NHH

INSTITUTT FOR FORETAKSØKONOMI

DEPARTMENT OF FINANCE AND MANAGEMENT SCIENCE

FOR 12 2007

ISSN: 1500-4066

MARCH 2007

Discussion paper

Simplifying and generalizing some efficient frontier and CAPM related results

BY

STEINAR EKERN

Norges Handelshøyskole

NORWEGIAN SCHOOL OF ECONOMICS AND BUSINESS ADMINISTRATION

Simplifying and generalizing some efficient frontier and CAPM related results

by
Steinar Ekern
Department of Finance and Management Science
NHH - Norwegian School of Economics and Business Administration
NHH, N-5045 Bergen, Norway
steinar.ekern@nhh.no

First draft: February 26, 2007
This version: September 14, 2007

Discussion Paper 2007/12
Department of Finance and Management Science
NHH - Norwegian School of Economics and Business Administration

Abstract

This paper simplifies, generalizes, extends, surveys and unifies results related to the efficient frontier in portfolio analysis and to asset pricing formulations of the Capital Asset Pricing Model (CAPM) type. It derives the composition and properties of many central portfolios in portfolio analysis. It also discusses and provides several CAPM type formulations involving different portfolios. In particular, the tangency portfolio properties are presented in an instructive and very simple way, focusing on similarities in going from the global minimum variance portfolio via a null index portfolio whose zero beta portfolio has a zero expected return. The Non-frontier zero beta, the Null index and the Augmented frontier CAPM versions supplement standard CAPM formulations. More importantly, the GMVP and the Benchmark versions of the CAPM do not rely on any zero beta portfolio, but require two betas.

JEL classifications: G11, G12, G10

Keywords: CAPM types, Roll's approach, tangency portfolio, GMVP, benchmark

1. Introduction

This paper simplifies, generalizes, extends, surveys and unifies results related to the efficient frontier in portfolio analysis and to asset pricing expressions similar to the Capital Asset Pricing Model (CAPM) and its "cousin", the Security Market Line (SML)[1]. Drawing on properties of the global mean variance portfolio and on the frontier portfolio whose zero beta portfolio has a mean of zero, a systematic pattern evolves, yielding simple and easy to remember expressions for the frontier tangency portfolio that is essential to both standard portfolio analysis and to the CAPM. Furthermore, several CAPM-type generalizations, including replacing the zero beta portfolio by the global mean variance portfolio or an arbitrary non-frontier benchmark, are developed or collected from other sources. The basic analytical approach builds on the first-order optimality conditions for a risk minimizing frontier portfolio from an arbitrary number of securities.

Both portfolio analysis and CAPM are standard topics in core master level finance courses, typically being allocated 20-30 % of class time, but relying heavily on diagrams and simple numerical examples[2]. Generally, students learn to compute means and variances of portfolios of two risky assets, trace the risk-return frontier, find the global minimum variance portfolio, compute the capital allocation line (CAL) of one risky and one risk free asset, identify the CAL with the highest slope as the capital market line (CML), and note the tangency portfolio of risky assets as the one preferred portfolio of risky assets only. Generalizations to more risky assets may be indicated, but actual computations for $N > 2$ are rare, especially for $N > 3$, except for

[1] Markowitz (1952) and Markowitz (1959) pioneered modern portfolio analysis. The CAPM is often attributed to Sharpe (1964), Lintner (1965), and Mossin (1966). The SML is a graphical portrayal of the linear expected return - beta relationship. Rubinstein (2006) describes the historical developments.
[2] See Womack (2001) for a breakdown of how various top US schools distribute class sessions to different topics. Dominating textbooks in the MBA core finance market are Bodie *et al*. (2008),

illustrative spreadsheet computations. Even computing (and particularly deriving and remembering) the weights of the tangency portfolio in the two risky asset case, may be outside the reach of most MBA students (and possibly some of their professors as well) without using available spreadsheet programs[3]. Turning to CAPM pricing, the formulas for the standard CAPM with a risk free asset are invariably presented, often supplemented by the zero beta CAPM formulation. Derivations of the CAPM formulas, and of the composition of the zero beta portfolio, are often omitted from the main text or left as elective exercises, leaving most students to accept them on faith and verify them on numerical examples.

But there is no need to throw in the towel and avoid formal analyses of more complex problems involving more than two risky assets. With a refreshed tool box and some tolerance for analytical formulations, solving complex problems may be surprisingly easy and yield convenient general results. More advanced graduate finance courses typically provide deeper analyses as well as some understanding of unifying principles[4]. Based on means, variances and covariances of the returns of $N \geq 2$ securities, a portfolio frontier may be identified, such that each frontier portfolio minimizes risk (standard deviation or variance of return) for any particular given level of expected return. A powerful result states that any portfolio of two frontier portfolios is itself a frontier portfolio. In principle the choice of frontier portfolios from which to generate the frontier is arbitrary. It is convenient to have a pair of portfolios that can be easily interpreted and whose properties can be identified,

---

Brealey and Myers (2006), and Ross *et al*. (2005), supplemented by others including Copeland *et al.* (2005), Elton *et al*. (2007), Grinblatt and Titman (2002), and Sharpe *et al.* (1999).

[3] Bodie *et al.* (2008) give in their equation (7.13) a seemingly quite complicated formula for tangency portfolio weights with two risky assets and a risk free one. Arnold *et al*. (2006) give the GMVP weights as simple functions of an intermediate factor, and tangency weights as simple functions of another rather complicated and hard to interpret factor.

[4] Textbooks such as Danthine and Donaldson (2005), and Huang and Litzenberger (1988) are at an upper master level and may facilitate transition to doctoral level literature such as the classical Ingersoll (1987) textbook.

and preferably such that the composition (asset weights) and properties are given in closed form. Some alternative pairs of portfolios include:

- A frontier portfolio with weights $\mathbf{g}$ having zero mean return and a corresponding frontier portfolio with weights $\mathbf{g}+\mathbf{h}$ having unit mean return, where the portfolio $\mathbf{h}$ is an arbitrage portfolio (with weights summing to zero) with unit mean return.
- The global minimum variance portfolio GMVP (hereafter abbreviated to *G*) and the null index frontier portfolio N whose uncorrelated zero beta frontier portfolio has zero mean, such that in a mean-standard deviation space the tangent to the frontier at N passes through origo.
- Any arbitrary risky frontier portfolio P and its corresponding frontier uncorrelated zero beta portfolio $Z(P)$.
- In case of a risk free security with certain return r_f, the risk free security and the risky frontier tangency portfolio T span the augmented efficient frontier originating at r_f and passing through T.
- More generally, the risk free security and any arbitrary portfolio P on the tangent to the risky frontier (in mean-standard deviation space) will span the augmented efficient frontier.

