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Cooperative Game Theory and Optimization in Collaborative Logistics

A Study of Coalition Structures and Cost Allocation Methods

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Preface

This thesis was written as part of the master in Economic Analysis (ECO) at the Norwegian School of Economics and Business Administration (NHH).

Through the course Economic Decision Models we were introduced to mathematical programming and AMPL. The motivation behind this paper was an interest to learn more about Operations Research, and the use of mathematical tools to provide optimal solutions to complex decision-making problems. The choice of topic is motivated by the high potential of efficiency improvements through collaboration in logistics, and is inspired by research papers written by our guiding professor, Mario Guajardo. The process of writing this thesis has been both challenging and educational. We have experienced the difficulties of designing models that accurately describe real world problems, and the challenge of finding the balance between the accuracy of a model and its tractability. Throughout the process we have encountered surprising findings in acknowledged methods, and we have realized the importance of studying results from a critical point of view.

We would like to thank Mario Guajardo for his helpful guidance and accessibility throughout the work on this thesis. His enthusiasm and knowledge on the subject have led to fruitful discussions and inspired interesting questions to address.

Bergen, 20.06.2014

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Abstract

Collaboration among different agents is an effective way to improve logistic operations. When companies transport items of similar assortments within the same regions, there are high potential of efficiency improvements. Large volumes and long distances, in combination with increasing fuel prices and environmental concern, enhance the importance of exploiting synergies between companies. In the context of collaboration, two essential questions arise: (1) Which coalitions can be expected to form? (2) How should the companies allocate the achieved benefits?

We aim to find optimal solutions for both questions by formulating linear programming models using Operations Research. We address the first question by finding how the companies should group in order to achieve the highest savings, both from a social planner point of view and from the individual company's point of view. Assuming that optimal coalition structures are identified, the next question concerns how the members in a coalition should distribute the cost savings. There are numerous of acknowledged allocation methods aiming to find a fair solution to this question. We investigate five different methods, and find striking results in terms of disparities between the allocations. For one company, the relative saving achieved ranges from 2.96 % to 18.75 % depending on which method we implement. We also discuss fundamental questions considering the aspect of fairness in the different allocation methods.

When approaching the question from the perspective of each company, we combine concepts of the different allocation methods and construct a new model to find optimal coalition structures. Based on different assumptions regarding the behavior of companies, the model provides two sets of coalition structures.

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Notations

K: set of coalitions

N: set of all players

M: set of players in the formed coalition

cost_k: total cost of coalition k

scost_j: stand − alone cost of player j

$\mathrm{u}_{j,k}$: cost allocated to player j in coalition k

m: maximum number of players in a coalition

CP_k: constructive power of coalition k

BP_k: blocking power of coalition k

$\mathrm{pair}_{k,b}$: pair of coalition k and its complementary coalition M \ k

1. Introduction

Evidence shows that collaboration among different agents in logistics may lead to significant cost savings. Several industries identify substantial benefits and improved efficiency through collaboration. Lynch (2001, p.2) refers to instances where shippers and carriers estimate 10 % to 12.3 % cost reduction by collaborating with another community member. Also in industries concerning inventory management (Guajardo et al., 2013), transportation (Frisk et al., 2010; Lozano et al., 2013), vehicle manufacturing and retail distribution (Akintoye, McIntosh, & Fitzgerald, 2000), supply chain collaboration has beneficially been applied. By opening up for possibilities of collaboration, two essential questions arise; which companies should optimally collaborate and how should the joint benefits be divided among the participants (Frisk et al, 2010). Several studies combining cooperative game theory and Operations Research models investigate how the cost should be allocated among companies within a coalition. Previous literature within this field often assume that the grand coalition will form, arguing that the grand coalition is the most efficient coalition. However, for various reasons, such as political issues or difficulties coordinating large coalitions, the grand coalition might be impossible or non-optimal to attain. These potential issues motivate a new perspective of collaborative logistics; the study of optimal coalition structures. In game theory, coalition structure refers to formed coalitions that define a partition of the set of players (Aumann & Dreze, 1974). Even though the subject of coalition structure has received significant attention in game theory, the topic is far less investigated in literature using Operations Research models. Some relevant studies concerning coalition structures in Operations Research are Axelrod, Mitchell, Thomas, Bennett and Bruderert (1995), Leng and Parlar (2009) and Nagarajan and Sošić (2009).

We use cooperative game theory with transferable utility to investigate how the overall cost should be distributed among participants within a coalition. To deal with the complications that may block the grand coalition from forming, we incorporate a condition on the maximum number of participants within a coalition. In the following analysis, we investigate optimal coalition structures considering two different perspectives. In the first approach we aim to find the social welfare maximizing coalition structure, i.e. we minimize the total cost of the formed coalitions. In this section we construct five different models for allocating cost within the optimal coalition structures; equal profit method, the nucleolus, the modiclus, the simplified modiclus and the proportional nucleolus. In the second approach we minimize the cost from each company's perspective, i.e. we minimize individual costs. In this case, the optimal coalition structure will depend upon for which player we minimize the cost. In order to determine a valid coalition structure for each cardinality *m*, we introduce different concepts for finding the priority of companies.

The models are implemented in two different applications. The main emphasis throughout the paper concerns a case study of collaborative forest transportation presented by Frisk et al. (2010). This case includes eight transportation companies operating in southern Sweden. We find that collaboration among different companies results in total cost savings up to 8.64 %. To demonstrate that the constructed models are applicable in other collaborative settings, we implement the models in a case study of collaboration through an inventory pool of spare parts for oil and gas operations.

We approach our research question from both a theoretical and practical point of view. To our knowledge, previous literature is mainly focused on the mathematical background of the relevant theories. We aim to provide a deeper understanding of the intuition behind the models. In the practical aspect, we formulate general models to make the methods applicable to other situations concerning collaborative logistics. As our research questions address coalition structures and cost allocation methods by utilizing cooperative game theory and Operations Research models, we believe our work contribute with insights in a relatively sparse research field. All models we formulate are implemented in AMPL and solved by using software CPLEX 10.0.

The remainder of this paper is organized as follows. Chapter 2 summarizes the relevant concepts in game theory including definitions and properties. In chapter 3, we introduce the cost allocation methods we implement in our two case studies. The case study in forest transportation and concepts relevant for the further analysis are presented in chapter 4. In chapter 5, we formulate and implement models to find the optimal coalition structures and cost allocations in the total cost minimizing problem. In chapter 6, we address the individual cost minimizing problem. In chapter 7, we discuss the results and compare the two approaches. In chapter 8, we implement the models in a case concerning inventory pooling of spare parts. Finally we present our conclusions and discuss questions for further research.

2. Game Theory

Game theory concerns the study of strategic decision making when several parties are involved. More formally game theory is defined as “the study of mathematical models of conflict and cooperation between intelligent rational decision-makers” (Myerson, 1997, p.1). Game theory provides general mathematical techniques for analyzing situations where two or more individuals make decisions that will affect the other party’s welfare. Such problems arise in a wide range of areas within for example economics, political science and psychology. The fundamental issue in all applications considers situations of zero-sum games. One person’s gain offsets the net losses to the other parties (Myerson, 1997).

The *game* refers to any social situation that includes one or more individuals - commonly labeled as players. There are two basic assumptions about the players; they are rational and intelligent. The rationality condition states that each player chooses the alternative that maximizes his or her expected utility. By declaring that the players are intelligent, we assume that the players have excess to all relevant information and can make decisions concerning the whole game (Myerson, 1997).

Cooperative games consist of two ingredients; players and payoffs. In contrast to non-cooperative games, cooperative game theory considers the set of joint actions that any group of players can take. The theory addresses the potential benefits of collaboration, which coalitions will form and how to divide the outcome in order to achieve a stable solution. Cooperative game theory distinguishes games with transferable payoff from those with nontransferable payoff. A game with transferable utility considers the total payoff achievable by the joint action of a given coalition. The coalition is then free to transfer payoff between its participants (Myerson, 1997).

Cooperative game theory will be illustrated with the following three-player shoes game (Hendrikse, 2003). Suppose player 1 has a left shoe, and player 2 and 3 both have a right shoe. The set of players is $N = \{1, 2, 3\}$, which give the following eight subsets:

$v(\phi) = 0$

$v(1) = 0$

$v(2) = 0$

$v(3) = 0$

$v(12) = 1$

$v(13) = 1$

$v(23) = 0$

$v(123) = 1$

The empty set ϕ gives no value, as it does not have any shoes. Having just one shoe, in cases where the players operate alone, is also associated with no value. Finally, in subsets where players possess a complete pair of shoes, value one is achieved. There are different methods for distributing the value obtained by the subsets. All methods aim to find a fair allocation that ensures that players have incentives to collaborate. However, the methods we utilize in this paper all incorporate different aspects for allocating the value in a coalition. Whereas some methods distribute the value equally among the players, other methods base the allocation on the associated power within the game. In the three-player shoe illustration, player 1 is a veto player possessing all power in the valuable coalition. An allocation method solely based on power would assign all value to player 1, and no value to the other players. On the other hand, both player 2 and 3 can prevent player 1 from getting any value by forming subset $\{2,3\}$. Some allocation methods therefore argue that blocking power also has to be considered. Such methods would assign ½ of the value generated to player 1 and ¼ to each player 2 and 3 (Sudhölter, 1997).

Whereas game theory focuses on the mathematical theory of interactive decisions, Operations Research models provide analytical instruments to formulate and solve decision models. In real world problems, many decisions are interactive, i.e. the result of one player's decision also depends on other players' choices. Cooperative game theory is therefore well suited to deal with these interactions (Fiestras-Janeiro, García-Jurado, Meca & Mosquera, 2011). In logistics there are high potentials for cost savings if players cooperate. In such cases, the common cost has to be distributed among the participants within the coalition such that all the players are willing to cooperate. In this paper we apply the concept of cooperative game theory within Operations Research models to investigate how the cost achieved in coalitions should be allocated between the players.

2.1 Definitions and Properties

An *n*-persons game is defined by a pair (*N*,*c*), where *N* is the set of players and *c* is the cost function obtained by every set of players. Players can form coalitions to take advantage of benefits from collaboration. The set *N* is the *grand coalition* and occurs if all players act together. We refer to $m = |N|$ as the cardinality of *N*, which states the maximum number of players participating in a coalition (Guajardo & Rönnqvist, 2014).

The cost function, $cost_k$, is referred to as the *characteristic function* of the game, and represents the optimal cost of each coalition k. The cost function associates a real number, $cost_k$,with every possible *subset* of *N*; we assume $cost_k$,≥ 0 for all *k*. In an empty set, $cost_\emptyset$ is zero and for any other $k \subseteq N$, $cost_k$ is the cost incurred by coalition k (Aumann & Dreze, 1974). We define a *sub-coalition k* as a group of players initially from the same coalition forming a smaller coalition. The cost of sub-coalitions containing only one player is referred to as the player's *stand-alone cost*, i.e. the cost the player is faced when operating alone.

Each solution concept that provides a cost allocation is said to satisfy some properties, i.e. criteria of fairness. The cost allocated to player *j* is defined by u_j, for each player $j \in N$. We assume $u_j \geq 0$ for all *j*. A cost allocation vector $u = (u_1, u_2,..., u_n)$ is in the *core* of the game (Gillies, 1959) if it satisfies the following constraints:

$$\textstyle\sum_{j\in k} u_j \leq cost_k \qquad \forall k \subset N \tag{2.1}$$

$$\sum_{j\in N} u_j = cost_N \tag{2.2}$$

Constraints (2.1) state the *rationality condition*, which assures that the cost allocated from the grand coalition is less than, or equal to, the participants' obtainable cost in other subsets *k*. This condition is referred to as *weak stability*. Constraint (2.2) corresponds to the *efficiency condition*, which implies that the costs allocated to all players add up to the expected cost obtained in the grand coalition. All vectors *u* that satisfy constraints (2.1) and (2.2) are defined as the core of the game. An allocation that belongs to the core is said to be *stable*, which means that no players or subsets have incentives to deviate from the coalition.

A *coalition structure* is defined as the formed coalitions within each boundary *m* (Aumann & Dreze, 1974). In contrast to simply assuming that the grand coalition will form, we consider different groups of players that may form different coalitions. Under this assumption it is relevant to study the stability in the whole structure, not only the stability within each of the formed coalitions. This is referred to as *strong stability*. The condition states that no group of players, whether from the same coalition or from different ones, can deviate and form new coalition(s) such that all participants will be better off (Aumann, 1959; Hart & Kurz., 1983).

The cost function is said to be *monotone*, which implies that if one additional company gets included in the coalition, the total cost will never decrease. This principle implies that if some players' contribution to an allocation increases, than the player's allocation should not decrease. Monotonicity thus provides a simple characterization of a fair division in cost allocation problems (Young, 1985).

Another common assumption made in cooperative game theory, defines the cost function to be *superadditive*. The condition states that the value of a union of disjoint coalitions is at least as good as the sum of the coalitions' separate values, i.e. $cost_{k \cup b} \leq cost_k + cost_b$, for all disjoint coalitions k, $b \subseteq N$. According to this principle, the outcome is better (or at least not worse) the more players that collaborate; hence it will always be profitable to form larger coalitions (Young, 1985).

3. Cost Allocation Methods

Allocating cost is not a simple task and sometimes providing a fair distribution requires use of advanced methods. Consider the simple case of three players; A, B and C. When operating alone they are faced with costs of 10, 40 and 80, while in collaboration they achieve a total cost of 105. Assuming that the three players are identical in terms of efficiency, a fair allocation should treat the players equally. But what does it mean to "treat the players equally"? The most straightforward allocation is to simply divide the obtained cost equally among the participants. This results in equal costs of 35 to each player. Due to the differences in stand-alone cost this is obviously not a fair solution, and player A will object. Another solution is to divide the achieved cost savings equally, which gives each player an equal absolute saving of 8. This method provides a better allocation, as none of the players are allocated a higher cost than they obtain by operating alone. However, the benefit of the collaboration is of greater magnitude for player A than for player C, as he attains much higher relative saving. A third alternative is to divide the cost proportionally to the player's stand-alone costs. This results in equal relative savings of 19 % to all players.

Table 3.0.1. Basic allocation methods

	Method 1			Method 2			Method 3		
	Cost	Saving	Saving, %	Cost	Saving	Saving, %	Cost	Saving	Saving, %
A	35	-25	-250 %	2	8	83 %	8	2	19 %
B	35	5	13 %	32	8	21 %	32	8	19 %
C	35	45	56 %	72	8	10 %	65	15	19 %

All three methods are simplistic and none of them are able to account for the possibility of two players collaborating. Situations where we have to consider sub-coalitions and differences in efficiency require more complex methods in order to find a fair allocation. In the following we will further explain method 3, which is known as the *proportional cost allocation method*. Furthermore we will introduce five more complex allocation methods. EPM and the proportional nucleolus are based on the same principle as method 3, while the nucleolus, the modiclus and the simplified modiclus follow the same principle as method 2.

3.1 The Proportional Cost Allocation Method

The proportional cost allocation method assigns the cost obtained in the coalition, $cost_k$, such that all participants achieve equal savings relative to their stand-alone cost, $scost_j$. Each player pays a share of the total cost weighted by their stand-alone cost relative to the sum of stand-alone costs of all players:

$$u_{j,k} = \frac{scost_j}{\sum_{j \in k} scost_j} * cost_k \qquad \forall j \in N, k \in K$$

Companies often prefer this model, as it is easy to understand, easy to show and easy to compute. However, the proportional cost allocation method does not assure stable allocations, i.e. the cost allocated might be greater than the cost obtained in sub-coalitions (Frisk et al., 2010).

3.2 Equal Profit Method

Equal profit method (EPM) is a cost allocation principle proposed by Frisk et al. (2010). The method minimizes the maximum difference in pairwise relative savings among the participants within a coalition. The relative savings of player *j* is defined as:

$$\frac{scost_j - u_j}{scost_j} = 1 - \frac{u_j}{scost_j} \qquad \forall j \in N$$

EPM minimizes the pairwise difference in relative savings *f*.

$$f \geq \frac{u_i}{scost_i} - \frac{u_j}{scost_j} \qquad \forall j, i \in N$$

To ensure that the participants are better off collaborating in the grand coalition rather than forming sub-coalitions, rationality constraints are incorporated. The rationality condition limits the sum of the costs allocated to the players in the grand coalition to be less than, or equal to, the participant's opportunity cost in other sub-coalitions. All participants are therefore willing to cooperate in the optimal coalition. If at least one possible solution exists, thus if the core is not empty, the solution found is proven to be stable (Frisk et al., 2010).

3.3 The Nucleolus

The nucleolus is another method for allocating costs, and was introduced by Schmeidler in 1969 (Solymosi & Raghavan, 1994). The method aims to maximize the worst ”satisfaction” among all coalitions and allocates the cost according to the constructive power of each coalition. The satisfaction of a coalition is expressed as the excess between the cost obtained by the coalition and the sum of allocated costs to the participants in the coalition (Schmeidler, 1969).

$$e(k,u) = cost_k - \sum_{j \in k} u_j \qquad \forall k \in K$$

The first term on the right hand side is the total cost of coalition k if they act unilaterally. The second term is the sum of costs allocated to the players participating in coalition k if they join the grand coalition. The bigger the excess, $e(k,u)$, the more satisfied is the coalition with the allocation u. As an attempt to treat all coalitions as equal as possible, the nucleolus lexicographically maximizes the minimum excess of all coalitions. This provides a cost allocation, u_j, which has the lexicographically greatest associated excess vector. An allocation belongs to the core if and only if $e(k,u) \geq 0$ for all coalitions k. That is, in the core, joint action is better than unilateral action for all coalitions (Schmeidler, 1969).

3.4 The Modiclus

While the nucleolus maximizes the lowest excesses, the modiclus aims to minimize the largest difference in excesses within pairs of coalitions. The excess of a coalition is defined in the same way, i.e. the difference between the cost obtained by a coalition and the sum of allocated cost to the participants containing the coalition. The difference of excesses between two coalitions, k and b, may be regarded as the mutual *envy* between coalition k and b. By *pairs* of coalitions we mean all possible combinations of coalition k and b, i.e. we consider $(k,b) \in (2^N - 2) \times (2^N - 2)$ pairs. This results in high computational complexity. The modiclus is constructed to lexicographically minimize the maximum envy, $e(k,b,u)$ (Sudhölter, 1997):

$$e(k,b,u) \geq \left(cost_k - \sum_{j\in k} u_j\right) - \left(cost_b - \sum_{j\in b} u_j\right) \qquad \forall k,b \in K$$

In order to minimize *Envy,* the modiclus allocates relatively lower cost to efficient coalitions and relatively higher cost to inefficient coalitions, i.e. the model considers both constructive power and blocking power.

