

Dynamic portfolio optimization with transaction costs and state-dependent drift[☆]

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Abstract

We present an efficient numerical method to determine optimal portfolio strategies under time- and state-dependent drift and proportional transaction costs. This scenario arises when investors have behavioral biases or the actual drift is unknown and needs to be estimated. The numerical method solves dynamic optimal portfolio problems for time-horizons of up to 40 years. It is applied to measure the value of information and the loss from transaction

[☆]The paper benefitted from the first and third authors' visits to the Hausdorff Research Institute for Mathematics at the University of Bonn in the framework of the Trimester Program *Stochastic Dynamics in Economics and Finance*.

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costs using the indifference principle.

Keywords: Portfolio choice, state-dependent drift, transaction costs, numerical methods

JEL: C61, G11

1. Introduction

Numerical methods for dynamic portfolio optimization under proportional transaction costs typically assume that the drift of the risky asset is constant. However, a non-constant, state-dependent and/or time-varying drift enters the dynamic portfolio problem in many scenarios. For instance, if the drift is constant but unobservable to the investor, it can be estimated with the Kalman-Bucy filter. This leads to a portfolio optimization problem where the drift depends on time and the most currently observed stock price, see [Rogers \(2001\)](#), [Lundtofte \(2006\)](#), [Danilova et al. \(2010\)](#). The drift is also state-dependent when contrarian investors optimize portfolios under the assumption that prices are mean-reverting; for instance when an investor is a victim of the Gambler's fallacy, see, e.g., [Shefrin \(2008\)](#). Similarly, investors who aim at following market trends will include a state-dependent drift in their portfolio optimization.

It turns out that in these cases an investor's optimal trading strategy strongly depends on the forecasting function used to predict asset prices. This poses a numerically demanding problem which is addressed here. Our paper proposes an efficient numerical method to solve finite-horizon portfolio optimization problems with transaction costs and time- and state-dependent drift. The method has time-complexity of $O(N^{2.5})$ whereas a discrete-time

dynamic programming algorithm that directly solves the value function of the optimization problem, see, e.g., [Davis et al. \(1993, Eq. \(6.5\)\)](#) or [Monoyios \(2004, Eq. \(40\)\)](#), has time-complexity $O(N^5)$. Our method allows us, for instance, to study 40-year investment horizons with time steps of 4-day length on a basic laptop computer.

There are several numerical methods for solving the dynamic portfolio problem with a constant drift under proportional transaction costs. [Davis et al. \(1993\)](#) proposed a backward recursive dynamic programming method which has seen a number of improvement in recent years. For instance, [Monoyios \(2004\)](#) provides an analytical approximation to the optimal decision in the final period which allows starting recursion from a smaller range of stock holdings thereby increasing numerical efficiency.¹ [Zakamouline \(2005, 2006\)](#) proposes bounds on stock holdings which reduces computing time to determine optimal holdings on a given discretization of the state space. Another method is to solve the Hamilton-Jacobi-Bellman (HJB) equations of optimization problems by appropriate finite difference schemes, see, for example, [Chellathurai and Draviam \(2007\)](#) and [Herzog et al. \(2013\)](#). In addition, [Atkinson and Quek \(2012\)](#) derive first order conditions for the optimal strategy and approximate the solution using perturbation analysis. [Lensberg and Schenk-Hoppé \(2013\)](#) use genetic programming algorithm to derive analytic approximations of trading strategies in a feedback form. These algorithms

¹The behavior of optimal stock holdings close to the terminal time matters for the proper initialization of the backward induction. Since the benefit of portfolio rebalancing quickly decreases close to the terminal time, the no-trade region increases dramatically which causes numerical issues, see, e.g., [Monoyios \(2004\)](#).

work well for short time-horizons, typically less than one year, and when the number of periods is small. Our paper fills this gap in the literature by proposing a method that works for non-constant drift and long time-horizons.

The main challenges arising from a state-dependent drift, specifically for long time-horizons, are that the search for an optimal decision has to be carried out for all nodes of an approximating binomial tree and that the state-dependent no-trade region requires widening the range of stock holdings. This increases the likelihood of over- and underflow arising for the exponential utility function (i.e., values become too large or too small to be represented on a computer) as pointed out by [Clewlow and Hodges \(1997\)](#), see also [Zakamouline \(2006\)](#). For a constant drift, in contrast, the no-trade region (in terms of the amount of wealth invested in stocks) is independent of stock prices at time t . One only needs to search for the no-trade region for a single node of the a binomial tree at time t , see [Monoyios \(2004, p. 902\)](#).

To overcome the challenges, we develop a fast and accurate approximation to the optimization problem, which is achieved by combining four aspects in our approach: (a) reducing dimensionality, (b) scaling the objective function, (c) carrying out local rather than global searches for optimal trading decisions, and (d) non-equidistant discretization of the state space. As in most papers in this field, we solve the model on the assumption that investors have constant absolute risk aversion (CARA) utility (i.e., a negative exponential utility function) and that there is only one risky and one riskfree asset.

We apply the numerical method in a detailed study of the true costs of market frictions using the indifference principle. The analysis reaps the full benefit of the approach because measuring these costs involves taking

averages over many realizations of the parameter value of the stock price drift. For each realization, one has to calculate optimal trading strategies and carry out Monte Carlo simulations. In general, we find that the optimal investment behavior strongly depends on whether the drift is state-dependent. A state-dependent drift leads to a more volatile no-trade region than that with a constant drift under proportional transaction costs which, in turn, entails more aggressive trading.

First, we measure the value of information by comparing realized utility of different types of investors. It turns out that information is most valuable to the least risk-averse investor, and that cautious trend-followers do almost as well as investors who estimate the drift from observations.

Second, we measure the loss in utility due to transaction costs as the indifference price of an investor. This is the maximum amount of money an investor is willing to pay up front to avoid incurring transaction costs of a certain size. It turns out that in general the loss in utility due to proportional transaction costs is about twice as large as the direct expenses incurred. From a welfare perspective, transaction costs are perceived as most detrimental by naive investors who do not revise their initial estimates of the drift at time-horizons longer than six years. In the long run naive investors are the most active traders and usually hold wrong beliefs. At time-horizons shorter than five years, transaction costs strongly affect the learning investor as his estimate of the drift varies drastically in the short run.

Third, we examine the impact of the length of the investment time-horizon. Although uncertainty about the true drift cannot be removed over a finite time-horizon, learning about the drift reduces the loss in utility due

to the uncertain drift (comparing with that with known drifts) by 1/3rd in one year and by 4/5th in ten years compared to a naive investor. Learning also reduces the loss in utility caused by transaction costs (comparing with that without transaction costs) by 1/2 over a 10-year time-horizon.

Section 2 presents the model. The numerical method is explained in detail in Section 3 and applied in Section 4 to quantify the economic costs of transaction costs and investment under various assumption on the dynamics of a state-dependent drift. Section 5 concludes.

2. Model

We consider an investor who maximizes utility from wealth at a terminal time by trading in two assets. A riskfree bond with a constant, continuously compounded interest rate r , and a risky stock. The investor assumes that the dynamics of the stock price $S(t)$ is given by a stochastic differential equation of the form

$$dS(t) = \mu(t, S(t))S(t)dt + S(t)\sigma dW(t), \quad S(0) = S_0 \quad (1)$$

with a constant volatility $\sigma > 0$ and standard, one-dimensional Brownian motion $W(t)$. The function $\mu(t, S)$ is the drift of the stock price. If this function is a constant, the model reduces to a standard geometric Brownian motion (the Black-Scholes model). We are interested in the case where the drift function is time- and state-dependent.

We consider a situation in which the true dynamics of the stock price may be unknown to investors: The actual drift is a random variable $m(\omega)$ which is determined at the initial time and fixed over the entire investment horizon

but unobservable to investors. The true stock price dynamics is given by

$$dS(t) = m(\omega)S(t)dt + \sigma S(t)dW(t). \quad (2)$$

If the structure of the price dynamics is known (but the realization of the drift is not), one can use observed stock prices to estimate the value of $m(\omega)$. Assume that $m(\omega)$ is independent of the Brownian motion W and normally distributed with mean μ_0 and variance $\gamma_0 > 0$. Then the Kalman-Bucy filter gives the estimate

$$\mu^L(t, S(t)) = \frac{\gamma_0 \sigma^2}{\sigma^2 + \gamma_0 t} \left(\frac{\mu_0}{\gamma_0} + \frac{t}{2} + \frac{1}{\sigma^2} \log(S(t)/S_0) \right) \quad (3)$$

This estimate takes the form $\mu(t, S(t))$, and hence entails a dynamics as defined in (1).

