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Heterogeniety and limited stock market participation

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Abstract

We derive the equilibrium interest rate and risk premiums using recursive utility with heterogeneity in a continuous time model. We solve the associated sup-convolution problem, and obtain explicit closed form solutions. The heterogeneous two-agent model is calibrated to the data of Mehra and Prescott (1985) assuming the market portfolio is not a proxy of the wealth portfolio. This results in plausible values for the preference parameters of the two agents under various assumptions for the wealth portfolio.

KEYWORDS: The equity premium puzzle, the risk-free rate puzzle, recursive utility, the stochastic maximum principle, heterogeneity, limited market participation

JEL-Code: G10, G12, D9, D51, D53, D90, E21.

1 Introduction

We consider recursive utility with heterogeneity to analyze the standard rational expectations model of Lucas (1978). We solve the resulting sup-convolution problem, and find explicit formulas for the risk aversion and the equilibrium interest rate. The resulting representative agent utility is not of the standard recursive type, and can be considered as a generalized recursive utility function.

The resulting model we adapt to address the problem with recursive utility that the market portfolio is not a good proxy of the wealth portfolio.

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As an example we calibrate the model to the US-data used by Mehra and Prescott (1985) under various assumptions related to the wealth portfolio.

It is well known that in the economy covered by this data set only a certain fraction of the population owned stock. According to Vissing-Jørgensen (1999) this fraction was of the order of 8-9 per cent for a large part of the period considered. We suggest to let one agent (agent 2) represent the fraction that participates in the stock market, and the other agent (agent 1) the non-participating part of the population.

We then suppose we can view exogenous income streams as dividends of some shadow asset, in which case our model structure is valid if the market portfolio is expanded to include the new asset. However, as long as the latter is not really "traded", the return to the wealth portfolio is not readily observable or estimable from available data. Still we get a good impression of how the model fits under various assumptions. Different scenarios are considered, where the income portfolio of agent 1 is supposed to have volatilities lying between the volatility of the growth rate of aggregate consumption and the return rate of the market portfolio, and to have various correlations with the market portfolio.

The resulting calibrations yield plausible values for the parameters of the two recursive utility functions. Both agents are shown to prefer early resolution of uncertainty to late, and both agents have EIS larger than one. Also the two agents seem to be remarkably similar with regards to preferences.

We consider the basic model developed by Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994), which elaborate the foundational work by Kreps and Porteus (1978) and Epstein and Zin (1989) of recursive utility in dynamic models. Guevenen (2009) considers a discrete-time model, in which he uses Epstein-Zin utility in a heterogeneous model. He studies a richer economy than ours, and rely on numerical solutions of the model for base case values of the parameters. In contrast, we give exact formulas for the risk premiums as well as for the equilibrium interest rate for the heterogeneous model.

There is by now a long standing literature that has been utilizing recursive preferences[1].

The paper is organized as follows: In Section 2 we present a brief intro-

[1]We mention Avramov and Hore (2007), Avramov et al. (2010), Eraker and Shaliastovich (2009), Hansen, Heaton, Lee, Roussanov (2007), Hansen and Scheinkman (2009), Wacther (2012), Bansal and Yaron (2004), Campbell (1996), Bansal and Yaron (2004), Kocherlakota (1990 b), and Ai (2012) to name some important contributions. Related work is also in Browning et al. (1999), and on consumption see Attanasio (1999). A few exceptions to late resolution exist in this literature. Bansal and Yaron (2004) study a richer economic environment than we employ.

duction to recursive utility along the lines of Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994). In Section 3 we set down the first order conditions, where we use the stochastic maximum principle. In Section 4 we we present the model for the financial market. In Section 5 we find risk premiums and the interest rate in terms of primitives of the one-agent model, and we connect the wealth portfolio to the primitives of the model. In Section 6 we consider the multi agent problem, and derive the equilibrium risk premiums and the interest rate for the heterogeneous model. In Section 7 we calibrate the model to the US-data, and Section 8 concludes. Two proofs are relegated to an Appendix.

2 Recursive Stochastic Differentiable Utility

In this section we recall the essentials of recursive, stochastic, differentiable utility along the lines of Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994).

We are given a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, t \in [0,T], P)$ satisfying the 'usual' conditions, and a standard model for the stock market with Brownian motion driven uncertainty, N risky securities and one riskless asset (Section 4 provides more details). Consumption processes are chosen form the space L of square integrable, progressively measurable processes with values in $\mathbb{R}_+$.

The stochastic differential utility $U : L \to \mathbb{R}$ is defined as follows by two primitive functions: $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ are the reals.

The function $f(t, c_t, V_t, \omega)$ corresponds to a felicity index at time t, and A corresponds to a measure of absolute risk aversion (of the Arrow-Pratt type) for the agent. In addition to current consumption c_t, the function f also depends on utility V_t, and it may depend on time t as well as the state of the world $\omega \in \Omega$.

The utility process V for a given consumption process c, satisfying $V_T = 0$, is given by the representation

$$V_t = E_t\Big\{ \int_t^T \big(f(t, c_s, V_s) - \frac{1}{2} A(V_s)\, Z(s)'Z(s)\big) ds \Big\}, \quad t \in [0,T] \qquad (1)$$

where $E_t(\cdot)$ denotes conditional expectation given $\mathcal{F}_t$ and $Z(t)$ is an $\mathbb{R}^d$-valued square-integrable progressively measurable volatility process, part of the primitives of the model. Here d is the dimension of the Brownian motion B_t. We think of V_t as the remaining utility for c at time t, conditional on current information $\mathcal{F}_t$, and $A(V_t)$ is penalizing for risk.

Recall the *time-less* situation with a mean zero risk X having variance σ^2, where the certainty equivalent m is defined by $Eu(w+X) := u(w-m)$ for a constant wealth w. Then the Arrow-Pratt approximation to m, valid for "small" risks, is given by $m \approx \frac{1}{2}A(w)\sigma^2$, where $A(\cdot)$ is the absolute risk aversion associated with u. We would expect this analogy to work well in a continuous-time model with Brownian driven uncertainty.

If, for each consumption process c_t, there is a well-defined utility process V, the stochastic differential utility U is defined by $U(c) = V_0$, the initial utility. The pair (f, A) generating V is called an aggregator.

Since $V_T = 0$ and $\int Z(t)dB_t$ is assumed to be a martingale, (1) has the stochastic differential equation representation

$$dV_t = \Big(- f(t, c_t, V_t) + \frac{1}{2}A(V_t)\, Z(t)'Z(t)\Big)\, dt + Z(t)\, dB_t. \tag{2}$$

We think of A as associated with a function $h : \mathbb{R} \to \mathbb{R}$ such that $A(v) = -\frac{h''(v)}{h'(v)}$, where h is two times continuously differentiable. U is monotonic and risk averse if $A(\cdot) \geq 0$ and f is jointly concave and increasing in consumption. A may also depend on time t.

The preference ordering represented by recursive utility is usually assumed to satisfy A1: Dynamic consistency, in the sense of Johnsen and Donaldson (1985), A2: Independence of past consumption, and A3: State independence of time preference (see Skiadas (2009a)).

In this paper we consider heterogeneity with two different utility functions of the following type: They have the Kreps-Porteus representation, which corresponds to an aggregator with a CES specification

$$f(c, v) = \frac{\delta}{1-\rho} \frac{c^{1-\rho} - v^{1-\rho}}{v^{-\rho}} \quad \text{and} \quad A(v) = \frac{\gamma}{v}. \tag{3}$$

Here $\rho \geq 0, \rho \neq 1, \delta \geq 0, \gamma \geq 0, \gamma \neq 1$ The elasticity of intertemporal substitution in consumption $\psi = 1/\rho$. The parameter ρ is the time preference parameter. This representation results in the desired, partial disentangling of γ from ρ.

An ordinally equivalent specification also exists, but we do not make any use of it in this paper. Its main purpose was originally to show existence to the solution of the associated backward stochastic differential equation (BSDE) (2). A uniqueness and existence proof can be found in Duffie and Lions (1992). The ordinally equivalent version is more readily analyzed using dynamic programming, and was analyzed by Duffie and Epstein (1992a). However it is the above version in (3) that gives the most unambiguous separation of risk preference from time substitution, which is the one we analyze in this paper. In doing so we use the stochastic maximum principle. This approach was first used to solve the problem in Aase (2014a,b) for the one agent model.

2.1 Homogeniety

The following result is made use of ($U = U_1$). For a given consumption process c_t we let $(V_t^{(c)}, Z_t^{(c)})$ be the solution of the BSDE

$$\begin{cases} dV_t^{(c)} = \big(- f(t, c_t, V_t^{(c)}) + \frac{1}{2} A(V_t^{(c)})\, Z(t)'^{(c)} Z(t)^{(c)} \big)\, dt + Z(t)^{(c)}\, dB_t \\ V_T^{(c)} = 0 \end{cases} \tag{4}$$

Theorem 1 *Assume that, for all $\lambda > 0$,*
(i) $\lambda\, f(t,\, c,\, v) = f(t,\, \lambda c,\, \lambda v)$; $\forall\ t,\ c,\ v,\ \omega$
(ii) $A(\lambda v) = \frac{1}{\lambda} A(v)$; $\forall\ v$
Then

$$V_t^{(\lambda c)} = \lambda V_t^{(c)} \text{ and } Z_t^{(\lambda c)} = \lambda Z_t^{(c)},\ t \in [0, T]. \tag{5}$$

A proof can be found in Aase (2014).