3.5 The Simplified Modiclus

The simplified modiclus is a method introduced by Tarashnina (2010), and was constructed to avoid the high computational complexity of the modiclus. The constructive power and blocking power of each coalition is now calculated between pairs of a coalition and its *complementary*. By the complementary coalition we mean the coalition b containing all players not participating in coalition *k*, i.e. we have $(k, N\backslash k) \in 2^N - 1$ pairs. The excess of coalition *k* is defined as follows;

$$e(k,b,u) = \frac{1}{2}cost_k + \frac{1}{2}\left(cost_N - cost_{N\backslash k}\right) - \sum_{j\in k} u_j \qquad \forall k,b \in K$$

The first term on the right hand side is the cost of coalition *k*, which measures its constructive power. The second term is the difference between the cost of the grand coalition and the cost of the complementary coalition of *k*. This term takes into account the blocking power of coalition *k*. The lowest excess $e(k,b,u)$ is lexicographically maximized following the same logic as in the nucleolus (Tarashnina, 2010).

3.6 The Proportional Nucleolus

The proportional nucleolus is a solution concept introduced by Young et al. (1980). The method is similar to the nucleolus as it lexicographically maximizes the satisfaction of the least satisfied coalition. The excess in the proportional nucleolus is defined as the difference between the cost of coalition k and the sum of the allocated costs to the participants of *k*, divided by the cost of coalition *k* (Lemaire, 1984).

$$e(u,k,j) = \frac{(cost_k - \sum_{j\in k} u_j)}{cost_k} \qquad \forall k \in K$$

The proportional nucleolus allocates the costs according to the constructive power of each player. While the nucleolus measures the excess in absolute value, the proportional nucleolus measures the excess relative to the cost of each coalition. It has been proven by Megiddo (1974) that the nucleolus does not satisfy the desirable property of monotonicity, cf. chapter 2.1. However, it has been showed that the proportional nucleolus satisfies monotonicity, which is viewed as a major advantage of this method compared to the nucleolus (Zhou, 1991).

4. A Case Study in Forest Transportation

The models we formulate are applicable for all cases where we have a set N of players and information on costs for all coalitions k. We focus mainly on a case in forest transportation introduced by Frisk et al. (2010), concerning eight companies with operations in southern Sweden. Transportation costs account for a large proportion of the total costs in the forest industry. Many companies operate in the same region and volumes of the same assortment are often transported in opposite directions; hence there is high potential of improving transportation efficiency. Companies that collaborate can better utilize transport capacity by bartering and/or backhauling. Wood bartering implies that destinations between supply and demand nodes are changed, such that a company closer to the location of the demand nodes supplies the goods on the authority of another company. Backhauling refers to when a truck that has carried one load between two points carries another load on its return, such that unloaded distance is decreased. Both bartering and backhauling improve transportation efficiency and lead to decreased total costs.

The forestry case consists of information on costs for eight players and 255 possible coalitions. The stand-alone costs of the players vary to a great extent, from 330 to 14860.

Figure 3.6.1. Maps describing the eight companies

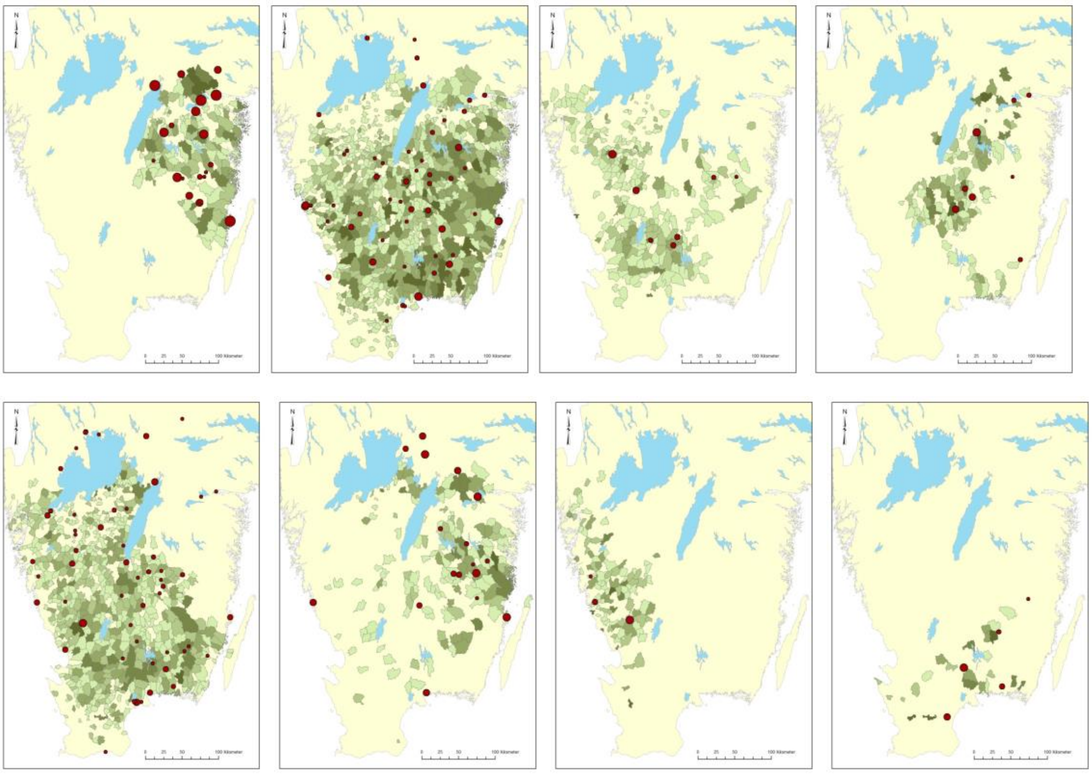


Figure 3.6.1 describes each of the eight forest transportation companies operating in southern Sweden (D'Amours & Rönnqvist, 2011). The green areas display supply areas and the red circles describe industries with demand. The companies operate with different volumes which is reflected in their stand-alone costs (Frisk et al., 2010). Based on the substantial differences in volume of each company, equal amounts of cost savings will be of greater significance for smaller players. When analyzing the outcome of the different allocation methods, we therefore find it reasonable to focus on the cost savings attained by each player relative to its stand-alone cost.

We assume that the players are equally efficient when they operate alone, hence differences in efficiency is only apparent when comparing coalitions. This assumption is essential to the discussion of fairness in following sections. In the following we discuss some concepts of great relevance for the cost allocation methods we investigate.

4.1 Additivity

Additivity implies that the cost obtained in collaboration will be less than the sum of the cost of players acting separately, cf. chapter 2.1. In the main, the property of additivity is satisfied in the forestry case. For instance, the cost achieved in coalition {C1, C2, C3} is less than the sum of the cost obtained by coalition {C1, C2} and {C3}, i.e. 22080 < 18300 + 4740. The same reasoning applies to all coalitions. One exception occurs for coalition {C7, C8} where the cost obtained by collaboration is greater than the sum of the stand-alone costs, i.e. 2220 > 1880 + 330.

4.2 Constructive Power

We define the constructive power of a coalition as the relative saving the coalition obtains.

$$CP_k = 1 - \frac{cost_k}{\sum_{j \in k} scost_j} \qquad \forall k \in K$$

In order to analyze the results of the allocation models presented in chapter 3, we wish to find a measure of each *player's* constructive power. The constructive power of each player is calculated by taking the average of the relative savings of all coalitions the player participates in. The higher relative savings obtained by the coalitions where player *j* participates, the higher is the constructive power of player *j*.

Figure 4.2.1. Average constructive power within the grand coalition

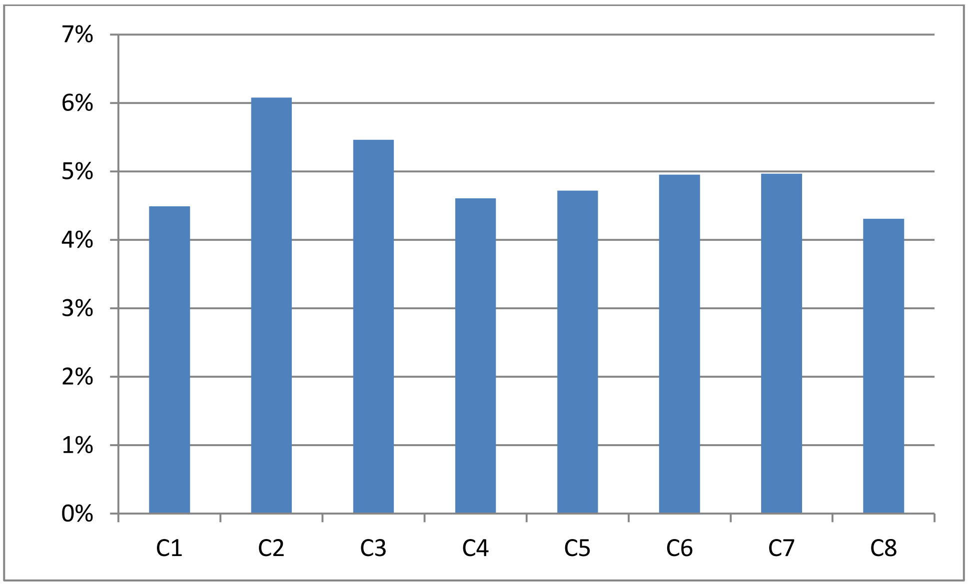


Figure 4.2.1 illustrates the average constructive power of all sub-coalitions each player participates in. Since all possible cub-coalitions are taken into account, the measures of constructive power apply for the grand coalition. Player C2 and C3 are the strongest players in terms of constructive power, with averages of 6.05 % and 5.44 %, respectively. Player C1, C4 and C8 have the lowest constructive powers, with averages of 4.48 %, 4.60 % and 4.31 %, respectively.

4.3 Blocking Power

The blocking power of coalition *k* is measured by the cost of the grand coalition minus the cost of *k*'s complementary coalition, relative to the cost of coalition *k*. Blocking power can therefore be interpreted as the contribution of the coalition *k* to the grand coalition;

$$BP_k = 1 - \frac{cost_N - cost_{N\backslash k}}{cost_k} \qquad \forall k \in K$$

As for constructive power, we wish to find the blocking power belonging to each *player*. The blocking power of each player is measured by calculating the average of blocking power of all coalitions the player participates in. The higher blocking power in the coalitions where player *j* participates, the higher is the blocking power of player *j*. The blocking power can be considered as an indication of each player's contribution to the grand coalition, i.e. how important each player is within its coalition. The averages of blocking power for each player in the grand coalition are illustrated in figure 4.3.1.

Figure 4.3.1. Average blocking power within the grand coalition

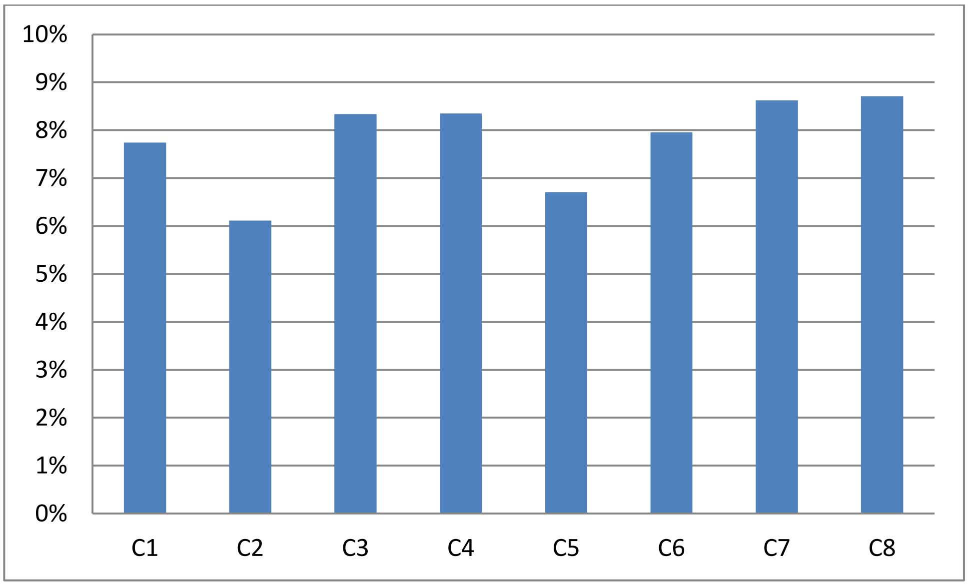


Figure 4.3.1 illustrates the differences in blocking power among the players. Contrary to the measures of constructive power, player C8 has the highest blocking power with an average of 8.71 %, while player C2 is measured with the lowest average blocking power of 6.11 %.

5. Total Cost Minimization

Superadditivity is a widely used assumption in cost allocation methods. In games where superadditivity is satisfied, the grand coalition will always be the social welfare maximizing coalition. In these instances, coalition structure generation is in principle trivial (Sandholm, Larson, Andersson, Shehory, & Tohmé, 1999).

However, even in cases where superadditivity is satisfied, there might be difficulties of managing large coalitions in practice. We consider the case where non-included managerial costs of conforming large coalitions, laws or political issues make it impossible or non-optimal to form the grand coalition. A relevant subject in such cases concerns which coalitions that optimally should form.

5.1 Coalition Structure

In order to address the question of optimal coalition structures, we introduce the upper bound, *m,* of maximum participants in a coalition. To find the coalition structure that maximizes social welfare, we replicate the mixed programming model from Guajardo and Rönnqvist (2014). In the following, this model will be referred to as *Model 1.* We start by introducing the notation on sets, parameters and decision variables.

Sets

N: set of players

K: set of coalitions

Parameters

$cost_k$: total cost of coalition k

$$\alpha_{j,k} = \begin{matrix} 1 & \text{if player } j \text{ belongs to coalition } k \\ 0 & \sim \end{matrix}$$

m: maximum number of players in a coalition

Variables

$$x_k = \begin{matrix} 1 & \text{if coalition } k \text{ is formed} \\ 0 & \sim \end{matrix}$$

$u_{j,k}$: cost allocated to player j in coalition k

The following model aims to find a coalition structure that minimizes total cost, such that all coalitions are stable.

Objective function

$$\text{minimize} \sum_{j \in N} \sum_{k \in K} u_{j,k} \tag{5.1}$$

Constraints

$$\sum_{j \in N} \alpha_{j,k} * u_{j,k} \leq cost_b * x_k \qquad \forall k, b \in K \tag{5.2}$$

$$\sum_{j \in N} \alpha_{j,k} * u_{j,k} = cost_k * x_k \qquad \forall k \in K \tag{5.3}$$

$$\sum_{k \in K} \alpha_{j,k} * x_k = 1 \qquad \forall j \in N \tag{5.4}$$

$$\sum_{j \in N} \alpha_{j,k} * x_k \leq m \qquad \forall k \in K \tag{5.5}$$

$$x_k \in \{0,1\}, \; u_{j,k} \geq 0 \qquad \forall j \in N, k \in K \tag{5.6}$$

The objective function (5.1) minimizes the total cost allocated among the players. Constraints (5.2) are rationality conditions assuring weak stability for all formed coalitions. These conditions assure that players in the formed coalition have no incentives to deviate and create their own coalition. Constraints (5.3) correspond to the efficiency condition, stating that the sum of costs allocated to all players in the formed coalition k equals the total cost of coalition k. Constraints (5.4) state that each player is assigned to one and only one coalition. Constraints (5.5) set an upper bound on the maximum cardinality of participants allowed in each coalition and constraints (5.6) express the nature of the variables.

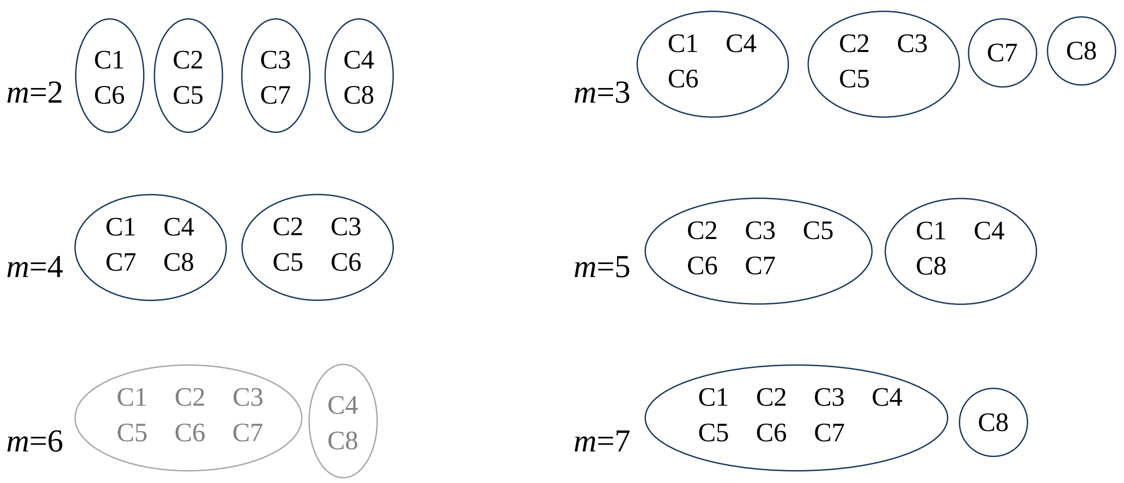
The grand coalition includes all players and its subsets will therefore involve all possible coalitions. In this case the weak rationality conditions (5.2) are sufficient to assure a stable allocation. However, in order to assure stability in smaller coalitions, the model must also concern potential collaborations with players outside the formed coalition. This is incorporated by the strong stability condition (5.7).

$$\sum_{j \in N} \left(\alpha_{j,k} * \sum_{b \in K} u_{j,b} \right) \leq cost_k \qquad \forall k \in K \tag{5.7}$$

Constraints (5.7) assure that no players, either from the same or from different coalitions, have incentives to form their own coalition. This is a more restrictive condition than the weak stability conditions, thus constraints (5.2) become superfluous~~.~~ For illustrative purposes, we choose to include them in the model. It is proven that if the model is feasible when including the strong stability conditions (5.7), the optimal objective value is equal to the optimal objective value of the model when only using the weak stability conditions (5.2) (Guajardo & Rönnqvist, 2014). As long as there are feasible solutions, the model provides the same total cost and coalition structure when including or excluding the strong stability conditions. For further explanation of the relationship between the two conditions, see Guajardo and Rönnqvist (2014).

The boundary *m* of the maximum participations within a coalition, limits the possibilities of formed coalitions. We implement the model in AMPL and solve for *m* ranging from 1 to 8. The optimal coalition structures are shown in figure 5.1.1.

Figure 5.1.1. Coalition structures when minimizing total cost



For *m*=1 no collaborations are allowed, thus all players operate alone. By allowing all players to collaborate, i.e. *m*=8, the grand coalition is formed. For *m*=6, the model is infeasible when including the strong stability condition. When allowing for maximum six players to collaborate, there will always be incentives for some of the players to deviate from their coalition to create coalitions with other players. Without the strong stability condition, the coalition structure {C1, C2, C3, C5, C6, C7} {C4, C8} minimizes the total cost and satisfies the weak stability conditions.

The calculated constructive and blocking power in chapter 4.2 and 4.3 are based on the grand coalition and will therefore not apply for the coalition structures found in *Model 1*. In order to find indications of power that correspond with the allocations in the following chapters, we measure the constructive power and blocking power of each player within each of the formed coalitions. This way we are able to compare the power of each player with the power of the player(s) it collaborates with.