Investors who are not aware of the structure of the price dynamics make forecasts in sub-optimal ways. We will consider two specific types of investors. The first is a naive investor who assumes that the dynamics is given by (2) with $m(\omega) = \mu_0$ (its mean). The second type of investor suffers from a behavioral bias and estimates the value of the drift as:

$$\mu^a(t, S(t)) = \mu_0 + a \arctan \left((\mu_0 - \sigma^2/2)t - \log(S(t)/S_0) \right). \quad (4)$$

We refer to the parameter ‘ a ’ as the investor’s *sentiment*. It measures the investor’s confidence in his initial estimate μ_0 . If the parameter a is positive then the investor believes that the price will revert to the predicted mean: A higher than predicted return is forecast to lead to a drift smaller than μ_0 . The investor’s decision is contrarian. It can be interpreted as the result of overconfidence about the ability to predict the stock price dynamics. If the parameter a is negative, the investor will revise the initial estimate upwards

if the returns are higher than predicted (resp. downwards if returns are lower than μ_0). The investor is a trend follower; he places more trust in the market's view about stock price dynamics than his own view.

We define

Definition 2.1. ***Informed** investors observe the realization of the random drift $m(\omega)$ at the initial time.*

***Learning** investors use (3) to estimate the realization of the random drift $m(\omega)$.*

***Naive** investors assume $m(\omega) = \mu_0$ and do not revise the estimate.*

***Biased** investors use (4) to forecast stock prices.*

Trading in the stock incurs proportional transaction costs $\lambda \in [0, 1)$. A purchase of y shares at time t costs $y(1 + \lambda)S(t) \geq yS(t)$ while a seller of y shares receives only $y(1 - \lambda)S(t) \leq yS(t)$. It is customary (see for instance [Davis et al. \(1993\)](#)) to describe the investor's trading strategy via two non-decreasing right-continuous processes $L(t)$ and $M(t)$ representing, respectively, the cumulative number of shares bought and sold over time interval $[0, t]$. The dynamics of portfolio positions $(x(t), y(t))$, where $x(t)$ is the value of bonds held and $y(t)$ is the number of shares, is given by the differential equations

$$dx(t) = rx(t)dt - (1 + \lambda)S(t)dL(t) + (1 - \lambda)S(t)dM(t),$$

$$dy(t) = dL(t) - dM(t).$$

Given an initial position (x_0, y_0) , the investor maximizes the expected utility of the wealth at time $T > 0$ obtained by following a trading strategy

$(L(t), M(t))$:

$$\max_{(L,M)} \mathbb{E}\{U(x(T) + y(T)S(T))\}.$$

We impose two standard assumptions: there are no liquidation costs of the portfolio at the terminal time T and the investor has CARA utility function given by

$$U(w) = -\exp(-\alpha w), \tag{5}$$

where α is the risk aversion coefficient.

In the case of an informed investor, this utility maximization problem is classical (see the discussion above). The same is true for a naive investor. For learning investors one can show that it is optimal to estimate the true drift using (3) and to solve the optimization problem under the stock price dynamics given by (1) with $\mu(t, S(t)) = \mu^L(t, S(t))$.² Biased investors' optimization problem mimics behavioral decision making.

Stochastic differential equations with drift of the form (3) or (4) do not satisfy the standard conditions for existence and uniqueness of solution. We therefore provide a result that establishes existence of a unique solution in both cases.

²The justification of this reasoning is based on two concepts: the separation principle (Fleming and Rishel 1975, Theorem 11.2) and a Kalman-Bucy filter, see Øksendal (2003, Chapter 6). Denoting by \mathbb{G} the filtration generated by the stock prices, the separation principle states that the original optimization problem is equivalent to the one with the drift replaced by its best estimate in the squared error sense, i.e., the conditional expectation $\mathbb{E}(m|\mathbb{G}_t)$. The theory of Kalman-Bucy filtering justifies the formula (3) for this conditional expectation.

Lemma 2.2. *Assume that $\mu : [0, T] \times (0, \infty) \rightarrow \mathbb{R}$ is a continuous function that satisfies a logarithmic growth condition*

$$|\mu(u, S)| \leq M(1 + |\log(S)|), \quad S > 0, \quad u \in [0, T],$$

and a logarithmic Lipschitz condition

$$|\mu(u_1, S_1) - \mu(u_2, S_2)| \leq M|\log(S_1) - \log(S_2)|,$$

where $S_1, S_2 > 0$, $u_1, u_2 \in [0, T]$, for some constant $M > 0$. Then there is a unique strong solution to the stochastic differential equation (1) for every initial condition $S > 0$.

Proof. Øksendal (2003, Theorem 5.2.1) implies that under the assumptions of the lemma there is a unique strong solution to the stochastic differential equation

$$dZ(u) = \left(\mu(u, e^{Z(u)}) - \frac{\sigma^2}{2} \right) du + \sigma dW(u), \quad Z(t) = 0. \quad (6)$$

By Itô's formula the process $S(u) = S(t)e^{Z(u)-Z(t)}$, $u \geq t$, satisfies (1), i.e., it is a strong solution to this equation. To prove uniqueness, assume that there is another strong solution to (1), denoted by $\bar{S}(u)$, $u \geq t$, with $\bar{S}(t) = S(t)$ and $\bar{S}(u) \neq S(u)$ for $u > t$. Define $\bar{Z}(u) = \log(\bar{S}(u)/\bar{S}(t))$. Again, by Itô's formula $\bar{Z}(u)$ satisfies (6) and is different from $Z(u)$. This contradicts the uniqueness of the solution to (6). \square

Denote by $V(t, s, x, y)$ the value function corresponding to the utility optimization problem. This is the highest expected utility achievable by an investor whose portfolio at time t consisting of x units of cash and y shares

of the risky stock priced at $S(t) = s$:

$$V(t, s, x, y) = \sup_{(L(u), M(u))} \mathbb{E}\{U(x(T)+y(T)S(T)) | S(t) = s, x(t) = x, y(t) = y\}.$$

In the simplest case when the drift function is constant, $\mu(t, s) \equiv \bar{\mu}$ (a constant), the value function is characterized as a unique continuous viscosity solution of an HJB equation, see [Davis et al. \(1993\)](#):³

$$\max \left\{ \begin{aligned} &V_y - (1 + \lambda)sV_x; && -V_y + (1 - \lambda)sV_x; \\ &V_t + rxV_x + \bar{\mu}sV_s + \frac{\sigma^2}{2}s^2V_{ss} \end{aligned} \right\} = 0 \quad (7)$$

with the terminal condition $V(T, s, x, y) = U(x + ys)$ (subscripts in (7) denote partial derivatives). Solving this equation is usually carried out using numerical approximation. For general drift functions, a HJB representation is not known. We therefore take a different route to study optimal investment when the drift function is state- and time-dependent. In this paper, approximations are designed for the control problem itself.

3. Numerical Approach

We present a direct approach to solve the utility optimization problem for time- and state-dependent drift. The stock price model will be discretized in both time and space, and the approach invokes Bellman's dynamic programming principle. Similar to the pricing of options, the programming starts from the final time and works recursively backwards in time until it reaches the initial time.

³This result requires a restriction of the set of available trading strategies $(L(t), M(t))$: the liquidation value at any time must be greater than or equal to a fixed constant.

