left(\frac{n}{m}\right)^{y_2}+\frac{y_2\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$
$$-\frac{z_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$
$$=\frac{y_2\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu}-\frac{\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu}+\frac{y_2 k}{r+\lambda}-\frac{y_2(\lambda n+k)}{r+\lambda}$$
$$+\frac{y_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}-\frac{z_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

$$(y_2-y_1)l_{41}\left(\frac{n}{m}\right)^{y_1}+\frac{(y_2-z_1)\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$
$$=\frac{(y_2-1)\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu}-\frac{y_2\lambda n}{r+\lambda}+\frac{(y_2-z_2)\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

The $l_{42}$ term disappears. I solve for $l_{41}$ and get the solution

$$l_{41} = \frac{1}{(y_2 - y_1)\left(\frac{n}{m}\right)^{y_1}} \left( \frac{(y_2 - 1)\lambda m \left(\frac{n}{m}\right)}{r + \lambda - \mu} - \frac{y_2 \lambda n}{r + \lambda} - \frac{(y_2 - z_1)\lambda \beta_1 \left(\frac{n}{m}\right)^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} + \frac{(y_2 - z_2)\lambda \beta_4 \left(\frac{n}{m}\right)^{z_2}}{r + \lambda - z_2 \mu - \frac{1}{2} z_2 (z_2 - 1)\sigma^2} \right)$$

**Deriving the equation for $l_{42}$**

$L_3(S_L) = l_{31} {S_L}^{x_1}$ when $V < S_L$

$L_4(S_L) = l_{41} {S_L}^{y_1} + l_{42} {S_L}^{y_2} - \frac{k}{r+\lambda} + \frac{\lambda \beta_1 {S_L}^{z_1}}{r+\lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2}$ when $S_L \leq V < \frac{n}{m}$

I set $L_3(S_L) = L_4(S_L)$

$$l_{31} {S_L}^{x_1} = l_{41} {S_L}^{y_1} + l_{42} {S_L}^{y_2} - \frac{k}{r + \lambda} + \frac{\lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2}$$

I set the 1st derivatives equal to each other $L_3'(S_L) = L_4'(S_L)$ and solve for $l_{31}$

$$x_1 l_{31} {S_L}^{x_1 - 1} = y_1 l_{41} {S_L}^{y_1 - 1} + y_2 l_{42} {S_L}^{y_2 - 1} + \frac{z_1 \lambda \beta_1 {S_L}^{z_1 - 1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2}$$

$$l_{31} = \frac{1}{x_1 {S_L}^{x_1 - 1}} \left( y_1 l_{41} {S_L}^{y_1 - 1} + y_2 l_{42} {S_L}^{y_2 - 1} + \frac{z_1 \lambda \beta_1 {S_L}^{z_1 - 1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} \right)$$

I insert for $l_{31}$ in $L_3(S_L) = L_4(S_L)$

$$\frac{{S_L}^{x_1}}{x_1 {S_L}^{x_1 - 1}} \left( y_1 l_{41} {S_L}^{y_1 - 1} + y_2 l_{42} {S_L}^{y_2 - 1} + \frac{z_1 \lambda \beta_1 {S_L}^{z_1 - 1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} \right) = l_{41} {S_L}^{y_1} + l_{42} {S_L}^{y_2} - \frac{k}{r + \lambda} + \frac{\lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2}$$

$$\frac{S_L}{x_1} \left( y_1 l_{41} {S_L}^{y_1 - 1} + y_2 l_{42} {S_L}^{y_2 - 1} + \frac{z_1 \lambda \beta_1 {S_L}^{z_1 - 1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} \right) = l_{41} {S_L}^{y_1} + l_{42} {S_L}^{y_2} - \frac{k}{r + \lambda} + \frac{\lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2}$$

I multiply by $x_1$ to remove the brackets

$$y_1 l_{41} {S_L}^{y_1} + y_2 l_{42} {S_L}^{y_2} + \frac{z_1 \lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1\mu - \frac{1}{2} z_1(z_1 - 1)\sigma^2}$$
$$= x_1 l_{41} {S_L}^{y_1} + x_1 l_{42} {S_L}^{y_2} - \frac{x_1 k}{r + \lambda} + \frac{x_1 \lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1\mu - \frac{1}{2} z_1(z_1 - 1)\sigma^2}$$