Remark Note that the system need not be Markovian in general, since we allow

$$f(t, c, v, \omega);\ (t, \omega) \in [0, T] \times \Omega$$

to be an adapted process, for each fixed c, v.

Corollary 1 *Define $U(c) = V_0^{(c)}$. Then $U(\lambda c) = \lambda U(c)$ for all $\lambda > 0$.*

Notice that the aggregator in (3) satisfies the assumptions of the theorem.

3 The analysis of recursive utility using the stochastic maximum principle

In the following we indicate how to solve the consumer's optimization problem, using the stochastic maximum principle and forward/backward stochastic differential equations. The representative agent has utility function U and endowment process e, and his problem is to solve

$$\sup_{\tilde{c} \in L} U(\tilde{c})$$

subject to

$$E\Big\{ \int_0^T \tilde{c}_t \pi_t dt \Big\} \le E\Big\{ \int_0^T e_t \pi_t dt \Big\}.$$

We denote the optimal solution by c. Here $V_t = V_t^{\tilde{c}}$ and (V_t, Z_t) is the solution of the backward stochastic differential equation (BSDE)

$$\begin{cases} dV_t = -\tilde{f}(t, \tilde{c}_t, V_t, Z(t))\, dt + Z(t)\, dB_t \\ V_T = 0. \end{cases} \tag{6}$$

where $\tilde{f}$ is given in (3), i.e.,

$$\tilde{f}(t, \tilde{c}_t, V_t, Z(t)) := f(\tilde{c}_t, V_t) - \frac{1}{2} A(V_t)\, Z(t)' Z(t).$$

For $\alpha > 0$ we define the Lagrangian

$$\mathcal{L}(\tilde{c}; \lambda) = U(\tilde{c}) - \alpha\, E\Big(\int_0^T \pi_t(\tilde{c}_t - e_t) dt \Big).$$

Important is here that the quantity $Z(t)$ is internalized. Market clearing will finally connect the market (wealth) portfolio to Z, the latter being parts of the primitives of the model.

Because of the generality of the problem, Aase (2014) utilize the stochastic maximum principle (see Pontryagin (1972), Bismut (1978), Kushner (1972), Bensoussan (1983), Øksendal and Sulem (2013), or Peng (1990)): We then have a forward backward stochastic differential equation (FBSDE) system consisting of the simple FSDE $dX(t) = 0; X(0) = 0$ and the BSDE (6). The Hamiltonian for this problem is

$$H(t, \tilde{c}, v, z, y) = y_t\, \tilde{f}(t, \tilde{c}_t, v_t, z_t) - \alpha\, \pi_t(\tilde{c}_t - e_t) \tag{7}$$

and the adjoint equation is

$$\begin{cases} dY_t = Y(t)\big(\frac{\partial \tilde{f}}{\partial v}(t, \tilde{c}_t, V_t, Z(t))\, dt + \frac{\partial \tilde{f}}{\partial z}(t, \tilde{c}_t, V_t, Z(t))\, dB_t\big) \\ Y_0 = 1. \end{cases} \tag{8}$$

If $c = c^*$ is optimal we therefore have

$$Y_t = \exp\Big(\int_0^t \{ \frac{\partial \tilde{f}}{\partial v}(s, c_s^*, V_s, Z(s)) - \frac{1}{2}\big(\frac{\partial \tilde{f}}{\partial z}(s, c_s^*, V_s, Z(s))\big)^2 \}\, ds + \int_0^t \frac{\partial \tilde{f}}{\partial z}(s, c_s^*, V_s, Z(s))\, dB(s) \Big) \quad a.s. \tag{9}$$

Accordingly the adjoint variable Y is determined from the primitives of the model. Despite the fact that the introduction of this variable will, in general, complicate the problem, in this case it works well as we shall see (for details see Aase (2014)).

Maximizing the Hamiltonian with respect to $\tilde{c}$ gives the first order equation

$$y\,\frac{\partial \tilde{f}}{\partial \tilde{c}}(t, c^*, v, z) - \alpha\, \pi = 0$$

or

$$\alpha\, \pi_t = Y(t)\frac{\partial \tilde{f}}{\partial \tilde{c}}(t, c_t^*, V(t), Z(t)) \quad \text{a.s. for all } t \in [0, T]. \tag{10}$$

Notice that the state price deflator π_t at time t depends, through the adjoint variable Y_t, on the entire optimal paths (c_s, V_s, Z_s) for $0 \leq s \leq t$, which means that the economy does not display the usual Markovian structure.

For the representative agent equilibrium the optimal consumption process $c = c^*$ is the given aggregate consumption (the agent's endowment process) e in society, and for this consumption process the utility V_t at time t is optimal.

We now review the analysis related to the aggregator given by (3), but first we specify the model for the financial market.

4 The financial market

Having established the general recursive utility form of interest, in his section we specify our model for the financial market. The model is much like the one used by Duffie and Epstein (1992a), except that we do not assume any unspecified factors in our model.

Let $\nu(t) \in R^N$ denote the vector of expected rates of return of the N given risky securities in excess of the riskless instantaneous return r_t, and let $\sigma(t)$ denote the matrix of diffusion coefficients of the risky asset prices, normalized by the asset prices, so that $\sigma(t)\sigma(t)'$ is the instantaneous covariance matrix for asset returns. Both $\nu(t)$ and $\sigma(t)$ are progressively measurable, ergodic processes.

The representative consumer's problem is, for each initial level w of wealth to solve

$$\sup_{(c,\varphi)} U(c) \tag{11}$$

subject to the intertemporal budget constraint

$$dW_t = \big(W_t(\varphi_t' \cdot \nu(t)) + r_t) - c_t\big)dt + W_t \varphi_t' \cdot \sigma(t) dB_t. \tag{12}$$

Here $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \cdots, \varphi_t^{(N)})$ are the fractions of total wealth W_t held in the risky securities.

Market clearing requires that $\varphi_t' \sigma(t) = (\delta_t^W)' \sigma(t) = \sigma_W(t)$ in equilibrium, where $\sigma_W(t)$ is the volatility of the return on the wealth portfolio, and δ_t^W are the fractions of the different securities, $j = 1, \cdots, N$ held in the value-weighted wealth portfolio. That is, the representative agent must hold the wealth portfolio in equilibrium, by construction.

Generally one can not assumed that all income is investment income. In the above we have assumed that one can view exogenous income streams as dividends of some shadow asset, in which case our model is valid if the market portfolio is expanded to include the new asset. In reality the latter is not traded, so the return to the wealth portfolio is not readily observable or estimable from available data. We indicate how the model may be slightly adjusted under various assumptions, when the market portfolio is not a proxy for the wealth portfolio.

5 The analysis of the recursive model

For our model, we now turn our attention to pricing restrictions relative to the given optimal consumption plan. The first order conditions are given by

$$\alpha \, \pi_t = Y_t \, \frac{\partial f}{\partial c}(c_t, V_t) \qquad \text{a.s. for all } t \in [0, T] \tag{13}$$

where f is given in (3). The volatility $Z(t)$ and the utility process V_t satisfiy the following dynamics

$$dV_t = \Big(- \frac{\delta}{1-\rho} \frac{c_t^{1-\rho} - V_t^{1-\rho}}{V_t^{-\rho}} + \frac{1}{2} \frac{\gamma}{V_t} Z'(t) Z(t) \Big) dt + Z(t) dB_t \tag{14}$$

where $V(T) = 0$. This is the backward equation for the ordinal model.

Aggregate consumption $c = e$ is exogenous in the Lucas model, with dynamics on of the form

$$\frac{dc_t}{c_t} = \mu_c(t) \, dt + \sigma_c(t) \, dB_t, \tag{15}$$

where $\mu_c(t)$ and $\sigma_c(t)$ are measurable, $\mathcal{F}_t$ adapted stochastic processes, satisfying appropriate integrability properties. We assume these processes to be ergodic, so that we may 'replace' (estimate) time averages by state averages.

The function $\tilde{f}$ of Section 3 is given by

$$\tilde{f}(t, c, v, z) = f(c, v) - \frac{1}{2} A(v) z' z,$$

and since $A(v) = \gamma/v$, from (8) the adjoint variable Y has dynamics

$$dY_t = Y_t\big(\{\frac{\partial}{\partial v}f(c_t, V_t) + \frac{1}{2}\frac{\gamma}{V_t^2}\, Z'(t)Z(t)\}\, dt - A(V_t)Z(t)\, dB_t\big), \tag{16}$$

where $Y(0) = 1$. From the FOC in (13) we get the dynamics of the state price deflator. We use the notation $Z(t)/V(t) = \sigma_V(t)$, valid for $V \neq 0$. By Theorem 1 the term $\sigma_V(t)$ is homogeneous of order zero in c. Sufficient assumptions guaranteeing existence and uniqueness of the solution of the stochastic maximum principle are the same as the ones for the BSDE (6).

Given V_t and $\sigma_V(t)$, we then seek the determination of risk premiums and the short rate in equilibrium. Notice that Y is an unbounded variation process, so by Ito's lemma

$$d\pi_t = f_c(c_t, V_t)\, dY_t + Y_t\, df_c(c_t, V_t) + dY_t df_c(c_t, V_t). \tag{17}$$

From this one can show that

$$\sigma_\pi(t) = \pi_t\big((\rho - \gamma)\sigma_V(t) - \rho\sigma_c(t)\big), \tag{18}$$

where $\sigma_V(t)$ is the volatility of the growth rate of the utility process V, here a primitive of the model.

This relationship gives the connection between "prices" and primitives of the model, which are utility and consumption, the latter because we consider a pure exchange economy, where aggregate consumption is given exogenously.