Merton (1972) and Roll (1977) are seminal works on frontier portfolios and their properties, including implied asset pricing relationships. Intermediate and advanced textbooks mostly seem to adopt Merton's approach, with the portfolios $\mathbf{g}$ and $\mathbf{g}+\mathbf{h}$ as primary building blocs. But Roll's approach, especially when focusing on the GMVP G and the nullindex portfolio N, has some advantages with respect to illustrations and convenient extensions. Roll's framework will therefore be the basis for this essentially self contained paper[5]. In particular, I'll show that going from the GMVP via N to T, reveals a particular nice pattern of the respective weights, means, variances, mean-variance ratios and slopes of the efficient frontier, as illustrated by Table 1. The composition of the tangency portfolio T has a neat and easy to remember closed form representation, and its mean and variance are simple ratios of adjusted information coefficients similar to those of G and N.

Next turn from building up the portfolio frontier from individual securities to the dual problem of pricing any arbitrary asset (security or portfolio) based on information about the frontier. Then there is a well-known theoretical linear relationship between the expected return and risk measured by, e.g., the product of beta and a "price of risk" measure. Beta depends on the correlation between an asset and a particular primary portfolio, and the "price of risk" may be defined as the difference in expected return of a primary and a secondary portfolio. The expected return of a primary portfolio appears as the first term in the "price of risk", and the fraction defining any asset beta has the primary portfolio's return covariance with the asset return as the numerator and the primary portfolio's return variance as the denominator. The expected return of a secondary portfolio appears as a constant and its negative value as the second term in the "price of risk".

Alternative portfolio pairs of primary and secondary portfolios used for generating various CAPM-type relationships, include respectively[6]:

- The market portfolio M and the risk free rate asset, in the standard Sharpe-Lintner-Mossin Equilibrium CAPM (18).

[5] Proofs are available from the author.
[6] The various CAPM types are formally stated in equations whose number is given in parentheses, and listed in Table 2.

- Any arbitrary frontier portfolio P and its corresponding frontier uncorrelated zero beta portfolio $Z(P)$, in the Zero beta CAPM (21).
- Any arbitrary frontier portfolio P and any of its corresponding nonfrontier uncorrelated zero beta portfolios with the same mean as the frontier $Z(P)$, in a Non-frontier Zero beta CAPM (21).
- The null index portfolio N and any portfolio having zero mean, including portfolio $\mathbf{g}$, in the Null index zero beta CAPM (22).
- Any arbitrary efficient portfolio P on the augmented risky frontier, and the risk free asset, in the Augmented frontier CAPM[7] (24).
- The risky assets tangency portfolio T and the risk free rate asset, in the standard Tangency CAPM (25).
- Any inefficient risky assets portfolio I and its uncorrelated frontier portfolio $Z(I)$, in an Inefficient portfolio CAPM[8] (28).
- Any arbitrary frontier portfolio P and the global minimum variance portfolio G, in the GMVP CAPM (26).
- Any arbitrary frontier portfolio P and any arbitrary (not necessarily frontier) benchmark portfolio B, in the Benchmark CAPM (27).

This paper introduces the two latter CAPM-type relations, replacing the zero beta frontier portfolio with either the GMVP or any arbitrary and not necessarily frontier benchmark portfolio. In both cases it will be necessary to provide not only the

[7] The augmented risky frontier coincides with the capital market line (CML), assuming a risk free asset, and provided all available assets are included in the investment universe from which the risky frontier has been constructed. An augmented frontier portfolio is then equivalently a portfolio lying on the CML.
[8] An inefficient portfolio here refers to a risky portfolio with a higher risk than a frontier risky portfolio with the same expected return, and not to a portfolio on the risky frontier below the GMVP.

asset beta with respect to the primary portfolio, but also the respective secondary portfolio's beta with respect to the primary portfolio.

The rest of the paper is organized as follows: Section 2 provides notation and the basic framework. Section 3 contains results on portfolio frontier relationships. Section 4 presents a set of CAPM-type relationships. Section 5 concludes.

2. Notation and basic framework

Consider $N \geq 2$ linearly independent and thus non redundant securities, each with a stochastic net rate of return r_i $\left(i = 1, 2, ..., N\right)$. At least two securities have different expected returns $\mu_i = E\left(r_i\right)$. The vector of the securities' expected returns is $\boldsymbol{\mu}$. Their variance-covariance return matrix $\mathbf{V}$ is symmetric and positive definite, such that the inverse covariance matrix $\mathbf{V}^{-1}$ exists. A portfolio of risky assets is defined by its weight vector $\mathbf{w}$ of proportions invested in the risky assets, summing to unity, such that $\mathbf{w}'\mathbf{1} = 1$, where $\mathbf{1}$ is a summation vector of ones, and primes denote vector or matrix transposition. Short selling is allowed, such that some securities may have negative weights in a portfolio. Subscripts identify different portfolios. An arbitrary portfolio P fully invested in risky assets has mean $\mu_P = \mathbf{w}_P'\boldsymbol{\mu}$ and variance $\sigma_P^2 = \mathbf{w}_P'\mathbf{V}\mathbf{w}_P$. The covariance between arbitrary portfolios P and Q is $\sigma_{PQ} = \mathbf{w}_P'\mathbf{V}\mathbf{w}_Q$. A risk free security, if it exists, has a net rate r_f.

