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## Discussion paper

# Making Bank: Why High Bank Leverage is Optimal – for the Bank's Shareholders

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## Abstract

We create a structural credit model to calculate the optimal capital structure for a bank that provides asset backed loans, such as corporate loans and mortgages. The bank’s assets are loans, which means that the bank’s exposure to risk is mitigated by the borrower’s equity. We capture the effect of this mitigation by including the borrower’s leverage, in addition to its asset volatility, as the sources of risk for the bank. Our results contribute a quantitative explanation for the high levels of bank leverage observed in practice. When unconstrained by regulation, the bank’s shareholders find it optimal, for reasonable values of borrower risk parameters, to select a bank leverage close to 100%.

# 1 Introduction

We create a structural credit model to calculate the optimal capital structure for a bank that provides asset backed loans, such as corporate loans and mortgages. Lending banks differ from regular companies in that their assets are loans, which are fundamentally distinct from other assets in the following ways. First, their market value is capped by the risk free value of their promised payments. Second, their exposure to the risk in the assets they finance is mitigated by the borrowers' equity. We capture both these properties in our model. We include the borrowers' leverage as a source of bank asset risk, in addition to the borrowers' asset volatility. The model may be applied to any lender that makes asset backed loans, where the value of the financed asset exceeds the value of the loan.

Our results show that it is optimal for the bank's shareholders, applying tradeoff theory (Miller, 1977) and assuming standard capital market frictions, to select a bank leverage close to 100%, which is equivalent to a situation where equity capital does not represent a restriction on a bank's total asset size. This result reconciles with the high bank leverage observed in practice – historically between 87% and 95% in US banks (Gornall and Strebulaev, 2013). Table 1 shows that, globally, loans make up more than 70% of banks' total assets and banks typically have a leverage ratio of approximately 80%. Introducing funding constraints on the bank, such as debt covenants, causes the optimal bank leverage to drop relative to the unconstrained optimum. Note that our model does not include properties of regulatory risk based capital requirements commonly applied to banks.

**Table 1. Global banks' lending and leverage**

Variable	Mean	Q1	Median	Q3
Loans and debt instruments scaled by total assets:				
- <i>Sample</i>	73%	61%	72%	83%
- <i>Weighted by total assets</i>	69%			
Bank deposits and other debt scaled by total assets:				
- <i>Sample</i>	79%	72%	82%	89%
- <i>Weighted by total assets</i>	82%			

*Source:* SNL Financial, data for 2,347 banks globally, 31 Dec 2014.

If we let the borrower's leverage approach 1, our bank capital structure results converge to the results from Leland (1994), in which the bank's asset value follows a geometric Brownian motion.

Thus, using a geometric Brownian motion to model a bank's asset value is the same as assuming that the bank only makes loans to borrowers with zero equity. The unlikely existence of a bank that only lends to such high risk borrowers explains why we cannot expect the standard Leland (1994) results for regular companies to apply to banks.

In our model the bank has one borrower at a time, which allows for a direct and transparent link between borrower and bank risk. This assumption also implies that any risk reduction from a bank's loan portfolio diversification is not explicitly included. However, even with this unrealistically risky asset we find a high optimal bank leverage. A lower and more realistic asset risk for a portfolio of loans would lead to an even higher optimal leverage. We do not include the bank's role as a liquidity provider as an explanation of their optimal capital structure (DeAngelo and Stulz, 2015). In the tradition of structural credit risk models, we further disregard any bank/borrower relationship lending features driven by information asymmetries like covenants or collateral. We exclude any investments as well as fee generating off balance sheet business, which clearly could be modeled, as additional components of the bank's assets, using geometric Brownian motions. The lending bank's asset risk is entirely derived from its borrower's asset risk and leverage, as is natural for a financial intermediary.

In finite horizon structural credit risk models, starting with Merton (1974), quantities of interest (probability of default, debt value, etc.) are functions of a fixed future date,  $T$ . The economic interpretation of  $T$  is unclear, it could be a date of some large audit, or a date on which the firm plans to stop operating. In certain contexts, like when analyzing optimal capital structure, the standard interpretation of  $T$  as debt maturity is economically uninteresting because the debt contracts of solvent borrowers are typically renewed. In contrast, we apply an infinite time horizon and model all securities as perpetual instruments, cf. the approach of Black and Cox (1976) and Leland (1994) among others. In these models, default occurs the first time that the debtor's (bank or borrower) asset value drops below a default threshold. We measure the riskiness of borrower and bank assets by their state price of default, which is a primitive security that pays 1 at default (Arrow and Debreu, 1954). We show that the classical optimal default threshold (Black and Cox, 1976) and capital structure (Leland, 1994) models can be applied to banks by replacing the standard geometric Brownian motion based state price of default with the lending bank's state price of default.

We explicitly model the bank's assets as loans, which is similar to the approaches of Dermine and Lajeri (2001), Chen, Ju, Mazumdar, and Verma (2006) and Nagel and Purnanandam (2015). Unlike in the above articles, we model both the borrowers' and the bank's assets and liabilities with an infinite time horizon. Several articles study the reasons for high bank leverage in an infinite time horizon setting, but, as opposed to our model, use a geometric Brownian motion to model the bank's asset value. Harding, Liang, and Ross (2013) add the effects of deposit insurance to the Leland (1994) setup. Sundaresan and Wang (2014) add a layer of subordinated debt to the bank's funding mix to explain the bank's endogenous choice of leverage. Nagel and Purnanandam (2015) tackle the unsuitability of the geometric Brownian motion for modeling a bank's asset value by using a pool of loans to calculate bank default risk. Gornall and Strebulaev (2013) model the bank's and borrower's capital structure decision as a joint optimization problem. They argue that high bank leverage is observed because of the low volatility in bank's assets. This lower volatility is explained by the bank's position as senior creditor among the borrower's liabilities, and also due to loan portfolio diversification effects.