I collect and draw together similar terms

$$y_1 l_{41} {S_L}^{y_1} - x_1 l_{41} {S_L}^{y_1} + y_2 l_{42} {S_L}^{y_2} - x_1 l_{42} {S_L}^{y_2} + \frac{z_1 \lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1\mu - \frac{1}{2} z_1(z_1 - 1)\sigma^2}$$
$$- \frac{x_1 \lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1\mu - \frac{1}{2} z_1(z_1 - 1)\sigma^2} = -\frac{x_1 k}{r + \lambda}$$

$$(y_1 - x_1) l_{41} {S_L}^{y_1} + (y_2 - x_1) l_{42} {S_L}^{y_2} + \frac{(z_1 - x_1) \lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1\mu - \frac{1}{2} z_1(z_1 - 1)\sigma^2} = -\frac{x_1 k}{r + \lambda}$$

I solve for $l_{42}$ and get the solution

$$l_{42} = -\frac{1}{(y_2 - x_1){S_L}^{y_2}} \left( (y_1 - x_1) l_{41} {S_L}^{y_1} + \frac{(z_1 - x_1) \lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1\mu - \frac{1}{2} z_1(z_1 - 1)\sigma^2} + \frac{x_1 k}{r + \lambda} \right)$$

**Deriving the equation for $l_{52}$**

$$L_4\left(\frac{n}{m}\right) = l_{41}\left(\frac{n}{m}\right)^{y_1} + l_{42}\left(\frac{n}{m}\right)^{y_2} - \frac{k}{r+\lambda} + \frac{\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2} \qquad \text{when } S_L \le V < \frac{n}{m}$$

$$L_5\left(\frac{n}{m}\right) = l_{52}\left(\frac{n}{m}\right)^{y_2} + \frac{\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu} - \frac{\lambda n + k}{r+\lambda} + \frac{\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} \qquad \text{when } V \ge \frac{n}{m}$$

I set $L_4\left(\frac{n}{m}\right) = L_5\left(\frac{n}{m}\right)$

$$l_{41}\left(\frac{n}{m}\right)^{y_1}+l_{42}\left(\frac{n}{m}\right)^{y_2}-\frac{k}{r+\lambda}+\frac{\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

$$=l_{52}\left(\frac{n}{m}\right)^{y_2}+\frac{\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu}-\frac{\lambda n+k}{r+\lambda}+\frac{\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

I set the 1st derivatives equal to each other $L_4'\left(\frac{n}{m}\right)=L_5'\left(\frac{n}{m}\right)$ and solve for $l_{41}$

$$y_1 l_{41}\left(\frac{n}{m}\right)^{y_1-1}+y_2 l_{42}\left(\frac{n}{m}\right)^{y_2-1}+\frac{z_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1-1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

$$=y_2 l_{52}\left(\frac{n}{m}\right)^{y_2-1}+\frac{\lambda m}{r+\lambda-\mu}+\frac{z_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

$$l_{41}=\frac{1}{y_1\left(\frac{n}{m}\right)^{y_1-1}}\left(y_2 l_{52}\left(\frac{n}{m}\right)^{y_2-1}-y_2 l_{42}\left(\frac{n}{m}\right)^{y_2-1}+\frac{\lambda m}{r+\lambda-\mu}\right.$$

$$\left.-\frac{z_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1-1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}+\frac{z_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}\right)$$

I insert for $l_{41}$ in $L_4\left(\frac{n}{m}\right)=L_5\left(\frac{n}{m}\right)$

$$\frac{\left(\frac{n}{m}\right)^{y_1}}{y_1\left(\frac{n}{m}\right)^{y_1-1}}\left(y_2 l_{52}\left(\frac{n}{m}\right)^{y_2-1}-y_2 l_{42}\left(\frac{n}{m}\right)^{y_2-1}+\frac{\lambda m}{r+\lambda-\mu}-\frac{z_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1-1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}\right.$$

$$\left.+\frac{z_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}\right)+l_{42}\left(\frac{n}{m}\right)^{y_2}-\frac{k}{r+\lambda}$$

$$+\frac{\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

$$=l_{52}\left(\frac{n}{m}\right)^{y_2}+\frac{\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu}-\frac{\lambda n+k}{r+\lambda}+\frac{\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

$$\frac{\left(\frac{n}{m}\right)}{y_1}\left(y_2 l_{52}\left(\frac{n}{m}\right)^{y_2-1} - y_2 l_{42}\left(\frac{n}{m}\right)^{y_2-1} + \frac{\lambda m}{r+\lambda-\mu} - \frac{z_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1-1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}\right.$$

$$\left. + \frac{z_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2-1}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}\right) + l_{42}\left(\frac{n}{m}\right)^{y_2} - \frac{k}{r+\lambda}$$

$$+ \frac{\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

$$= l_{52}\left(\frac{n}{m}\right)^{y_2} + \frac{\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu} - \frac{\lambda n + k}{r+\lambda} + \frac{\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