A frontier portfolio is the risky portfolio that minimizes the variance among all portfolios having the same targeted expexted return $\bar{\mu}$. It satisfies the portfolio optimality condition

$$\mathbf{V}\mathbf{w} = \lambda\boldsymbol{\mu} + \gamma\mathbf{1} \quad (1)$$

where λ and γ are Lagrange multipliers associated with the portfolio mean and weight sum constraints, respectively. A frontier portfolio therefore has the weight vector in risky assets of

$$\mathbf{w} = \lambda \mathbf{V}^{-1}\boldsymbol{\mu} + \gamma \mathbf{V}^{-1}\mathbf{1} \tag{2}$$

Following Roll (1977), it will be useful to define the information constants[9]

$$a \equiv \boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu} \tag{3a}$$

$$b \equiv \boldsymbol{\mu}'\mathbf{V}^{-1}\mathbf{1} \tag{3b}$$

$$c \equiv \mathbf{1}'\mathbf{V}^{-1}\mathbf{1} \tag{3c}$$

$$d \equiv ac - b^2 > 0 \tag{3d}$$

Premultiplying (2) by the transposed mean vector $\boldsymbol{\mu}'$ and next by the summation vector $\mathbf{1}'$, and solving the resulting two linear equations for the Lagrange multipliers,

$$\lambda = \lambda(\bar{\mu}) = \frac{c\bar{\mu} - b}{d} \tag{4a}$$

$$\gamma = \gamma(\bar{\mu}) = \frac{a - b\bar{\mu}}{d} \tag{4b}$$

The risky portfolio frontier is the set of all risky frontier portfolios. Any frontier portfolio P (without any additional restrictions on the weights $\mathbf{w}_P$ beyond summing to unity) satisfies the mean-variance relation

$$\sigma_P^2 = \frac{a - 2b\mu_P + c\mu_P^2}{d} \tag{5}$$

giving the risk of any frontier portfolio with a stipulated expected return μ_P.

Furthermore, P also satisfies the mean-covariance relation

$$\sigma_{PQ} = \frac{a - b\mu_P - b\mu_Q + c\mu_P\mu_Q}{d} \tag{6}$$

for the covariance between the frontier portfolio P and any arbitrary asset Q (not necessarily a frontier portfolio) with weight vector $\mathbf{w}_Q$. This expression can be used in computing asset betas and for finding pairs of uncorrelated portfolios.

If a risk free security is available, it will be on a new and augmented frontier. Any portfolio may then consist of both the risk free security and the N risky securities. The proportions invested in risky assets no longer necessarily sum to one. An augmented frontier portfolio minimizes risk for a given level of expected return, allowing for some risk free investments. All portfolios on the risk-minimizing augmented frontier will have risky portfolio weights not summing to unity, except the so-called tangency portfolio T that is both on the augmented frontier originating at the risk free security, and on the original frontier of the risky assets only.

3. Frontier portfolios and relations

The global minimum variance frontier portfolio GMVP denoted by G, can be easily found by minimizing (one half) the variance, subject to weights summing to unity, yielding:

Weight vector: $$\mathbf{w}_G = \left(\frac{1}{c}\right)\mathbf{V}^{-1}\mathbf{1} \tag{7a}$$

Mean: $$\mu_G = \frac{b}{c} \tag{7b}$$

Variance: $$\sigma_G^2 = \frac{1}{c} \tag{7c}$$

Covariance: $$\sigma_{GQ} = \frac{1}{c} \tag{7d}$$

---

[9] Merton (1972) and his followers generally use the notation A for Roll's b, and B for Roll's a. Adding to the confusion, later in Section 3 I define adjusted constants A and B related to but not equal to Roll's a and b, respectively.

The weight of any arbitrary security i in the GMVP is thus simply the sum of the elements of the i^{th} row of the inverse covariance matrix, divided by the sum of all elements of $\mathbf{V}^{-1}$. The inverse sum of all such elements is both the return variance of G and the covariance between the returns of G and any arbitrary (possibly non-frontier) asset Q.

Substitution of the evaluated Lagrange multipliers into the weight equation for any frontier portfolio, gives the well-known result that any portfolio of two frontier portfolios is itself a frontier portfolio. The risky frontier may be generated by

$$\mathbf{w} = \mathbf{g} + \bar{\mu}\mathbf{h} \tag{8a}$$

Here the weight vector

$$\mathbf{g} \equiv \frac{1}{d}\left(a\mathbf{V}^{-1}\mathbf{1} - b\mathbf{V}^{-1}\boldsymbol{\mu}\right) \tag{8b}$$

sums to one and has an expected return of zero. The arbitrage portfolio weight

$$\mathbf{h} \equiv \frac{1}{d}\left(c\mathbf{V}^{-1}\boldsymbol{\mu} - b\mathbf{V}^{-1}\mathbf{1}\right) \tag{8c}$$

sums to zero with an expected return of one. The sum $\mathbf{g} + \mathbf{h}$ is a frontier portfolio with expected return one[10]. Thus, equation (8a) will yield a frontier portfolio whose return has a mean $\bar{\mu}$, and variance according to the mean-variance relation (5). Unfortunately, it would be rather difficult to try to remember and even interpret the compositions of the generating portfolios $\mathbf{g}$ and $\mathbf{g} + \mathbf{h}$.

An alternative approach for generating the efficient frontier would be to use the GMVP matched with a suitable companion frontier portfolio. Rather than using the mean zero frontier portfolio $\mathbf{g}$ itself, it may be convenient to use the frontier portfolio N that is uncorrelated with $\mathbf{g}$. The covariance between the returns of the frontier portfolios $\mathbf{g}$ and N is thus zero. Recalling beta as covariance divided by variance, frontier portfolios $\mathbf{g}$ and N may be referred to as zero beta portfolios of each other.

In general, any arbitrary frontier portfolio P except the GMVP has a unique uncorrelated frontier portfolio or zero beta portfolio, which may be denoted $Z(P)$, with mean $\mu_{Z(P)}$ or μ_Z for short. In a mean-standard deviation diagram, the tangent to the frontier at P intersects the expected return axis at $\mu_{Z(P)}$, such that the frontier zero beta portfolio $Z(P)$ is located on the frontier with a mean equal to the tangent's intercept. Alternatively, in a mean-variance diagram, a ray through the GMVP and P intersects the expected return axis at $\mu_{Z(P)}$. Given a frontier zero beta portfolio with mean μ_Z, from the means-covariance relation (6) the mean of the corresponding frontier portfolio P is given as

$$\mu_P = \frac{b\mu_Z - a}{c\mu_Z - b} \tag{9a}$$

The corresponding weight of the frontier portfolio P is

$$\mathbf{w}_P = \left(\frac{1}{b - c\mu_Z}\right)\mathbf{V}^{-1}\left(\boldsymbol{\mu} - \mu_Z \mathbf{1}\right) \tag{9b}$$

Interchanging the subscripts P and Z, the mean and weight of the zero beta portfolio $Z(P)$ is readily available from the expected return of P.