Figure 5.1.2. Average constructive power within formed coalitions

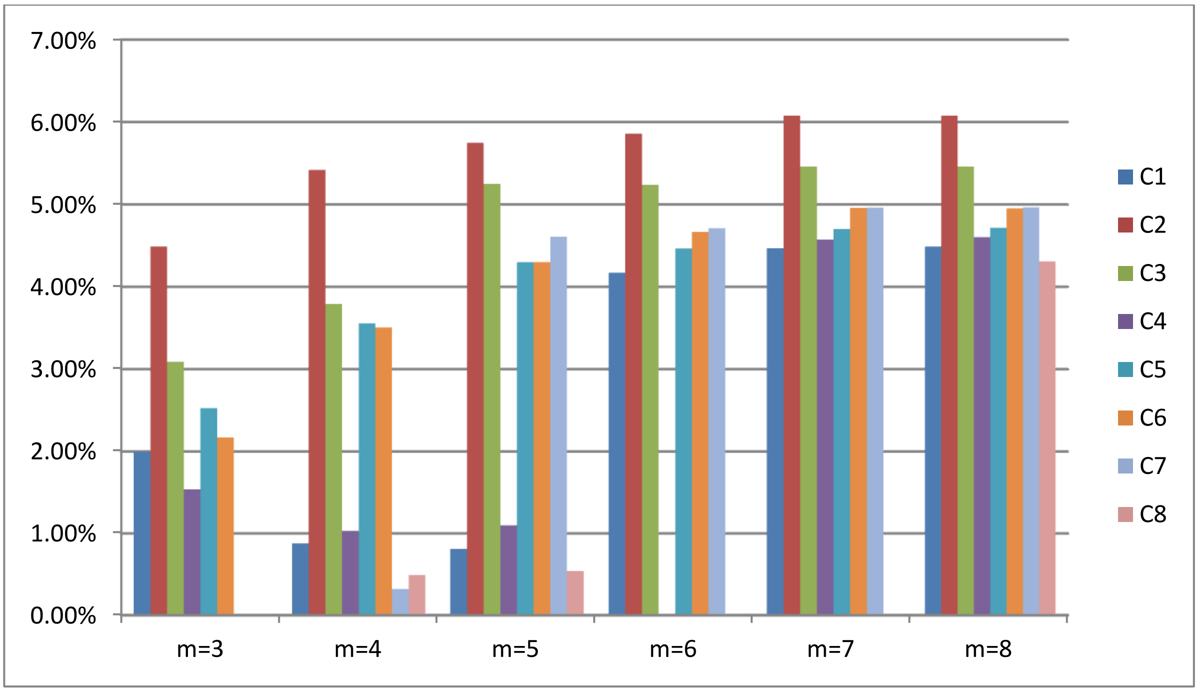
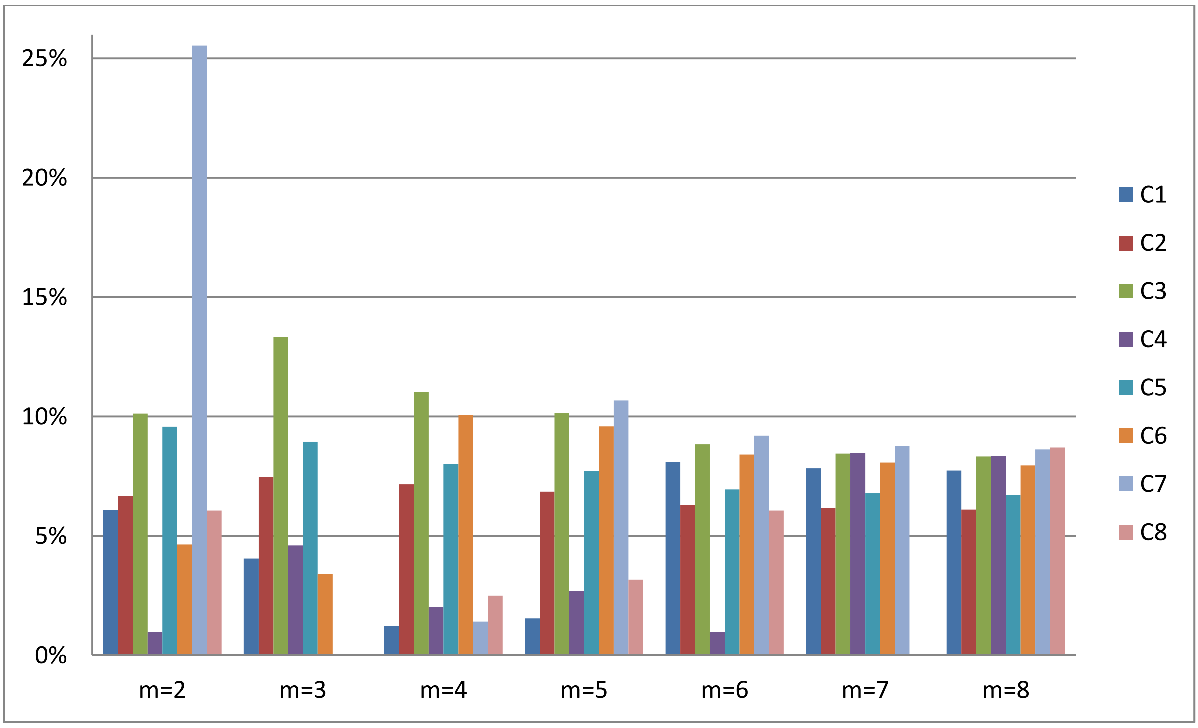


Figure 5.1.2 gives an indication of the constructive power each player possesses within the formed coalitions. For *m*=2 there is no constructive power to consider since there is no savings when the players operate alone. In the main, player C2 and C3 possess the highest constructive power. This means that player C2 and C3 are involved in coalitions that obtain high relative savings and further implies that these players are efficient in collaborations. The largest variation in constructive power occurs between the different coalitions, where

bigger coalitions in general obtain higher relative savings. Consequently, comparisons of constructive power across different coalitions may give wrong conclusions. The measures do however give a good indication of the balance of power within the formed coalitions. The measures of constructive power for *m*=8 are the same as in chapter 4.2.

Figure 5.1.3 gives an indication of the blocking power each player possesses in the formed coalitions. The blocking power is calculated as explained in chapter 4.3, only in this case we consider each formed coalition as the grand coalition.

Figure 5.1.3. Average blocking power within formed coalitions



Player C7 stands out for *m*=2 by having extremely high blocking power. This is a result of coalition {C3, C7} achieving high cost savings in addition to player C7 having low stand-alone cost. Player C2 has the lowest blocking power in all the coalitions he participates in. Player C4 has in general low blocking power, but he also collaborates with other players possessing low blocking power. The measures of blocking power for *m*=8 are the same as in chapter 4.3.

Both constructive power and blocking power of each player are averages over all coalitions the player participates in. This calculation method gives imprecise measures, but still an adequate indication of the power possessed by each player. Constructive power and blocking power are more accurately taken into account in the allocation methods we utilize in the following chapters. Our purpose with calculating measures of power is not to introduce a new solution concept for allocating costs, but to be able to analyze the results of the allocation methods and understand the intuition behind the models. For this purpose, figure 5.1.2 and 5.1.3 give sufficiently precise measures of constructive power and blocking power possessed by each player.

By implementing *Model 1*,the coalition structures that minimize total costs are determined. The next question to address is how the costs should be allocated between the participants. When allocating the costs among the players, the coalition structures are considered as settled, i.e. there is no possibility for players in one coalition to collaborate with players in other coalitions. This implies that the formed coalitions can be regarded as grand coalitions, hence the weak rationality condition is sufficient to provide a stable allocation. This assumption is based on the idea that players do have the possibility to refuse to collaborate, but not the possibility to initiate collaboration with players outside the formed coalition. In the following sections we investigate five different cost allocation methods to find the allocated cost to the players. The models are formulated to allocate the cost of the grand coalition. We create new datasets for the coalitions determined by *Model 1*, where the set of players, *N*, includes only the participants of the formed coalition. We denote the new sets of players as *M*. This enables us to generalize the model formulations, and implement the same model for all formed coalitions.

5.2 Equal Profit Method

Equal profit method aims to find a stable allocation that minimizes the largest pairwise difference in relative savings among the participants in a coalition, cf. chapter 3.2. To determine the EPM cost allocation in the coalition structures found in *Model 1*, we implement the following model in AMPL:

Sets

M: set of players in formed coalition

K: set of coalitions

Parameters

$$\alpha_{j,k} = \begin{cases} 1 & \text{if player } j \text{ belongs to coalition } k \\ 0 & \sim \end{cases}$$

$cost_k$: total cost of coalition k

$scost_j$: stand − alone cost of player j

Variables

u_j: cost allocated to player j

f: difference in relative saving

Objective function

$$\text{minimize } f \tag{5.8}$$

Constraints

$$f \geq \frac{u_i}{scost_i} - \frac{u_j}{scost_j} \qquad \forall i, j \in M \tag{5.9}$$

$$\sum_{j \in M} u_j * \alpha_{j,k} \leq cost_k \qquad \forall k \in K \tag{5.10}$$

$$\sum_{j \in M} u_j * \alpha_{j,k} = cost_M \qquad \forall k \in K \tag{5.11}$$

$$u_j \geq 0 \qquad \forall j \in M \tag{5.12}$$

The objective function (5.8) minimizes the difference in relative savings *f*. Constraints (5.9) state that *f* is greater than, or equal to, the difference between the cost allocated to player *i* divided by his stand-alone cost and the cost allocated to player *j* divided by *j*'s stand-alone cost. Constraints (5.10) are weak rationality conditions, assuring a stable allocation. Constraints (5.11) correspond to the efficiency condition, stating that the total cost allocated to the players participating in the formed coalition is equal to the total cost of that coalition. Constraints (5.12) express the nature of the variables.

By running EPM for each coalition found by *Model 1*, we obtain the cost allocation shown in table 5.2.1.

Table 5.2.1. Cost allocations using EPM

m	C1	C2	C3	C4	C5	C6	C7	C8	TOTAL
1	3780	14860	4740	2070	10340	4960	1880	330	42960
2	3681	14276	4396	2053	9934	4829	1744	327	41240
3	3658	13852	4419	2003	9639	4799	1880	330	40580
4	3719	13697	4369	2037	9531	4572	1870	325	40120
5	3719	13583	4333	2037	9452	4534	1718	325	39700
6	3520	13583	4333	2053	9452	4534	1718	327	39520
7	3520	13551	4323	1888	9430	4523	1715	330	39280
8	3520	13551	4323	1888	9430	4523	1714	301	39250

The costs for *m*=1 are the stand-alone cost of each player. The column to the right shows the total cost for each cardinality *m*, which are the costs we minimize in *Model 1*. Since the stand-alone costs of the players differ to a great extent, we find it more informative to compare the cost savings relative to each player's stand-alone cost.

Table 5.2.2. Relative savings using EPM

m	C1	C2	C3	C4	C5	C6	C7	C8
2	2.63 %	3.93 %	7.25 %	0.83 %	3.93 %	2.63 %	7.25 %	0.83 %
3	3.24 %	6.78 %	6.78 %	3.24 %	6.78 %	3.24 %	0.00 %	0.00 %
4	1.62 %	7.82 %	7.82 %	1.62 %	7.82 %	7.82 %	0.53 %	1.62 %
5	1.62 %	8.59 %	8.59 %	1.62 %	8.59 %	8.59 %	8.59 %	1.62 %
6	6.88 %	8.59 %	8.59 %	0.83 %	8.59 %	8.59 %	8.59 %	0.83 %
7	6.88 %	8.80 %	8.80 %	8.80 %	8.80 %	8.80 %	8.80 %	0.00 %
8	6.88 %	8.81 %	8.81 %	8.81 %	8.81 %	8.81 %	8.81 %	8.81 %

EPM minimizes the difference in relative savings between the players *within* each coalition. This is evident when studying the relative savings presented in table 5.2.2. For *m*=2, *m*=3 and *m*=5, the method gives equal relative savings to all players within each coalition. For the remaining cardinalities the method assigns unequal relative savings within coalitions in order to satisfy rationality constraints. For *m*=4, player C7 attains 0.53 % relative savings in contrast to the other participants saving 1.62 %. Player C1, C4 and C8 can achieve relative savings of 1.62 % by excluding player C7 from the coalition; hence in order to provide a stable coalition they must attain at least 1.62 % when collaborating with C7. The same reasoning applies for $m \geq 6$, where player C1 attains lower relative savings than his collaborators. The largest difference of relative savings within a coalition occurs in the grand coalition, where player C1 attains relative saving of 6.88 % and the other players save 8.81 %.

5.3 The Nucleolus

The nucleolus aims to find a fair cost allocation by maximizing the satisfaction of the least satisfied coalition, cf. chapter 3.3. Strictly speaking, the formulated model is the *pre-nucleolus* as it does not incorporate individual rationality conditions. However, as we operate with games with non-empty cores, the pre-nucleolus and nucleolus coincide. We therefore refer to the model as the nucleolus. We implement the nucleolus in AMPL to allocate the cost between the players within the coalition structures found in *Model 1*.

Sets

M: set of players

K: set of coalitions

Parameters

$$\alpha_{j,k} = \begin{matrix} 1 & \text{if player } j \text{ belongs to coalition } k \\ 0 & \sim \end{matrix}$$

$cost_k$: total cost of coalition k

Variables

u_j: cost allocated to player j

r: minimum excess among all coalitions

Objective function

$$\text{maximize } r \tag{5.13}$$

Constraints

$$r \leq cost_k - \sum_{j \in M} \alpha_{j,k} * u_j \qquad \forall k \in K \tag{5.14}$$

$$\sum_{j \in M} \alpha_{j,k} * u_j = cost_M \qquad \forall k \in K \tag{5.15}$$

$$u_j \geq 0 \qquad \forall j \in M \tag{5.16}$$

The objective function (5.13) maximizes the minimum excess, *r*. Constraints (5.14) state that *r* is less than, or equal to, the cost saving of the least satisfied coalition. Constraints (5.15) correspond to the efficiency condition and constraints (5.16) express the nature of the variables.

In the following, we explain the process to find a unique cost allocation for the grand coalition. In the first run coalition 8 and 247 have the lowest excess, with an excess of 15. The coalitions with lowest excess in each run have positive dual values. When the optimal value of a dual variable is positive, the inequality constraint to this variable must hold with equality at any optimal solution, i.e. the associated constraints are binding (Guajardo & Jörnsten, 2014). We fix the excess for these coalitions by including the following constraints:

$$u_{C8} + 15 = cost_8$$

$$u_{C1} + u_{C2} + u_{C3} + u_{C4} + u_{C5} + u_{C6} + u_{C7} + 15 = cost_{247}$$

The equations above state that the cost allocated to player C8 is bound at 330-15=315. The sum of costs allocated to the other players must be equal to 38950-15=38935. This gives an infinite number of possible allocations, hence we run the model again to further restrict the problem. By fixing the excess of coalitions with positive dual values and removing these coalitions from the problem, the model continues to maximize the second lowest excess and thus improve the cost allocation.

With coalition 8 and 247 removed from the problem, the second run finds that coalition 1 and 254 have positive dual values, and an excess of 130. We include the following constraints:

$$u_{C1} + 130 = cost_1$$

$$u_{C2} + u_{C3} + u_{C4} + u_{C5} + u_{C6} + u_{C7} + u_{C8} + 130 = cost_{254}$$

The cost allocated to player C1 is fixed at 3780-130=3650. We extract coalition 1 and 254 from the problem and run the model for the third time. Coalition 30 and 251 have the lowest satisfaction of the remaining coalitions, with an excess of 132.5. Coalition 30 contains player C4 and C8, and since the cost allocated to player C8 is already set at 315, the cost allocated to player C4 is fixed at 2380-315-132.5= 1932.5. After the fourth run the final cost allocated to player C1, C4 and C8 is determined.

As more and more excesses get fixed, the feasible set reduces and the allocated cost to the players gets determined. In the forestry case this requires 48 rounds for the grand coalition. The process of computing the nucleolus is summarized in table 5.3.1.

Table 5.3.1. Computing the nucleolus manually for the grand coalition

Round	Fixed *k*	Excess, *r*	$j \in k$	Cost, u_j
2	8	15	{C8}	u_{C8} = 315
3	1	130	{C1}	u_{C1} = 3650
4	30	132.5	{C4, C8}	u_{C4} = 1932.5
11	7	215	{C7}	u_{C7} = 1665
16	13	315	{C1, C6}	u_{C6} = 4545
31	25	397.5	{C3, C7}	u_{C3} = 4077.5
48	5	495	{C5}	u_{C5} = 9845

The column to the left in table 5.3.1 shows the number of runs required to find the cost allocated to each player. After round 31 the costs are fixed to all players except C2 and C5. The following 16 rounds do not restrict the problem sufficiently to fix the cost of these players. For instance, round 35 where the lowest excess occurs for coalition 6 does not provide any new information since the cost of player C6 was indirectly fixed through coalition 13 in round 16. Other examples occur when we fix the excess of larger coalitions containing more than one player for which the cost allocated is unknown. As long as the coalition with the lowest excess contains both player C2 and C5, the constraint does not restrict the problem sufficiently. In round 48, coalition 5 has the lowest excess and the cost allocated to player C5 equals 10340-495=9845. The final costs are determined for all players except player C2. Hence, the cost allocated to player C2 is easily found by calculating the difference between the total cost of the grand coalition and the sum of costs allocated to the other players, which is 13220.

As seen from the explanation above, finding the nucleolus manually is a time-consuming process. In order to run the necessary rounds of the model automatically, we incorporate some modifications to the model. We add the following sets and parameters:

Sets

$iFixExcess_h$: set of coalitions for which the excess is fixed in round h

$AlliFixExcess_h$: set of all coalitions with fixed excess in round h

Parameters

nLP: round of LP running

$fExcess_h$: excess value fixed for each set of coalitions with fixed excess in round h

$$fExcess_h = cost_k - \sum_{j \in M} \alpha_{j,k} * u_j \quad \forall h \in 1..nLP, k \in iFixExcess_h \tag{5.17}$$

By stating $k \notin AlliFixExcess$ in constraints (5.14), all coalitions with fixed excess get automatically removed from the problem. We also add constraints (5.17), which fix the excess of the coalition(s) with lowest satisfaction in each round.

Table 5.3.2 shows the cost allocated to each player within the coalition structures found in *Model 1*.

Table 5.3.2. Cost allocations using the nucleolus

m	C1	C2	C3	C4	C5	C6	C7	C8	TOTAL
1	3780	14860	4740	2070	10340	4960	1880	330	42960
2	3665	14365	4500	2060	9845	4845	1640	320	41240
3	3655	13870	4220	2010	9820	4795	1880	330	40580
4	3740	13475	4240	2015	9845	4610	1875	320	40120
5	3740	13452	4073	2020	9830	4600	1665	320	39700
6	3650	13357	4083	2060	9845	4540	1665	320	39520
7	3650	13222	4083	1940	9850	4540	1665	330	39280
8	3650	13219	4078	1933	9845	4545	1665	315	39250

The nucleolus cost allocation keeps the satisfaction of each coalition as high as possible, and reflects the constructive power of each coalition. For m=2, the nucleolus allocates costs such that each player within the same coalition obtains equal cost savings in absolute value. This corresponds to the assumption that all players are equally efficient when they operate alone. For $m \geq 3$, differences in constructive power affect the allocation such that players with efficient sub-coalitions attain higher relative savings than players with less efficient sub-coalitions.