I multiply by $y_1$ to remove the brackets

$$y_2 l_{52}\left(\frac{n}{m}\right)^{y_2} - y_2 l_{42}\left(\frac{n}{m}\right)^{y_2} + \frac{\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu} - \frac{z_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

$$+ \frac{z_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} + y_1 l_{42}\left(\frac{n}{m}\right)^{y_2} - \frac{y_1 k}{r+\lambda}$$

$$+ \frac{y_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

$$= y_1 l_{52}\left(\frac{n}{m}\right)^{y_2} + \frac{y_1\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu} - \frac{y_1(\lambda n + k)}{r+\lambda} + \frac{y_1\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

I collect and draw together similar terms

$$y_2 l_{52}\left(\frac{n}{m}\right)^{y_2} - y_1 l_{52}\left(\frac{n}{m}\right)^{y_2} - y_2 l_{42}\left(\frac{n}{m}\right)^{y_2} + y_1 l_{42}\left(\frac{n}{m}\right)^{y_2} + \frac{z_2\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

$$- \frac{y_1\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} - \frac{z_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

$$+ \frac{y_1\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}$$

$$= \frac{y_1\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu} - \frac{\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu} + \frac{y_1 k}{r+\lambda} - \frac{y_1(\lambda n + k)}{r+\lambda}$$

$$(y_2-y_1)l_{52}\left(\frac{n}{m}\right)^{y_2}+(y_1-y_2)l_{42}\left(\frac{n}{m}\right)^{y_2}+\frac{(z_2-y_1)\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

$$+\frac{(y_1-z_1)\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}=\frac{(y_1-1)\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu}-\frac{y_1\lambda n}{r+\lambda}$$

I solve for $l_{52}$

$$l_{52}=\frac{1}{(y_2-y_1)\left(\frac{n}{m}\right)^{y_2}}\left(-(y_1-y_2)l_{42}\left(\frac{n}{m}\right)^{y_2}-\frac{(z_2-y_1)\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}\right.$$

$$\left.-\frac{(y_1-z_1)\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}+\frac{(y_1-1)\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu}-\frac{y_1\lambda n}{r+\lambda}\right)$$

I draw together the terms and get the solution for $l_{52}$

$$l_{52}=l_{42}+\frac{1}{(y_2-y_1)\left(\frac{n}{m}\right)^{y_2}}\left(-\frac{(z_2-y_1)\lambda\beta_4\left(\frac{n}{m}\right)^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}\right.$$

$$\left.-\frac{(y_1-z_1)\lambda\beta_1\left(\frac{n}{m}\right)^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}+\frac{(y_1-1)\lambda m\left(\frac{n}{m}\right)}{r+\lambda-\mu}-\frac{y_1\lambda n}{r+\lambda}\right)$$

# Appendix B - Equations for the optimal switching points

I will here derive the equations for the optimal switching points between and active and a passive investment project. If this switching point is above the switching point for an open and a closed mine in the production phase, $S_H > \frac{n}{m}$, the switching point will be denoted $S_H$. If $S_L < \frac{n}{m}$the switching point will be denoted $S_L$.

## Deriving of the equation for the optimal switching points $S_H$

To find the optimal switching point between an active and a passive investment project when $S_H > \frac{n}{m}$ I use the equilibrium equation found in chapter 3.1.

$$L_1(S_H) = L_2(S_H) = \beta_4 {S_H}^{z_2} + mS_H - n - \frac{k}{\lambda}$$

It is indifferent if I use $L_1(S_H)$ or $L_2(S_H)$ to derive the formula for the switching point because they are equal. I have used both to be certain that they give the same result.

**Deriving of the equation for the optimal switching point $\boldsymbol{S_H}$ using $\boldsymbol{L_1(S_H)}$**

The ordinary differential equation for $L_1(V)$ gave this solution

$$L_1(V) = l_{11} V^{x_1}$$

$$l_{11} = \frac{1}{(y_2 - x_1)V^{x_1}}\left(\frac{(y_2 - 1)\lambda mV}{r + \lambda - \mu} + \frac{(y_2 - z_2)\lambda\beta_4 V^{z_2}}{r + \lambda - z_2\mu - \frac{1}{2}z_2(z_2 - 1)\sigma^2} - \frac{y_2(\lambda n + k)}{r + \lambda}\right)$$

When $V = S_H$ the equilibrium equation is equal to the right hand side of the equation

$$l_{11} {S_H}^{x_1} = \beta_4 {S_H}^{z_2} + mS_H - n - \frac{k}{\lambda}$$

I insert for $l_{11}$

$$\frac{{S_H}^{x_1}}{(y_2 - x_1){S_H}^{x_1}}\left(\frac{(y_2 - 1)\lambda mS_H}{r + \lambda - \mu} + \frac{(y_2 - z_2)\lambda\beta_4 {S_H}^{z_2}}{r + \lambda - z_2\mu - \frac{1}{2}z_2(z_2 - 1)\sigma^2} - \frac{y_2(\lambda n + k)}{r + \lambda}\right)$$
$$= \beta_4 {S_H}^{z_2} + mS_H - n - \frac{k}{\lambda}$$