The general expressions (9a) and (9b) are not particularly simple or intuitive. However, consider the mean zero frontier portfolio $\mathbf{g}$ as being the mean zero frontier

[10] If returns are given in decimal form, then $\mu_{g+h} = 1.00$ corresponds to an expected portfolio return of 100%, which is probably far above the interesting part of the risky frontier.

portfolio of a particular frontier portfolio N to be called the null index portfolio. From setting $\mu_Z = 0$ in (9b), the frontier portfolio N has the weight vector

$$\mathbf{w}_N = \left(\frac{1}{b}\right)\mathbf{V}^{-1}\boldsymbol{\mu} \tag{10a}$$

This weight vector is very similar to the weight vector (7a) of the GMVP, replacing the summation vector $\mathbf{1}$ by the mean return vector $\boldsymbol{\mu}$, and correspondingly also replacing the information constant c by the information constant b according to the definitions of the two information constants in (3b) and (3c). Thus, the weight of the null index frontier portfolio N is quite simple. Its expected return is just

$$\mu_N = \frac{a}{b} \tag{10b}$$

from the general expression (9a) with μ_Z = 0, or by $\mathbf{w}_N\,'\boldsymbol{\mu}$. Its variance $\mathbf{w}_N\,'\mathbf{V}\mathbf{w}_N$ is

$$\sigma_N^2 = \frac{a}{b^2} \tag{10c}$$

Furthermore, the GMVP and the null index portfolio have the same mean-variance ratio $\frac{\mu_G}{\sigma_G^2} = b = \frac{\mu_N}{\sigma_N^2}$. Thus, in mean-variance space, a ray from the origin through the frontier null index portfolio N will pass through the GMVP. In mean-standard deviation space, the tangent to the efficient frontier at N intercepts the mean axis in origo. Thus, that the slope of the efficient frontier at N is

$$\frac{d\mu_P}{d\sigma_P \,|\, P = N} = \frac{\mu_N}{\sigma_N} = \sqrt{a} \tag{10d}$$

Hence, following Roll, it may be convenient to generate the whole portfolio frontier from the global minimum variance portfolio G and the null index portfolio N, both having weight vectors which are simple, easy to remember and of the same structure, as well as a common mean-variance ratio.

Hereafter suppose there is a risk free security, earning a net rate r_f. The portfolio optimality condition with a risk free security is then

$$\mathbf{Vw} = \lambda\left(\boldsymbol{\mu} - r_f \mathbf{1}\right) \tag{11}$$

replacing the previous first order condition (1). Any augmented frontier portfolio has a weight vector

$$\mathbf{w} = \lambda \mathbf{V}^{-1}\left(\boldsymbol{\mu} - r_f \mathbf{1}\right) \tag{12}$$

where the single Lagrange multiplier $\lambda = \lambda\left(\overline{\mu}\right)$ depends on the stipulated portfolio return $\overline{\mu}$. Let P be some arbitrary portfolio on the tangent in mean-standard deviation space, from the risk free rate to the frontier of risky assets only. Then P is an augmented frontier portfolio, and the required return may be set equal to its mean, such that $\overline{\mu} = \mu_P$. The Lagrange multiplier may be rewritten as $\lambda = \lambda\left(P\right)$, giving portfolio weights[11] $\mathbf{w}_P = \lambda\left(P\right)\mathbf{V}^{-1}\left(\boldsymbol{\mu} - r_f \mathbf{1}\right)$. Without loss of generality one may choose any arbitrary Lagrange multiplier and back out the corresponding augmented frontier portfolio P with weight $\mathbf{w}_P\left(\lambda\right)$. The easiest choice would be to set the Lagrange multiplier to one, yielding

$$\mathbf{w}_P = \mathbf{V}^{-1}\left(\boldsymbol{\mu} - r_f \mathbf{1}\right) \tag{13}$$

without regard to whether the resulting augmented frontier portfolio P is an interesting one on its own.

A more natural case would be to rescale the weights $\mathbf{w}_P$ to sum to one. Then the augmented frontier portfolio would also be a frontier portfolio with respect to the risky assets only, giving the tangency portfolio T from standard portfolio analysis. As

[11] Related expressions (3.18.1) in Huang and Litzenberger (1988), or equivalently (7.31) in Danthine and Donaldson (2005), give the risky assets weights of an arbitrary point on the augmented frontier.

the weight sum $\mathbf{1}'\mathbf{w}_P = \lambda(P)(b - cr_f)$, the Lagrange multiplier drops out when rescaling the previous weight $\mathbf{w}_P$ by its inverse weight sum. This tangency portfolio would then have portfolio weights given by

$$\mathbf{w}_T = \frac{1}{b - cr_f}\mathbf{V}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}) \quad (14)$$

It is easy to check that the tangency portfolio has a mean $\mu_T = \mathbf{w}_T{}'\boldsymbol{\mu} = \frac{br_f - a}{cr_f - b}$, which is similar to equation (9a) for the mean of a risky frontier portfolio with a zero beta frontier portfolio whose mean equals the risk free rate. Also, (14) corresponds to the weight vector (9b) for a risky frontier portfolio, with the risk free rate replacing the zero beta mean.

For a further simplification, define the excess return of any asset j as the difference between its net return and the risk free rate, such that its expected excess return $m_j \equiv \mu_j - r_f$. The vector $\mathbf{m}$ of expected excess returns for the N individual risky securities is therefore

$$\mathbf{m} \equiv \boldsymbol{\mu} - r_f\mathbf{1} \quad (15)$$

The RHS expression of (15) appears several places above, including in equations (11) through (14). By substitution, the optimality condition may be rewritten as

$$\mathbf{Vw} = \lambda\mathbf{m} \quad (11')$$

Any augmented frontier portfolio P has weights

$$\mathbf{w}_P = \lambda(P)\mathbf{V}^{-1}\mathbf{m} \quad (12')$$

Arbitrarily setting the Lagrange multiplier at unity yields

$$\mathbf{w}_P = \mathbf{V}^{-1}\mathbf{m} \quad (13')$$

Both latter expressions introduce a constant $H \equiv a - 2br_f + cr_f{}^2$, which in fact turns out to be identical

The latter expression is the main result of Feldman and Reisman (2003), using a different procedure and a different notation.