There is a large literature studying other bank features such as deposit insurance, contingent capital securities, executive compensation, and regulatory requirements. Dermine and Lajeri (2001) analyze the cost of deposit insurance by modeling a bank's assets as one loan, which is a zero coupon bond with a finite horizon, as is the bank's own debt. Mjøs and Persson (2010) analyze finite horizon claims on debt contracts with infinite horizons. Similar to their approach, we value claims using debt contracts as the underlying asset. Glasserman and Nouri (2012) analyze the case of contingent capital securities with a capital-ratio trigger and both partial and ongoing conversions. They use a geometric Brownian motion to model the bank's asset value. Pennacchi (2010) presents a structural credit risk model of a bank that issues contingent capital bonds. He uses a jump-diffusion process to model the returns on the bank's assets. Our model is general enough to be applied to the analysis of such bank features, but our focus in this paper is on a bank's optimal capital structure using standard static tradeoff theory.

While the discussion of a bank's capital structure is generally interesting, recent events have much increased the tension in this discussion. The researchers mentioned above have approached this discussion quantitatively, while other researchers, like Admati, DeMarzo, Hellwig, and Pfleiderer (2013), have approached it qualitatively. Most of them agree that banks were too highly

leveraged going into the crisis of 2007-2009, but there is no agreement on what the appropriate level of bank leverage over the business cycle is. We contribute to this literature by quantifying, in an internally consistent structural framework, the risk of a bank's default and the effects of different types of bank debt on a bank shareholders' capital structure decision.

Our focus is on the optimal capital structure of a lending bank, given the tax benefits of debt financing and bankruptcy costs, and thus we do not explicitly consider the borrower's optimal capital structure. This approach also allows for a study of the sensitivities of a bank's optimal leverage to alternative combinations of borrower asset volatility and leverage.

Black and Cox (1976) use the state price of default to value a perpetual debt contract. Leland (1994) combines the state price of default with tradeoff theory to calculate the optimal capital structure for a firm. Our approach builds on these models and contains their results as special cases, when borrower leverage approaches 1. This convergence is not obvious because the bank asset value process, even in this special case, does not converge to a geometric Brownian motion (see Figure 2). But the bank state price of default still converges to the geometric Brownian motion state price of default, which explains why the results coincide (see Proposition 5 and Figure 4).

## 2 The Model

Our model consists of a bank and its borrowers. A probability space  $(\Omega, \mathcal{F}, Q)$  is given. The set  $\Omega$  consists of all the possible states of the world. Here  $\mathcal{F}$  is a  $\sigma$ -algebra of subsets of  $\Omega$  and  $Q$  is the equivalent martingale measure. We also impose the standard frictionless, continuous time market assumptions, see, e.g., Duffie (2001). This assumption implies that all interest rates on debt and deposits exclude any transaction costs, share of bank fixed costs, or any bank profit, expected to be included in observed market rates.

### 2.1 A Borrower

A borrower finances itself through some combination of equity and debt. Its debt is in the form of a bank loan, which is modeled as a Black and Cox (1976) type perpetual debt contract that pays a continuous coupon. We disregard taxes and bankruptcy costs at the borrower level, since we are not optimizing the capital structure of the borrower. Let  $\tau(0)$  be the time of loan origination.

Let  $A_t$  be the asset value of the borrower, with time  $\tau(0)$  value  $A_{\tau(0)}$ , such that

$$dA_t = rA_t dt + \sigma A_t dW_t, \quad (1)$$

where  $\sigma$  is a constant and  $W_t$  is a standard Brownian motion under the equivalent martingale measure  $Q$ . Here,  $r$  is the constant, continuously compounded, risk-free rate of return. The borrower borrows  $\hat{B}$  from the bank, such that its leverage at loan origination

$$L = \frac{\hat{B}}{A_{\tau(0)}}, \quad (2)$$

where  $0 < L < 1$ . The borrower pays a constant continuous coupon rate,  $c$ , on its loan.

The borrower will default on its loan when its asset value ( $A_t$ ) first hits a threshold  $\bar{A}$ . The value of the loan can be expressed as a function of the borrower's asset value and the default threshold as (Black and Cox, 1976)

$$B(A_t) = \frac{c\hat{B}}{r} - \left( \frac{c\hat{B}}{r} - \bar{A} \right) \left( \frac{A_t}{\bar{A}} \right)^{-\gamma}, \quad (3)$$

where

$$\gamma = \frac{2r}{\sigma^2} > 0$$

and the bank recovers  $\bar{A}$  in the event of default. Expression 3 also includes the value of a risk-free perpetual coupon stream ( $\frac{c\hat{B}}{r}$ ), and the state price of the borrower's default,  $\left(\frac{A_t}{\bar{A}}\right)^{-\gamma}$ . The state price of the borrower's default (Arrow and Debreu, 1954) is the price of a security that pays 1 in the event of default by the borrower. The time  $t$  price of this security is

$$G_t = \left( \frac{A_t}{\bar{A}} \right)^{-\gamma}. \quad (4)$$

This  $G_t$  process takes values in the interval  $(0, 1]$ . Borrower default occurs when the process hits 1. Using Itô's lemma on Expression 1, one can show that

$$\frac{dG_t}{G_t} = r dt - \sigma \gamma dW_t, \quad (5)$$

where  $G_{\tau(0)} = \left( \frac{A_{\tau(0)}}{\bar{A}} \right)^{-\gamma}$ .



The borrower defaults at time  $\tau(1)$ , where

$$\tau(1) = \inf\{t \geq \tau(0) : A_t = \bar{A}\} = \inf\{t \geq \tau(0) : G_t = 1\}.$$

The borrower's default threshold ( $\bar{A}$ ) is determined endogenously in our model. The borrower optimally picks  $\bar{A}$  to minimize the value of debt (Expression 3) for a given face value of debt ( $\hat{B}$ ) and coupon rate ( $c$ ). Black and Cox (1976) show that

$$\bar{A} = \Psi \hat{B}, \tag{6}$$

where

$$\Psi = \frac{c}{r} \frac{\gamma}{\gamma + 1} < 1 \tag{7}$$

is the factor which the face value of the borrower's debt is reduced by to obtain the borrower's optimal default threshold. With this notation we can write the initial value of the state price of the borrower's default (Expression 5) as  $G_{\tau(0)} = (L\Psi)^\gamma = G$ .

The market value of the loan at origination equals its face value,  $B(A_{\tau(0)}) = \hat{B}$ , i.e. the loan is granted at par. This assumption, combined with Expressions 3 and 6, lets us numerically calculate the coupon rate ( $c$ ) paid by the borrower from the equation

$$\frac{c}{r} \left(1 - \frac{G}{\gamma + 1}\right) = 1. \tag{8}$$

Observe that  $G$  depends on  $c$  through  $\Psi$ .