I multiply by both sides with $(y_2 - x_1)$ to remove the brackets on the left hand side

$$\frac{(y_2-1)\lambda m S_H}{r+\lambda-\mu} + \frac{(y_2-z_2)\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} - \frac{y_2(\lambda n+k)}{r+\lambda}$$
$$= (y_2-x_1)\beta_4 {S_H}^{z_2} + (y_2-x_1)mS_H - (y_2-x_1)n - \frac{(y_2-x_1)k}{\lambda}$$

I collect and draw together similar terms

$$\frac{(y_2-1)\lambda m S_H}{r+\lambda-\mu} - (y_2-x_1)mS_H + \frac{(y_2-z_2)\lambda\beta_4 {S_H}^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} - (y_2-x_1)\beta_4 {S_H}^{z_2}$$
$$= \frac{y_2(\lambda n+k)}{r+\lambda} - (y_2-x_1)n - \frac{(y_2-x_1)k}{\lambda}$$

$$\left(\frac{(y_2-1)\lambda}{r+\lambda-\mu} - (y_2-x_1)\right)mS_H + \left(\frac{(y_2-z_2)\lambda}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} - (y_2-x_1)\right)\beta_4 {S_H}^{z_2}$$
$$= \frac{y_2(\lambda n+k)}{r+\lambda} - \frac{(y_2-x_1)\lambda n + (y_2-x_1)k}{\lambda}$$

I move all the terms to the left hand side and get an equation that is equal to zero

$$\left(\frac{(y_2-1)\lambda}{r+\lambda-\mu} - (y_2-x_1)\right)mS_H + \left(\frac{(y_2-z_2)\lambda}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} - (y_2-x_1)\right)\beta_4 {S_H}^{z_2}$$
$$- \frac{y_2(\lambda n+k)}{r+\lambda} + \frac{(y_2-x_1)(\lambda n+k)}{\lambda} = 0$$

This is the equation for the optimal switching point $S_H$. It is a non-linear equation and I will use Excel to solve for $S_H$ in the example in chapter 4.

**Deriving of the equation for the optimal switching point $S_H$ using $L_2(S_H)$**

The ordinary differential equation for $L_2(V)$ gave this solution

$$L_2(V) = l_{22}V^{y_2} + \frac{\lambda m V}{r+\lambda-\mu} - \frac{\lambda n+k}{r+\lambda} + \frac{\lambda\beta_4 V^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}$$

$$l_{22} = -\frac{1}{(y_2-x_1)V^{y_2}}\left(\frac{(1-x_1)\lambda m V}{r+\lambda-\mu} + \frac{(z_2-x_1)\lambda\beta_4 V^{z_2}}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2} + \frac{x_1(\lambda n+k)}{r+\lambda}\right)$$

When $V = S_H$ the equilibrium equation is equal to the right hand side of the equation

$$l_{22}{S_H}^{y_2} + \frac{\lambda m S_H}{r+\lambda-\mu} - \frac{\lambda n + k}{r+\lambda} + \frac{\lambda \beta_4 {S_H}^{z_2}}{r+\lambda - z_2\mu - \frac{1}{2}z_2(z_2-1)\sigma^2} = \beta_4 {S_H}^{z_2} + m S_H - n - \frac{k}{\lambda}$$

I insert for $l_{22}$

$$-\frac{{S_H}^{y_2}}{(y_2-x_1){S_H}^{y_2}}\left(\frac{(1-x_1)\lambda m S_H}{r+\lambda-\mu} + \frac{(z_2-x_1)\lambda\beta_4 {S_H}^{z_2}}{r+\lambda - z_2\mu - \frac{1}{2}z_2(z_2-1)\sigma^2} + \frac{x_1(\lambda n + k)}{r+\lambda}\right)$$
$$+ \frac{\lambda m S_H}{r+\lambda-\mu} - \frac{\lambda n + k}{r+\lambda} + \frac{\lambda \beta_4 {S_H}^{z_2}}{r+\lambda - z_2\mu - \frac{1}{2}z_2(z_2-1)\sigma^2}$$
$$= \beta_4 {S_H}^{z_2} + m S_H - n - \frac{k}{\lambda}$$