Let time be discretized in steps of length Δt with $\Delta t = T/N$ where N is the number of time steps. At each time-point the investor has to choose whether to trade and, if yes, how many units of stock to trade. The cash holdings are determined by the self-financing condition. The expected utility derived from each possible trading choice is determined by the value function. To select the trading decision that maximizes expected utility, the investor solves the maximization problem:

$$\begin{aligned}
V(t, s, x, y) = \max & \left\{ \underbrace{\mathbb{E} (V(t + \Delta t, S(t + \Delta t), e^{r\Delta t}x, y) | S(t) = s)}_{\text{benefit from not trading, } \Delta y = 0}, \right. \\
& \max_{\Delta y > 0} \underbrace{\mathbb{E} (V(t + \Delta t, S(t + \Delta t), e^{r\Delta t}(x - \Delta y \times s(1 + \lambda)), y + \Delta y) | S(t) = s)}_{\text{benefit from buying } \Delta y > 0 \text{ shares}}, \\
& \left. \max_{\Delta y > 0} \underbrace{\mathbb{E} (V(t + \Delta t, S(t + \Delta t), e^{r\Delta t}(x + \Delta y \times s(1 - \lambda)), y - \Delta y) | S(t) = s)}_{\text{benefit from selling } \Delta y > 0 \text{ shares}} \right\} \quad (8)
\end{aligned}$$

where the maximization is over the type of trade and the corresponding volume to be traded.

One might conjecture that the spatial discretization of the stock price process is complicated when its drift is state-dependent. However, one can use a standard binomial tree approximation due to Cox, Ross and Rubinstein and define adjusted probabilities for the up- and down-movement of the discretized stock price. The benefit is that the stock-price model is still a recombining tree. Specifically, we use the following binomial model. Define the coefficients $u = 1/d = e^{\sigma\sqrt{\Delta t}}$, and set the process

$$S(t + \Delta t) = \begin{cases} uS(t) & \text{with probability } p(t, S(t)) = [e^{\mu(t, S(t))\Delta t} - d] / [u - d] \\ dS(t) & \text{with probability } 1 - p(t, S(t)) \end{cases} \quad (9)$$

A natural discretization of the state space of money and stock holdings is given by the set $M_x \times M_y$ with $M_x = \{x_j : x_j = \underline{x} + j\delta x \leq \bar{x}, k \in \mathbb{N}\}$ and $M_y = \{y_k : y_k = \underline{y} + k\delta y \leq \bar{y}, k \in \mathbb{N}\}$ with given minimum and maximum values.

A direct algorithm for determining the value function and the optimal trading strategy proceeds as follows.

Define the value function at the terminal time as the realized utility. Set $V(T, s, x_j, y_k) := U(x_j + y_k s)$ for all values s of the discretized stock prices in period T and all portfolio holdings $(x_j, y_k) \in M_x \times M_y$.

For $t = T - \Delta t, \dots, 0$

For all values of the discretized stock price $s = S(t)$ at time t

For all values $(x_j, y_k) \in M_x \times M_y$

Given the functions $V(t + \Delta t, \dots)$, find the highest value in (8) obtained over all values Δy such that $y_k + \Delta y \in M_y$. $V(t + \Delta t, \dots)$ is approximated by linear interpolation since $\exp(r\Delta t)[x_j \mp \Delta y s(1 \pm \lambda)]$ is typically not an element of M_x . Denote the maximum value $V(t, s, x_j, y_k)$.

End For

End For

End For

The computational complexity of the direct method is of the order $O(N^2 \times M_x \times M_y \times M_y)$ or $O(N^5)$.⁴ The factor N^2 arises because the algorithm

⁴We let M_x and M_y linearly depend on time steps N to ensure that the grid sizes δx and δy approach 0 when Δt is close to 0 with increasing N . Letting step sizes grow at the same rate is not necessarily optimal, it depends on the accuracy/order of the approximations in

loops through all points on the stock price lattice, the factor $M_x \times M_y$ is due to the loop through the grid of portfolio holdings, and the final factor M_y comes from the Δy -search. This is slow; doubling the number of steps in all dimensions makes the computation time grow by a factor of 32.

The range of $M_x \times M_y$ is usually large in order to include optimal solutions for all possible states $(t, S(t))$ on the lattice. The above standard numerical method uses an equidistant grid and searches for optimal solutions in the whole set.

As a benchmark, suppose the direct algorithm is implemented in a high-level language such as *Matlab* on a typical laptop computer. Running through a binomial lattice with $T = 1$ year and time steps of 1 day takes 5 - 10 milliseconds. Using a grid of one million points $M_x = M_y = 100$ (think of this as percentage points of wealth in money resp. stock) would take about 2 hours. This is not a computationally feasible approach since reasonable outputs require high-resolution grids and thousands of simulations of a random drift.

Five measures are employed to reduce running time of simulations:

Reducing dimension. When measuring utility by the negative exponential function (5), the value function V can be written in the form

$$V(t, s, x, y) = H(t, s, y) \exp(-\alpha x \exp[r(T - t)]), \quad (10)$$

where $H(t, s, y)$ is defined by $H(t, s, y) := V(t, s, 0, y)$, see, e.g., [Davis et al. \(1993\)](#) or [Monoyios \(2004\)](#). This representation allows reducing the dimen-

the different dimensions. But it seems futile to hope for anything beyond linear. The free boundary nature of the problem also seems likely to rule out second order accuracy.

sion of the optimization problem by one. However, this measure carries a potential cost. Suppose an investor's money and stock holdings are large (in absolute terms) but offsetting in terms of value. Then the exponent of the exponential utility function implied by $H(t, s, y)$ will include the product of a very large stock holding and a large stock price. This can cause numerical over- or underflow errors in the computer program, which are dealt with by our following function $H(t, s, y)$ scale, along with local search and non-equidistant discretization that speeds up the program.

Function $H(t, s, y)$ scale. To handle the over- or underflow issues we scale the value function $H(t, s, y)$ by

$$G(t, s, y) := V(t, s, -ys, y) = H(t, s, y) \exp(\alpha y s \exp[r(T - t)]).$$

Then we solve a discrete time dynamic programming equation for the value function $G(t, s, y)$ similar to (8) with the terminal condition $G(T, s, y) = -1$.

Local Δy -search. The solution to $H(t, s, y)$ is known to have a particular structure. The space of stock holdings is split into three regions: buy, no-trade and sell. If the stock position is either in the buy or sell region, a trade is initiated that leads to a stock position on the closest boundary of the no-trade region. If the stock position is inside or on the boundary of the no-trade region, the investor does not change his stock position.

In the case of a constant drift (Monoyios 2004, p. 902) the upper boundary (above which one sells) and the lower boundary (below which one buys) of the no-trade region at a given time t can be both defined by market values of stock positions. It is therefore sufficient to determine the optimal trade in all time- t nodes with a node (t, S) to find the two boundaries.

With a state-dependent drift, this observation no longer holds true: If the forecast of the drift is revised depending on the current stock price, then the no-trade region will depend on this information. One therefore has to determine a no-trade region in each node (t, S) . This is computationally costly. A numerically efficient approach, which we implement, is to determine the boundaries of the no-trade region through searching over a local range of y . The local range denoted by $[\varphi_b(t, S), \varphi_s(t, S)]$ is determined by an appropriate extension of the boundaries at the successive nodes.

Non-equidistant y -discretization. The structure of optimal trading strategies suggests that it is not efficient to have an equidistant discretization of the y -space. The set of discretization points should be denser close to the boundaries of the no-trade region. We therefore use a symmetric, non-equidistant discretization.

The set is centered at Merton's closed-form solution for the case of a constant drift and no transaction costs, which is denoted by $\varphi_M(t, S)$. The value of drift μ is given by investors (possibly an actual value or an estimate). The non-equidistant grid has larger step-sizes away from the center $\varphi_M(t, S)$. For a given (t, S) -node and the local range $[\varphi_b(t, S), \varphi_s(t, S)]$, we first define the radius

$$\Phi(t, S) := \max \{ \varphi_M(t, S) - \varphi_b(t, S), \varphi_s(t, S) - \varphi_M(t, S) \}. \quad (11)$$

Then we define the set of discretization points as:

$$y(t, S, k) = \varphi_M(t, S) + \frac{\Phi(t, S)}{\left(\frac{M_y}{2}\right)^\varpi} \left(k - \frac{M_y}{2}\right) \left|k - \frac{M_y}{2}\right|^{\varpi-1}, \quad (12)$$

where

$$\varphi_M(t, S) = \frac{\mu - r}{e^{r(T-t)} \alpha \sigma^2 S}$$

is the [Merton \(1971\)](#) solution. The coefficient $\varpi > 1$ controls the level of dispersion.⁵ Numerical experiments have revealed that an appropriate choice of the coefficient ϖ is 1.6.⁶

Low-level language. Implementation in a low-level language, e.g., C++, gives a speed-up of a factor approximately 10.