The risk premium of any risky security with return process denoted by R is in general given by

$$\mu_R(t) - r_t = -\frac{1}{\pi_t}\, \sigma_\pi(t)\, \sigma_R(t). \tag{19}$$

By (18) it follows that this risk premium is

$$\mu_R(t) - r_t = \rho\, \sigma_c(t)\, \sigma_R(t) + (\gamma - \rho)\sigma_V(t)\, \sigma_R(t), \tag{20}$$

The volatility of the utility process V, is a primitive of the model, but not readily observed. Later we show how to link this quantity to observables.

The equilibrium short-term, real interest rate r_t is given in general given by the formula

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}. \tag{21}$$

The real interest rate at time t can be thought of as the expected exponential rate of decline of the representative agent's marginal value, which is π_t in equilibrium. From the relationship (17) we can also show that

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\rho(\rho+1)\sigma_c'(t)\sigma_c(t) - \\ \rho(\gamma-\rho)\sigma_{cV}(t) - \frac{1}{2}(\gamma-\rho)(1-\rho)\sigma_V'(t)\sigma_V(t). \quad (22)$$

We proceed to connect the wealth portfolio to the utility process V.

5.1 The volatility of of the wealth portfolio

In order to determine link the stochastic process of the wealth portfolio to primitives of the economy, first notice that the wealth at any time t is given by

$$W_t = \frac{1}{\pi_t} E_t\Big(\int_t^T \pi_s c_s^* \, ds \Big). \quad (23)$$

From Theorem 1 it follows that the nonordinal utility function $U(= U_1)$ is homogenous of degree one. By the definition of directional derivatives we have that

$$\bigtriangledown U(c^*; c^*) = \lim_{\alpha \downarrow 0} \frac{U(c^* + \alpha c^*) - U(c^*)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{U(c^*(1+\alpha)) - U(c^*)}{\alpha} \\ = \lim_{\alpha \downarrow 0} \frac{(1+\alpha)U(c^*) - U(c^*)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{\alpha U(c^*)}{\alpha} = U(c^*),$$

where the third equality uses that U is homogeneous of degree one. By the Riesz representation theorem it follows from the linearity and continuity of the directional derivative that

$$\bigtriangledown U(c^*; c^*) = E\Big(\int_0^T \pi_t c_t^* \, dt \Big) = W_0 \pi_0 \quad (24)$$

where W_0 is the wealth of the representative agent at time zero, and the last equality follows from (23) for $t = 0$. Thus $U(c^*) = \pi_0 W_0$.

From the market clearing condition in (12) it follows that $\varphi_t'\sigma(t) = (\delta_t^W)'\sigma(t) = \sigma_W(t)$ in equilibrium, where δ_t^W are the fractions of the different securities, $j = 1, 2, \cdots, N$, held in the value-weighted wealth portfolio.

Here we have made the assumption that we can view exogenous income streams as dividends of some shadow asset, so that our model is valid if the market portfolio is expanded to include the new asset. In this case $\varphi_t'\sigma(t) = (\delta_t^W)'\sigma(t) = \sigma_W(t)$ in equilibrium, where $\sigma_W(t)$ is the volatility of the return rate on the value-weighted wealth portfolio.

Let $V_t = V^{(c_t^*)}$ denote utility at time t for the optimal consumption. Since also V_t is homogeneous of degree one and continuously differentiable, by Riesz' representation theorem and the dominated convergence theorem, the same type of basic relationship holds here for the associated directional derivatives at any time t, i.e.,

$$\bigtriangledown V_t(c^*; c^*) = E_t\Big(\int_t^T \pi_s^{(t)}\, c_s^*\, ds \Big) = V_t$$

where $\pi_s^{(t)}$ for $s \geq t$ is the state price deflator at time $s \geq t$, as of time t. As for the discrete time model, with assumption A2, the consumption history in the adjoint variable Y_t is simply 'removed' from the state price deflator π_t, so that $\pi_s^{(t)} = \pi_s / Y_t$ for all $t \leq s \leq T$.

It is then the case that

$$V_t = \frac{1}{Y_t} \pi_t W_t. \tag{25}$$

This relationship connects the dynamics of W to the primitives in the economy. First rewrite this as

$$V_t Y_t = \pi_t W_t,$$

which, by the product rule gives

$$Y_t dV_t + V_t dY_t + dV_t dY_t = W_t d\pi_t + \pi_t dW_t + d\pi_t dW_t.$$

From this relationship we find the following connection between the various volatilities involved:

$$Y_t V_t \sigma_V(t) + V_t Y_t \sigma_Y(t) = W_t \pi_t \frac{\sigma_\pi(t)}{\pi_t} + \pi_t W_t \sigma_W(t),$$

or, using (25)

$$\sigma_V(t) + \sigma_Y(t) = \frac{\sigma_\pi(t)}{\pi_t} + \sigma_W(t).$$

From (16) and (18) we have

$$\sigma_V(t) - \gamma \sigma_V(t) = (\rho - \gamma)\sigma_V(t) - \rho \sigma_c(t) + \sigma_W(t).$$

Thus

$$\sigma_W(t) = (1 - \rho)\sigma_V(t) + \rho \sigma_c(t).$$

As promised, this links the main characteristic of the stochastic process of the market portfolio, its volatility process, to the corresponding volatility processes of utility and the aggregate consumption process. This result was shown in Aase (2014).

The interesting application of this is now obtained by turning this equality around, and expressing the 'unknown' $\sigma_V(t)$ in terms of the 'observables' $\sigma_W(t)$ and $\sigma_c(t)$ as follows:

$$\sigma_V(t) = \frac{1}{1-\rho}(\sigma_W(t) - \rho\sigma_c(t)). \tag{26}$$

This expression may now be inserted into (20) and (22), in which case the 'unobservable' $\sigma_V(t)$ becomes replaced by $\sigma_c(t)$ which we have estimates for, and $\sigma_W(t)$. The latter quantity may not be readily observable from available data, but we can at least present various scenarios for this quantity.

5.2 The optimal consumption

Later we will need the optimal consumption of each participant in the economy in order to formulate the budget constraints, which will determine the agent weights λ_i in a heterogeneous model. When the agent takes the state price π_t as given, then the optimal consumption c is given in terms of the state price π_t by

$$c_t = c_0 \pi_t^{-\frac{1}{\rho}} e^{\int_0^t (-\frac{\delta}{\rho} + \frac{1}{2\rho}(\gamma-\rho)(1-\gamma)\sigma_V'(s)\sigma_V(s))ds + \frac{1}{\rho}(\rho-\gamma)\int_0^t \sigma_V(s)\,dB_s}. \tag{27}$$

As we shall se below, c_0 depends on the agent weights, meaning that the budget constraints determine these weights (modulo a constant).

6 Heterogeneity in preferences

From the above results, it seems reasonable to study a model with heterogeneity containing two agents with recursive utility having different parameters. A one-agent model has been shown to fit the data in isolation with reasonable parameter values, in particular when the market portfolio is not considered as a proxy for the wealth portfolio (Aase (2015a,b)). We then naturally expect that a model with two different agents will explain the data even better. This allows us to present an economy consisting of two groups of people, one more exposed to stock market uncertainty than the other. This is what we formalize next.

6.1 The Arrow-Debreu economy

In this section we derive an Arrow-Debreu markets equilibrium in which each agent has a recursive utility function U_i of the type we have considered in this paper. As in Duffie (1986) there exists an implementation of such equilibria is the setting with security and spot markets only, given an appropriate set of admissible trading strategies and a spanning assumption on nominal cumulative dividend processes.

As we shall simply calculate the relevant equilibrium, we do not really employ the theorems for such equilibria to exist, but there is a theory for recursive preferences in this regard that should be consulted (Duffie, Geoffard and Skiadas (1994)).

The situation is as follows: Given an initial allocation $(e^1, e^2, \ldots, e^m) \in L^m$, an m-dimensional Ito process, with $e = \sum_i e^i$, an *equilibrium* is a feasible allocation $(c^1, c^2, \ldots, c^m)$ and a non-zero linear price functional $\Pi : L \to R$ such that, for all i, c^i solves the problem

$$\max_{c \in L} U_i(c) \quad \text{subject to} \quad \Pi(c) \leq \Pi(e^i) \tag{28}$$

By assuming there is no arbitrage possibilities in this market of Arrow-Debreu securities, the price functional is strictly positive on L, hence it is bounded, and thus also continuous. By the Riesz' Representation Theorem there is an element $\pi \in L$, the Riesz Representation, such that $\Pi(c) = E\big(\int_0^T \pi_t c_t\big)$ for any $c \in L$.

Under certain smoothness conditions on the aggregator (f, A), there exists an Arrow-Debreu equilibrium $(\Pi, (c^1, c^2, \ldots, c^m))$ having the following properties:

(i) $(c^1, c^2, \ldots, c^m)$ is Pareto optimal.

(ii) For each i, U_i has a gradient at c^i with a Riesz Representation $\pi_i(c^i)$ given by the stochastic maximum principle in (10). The FOC in the heterogeneous model is now

$$Y_t^i \; \frac{\partial f^i}{\partial c}(c_t^i, V_t^i) = \alpha_i \pi_t \quad \text{ a.s. for all } t \in [0, T],\ i = 1, 2, \cdots. \tag{29}$$

(iii) The state price deflator $\pi_t = \lambda_i \pi_i(t)$, $i = 1, 2, \cdots$.

The conditions (ii) and (iii) can be considered as a version of Borch's characterization of Pareto optimality in a one-period (time-less) setting (Borch (1960-62)).