## 2.2 The Bank

The bank's only assets are loans and its time 0 asset value is  $B$ . We postpone the introduction of capital market frictions at the bank level, until we optimize its capital structure in Section 3. The bank only lends to one borrower, of the type described in Section 2.1, at a time. When this borrower defaults, the recovered amount is lent to the next borrower. All borrowers have the same constant volatility ( $\sigma$ ) and the same initial leverage ( $L$ ), when their loan is granted. The first loan originates at time  $\tau(0) = 0$  and the sequence of borrowers is indexed by  $j \geq 1$ . The loan pays a

continuous coupon stream of  $c\hat{B}^j$  until borrower  $j$  defaults, where  $\hat{B}^j$  is the face amount of the loan to borrower  $j$ . Upon default by borrower  $j$ , the bank reinvests the recovered amount ( $\bar{A}^j$ ) in a new loan to borrower  $j + 1$ , which means that the face value of the new loan ( $\hat{B}^{j+1}$ ) equals the amount recovered from the old loan. We disregard bankruptcy costs in the recovery/re-lending process. Since the bank selects borrowers with the same risk characteristics  $(\sigma, L)$ , both  $\hat{B}^j$  and  $\bar{A}^j$  are deterministic sequences in  $j$ . We can express the loan amount to borrower  $j + 1$  in terms of the bank's initial asset value ( $B$ ) and number of defaults ( $j$ ) as

$$\hat{B}^{j+1} = \bar{A}^j = B\Psi^j, \quad (9)$$

where  $\hat{B}^1 = B$ .

The relation between the bank and borrower at time  $t$  is shown in Table 2. This is similar to the approach of Dermine and Lajeri (2001). Equity is conventionally valued as a residual claim on assets after debt is serviced.

**Table 2. Relation between bank and borrower at time  $t$ , in Section 2.2.** The borrower's asset's value  $A_t^j$  evolves according to Expression 1. The value of the borrower's loan  $D_t^j$  is calculated by Expression 3. This loan is also the bank's only asset, whose value is denoted  $B_t$ . The bank's debt is denoted  $D_t(B)$  and its value is calculated in Section 2.2.2. Equity is valued as a residual claim on assets after debt is serviced.

Borrower $j$ balance sheet		----->	Bank balance sheet	
$A_t^j$	$D_t^j = B(A_t^j) \bullet$		$B_t = D_t^j$	$D_t(B)$
	$E_t^j = A_t^j - D_t^j$			$E_t(B) = B_t - D_t(B)$
$A_t^j$	$A_t^j$		$B_t$	$B_t$

**2.2.1 The bank's asset value process.** Since the bank's assets, at any point in time, consist of only one loan, its asset value must be equal to the value of that loan. While the first borrower is solvent,  $t < \tau(1)$ , the bank's assets can be valued using Expression 3 as

$$\begin{aligned} B_t &= B(A_t) \\ &= \frac{c\hat{B}^1}{r} - \left( \frac{c\hat{B}^1}{r} - \bar{A} \right) \left( \frac{A_t}{\bar{A}} \right)^{-\gamma}. \end{aligned}$$

Using Expressions 4 and 6 we get

$$B_t = \frac{c\hat{B}^1}{r} \left( 1 - \frac{G_t}{\gamma + 1} \right). \quad (10)$$

To obtain a general expression for the bank's asset value process, valid for any time and borrower, we need expressions for the face value of the loan outstanding at time  $t$  and the state price of default of the borrower at time  $t$ . For this we consider the arithmetic Brownian motion  $Y_t$ , whose dynamics are

$$dY_t = \nu dt - dW_t, \quad (11)$$

where

$$Y_0 = 0.$$

Here

$$\nu = \frac{r}{\sigma\gamma} - \frac{\sigma\gamma}{2}$$

and

$$d = \frac{1}{\sigma\gamma} \log \left( \frac{1}{G} \right).$$

The constant  $d$  is the borrower's normalized distance to default (Merton, 1974), at loan origination. In particular,  $\log \left( \frac{1}{G} \right)$  is a measure of the distance to default in terms of the borrower's state price of default, and  $\sigma\gamma$ , the volatility of the  $G$  process, is the normalization factor. Observe that  $G_t$  in Expression 4 can be written as  $G_t = Ge^{\sigma\gamma Y_t}$ , which means that  $Y_t$  determines the dynamics of  $G_t$ . Define

$$\eta_t = \sup_{0 \leq s \leq t} Y_s. \quad (12)$$

The number of borrower defaults  $N_t$  up to time  $t$  is calculated from  $\eta_t$  as

$$N_t = \lfloor \eta_t / d \rfloor,$$

where  $\lfloor x \rfloor$  is the integer part of  $x \in \mathbb{R}^+$ .

All the information we need to calculate the value of the bank's assets ( $B_t$ ) is contained in  $Y_t$  and its functional,  $\eta_t$ , as well as the constant borrower's asset volatility and leverage ( $\sigma, L$ ) and the constant risk-free rate ( $r$ ).

**Proposition 1.** *The state price of default of the borrower at time  $t$ ,  $\Pi_t$  can be expressed in terms of  $Y_t$  and  $N_t$  as*

$$\Pi_t = Ge^{\sigma\gamma(Y_t - N_t d)}.$$

**Proposition 2.** *The time  $t$  value of the bank's assets is*

$$B_t = \frac{c}{r} B \Psi^{N_t} \left( 1 - \frac{\Pi_t}{\gamma + 1} \right). \quad (13)$$

Note that Expression 13 has the same structure as Expression 10. The face value of the loan to the borrower at time  $t$  is  $B\Psi^{N_t}$ .

Figure 1 shows a simulated path of how the arithmetic Brownian motion  $Y_t$  and its running maximum  $\eta_t$  might evolve over time, and how they drive the bank's asset value process ( $B_t$ ) through the borrowers' state price of default ( $\Pi_t$ ). Panel 1 shows one realization of  $Y_t$  (Expression 11) and  $\eta_t$ , both scaled by the borrower's normalized distance to default,  $d$ . Defaults occur every time  $\eta_t$  goes up one normalized distance to borrower default. This realization of the process contains two borrower defaults. Each default results in a new loan being granted by the bank, which causes the borrower's state price of default,  $\Pi_t$ , to reset to  $G$ . This effect can be seen in Panel 2 of Figure 1. Panel 3 of Figure 1 shows how the bank's asset value process,  $B_t$  (Expression 10), evolves in this scenario. This panel also shows the evolution of a geometric Brownian motion ( $S_t$ ) using the same input parameters and innovations as  $Y_t$ . We see that our bank's asset value process ( $B_t$ ) doesn't achieve the same extremes in value as the geometric Brownian motion ( $S_t$ ), even at the high amount of borrower leverage (75%) that we use in this example.