Numerical illustration. We use the following values of parameters as a base case for our numerical results: drift drawn from normal distribution with mean $\mu_0 = 0.15$ and variance $\gamma_0 = 0.04$, volatility $\sigma = 0.25$, proportional transaction cost rate $\lambda = 0.01$, initial stock price $S_0 = 15$, risk aversion $\alpha = 0.1$, interest rate $r = 0.03$, time-horizon $T = 1$ year, and discretization parameters $\Delta t = 0.01$, $M_y = 3,500$ and $\varpi = 1.6$.

Figure 1 demonstrates the joint effect of transaction costs and state-dependent drift. It shows one realization of the optimal trading strategy for a 40-year time-horizon. The effect is substantial as evidenced by the high variability of the boundaries of the no-trade region. The volatility of these

⁵If $\varpi = 1$, the grid degenerates to the equidistant discretization, while if ϖ is large, the points are too concentrative in the neighbor of the center.

⁶The maximum in (11) ensures that the y grid is wide enough. Although $\varphi_M(t, S)$ may not be the center of $[\varphi_b(t, S), \varphi_s(t, S)]$, it is the center of $[y(t, S, 0), y(t, S, M_y)]$ which would include a few extra points outside $[\varphi_b(t, S), \varphi_s(t, S)]$. The two end points $y(t, S, 0)$ and $y(t, S, M_y)$ correspond to the value of (12) with $k = 0$ and $k = M_y$ respectively. See [Wang \(2010, Sect. 3.6.4\)](#) for details.

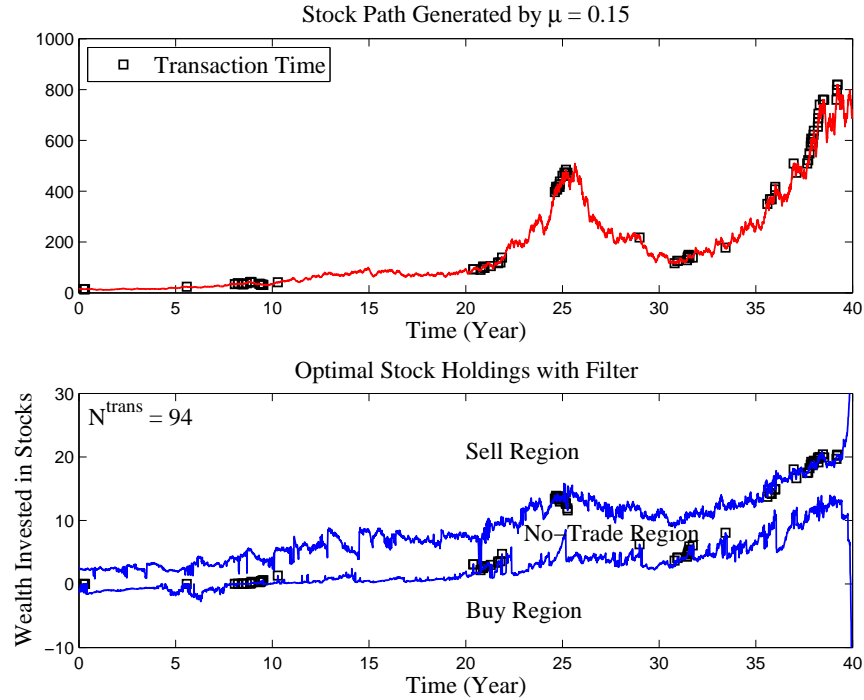


Figure 1: Dynamics of the no-trade region with state-dependent drift $\mu^L(t, S(t))$ within $T = 40$ years horizon. The squares indicate transaction times. N^{trans} is the total number of transactions.

boundaries reflects changes in the learning investor's estimate of the drift. For instance, the boundaries move downwards around 30 years in response to a fall in the stock price, and move upwards from about 35 years when the stock price recovers. With a known, constant drift, these boundaries (when measured in terms of the amount of wealth invested in stocks) are hyperbola-like curves that are independent of the stock price.

Comparison with Monoyios (2004)'s results. Verification of our method is carried out by comparing numerical results with those reported in Monoyios (2004). The comparison is for the simple case of a known, constant drift

which is considered in the latter paper. Table 1 reports the two boundaries of the no-trade region at the initial time for different transaction costs. We calculate results with our method under both equidistant and non-equidistant discretization. In all three scenarios and for different transaction costs the calculated boundaries coincide up to 3-4 significant digits.

The non-equidistant discretization requires fewer points on the y -grid than the equidistant discretization, which substantially shortens the runtime of the program. Our approach works efficiently because we take state-dependent non-equidistant discretization on a small local range of y -values. The discretization equation (12) produces a great number of points with the precision up to 0.0001 around the area where the no-trade region is most probably located. The distance between grid points increases gradually towards the two end-points of the local range of y -values.⁷ As a result, it suffices to set $M_y = 3,500$ to achieve similar results to those obtained by the standard equidistant discretization that requires from about 0.27 million to about 2.38 million grid points, depending on the full range of y -values, see the last row in Table 1.

We also compare the performance of non-equidistant and equidistant discretizations in the case of the state-dependent drift $\mu^L(t, S(t))$. Figure 2 shows that the most stable results are obtained under the non-equidistant discretization. The precision of the approximation increases gradually as the number of time steps increases. Equidistant discretizations exhibit a more volatile behavior.

⁷See Wang (2010, Figures 3.5 and 3.6) for an example of the frequency histogram and the diagram of varying precision of y -values.

| | $\lambda=0.005$ | $\lambda=0.01$ | $\lambda=0.02$ | $\lambda=0.03$ |
|-----------------|-------------------|-------------------|-------------------|-------------------|
| Monoyios | [0.3866, 0.5780] | [0.3499, 0.6197] | [0.2702, 0.7196] | [0.1813, 0.8243] |
| Equidistant | [0.3870, 0.5772] | [0.3510, 0.6193] | [0.2708, 0.7137] | [0.1851, 0.8161] |
| Non-equidistant | [0.3864, 0.5763] | [0.3527, 0.6209] | [0.2720, 0.7177] | [0.1826, 0.8113] |
| Range of y | [-10.748, 16.213] | [-24.622, 30.443] | [-56.800, 64.677] | [-109.43, 128.71] |

Table 1: Boundaries of no-trade region at $t = 0$. The binomial lattice is as in [Monoyios \(2004, p. 896\)](#). The first row and the parameters are taken from [Monoyios \(2004, Table 1\)](#): $r = 0.1$, $\Delta t = 0.02$, $\mu = 0.15$ (known drift $\gamma_0 = 0$), $\sigma = 0.25$, $S_0 = 15$, $\alpha = 0.1$, $T = 1$ year. The second row uses the equidistant discretization with $\Delta y = 0.0001$, while the third row uses the non-equidistant discretization (12) with $M_y = 3,500$ and $\varpi = 1.6$. The last row presents the ranges of y grid determined by equations (A.2) and (A.5) in [Monoyios \(2004\)](#).

We finally consider the relationship between computation time and numerical accuracy. Figure 3 shows results for the non-equidistant discretization with local search in the case of the state-dependent drift $\mu^L(t, S(t))$. All observations are close to a straight line with slope -0.4 (taking logarithms of both variables): The approximation error is $O(1/N)$ and computational complexity is $O(N^{1/0.4})$. This means that to halve the numerical error, computing time is increased by a factor of $2^{(1/0.4)} \approx 5.7$. The numerical result demonstrates that our algorithm has the time-complexity around $O(N^{2.5})$ while the direct method is $O(N^5)$.