Equality in the budget constraints determine the constants α_i as a function of the preferences of the agents and the joint probability distributions of the initial endowments and the state price deflator. The Lagrange multipliers in each agent's optimization problem, $\alpha_i = \frac{1}{\lambda_i}$ for $i = 1, 2, \ldots, m$, where λ_i are the agent weights appearing in the function $U_\lambda(c) = \sum_{i=1}^m \lambda_i U_i(c^i)$. This function be thought of as the utility function of the representative agent, here as a generalized recursive utility function, where $c_t := \sum_{i=1}^m c_t^i = \sum_{0=1}^m e_t^i := e_t$. In the following section we confine ourselves to the case of $m = 2$.

The fourth property of an equilibrium determines the agent weights λ_i from the budget constraints, modulo a constant, as follows

(iv) The constants λ_i are determined from

$$E\big(\int_0^T c_t^{(i)} \pi_t dt\big) = E\big(\int_0^T e_t^{(i)} \pi_t dt\big), \quad i = 1, 2,$$

where

$$c_t^{(i)} = c_0^{(i)} \pi_i(t)^{-\frac{1}{\rho}} e^{\int_0^t (-\frac{\delta_i}{\rho_i} + \frac{1}{2\rho_i}(\gamma_i - \rho_i)(1-\gamma_i)\sigma'_{V_i}(s)\sigma_{V_i}(s))ds + \frac{1}{\rho_i}(\rho_i - \gamma_i)\int_0^t \sigma_{V_i}(s)\, dB_s}$$

and

$$c_0^{(i)} = (\lambda_i \delta_i)^{\frac{1}{\rho_i}} U_i(c^{(i)}), \quad i = 1, 2.$$

The latter follows from the first order conditions, since

$$c_t^{(i)} = (\delta_i\, Y_t^{(i)})^{\frac{1}{\rho_i}}\, V_i(t)\, (\alpha_i \pi_t)^{-\frac{1}{\rho_i}}, \quad i = 1, 2,$$

and the above expression follows assuming $\pi_0 = 1$, recallling that $U_i(c^{(i)}) = V_i(0)$, $i = 1, 2$.

6.2 Heterogeneity with U_1 and U_2

We imagine that the market consists of two groups of people, both with recursive preferences which can have different parameters, and we want to characterize the resulting Pareto optimal equilibrium.

We denote $c_t^{(i)}/c_t$ as the fraction of the aggregate consumption consumed by agent i at time t, $i = 1, 2$. We show how to determine the optimal consumptions $c_t^{(i)}$ of the two agents below.

For the markets to clear, it must be the case that

$$W_t \varphi^W(t)\sigma(t) = \big(W_1(t)\varphi_1(t) + W_2\varphi_2(t)\big)\sigma(t),$$

where $W_i(t)$ is the wealth of agent i at time t, and $\varphi_i(t)$ is agent i's fraction in the different securities held at time t in equilibrium, $i = 1, 2$. We then use the notation $\sigma_{W_i}(t) = \varphi_i(t)\sigma(t)$, $i = 1, 2$.

For the heterogeneous economy the following relationship between the directional derivatives holds: Assuming c, $c^{(1)}$ and $c^{(2}$ optimal

$$\bigtriangledown U(c; c) = \lambda_1 \bigtriangledown U_1(c^{(1)}; c^{(1)}) + \lambda_2 \bigtriangledown U_2(c^{(2)}; c^{(2)}) =$$

$$\lambda_1 U_1(c^{(1)}) + \lambda_2 U_2(c^{(2)})$$

where the equality follows from homogeneity of U_1 and U_2. Accordingly U is also homogeneous of degree one in consumption. Recall that

$$W_i(0) = \frac{1}{\pi_0} E\big(\int_0^T c_t^{(i)} \pi_t dt \big), \quad i = 1, 2,$$

and that $\pi_i(t) = \alpha_i \pi_t$, $i = 1, 2$. Now

$$\triangledown U_i(c^{(i)}; c^{(i)}) = E\big(\int_0^T c_t^{(i)} \pi_i(t)\, dt \big) = \alpha_i W_i(0) \pi_0, \quad i = 1, 2,$$

It follows that

$$\triangledown U(c; c) = W_1(0)\pi_0 + W_2(0)\pi_0 = W(0)\pi_0$$

since $\lambda_i\, \alpha_i = 1$, $i = 1, 2$.

Moving to time $t > 0$ we have

$$W_i(t) = \frac{1}{\pi_t} E_t\big(\int_t^T c_s^{(i)} \pi_s ds \big), \quad i = 1, 2.$$

Also it follows from (25) in Section 5.1 that the linear functional $\triangledown V_i(t)(\cdot, \cdot)$ has Riesz-representation $\pi_i^{(t)}(s) = \pi_i(s)/Y_i(t)$ for $s \geq t$, so that

$$\triangledown V_i(t)(c^{(i)}; c^{(i)}) = E_t\Big(\int_t^T \pi_i^{(t)}(s) c_s^{(i)} ds \Big) = \frac{\alpha_i \pi_t}{Y_i(t)} W_i(t) = \frac{\pi_i(t)}{Y_i(t)} W_i(t).$$

Accordingly

$$\triangledown V_t(c; c) = \lambda_1 \triangledown V_1(t)(c^{(1)}; c^{(1)}) + \lambda_2 \triangledown V_2(t)(c^{(2)}; c^{(2)}) =$$

$$\lambda_1 V_1(t)(c^{(1)}) + \lambda_2 V_2(t)(c^{(2)}) = \lambda_1 \frac{\pi_1(t) W_1(t)}{Y_1(t)} + \lambda_2 \frac{\pi_2(t) W_2(t)}{Y_2(t)}$$

where the second equality follows from of homogeneity, the third from the above. Thus

$$V_i(t) = \frac{\pi_i(t) W_i(t)}{Y_i(t)}, \quad i = 1, 2. \tag{30}$$

As in Section 5.1, this leads directly to

$$\sigma_{V_i}(t) = \frac{\sigma_{\pi_i}(t)}{\pi_i(t)} + \sigma_{W_i}(t) - \sigma_{Y_i}(t), \quad i = 1, 2, \tag{31}$$

This means that the wealth processes of the agents are endogeneized as follows

$$\sigma_{W_i}(t) = (1 - \rho_i)\sigma_{V_i}(t) + \rho_i \sigma_{c_i}(t), \quad i = 1, 2.$$

Also

$$\sigma_{V_i}(t) = \frac{1}{1-\rho_i}\big(\sigma_{W_i}(t) - \rho_i \sigma_{c_i}(t)\big), \quad i = 1,2. \tag{32}$$

We now have a homogeneous V_t, a well defined π_t and W_t. Does this mean there is a well defined adjoint process Y_t for the representative agent? If so, has this agent a recursive utility of the type we consider? A little algebra shows that such Y must satisfy

$$Y_t = \frac{Y_1(t)Y_2(t)W(t)}{W_1(t)Y_2(t) + W_2(t)Y_1(t)},$$

which means that the representative agent is not in this class[2].

6.3 The risk premium with heterogeneity

Our first result concerns the risk premium in the heterogeneous economy:

Theorem 2 *The risk premium of a risky asset denoted R has the following representation*

$$\mu_R(t) - r_t = \frac{1}{\bar{\psi}_t}\Big(\sigma_c(t)\sigma_R(t) + \big(\frac{c_t^{(1)}}{c_t}\big)\frac{\gamma_1 - \rho_1}{\rho_1}\,\sigma_{V_1}(t)\sigma_R(t) + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\gamma_2 - \rho_2}{\rho_2}\,\sigma_{V_2}(t)\sigma_R(t)\Big), \tag{33}$$

where the average value of the population EIS ($\psi_i = 1/\rho_i$) is

$$\bar{\psi}_t := \frac{1}{\rho_1}\Big(\frac{c_t^{(1)}}{c_t}\Big) + \frac{1}{\rho_2}\Big(\frac{c_t^{(2)}}{c_t}\Big), \tag{34}$$

the average time preference is

$$\bar{\rho}_t := \rho_1\big(\frac{c_t^{(1)}}{c_t}\big) + \rho_2\big(\frac{c_t^{(2)}}{c_t}\big),$$

and where $V_i(t)$ are given in (32).

For proof of Theorem 2, see the Appendix.

Remarks

1) Since the harmonic mean is smaller than or equal to the arithmetic mean,

[2] This is not to be expected; for the conventional model with CRRA utility, the solution of the sup-convolution problem is not of the CRRA-type either.

it follows that $1/\bar{\psi}_t \leq \bar{\rho}_t$.
2) In the special situation that $\varphi_1(t) = \varphi_2(t)$ for all t, both agents hold the wealth portfolio in the same proportions relative to their own wealth in equilibrium, in which case $\sigma_{W_i}(t) = \sigma_W(t)$ for $i = 1, 2$ in (32).
3) When $c_t^{(1)} = c_t$ for all t a.s., then (33) reduces to the risk premium in Section 5 derived for one single agent. When $\gamma_1 = \rho_1$ and $\gamma_2 = \rho_2$, then (33) reduces to the risk premium of the conventional model with two heterogeneous agents.