I multiply by both sides with $(y_2 - x_1)$ to remove the brackets

$$-\frac{(1-x_1)\lambda m S_H}{r+\lambda-\mu} - \frac{(z_2-x_1)\lambda\beta_4 {S_H}^{z_2}}{r+\lambda - z_2\mu - \frac{1}{2}z_2(z_2-1)\sigma^2} - \frac{x_1(\lambda n + k)}{r+\lambda} + \frac{(y_2-x_1)\lambda m S_H}{r+\lambda-\mu}$$
$$- \frac{(y_2-x_1)(\lambda n + k)}{r+\lambda} + \frac{(y_2-x_1)\lambda\beta_4 {S_H}^{z_2}}{r+\lambda - z_2\mu - \frac{1}{2}z_2(z_2-1)\sigma^2}$$
$$= (y_2-x_1)\beta_4 {S_H}^{z_2} + (y_2-x_1) m S_H - (y_2-x_1) n - \frac{(y_2-x_1)k}{\lambda}$$

I collect and draw together similar terms

$$\frac{(y_2-x_1)\lambda m S_H}{r+\lambda-\mu} - \frac{(1-x_1)\lambda m S_H}{r+\lambda-\mu} - (y_2-x_1) m S_H + \frac{(y_2-x_1)\lambda\beta_4 {S_H}^{z_2}}{r+\lambda - z_2\mu - \frac{1}{2}z_2(z_2-1)\sigma^2}$$
$$- \frac{(z_2-x_1)\lambda\beta_4 {S_H}^{z_2}}{r+\lambda - z_2\mu - \frac{1}{2}z_2(z_2-1)\sigma^2} - (y_2-x_1)\beta_4 {S_H}^{z_2}$$
$$= \frac{x_1(\lambda n + k)}{r+\lambda} + \frac{(y_2-x_1)(\lambda n + k)}{r+\lambda} - (y_2-x_1) n - \frac{(y_2-x_1)k}{\lambda}$$

$$\frac{(y_2-1)\lambda m S_H}{r+\lambda-\mu} - (y_2-x_1) m S_H + \frac{(y_2-z_2)\lambda\beta_4 {S_H}^{z_2}}{r+\lambda - z_2\mu - \frac{1}{2}z_2(z_2-1)\sigma^2} - (y_2-x_1)\beta_4 {S_H}^{z_2}$$
$$= \frac{y_2(\lambda n + k)}{r+\lambda} - \frac{(y_2-x_1)\lambda n + (y_2-x_1)k}{\lambda}$$

$$\left(\frac{(y_2-1)\lambda}{r+\lambda-\mu}-(y_2-x_1)\right)mS_H+\left(\frac{(y_2-z_2)\lambda}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}-(y_2-x_1)\right)\beta_4{S_H}^{z_2}$$
$$=\frac{y_2(\lambda n+k)}{r+\lambda}-\frac{(y_2-x_1)\lambda n+(y_2-x_1)k}{\lambda}$$

I move all the terms to the left hand side and get an equation that is equal to zero

$$\left(\frac{(y_2-1)\lambda}{r+\lambda-\mu}-(y_2-x_1)\right)mS_H+\left(\frac{(y_2-z_2)\lambda}{r+\lambda-z_2\mu-\frac{1}{2}z_2(z_2-1)\sigma^2}-(y_2-x_1)\right)\beta_4{S_H}^{z_2}$$
$$-\frac{y_2(\lambda n+k)}{r+\lambda}+\frac{(y_2-x_1)\lambda n+(y_2-x_1)k}{\lambda}=0$$

This is the equation for the optimal switching point $S_H$. It is identical to the equation I got when I used $L_1(S_H)$.

# Deriving of the equation for the optimal switching point $S_L$

To find the optimal switching point between an active and a passive investment project when $S_L<\frac{n}{m}$ I use the equilibrium equation found in chapter 3.1.

$$L_3(S_L)=L_4(S_L)=\beta_1{S_L}^{z_1}-\frac{k}{\lambda}$$

It is indifferent if I use $L_3(S_L)$ or $L_4(S_L)$ to derive the formula for the switching point because they are equal. I have used both to be certain that they give the same result.