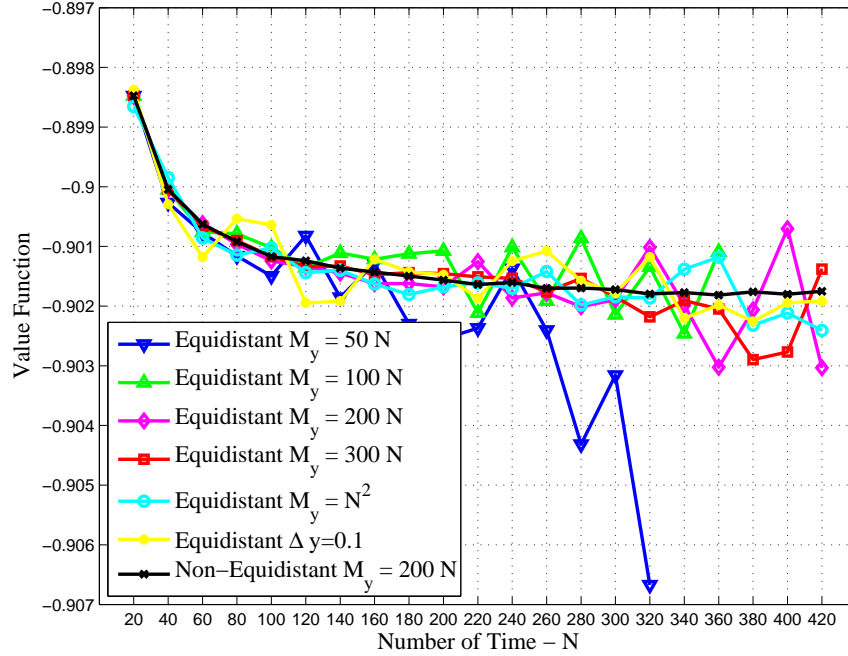


Figure 2: Value functions at initial time versus the number of time N with the state-dependent drift $\mu^L(t, S(t))$, where $N = 20 + i \times 20$, $i = 0, 1, \dots, 20$. Other parameters are the same as the base case.

4. Results

The numerical solution technique is applied to study the effects of transaction costs and uncertainty about the drift over investment time-horizons of up to 10 years. We consider the four types of investors introduced in Section 2: informed investors (knowing the actual value of the randomly drawn drift), learning investors (learning about the true value of the drift through filtering), naive investors (no revision of initial estimate of the drift) and bi-ased investors (varying from strongly trend-following to strongly contrarian, depending on the value of sentiment parameter a).

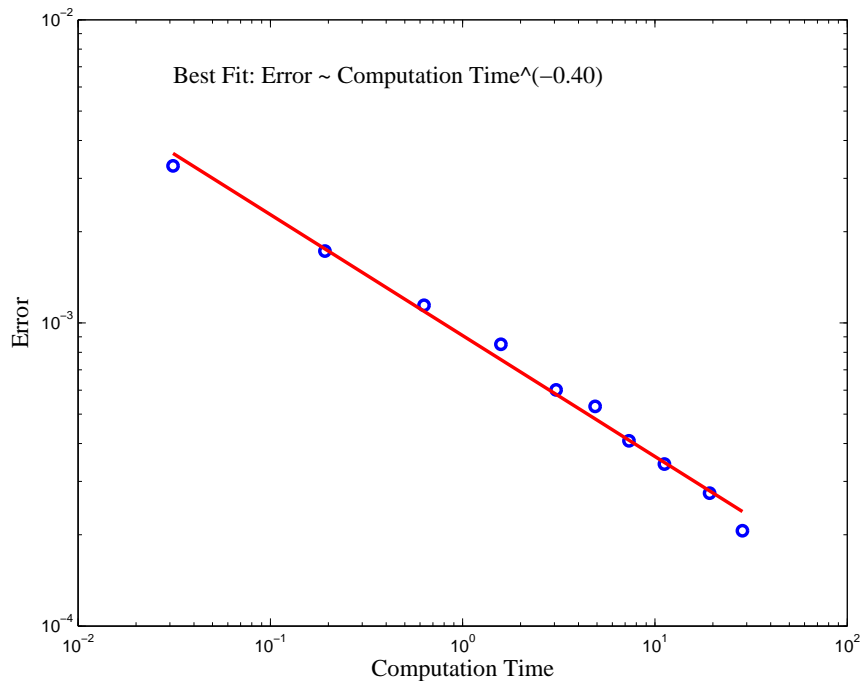


Figure 3: Convergence with non-equidistant discretization with the state-dependent drift $\mu^L(t, S(t))$. The x-axis shows the physical computation time, and the y-axis reports the absolute error of the calculated initial value function V . The error is defined as $|V - \hat{V}|$ and \hat{V} is taken as the approximation of the true value of value function. The value of \hat{V} is obtained using a lattice with a sufficiently small step length Δt .

Our numerical results provide three main insights of practical relevance:

- Not knowing the true stock price dynamics leads to large losses in utility for less risk-averse investors, strongly biased investors, and naive investors (in decreasing order).
- Learning generally reduces the loss in utility caused by uncertainty about the true drift.

- Lower trading volumes due to transaction costs explain about half of the total loss in utility. The other half is caused by transaction-cost payments.

When comparing the choices of different investors that are in the same situation or of identical investors that are in different situations, one has to take into account two aspects. First, quantifying an investor's gain or loss should be done using monetary units: This allows expressing differences in utility as the value of contract that, for instance, provides the investor with information about the drift or frees an investor from having to pay transaction costs. These values are defined as the amount of wealth that an investor has to pay (needs to receive) at initial time in order to be indifferent between two situations. Second, naive investors and investors with biases make trading decisions that are not optimal. Such an investor will see his average realized utility being lower than the one expected *ex ante*. We therefore take realized rather than perceived utility when measuring losses relative to an informed investor in monetary equivalents.

Section 4.1 considers the value of knowing the realization of the drift and the true stock price dynamics ('value of information,' for short) and Section 4.2 analysis the true (economic) cost of proportional transaction costs.

4.1. Value of information

For each investor type, the average realized utility is given by

$$R(x) := E_\mu \bar{U}_\mu(x)$$

where x is the initial money endowment (the initial share endowment is zero). E_μ denotes expectation with respect to μ which has the distribution

$\mathcal{N}(\mu_0, \gamma_0)$. The realized utility \bar{U}_μ is determined by the realized stock price path, the investor's realized trading strategy (L, M) , and the utility function U defined in (5):

$$\bar{U}_\mu = \mathbb{E}\{U(x(T) + y(T)S(T)) \mid (L, M)\}.$$

Since the average utility an investor expects to achieve cannot be lower than the actually realized one, one has

$$R(x) \leq E_\mu V_\mu(0, S_0, x, 0),$$

where $V_\mu(0, S_0, x, 0)$ is the value of expected utility. For naive and biased investors, the inequality will, in general, be strict as these investors make incorrect assumptions about the stock price dynamics, overestimating the utility their trading strategy will deliver. However, an informed investor's average realized utility satisfies

$$R^F(x) = E_\mu V_\mu^F(0, S_0, x, 0),$$

where $V_\mu^F(0, S_0, x, 0)$ is the expected utility which the investor maximizes under knowledge of the value of μ . For a learning investor, who always uses μ_0 as prior for the drift estimate at the initial time,

$$R^L(x) = V^L(0, S_0, x, 0).$$

The monetary value of being informed rather than having to learn the true drift over time from observations is:

$$\mathbb{I}E^L(x) = \sup\{c \geq 0 \mid R^L(x) \leq R^F(x - c)\}. \quad (13)$$

This maximum amount a learning investor can pay to obtain the true value of μ without being worse off can be interpreted as a *information equivalent* (IE). If the realization of the randomly drawn drift could be purchased then $\text{IE}^L(x)$ were the highest price a learning investor is willing to pay to be certain about the value μ . Since the utility function (5) is CARA, the measure defined in (13) is actually independent of the monetary endowment x .