6.4 The equilibrium interest rate with heterogeneity

Our next result concerns the equilibrium interest rate in the heterogeneous economy:

Theorem 3 *The equilibrium short rate for the heterogeneous model is given by the following expression*

$$
\begin{aligned}
r_t &= \bar{\delta}_t^{(\rho)} + \frac{1}{\bar{\psi}_t}\,\mu_c(t) \\
&- \frac{1}{2}\frac{1}{\bar{\psi}_t}\Big\{\frac{1}{\bar{\psi}_t^2}\Big(\big(\frac{c_t^{(1)}}{c_t}\big)\frac{1}{\rho_1}\big(\frac{1+\rho_1}{\rho_1}\big) + \frac{c_t^{(2)}}{c_t}\frac{1}{\rho_2}\big(\frac{1+\rho_2}{\rho_2}\big)\Big)\Big(\sigma_c(t) + \big(\frac{c_t^{(1)}}{c_t}\big)\frac{\gamma_1-\rho_1}{\rho_1}\,\sigma_{V_1}(t) \\
&\qquad + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\gamma_2-\rho_2}{\rho_2}\,\sigma_{V_2}(t)\Big)^2\Big\} \\
&- \frac{1}{\bar{\psi}_t^2}\Big\{\big(\frac{c_t^{(1)}}{c_t}\big)\frac{\rho_1-\gamma_1}{\rho_1^2}\Big(\sigma_c(t) + \big(\frac{c_t^{(1)}}{c_t}\big)\frac{\gamma_1-\rho_1}{\rho_1}\,\sigma_{V_1}(t) + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\gamma_2-\rho_2}{\rho_2}\,\sigma_{V_2}(t)\Big)\sigma_{V_1}(t) \\
&+ \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\rho_2-\gamma_1}{\rho_2^2}\Big(\sigma_c(t) + \big(\frac{c_t^{(1)}}{c_t}\big)\frac{\gamma_1-\rho_1}{\rho_1}\,\sigma_{V_1}(t) + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\gamma_2-\rho_2}{\rho_2}\,\sigma_{V_2}(t)\Big)\sigma_{V_2}(t)\Big\} \\
&- \frac{1}{2}\frac{1}{\bar{\psi}_t}\Big\{\big(\frac{c_t^{(1)}}{c_t}\big)\frac{\rho_1-\gamma_1}{\rho_1}\,\gamma_1\,\frac{\rho_1-1}{\rho_1}\,\sigma_{V_1}(t)^2 + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\rho_2-\gamma_2}{\rho_2}\,\gamma_2\,\frac{\rho_2-1}{\rho_2}\,\sigma_{V_2}(t)^2\Big\},
\end{aligned}
\tag{35}
$$

where the population impatience rate is given by

$$
\bar{\delta}_t^{(\rho)} := \frac{1}{\bar{\psi}_t}\sum_{i=1}^{2}\big(\frac{c_t^{(i)}}{c_t}\big)\big(\frac{\delta_i}{\rho_i}\big). \tag{36}
$$

For proof of Theorem 3, see the Appendix.
Remarks

1) When $\delta_1 = \delta_2 := \delta$, then $\bar{\delta}_t^{(\rho)} = \delta$ for all t.
2) When $c_t^{(1)} \equiv c_t$, then (35) reduces to the interest rate presented in Section 5 for the single agent economy. When $\gamma_1 = \rho_1$ and $\gamma_2 = \rho_2$, then (35) reduces to the equilibrium interest rate in the conventional model with two heterogeneous agents.

In the above, referring to the representative agent (U, e), where $U = \lambda_1 U_1 + \lambda_2 U_2$ and $e = e_1 + e_2$, the quantity $\bar{\rho}_t$ can be interpreted as time preference, $\bar{\psi}_t$ the EIS, and $\bar{\delta}_t^{(\rho)}$ the impatience rate of this agent, where $1/\bar{\psi}_t \leq \bar{\rho}_t$.

6.5 The conventional model with heterogeneity.

We obtain the conventional model by setting $\gamma_1 = \rho_1$ and $\gamma_2 = \rho_2$. From (33) this gives the equilibrium risk premium

$$\mu_R(t) - r_t = \frac{1}{\bar{\psi}_t}\big(\sigma_c(t)\sigma_R(t)\big), \tag{37}$$

where

$$\bar{\psi}_t := \frac{1}{\gamma_1}\Big(\frac{c_t^{(1)}}{c_t}\Big) + \frac{1}{\gamma_2}\Big(\frac{c_t^{(2)}}{c_t}\Big),$$

and the equilibrium short term interest rate follows from (35)

$$r_t = \bar{\delta}_t^{(\gamma)} + \frac{1}{\bar{\psi}_t}\,\mu_c(t) - \frac{1}{2}\frac{1}{\bar{\psi}_t^3}\Big(\frac{c_t^{(1)}}{c_t}\,\frac{1}{\gamma_1}\big(\frac{1+\gamma_1}{\gamma_1}\big) + \frac{c_t^{(2)}}{c_t}\,\frac{1}{\gamma_2}\big(\frac{1+\gamma_2}{\gamma_2}\big)\Big)\sigma_c'(t)\sigma_c(t). \tag{38}$$

When $i = 1$ these expressions reduce to the standard ones for the conventional one-agent model

$$\mu_R(t) - r_t = \gamma\sigma_{c,R}(t)$$

and

$$r_t = \delta + \gamma\mu_c(t) - \frac{1}{2}\gamma(1+\gamma)\sigma_c'(t)\sigma_c(t).$$

From the expression for the risk premium we notice that the two-agent model has the same problem as the one-agent model to explain the large observed risk premium, since the covariance rate between the consumption growth rate and the return rate on the market portfolio is still the same, while the term

$$1/\bar{\psi}_t \leq \bar{\gamma}_t := \gamma_1\big(\frac{c_t^{(1)}}{c_t}\big) + \gamma_2\big(\frac{c_t^{(2)}}{c_t}\big),$$

making the problem even more difficult than for the one agent model, i.e., the two-agent model risk premium is not larger, and can be smaller. Thus we conclude that heterogeneity in itself does not solve the equity premium puzzle for the conventional model.

7 Some calibrations of the heterogeneous model

In these sections we give a description of some situations in which the empirical relevance of the above theory can be tested. This we do by calibrating our resulting model to market data. In Table 1 we present the Mehra and Prescott (1985) key summary statistics of the real annual return data related to the S&P-500, denoted by M, as well as for the annualized consumption data, denoted c, and the government bills, denoted b [3].

Since our development is in continuous time, we have carried out standard adjustments for continuous-time compounding, from discrete-time compounding. The results of these operations are presented in Table 1 [4]. This gives, e.g., the estimate $\hat{\kappa}_{M,c} = .4033$ for the instantaneous correlation coefficient $\kappa_{c,M}(t)$ between the consumption growth rate and the return on the S&P-500 index.

	Expectation	Standard dev.	Covariances
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{Mc} = .002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	

Table 1: Key US-data for the time period 1889-1978. Continuous-time compounding.

We interpret the risky asset as the value weighted market portfolio M corresponding to the S&P-500 index.

7.1 The base case for the parameters of the wealth portfolio

To demonstrate our results, assume that agent 1 (the public at large) consumes 80 per cent of the total consumption, $\sigma_{c_1} = \sigma_{c_2} = 0.0355$ and $\sigma_{W_1} =$

[3]There are of course newer data by now, but these retain the same basic features. If we can explain the data in Table 1, we can explain any of the newer sets as well. We obtained the full data set from R. Mehra.

[4]The overall changes are in principle small, and do not influence our comparisons to any significant degree, but are still important.

.04. We take $\sigma_{W_2} = \sigma_M = .1584$ in the calibrations. For the various correlations we assume the following: $\kappa_{W_1,M} = .5$, $\kappa_{c_1,M} = \kappa_{c_2,M} = .4$, $\kappa_{c_1,c_2} = .8$, $\kappa_{c_1,W_1} = \kappa_{c_2,W_1} = .4$.

One view is that the consumption growth of non-stockholders covaries with the stock return in the same way as the consumption growth of stockholders (e.g., Vissing-Jørgensen (1999)). There are also arguments why consumption growth of non-stockholders is less correlated with stock returns than that of stockholders. Below we adhere to the former view. Then we obtain:

	γ_1	ρ_1	γ_2	ρ_2
$\bar{\delta} = .000$	2.60	.50	1.76	.95
$\bar{\delta} = .010$	2.60	.50	2.24	.92
$\bar{\delta} = .015$	2.60	.50	2.48	.91
$\bar{\delta} = .020$	2.30	.55	2.24	.91
$\bar{\delta} = .025$	2.00	.75	2.09	.90
$\bar{\delta} = .030$	2.00	.75	2.37	.87
$\bar{\delta} = .035$	2.00	.75	2.65	.83
$\bar{\delta} = .040$	2.00	.75	2.92	.80
$\bar{\delta} = .045$	2.00	.65	3.11	.82
$\bar{\delta} = .050$	1.90	.65	3.17	.81

Table 2: Calibrations Consistent with Table 1;
$\sigma_{W_1} = .04$; $c_t^{(1)}/c_t = 4/5$.

	γ_1	ρ_1	γ_2	ρ_2
$\bar{\delta} = .000$	2.61	.51	1.80	.95
$\bar{\delta} = .010$	2.35	.53	1.80	.95
$\bar{\delta} = .015$	2.63	.71	2.00	.90
$\bar{\delta} = .020$	2.39	.73	2.00	.90
$\bar{\delta} = .025$	2.11	.77	2.00	.90
$\bar{\delta} = .030$	1.84	.81	2.00	.90
$\bar{\delta} = .035$	1.56	.85	2.00	.90
$\bar{\delta} = .040$	1.27	.90	2.00	.90
$\bar{\delta} = .045$	1.71	.51	1.90	.95
$\bar{\delta} = .050$	1.61	.51	1.90	.95

Table 3: Calibrations Consistent with Table 1;
$\sigma_{W_1} = .04$; $c_t^{(1)}/c_t = 4/5$.