Figure 2 shows the simulated probability density function of  $S_T$ , which has a log-normal distribution, and  $B_T$  for a defined set of parameters, at a fixed future time  $T=4$ . One can see that the value of the bank's assets ( $B_T$ ) does not have a log-normal distribution and is capped, unlike the value of the log-normally distributed asset ( $S_T$ ). One can also see that the lower the borrower's leverage ( $L$ ) the lower the dispersion of potential outcomes for the bank's asset value process. For

the lowest borrower leverage ( $L = 25\%$ ) it is very likely that there will be no defaults within this 4 year window. As borrower leverage increases ( $L = 50\%, 75\%$ ) the likelihood of default in the 4 year window increases, which can be seen in the multiple modes of the corresponding distributions. When we let borrower leverage approach 1 ( $L = 99.9999\%$ ) the distribution of possible outcomes becomes smoother and more dispersed, relative to the outcomes in the lower borrower leverage cases. Increased leverage smoothens the left side of the distribution because defaults are more frequent. However, the distribution of the high leverage outcome is not log-normal and the value of the loan to the high leverage borrower is still capped.

**2.2.2 The bank's debt.** We also model the bank's debt as Black and Cox (1976) type perpetual debt that pays a continuous coupon. Like the borrower, the bank defaults on its debt the first time the value of its assets ( $B_t$ ) crosses some arbitrary threshold  $\bar{B} (< B)$ . The bank defaults at time

$$\tau_{\bar{B}} = \inf\{t \geq 0 : B_t = \bar{B}\}.$$

We define

$$m_t = \inf_{0 \leq s \leq t} B_s$$

as the minimum value of the bank's assets up to time  $t$ .

The distribution of the bank's default time can then be expressed in terms of the distribution of the minimum value process as

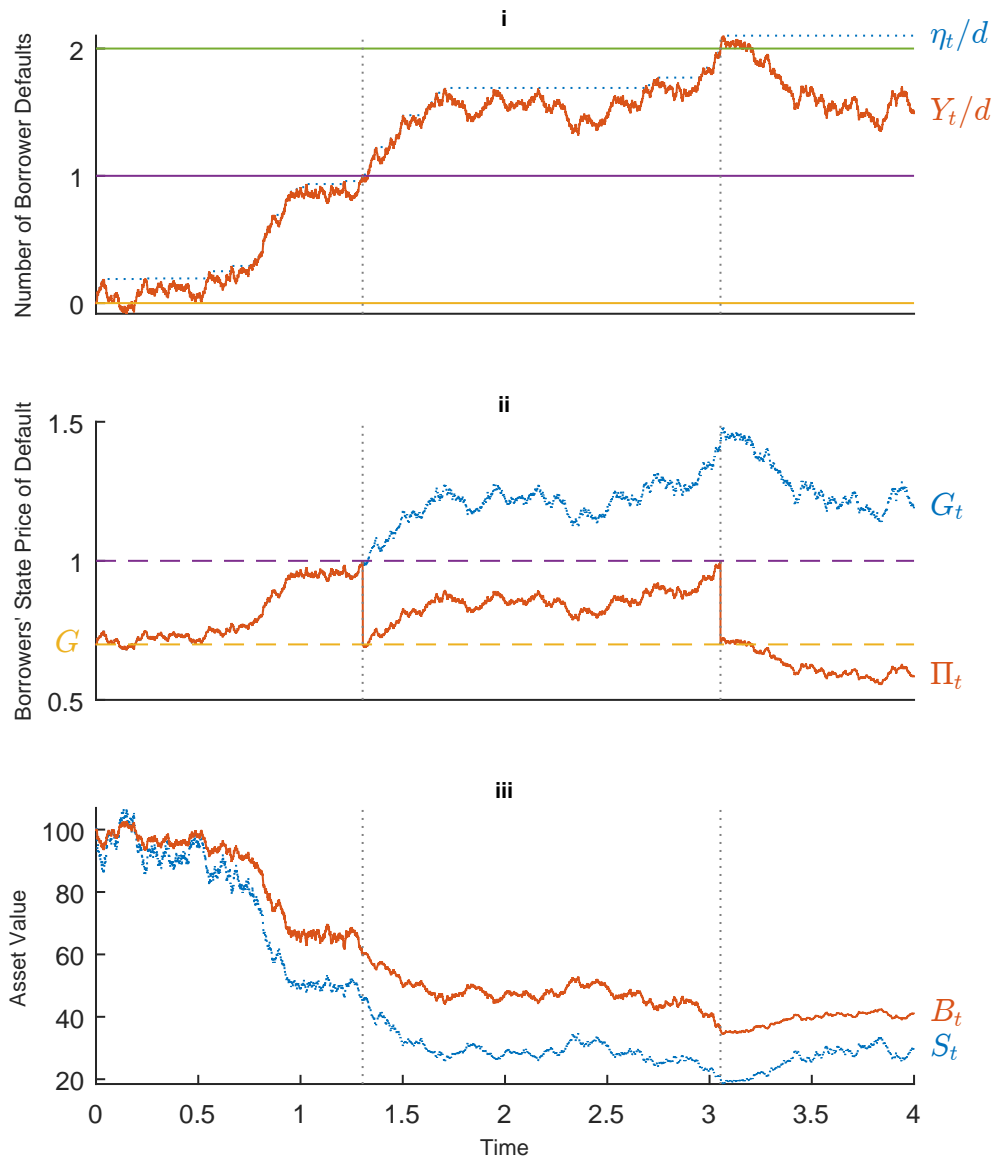
$$Q(\tau_{\bar{B}} < t) = Q(m_t < \bar{B}). \tag{14}$$

The following proposition shows that the above probability can also be expressed in terms of  $\eta_t$  in Expression 12, the supremum of an arithmetic Brownian motion, for which the probability distribution is known (see e.g. Harrison, 1985).