One new and instructive contribution of the current paper may be to rewrite the results for the tangency portfolio in a simple way which is also easy to remember, drawing on similarities with the global minimum variance portfolio G and the null index portfolio N. First, define adjusted information constants in terms of expected excess return vector $\mathbf{m}$, similarly to previous terms using expected return $\boldsymbol{\mu}$:

$$A \equiv \mathbf{m}'\mathbf{V}^{-1}\mathbf{m} \tag{5a'}$$

$$B \equiv \mathbf{m}'\mathbf{V}^{-1}\mathbf{1} \tag{5b'}$$

These new information constants are written in capitals to distinguish them from their previous counterparts written in lower-case letters, and which are based on the expected return vector $\boldsymbol{\mu}$ rather than the expected excess return vector $\mathbf{m}$.

The tangency portfolio T is also on the augmented portfolio frontier, such that $\mathbf{w}_T = \lambda(T)\mathbf{V}^{-1}\mathbf{m}$, summing to one. Premultiplying by the transposed summation vector, and solving for the Lagrange multiplier $\lambda(T) = \frac{1}{B}$ when using the definition of the adjusted information constant B, the tangency portfolio's weight can be written as simply

$$\mathbf{w}_T = \left(\frac{1}{B}\right)\mathbf{V}^{-1}\mathbf{m} \tag{14'}$$

The tangency portfolio has an expected excess return $m_T = \mathbf{w}_T'\mathbf{m}$, yielding

$$m_T = \frac{A}{B} \tag{16a}$$

Subtracting a constant has no effect on variances, such that the variance of excess returns equals the variance $\mathbf{w}_T'\mathbf{V}\mathbf{w}_T$ of returns, giving

---

to an adjusted information constant A to be defined shortly in equation (5a').

$$\sigma_T^2 = \frac{A}{B^2} \tag{16b}$$

The slope of the tangent passing through the risk free rate, commonly referred to as the Sharpe ratio, is

$$\frac{d\mu_P}{d\sigma_P | P = T} = \frac{dm_P}{d\sigma_P | P = T} = \frac{m_T}{\sigma_T} = \sqrt{A} \tag{16c}$$

whether the vertical axis measures mean return μ_P or mean excess return m_P.

Comparing the latter four equations for the tangency portfolio T with the corresponding equations (10a)-(10d) for the null index frontier portfolio N, shows that the expressions are indeed very similar. The mean return vector $\boldsymbol{\mu}$ has been replaced by the expected excess return vector $\mathbf{m}$, both in the resulting weight and in the adjusted information coefficients A and B. These adjusted information coefficients A and B replace the initial information coefficient a and b, both in the weight vector, the mean, the variance, the mean-variance ratio, and the slope of the efficient frontier at the respective tangency points. The tangent intercepts are at origo, when the vertical axis measures mean excess return m_P for the (traditional) tangency portfolio T and mean return μ_P for the null index portfolio N.

Thus, going from G via N to T, reveals a systematic pattern making it easy and simple to derive, formulate and remember convenient closed form expressions for the composition of the tangency portfolio and its two first moments, as well as the mean-variance ratios and the mean-standard deviation slopes of the efficient frontier[12].

Insert Table 1 about here

[12] The multi assets tangency weight expression (14') is definitely simpler than its two assets risky counterparts referred to in footnote 3.

Mean-variance portfolio analysis is traditionally used and taught for total returns or excess returns over the risk free rate. Practitioners are often more interested in the differential return of some managed return relative to some prespecified benchmark return, sometimes called a tracking error or active return[13]. Some agents consider positive expected tracking error "good", and variance (or volatility) of tracking error as "bad". Roll (1992) formalized TEV analysis, aimed at minimizing tracking error variance for a given level of expected tracking error, by mimicking his approach to traditional MV analysis, but here by identifying optimal differential (or hedging or arbitrage) portfolios whose weights $\mathbf{x} \equiv \mathbf{w}_P - \mathbf{w}_B$ sum to zero. Such analysis is beyond the scope of this paper, but we may note some somewhat surprising features indicating that familiarity with traditional MV analysis is useful for TEV analysis as well[14]:

- The optimal differential portfolios are independent of the benchmark. They may be expressed as the difference between the weights of the null index portfolio N in (10a) and the GMVP G in (7a), scaled by "performance" as given by the ratio of the targeted expected tracking error $\overline{\mu}^B \equiv \mu_P - \mu_B$ to the difference between expected returns on N from (10b) and on G from (7b):

$$\mathbf{x} = \left(\mathbf{w}_N - \mathbf{w}_G\right)\frac{\overline{\mu}^B}{\mu_N - \mu_G} \qquad (17a)$$

- Equivalently, the weights of the TEV frontier portfolios are proportional to the $\mathbf{h}$ vector as given in (8c) and which has unit mean and weights summing to zero:

$$\mathbf{x} = \overline{\mu}^B \mathbf{h} \qquad (17b)$$

[13] The terminology may be confusing, as tracking error in the literature may be used for both the stochastic differential return and for its volatility (standard deviation),

- The TEV-efficient frontier portfolios are on straight lines through the origin in differential return mean-standard deviation space, and with slopes equal to the slopes of the asymptotes to the total return portfolio frontier. Loosely speaking, the benchmark is analogous to the risk free rate in standard mean-variance analysis.
- The risk minimizing TEV implies a total portfolio with a total return beta exceeding one, such that when the benchmark performs badly, the managed portfolio does even worse[15].

4. CAPM type relations

According to the Standard Capital Asset Pricing Model (CAPM), the expected return $E\left(r_j\right)$ on any asset indexed by j can be given in terms of the risk free rate r_f, the expected return $E\left(r_M\right)$ on the "market portfolio" M, and its systematic risk beta:

$$E\left(r_j\right) = r_f + \left[E\left(r_M\right) - r_f\right]\beta_{jM} \tag{18}$$

The asset beta with respect to the market portfolio, is the covariance between the returns divided by the variance of the market portfolio return:

$$\beta_{jM} \equiv \frac{\text{Cov}\left(r_j, r_M\right)}{\text{Var}\left(r_M\right)} \tag{19}$$

The Standard CAPM is an equilibrium model, such that markets clear when all agents hold the market portfolio of risky assets, possibly in combination with the risk-free security. Formally identical expressions may also be derived from efficiency

---

[14] Expression (17a) shows how the GMVP and the null index portfolio may be used to trace out the tracking error frontier. Alternatively, expression (17b) for the TEV frontier portfolios is analogous to (8a) for the total risky frontier, but omits the zero mean frontier portfolio $\mathbf{g}$.