As the value functions of these two investors satisfy (10), one finds

$$\text{IE}^L = \frac{1}{\alpha} \exp(-rT) \log (H^L / E_\mu H_\mu^F)$$

with H^L and H_μ^F corresponding to the reduced form value functions of the learning investor and the informed investor. The information equivalent IE^L is approximated numerically using the value functions derived in Section 3. An approximation \hat{H}^F of the expected value $E_\mu H_\mu^F$ is calculated as follows:

1. Draw independently M_μ values from the distribution $\mathcal{N}(\mu_0, \gamma_0)$. These are realizations of the drift.
2. For each random draw μ_i , calculate the value function $H_{\mu_i}^F$ by solving the portfolio optimization problem (8) with (10).
3. Calculate

$$\hat{H}^F = \frac{1}{M_\mu} \sum_i^{M_\mu} H_{\mu_i}^F.$$

Similar to (13), we can approximate the monetary value of being an informed investor rather than a naive investor or a biased investor. For naive and biased investors, one first needs to solve the optimization problem to determine their trading strategies. Using these strategies one can determine realized utility in a Monte Carlo simulation. To obtain the average realized utility one

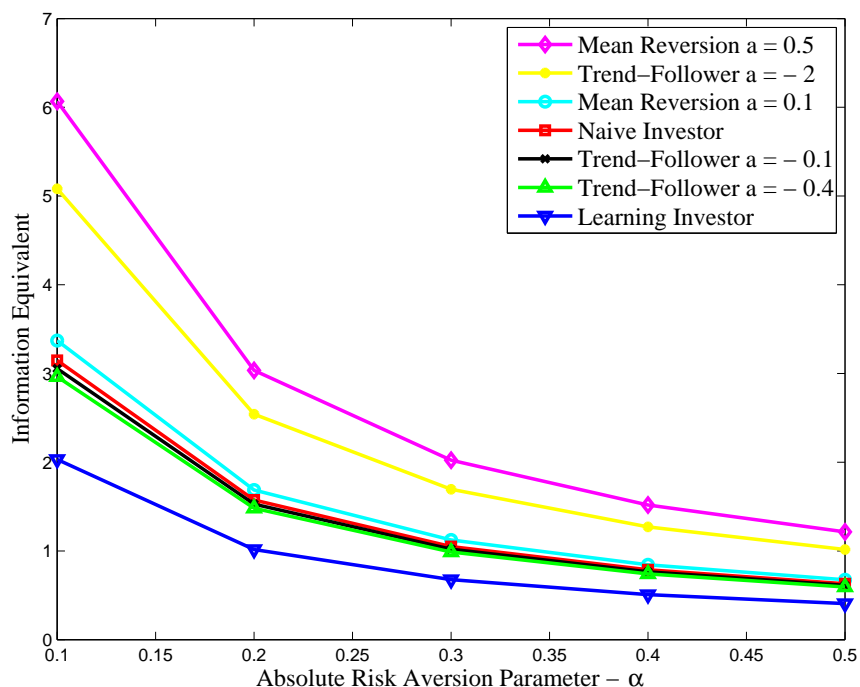


Figure 4: Information equivalents for different levels of risk aversion.

has to repeat this procedure for many independent draws of μ . In addition, these calculations have to be carried out for different levels of risk aversion and, if the investor is biased, for different degrees of sentiment. The efficient numerical method introduced in Section 3 allows us to perform these simulations in a matter of hours.

Figure 4 depicts information equivalents for different levels of risk aversion and different investor types. The lowest values are obtained for a learning investor. This observation confirms that empirical estimation of the drift using a filter is beneficial. The highest values are associated with aggressive trend-followers and contrarian investors while less aggressive ones have information equivalents close to that of the naive investor.

Information equivalents are decreasing in the risk aversion: more risk-averse investors of any type, receive lower benefits from knowing the true drift. For instance, the more risk-averse investors with $\alpha = 0.5$ are only willing to pay from 1/6th to 1/4th as much as the less risk-averse investors with $\alpha = 0.1$ to remove uncertainty about the actual drift. At first sight this might be surprising as higher risk-aversion is generally associated with higher willingness to pay in order to avoid risk. The opposite is true here as higher risk aversion leads to less investment in the stock, see also [Liu and Loewenstein \(2002\)](#); [Liu \(2004\)](#); [Muthuraman and Kumar \(2006\)](#). [Cvitanić et al. \(2006\)](#) also find that the certainty equivalents that they examine achieve the highest values for the lowest risk aversion in different setups.

The sentiment parameter a has a marked impact on information equivalents, cf. [Figure 4](#), which warrants a more detailed analysis. [Figure 5](#) shows the result for the information equivalent with the sentiment parameter a in [\(4\)](#) varying between -2 (strongly trend-following) and 0.5 (strongly contrarian). The information equivalent is a U-shaped function of a . Its minimum is obtained for a mildly trend-following investor. For the parameter values considered here, the minimum is obtained for a degree of sentiment $a \approx -0.4$. Mild trend-following therefore mimics the optimal filtering. As a result, the trading strategy of an investor whose estimate of the drift is derived from cautious interpolation of an observed (short-term) trend, is close to that of a learning investor.

The effect of the investment time-horizon on the information equivalent is studied in [Figure 6](#). It shows annualized information equivalents which are defined as $IE \cdot e^{rT}/T$ as IE is defined at the initial time. First, the

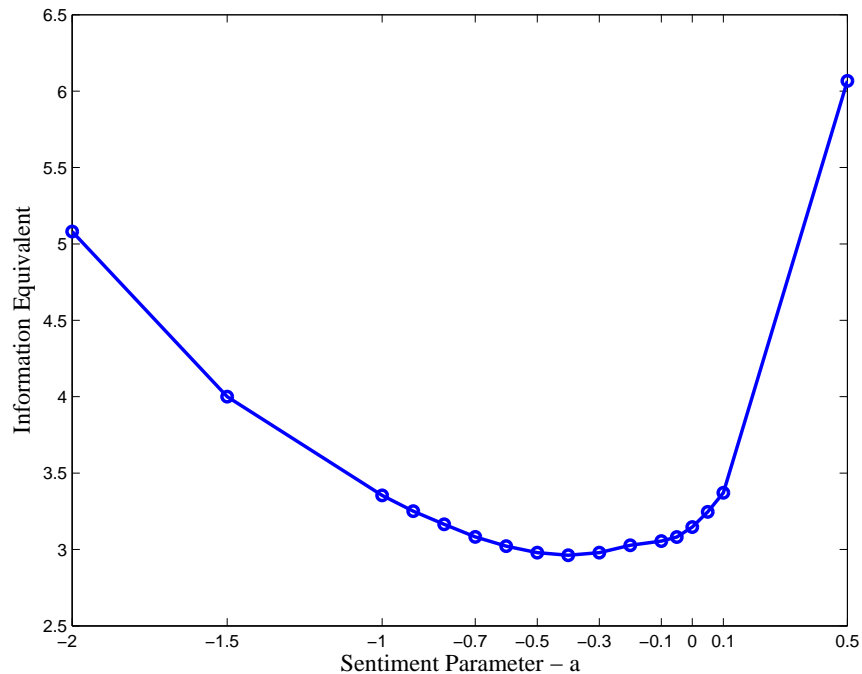


Figure 5: Information equivalents of biased investors with different values of parameter a (see (4)): naive investor ($a = 0$), trend-follower ($a < 0$) and contrarian investor ($a > 0$).

naive investor has more to gain from knowing the true drift than the learning investor, and the annualized benefit is fairly constant across different investment horizons. In contrast, the annualized information equivalent of a learning investor is decreasing in the investment horizon. This reflects the gain in knowledge from filtering which reduces conditional variance when the investment horizon increases. It also provides a hedge against unfavorable realizations of the drift (Brennan 1998).