**Proposition 3.**  $Q(m_t < \bar{B}) = Q(\eta_t > n_{\bar{B}} \cdot d)$ .

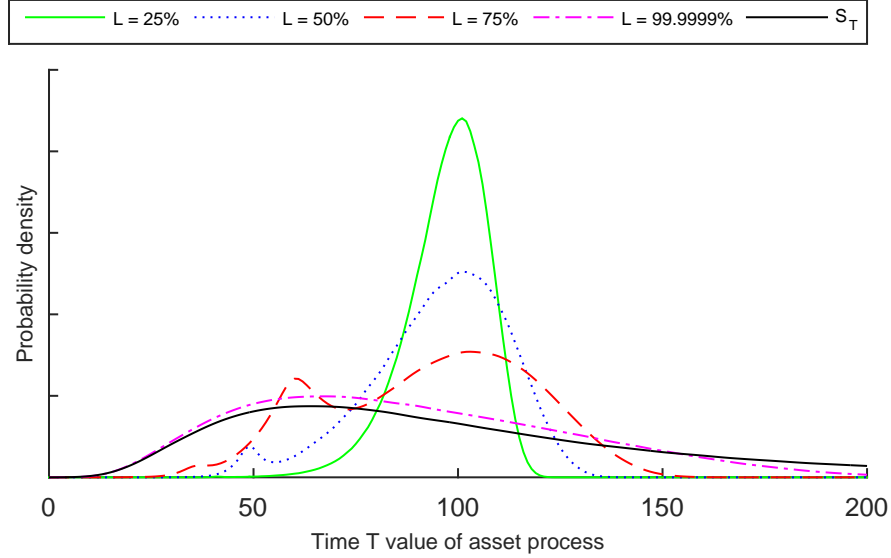
Here,

$$n_{\bar{B}} = M_{\bar{B}} + \left( 1 - \frac{\log \left[ (\gamma + 1) \left( 1 - \frac{r}{c} \frac{\bar{B}}{B \Psi^{M_{\bar{B}}}} \right) \right]}{\log G} \right), \tag{15}$$



**Figure 1.**

One realization of the  $Y_t$  process and its effect, through the borrowers' state price of default  $\Pi_t$ , on the bank's asset value process  $B_t$ . For contrast, the lowest panel includes a geometric Brownian motion,  $S_t$ , which is subjected to the same innovations as  $Y_t$ .  $T = 4$ ,  $\sigma = 30\%$ ,  $L = 75\%$ ,  $r = 2\%$ ,  $S_0 = B_0 = 100$ . The vertical, grey, dotted lines show borrower defaults.



**Figure 2.** Simulated probability density function of  $S_T$  (log-normal distribution) and  $B_T$  (for different levels of borrower leverage,  $L$ ).  $T = 4$ ,  $\sigma = 30\%$ ,  $r = 2\%$ ,  $S_0 = B_0 = 100$ .

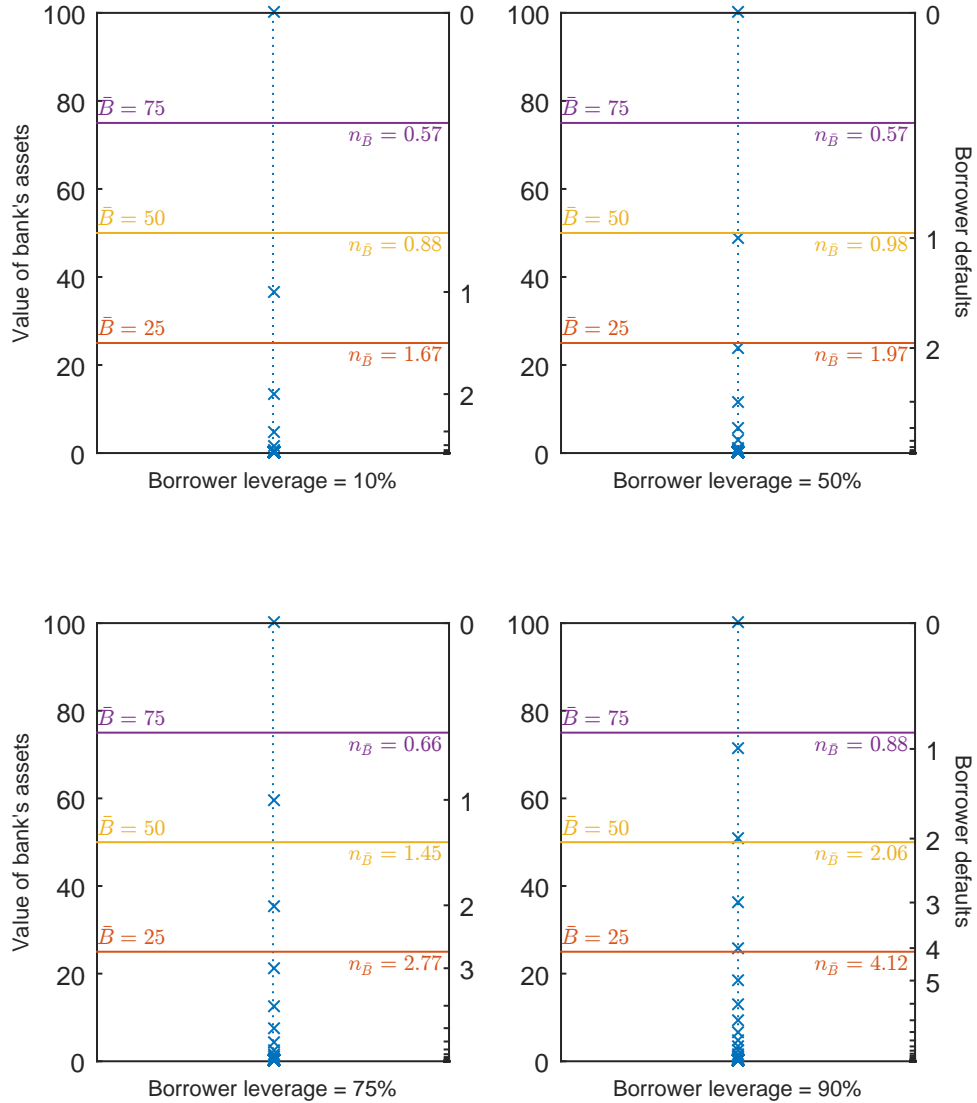
where

$$M_{\bar{B}} = \max\{n : B\Psi^n > \bar{B}\} = \left\lfloor \frac{\log \bar{B} - \log B}{\log \Psi} \right\rfloor.$$