In Table 2 we consider a situation where γ_2 and ρ_2 are determined as γ_1

and ρ_1 are pre-determined, while $\bar{\delta}_t^{(\rho)}$ vary between zero and five per cent, such that (33) matches the estimated equity premium of 5.98% and (35) matches the short rate of .0080 for the period considered, together with the rest of the summary statistics of Table 1, and the above choice for the rest of the parameters.

In Table 3 we similarly determine γ_1 and ρ_1 as γ_2 and ρ_2 are pre-determined, while $\bar{\delta}_t^{(\rho)}$ varies as before.

Typically for both agents $\gamma_i > \rho_i$ so that they both prefer early resolution of uncertainty to late. Also $\rho_i < 1$, for $i = 1, 2$, so the EIS-parameters $\psi_i > 1$. The values of the parameters of the utility functions in both tables seem rather plausible.

We can also find calibrated values where $\rho_i > 1$ for one of the agents (not shown in the tables), but this we find less plausible.

From the tables it is unclear which of the agents is the more risk averse, since this varies with the values of $\bar{\delta}_t^{(\rho)}$. Because of the nature of this latter parameter, we do not distinguish between δ_1 and δ_2, but the tables can of course be used to do just that. From the tables it follows that the EIS of agent 1 is the largest of the two. We may think of agent 1 representing "the public at large". Thus, the part of the population not in the stock market seem better able to tolerate deterministic variations in consumption across time. This could be because this group is not that much exposed to risk as group 2.

One would perhaps presume that agent 1 should have the higher risk aversion of the two, since this agent avoids the stock market. This argument hinges on a self-selection perspective that we don not assume here. For certain values of $\bar{\delta}_t^{(\rho)}$ agent 1 is more risk averse than agent 2, for example for an (overall) impatience rate in the range from 0.0 to 2.5 per cent in Table 3, and for an impatience rate in the range from 0.0 to 2.0 per cent in Table 2. These are, perhaps, the most plausible ranges for the impatience rates.

The preference "parameters" of the representative agent can also be calculated. As an example, for the the fourth row of Table 2 we obtain the values:

$$\bar{\rho}_t = .62, \quad \bar{\psi}_t = 1.67, \quad \bar{\gamma}_t = 2.28 \quad \text{and} \quad \bar{\delta}_t^{(\rho)} = .02,$$

so the representative agent has $\bar{\gamma}_t = 2.28 > \bar{\rho}_t = .62$. Also $1/\bar{\psi}_t = 0.60 < \bar{\rho}_t = 0.62$.

The representative agent of the fourth row in Table 3 has parameters

$$\bar{\rho}_t = .76, \quad \bar{\psi}_t = 1.32, \quad \bar{\gamma}_t = 2.31 \quad \text{and} \quad \bar{\delta}_t^{(\rho)} = .02.$$

Here $\bar{\gamma}_t = 2.31 > \bar{\rho}_t = .76$, and $1/\bar{\psi}_t = 0.757 < \bar{\rho}_t = 0.760$.

In conclusion, the calibrations in Tables 2 and 3 present us with rather reasonable scenarios for the parameters of the two utility functions, where agent 1 has the highest EIS of the two, and is also most risk averse for plausible values of the impatience rates. Figure 1 illustrates for the points corresponding to $\bar{\delta}_t^{(\rho)} = .020$ in Table 3. The middle point represents the corresponding representative agent in the last row above.

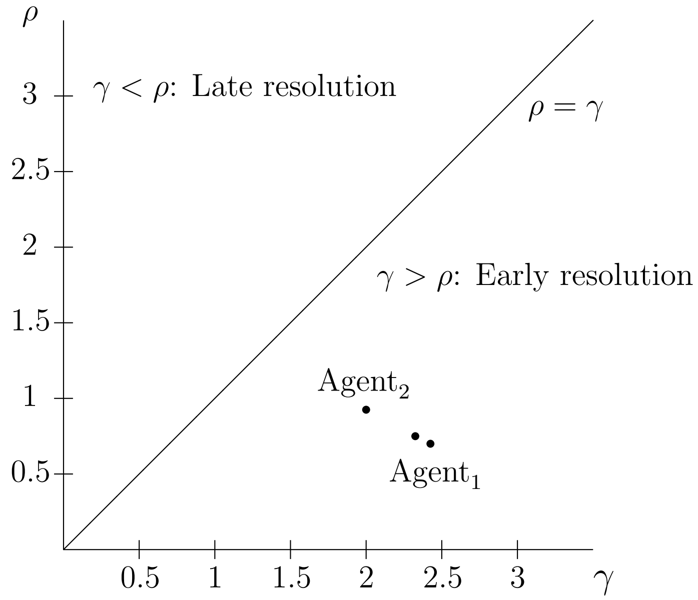

Figure 1: Calibration points in the (γ, ρ)-space

7.2 Some other values for the parameters characterizing the wealth portfolio

Next we try some other values for σ_{W_1} and $\kappa_{W_1,M}$. We start with $\sigma_{W_1} = .08$ and $\kappa_{W_1,M} = .7$. As before we take $\sigma_{W_2} = \sigma_M = .1584$ in the calibrations. For the various correlations we assume the following: $\kappa_{c_1,M} = \kappa_{c_2,M} = .4$, $\kappa_{c_1,c_2} = .8$, $\kappa_{c_1,W_1} = \kappa_{c_2,W_1} = .4$, the same as in the last section. The results of the calibrations are shown in tables 4 and 5.

In Table 4 we consider a situation where γ_1 and ρ_1 are determined as γ_2 and ρ_2 are pre-determined, while $\bar{\delta}_t^{(\rho)}$ varies between zero and five per cent,

	γ_1	ρ_1	γ_2	ρ_2
$\delta = .000$	1.64	.32	2.20	.95
$\delta = .010$	1.56	.33	2.20	.95
$\delta = .015$	1.52	.33	2.20	.95
$\delta = .020$	1.48	.33	2.20	.95
$\delta = .025$	1.43	.34	2.20	.95
$\delta = .030$	1.39	.34	2.20	.95
$\delta = .035$	1.35	.34	2.20	.95
$\delta = .040$	1.31	.35	2.20	.95
$\delta = .045$	1.27	.35	2.20	.95
$\delta = .050$	1.23	.35	2.20	.95

Table 4: Calibrations Consistent with Table 1;
$\sigma_{W_1} = .08$; $c_t^{(1)}/c_t = 4/5$.

such that (33) matches the estimated equity premium of 5.98% and (35) matches the short rate of .0080 for the period considered, together with the rest of the summary statistics of Table 1, and the above choice for the rest of the parameters.

In Table 5 we similarly determine γ_2 and ρ_2 as γ_1 and ρ_1 are pre-determined, while $\bar{\delta}_t^{(\rho)}$ varies as before.

Typically for both agents $\gamma_i > \rho_i$ so that they both prefer early resolution of uncertainty to late. Also $\rho_i < 1$, so that the EIS-parameters $\psi_i > 1$. Again, these values all seem rather plausible.

From both tables we notice that agent 2 is more risk averse. In Table 4 agent 2 has lower EIS that agent 1, while in Table 3 this is reversed.

For the agent not in the stock market, one would perhaps presume the higher risk aversion of the two. In contrast to the results of the previous section, our calibrations for these parameter values indicate the opposite. There could of course be several reasons other than differences in risk aversion why some consumers do not invest in the stock market for the period considered. For many ordinary people the stock market may have appeared as something for the elite, that did not concern them. Also ordinary people may not have much to invest in this market. On the other hand, many ordinary people like gambling (e.g., they visit Las Vegas). [5]

From the discrete time model we know that the impatience rate of the agent increases with the return rate on the wealth portfolio in similar calibrations. Recalling the expression for the impatience rate of the representative

[5] However, "gambling" at this level may often be attributed to entertainment, hence consumption, and have little to do with risk aversion.

	γ_1	ρ_1	γ_2	ρ_2
$\delta = .000$	1.40	.95	2.04	.20
$\delta = .010$	1.40	.95	1.97	.25
$\delta = .015$	1.40	.95	1.89	.30
$\delta = .020$	1.50	.95	1.95	.12
$\delta = .025$	1.50	.95	1.93	.12
$\delta = .030$	1.50	.95	1.91	.13
$\delta = .035$	1.50	.95	1.89	.14
$\delta = .040$	1.50	.95	1.86	.14
$\delta = .045$	1.50	.95	1.83	.15
$\delta = .050$	1.50	.95	1.79	.16

Table 5: Calibrations Consistent with Table 1;
$\sigma_{W_1} = .08$; $c_t^{(1)}/c_t = 4/5$.

agent,

$$\bar{\delta}_t^{(\rho)} = \frac{1}{\bar{\psi}_t} \sum_{i=1}^{2} (\frac{c_t^{(i)}}{c_t})(\frac{\delta_i}{\rho_i}),$$

we notice that the agent in the stock market obtains a large weight on his impatience rate δ_2 when ρ_2 is small, which may be an explanation for the good fit for relatively large values of $\bar{\delta}_t^{(\rho)}$ in Table 5.

In conclusion, the calibrations in Tables 4 and 5 present us with two scenarios, one in which agent 1 has an EIS of around 2.9, while agent 2 has an EIS of around 1.05, the other where agent 1 has an EIS of 1.05 while agent 2 varies between 4 and 8. In both tables agent 2 is the most risk averse of the two. Both scenarios, and all the parameters seem plausible.

Compared to the results of the last section, the latter were slightly more consistent, but the scenarios in both sections yield plausible results.