**Deriving of the equation for the optimal switching point $S_L$ using $L_3(S_L)$**

The ordinary differential equation for $L_3(V)$ gave this solution

$$L_3(S_L)=l_{31}V^{x_1}$$

$$l_{31}=\frac{1}{(y_2-x_1)V^{x_1}}\left((y_2-y_1)l_{41}V^{y_1}-\frac{y_2k}{r+\lambda}+\frac{(y_2-z_1)\lambda\beta_1V^{z_1}}{r+\lambda-z_1\mu-\frac{1}{2}z_1(z_1-1)\sigma^2}\right)$$

When $V=S_L$ the equilibrium equation is equal to the right hand side of the equation

$$l_{31}{S_L}^{x_1}=\beta_1{S_L}^{z_1}-\frac{k}{\lambda}$$

I insert for $l_{31}$

$$\frac{{S_L}^{x_1}}{(y_2 - x_1){S_L}^{x_1}}\left((y_2 - y_1)l_{41}{S_L}^{y_1} - \frac{y_2 k}{r+\lambda} + \frac{(y_2 - z_1)\lambda\beta_1{S_L}^{z_1}}{r + \lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2}\right) = \beta_1{S_L}^{z_1} - \frac{k}{\lambda}$$

I multiply by both sides with $(y_2 - x_1)$ to remove the brackets

$$(y_2 - y_1)l_{41}{S_L}^{y_1} - \frac{y_2 k}{r+\lambda} + \frac{(y_2 - z_1)\lambda\beta_1{S_L}^{z_1}}{r + \lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2} = (y_2 - x_1)\beta_1{S_L}^{z_1} - \frac{(y_2 - x_1)k}{\lambda}$$

I collect and draw together similar terms

$$(y_2 - y_1)l_{41}{S_L}^{y_1} + \frac{(y_2 - z_1)\lambda\beta_1{S_L}^{z_1}}{r + \lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2} - (y_2 - x_1)\beta_1{S_L}^{z_1} = \frac{y_2 k}{r+\lambda} - \frac{(y_2 - x_1)k}{\lambda}$$

$$(y_2 - y_1)l_{41}{S_L}^{y_1} + \left(\frac{(y_2 - z_1)\lambda}{r + \lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2} - (y_2 - x_1)\right)\beta_1{S_L}^{z_1}$$
$$= \frac{y_2 k}{r+\lambda} - \frac{(y_2 - x_1)k}{\lambda}$$

I move all the terms to the left hand side and get an equation that is equal to zero

$$(y_2 - y_1)l_{41}{S_L}^{y_1} + \left(\frac{(y_2 - z_1)\lambda}{r + \lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2} - (y_2 - x_1)\right)\beta_1{S_L}^{z_1} - \frac{y_2 k}{r+\lambda}$$
$$+ \frac{(y_2 - x_1)k}{\lambda} = 0$$

This is the equation for the optimal switching point $S_L$. It is also a non-linear equation. There is no need to insert for $l_{41}$ as it is dependent on $\frac{n}{m}$ and not on $S_L$. In this equation $l_{41}$ is only a constant.

**Deriving of the equation for the optimal switching point $S_L$ using $L_4(S_L)$**

The ordinary differential equation for $L_4(V)$ gave this solution

$$L_4(V) = l_{41}V^{y_1} + l_{42}V^{y_2} - \frac{k}{r+\lambda} + \frac{\lambda\beta_1 V^{z_1}}{r + \lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2}$$

$$l_{42} = -\frac{1}{(y_2 - x_1)V^{y_2}} \left( (y_1 - x_1) l_{41} V^{y_1} + \frac{(z_1 - x_1)\lambda \beta_1 V^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} + \frac{x_1 k}{r + \lambda} \right)$$

When $V = S_L$ the equilibrium equation is equal to the right hand side of the equation

$$l_{41} {S_L}^{y_1} + l_{42} {S_L}^{y_2} - \frac{k}{r + \lambda} + \frac{\lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} = \beta_1 {S_L}^{z_1} - \frac{k}{\lambda}$$

I insert for $l_{42}$

$$l_{41} {S_L}^{y_1} - \frac{{S_L}^{y_2}}{(y_2 - x_1){S_L}^{y_2}} \left( (y_1 - x_1) l_{41} {S_L}^{y_1} + \frac{(z_1 - x_1)\lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} + \frac{x_1 k}{r + \lambda} \right)$$
$$- \frac{k}{r + \lambda} + \frac{\lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} = \beta_1 {S_L}^{z_1} - \frac{k}{\lambda}$$

I multiply by both sides with $(y_2 - x_1)$ to remove the brackets

$$(y_2 - x_1) l_{41} {S_L}^{y_1} - (y_1 - x_1) l_{41} {S_L}^{y_1} - \frac{(z_1 - x_1)\lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} - \frac{x_1 k}{r + \lambda} - \frac{(y_2 - x_1)k}{r + \lambda}$$
$$+ \frac{(y_2 - x_1)\lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} = (y_2 - x_1)\beta_1 {S_L}^{z_1} - \frac{(y_2 - x_1)k}{\lambda}$$