Also,  $M_{\bar{B}}$  is the maximum number of borrower defaults that the bank can endure before it defaults. The bank defaults when its asset value falls below the threshold  $\bar{B}$ , and  $n_{\bar{B}}$  can be interpreted as the bank's distance to default measured in number of sequential borrower defaults. Note that  $n_{\bar{B}}$  is an invertible function of  $\bar{B}$ , that is for a given number of borrower defaults  $n$ , the bank's default threshold  $\bar{B}$  can be expressed as a function of  $n$ . An example of this calculation can be seen in Figure 3. The figure shows that the frequency of borrower defaults is increasing in borrower leverage. It also shows that  $n_{\bar{B}}$ , the number of borrower defaults before the bank defaults, is falling in the bank's own default threshold,  $\bar{B}$ .

The distribution of  $\eta_t$  is known, which implies that  $\tau_{\bar{B}}$  has an inverse Gaussian (IG) distribution (Schrödinger, 1915),

$$\tau_{\bar{B}} \sim \text{IG} \left( \frac{n_{\bar{B}} \cdot d}{\nu}, (n_{\bar{B}} \cdot d)^2 \right),$$



**Figure 3.** The bank's distance to default ( $n_{\bar{B}}$ ) in terms of number of borrower defaults by different levels of borrower leverage. Borrower  $\sigma = 30\%$ .  $\bar{B}$  represents different, arbitrary, bank default thresholds. The bank's initial asset value is 100. The markers indicate the bank's asset value at sequential borrower defaults.



where  $IG(\mu, \lambda)$  indicates an inverse Gaussian distribution with mean  $\mu$  and shape parameter  $\lambda$ .

This formulation allows us to calculate the state price of the bank's default in the following proposition.

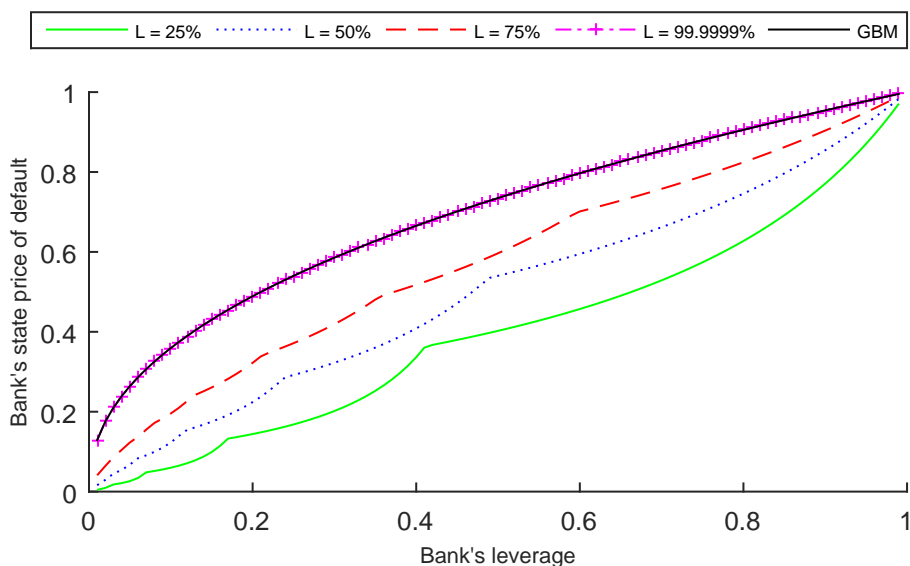
**Proposition 4.** *The time 0 value of the state price of the bank's default is*

$$\Pi_{\bar{B}} = G^{n_{\bar{B}}}.$$

An example can be seen in Figure 4 for various levels of bank leverage. The following proposition states that, as borrower leverage approaches one, the state price of the bank's default converges to the standard state price of default from using a geometric Brownian motion to model the bank's asset value.

**Proposition 5.**

$$\lim_{L \nearrow 1} \Pi_{\bar{B}} = \left( \frac{B}{\bar{B}} \right)^{-\gamma}.$$



**Figure 4.** State price of the bank's default at various bank and borrower leverage (L) levels, as well as assuming that the bank's asset value follows a geometric Brownian motion.  $r = 2\%$ .  $\sigma = 30\%$ .

**2.2.3 Optimal bank default threshold for a given amount of bank leverage.** Assume that the perpetual debt issued by the bank has a face value of  $F$ , on which the bank pays a continuous interest of  $i$ . The bank's shareholders maximize their time 0 equity value  $E(B)$  by solving the following

$$E(B) = \sup_{\tau} \mathbb{E} \left[ \int_0^{\tau} (cB\Psi^{N_t} - iF)e^{-rt} dt \right]. \quad (16)$$

This problem has been studied extensively when the firm's operating cash flow follows a geometric Brownian motion (Duffie and Lando, 2001). The solution to this problem is given in the following proposition. We first define  $n^* = \min\{j \geq 1 : cB\Psi^j < iF\}$  with solution

$$n^* = \left\lceil \frac{\log(iF) - \log(cB)}{\log \Psi} \right\rceil, \quad (17)$$

where  $\lceil x \rceil$  is the integer part of  $x + 1$ , where  $x \in \mathbb{R}^+$ .

The bank's cash inflows are stepwise decreasing in time, as the loan to each new borrower is smaller than the loan to the previous, defaulted, borrower. After  $n^*$  borrower defaults, the bank's interest income falls below its fixed interest expense. Denote the optimal stopping time in Problem 16 as  $\tau_{\bar{B}^*}$  and the default time of borrower  $j$  as  $\tau(j)$  (See Appendix A).