7.2.1 Increasing $\sigma_{W_1}(t)$ to .10

In tables 6 and 7 the volatility $\sigma_{W_1}(t) = .10$. The results are similar to those of tables 4 and 5, except that in Table 7 the risk aversion of agent 1 is higher than in the corresponding Table 5. This may reflect that agent 1 now requires a higher rate of return on the part of the wealth portfolio that he/she invests in.

	γ_1	ρ_1	γ_2	ρ_2
$\delta = .000$	1.44	.31	2.20	.95
$\delta = .010$	1.37	.32	2.20	.95
$\delta = .015$	1.33	.32	2.20	.95
$\delta = .020$	1.30	.32	2.20	.95
$\delta = .025$	1.26	.33	2.20	.95
$\delta = .030$	1.23	.33	2.20	.95
$\delta = .035$	1.20	.34	2.20	.95
$\delta = .040$	1.16	.34	2.20	.95
$\delta = .045$	1.10	.41	2.40	.93
$\delta = .050$	1.06	.41	2.40	.93

Table 6: Calibrations Consistent with Table 1; $\sigma_{W_1} = .10$; $c_t^{(1)}/c_t = 4/5$.

7.2.2 The market portfolio as a proxy for the wealth portfolio.

Finally we test out the situation, often taken for granted, that the market portfolio may serve as a proxy for the wealth portfolio. In this situation the choice of the parameters are as follows:

$\sigma_{W_1} = \sigma_{W_2} = \sigma_M = .1584$, and $\kappa_{W_1,M} = \kappa_{W_2,M} = 1.0$. As before we take $\kappa_{c_1,M} = \kappa_{c_2,M} = .4$, $\kappa_{c_1,c_2} = .8$, $\kappa_{c_1,W_1} = \kappa_{c_2,W_1} = .4$, the same as in the last sections. The results of the calibrations are shown in tables 8 and 9.

These calibrations essentially present us with two scenarios, one in which agent 1 has relative risk aversion less than one, while agent 2 has risk aversion above two, the other where agent 1 has an risk aversion above two while agent 2 has risk aversion a little above one. In both scenarios the agents prefer early resolution of uncertainty to late, and both agents have EIS larger than one. Both scenarios represent reasonable values of the preference parameters, although we think that best fit is still represented in the base case of Section 7.1. For example does the EIS of agent 2 in Table 9 seem a bit large, and the relative risk aversions of agent 1 in Table 8 seems a bit low.

Our results are supported by calibrating the one-agent model with recursive utility to market data using a continuous-time approach (Aase (2014)), as well as in the discrete time model of the Epstein-Zin-type (Aase (2013).

A variety of other scenarios are possible. We have varied the parameters within feasible ranges, and demonstrated that the recursive model with heterogeneity can explain the data with reasonable parameter values for the utility functions.

	γ_1	ρ_1	γ_2	ρ_2
$\delta = .000$	2.20	.95	2.16	.01
$\delta = .010$	2.00	.85	1.92	.14
$\delta = .015$	2.00	.85	1.88	.15
$\delta = .020$	2.00	.85	1.02	.57
$\delta = .025$	2.00	.85	1.18	.45
$\delta = .030$	1.00	.93	2.38	.85
$\delta = .035$	1.00	.93	2.76	.80
$\delta = .040$	1.70	.90	1.60	.24
$\delta = .045$	1.75	.90	1.73	.16
$\delta = .050$	1.75	.90	1.68	.17

Table 7: Calibrations Consistent with Table 1;
$\sigma_{W_1} = .10$; $c_t^{(1)}/c_t = 4/5$.

8 Conclusions

We have considered recursive utility with heterogeneity to analyze the standard rational expectations model of Lucas (1978). In this setting we derive equilibrium risk premiums of risky securities and the equilibrium interest rate. The resulting model we adapt to address the problem that the market portfolio may not be a reliable proxy of the wealth portfolio. As an example we calibrate the model to the US-data used by Mehra and Prescott (1985).

It is well known that in the economy covered by these data only a certain fraction of the population owned stock. According to Vissing-Jørgensen (1999) this fraction was of the order of 8-9% for a large part of the period considered. We suggest to let agent 2 represent the fraction that participates in the stock market, and agent 1 the non-participating part of the population.

We then suppose we can view exogenous income streams as dividends of some shadow asset, in which case our model structure is valid if the market portfolio is expanded to include the new asset. However, as long as the latter is not really "traded", the return to the wealth portfolio is not readily observable or estimable from available data. Still we get a good impression of how the model fits under different, realistic assumptions. Various scenarios are considered, where the income portfolio of agent 1 is supposed to have volatilities lying between the volatility of aggregate consumption and the market portfolio, and to have either low, or high correlations with the market portfolio.

The resulting calibrations to market data gives plausible values for the

	γ_1	ρ_1	γ_2	ρ_2
$\delta = .000$	.88	.34	2.40	.93
$\delta = .010$	.82	.30	2.20	.95
$\delta = .015$	.81	.30	2.20	.95
$\delta = .020$	.86	.66	2.20	.90
$\delta = .025$	.85	.67	2.20	.90
$\delta = .030$	.90	.80	2.50	.85
$\delta = .035$	.90	.83	2.50	.85
$\delta = .040$	.90	.86	2.50	.85
$\delta = .045$	.92	.90	2.50	.85
$\delta = .050$	.94	.93	2.50	.85

Table 8: Calibrations Consistent with Table 1;
$\sigma_{W_1} = .1584$; $c_t^{(1)}/c_t = 4/5$.

parameters of the utility functions. The data are consistent with a situation where both agents prefer early resolution of uncertainty to late, and where both agents have EIS larger than one. The relative risk aversions of the agents are of moderate size, as are both the time preferences and the impatience rates. The preference parameters did not change much when the population parameters changed. The situation with the lowest volatility for the wealth portfolio of the agent not in the stock market did fit the data particularly well.

We have not made any specific hypotheses about differences between the agents in terms of the parameters in the preferences. These parameters are in relative terms, and should be unaffected by e.g., wealth levels. Since our analysis produce explicit, closed form expressions for risk premiums as well as the short rate, researchers may find these results as good starting points for testing various hypotheses of this kind.

9 Appendix

THE RISK PREMIUMS AND THE SHORT RATE FOR THE HETEROGENEOUS MODEL. PROOFS.
In this section we prove theorems 2 and 3, the results for the risk premiums and the short rate for heterogeneous model.

The methods used here are somewhat different from the ones used in the rest of the paper, since we have to find the optimal consumption for both agents separately. To this end, consider the expression for the aggregate

	γ_1	ρ_1	γ_2	ρ_2
$\delta = .000$	2.20	.50	1.40	.13
$\delta = .010$	2.20	.50	1.29	.14
$\delta = .015$	2.20	.50	1.24	.15
$\delta = .020$	2.20	.50	1.17	.16
$\delta = .025$	2.20	.50	1.10	.17
$\delta = .030$	2.20	.55	1.18	.14
$\delta = .035$	2.30	.50	1.14	.14
$\delta = .040$	2.30	.50	1.09	.15
$\delta = .045$	2.30	.50	1.02	.15
$\delta = .050$	2.40	.50	1.14	.12

Table 9: Calibrations Consistent with Table 1;
$\sigma_{W_1} = .1584$; $c_t^{(1)}/c_t = 4/5$.

consumption $c_t = c_t^{(1)} + c_t^{(2)}$, where $c^{(i)}$ is optimal for agent i, $i = 1, 2$. It follows from the first order conditions in (29) that

$$c_t = (\delta_1\, Y_t^{(1)})^{\frac{1}{\rho_1}}\, V_1(t)\, (\alpha_1\pi_t)^{-\frac{1}{\rho_1}} + (\delta_2\, Y_t^{(2)})^{\frac{1}{\rho_2}}\, V_2(t)\, (\alpha_2\pi_t)^{-\frac{1}{\rho_2}}. \tag{39}$$

The state price deflator π has dynamics

$$d\pi_t = \mu_\pi(t)dt + \sigma_\pi(t)dB_t \tag{40}$$

We now develop the dynamic equation for the aggregate consumption. From the above we get, using Ito's lemma

$$dc_t = c_t\mu_c(t)dt + c_t\sigma_c(t)dB_t =$$

$$d\big((\delta_1 Y_t^{(1)})^{\frac{1}{\rho_1}}\big)V_1(t)(\alpha_1\pi_t)^{-\frac{1}{\rho_1}} + \big(\delta_1 Y_t^{(1)}\big)^{\frac{1}{\rho_1}} d(V_1(t))(\alpha_1\pi_t)^{-\frac{1}{\rho_1}}$$

$$+\big(\delta_1 Y_t^{(1)}\big)^{\frac{1}{\rho_1}} V_1(t) d(\alpha_1\pi_t^{-\frac{1}{\rho_1}}) + \big(\delta_1 Y_t^{(1)}\big)^{\frac{1}{\rho_1}} d(V_1(t)\alpha_1\pi_t^{-\frac{1}{\rho_1}})$$

$$+(\alpha_1\pi_t)^{-\frac{1}{\rho_1}} d\Big(\big(\delta_1 Y_t^{(1)}\big)^{\frac{1}{\rho_1}}(V_1(t)\big) + (V_1(t))d\big(\delta_1 Y_t^{(1)}\big)^{\frac{1}{\rho_1}}\alpha_1\pi_t^{-\frac{1}{\rho_1}}\Big)$$

$$d\big((\delta_2 Y_t^{(2)})^{\frac{1}{\rho_2}}\big)V_2(t)(\alpha_2\pi_t)^{-\frac{1}{\rho_2}} + \big(\delta_2 Y_t^{(2)}\big)^{\frac{1}{\rho_2}} d(V_2(t))(\alpha_2\pi_t)^{-\frac{1}{\rho_1}}$$

$$+\big(\delta_2 Y_t^{(2)}\big)^{\frac{1}{\rho_2}} V_2(t) d(\alpha_2\pi_t^{-\frac{1}{\rho_2}}) + \big(\delta_2 Y_t^{(2)}\big)^{\frac{1}{\rho_2}} d(V_2(t)\alpha_2\pi_t^{-\frac{1}{\rho_2}})$$

$$+(\alpha_2\pi_t)^{-\frac{1}{\rho_2}} d\Big(\big(\delta_2 Y_t^{(2)}\big)^{\frac{1}{\rho_2}}(V_2(t)\big) + (V_2(t))d\big(\delta_2 Y_t^{(2)}\big)^{\frac{1}{\rho_2}}\alpha_2\pi_t^{-\frac{1}{\rho_2}}\Big)$$