[15] As stated in the ingress to Roll (1992): "Minimizing the volatility of tracking error will not produce a more efficient managed portfolio."

analysis, based on the first order optimality conditions. The term CAPM-type will be used for relations, which do not necessarily require equilibrium for all assets.

This section neither invokes market clearing equilibrium conditions nor assumes that all possible assets are included in an agent's investment universe, but takes portfolio optimality conditions as a starting point. Suppose first that there is no risk free security. Consider three assets: An arbitrary frontier portfolio P, its corresponding uncorrelated frontier zero beta portfolio $Z(P)$, and finally some arbitrary risky asset (security or portfolio, and possibly off the frontier) indexed by j. From (1), the first order condition for portfolio P is

$$\mathbf{V}\mathbf{w}_P = \lambda(P)\boldsymbol{\mu} + \gamma(P)\mathbf{1} \tag{20}$$

Premultiplying (20) by the transposed weights vectors of the three assets (i.e., by $\mathbf{w}_P\,'$, $\mathbf{w}_{Z(P)}\,'$ and $\mathbf{w}_j\,'$), recognizing terms, eliminating the Lagrange multipliers, and rearranging, yields the Zero beta CAPM:

$$E(r_j) = E\left(r_{Z(P)}\right) + \left[E(r_P) - E\left(r_{Z(P)}\right)\right]\beta_{jP} \tag{21}$$

Here the asset beta is specific for the particular asset and is defined with respect to the arbitrary frontier portfolio P, i.e., $\beta_{jP} \equiv \dfrac{\operatorname{Cov}(r_j, r_P)}{\operatorname{Var}(r_P)}$. The intercept and the "price of risk" are common for all assets.

The Zero beta CAPM (21) may be generalized slightly, by not requiring the zero beta portfolio $Z(P)$ to be on the frontier of the risky assets, as that assumption has not been used in the derivation of (21). All portfolios having the same mean

$\mu_Z = E\left(r_{Z(P)}\right)$ as the frontier zero beta portfolio, are themselves uncorrelated with the frontier portfolio P and thus eligible to be used in a Non-frontier zero beta CAPM[16].

The Null index zero beta CAPM is a simplifying special case, using the null index portfolio N whose zero beta portfolios, such as e.g. $\mathbf{g}$ as defined in (8b), has a mean of zero. The Standard zero beta CAPM is replaced by

$$E\left(r_j\right) = E\left(r_N\right)\beta_{jN} \tag{22}$$

Substituting the mean, variance, and weight vector of the null index portfolio, verifies that $E\left(r_j\right) = \mathbf{w}_j{}'\boldsymbol{\mu}$ [17].

If there is a risk free security, then the relevant first order condition (11) may be evaluated at any arbitrary point P on the augmented frontier, giving

$$\mathbf{V}\mathbf{w}_P = \lambda\left(P\right)\left(\boldsymbol{\mu} - r_f\mathbf{1}\right) \tag{23}$$

Premultiplying (23) by the transposed weight vectors $\mathbf{w}_p{}'$ of the arbitrary augmented frontier portfolio and $\mathbf{w}_j{}'$ of the arbitrary asset, recognizing terms, eliminating the single Lagrange multiplier, and reorganizing, now yields the Augmented frontier CAPM

$$E\left(r_j\right) = r_f + \left[E\left(r_P\right) - r_f\right]\beta_{jP} \tag{24}$$

Note that the portfolio P does not need to be on the frontier of risky assets only, as long as it is on the augmented frontier, i.e., on the CML[18].

One obvious special case is where the reference portfolio on the augmented frontier is also on the standard frontier of risky assets only, and hence must be the tangency portfolio T. In the Tangency CAPM,

---

16 Elton *et al*. (2007:310) comment that it makes sense to use the least risky zero beta portfolio.
17 It is thus a matter of convenience or data availability, whether to use the Null index zero beta formulation or the implied equivalent weighted average of means formulation.
18 Feldman and Reisman (2003) state this result in their Lemma 1.

$$E\left(r_j\right)=r_f+\left[E\left(r_T\right)-r_f\right]\beta_{jT} \tag{25}$$

This formulation is not only a special case of the Augmented portfolio CAPM (with the augmented frontier portfolio specialized to the tangency portfolio), but also a generalization of the Standard CAPM (with the equilibrium market portfolio replaced by the assumed efficient tangency portfolio).

All the CAPM type results above are more or less well known by themselves or rather trivial extensions. Next, this paper points out a couple of possible further generalizations.

The GMVP has the same covariance (7d) with any asset, being equal to its own variance (7c). Hence, it cannot be uncorrelated with any asset whatsoever, and thus has no zero beta portfolio $Z\left(G\right)$. Still, a CAPM-type relation may be developed. In the derivation of the Zero beta CAPM, replace the premultiplication of the optimality condition (20) by the transposed zero beta portfolio $\mathbf{w}_{Z(P)}'$ by the transposed GMVP weight $\mathbf{w}_G'$ from (7a), and otherwise proceed as before. After some boring algebraic manipulations, one obtains the GMVP CAPM formulation[19]

$$E\left(r_j\right)=E\left(r_G\right)+\left[E\left(r_P\right)-E\left(r_G\right)\right]\frac{\beta_{jP}-\beta_{GP}}{1-\beta_{GP}} \tag{26}$$

Here, the mean of the GMVP has replaced the mean of the zero beta portfolio both in the intercept and the "price of risk" terms. Consequently, the beta term has to be adjusted. The GMVP's beta with respect to the frontier portfolio P, i.e., $\beta_{GP}\equiv\frac{\text{Cov}\left(r_G,r_P\right)}{\text{Var}\left(r_P\right)}=\frac{\text{Var}\left(r_G\right)}{\text{Var}\left(r_P\right)}$, enters the fraction with negative sign both in the

[19] A somewhat related extended CAPM formulation, using the market portfolio with the GMVP and without a risk free asset, can be found in van Zijl (1987).

numerator and denominator. The denominator is always positive for any frontier $P \neq G$, as the GMVP by definition has the smallest possible variance.