**Proposition 6.** *The solution to Problem 16 is*

$$\tau_{\bar{B}^*} = \tau(n^*),$$

where  $\tau(j)$  is defined in Expression 35 and  $n^*$  is defined in Expression 17 and

$$E(B) = B - \left\{ \frac{iF}{r} - \left( \frac{iF}{r} - \bar{B}^* \right) G^{n^*} \right\}$$

This result implies that the bank optimally defaults at the same time as borrower number  $n^*$ . The bank's optimal default threshold is

$$\bar{B}^* = B\Psi^{n^*}. \quad (18)$$

The time 0 value of the bank's own debt is residually calculated as  $D(B) = B - E(B)$ , or

$$D(B) = \frac{iF}{r} - \left( \frac{iF}{r} - \bar{B}^* \right) G^{n^*}. \quad (19)$$

Expression 19 and Expression 3 (the time 0 value of the borrower's debt) only differ in that the bank's state price of default replaces the borrower's state price of default. If the debt is issued at par, that is  $D(B) = F$ , then one could numerically solve for the continuous interest rate paid by the bank on its debt as

$$i = r \left( \frac{1 - \frac{(G\Psi)^{n^*}}{L_B}}{1 - G^{n^*}} \right), \quad (20)$$

where

$$L_B = \frac{F}{B}$$

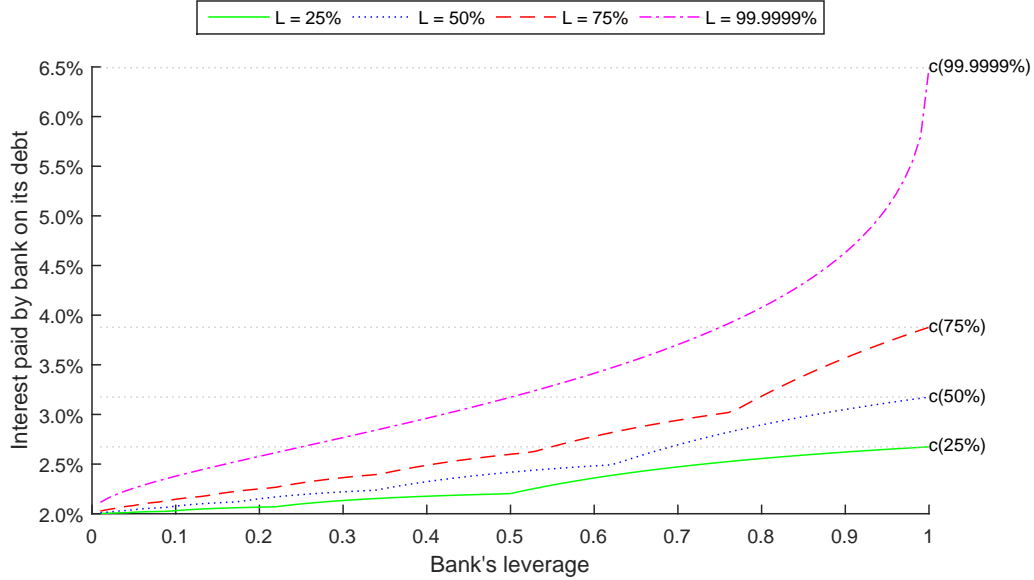
is the initial leverage at the bank. Note that  $n^*$  is a function of  $i$  and  $L_B$  in Expression 20. It can be shown that  $i$  is bounded below by  $r$  and bounded above<sup>1</sup> by  $c$ . Figure 5 shows an example of how the bank's cost of debt varies with its leverage and borrower type, as defined by their leverage. It illustrates that the cost of debt,  $i$ , for a fully levered bank equals the borrower's interest cost,  $c$ .

### 3 The optimal capital structure of a bank in the presence of capital market frictions

We now derive the optimal capital structure for a bank whose shareholders are free to maximize their equity value in the presence of standard capital market frictions. We use the tradeoff theory of optimal capital structure, which balances the tax benefit of debt against bankruptcy costs. In this section, we disregard bank regulation, deposit insurance, and any other bank specific institutional limitations on the bank's capital structure. Following Modigliani and Miller (1958), the capital structure of a firm is irrelevant in the absence of frictions like taxes and bankruptcy costs. We use the framework from Leland (1994) in this section, but let the value of the bank's assets follow the process in Expression 13 rather than a geometric Brownian motion. The bank's optimal capital

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<sup>1</sup>Observe that  $c = r \frac{c(1-G)}{1-G} = r \frac{1-G\Psi}{1-G} > r \frac{1 - \frac{(G\Psi)^{n^*}}{L_B}}{1-G^{n^*}} = i$ . The second equality follows from the par condition (8), see e.g. the proof of Proposition 6.



**Figure 5.** Interest paid by the bank for various values of its initial leverage, by different levels of borrower leverage ( $L$ ). Borrower  $\sigma = 30\%$ .  $c(L)$  is the coupon paid by a borrower with leverage  $L$ .

structure from using a geometric Brownian motion is recovered as a special case in our model, when we let borrower leverage approach 1.

Assume, as in the previous section, that the perpetual debt issued by the bank has a face value of  $F$ , on which the bank pays a continuous interest of  $i$ . Unlike the previous sections, we now include the standard capital market frictions of taxes at a rate of  $0 \leq \theta < 1$ , and a bankruptcy cost of  $0 \leq \alpha < 1$ . The bank now gets a tax deduction on interest payments equal to  $\theta i F$  and the lenders to the bank face a bankruptcy cost  $\alpha \bar{B}$ , where  $\bar{B}$  is the bank's default threshold. We assume no bankruptcy costs for borrowers and that the bank receives  $cB\Psi^{N_t}$  as an after tax interest income at time  $t$ .

For some bank default threshold  $\bar{B}$  with corresponding number of borrower defaults  $n_{\bar{B}}$ , where  $\bar{B} = B\Psi^{n_{\bar{B}}}$ , the time 0 value of the tax benefit for the bank from debt financing is

$$TB(n_{\bar{B}}) = \frac{\theta i F}{r} (1 - G^{n_{\bar{B}}}).$$

Similarly, we can write the time 0 value of the bankruptcy cost incurred by the bank's creditors as

$$BC(n_{\bar{B}}) = \alpha \bar{B} G^{n_{\bar{B}}} = \alpha B (\Psi G)^{n_{\bar{B}}}.$$

The total time 0 enterprise value of the bank is the sum of its asset value and the tradeoff value, which is defined as the difference between the tax benefit and bankruptcy cost,

$$V(n_{\bar{B}}) = B + TO(n_{\bar{B}}),$$

where

$$TO(n_{\bar{B}}) = TB(n_{\bar{B}}) - BC(n_{\bar{B}}) = \frac{\theta i F}{r} (1 - G^{n_{\bar{B}}}) - \alpha B (\Psi G)^{n_{\bar{B}}}. \quad (21)$$