I collect and draw together similar terms

$$(y_2 - x_1) l_{41} {S_L}^{y_1} - (y_1 - x_1) l_{41} {S_L}^{y_1} + \frac{(y_2 - x_1)\lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2}$$
$$- \frac{(z_1 - x_1)\lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} - (y_2 - x_1)\beta_1 {S_L}^{z_1}$$
$$= \frac{x_1 k}{r + \lambda} + \frac{(y_2 - x_1)k}{r + \lambda} - \frac{(y_2 - x_1)k}{\lambda}$$

$$(y_2 - y_1) l_{41} {S_L}^{y_1} + \frac{(y_2 - z_1)\lambda \beta_1 {S_L}^{z_1}}{r + \lambda - z_1 \mu - \frac{1}{2} z_1 (z_1 - 1)\sigma^2} - (y_2 - x_1)\beta_1 {S_L}^{z_1} = \frac{y_2 k}{r + \lambda} - \frac{(y_2 - x_1)k}{\lambda}$$

$$(y_2 - y_1)l_{41}{S_L}^{y_1} + \left(\frac{(y_2 - z_1)\lambda}{r + \lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2} - (y_2 - x_1)\right)\beta_1{S_L}^{z_1} = \frac{y_2 k}{r + \lambda} - \frac{(y_2 - x_1)k}{\lambda}$$

I move all the terms to the left hand side and get an equation that is equal to zero

$$(y_2 - y_1)l_{41}{S_L}^{y_1} + \left(\frac{(y_2 - z_1)\lambda}{r + \lambda - z_1\mu - \frac{1}{2}z_1(z_1 - 1)\sigma^2} - (y_2 - x_1)\right)\beta_1{S_L}^{z_1} - \frac{y_2 k}{r + \lambda} + \frac{(y_2 - x_1)k}{\lambda} = 0$$

This is the equation for the optimal switching point $S_L$. It is identical to the equation I got when I used $L_3(S_L)$.

# Appendix C – Analyse of the Switching Points

Because the equation for the optimal switching point between an active and a passive investment project, $S_H$, is non-linear, it will result in two switching. This is the case in all the scenarios in my analysis. I will refer to the different switching points as the lowest and the highest switching point. The switching points are shown in table 6. The lowest switching points when $T = 10$ and $k = 2$ are less than $\frac{n}{m}$ and the equation for $S_L$ will therefore apply.

When I use the lowest value for the switching point in the cases where expected time to completion changes, the switching point will lower when the expected time to completion is higher, and vice versa. This is shown in figure 8. This is not consistent with theory. When the expected time to completion is higher there is lower probability of completion, there are higher on-going investment costs, and the value of the investment project is lower. Switching to a passive investment project should occur at a higher value of the outcome. I find that the lowest switching point is not valid. When I use the highest switching point, the behavior is consistent with theory, see figure 6 in chapter 4.2.

The same will happen when on-going investment costs until completion are higher; when using the lowest switching point, the switching point is lower when the on-going investment costs are higher, and vice versa. See figure 9. When the investment project cost more, switching to a passive state should happen at a higher level. This is the case if I use the highest switching point.

The results from using the lowest switching point are inconsistent with theory, and my conclusion is that the lowest switching points are not valid. I choose always to use the highest switching points.

| Switching points | | |
|---|---|---|
| | Lowest | Highest |
| $T$ = 5, $k$ = 1, $r$ = 5% | 0,61 | 2,62 |
| $T$ = 1 | 1,00 | 1,72 |
| $T$ = 10 | 0,17 | 3,44 |
| $k$ = 0,2 | 1,01 | 1,26 |
| $k$ = 2 | 0,20 | 4,00 |