For a further generalization, start with the Zero beta CAPM derivation, but replace the zero beta portfolio with some arbitrary benchmark B with return r_B, which is not necessarily a frontier portfolio. By premultiplying the first order condition (20) by the transposed benchmark weight $\mathbf{w}_B{}'$, and then as usual eliminating the Lagrange multipliers, recognizing terms, and doing some further algebraic manipulations, the Benchmark CAPM

$$E\left(r_j\right) = E\left(r_B\right) + \left[E\left(r_P\right) - E\left(r_B\right)\right] \frac{\beta_{jP} - \beta_{BP}}{1 - \beta_{BP}} \qquad (27)$$

is obtained[20]. It corresponds to the GMVP CAPM (26), replacing the GMVP G by the benchmark B. To avoid division by zero, the correlation coefficient must satisfy $\rho_{BP} \neq \frac{\sigma_P}{\sigma_B}$. A sufficient, but not necessary, condition is thus that the benchmark B has a smaller standard deviation than the frontier portfolio P.

Insert Table 2 about here

All various CAPM formulations in this section may be derived from applying appropriate first order portfolio optimality conditions to a suitable frontier (or augmented frontier) portfolio, followed by creative mathematical operations. A different Inefficient portfolio CAPM approach focuses on an inefficient (that is, a non-frontier portfolio) I as a central portfolio for pricing. Diacogiannis and Feldman (2006) decompose the return r_I of an arbitrary inefficient portfolio I as $r_I = r_P + r_e$, when reformulated in the current notation. Here P is the frontier portfolio having the same mean and a smaller standard deviation than the inefficient portfolio I, whereas e is an arbitrage portfolio, with mean zero, weights summing to zero, and being uncorrelated with the frontier portfolio P. The Standard zero beta CAPM may then be reformulated in terms of the inefficient portfolio and the arbitrage portfolio as

$$E\left(r_j\right)=E\left(r_{Z(I)}\right)+\left[E\left(r_I\right)-E\left(r_{Z(I)}\right)\right]\frac{\sigma_I^2}{\sigma_P^2}\beta_{jI}+\left[E\left(r_I\right)-E\left(r_{Z(I)}\right)\right]\frac{\sigma_e^2}{\sigma_P^2}\beta_{je} \quad (28)$$

The major advantage of the Inefficient CAPM is probably not a computational one but rather pointing out the various fallacies in using a non-frontier proxy I for a frontier portfolio P in an otherwise Standard zero beta CAPM.

5. Conclusions

This paper has aimed at going beyond elementary portfolio analysis and standard CAPM formulations, as experienced by a majority of master students in finance core courses. It has surveyed, extended, generalized and simplified applicable further results, using a unified Roll-Merton approach. All the approaches are consistent and equivalent, in the sense that they provide the same efficient frontier and/or the same expected asset returns. The various formulations are based on properties of different portfolios. No advanced methods beyond college level matrix algebra and elementary optimization have been used, assuming familiarity with basic probability theory. The focus has been on applicable concepts and insight, whereas implementation would typically require spreadsheets or more advanced computational tools to perform the matrix operations for $N \geq 3$.

From the first-order portfolio optimality conditions as starting points, the composition and properties of various portfolios related to the efficient frontier have

[20] Bodie *et a*l. (2008) present in their equation (9.11) a related result, but with the additional restriction not imposed here that the benchmark should be a frontier portfolio.

been derived: The global minimum variance portfolio, two feasible but hard to interpret generating portfolios, an arbitrary frontier portfolio, its frontier uncorrelated zero beta portfolio, a null index portfolio whose zero beta portfolio also has zero mean, an arbitrary augmented portfolio on the CML, and the familiar tangency portfolio.

Furthermore, the paper has discussed and provided several CAPM type formulations involving different properties of various portfolios: The standard Equilibrium CAPM, the standard Zero beta CAPM, the Non-frontier zero beta CAPM, the Null index zero beta CAPM, the Augmented frontier CAPM, the Tangency CAPM, the GMVP CAPM, the Benchmark CAPM, and the Inefficient portfolio CAPM.

In particular, the composition and the first two moments of the tangency portfolio and its associated mean-variance ratio and Sharpe ratio, have been provided in an instructive and simple way, using adjusted information coefficients, and focusing on analogies in going from the global minimum variance portfolio via the null index portfolio to the tangency portfolio. The Non-frontier zero beta CAPM and the Null index zero beta CAPM are fairly trivial extensions. The Augmented frontier CAPM easily drops out from the basic approach. More importantly, the GMVP and Benchmark versions of the CAPM may be useful extensions, at best hard to find elsewhere.

References

Arnold, T, L.A. Nail, and T. D. Nixon (2006), "Getting more out of two-assets portfolios", *Journal of Applied Finance, 16* (1, Spring/Summer), 72-81.

Bodie, Z., A. Kane and A. J. Marcus (2008), *Investments, 7th Ed.*, McGraw-Hill

Brealey, R. A., S. C. Myers, and F. Allen (2006), *Principles of corporate finance, 8th Ed.*, McGraw-Hill

Copeland, T. E., J. F. Weston, and K. Shastri (2005), *Financial theory and corporate policy, 4th Ed.*, Addison-Wesley

Danthine, J.-P. and J. Donaldson (2005), *Intermediate financial theory, 2nd Ed.*, Elsevier Academic Press

Diacogiannis, G. and D. Feldman (2006), "The CAPM for inefficient portfolios", unpublished working paper, available at SSRN: http://ssrn.com/abstract=893702

Feldman, D. and H. Reisman (2003), "Simple construction of the efficient frontier", *European Financial Management, 9*(2), 251-259

Elton, E. J., M. J. Gruber, S. J. Brown, and W. N. Goetzmann (2007), *Modern portfolio theory and investment analysis, 7th Ed.*, Wiley

Grinblatt, M. and S. Titman (2002), *Financial markets and corporate strategy, 2nd Ed.*, McGraw-Hill

Huang, C-f. and R. H. Litzenberger (1988), *Foundations for financial economics*, North-Holland

Ingersoll, J. E., Jr. (1987), *Theory of financial decision making*, Rowman & Littlefield