Table 5.3.3. Relative savings using the nucleolus

m	C1	C2	C3	C4	C5	C6	C7	C8
2	3.04 %	3.33 %	5.06 %	0.48 %	4.79 %	2.32 %	12.77 %	3.03 %
3	3.31 %	6.66 %	10.97 %	2.90 %	5.03 %	3.33 %	0.00 %	0.00 %
4	1.06 %	9.32 %	10.55 %	2.66 %	4.79 %	7.06 %	0.27 %	3.03 %
5	1.06 %	9.47 %	14.08 %	2.42 %	4.93 %	7.26 %	11.44 %	3.03 %
6	3.44 %	10.11 %	13.87 %	0.48 %	4.79 %	8.47 %	11.44 %	3.03 %
7	3.44 %	11.02 %	13.87 %	6.28 %	4.74 %	8.47 %	11.44 %	0.00 %
8	3.44 %	11.04 %	13.98 %	6.64 %	4.79 %	8.37 %	11.44 %	4.55 %

Table 5.3.3 shows the relative savings for each player when m ranges from 2 to 8. In general, player C2, C3 and C7 attain high relative savings for all m. For m=2, the difference in relative savings within the formed coalitions is solely based on different stand-alone costs among the players. For m=3, player C7 and C8 are not included in any collaboration, hence their relative savings equal zero. Normally players benefit from collaborating, yet player C7

and C8 are better off operating alone, cf. chapter 4.1. For *m*=4, the main differences in relative savings occur between the two coalitions formed; the players in coalition {C1, C4, C7, C8} obtain low relative saving, whereas the more efficient coalition, {C2, C3, C5, C6}, provides higher relative savings to its participants. This trend applies to all coalition structures, and correspond with the pattern observed in average constructive power, cf. figure 5.1.2. However, by studying the results in greater detail, the relative savings obtained *within* coalitions do not entirely reflect the measure of constructive power among the players. For instance, player C3 obtains the highest relative savings in all coalitions for $m \geq 3$, in which player C2 is measured to have the highest constructive power. This inconsistency will be further elaborated in chapter 5.6.

5.4 The Modiclus

The modiclus minimizes the difference in excesses within all pairs of coalitions, the so-called *Envy*, cf. chapter 3.4. The method considers both constructive power and blocking power, even though the excess of each coalition is calculated the same way as in the nucleolus. We modify the run-codes used in the nucleolus, and implement the following linear programming model in AMPL for lexicographically minimizing the largest envy:

Sets

M: set of players

K: set of coalitions

$iFixEnvy_h$: set of coalitions for which $Envy$ is fixed in round h

$AlliFixEnvy_h$: set of all coalitions with fixed $Envy$ in round h

Parameters

$$\alpha_{j,k} = \begin{matrix} 1 & \text{if player } j \text{ belongs to coalition } k \\ 0 & \sim \end{matrix}$$

$cost_k$: total cost of coalition k

nLP: round of LP running

$fEnvy_h$: $Envy$ value fixed for each set of pairs with fixed $Envy$ in round h

Variables

u_j: cost allocated to player j

$Envy$: maximum envy among all pairs

Objective function

$$\text{minimize } Envy \tag{5.18}$$

Constraints

$$Envy \geq (cost_k - \sum_{j \in M} \alpha_{k,j} * u_j) - (cost_b - \sum_{j \in M} \alpha_{b,j} * u_j) \qquad \forall k, b \in K, k, b \notin iFixEnvy_h \tag{5.19}$$

$$\sum_{j \in M} \alpha_{k,j} * u_j = cost_M \qquad \forall k \in K \tag{5.20}$$

$$fEnvy_h = cost_k \sum_{j \in M} \alpha_{k,j} * u_j - (cost_b - \sum_{j \in M} \alpha_{b,j} * u_j) \quad \forall h \in 1..nLP, \ \ k \in iFixEnvy_h \tag{5.21}$$

$$u_j \geq 0 \qquad \forall j \in M \tag{5.22}$$

The objective function (5.18) minimizes *Envy*. Constraints (5.19) state that *Envy* is greater than, or equal to, the difference in excess within all pairs of coalitions (k, b). Stating $b, k \notin iFixEnvy_h$, ensures that the condition do not account for the pairs with fixed *Envy*. Constraints (5.20) correspond to the efficiency condition and constraints (5.21) fix *Envy* of the pairs of coalitions with highest envy each round. Constraints (5.22) express the nature of the variables.

The process to minimize the *Envy* between the pairs in the grand coalition is explained in the following. For *m*=8 there are 254 * 254 = 64.516 different pairs of coalitions, which results in high computational complexity. The first run gives positive dual values for the pairs {20, 247}, {20, 8}, {183, 247} and {183, 8}. Coalition 247 and 8 envy coalition 20 and 183 by 1247.5, which is the highest *Envy* among the pairs. By running the model a second time, excluding the pairs with fixed *Envy*, the highest *Envy* equals 1242.5.

Table 5.4.1. First and second run of the modiclus

k	***j* ∈ *k***	**Excess**	***b***	***j* ∈ *b***	**Excess**	***Envy***
20	{C1, C7}	1262.5	247	{C1, C2, C3, C4, C5, C6, C7}	15	1247.5
20	{C1, C7}	1262.5	8	{C8}	15	1247.5
183	{C1, C3, C4, C5, C6}	1262.5	247	{C1, C2, C3, C4, C5, C6, C7}	15	1247.5
183	{C1, C3, C4, C5, C6}	1262.5	8	{C8}	15	1247.5
214	{C3, C4, C5, C6, C8}	1257.5	8	{C8}	15	1242.5
20	{C1, C7}	1262.5	254	{C2, C3, C4, C5, C6, C7, C8}	20	1242.5

The procedure of fixing the highest *Envy* and removing the associated pairs continues until the final cost allocation is determined. The *Envy* in modiclus is symmetric within all pairs, i.e. the best solution the model can achieve is *Envy* equal to zero. Table 5.4.2 shows the final cost allocation within the coalition structures found in *Model 1*.

Table 5.4.2. Cost allocations using the modiclus

m	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**	**TOTAL**
1	3780	14860	4740	2070	10340	4960	1880	330	42960
2	3665	14365	4500	2060	9845	4845	1640	320	41240
3	3635	13930	4190	2010	9790	4815	1880	330	40580
4	3740	13714	4118	2015	9728	4610	1875	320	40120
5	3735	13606	4063	2025	9713	4573	1665	320	39700
6	3650	13585	4090	2060	9715	4515	1585	320	39520
7	3600	13596	4134	1890	9715	4455	1560	330	39280
8	3540	13660	4193	1838	9775	4401	1528	315	39250

As presented in table 5.4.2, the cost allocated to each player is of various sizes. For m=2 there are no constructive or blocking power to consider, and the cost savings are divided equally within the coalitions. For $m \geq 3$, the allocated costs depend on the constructive and blocking power of each player. In order to compare the results between the players, the different outcomes are discussed in terms of relative savings.

Table 5.4.3. Relative savings using the modiclus

m	C1	C2	C3	C4	C5	C6	C7	C8
2	3.04 %	3.33 %	5.06 %	0.48 %	4.79 %	2.32 %	12.77 %	3.03 %
3	3.84 %	6.26 %	11.60 %	2.90 %	5.32 %	2.92 %	0.00 %	0.00 %
4	1.06 %	7.71 %	13.13 %	2.66 %	5.92 %	7.06 %	0.27 %	3.03 %
5	1.19 %	8.43 %	14.29 %	2.17 %	6.07 %	7.81 %	11.44 %	3.03 %
6	3.44 %	8.58 %	13.71 %	0.48 %	6.04 %	8.97 %	15.69 %	3.03 %
7	4.76 %	8.51 %	12.78 %	8.70 %	6.04 %	10.18 %	17.02 %	0.00 %
8	6.35 %	8.08 %	11.53 %	11.23 %	5.46 %	11.27 %	18.75 %	4.55 %

Table 5.4.3 shows the relative savings of each player using the modiclus. Compared to the results by using the nucleolus, cf. table 5.3.3, player C2 attains significantly less savings when blocking power is taken into account. Player C3 attains the highest relative savings when m=3, m=4 and m=5 whereas player C7 achieves the highest relative savings for $m \geq 6$. All these observations are compatible with the measures of blocking power for each player, cf. figure 5.1.3. The largest disparity between the results of the nucleolus and the modiclus occurs for the allocations in the grand coalition. Player C1, C4, C6 and C7 get significantly higher savings in the grand coalition by using the modiclus while player C2, C3 and C5 obtain lower cost savings.

5.5 The Simplified Modiclus

Similar to the nucleolus, the simplified modiclus lexicographically maximizes the excess of the least satisfied coalition, cf. chapter 3.5. However, the simplified modiclus base the excess on the average of constructive power and blocking power of each coalition.

We incorporate some minor modifications to the automatic run-codes used in previous models and implement the following linear programming model in AMPL:

Sets

M: set of players

K: set of coalitions

$iFixExcess_h$: set of coalitions for which the excess is fixed in round h

$AlliFixExcess_h$: set of all coalitions with fixed excess in round h

Parameters

$\alpha_{j,k} = \begin{matrix} 1 & \text{if player } j \text{ belongs to coalition } k \\ 0 & \sim \end{matrix}$

$cost_k$: total cost of coalition k

$pair_{k,b}$: $\begin{matrix} 1 & \text{if player } k \text{ and player } b \text{ is a pair} \\ 0 & \sim \end{matrix}$

nLP: round of LP running

$fExcess_h$: excess value fixed for each set of coalitions with fixed excess in round h

Variables

u_j: cost allocated to player j

r: minimum excess among all coalitions

Objective function

$$\text{maximize } r \tag{5.23}$$

Constraints

$$r \leq \frac{1}{2}cost_k + \frac{1}{2}(cost_M - cost_b) - \sum_{j \in M} \alpha_{j,k} * u_j$$
$$\forall k, b \in K,\ pair[b,k] = 1,\ \ k \notin iFixExcess_h \tag{5.24}$$

$$\sum_{j \in M} \alpha_{j,k} *\ u_j = cost_M \qquad \forall k \in K \tag{5.25}$$

$$fExcess_h = \frac{1}{2}cost_k + \frac{1}{2}(cost_M - cost_b) - \sum_{j \in M} \alpha_{j,k} * u_j$$
$$\forall h \in 1..nLP,\ \ k \in iFixExcess_h,\ \ b \in K,\ pair[b,k] = 1 \tag{5.26}$$

$$u_j \geq 0 \quad \forall j \in M \tag{5.27}$$

The objective function (5.23) maximizes the excess r. Constraints (5.24) define the excess r as the average of the cost of coalition k and the difference between the cost of the formed coalition and the cost of coalition b, minus the sum of the allocated costs. Stating that

$pair[b,k]=1$ ensures that the condition only accounts for pairs of complementary coalitions. Constraints (5.25) correspond to the efficiency condition. Constraints (5.26) fix the excess of the coalitions with lowest excess in each round and constraints (5.27) express the nature of the variables.

The difference between the cost of the formed coalition and the complementary coalition can be interpreted as k's contribution to the formed coalition, i.e. the blocking power of coalition k. If k leaves the formed coalition, the complementary coalition gets $cost_{M\setminus k}$, while they can achieve $cost_M$ if they all act together. Hence, for any coalition k it is beneficial to have a small difference between $cost_M$ and $cost_{M\setminus k}$, which means that k has high blocking power. The property of additivity implies that k's share of the cost in the formed coalition will be lower than k's cost by operating alone, i.e. $\sum_{j\in M}\alpha_{j,k} * u_j \leq cost_k$. The same logic applies for the complementary coalition, meaning that $cost_{M\setminus k}$ will be greater than the allocated cost to the players in the complementary coalition. These two effects lead to a smaller measure of contribution to formed coalition than the cost for each coalition. This implies that the average used in the simplified modiclus will be smaller than the actual cost used in the nucleolus.

$$\frac{1}{2}cost_k + \frac{1}{2}\left(cost_M - cost_{M\setminus k}\right) < cost_k$$

This means that condition (5.24) is stricter than condition (5.14), which further implies that the excess r is smaller in the simplified modiclus than in the nucleolus.

The process of finding a unique cost allocation for the grand coalition is explained in the following. We run the model for m=8, and find the maximum satisfaction of the least satisfied coalitions.

Table 5.5.1. First and second run of the simplified modiclus

k	***j* ∈ *k***	***r***	***b***	***j* ∈ *b***	***r***
129	{C2, C3, C4, C6}	-158	126	{C1, C5, C7, C8}	158
144	{C2, C5, C6, C7}	-158	111	{C1, C3, C4, C8}	158
169	{C1, C2, C3, C5, C8}	-158	86	{C4, C6, C7}	158
228	{C1, C2, C3, C6, C7, C8}	-158	27	{C4, C5}	158
231	{C1, C2, C4, C5, C7, C8}	-158	24	{C3, C6}	158
253	{C1, C3, C4, C5, C6, C7,C8}	-158	2	{C2}	158
132	{C2, C3, C5, C6}	-157	123	{C1, C4, C7, C8}	157
205	{C2, C3, C5, C6, C8}	-157	50	{C1, C4, C7}	157

Table 5.5.1 shows the coalitions for whom we fix the excess in the second and the third round, i.e. the coalitions with positive dual values. The coalitions *k* to the left in the table are the six coalitions that are least satisfied with the cost allocation, while their complementary coalitions *b* have most satisfaction. Since the constructive power and the blocking power are weighted equally, the total excess within each pair of a coalition and its complement will always add up to zero. The positive excess in one coalition is offset by the same amount of negative excess in the complementary coalition. As the objective is to maximize the minimum excess, the best possible solution the model can achieve is an excess equal to zero.

The following procedure is similar to the nucleolus. As more and more excesses get fixed, the model provides an improved solution until the final cost allocation is determined. Table 5.5.2 shows the cost allocated to each player in the coalition structures found in *Model 1*.

Table 5.5.2. Cost allocations using the simplified modiclus

m	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**	**TOTAL**
1	3780	14860	4740	2070	10340	4960	1880	330	42960
2	3665	14365	4500	2060	9845	4845	1640	320	41240
3	3652	13909	4200	1997	9800	4812	1880	330	40580
4	3738	13650	4185	2013	9780	4555	1877	322	40120
5	3738	13650	4100	2023	9726	4540	1605	318	39700
6	3566	13616	4131	2060	9711	4496	1620	320	39520
7	3561	13581	4124	1874	9709	4483	1618	330	39280
8	3562	13570	4122	1869	9708	4491	1618	310	39250

For m=2, there are no constructive or blocking power to consider and the costs allocated to the players equal the cost in the nucleolus and the modiclus. For $m \geq 3$ the cost allocations reflect the performance of each player. In order to compare the outcome for each player, we express the cost savings relative to stand-alone costs.

Table 5.5.3. Relative savings using the simplified modiclus

m	C1	C2	C3	C4	C5	C6	C7	C8
2	3.04 %	3.33 %	5.06 %	0.48 %	4.79 %	2.32 %	12.77 %	3.03 %
3	3.39 %	6.39 %	11.39 %	3.54 %	5.22 %	2.99 %	0.00 %	0.00 %
4	1.10 %	8.14 %	11.71 %	2.74 %	5.42 %	8.17 %	0.18 %	2.52 %
5	1.10 %	8.14 %	13.50 %	2.25 %	5.95 %	8.47 %	14.63 %	3.54 %
6	5.67 %	8.37 %	12.85 %	0.48 %	6.09 %	9.35 %	13.81 %	3.03 %
7	5.79 %	8.60 %	12.99 %	9.49 %	6.11 %	9.62 %	13.96 %	0.00 %
8	5.78 %	8.67 %	13.05 %	9.71 %	6.11 %	9.47 %	13.96 %	6.06 %

Table 5.5.3 shows the relative savings of each player using the simplified modiclus allocation method. As the simplified modiclus base the cost allocation on the constructive and blocking power of each player, the relative savings are similar to the results by using modiclus, cf. table 5.4.3. The modiclus emphasizes blocking power to a further extent than the simplified modiclus; hence the relative savings shown in table 5.5.3 tend to lie between the allocations provided by the nucleolus and the modiclus. Player C3 and C7 still attain high relative savings in most of the formed coalitions due to their high measure of constructive and blocking power. However, for m=4, player C7 participates in the least efficient coalition, with player C1, C4 and C8. In this coalition, player C7 has the lowest constructive and blocking power, which explains his correspondingly low relative saving. Player C1 and C8 generally attain low relative savings due to their participation in the least efficient coalitions.

5.6 Discussion of the Nucleolus

The nucleolus allocates the costs such that the minimum excess is lexicographically maximized. The model operates with excess in absolute values, and does not adjust for the differences in volumes among the players. Consequently, this gives higher relative savings to players with low stand-alone costs. As mentioned in chapter 4, the differences in stand-alone cost between the players in the forestry case are due to different volumes, and not differences in efficiency. In order to provide cost savings of equal significance to all players, we therefore find it crucial to consider the *relative* savings the players achieve, and not the absolute savings.

In previous literature (Frisk et al., 2010) the nucleolus has been computed in the forestry case for *m*=8, with the same results as we present in table 5.3.2. By studying the data, we find that this method benefits the smaller players, especially player C7 and C8. To give a comprehensive illustration of the relevance of adjusting for different volumes, we use the dataset from the coalition containing player C2, C3 and C5. This coalition is formed when minimizing total cost for *m*=3. By using a smaller dataset we can easier see the basis of the allocations.

Table 5.6.1. Data for coalition 59

$\boldsymbol{k}$	$\boldsymbol{j \in k}$	$\sum_{j \in k} \boldsymbol{scost}$	$\boldsymbol{Cost_k}$	**Saving**	**Constructive Power**
2	{C2}	14860	14860	0	0.00 %
3	{C3}	4740	4740	0	0.00 %
5	{C5}	10340	10340	0	0.00 %
16	{C2, C3}	19600	18610	990	5.05 %
18	{C2, C5}	25200	24210	990	3.93 %
23	{C3, C5}	15080	14910	170	1.13 %
59	{C2, C3, C5}	29940	27910	2030	6.78 %

Table 5.6.1 compares the cost of the sub-coalitions of coalition 59 against the sum of the stand-alone costs of the participating players. The sub-coalitions where player C2 participate have an average relative saving of 4.49 %, while the coalitions where player C3 and C5 participate have an average relative saving of 3.09 % and 2.53 %, respectively. We interpret this as player C2 having high constructive power, and would therefore expect C2 to be allocated a low cost relative to his stand-alone cost.