The information equivalent is strictly positive even at a 10-year investment horizon. The lesson is that the true drift is difficult to estimate and one

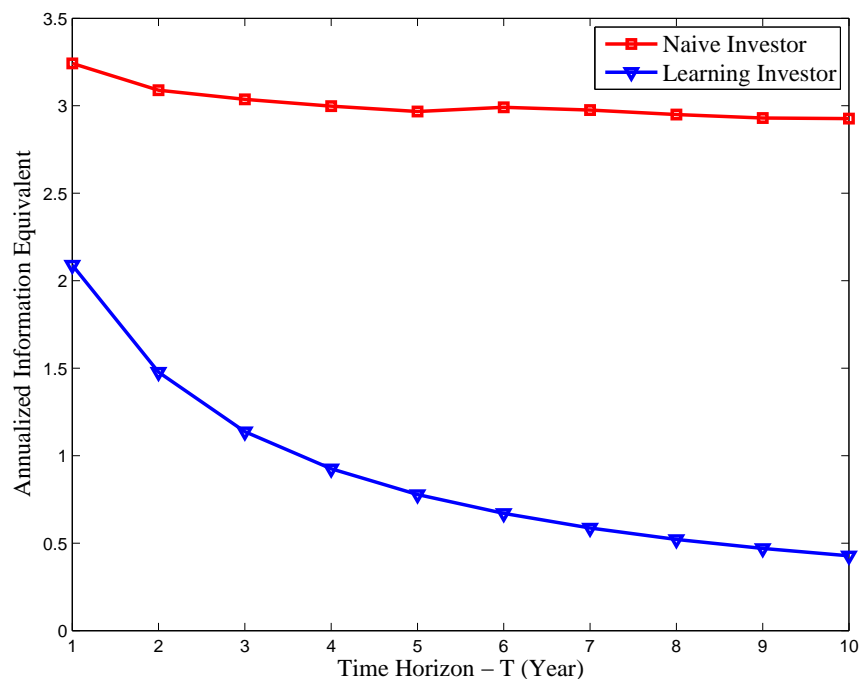


Figure 6: Annualized information equivalents for different investment horizons. Learning investor (triangles) and naive investor (squares).

cannot eliminate uncertainty about the drift within this finite time-horizon. Therefore, learning about the drift via filtering has benefits even in the long run as the estimation error decreasing slowly with time. Previous studies of the case without transaction costs find substantial utility gains when investors take an optimal dynamic strategy with learning, see [Xia \(2001\)](#) and [Cvitanic et al. \(2006\)](#).⁸ Our results under transaction costs show that a naive

⁸Using the optimal strategy with learning, [Xia \(2001\)](#) shows that investors can improve their welfare by from 15% to 100%, and [Cvitanic et al. \(2006\)](#) report that certainty equivalents increase from 2.93% to 215.73%. A quantitative comparison between their results and ours is inappropriate sine the models and values of parameters are substantially

investor with a 1-year horizon can reduce the loss due to uncertainty about the true drift by 1/3rd when adopting a filtering strategy; with a 10-year horizon the loss is reduced by 4/5th.

4.2. Transaction costs

Trading strategies are usually very sensitive to transaction costs. We therefore explore the impact of these costs in detail. Figure 7 shows the utility of a learning investor under different transaction cost scenarios. The top graph (horizontal line) is the benchmark case of no transaction costs. The bottom graph is the utility of a learning investor who incurs proportional transaction costs. Utility is decreasing as the proportional transaction cost increases which coincides with findings in previous studies (see, e.g. [Gennotte and Jung 1994](#); [Balduzzi and Lynch 1999](#)). In the range 0.5% to 2% the loss in utility is approximately linear.

This loss in utility is caused by two effects of transaction costs: (a) a direct effect due to the additional expense incurred and (b) an indirect effect due to less trading on the asset allocation. In our model, we can quantify these two separate effects by stripping out the first one by reimbursing all transaction costs (with interest) at the final period. Of course the investor optimizes the dynamic portfolio strategy without knowing about this reimbursement as otherwise the scenario is identical to the no transaction cost case. The result is the middle graph in Figure 7 which is about halfway (except those below halfway cases for the small cost rate $\lambda < 0.01$) between the zero-cost and positive-cost without reimbursement case.

different.

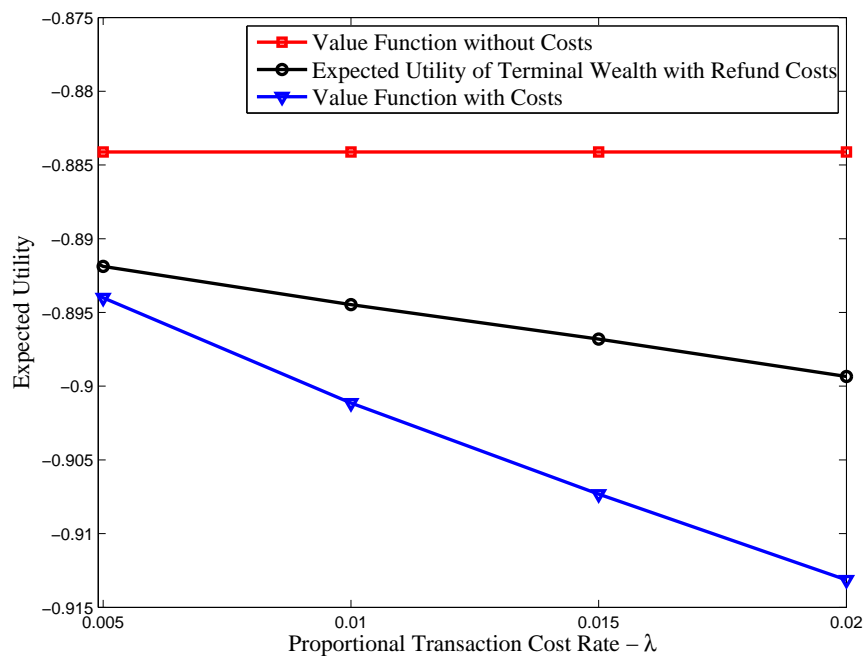


Figure 7: Maximum expected utility of a learning investor who faces no transaction costs (top graph) and positive transaction costs with reimbursement (middle graph) and without reimbursement (bottom graph).

The difference between the reimbursement and the zero-cost benchmark case is the deadweight loss from the proportional transaction cost. It measures the true economic cost of this friction. We find that the total effect of the transaction cost is about twice (except the small cost rate $\lambda < 0.01$) as large as the loss in utility due to less trading resulting from transaction costs. The implications are that freely re-balancing portfolio significantly contributes to investors' expected utilities, and less re-balancing with costs than that without costs brings about half (or more than half for $\lambda < 0.01$) of the total loss in utility of this friction.

The welfare impact of transaction costs is analyzed in more detail. To capture the value from investing in a market without transaction costs, we denote the corresponding gain to an investor of type \cdot as

$$\text{TE}(\lambda) = \sup\{c \geq 0 \mid E_\mu V_{\mu,\lambda}(0, S_0, x, 0) \leq E_\mu V_{\mu,\lambda=0}(0, S_0, x - c, 0)\}, \quad (14)$$

where $V_{\mu,\lambda}(0, S_0, x, 0)$ is the value of expected utility which an investor maximizes according to his perspective of the drift, i.e. the drift he takes when solving his portfolio optimization problem, under proportional transaction costs at a rate λ . The reasons of using expected utilities are explained later. The CARA utility function (5) implies that the measure is independent of the monetary endowment x . $\text{TE}(\lambda)$ is the maximum price an investor of a given type is willing to pay to avoid incurring transaction costs at a rate λ . This justifies the notion transaction-cost equivalent (TE). As the value function satisfies (10), one has

$$\text{TE}(\lambda) = \frac{1}{\alpha} \exp(-rT) \log(E_\mu H_{\mu,\lambda} / E_\mu H_{\mu,\lambda=0}),$$

where $H_{\mu,\lambda}$ is the reduced form value function when the transaction cost rate is λ . Only for an informed investor, does the value function depend on μ following the distribution $\mathcal{N}(\mu_0, \gamma_0)$. For all other types, one can drop the expected value operator E_μ and the subscript μ of the value function.