*TABLE 6: Comparison of the switching points*

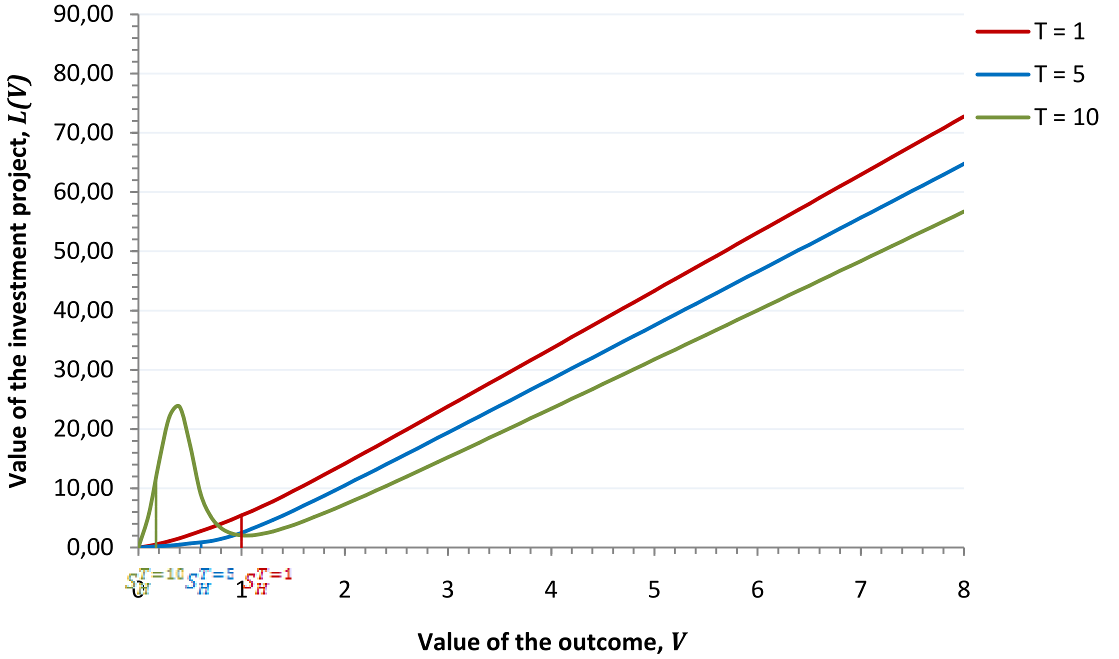


*FIGURE 8: Value of the investment project when using the lowest switching point for different values of T. T = 5 represents the base case scenario and has a switching point $S_H^{T=5}$ = 0,61. When T = 1 the value of the investment project rises, and the threshold level also rises, $S_H^{T=1}$ = 1,00. When T = 10 the threshold level falls to $S_L^{T=10}$ =0,17.*

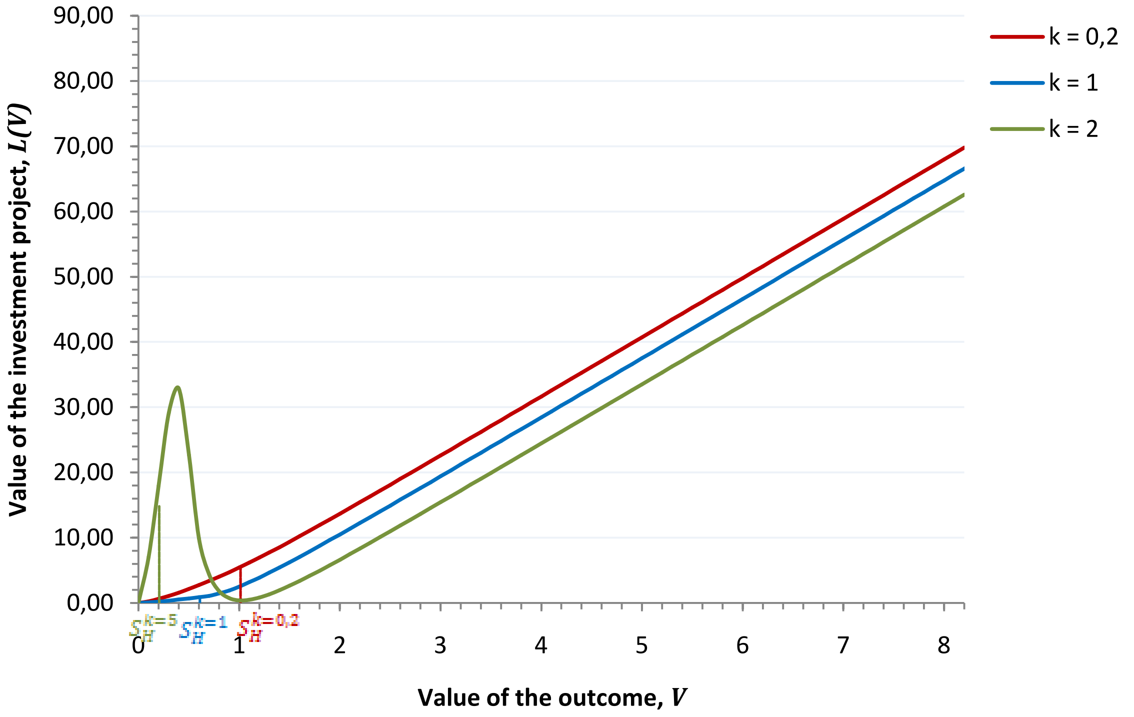


*FIGURE 9: Value of the investment project when the on-going investment costs until completion changes. k = 1 represents the base case scenario with switching point $S_H^{k=1}$ = 0,61. When k = 0,2 the threshold level increases to $S_H^{k=0,2}$ = 1,01 and when k = 2 the threshold level drops to $S_L^{k=2}$ = 0,20.*