The nucleolus aims to keep all coalitions as equally satisfied as possible, by lexicographically maximizing the minimum excess:

$$e(u,k) = cost_k - \sum_{j \in k} u_j \qquad \forall k \in K$$

In the first run coalition 3 and 18 have the lowest excess, with an excess of 520. When fixing these excesses, player C3 is allocated a cost of 4220 and player C2 and C5 are in total allocated 23690.

$$u_{C3} + 520 = cost_3$$

$$u_{C2} + u_{C5} + 520 = cost_{18}$$

The coalitions have the same excess in absolute value and according to the nucleolus these coalitions are equally satisfied. However, the excess of coalition 3 gives a relative saving of $\frac{520}{4740} = 10{,}97\ \%,$ while coalition 18 attains a relative saving of $\frac{520}{24210} = 2{,}15\ \%$; hence it is difficult to argue that they have equal satisfaction. The results from running the nucleolus on coalition 59 are presented in table 5.6.2.

Table 5.6.2. Cost allocation and savings using the nucleolus for *k*=59

	C2	**C3**	**C5**	**TOTAL**
Cost, u_j	13870	4220	9820	27910
Excess, r	990	520	520	2030
Relative Saving	6.66 %	10.97 %	5.03 %	6.78 %

The excesses in absolute measures correspond with the absolute savings each coalition achieves. Player C3 attains a relative saving of almost 11 %, while C2 has a relative saving of less than 7 %. These results are not compatible with our definition of constructive power and the allocation seems to be unfair. As we focus on the relative savings, maximizing the excess in absolute value gives an imbalanced cost allocation, which benefits players with low volume. This explains the inconsistency between constructive power and the nucleolus allocations seen in table 5.3.3. In order to adjust for different volumes, the excess of each coalition is divided by the corresponding cost:

$$e(u,k) = \frac{cost_k - \sum_{j \in k} u_j}{cost_k} \qquad \forall k \in K$$

When lexicographically maximizing the *relative* excess of each coalition, we get the following results:

Table 5.6.3. Cost allocation and savings based on relative excess for *k*=59

	C2	C3	C5	TOTAL
Cost, u_j	13493	4501	9916	27910
Excess, r	1367	239	424	2030
Relative Saving	9,20 %	5,04 %	4,10 %	6,78 %

The relative savings shown in table 5.6.3 correspond better to the constructive power of each coalition. The coalitions where player C2 participates have high constructive power, and player C2 therefore attains high relative saving.

In our interpretation, this approach provides a fairer cost allocation than the nucleolus when the players differ in volume. The nucleolus based on excess in relative measures is a cost allocation method known as *the proportional nucleolus*. Despite its desirable properties when players differ in volume, the method seems to be less known and less used than the nucleolus.

5.7 The Proportional Nucleolus

The proportional nucleolus lexicographically maximizes the satisfaction of the least satisfied coalition based on relative excess, cf. chapter 3.6. To find the proportional nucleolus cost allocation in the coalition structures found in *Model 1*, we implement the following linear programming model in AMPL:

Sets

M: set of players

K: set of coalitions

$iFixExcess_h$: set of coalitions for which the excess is fixed in round h

$AlliFixExcess_h$: set of all coalitions with fixed excess in round h

Parameters

$$\alpha_{j,k} = \begin{matrix} 1 & \text{if player } j \text{ belongs to coalition } k \\ 0 & \sim \end{matrix}$$

$cost_k$: total cost of coalition k

nLP: round of LP running

$fExcess_h$: excess value fixed for each set of coalitions with fixed excess in round h

Variables

u_j: cost allocated to player j

r: minimum excess among all coalitions

Objective function

$$\text{maximize } r \tag{5.28}$$

Constraints

$$r \leq \frac{(cost_k - \sum_{j\in M} \alpha_{j,k} * u_j)}{cost_k} \qquad \forall k \in K, k \notin iFixExcess_h \tag{5.29}$$

$$\sum_{j\in M} \alpha_{j,k} * u_j = cost_M \qquad \forall k \in K \tag{5.30}$$

$$fExcess_h = \frac{(cost_k - \sum_{j\in M} \alpha_{j,k} * u_j)}{cost_k} \qquad \forall k \in K, k \notin iFixExcess_h \tag{5.31}$$

$$u_j \geq 0 \qquad \forall j \in M \tag{5.32}$$

The objective function (5.28) maximizes the excess, r. Constraints (5.29) state that r is less than, or equal to, the relative cost saving of all coalitions. Constraints (5.30) correspond to the efficiency condition and constraints (5.31) fix the excess of the coalitions with the lowest excess. Constraints (5.32) express the nature of the variable.

The results obtained from implementing the model in AMPL are shown in table 5.7.1.

Table 5.7.1. Cost allocations using the proportional nucleolus

m	C1	C2	C3	C4	C5	C6	C7	C8	TOTAL
1	3780	14860	4740	2070	10340	4960	1880	330	42960
2	3681	14276	4396	2053	9934	4829	1744	327	41240
3	3645	13492	4501	2047	9916	4769	1880	330	40580
4	3735	13035	4415	2009	9865	4854	1878	329	40120
5	3726	13065	4179	2025	9755	4765	1856	329	39700
6	3754	13016	4126	2053	9754	4646	1844	327	39520
7	3719	12952	4093	2019	9735	4606	1826	330	39280
8	3714	12940	4091	2003	9733	4615	1824	330	39250

For *m*=2, the cost allocation equals the allocation using EPM, and the players within each coalition are allocated costs proportionally to their stand-alone costs. For $m \geq 3$, the costs obtained using the proportional nucleolus reflect the constructive power of each player and no longer equal the EPM allocation. Players with efficient sub-coalitions have a better strategic position than players with less efficient alternatives, and are therefore allocated a smaller cost using the proportional nucleolus.

Table 5.7.2. Relative savings using the proportional nucleolus

m	C1	C2	C3	C4	C5	C6	C7	C8
2	2.63 %	3.93 %	7.25 %	0.83 %	3.93 %	2.63 %	7.25 %	0.83 %
3	3.58 %	9.20 %	5.04 %	1.13 %	4.10 %	3.85 %	0.00 %	0.00 %
4	1.20 %	12.28 %	6.86 %	2.96 %	4.59 %	2.13 %	0.13 %	0.25 %
5	1.43 %	12.08 %	11.83 %	2.18 %	5.66 %	3.92 %	0.78 %	0.33 %
6	0.70 %	12.41 %	12.95 %	0.83 %	5.67 %	6.33 %	1.92 %	0.83 %
7	1.61 %	12.83 %	13.66 %	2.48 %	5.86 %	7.13 %	2.88 %	0.00 %
8	1.75 %	15.94 %	13.70 %	3.22 %	5.87 %	6.96 %	2.96 %	0.08 %

Table 5.7.2 shows the relative savings for all players when allocating costs by using the proportional nucleolus. Compared to the nucleolus, the results by using proportional nucleolus are more compatible with the measures of constructive power of each player, cf. figure 5.1.2. By adjusting for volume, player C2 is rewarded for his high constructive power by attaining higher relative savings. Player C3 still achieves high relative savings, even though the results are slightly reduced compared to the nucleolus. Player C1, C4 and C8

attain less savings by using the proportional nucleolus, which correspond to their low constructive power. However, the biggest gap between the nucleolus and the proportional nucleolus appear for player C7. By using the nucleolus, player C7 attains relative savings as high as 11.44 % for $m \geq 6$, while the cost allocated in the proportional nucleolus only generates relative savings from 1.92 % to 2.96 % for the same cardinalities. The results demonstrate that smaller players are worse off by adjusting for volume.

5.8 Discussion of Fairness

In previous sections we have derived cost allocations within the different coalition structures using EPM, the nucleolus, the modiclus, the simplified modiclus and the proportional nucleolus. To give a more comprehensive comparison of the different methods we focus on the cost allocated in the grand coalition.

Figure 5.8.1. Relative savings using different allocation methods for *m*=8

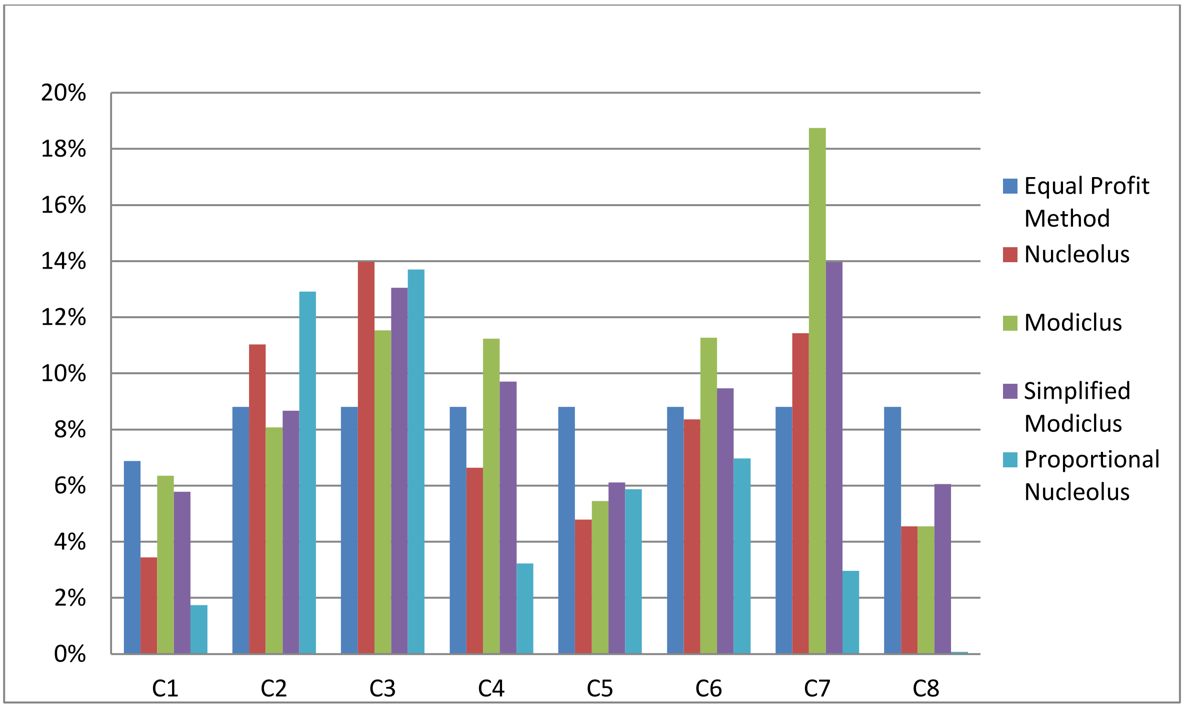


Figure 5.8.1 compares the relative savings obtained by the different allocation methods when *m*=8, i.e. the grand coalition is formed. The nucleolus and the modiclus agree on the cost allocated to player C8, but the rest of the costs differ significantly. EPM gives equal relative saving to all players, except for C1. Player C1, C5 and C8 attain the highest cost savings by

using EPM. This implies that they have low contribution in the coalitions they participate in, and are therefore better off with an equal allocation. Compared to EPM, the nucleolus gives higher relative savings to player C2, C3 and C7, due to their high constructive power. The modiclus favors player C3, C4, C6 and C7 which implies that these players obtain relative high blocking power.

Player C1, C4, C6, C7 and C8, who operate with low volumes, are clearly worst off by using the proportional nucleolus. As the method adjusts for volume when the cost savings are allocated, the small players will no longer benefit by having low stand-alone costs. Compared to using the nucleolus, player C2 and C5 are the only players that will benefit by using the proportional nucleolus method in grand coalition. These players operate with largest volumes. The nucleolus, the simplified modiclus and the modiclus roughly follow the same pattern when distributing the cost savings among the players. This may indicate that also the simplified modiclus and the modiclus provide imbalanced results for players with different volumes. The relative savings obtained for player C7 ranges from 2.96 % to 18.75 % depending on which allocation method is chosen. These extremes appear by using the proportional nucleolus and the modiclus. We interpret this as a consequence of player C7 having low constructive power, high blocking power and a very low volume.

By comparing the results between EPM and the proportional nucleolus, we can directly observe how differences in constructive power among the players affect their obtained savings. The disparities between the two allocations are completely due the constructive power of each player. Player C2 and C3 are the only players who attain higher relative savings by using the proportional nucleolus, which is explained by their high constructive power.

As seen in the results above, the different allocation methods provide very different cost allocation among the players. It is a difficult, if not impossible, task to determine which allocation method provides the fairest solution. For m=2 the stand-alone costs are the only available information about the players; hence it is impossible to say which player is more efficient or has the most power. As the players have equal contribution to the coalition, it seems reasonable to divide the achieved cost savings equally within each coalition. The nucleolus, the modiclus and the simplified modiclus follow this approach, and both players within each coalition attain the same saving in absolute value. It can, however, be argued that a fair allocation should *benefit* the players equally. Dividing the absolute cost savings equally results in a much greater benefit for players with low stand-alone cost than for players with high stand-alone cost. EPM and the proportional nucleolus avoid this issue by allocating the costs proportional to stand-alone costs, such that both players within each coalition attain the same relative saving. There is, however, no definite answer to which method provides the fairest solution, i.e. whether the savings should be considered in absolute terms or relative terms.

The concept of fair allocation is even more complex for $m \geq 3$. In addition to decide whether to consider savings in absolute terms or relative terms, we now have to take into account potential differences in efficiency among the players. When some players contribute to the coalition more than others, it is plausible to assume that a fair allocation should allocate less cost to these players. The nucleolus, the proportional nucleolus and, to some degree, EPM base the allocations on constructive power. This implies that only the most efficient coalitions are associated with binding constraints, thus affect the cost allocation. In this way players with high contribution are rewarded by attaining a relatively low cost.

It can be argued that a fair allocation method should consider not only the cost savings a coalition k can achieve, but also the loss in cost saving other players experience if k deviates from the formed coalition. The modiclus and the simplified modiclus follow this reasoning by allocating the costs according to both constructive power and blocking power. When we consider blocking power, the least efficient coalitions will give binding constraints. In the modiclus, inefficient coalitions affect the allocation through the minimization of the difference in excess within all pairs of coalitions. In an equal allocation, the excess of inefficient coalitions is higher than the excess of efficient coalitions. In order to level the excesses, the model both allocates *less* cost to players in efficient coalitions and *more* costs to players in inefficient coalitions. In the simplified modiclus, inefficient coalitions affect the solution through the second term of the excess-formula; $\frac{1}{2}\left(cost_N - cost_{N \setminus k}\right)$. This term is low when the coalition $N \setminus k$ is inefficient; hence the constraint is likely to be binding. In this way considering blocking power leads to higher allocated cost to players with inefficient sub-coalitions.

The modiclus and the simplified modiclus both reward efficient coalitions and punish inefficient coalitions, while the nucleolus and the proportional nucleolus only reward the efficient coalitions. The main features of the allocation methods we have discussed are summarized in table 5.8.1.

Table 5.8.1. Properties of the allocation methods

Method	Adjusting for Volume	Constructive Power	Blocking Power
EPM	YES	(YES)	NO
Nucleolus	NO	YES	NO
Modiclus	NO	YES	YES
Simplified Modiclus	NO	YES	YES
Proportional Nucleolus	YES	YES	NO

There are positive and negative attributes concerning all methods. It can be argued that EPM considers constructive power, as the method satisfies rationality constraints. However, the surplus of cost savings after the rationality constraints are satisfied is divided equally among the players, proportional to their stand-alone costs. In situations where the cost savings in the formed coalition are sufficiently high, the rationality constraints will not be binding. This results in equal relative savings to all players – regardless of their constructive power. Consequently, whether EPM considers constructive power or not, depends on the magnitude of the benefits from collaboration in the formed coalition. In most of the formed coalitions in the forestry case, the rationality constraints are not binding, hence the EPM allocation does not reflect the player's constructive power. In one way, it can be argued that EPM provides a fair allocation since the players achieve relative savings as similar as possible. However, as we have seen the players differ in terms of efficiency and contribution to coalitions. Treating all players equally, without considering constructive power and blocking power, might therefore not be the fairest solution.

EPM and the proportional nucleolus are the only methods that operate with cost savings in relative terms when allocating costs. In our opinion, adjusting for volume is crucial when the volumes differ significantly among the players. We therefore suspect the nucleolus, the modiclus and the simplified modiclus to give imbalanced results.

Although most important, fairness is not the only aspect that should be considered when choosing an allocation method. When it comes to the acceptance of cost allocation among companies, EPM has some desirable properties. The method is easier to understand and is therefore more acceptable for the planners. EPM is the only method that directly assures a stable solution, since it incorporates rationality constraints. However, stability is likely to follow in the other allocation methods as well, since they allocate the costs according to constructive and/or blocking power. Another aspect worth considering is the computational complexity of the models. The modiclus measures the difference in excess between all possible pairs of coalitions, and therefore involves high computational complexity. By considering the difference in excess only between pairs of complementary coalitions, the simplified modiclus inherits convenient properties from the modiclus, but avoids its high computational complexity.

6. Individual Cost Minimization

In the previous analysis we have found the coalition structure for each upper bound *m* by minimizing the total cost of all coalitions. These coalition structures are optimal from a social planner point of view, where the aim is to maximize social welfare. It can be argued that the goal of any individual player is to minimize its own cost, without considering the potential cost-increase this may lead to for other players. In the following chapter we focus on the company's perspective, where the aim is to minimize each individual cost while keeping stability for the players conformed in the coalition.

6.1 Model 2

In order to find optimal coalition structures in the individual cost minimizing problem we construct the following model in AMPL, referred to as *Model 2*. We start by introducing the notation on sets, parameters and decision variables.