In contrast to the above study of the value of information IE , we compare here one investor (rather than two investors when calculating IE) in two different situations with and without transaction costs. We do not need to distinguish here whether the drift he takes is the true drift or not. In addition, average realized utility can be actually increased by transaction costs. In fact, investors who maximize utility under incorrect assumption of the

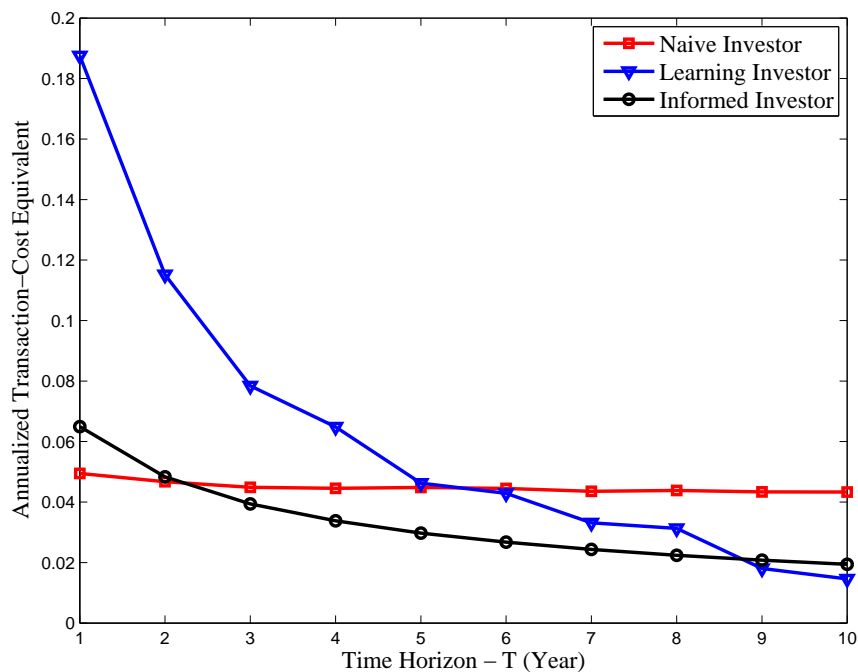


Figure 8: Annualized transaction-cost equivalents for different investment horizons. Learning investor (triangles), naive investor (squares) and informed investor (circles). The proportional transaction cost rate is given by the base case as $\lambda = 0.01$.

stock price dynamics make subjectively optimal but objectively sub-optimal trading decisions. When the cost discourages investors from making trades that are objectively sub-optimal, transaction costs can increase realized average utility. Perceived utility however will never increase when transaction costs increase. We therefore use perceived expected utility rather than average realized utility.

Figure 8 shows the effect of proportional transaction costs on three investor types. The transaction-cost equivalents are annualized to allow a

meaningful comparison of investment over different time-horizons. This measure is approximately constant for the naive investor but slowly decreasing for the informed investor and rapidly decreasing for the learning investor. For time-horizons of up to 5 years, the learning investor is the one most strongly affected by transaction costs. At a 1-year time-horizon, his willingness to pay to avoid 1% transaction costs is almost 4 times that of the naive investor and 3 times that of an informed investor. In the short run, the estimate of the actual drift is inaccurate and can vary drastically, see also [Lundtofte \(2006, 2008\)](#) for a related discussion.⁹ This increases the investor's incentive to trade and leads to higher transaction costs. Therefore a learning investor is the most keen to remove these costs.

At longer time-horizons, the naive investor has the most to gain from the absence of transaction costs as the misspecification of the drift leads to excess trading compared to investors who either know or have learned enough about the actual drift. For a learning investor, trading is slightly contrarian, which leads to the lowest transaction-cost equivalent. For instance, a sudden sharp drop (rise) in the stock price leads to a stock purchase (sale) from the informed investor in order to keep holdings in the no-trade region. A learning investor would at the same time lower (increase) the estimate of the drift and therefore tends to make a smaller trade, incurring lower transaction costs. As a result, the learning investor reduces the loss in utility implied by TE by about 1/2 over a 10-year time-horizon compared with the naive investor.

⁹At short time-horizons of less than 5 years, the conditional variance of the filter, which decreases with time, is relatively large compared to those over time-horizons of 10 years and more.

The observation on the benefit of learning mirrors those by [Cvitanic et al. \(2006\)](#) who, in the case without transaction costs, find substantial utility gains resulting from the optimal strategy with filtering, especially for long horizons.

5. Conclusion

The efficient numerical algorithm introduced in the paper allows us to solve portfolio optimization problems with state-dependent drift and long time-horizons in the presence of proportional transaction costs. We apply the method to explore scenarios in which investors (a) use past stock prices to learn about the true (but unknown) drift, (b) react to stock price movements as trend-followers or contrarians, or (c) are naive and ignore information that is revealed over time.

The numerical results show that forecasting behavior has a strong impact on trading in the presence of transactions costs. Using the indifference principle, we quantify the value of information and the welfare effect of transaction costs. Information is most valuable to the least risk-averse investor, and transaction costs are most detrimental to naive investors. The total loss in utility from proportional transaction costs is generally about twice as large as the direct cost incurred. In general, learning reduces the losses in utility due to the uncertain drift and transaction costs, especially for long horizons.

References

Atkinson, C., Quek, G., 2012. Dynamic portfolio optimization in discrete-time with transaction costs. *Applied Mathematical Finance* 19, 265–298.

- Balduzzi, P., Lynch, A.W., 1999. Transaction costs and predictability: Some utility cost calculations. *Journal of Financial Economics* 52, 47–78.
- Brennan, M.J., 1998. The role of learning in dynamic portfolio allocations. *European Finance Review* 1, 295–306.
- Chellathurai, T., Draviam, T., 2007. Dynamic portfolio selection with fixed and/or proportional transaction costs using non-singular stochastic optimal control theory. *Journal of Economic Dynamics and Control* 31, 2168–2195.
- Clelow, L., Hodges, S., 1997. Optimal delta-hedging under transactions costs. *Journal of Economic Dynamics and Control* 21, 1353–1376.
- Cvitanic, J., Lazrak, A., Martellini, L., Zapatero, F., 2006. Dynamic portfolio choice with parameter uncertainty and the economic value of analysts' recommendations. *Review of Financial Studies* 19, 1113–1156.
- Danilova, A., Monoyios, M., Ng, A., 2010. Optimal investment with inside information and parameter uncertainty. *Mathematics and Financial Economics* 3, 13–38.
- Davis, M.H.A., Panas, V.G., Zariphopoulou, T., 1993. European option pricing with transaction costs. *SIAM Journal on Control and Optimization* 31, 470–493.
- Fleming, W.H., Rishel, R.W., 1975. *Deterministic and Stochastic Optimal Control*. Springer, New York.

- Gennotte, G., Jung, A., 1994. Investment strategies under transaction costs - the finite-horizon case. *Management Science* 40, 385–404.
- Herzog, R., Kunisch, K., Sass, J., 2013. Primal-dual methods for the computation of trading regions under proportional transaction costs. *Mathematical Methods of Operations Research* 77, 101–130.
- Lensberg, T., Schenk-Hoppé, K.R., 2013. Hedging without sweat: a genetic programming approach. *Quantitative Finance Letters* 1, 41–46.
- Liu, H., 2004. Optimal consumption and investment with transaction costs and multiple risky assets. *Journal of Finance* 59, 289–338.
- Liu, H., Loewenstein, M., 2002. Optimal portfolio selection with transaction costs and finite horizons. *Review of Financial Studies* 15, 805–835.
- Lundtofte, F., 2006. The effect of information quality on optimal portfolio choice. *Financial Review* 41, 157–185.
- Lundtofte, F., 2008. Expected life-time utility and hedging demands in a partially observable economy. *European Economic Review* 52, 1072–1096.
- Merton, R.C., 1971. Optimal portfolio and consumption rules in a continuous-time model. *Journal of Economic Theory* 3, 373–413.
- Monoyios, M., 2004. Option pricing with transaction costs using a Markov chain approximation. *Journal of Economic Dynamics and Control* 28, 889–913.
- Muthuraman, K., Kumar, S., 2006. Multidimensional portfolio optimization with proportional transaction costs. *Mathematical Finance* 16, 301–335.

- Øksendal, B., 2003. *Stochastic Differential Equations: An Introduction with Applications*. 6th ed., Springer, New York.
- Rogers, L.C.G., 2001. The relaxed investor and parameter uncertainty. *Finance and Stochastics* 5, 131–154.
- Shefrin, H., 2008. *A Behavioral Approach to Asset Pricing Theory*. 2nd ed., Academic Press, San Diego.
- Wang, H., 2010. *Optimal Portfolio Choice under Partial Information and Transaction Costs*. University of Leeds, PhD thesis, <http://ssrn.com/abstract=1664427>.
- Xia, Y., 2001. Learning about predictability: The effects of parameter uncertainty on dynamic asset allocation. *Journal of Finance* 56, 205–246.
- Zakamouline, V., 2005. A unified approach to portfolio optimization with linear transaction costs. *Mathematical Methods of Operations Research* 62, 319–343.
- Zakamouline, V., 2006. European option pricing and hedging with both fixed and proportional transaction costs. *Journal of Economic Dynamics and Control* 30, 1–25.