The dynamics of $\pi_t^{-\frac{1}{\rho_i}}$ is needed next. By Ito's lemma we get

$$d\pi_t^{-\frac{1}{\rho_i}} = \Big(-\frac{1}{\rho_i}\pi_t^{-\frac{1}{\rho_i}-1}\mu_\pi(t) + \frac{1}{2}\frac{1}{\rho_i}(\frac{1}{\rho_i}+1)\pi_t^{-(\frac{1}{\rho_i}+2)}\sigma_\pi^2(t)\Big)dt \\ - \frac{1}{\rho_i}\pi_t^{-\frac{1}{\rho_i}-1}\sigma_\pi(t)\, dB_t. \quad (41)$$

We now use the dynamics of the utility processes $V_i(t)$ and the adjoint processes Y_t^i which can be found in Section 3 together with the dynamics for $\pi_t^{-\frac{1}{\rho_i}}$ given in (41).

Going back to the dynamics for the aggregate consumption process c, we use the notation

$$c_t\mu_c(t) = c_t^{(1)}\mu_c^{(1)}(t) + c_t^{(2)}\mu_c^{(2)}(t)$$

and

$$c_t\sigma_c(t) = c_t^{(1)}\sigma_c^{(1)}(t) + c_t^{(2)}\sigma_c^{(2)}(t)$$

This results in a stochastic differential equation for c. Using the expressions given above for the optimal consumptions of the two agents $c_t^{(1)}$ and $c_t^{(2)}$ respectively, this reduces to the following

$$dc_t = c_t\mu_c(t)dt + c_t\sigma_c(t)dB_t =$$

$$\big\{c_t^{(1)}\big[-\frac{\delta_1}{\rho_1} + \frac{1}{2}\frac{\gamma_1}{\rho_1}\sigma_{V_1}^2(t) + \frac{1}{2}\frac{1}{\rho_1}(\frac{1}{\rho_1}-1)\gamma_1^2\sigma_{V_1}^2(t) + \frac{1}{2}\gamma_1\sigma_{V_1}^2(t)$$

$$-\frac{1}{\rho_1}(\pi_t^{-1}\mu_\pi(t)) + \frac{1}{2}\frac{1}{\rho_1}(\frac{1}{\rho_1}+1)(\pi_t^{-1}\sigma_\pi(t))^2 - \frac{\gamma_1}{\rho_1}\sigma_{V_1}^2(t) + \frac{\gamma_1-\rho_1}{\rho_1^2}(\pi_t^{-1}\sigma_\pi(t))\sigma_{V_1}(t)\big]$$

$$+c_t^{(2)}\big[-\frac{\delta_2}{\rho_2} + \frac{1}{2}\frac{\gamma_2}{\rho_2}\sigma_{V_2}^2(t) + \frac{1}{2}\frac{1}{\rho_2}(\frac{1}{\rho_2}-1)\gamma_2^2\sigma_{V_2}^2(t) + \frac{1}{2}\gamma_2\sigma_{V_2}^2(t)$$

$$-\frac{1}{\rho_2}(\pi_t^{-1}\mu_\pi(t)) + \frac{1}{2}\frac{1}{\rho_2}(\frac{1}{\rho_2}+1)(\pi_t^{-1}\sigma_\pi(t))^2 - \frac{\gamma_2}{\rho_2}\sigma_{V_2}^2(t) + \frac{\gamma_2-\rho_2}{\rho_2^2}(\pi_t^{-1}\sigma_\pi(t))\sigma_{V_2}(t)\big]\big\}dt$$

$$+\big\{c_t^{(1)}\big[\frac{\rho_1-\gamma_1}{\rho_1})\sigma_{V_1}(t) - \frac{1}{\rho_1}(\pi_t^{-1}\sigma_\pi(t))\big] + c_t^{(2)}\big[\frac{\rho_2-\gamma_2}{\rho_2}\sigma_{V_2}(t) - \frac{1}{\rho_2}(\pi_t^{-1}\sigma_\pi(t))\big]\big\}dB_t$$

Using this representation and applying diffusion invariance, we obtain two relationships from which we can determine $\pi_t^{-1}\sigma_\pi(t)$ and $\pi_t^{-1}\mu_\pi(t)$ in terms of the primitives of the economy. The first equation determines the diffusion of the state price deflator: It is

$$-\frac{\sigma_\pi(t)}{\pi_t} = \frac{1}{\bar{\psi}_t}\Big(\sigma_c(t) + (\frac{c_t^{(1)}}{c_t})\frac{\gamma_1-\rho_1}{\rho_1}\sigma_{V_1}(t) + (\frac{c_t^{(2)}}{c_t})\frac{\gamma_2-\rho_2}{\rho_2}\sigma_{V_2}(t)\Big)$$

where

$$\bar{\psi}_t := \frac{1}{\rho_1}\Big(\frac{c_t^{(1)}}{c_t}\Big) + \frac{1}{\rho_2}\Big(\frac{c_t^{(2)}}{c_t}\Big).$$

From this we obtain the risk premium of any risky asset, denoted R, as follows

$$\mu_R(t) - r_t = \frac{1}{\bar{\psi}_t}\Big(\sigma_c(t)\sigma_R(t) + \big(\frac{c_t^{(1)}}{c_t}\big)\frac{\gamma_1-\rho_1}{\rho_1}\sigma_{V_1}(t)\sigma_R(t) + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\gamma_2-\rho_2}{\rho_2}\sigma_{V_2}(t)\sigma_R(t)\Big). \quad (42)$$

This proves (33).

Turning to the equilibrium short rate, from the drift of the aggregate consumption we obtain that

$$\bar{\psi}_t\, r_t = \bar{\psi}_t(-\pi_t^{-1}\mu_\pi(t)) = \Big(\big(\frac{c_t^{(1)}}{c_t}\big)\frac{\delta_1}{\rho_1} + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\delta_2}{\rho_2}\Big) + \mu_c(t)$$
$$-\frac{1}{2}\Big\{\big(\frac{c_t^{(1)}}{c_t}\big)\frac{\rho_1-\gamma_1}{\rho_1}\gamma_1\frac{\rho_1-1}{\rho_1}\sigma_{V_1}^2(t) + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\rho_2-\gamma_2}{\rho_2}\gamma_2\frac{\rho_2-1}{\rho_2}\sigma_{V_2}^2(t)\Big\}$$
$$-\Big\{\frac{1}{2}\big(\frac{c_t^{(1)}}{c_t}\big)\frac{1}{\rho_1}\big(\frac{1}{\rho_1}+1\big)\frac{1}{\bar{\psi}_t^2}\Big(\sigma_c(t) + \big(\frac{c_t^{(1)}}{c_t}\big)\frac{\gamma_1-\rho_1}{\rho_1}\sigma_{V_1}(t) + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\gamma_2-\rho_2}{\rho_2}\sigma_{V_2}(t)\Big)^2$$
$$+\frac{1}{2}\big(\frac{c_t^{(2)}}{c_t}\big)\frac{1}{\rho_2}\big(\frac{1}{\rho_2}+1\big)\frac{1}{\bar{\psi}_t^2}\Big(\sigma_c(t) + \big(\frac{c_t^{(1)}}{c_t}\big)\frac{\gamma_1-\rho_1}{\rho_1}\sigma_{V_1}(t) + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\gamma_2-\rho_2}{\rho_2}\sigma_{V_2}(t)\Big)^2\Big\}$$
$$-\Big\{\big(\frac{c_t^{(1)}}{c_t}\big)\frac{\rho_1-\gamma_1}{\rho_1^2}\frac{1}{\bar{\psi}_t}\Big(\sigma_c(t) + \big(\frac{c_t^{(1)}}{c_t}\big)\frac{\gamma_1-\rho_1}{\rho_1}\sigma_{V_1}(t) + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\gamma_2-\rho_2}{\rho_2}\sigma_{V_2}(t)\Big)\sigma_{V_1}(t)$$
$$+\big(\frac{c_t^{(2)}}{c_t}\big)\frac{\rho_2-\gamma_2}{\rho_2^2}\frac{1}{\bar{\psi}_t}\Big(\sigma_c(t) + \big(\frac{c_t^{(1)}}{c_t}\big)\frac{\gamma_1-\rho_1}{\rho_1}\sigma_{V_1}(t) + \big(\frac{c_t^{(2)}}{c_t}\big)\frac{\gamma_2-\rho_2}{\rho_2}\sigma_{V_2}(t)\Big)\sigma_{V_2}(t)\Big\}.$$

From this it follows that r_t is given by (35).

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