Sets

N: set of players

K: set of coalitions

Parameters

$\alpha_{j,k} = \begin{matrix} 1 & \text{if player } j \text{ belongs to coalition } k \\ 0 & \sim \end{matrix}$

$cost_k$: total cost of coalition k

$scost_j$: stand − alone cost of player j

$size_k$: number of players in coalition k

m: maximum number of players allowed

$\hat{u}_{j,k}$: cost allocated in alternative coalitions (proportional)

Variables

$x_k = \begin{matrix} 1 & \text{if coalition } k \text{ is formed} \\ 0 & \sim \end{matrix}$

$w_j = \begin{matrix} 1 & \text{if player } j \text{ participates in formed coalition} \\ 0 & \sim \end{matrix}$

$u_{j,k}$: cost allocated to player j in coalition k

Objective function

$$\text{minimize} \sum_{k \in K} u_{\bar{j},k} \qquad \forall \bar{j} \in \{C1, .., C8\} \tag{6.1}$$

Constraints

$$\hat{u}_{j,k} * \sum_{i \in N} scost_i * \alpha_{i,k} = cost_k * scost_j * \alpha_{j,k} \qquad \forall j \in N, k \in K \tag{6.2}$$

$$\sum_{j \in N} \alpha_{j,k} * \hat{u}_{j,k} = cost_k \qquad \forall k \in K \tag{6.3}$$

$$\sum_{k \in K} u_{j,k} * \alpha_{j,b} \leq \hat{u}_{j,b} + (1 - w_j) * T + \alpha_{\bar{j},b} * T$$
$$\forall j \in N, b \in K: \; size_b \leq m, \bar{j} \in \{C1, .., C8\} \tag{6.4}$$

$$\alpha_{j,k} * x_k + \alpha_{\bar{j},k} * x_k \leq 1 + w_j \qquad \forall k \in K, j \in N, j \neq \bar{j}, \bar{j} \in \{C1, .., C8\} \tag{6.5}$$

$$w_j \leq \sum_{k \in K} \alpha_{j,k} * \alpha_{\bar{j},k} * x_k \qquad \forall j \in N, j \neq \bar{j}, \bar{j} \in \{C1, .., C8\} \tag{6.6}$$

$$\sum_{j \in N} \alpha_{j,b} * u_{j,k} \leq cost_b * x_k \qquad \forall k, b \in K \tag{6.7}$$

$$\sum_{j \in N} \alpha_{j,k} * u_{j,k} = cost_k * x_k \qquad \forall k \in K \tag{6.8}$$

$$\sum_{k \in K} \alpha_{j,k} * x_k = 1 \qquad \forall j \in N \tag{6.9}$$

$$\sum_{j \in N} \alpha_{j,k} * x_k \leq m \qquad \forall k \in K \tag{6.10}$$

$$x_k \in \{0,1\}, w_j \in \{0,1\}, u_{j,k} \geq 0, \hat{u}_{j,k} \geq 0 \qquad \forall j \in N, k \in K \tag{6.11}$$

The objective function (6.1) minimizes the cost of player $\bar{J}$, where $\bar{J}$ represents each of the eight players in turn. Constraints (6.2) allocate the total cost of coalition k between the players participating in k, using the proportional cost allocation method, cf. chapter 3.1. These constraints allocate the costs of all possible coalitions, not only the coalition actually formed. The idea behind this operation is to get a base-allocation, $\hat{u}_{j,k}$, where all players have equal relative savings.

Constraints (6.3) correspond to the efficiency condition for the alternative cost, $\hat{u}_{j,k}$, which assures that the sum of the costs allocated to the players in each coalition k equals the total cost of coalition k. Constraints (6.4) are rationality conditions and provide strong stability for the players who join the coalition player $\bar{J}$ initiates. These constraints assure that the players in the formed coalition are allocated cost, $u_{j,k}$, less than, or equal to, their lowest alternative cost, $\hat{u}_{j,k}$. The binary decision variable, w_j, takes value one if player j joins the formed coalition, and zero otherwise. T is a large number (in this case 50000) which ensures that the cost allocated, $u_{j,k}$, is less than the alternative cost only for the players who joins the coalition. The parameter $\alpha_{\bar{J},b}$ is one for all coalitions b where the initiating player participates. By including the term $\alpha_{\bar{J},b} * T$, the rationality conditions account only for the alternative coalitions that do not include the initiating player. This method opens up for a non-proportional cost allocation in the formed coalition, where the most efficient players get rewarded by attaining a relatively lower cost.

Constraints (6.5) and (6.6) are logical relationships that define w_j to be one if and only if player j joins the coalition initiated by player $\bar{J}$. Constraints (6.7) are rationality conditions assuring weak stability for the formed coalition. These constraints state that the participants in the formed coalition cannot attain a lower cost by forming a smaller coalition. Constraints (6.8) correspond to the efficiency condition for the variable $u_{j,k}$ which assures that the sum of the cost allocated between the players in the formed coalition equals the total cost of that coalition . Constraints (6.9) are logical constraints making sure that each player is assigned to one and only one coalition. Constraint (6.10) states the maximum number, m, of participants allowed in each coalition.

The total cost minimization-problem in the previous chapter provides one unique solution for each cardinality *m*. It is more problematic to find optimal coalition structures when the objective is to minimize individual costs. In a game where a single player motivates the formation of a coalition, the final structure will depend on who this player is. In other words, the coalition structures will change according to the cost-minimizing player. This is evident when the maximum cardinality of participants in a coalition is small. For instance, when *m*=2 player C1 prefers to collaborate with player C4, while C4 prefers to collaborate with C8 and so on. Different preferences make it challenging to determine a final coalition structure when minimizing the individual costs. The questions of which coalitions are likely to form and which player's preferences should be prioritized need to be addressed.

We investigate two different approaches in order to derive final coalition structures. The first approach is based on the assumption that when all players in a potential coalition have this particular coalition as their cost-minimizing alternative, the coalition will form. We refer to this occurrence as the players having *mutual preferences*. The second approach is based on the assumption that some players will possess more power than others, thus will be in a position to compel their own preferences. In both approaches we utilize *Model 2* to find optimal coalition structures and allocate the costs among the players.

6.2 Coalition Structures based on Mutual Preferences

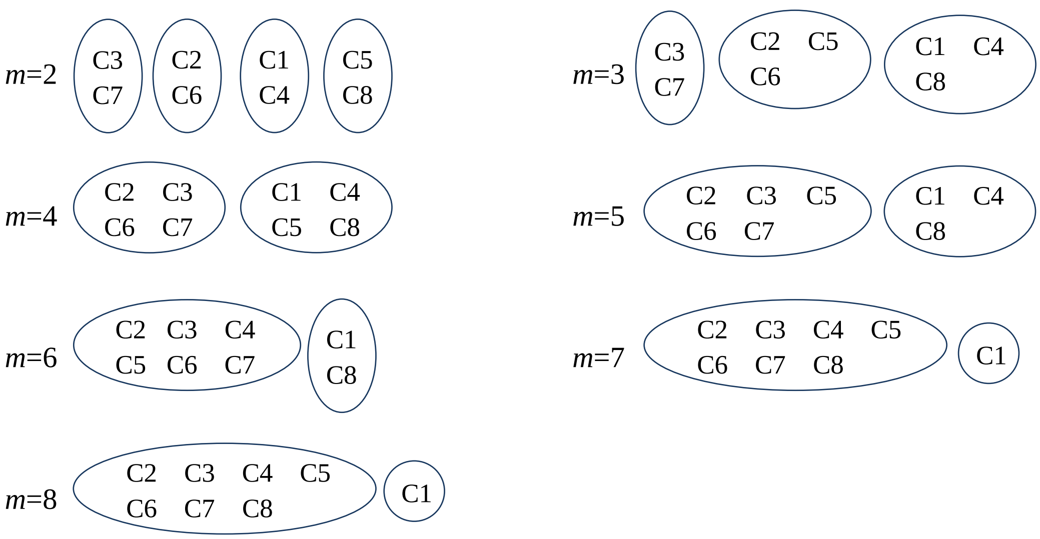
As mentioned above, it is difficult to predict the coalition structures when the players have different preferences in which coalitions they want to realize. However, there are some cases where all participants have the same coalition as their best option, i.e. the players have mutual preferences. In this chapter we assume that a coalition will form if and only if all participants have mutual preferences. This approach is referred to as *Method 2.1*. We implement *Model 2* in AMPL, and minimize the cost for each player in turn to find their cost-minimizing alternative under each upper bound *m*.

Table 6.2.1. Coalitions preferred in each player's minimization problem

m	min u_{C1}	min u_{C2}	min u_{C3}	min u_{C4}	min u_{C5}	min u_{C6}	min u_{C7}	min u_{C8}
2	11	16	25	30	18	13	25	8
3	46	60	25	75	59	60	25	82
4	95	135	135	75	132	135	135	8
5	170	204	204	201	204	204	204	8
6	222	240	240	240	240	240	240	8
7	247	254	254	254	254	254	254	254
8	255	254	254	254	254	254	254	254

Table 6.2.1 shows each player's preferred coalition when minimizing its own costs. In coalition 25, 135, 204, 240 and 254, all participants have mutual preferences; hence these coalitions are formed. In further analysis, we assume that the collaborative game appears sequentially. Players with mutual preferences form coalitions and thus get removed from the continuing game. As more players are removed, fewer alternatives remain for the players still available. Consequently, the preferences of the remaining players change and new coalitions form. This procedure continues until the final coalition structure is settled. Figure 6.2.1 illustrates the coalition structures based on mutual preferences.

Figure 6.2.1. Coalition structures based on mutual preferences



In the main, the coalition structures found when minimizing individual cost differ from the coalition structures found when minimizing total cost. In cases where companies minimize individual cost, each player will prefer coalitions that are optimal for them, not necessarily beneficial to the whole game. However, there are three coalitions that form in both approaches; coalition {C3, C7} for *m*=2 and coalition {C2, C3, C5, C6, C7} and {C1, C4, C8} for *m*=5.

The preferences of each player are based on the individual cost minimizing problem, i.e. each player requires the whole gain after satisfying rationality constraints. Naturally, all players cannot attain the whole surplus at the same time, thus we must find a method for allocating the benefit of collaborations. In this chapter we distribute the cost savings attained from collaboration as equal as possible within the coalitions. In order to find a stable solution where the differences in relative savings are minimized, we implement the EPM model formulated in chapter 5.2. Table 6.2.2 shows the allocated costs to each player within each coalition found by considering mutual preferences.

Table 6.2.2. Cost allocation in coalitions based on mutual preferences

m	C1	C2	C3	C4	C5	C6	C7	C8	TOTAL
1	3780	14860	4740	2070	10340	4960	1880	330	42960
2	3728	14275	4396	2042	10330	4765	1744	330	41610
3	3719	14007	4396	2037	9747	4675	1744	325	40650
4	3718	13657	4356	2036	10171	4559	1728	325	40550
5	3719	13582	4333	2037	9452	4534	1718	325	39700
6	3771	13551	4323	1888	9430	4523	1715	329	39530
7	3780	13551	4323	1888	9430	4523	1714	301	39510
8	3780	13551	4323	1888	9430	4523	1714	301	39510

The costs for $m=1$ represent the stand-alone cost of each player. For $m \geq 2$ the allocated cost to each player is a result of which coalition the player participates in. EPM aims to allocate the costs such that all participants within a coalition attain equal relative saving, which is evident by studying table 6.2.3.

Table 6.2.3. Relative savings in coalitions based on mutual preferences

m	C1	C2	C3	C4	C5	C6	C7	C8
2	1.37 %	3.94 %	7.25 %	1.37 %	0.09 %	3.94 %	7.25 %	0.09 %
3	1.62 %	5.74 %	7.25 %	1.62 %	5.74 %	5.74 %	7.25 %	1.62 %
4	1.63 %	8.09 %	8.09 %	1.63 %	1.63 %	8.09 %	8.09 %	1.63 %
5	1.62 %	8.59 %	8.59 %	1.62 %	8.59 %	8.59 %	8.59 %	1.62 %
6	0.24 %	8.80 %	8.80 %	8.80 %	8.80 %	8.80 %	8.80 %	0.24 %
7	0.00 %	8.81 %	8.81 %	8.81 %	8.81 %	8.81 %	8.81 %	8.81 %
8	0.00 %	8.81 %	8.81 %	8.81 %	8.81 %	8.81 %	8.81 %	8.81 %

Table 6.2.3 shows the relative savings obtained by each player using EPM. The cost saving in each coalition is sufficiently large to satisfy all rationality constraints with additional surplus, i.e. none of the rationality constraints are binding. All players within each coalition therefore attain equal relative savings.

6.3 Coalition Structures based on Power Delegation

As previously stated, some players are more crucial than others in terms of contribution to the coalition. Thus, it is likely to assume that some players will possess more power than others. In this section we assume that the most powerful player will be in the position to minimize cost, and thereby initiate coalitions based on own preferences. This approach is referred to as *Method 2.2.* We define power as a weighted average of constructive and blocking power under each upper bound, *m*. Table 6.3.1 shows the priority of cost-minimizing players for each *m*.

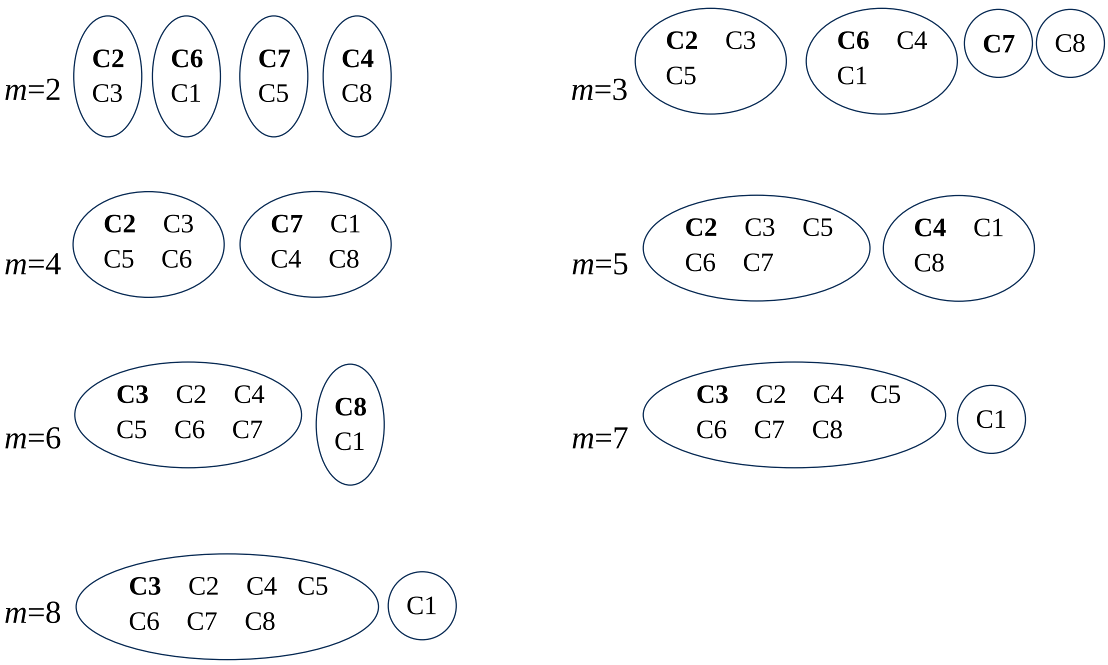
Table 6.3.1. The priority of cost-minimizing players for each *m*

m	First	Second	Third	Forth	Fifth	Sixth	Seventh	Eight
2	C2	C3	C7	C6	C4	C5	C1	C8
3	C2	C3	C7	C6	C4	C5	C1	C8
4	C2	C3	C7	C6	C4	C8	C1	C5
5	C2	C3	C7	C6	C4	C8	C1	C5
6	C3	C2	C7	C6	C4	C8	C1	C5
7	C3	C7	C8	C2	C6	C4	C1	C5
8	C3	C2	C7	C6	C4	C8	C1	C5

We implement *Model 2* in AMPL in order to find a unique coalition structure under each upper bound *m*. The process of minimizing costs of the most powerful players is explained in the following.

According to the priority derived in table 6.3.1, player C2 is the first player to minimize costs for all $m \leq 5$. After satisfying the rationality constraints to the players conformed in the coalition, the cost-minimizing player attains the remaining surplus from the collaboration. For *m*=2, player C2 minimizes cost by collaborating with player C3. Assuming that coalition {C2, C3} is stable; the two players are removed from the game. Following a sequential approach, player C7 has the highest measure of power among the remaining players, and is therefore the next player to minimize cost. However, due to the requirement of satisfying rationality constraints, player C7 does not benefit from initiating a coalition in this stage of the game. Consequently, player C6 gets in the position to minimize cost, and initiates a coalition with player C1. With coalition {C1, C6} removed from the game, player C7 affords to satisfy the rationality constraints of player C5. This procedure continues until all players are settled in coalitions. This method of determining the final coalition structure applies for all upper bounds, *m*. For $m \geq 6$, player C3 possesses most power, and is therefore the first cost-minimizing player. The final coalition structure for each upper bound *m* is illustrated in figure 6.3.1.

Figure 6.3.1. Coalition structures based on power delegation



The player written in bold is the cost-minimizing player in each coalition, i.e. the player that possesses the most power in each step. The coalition structures found in this approach are based on the preferences of the most powerful players, which neither considers the total game nor conform coalitions based on mutual preferences. However, the coalition structures derived by delegating power match the solutions found in the two other approaches remarkably well. For *m*=2, coalition {C1, C6} and {C4, C8} coincide with the optimal coalitions found by minimizing total cost. For *m*=3 and *m*=4, the whole coalition structures found by delegating power coincide with the total cost minimizing problem. For *m*=5, all three approaches provide the same coalition structure. For $m \geq 6$, the coalition structures based on power delegation differ from the total cost minimizing structure, but coincide with the coalition structure based on mutual preferences.

The individual cost-minimizing problem does not emphasize the aspect of fairness and all players are assumed to act in their own self-interest. We do not use any particular method to allocate the cost attained in the coalitions found by delegating power. As the most powerful player initiates the coalitions, this player is assumed to receive the whole surplus after satisfying the rationality constraints to the conformed players. Table 6.3.1 shows the cost assigned to each player within the coalition structures found by delegating power.

Table 6.3.1. Cost allocation in coalitions based on power delegation

m	C1	C2	C3	C4	C5	C6	C7	C8	TOTAL
1	3780	14860	4740	2070	10340	4960	1880	330	42960
2	3728	14214	4396	2050	10265	4782	1825	330	41590
3	3719	13600	4396	2037	9913	4705	1880	330	40580
4	3719	13216	4396	2037	9873	4684	1870	325	40120
5	3771	12980	4396	1980	9819	4681	1744	329	39700
6	3771	13828	3688	1926	9622	4616	1750	329	39530
7	3780	13828	3681	1926	9622	4616	1750	307	39510
8	3780	13828	3681	1926	9622	4616	1750	307	39510

The costs assigned to each player depend on the rationality constraints as well as the distribution of power among the participants. Table 6.3.2 shows the relative saving each player obtains in the different coalitions.

Table 6.3.2. Relative savings in coalitions based on power delegation

m	C1	C2	C3	C4	C5	C6	C7	C8
2	1.38 %	4.35 %	7.26 %	0.97 %	0.73 %	3.59 %	2.93 %	0.00 %
3	1.61 %	8.47 %	7.26 %	1.59 %	4.13 %	5.14 %	0.00 %	0.00 %
4	1.61 %	11.06 %	7.26 %	1.59 %	4.52 %	5.56 %	0.53 %	1.52 %
5	0.24 %	12.65 %	7.26 %	4.35 %	5.04 %	5.63 %	7.23 %	0.30 %
6	0.24 %	6.94 %	22.19 %	6.96 %	6.94 %	6.94 %	6.91 %	0.30 %
7	0.00 %	6.94 %	22.34 %	6.96 %	6.94 %	6.94 %	6.91 %	6.97 %
8	0.00 %	6.94 %	22.34 %	6.96 %	6.94 %	6.94 %	6.91 %	6.97 %

For $m \geq 4$, we clearly see the advantage of being the cost-minimizing player. The achieved cost savings are in general higher for larger coalitions, but this also involves that the rationality constraints are more costly to satisfy. In the forestry case, the first effect outweighs the second, and the cost-minimizing player obtains high relative savings. However, the exceptionally high relative savings of player C3 when he minimizes own cost for $m \geq 6$ is explained by his relatively low stand-alone cost. The similarity in relative savings obtained by the included participants is a consequence of the rationality constraints being based on proportionally allocated costs.