Lintner, J. (1965), "The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets", *Review of Economics and Statistics, 47* (1, February), 13-37

Markowitz, H. (1952), "Portfolio selection", *Journal of Finance, 7* (1, March), 77-91

Markowitz, H. M. (1959), *Portfolio selection: Efficient diversification of investments*, Cowles Foundation Monograph # 16, Wiley

Merton, R. (1972), "An analytical derivation of the efficient frontier", *Journal of Financial and Quantitative Analysis, 7(4)*, 1851-1872

Mossin, J. (1966), "Equilibrium in a capital asset market", *Econometrica, 34* (4, October), 768-783

Roll, R. (1977), "A critique of the asset pricing theory's tests", *Journal of Financial Economics, 4(2)*, 129-176

Roll, R. (1992), "A mean/variance analysis of tracking error", *Journal of Portfolio Management*, *18*(4, Summer), 13-23

Ross, S. A., R. W. Westerfield and J. F. Jaffe (2005), *Corporate finance*, *7th Ed.*, McGraw-Hill

Rubinstein, M. (2006), *A history of the theory of investments: My annotated bibliography*, Wiley

Sharpe, W. F. (1964), "Capital asset prices: A theory of market equilibrium under conditions of risk", *Journal of Finance*, *19* (3, September), 425-442

Sharpe, W. F., G. J. Alexander and J. F. Bailey (1999), *Investments, 6th Ed.*, Prentice Hall

van Zijl, T. (1987), "Risk decomposition: Variance or standard deviation - a reexamination and extension", *Journal of Financial and Quantitative Analysis*, *22*(2, June), 237-247

Womack, K. L. (2001), "Core finance courses in the top MBA programs in 2001", Tuck School of Business at Dartmouth Working Paper No. 01-07, available at SSRN: http://papers/ssrn/com/abstract=291973

Table 1

Properties of three central portfolios

	Global minimum variance portfolio *G*	Null index portfolio *N*	Risky assets tangency portfolio *T*
Mean	$\mu_G = \frac{b}{c}$	$\mu_N = \frac{a}{b}$	$m_T = \frac{A}{B}$
Variance	$\sigma_G^2 = \frac{1}{c}$	$\sigma_N^2 = \frac{a}{b^2}$	$\sigma_T^2 = \frac{A}{B^2}$
Weight	$\mathbf{w}_G = \left(\frac{1}{c}\right)\mathbf{V}^{-1}\mathbf{1}$	$\mathbf{w}_N = \left(\frac{1}{b}\right)\mathbf{V}^{-1}\boldsymbol{\mu}$	$\mathbf{w}_T = \left(\frac{1}{B}\right)\mathbf{V}^{-1}\mathbf{m}$
Mean-variance ratio	$\frac{\mu_G}{\sigma_G^2} = b$	$\frac{\mu_N}{\sigma_N^2} = b$	$\frac{m_T}{\sigma_T^2} = B$
Slope of efficient frontier	$\frac{d\mu_P}{d\sigma_P \mid P = G} = \infty$	$\frac{d\mu_P}{d\sigma_P \mid P = N} = \frac{\mu_N}{\sigma_N} = \sqrt{a}$	$\frac{d\mu_P}{d\sigma_P \mid P = T} = \frac{m_T}{\sigma_T} = \sqrt{A}$

The global minimum variance portfolio G has the smallest variance of all portfolios fully invested in the risky assets. The null index portfolio N is the risky assets frontier portfolio that is uncorrelated with all portfolios having a zero expected return. The risky asset tangency portfolio T is the portfolio consisting of risky assets only being on the tangent from the risk free rate to the risky frontier in mean-standard deviation space. Mean is the expected return on the portfolios G and N, whereas it is the expected excess return above the risk free rate for the tangency portfolio T. Variance is the variance of the portfolio's return. Weight is the vector of investment proportions in the risky assets. Mean-variance ratio is the ratio of mean to variance. Slope of efficient frontier applies with mean along the vertical axis and standard deviation along the horizontal axis. Evaluated at the tangency portfolio T, this slope is also the maximal Sharpe ratio. Following Roll (1977), the information constants are defined as $a \equiv \boldsymbol{\mu}'\mathbf{V}^{-1}\boldsymbol{\mu}$, $b \equiv \boldsymbol{\mu}'\mathbf{V}^{-1}\mathbf{1}$ and $c \equiv \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$. By replacing the vector $\boldsymbol{\mu}$ of assets' expected returns by the corresponding vector $\mathbf{m}$ of assets' expected returns above the risk free rate, the adjusted information constants introduced in this paper are similarly defined by $A \equiv \mathbf{m}'\mathbf{V}^{-1}\mathbf{m}$ and $B \equiv \mathbf{m}'\mathbf{V}^{-1}\mathbf{1}$.

Table 2

Different CAPM type formulations

CAPM type	Equation number	Primary portfolio	Secondary portfolio	Additional beta required
Standard	18	Market M	Risk free rate r_f	None
Zero beta	21	Arbitrary frontier P	Frontier uncorrelated $Z(P)$	None
Non-frontier zero beta	21	Arbitrary frontier P	Corresponding uncorrelated $Z(P)$	None
Null index zero beta	22	Null index N	Zero mean zero beta $Z(N)$	None
Augmented frontier (CML)	24	Arbitrary augmented frontier P	Risk free rate r_f	None
Standard tangency	25	Risky assets tangency T	Risk free rate r_f	None
Inefficient portfolio	28	Risky assets inefficient I	Risk free rate r_f	Asset wrt arbitrage portfolio
Global minimum variance	26	Arbitrary frontier P	GMVP G	GMVP wrt primary portfolio
Benchmark	27	Arbitrary frontier P	Arbitrary benchmark B	Benchmark wrt primary portfolio

CAPM type formulations may all be written in the general linear format "constant plus price of risk times (adjusted) asset beta(s)". Equation number refers to number in text. The primary portfolio is the portfolio whose expected return is the first term in the "price of risk", and with respect to which any asset's beta is computed. The secondary portfolio is the portfolio whose expected return is the constant term and its negative value is the second term in the "price of risk". Asset betas are defined as the covariance between the returns of the asset and the primary portfolio, divided by the variance of the primary portfolio's return. The rightmost column indicates additional betas required in the respective CAPM types.