6.4 Discussion of *Model 2*

In order to provide stable allocations, *Model 2* satisfies strong rationality constraints based on each player's alternative cost. As stated, the model allocates costs in alternative coalitions by using the proportional cost allocation method. Nevertheless, allocating the alternative cost equally between the players carries a potential weakness of the model. The method does not provide a stable solution, and it can be discussed whether these costs reflect the actual alternatives a player has or not.

One way to avoid this problem is to formulate strong stability condition based on the total cost of the alternative coalitions instead of the allocated cost to each player.

$$\sum_{i \in N} \left(\alpha_{i,k} \sum_{k \in K} u_{i,k} \right) \leq cost_b \cdot w_j + T \cdot (1 - w_j) \qquad \forall b \in K, j \in N, size_b \leq m, \alpha_{j,b} = 1 \qquad (6.12)$$

By replacing the strong rationality constraints (6.4) by constraints (6.12), in addition to some other minor modifications, we can circumvent to allocate the cost in alternative coalitions. Constraints (6.12) provide strong stability for the players who join the coalition of player $\bar{J}$, by stating that the sum of allocated costs to all participants of coalition b is less than, or equal to, the total cost of coalition *b*. This method is referred to as *M2* and is proposed by Guajardo and Rönnqvist (2013).

The strong stability constraints based on the total cost of each coalition are stricter than the strong stability constraints based on the allocated alternative cost to each player. In *Model 2* the total cost of each coalition is allocated by using the proportional cost allocation method, and the model satisfies rationality constraints based on equal relative savings among the participants in all alternative coalitions. In *M2* the cost of the alternative coalitions are not allocated beforehand. In order to ensure stability, the initiating player must satisfy the strictest possible rationality constraints for the players included in his coalition. This means that he must satisfy the rationality constraints assuming that these players will attain the whole excess of achieved cost savings in the alternative coalitions. As an example we consider the case where *m*=6 and player C1 minimizes his cost. *M2* finds it optimal for C1 to collaborate with player C2, C3, C5, C6 and C7, while player C4 and C8 form their own coalition. Coalition {C4, C8} achieves a total cost of 2380. In *M2*, player C1 must ensure that his collaborators are allocated less than they attain by collaborating with C4 and/or C8 instead, assuming that C4 and C8 attain the same costs as before. For instance, C2 must be allocated less than the total cost of coalition {C2, C4, C8} minus the cost of {C4, C8}. This gives the equation: $u_{c2} \leq cost_{66} - (u_{C4} + u_{C8})$, which gives $u_{c2} \leq 16960 - 2380 = 14580$. The same logic applies for all coalitions and all players. By using the dual function in AMPL, coalition *k*=240 where all players but C1 and C8 collaborate, is the only coalition with positive dual value. This gives the equation $u_{C2} + u_{C3} + u_{C5} + u_{C6} + u_{C7} \leq cost_{240} - u_{C4}$, which gives $u_{C2} + u_{C3} + u_{C5} + u_{C6} + u_{C7} \leq 35430 - 2050 = 33380$. Consequently, the total cost allocated to player C2, C3, C5, C6 and C7 must be less than, or equal to, 33380. The total cost of the coalition {C1, C2, C3, C5, C6, C7} is 37140, hence player C1 is left with a cost of 3760. In all alternative coalitions player C4 and C8 attain total cost of 2380. This means that the collaborators of C1 retain the whole gain in alternative coalitions. Additivity implies that any coalition containing six players will be more efficient than any coalition containing only two players. It can therefore be argued as an unrealistic assumption that C4 is allocated the same cost as before. This method gives strict conditions for the player who is minimizing its cost, in this case player C1.

As seen from the example, the strict rationality conditions in *M2* may gain the included players in a coalition, but is a disadvantage for the cost-minimizing player. The price of assuring stability may give incentives for players to speculate and wait for another player to make the first move. However, there is no guarantee that the first mover will act in their favor and they might not get included in the formed coalition. In some cases it is too expensive for the cost-minimizing player to satisfy the rationality conditions of the other players; hence no coalitions are formed. This results in less collaboration and loss of cost savings – which is negative for all parties.

We avoid this issue in *Model 2* by ensuring equal relative savings in all alternative coalitions. The cost-minimizing player must satisfy rationality conditions, assuming that the cost in other coalitions is divided among the players proportionally to their stand-alone costs. In *Model 2*, player C4 would attain a lower cost than 2050 in coalition k=240, which in turn gives player C1 the possibility to allocate a higher cost than 33380 to the other players. Hence, the rationality conditions are easier to satisfy for the cost-minimizing player. This is apparent when we compare the results of *Model 2* and *M2*. For m=6, *Model 2* gives coalitions containing six players when player C1, C2, C3, C4, C5, C6 and C7 minimize their cost. Only player C8 cannot afford to initiate collaboration. In *M2*, player C1 is the only player who can afford to initiate a coalition containing six players. Player C4 affords to initiate collaboration with player C8, i.e. a coalition containing two players. The other players cannot afford to initiate coalitions when they minimize their cost, thus they are allocated their stand-alone cost. This is obviously not an optimal coalition structure, and all players would benefit if the rationality conditions were less strict and collaborations would form.

As explained above it may be beneficial to allocate the costs of alternative collaborations before satisfying rationality conditions. Finding realistic base-allocations is however challenging and the issue could be further explored. Using the proportional cost allocation method may result in unfair allocations as the costs are divided without considering the player’s contribution. However, the proportional cost allocation is often preferred by decision-makers because of its easiness to understand and compute (Frisk et al., 2010).

Method 2.1 assumes that in situations where all players have mutual preferences in forming a coalition – the coalition is formed. This is a reasonable assumption, as the players act according to their own self-interest. However, the method of determining the preferences of each player carries a potential weakness. The optimal collaboration for a player is found by minimizing the player's cost. This involves that after the rationality constraints of the collaborators are satisfied, the cost-minimizing player retains the excess of achieved cost savings. This accounts for all players in the formed coalition; hence all participants expect to retain the excess of the achieved cost savings. Naturally, the same amount of excess cannot be distributed more than once and the players must find a way to allocate the costs. It is, however, no guarantee that the preferences of the players remain the same when the achieved cost savings are distributed among all participants. The problem of finding each player's preferences based on the final cost allocation remains open for further research.

Method 2.2 assumes that a single player is in the position to compel his own preferences and thereby initiate his cost-minimizing coalition. As the potential benefits of being the cost-minimizing player are substantial, the method requires an accurate measure for determining which player should be prioritized. In some instances it is expected that the biggest player initiate collaborations. Other relevant aspects may be strategic positions in the market, collaborative alliances in the history and financial strengths. We assume that the most *powerful* player is prioritized, and base our calculations on the measure of constructive and blocking power. These measures are averages of all coalitions each player participates in, and may therefore give imprecise results. Pushed to extremes, this may result in incorrect ranking of players.

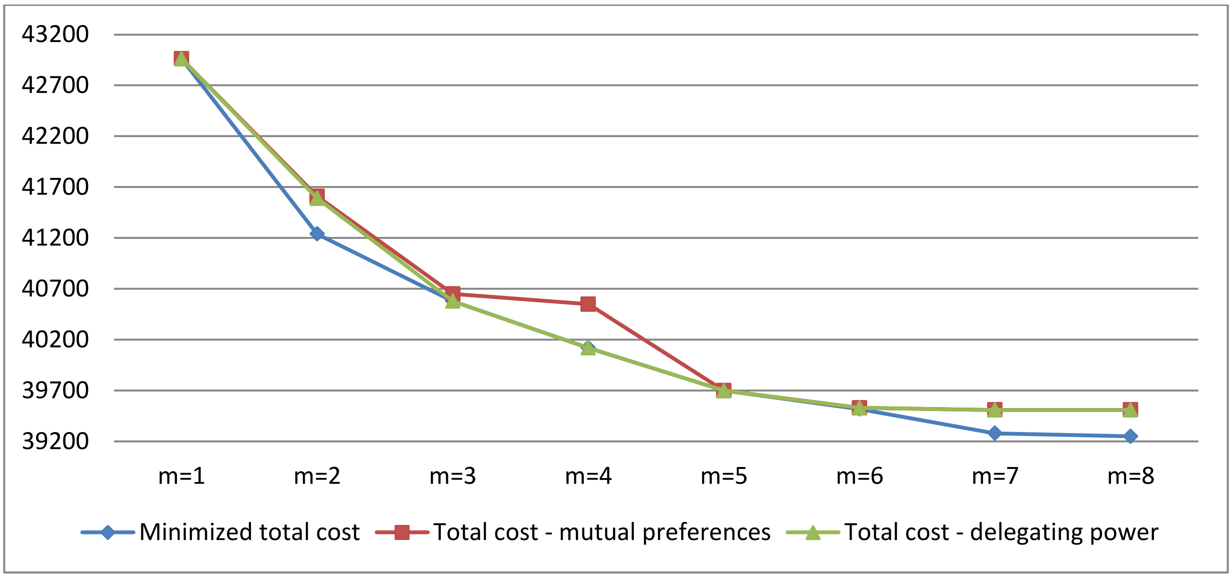
Even if the ranking of players is reasonable, the concept of delegating all power to a single player in turn carries another potential weakness. As the most powerful player initiates the coalition providing highest cost savings, the coalition formed will likely include other efficient players, i.e. players with relatively high power. The conformed players accept to collaborate as long as their rationality constraints are satisfied. However, these players will not get in the position to initiate their best alternative, even though they might be the second or third most powerful player. Consequently, less powerful players will have the advantage of being the next player to minimize cost.

7. Total cost minimization vs. individual cost minimization

In this paper we have studied the potential for collaboration among companies. In chapter 5 we approach the subject by minimizing total cost. Here we find a coalition structure for each cardinality *m* that generates lowest obtainable cost within the total game. This solution is derived from a social planner point of view and we find the social welfare maximizing solution. However, as the players are competitors, it is expected that each company will be concerned about minimizing its own cost, rather than the total cost of the game. Choosing the cost minimizing coalition from an egocentric perspective, not considering the inconvenience this may lead to for other players, could result in both a lower individual cost and a competitive advantage. However, there are issues concerning the availability of the information needed in order to initiate individual cost minimizing coalitions (Guajardo & Rönnqvist, 2014). Unwillingness to share sensitive information that other players might exploit, can complicate the forming of coalitions. One possible solution is to engage an impartial third party to gather the necessary information and come up with suggestions on how to implement the collaborations among the players. An *impartial* third party implies that the objective will be to minimizing the total costs, as we did in chapter 5.

Model 1, *Method 2.1* and *Method 2.2* represent three different approaches for finding optimal coalition structures. The total costs depend solely on the coalition structures, and will not be influenced by the cost allocations. The total costs for each cardinality *m*, obtained by the different approaches, are shown in figure 6.5.1.

Figure 6.4.1. Comparing total cost for each cardinality *m*



Deviation from the total cost-minimizing solution may be interpreted as the social welfare loss. The welfare loss is in general higher in the coalition structures based on mutual preferences. For *m*=2 this approach provides a solution with a welfare loss of 370, while the welfare loss in the approach of delegating power is 350. For *m*=3 and *m*=4 the approach of delegating power gives the total cost-minimizing solution, while considering mutual preferences gives welfare losses of 70 and 430, respectively. For *m*=5, all three approaches find the same coalition structure to be optimal, thus provide the solution which gives lowest achievable total cost. This implies that the coalition structure {C1, C4, C8} {C2, C3, C5, C6, C7} is optimal both from the social planner point of view and from the individual player's point of view. It is plausible to assume that these coalitions will form when the maximum allowed participants in a coalition is five. For *m*=6, *m*=7 and *m*=8, the approaches of considering mutual preferences and delegating power provide the same total costs, which compared to the optimal solution gives welfare losses of 10, 230 and 260, respectively.

Roughly speaking, the three approaches provide nearly the same total costs. The highest welfare loss, occurred when considering mutual preferences for *m*=4, represents an increase in total cost of 1.07 % compared to the optimal solution. This may be regarded as a trivial welfare loss and may further imply that the solution obtained when each player acts egocentric is constructive to the society as a whole. On average considering mutual preferences gives an increase in total cost of 0.49 %, while the approach of delegating power gives an increase in total cost of 0.30 %.

8. Other Applications

Cost allocation problems arise in many real life situations where benefits are achieved from collaboration. Examples include group purchasing, capacity and inventory information sharing and resource pooling. Specific industries that emphasize the benefits of collaboration are grocery distribution, shipping companies concerned with effective route scheduling, and railway transportation (Lozano et al, 2013). Besides the broad use in collaborative logistics and supply chain management, there are high potential for collaborative activities within the health care industry as well as in various service sectors. Health-care institutions identify not only financial benefits through collaboration, but argue for better educational systems, synergy effects through joint research activities and enhanced quality and performance (Lasker, 1997). Within the service sector, pooling of employers and resource networks might lead to higher productivity and cost-effectiveness. As an example of cases where the cost allocation methods are applicable, we implement the models in a case concerning inventory pool of spare parts introduced by Guajardo and Rönnqvist (2014).

8.1 Inventory Pool of Spare Parts

Holding inventory of spare parts is essential to cope with failures in production as well as managing demand fluctuations. When different inventory plants use the same equipment, collaboration through pooling of spare parts may lead to significant cost savings. We consider an inventory pool of spare parts where the members require different target service levels. The members will in this case represent different demand classes, which complicate the cost allocation between the members.

We consider a case containing four players; A, B, C and D. The average demand, target service level and stand-alone cost of each player are displayed in the table below.

Table 8.1.1. Data for the inventory case

N	Average Demand	Service Level	Cost
A	0.10	99 %	2020
B	0.20	95 %	2041
C	0.30	90 %	2064
D	0.40	85 %	2089

Dependent on each member's target service level, optimal base-stock levels S are computed, cf. appendix B. If a player acts independently, it will set its base stock level S and run its inventory separately from the other players. By alternatively collaborating, one single inventory will serve their total demand. In the context of collaboration it is beneficial for the players to have as similar target service levels as possible. The inventory carrying cost increases in base-stock level; hence it is favorable for the coalitions to have low base stock levels. However, the base stock level must be sufficiently high to satisfy each player's target service level. The target service level of player A is 99 %, which means that in all coalitions containing player A, the optimal service level must be at least 99 %. Player D has a target service level of 85 %, hence he will have excessive service level in collaborations with player A, carrying an unnecessary high cost. For more background on inventory cost and service level we refer to Guajardo and Rönnqvist (2014).

8.2 Cost Allocation in the Grand Coalition

We assume the grand coalition {A, B, C, D} is formed, and address the question of how the expected inventory cost of the pooling system should be allocated among the players. By constructing a dataset containing the expected cost of all possible coalitions, the cost allocation models formulated in chapter 5 are directly applicable.

We implement EPM, the nucleolus, the modiclus, the simplified modiclus and the proportional nucleolus to allocate the expected cost of the grand coalition, and obtain the following results:

Table 8.2.1. Cost allocation grand coalition

Method	A	B	C	D	TOTAL
Stand-alone cost	2020	2041	2064	2089	8214
EPM	1279	1292	1307	1323	5201
Nucleolus	1269	1290	1311	1332	5201
Modiclus	1516	1041	1062	1583	5201
Simplified Modiclus	1350	1207	1229	1416	5201
Proportional Nucleolus	1271	1290	1310	1330	5201

Table 8.2.1 shows the stand-alone costs and the cost obtained by each player using five different allocation methods. The total cost allocated to the players is reduced from 8214 to 5201 when they collaborate. This gives a total cost saving of 36.7 % and demonstrates high efficiency improvements from collaboration. There are minor differences between the allocated costs using EPM and the nucleolus. This implies that the constructive powers of the coalitions are fairly similar. The stand-alone costs of the players are approximately the same; hence the obtained results using the nucleolus roughly coincide with the obtained results using the proportional nucleolus. The question of whether we should consider savings in absolute terms or relative terms is therefore not an issue in this case. We observe bigger variation in costs allocated when using the modiclus and the simplified modiclus, and we analyze the results in terms of the relative savings obtained.

Table 8.2.2. Relative savings grand coalition

Method	A	B	C	D
EPM	36.7 %	36.7 %	36.7 %	36.7 %
The Nucleolus	37.2 %	36.8 %	36.5 %	36.2 %
The Modiclus	25.0 %	49.0 %	48.5 %	24.2 %
The Simplified Modiclus	33.2 %	40.9 %	40.5 %	32.2 %
The Proportional Nucleolus	37.1 %	36.8 %	36.5 %	36.3 %

Table 8.2.2 shows the relative savings each player obtains by using the different allocation methods. EPM provides equal relative saving to all players. When considering the player's constructive power player A obtains a slightly higher relative saving, which implies that A has relatively more efficient sub coalitions. Modiclus allocates higher savings to player B and C and lower savings to A and D. This is due to the higher blocking power of player B and C, cf. table 8.2.3. The allocation using the simplified modiclus appears as a weighted average of the nucleolus and the modiclus, weighted by approximately 0.67 and 0.33 respectively.

Table 8.2.3. Average constructive and blocking power of each player

	A	B	C	D
Average CP	26.30 %	29.70 %	29.67 %	26.17 %
Average BP	33.81 %	35.90 %	35.74 %	33.35 %

Table 8.2.3 shows the average constructive power and the average blocking power of each player. Player B and C have on average higher constructive power than player A and D. This is not compatible with the results obtained by the nucleolus. This is due to the earlier mentioned weakness of using average measures. Most of the coalitions have constructive power ranging from 24 % to 33 %, cf. appendix B. Due to the large differences in average demand and target service level of player A and D, coalition {A, D} only achieve a saving of 0.22 %. This inefficient collaboration lowers the average constructive power of player A and D significantly. However, the constraints associated with low outliers will not be binding in the nucleolus model; hence they will not affect the cost allocation. When calculating the constructive power without coalition {A, D}, we find that A has an average of 30.64 %, which is more compatible with the results provided by nucleolus.

The low saving of coalition {A, D} results in extremely high blocking power for coalition {B, C}. This is reflected in the average blocking power, and is even more evident when comparing the cost allocation using nucleolus, modiclus and simplified modiclus. As with constructive power, only the coalitions with the highest blocking power will be associated with binding constraints. The effect of the particularly high blocking power of coalition {B, C} is therefore more evident in the cost allocation, than in the averages of blocking power.

Measuring the constructive and blocking power for each player as the average of constructive and blocking power of all coalitions they participate in, results in an underestimation of high outliers and overestimation of low outliers.

8.3 Coalition Structures

To address the question of different coalition structures we include the upper bound, m, of the number of players in each coalition. We run the three models for predicting coalition structure for $m=2$ and $m=3$. *Model 1* finds the optimal coalition structure from a social planner point of view, i.e. the total cost minimizing solution, cf. chapter 5.1. *Method 2.1* predicts the coalition structures based on mutual preferences when each player minimizes his cost, cf. chapter 6.2. *Method 2.2* provide the optimal solution when the most powerful player minimizes his cost, cf. chapter 6.3.

Figure 8.3.1. Coalition structures in the inventory case

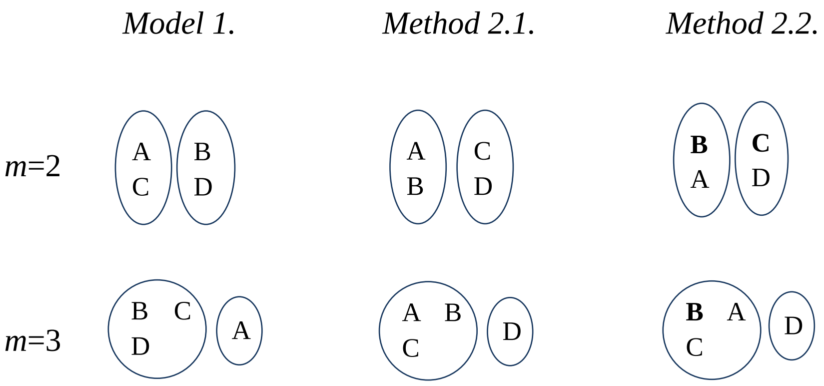


Figure 8.3.1 shows the optimal coalition structures obtained from the models presented in chapter 5 and 6. The total cost obtained from *Model 1* is 6205 for *m*=2 and 6203 for *m*=3. The additional saving of increasing *m* from 2 to 3 is in other words marginal. The two methods in the individual cost minimizing problem both provide the same coalition structure. Surprisingly, in this case the total cost increase from 6207 to 6209 when *m* increases from 2 to 3. In the inventory case the significant cost savings occur when *m* goes from 1 to 2 and from 3 to 4. This is due to generally high savings by collaboration and when *m*=3 one player is forced to operate alone.

Allocating the costs in the coalitions provided by *Model 1* gives the same pattern as the grand coalition allocation. EPM, the nucleolus and the proportional nucleolus will give similar relative savings to all players, while the modiclus and the simplified modiclus allocate less cost to player B and C. In the coalitions based on mutual preferences the obtained relative savings using EPM will be fairly equal for all players within each coalition. In the coalitions based on power, player B has an advantage from being the cost minimizing player. However, all coalitions are nearly equally efficient and after satisfying the rationality constraints of the included players, there is not much additional surplus for player B to obtain. Hence, the relative saving of each player will be similar within each coalition in this case as well. For *m*=3 player A and D are the losing parties as they are forced to operate alone.

Conclusion

By using principles in cooperative game theory, we have addressed some essential questions relevant for collaborative logistics. We have investigated how companies should group in order to achieve the highest benefits of collaboration, and how the joint costs should be allocated among the collaborators. We aimed to find optimal coalition structures by minimizing total cost in part one and minimizing the cost of individual companies in part two. Assuming that optimal coalition structures are identified, the next question concerned how the companies in a coalition should allocate the cost savings. With the intent of finding a fair cost allocation to the total cost minimizing problem, we compared the results of five well-known models for allocating the joint cost of a coalition. Interestingly, we found significant disparities between the allocations provided by each model.

Comparing the results obtained from the different allocation methods raised a fundamental question regarding fairness. Should a fair allocation be based on savings in absolute terms or savings in relative terms? The relevance of adjusting for volume is directly reflected in the differences between the results when using the nucleolus and the proportional nucleolus. Larger companies benefit from considering savings in relative terms, while smaller companies are better off when considering savings in absolute terms.

In order to investigate and analyze the intuition behind the models, we constructed measures of each company's constructive and blocking power in the formed coalitions. When some companies contribute with more cost savings than others, it can be argued that a fair allocation should allocate less cost to these companies. The nucleolus, the proportional nucleolus and, to some degree, EPM base the allocation on constructive power, while the modiclus and the simplified modiclus consider both constructive power and blocking power for allocating the joint costs. Companies with particularly efficient sub-coalitions benefit the most by using the nucleolus and the proportional nucleolus. Considering blocking power punishes companies with particularly inefficient sub-coalitions, hence the modiclus and the simplified modiclus work in the disfavor of these companies.

In the total cost minimizing problem, we found a unique coalition structure for each cardinality m. It is, however, less straightforward to derive the final coalition structure when the objective is to minimize individual cost. We investigated two different approaches for determining the final coalition structure, for which we allocated the joint cost. The coalition structures obtained by minimizing individual costs provide approximately the same total cost as the optimal solution in the total cost minimizing problem. Even though companies aim to minimize individual costs, the solutions derived are not far from optimal to the society as a whole.

Although the approach by minimizing individual cost may be reasonable, the unwillingness to share sensitive information may hinder the parties from establishing collaborations. An interesting question for further research may be to investigate the incentives to reveal information on costs, such that optimal solutions could be derived from the perspective of individual companies.

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Appendix A. Coalitions in forestry case

k	*j* ∈ *k*	k	*j* ∈ *k*	k	*j* ∈ *k*
1	{ C1 }	43	{ C1 , C3 , C4 }	85	{ C4 , C5 , C8 }
2	{ C2 }	44	{ C1 , C3 , C5 }	86	{ C4 , C6 , C7 }
3	{ C3 }	45	{ C1 , C3 , C6 }	87	{ C4 , C6 , C8 }
4	{ C4 }	46	{ C1 , C3 , C7 }	88	{ C4 , C7 , C8 }
5	{ C5 }	47	{ C1 , C3 , C8 }	89	{ C5 , C6 , C7 }
6	{ C6 }	48	{ C1 , C4 , C5 }	90	{ C5 , C6 , C8 }
7	{ C7 }	49	{ C1 , C4 , C6 }	91	{ C5 , C7 , C8 }
8	{ C8 }	50	{ C1 , C4 , C7 }	92	{ C6 , C7 , C8 }
9	{ C1 , C2 }	51	{ C1 , C4 , C8 }	93	{ C1 , C2 , C3 , C4 }
10	{ C1 , C3 }	52	{ C1 , C5 , C6 }	94	{ C1 , C2 , C3 , C5 }
11	{ C1 , C4 }	53	{ C1 , C5 , C7 }	95	{ C1 , C2 , C3 , C6 }
12	{ C1 , C5 }	54	{ C1 , C5 , C8 }	96	{ C1 , C2 , C3 , C7 }
13	{ C1 , C6 }	55	{ C1 , C6 , C7 }	97	{ C1 , C2 , C3 , C8 }
14	{ C1 , C7 }	56	{ C1 , C6 , C8 }	98	{ C1 , C2 , C4 , C5 }
15	{ C1 , C8 }	57	{ C1 , C7 , C8 }	99	{ C1 , C2 , C4 , C6 }
16	{ C2 , C3 }	58	{ C2 , C3 , C4 }	100	{ C1 , C2 , C4 , C7 }
17	{ C2 , C4 }	59	{ C2 , C3 , C5 }	101	{ C1 , C2 , C4 , C8 }
18	{ C2 , C5 }	60	{ C2 , C3 , C6 }	102	{ C1 , C2 , C5 , C6 }
19	{ C2 , C6 }	61	{ C2 , C3 , C7 }	103	{ C1 , C2 , C5 , C7 }
20	{ C2 , C7 }	62	{ C2 , C3 , C8 }	104	{ C1 , C2 , C5 , C8 }
21	{ C2 , C8 }	63	{ C2 , C4 , C5 }	105	{ C1 , C2 , C6 , C7 }
22	{ C3 , C4 }	64	{ C2 , C4 , C6 }	106	{ C1 , C2 , C6 , C8 }
23	{ C3 , C5 }	65	{ C2 , C4 , C7 }	107	{ C1 , C2 , C7 , C8 }
24	{ C3 , C6 }	66	{ C2 , C4 , C8 }	108	{ C1 , C3 , C4 , C5 }
25	{ C3 , C7 }	67	{ C2 , C5 , C6 }	109	{ C1 , C3 , C4 , C6 }
26	{ C3 , C8 }	68	{ C2 , C5 , C7 }	110	{ C1 , C3 , C4 , C7 }
27	{ C4 , C5 }	69	{ C2 , C5 , C8 }	111	{ C1 , C3 , C4 , C8 }
28	{ C4 , C6 }	70	{ C2 , C6 , C7 }	112	{ C1 , C3 , C5 , C6 }
29	{ C4 , C7 }	71	{ C2 , C6 , C8 }	113	{ C1 , C3 , C5 , C7 }
30	{ C4 , C8 }	72	{ C2 , C7 , C8 }	114	{ C1 , C3 , C5 , C8 }
31	{ C5 , C6 }	73	{ C3 , C4 , C5 }	115	{ C1 , C3 , C6 , C7 }
32	{ C5 , C7 }	74	{ C3 , C4 , C6 }	116	{ C1 , C3 , C6 , C8 }
33	{ C5 , C8 }	75	{ C3 , C4 , C7 }	117	{ C1 , C3 , C7 , C8 }
34	{ C6 , C7 }	76	{ C3 , C4 , C8 }	118	{ C1 , C4 , C5 , C6 }
35	{ C6 , C8 }	77	{ C3 , C5 , C6 }	119	{ C1 , C4 , C5 , C7 }
36	{ C7 , C8 }	78	{ C3 , C5 , C7 }	120	{ C1 , C4 , C5 , C8 }
37	{ C1 , C2 , C3 }	79	{ C3 , C5 , C8 }	121	{ C1 , C4 , C6 , C7 }
38	{ C1 , C2 , C4 }	80	{ C3 , C6 , C7 }	122	{ C1 , C4 , C6 , C8 }
39	{ C1 , C2 , C5 }	81	{ C3 , C6 , C8 }	123	{ C1 , C4 , C7 , C8 }
40	{ C1 , C2 , C6 }	82	{ C3 , C7 , C8 }	124	{ C1 , C5 , C6 , C7 }
41	{ C1 , C2 , C7 }	83	{ C4 , C5 , C6 }	125	{ C1 , C5 , C6 , C8 }
42	{ C1 , C2 , C8 }	84	{ C4 , C5 , C7 }	126	{ C1 , C5 , C7 , C8 }

k	$j \in k$	k	$j \in k$	k	$j \in k$
127	{ C1 , C6 , C7 , C8 }	170	{ C1 , C2 , C3 , C6 , C7 }	213	{ C3 , C4 , C5 , C6 , C7 }
128	{ C2 , C3 , C4 , C5 }	171	{ C1 , C2 , C3 , C6 , C8 }	214	{ C3 , C4 , C5 , C6 , C8 }
129	{ C2 , C3 , C4 , C6 }	172	{ C1 , C2 , C3 , C7 , C8 }	215	{ C3 , C4 , C5 , C7 , C8 }
130	{ C2 , C3 , C4 , C7 }	173	{ C1 , C2 , C4 , C5 , C6 }	216	{ C3 , C4 , C6 , C7 , C8 }
131	{ C2 , C3 , C4 , C8 }	174	{ C1 , C2 , C4 , C5 , C7 }	217	{ C3 , C5 , C6 , C7 , C8 }
132	{ C2 , C3 , C5 , C6 }	175	{ C1 , C2 , C4 , C5 , C8 }	218	{ C4 , C5 , C6 , C7 , C8 }
133	{ C2 , C3 , C5 , C7 }	176	{ C1 , C2 , C4 , C6 , C7 }	219	{ C1 , C2 , C3 , C4 , C5 , C6 }
134	{ C2 , C3 , C5 , C8 }	177	{ C1 , C2 , C4 , C6 , C8 }	220	{ C1 , C2 , C3 , C4 , C5 , C7 }
135	{ C2 , C3 , C6 , C7 }	178	{ C1 , C2 , C4 , C7 , C8 }	221	{ C1 , C2 , C3 , C4 , C5 , C8 }
136	{ C2 , C3 , C6 , C8 }	179	{ C1 , C2 , C5 , C6 , C7 }	222	{ C1 , C2 , C3 , C4 , C6 , C7 }
137	{ C2 , C3 , C7 , C8 }	180	{ C1 , C2 , C5 , C6 , C8 }	223	{ C1 , C2 , C3 , C4 , C6 , C8 }
138	{ C2 , C4 , C5 , C6 }	181	{ C1 , C2 , C5 , C7 , C8 }	224	{ C1 , C2 , C3 , C4 , C7 , C8 }
139	{ C2 , C4 , C5 , C7 }	182	{ C1 , C2 , C6 , C7 , C8 }	225	{ C1 , C2 , C3 , C5 , C6 , C7 }
140	{ C2 , C4 , C5 , C8 }	183	{ C1 , C3 , C4 , C5 , C6 }	226	{ C1 , C2 , C3 , C5 , C6 , C8 }
141	{ C2 , C4 , C6 , C7 }	184	{ C1 , C3 , C4 , C5 , C7 }	227	{ C1 , C2 , C3 , C5 , C7 , C8 }
142	{ C2 , C4 , C6 , C8 }	185	{ C1 , C3 , C4 , C5 , C8 }	228	{ C1 , C2 , C3 , C6 , C7 , C8 }
143	{ C2 , C4 , C7 , C8 }	186	{ C1 , C3 , C4 , C6 , C7 }	229	{ C1 , C2 , C4 , C5 , C6 , C7 }
144	{ C2 , C5 , C6 , C7 }	187	{ C1 , C3 , C4 , C6 , C8 }	230	{ C1 , C2 , C4 , C5 , C6 , C8 }
145	{ C2 , C5 , C6 , C8 }	188	{ C1 , C3 , C4 , C7 , C8 }	231	{ C1 , C2 , C4 , C5 , C7 , C8 }
146	{ C2 , C5 , C7 , C8 }	189	{ C1 , C3 , C5 , C6 , C7 }	232	{ C1 , C2 , C4 , C6 , C7 , C8 }
147	{ C2 , C6 , C7 , C8 }	190	{ C1 , C3 , C5 , C6 , C8 }	233	{ C1 , C2 , C5 , C6 , C7 , C8 }
148	{ C3 , C4 , C5 , C6 }	191	{ C1 , C3 , C5 , C7 , C8 }	234	{ C1 , C3 , C4 , C5 , C6 , C7 }
149	{ C3 , C4 , C5 , C7 }	192	{ C1 , C3 , C6 , C7 , C8 }	235	{ C1 , C3 , C4 , C5 , C6 , C8 }
150	{ C3 , C4 , C5 , C8 }	193	{ C1 , C4 , C5 , C6 , C7 }	236	{ C1 , C3 , C4 , C5 , C7 , C8 }
151	{ C3 , C4 , C6 , C7 }	194	{ C1 , C4 , C5 , C6 , C8 }	237	{ C1 , C3 , C4 , C6 , C7 , C8 }
152	{ C3 , C4 , C6 , C8 }	195	{ C1 , C4 , C5 , C7 , C8 }	238	{ C1 , C3 , C5 , C6 , C7 , C8 }
153	{ C3 , C4 , C7 , C8 }	196	{ C1 , C4 , C6 , C7 , C8 }	239	{ C1 , C4 , C5 , C6 , C7 , C8 }
154	{ C3 , C5 , C6 , C7 }	197	{ C1 , C5 , C6 , C7 , C8 }	240	{ C2 , C3 , C4 , C5 , C6 , C7 }
155	{ C3 , C5 , C6 , C8 }	198	{ C2 , C3 , C4 , C5 , C6 }	241	{ C2 , C3 , C4 , C5 , C6 , C8 }
156	{ C3 , C5 , C7 , C8 }	199	{ C2 , C3 , C4 , C5 , C7 }	242	{ C2 , C3 , C4 , C5 , C7 , C8 }
157	{ C3 , C6 , C7 , C8 }	200	{ C2 , C3 , C4 , C5 , C8 }	243	{ C2 , C3 , C4 , C6 , C7 , C8 }
158	{ C4 , C5 , C6 , C7 }	201	{ C2 , C3 , C4 , C6 , C7 }	244	{ C2 , C3 , C5 , C6 , C7 , C8 }
159	{ C4 , C5 , C6 , C8 }	202	{ C2 , C3 , C4 , C6 , C8 }	245	{ C2 , C4 , C5 , C6 , C7 , C8 }
160	{ C4 , C5 , C7 , C8 }	203	{ C2 , C3 , C4 , C7 , C8 }	246	{ C3 , C4 , C5 , C6 , C7 , C8 }
161	{ C4 , C6 , C7 , C8 }	204	{ C2 , C3 , C5 , C6 , C7 }	247	{ C1 , C2 , C3 , C4 , C5 , C6 , C7 }
162	{ C5 , C6 , C7 , C8 }	205	{ C2 , C3 , C5 , C6 , C8 }	248	{ C1 , C2 , C3 , C4 , C5 , C6 , C8 }
163	{ C1 , C2 , C3 , C4 , C5 }	206	{ C2 , C3 , C5 , C7 , C8 }	249	{ C1 , C2 , C3 , C4 , C5 , C7 , C8 }
164	{ C1 , C2 , C3 , C4 , C6 }	207	{ C2 , C3 , C6 , C7 , C8 }	250	{ C1 , C2 , C3 , C4 , C6 , C7 , C8 }
165	{ C1 , C2 , C3 , C4 , C7 }	208	{ C2 , C4 , C5 , C6 , C7 }	251	{ C1 , C2 , C3 , C5 , C6 , C7 , C8 }
166	{ C1 , C2 , C3 , C4 , C8 }	209	{ C2 , C4 , C5 , C6 , C8 }	252	{ C1 , C2 , C4 , C5 , C6 , C7 , C8 }
167	{ C1 , C2 , C3 , C5 , C6 }	210	{ C2 , C4 , C5 , C7 , C8 }	253	{ C1 , C3 , C4 , C5 , C6 , C7 , C8 }
168	{ C1 , C2 , C3 , C5 , C7 }	211	{ C2 , C4 , C6 , C7 , C8 }	254	{ C2 , C3 , C4 , C5 , C6 , C7 , C8 }
169	{ C1 , C2 , C3 , C5 , C8 }	212	{ C2 , C5 , C6 , C7 , C8 }	255	{ C1 , C2 , C3 , C4 , C5 , C6 , C7 , C8 }

Appendix B. Coalitions in inventory case

K	N	Base stock level	Service level at the optimal	Estimated Cost	CP	BP
1	{A}	2	99.5 %	2020	0.00 %	49.60 %
2	{B}	2	98.2 %	2041	0.00 %	49.09 %
3	{C}	2	96.3 %	2064	0.00 %	48.64 %
4	{D}	2	93.8 %	2089	0.00 %	48.25 %
5	{A,B}	3	99.6 %	3060	24.65 %	32.88 %
6	{A,C}	3	99.2 %	3081	24.56 %	32.59 %
7	{A,D}	4	99.8 %	4100	0.22 %	48.80 %
8	{B,C}	3	98.6 %	3102	24.43 %	64.51 %
9	{B,D}	3	97.7 %	3124	24.36 %	32.14 %
10	{C,D}	3	96.6 %	3147	24.22 %	31.97 %
11	{A,B,C}	4	99.7 %	4120	32.73 %	24.47 %
12	{A,B,D}	4	99.4 %	4141	32.67 %	24.25 %
13	{A,C,D}	4	99.1 %	4162	32.58 %	24.07 %
14	{B,C,D}	4	98.7 %	4183	32.47 %	23.95 %
15	{A,B,C,D}	5	99.6 %	5201	36.68 %	-