### 3.1 Optimal bank default threshold for a given amount of bank leverage

The bank's shareholders maximize their time 0 equity value  $E_f(B)$  by solving

$$E_f(B) = \sup_{\tau} \mathbb{E} \left[ \int_0^{\tau} (cB \Psi^{N_t} - (1 - \theta)iF) e^{-rt} dt \right]. \quad (22)$$

The subscript  $f$  indicates the presence of frictions. The solution to this problem is given in the following proposition. We first define  $n_f^* = \min\{j \geq 1 : cB \Psi^j < (1 - \theta)iF\}$  with solution

$$n_f^* = \left\lceil \frac{\log[(1 - \theta)iF] - \log(cB)}{\log \Psi} \right\rceil. \quad (23)$$

Denote the optimal stopping time in Problem 22 as  $\tau_{\bar{B}_f^*}$ .

**Proposition 7.** *The solution to Problem 22 is*

$$\tau_{\bar{B}_f^*} = \tau(n_f^*),$$

where  $\tau(j)$  is defined in Expression 35 and  $n_f^*$  is defined in Expression 23, and

$$E_f(B) = B - \left\{ \frac{(1 - \theta)iF}{r} - \left( \frac{(1 - \theta)iF}{r} - \bar{B}_f^* \right) G^{n_f^*} \right\},$$

where  $\bar{B}_f^* = B \Psi^{n_f^*}$ .

Note that, by the definition of  $n_f^*$ , this solution is valid for  $cB\Psi^{n_f^*} < (1 - \theta)iF \leq cB\Psi^{n_f^*-1}$ .

The time 0 value of the bank's payments to its creditors, before any tax effects, is

$$\frac{iF}{r} - \left( \frac{iF}{r} - \bar{B}_f^* \right) G^{n_f^*}. \quad (24)$$

The time 0 value of the bank's tax deductions on interest payments is

$$TB(n_f^*) = \frac{\theta iF}{r} (1 - G^{n_f^*}),$$

which makes the net time 0 value of the bank's debt liability adjusted for the tax benefit equal to

$$D_f(B) = \frac{(1 - \theta)iF}{r} - \left( \frac{(1 - \theta)iF}{r} - \bar{B}_f^* \right) G^{n_f^*}, \quad (25)$$

which is also equal to  $B - E_f(B)$  (See Proposition 7).

The bank's creditors receive the payments whose value is given in Expression 24. But they also, in the event of the bank's default, pay the bankruptcy cost  $\alpha \bar{B}_f^*$ , which reduces the value of their claim. From their perspective the time 0 value of the loan to the bank, before any tax effects for them, is

$$D_f^c(B) = \frac{iF}{r} - \left( \frac{iF}{r} - (1 - \alpha)\bar{B}_f^* \right) G^{n_f^*}. \quad (26)$$

### 3.2 Optimal capital structure

The equity-maximizing default threshold for an unrestricted bank with a given capital structure is given in Proposition 7. We now solve for the capital structure that maximizes the enterprise value of the bank, which also maximizes its equity value. The enterprise value of the bank can be maximized by maximizing the tradeoff between the tax benefit of debt financing and the bankruptcy cost of default. As a reminder, this tradeoff is

$$TO(n_{\bar{B}}) = \frac{\theta iF}{r} (1 - G^{n_{\bar{B}}}) - \alpha B (\Psi G)^{n_{\bar{B}}}.$$

We know from the optimization problem in Section 3.1 that the optimal  $n_{\bar{B}}$  is an integer. That is, the bank defaults exactly when its last borrower defaults. We also know that the possible range of

$iF$  is given by  $cB\Psi^{n_f^*} < (1 - \theta)iF \leq cB\Psi^{n_f^*-1}$  at the optimum. For a given  $n_f^*$  the tradeoff benefit ( $TO$ ) is increasing in  $iF$ . In order to maximize  $TO$  we set  $iF$  to its maximum possible value

$$(iF)^* = \frac{cB\Psi^{n_f^*-1}}{1 - \theta}. \quad (27)$$

The optimal number of borrower defaults belongs to the set of natural numbers ( $n^* \in \mathbb{N}$ ), which makes the optimization a discrete problem. We address this problem by treating the number of borrower defaults as if it is continuous ( $n_{\bar{B}} \in \mathbb{R}^+$ ), and then testing the objective function at the integers above and below the solution from the continuous optimization. To represent the tradeoff as a function of the number of borrower defaults we say

$$TO(n_{\bar{B}}) = \frac{\theta}{r} \left[ \frac{cB\Psi^{(n_{\bar{B}}-1)}}{(1 - \theta)} \right] (1 - G^{n_{\bar{B}}}) - \alpha B(\Psi G)^{n_{\bar{B}}}.$$

**Proposition 8.** *Maximizing this tradeoff with respect to  $n_{\bar{B}}$  gives the integer optimal number of borrower defaults*

$$1 \leq n^* = \begin{cases} \lfloor n_{\bar{B}}^* \rfloor & \text{if } TO(\lfloor n_{\bar{B}}^* \rfloor) > TO(\lceil n_{\bar{B}}^* \rceil) \\ \lceil n_{\bar{B}}^* \rceil & \text{otherwise} \end{cases}$$

where

$$n_{\bar{B}}^* = \frac{\log \left[ \frac{\log \Psi}{\left( \frac{r(1-\theta)\alpha\Psi}{\theta c} + 1 \right) (\log \Psi + \log G)} \right]}{\log G}. \quad (28)$$

The optimal cash flow for the bank to dedicate to debt service is

$$(iF)^* = \frac{cB\Psi^{n^*-1}}{(1 - \theta)}, \quad (29)$$

which makes the optimum value of the bank's debt

$$D(B)^* = \frac{(iF)^*}{r} - \left( \frac{(iF)^*}{r} - (1 - \alpha)B\Psi^{n^*} \right) G^{n^*}, \quad (30)$$

and the optimum enterprise value of the bank

$$V(B)^* = B + \frac{\theta(iF)^*}{r} (1 - G^{n^*}) - \alpha B(\Psi G)^{n^*}. \quad (31)$$

### 3.3 Results

Panel 1 of Figure 6 shows the bank's optimal leverage, which is almost 1 for most reasonable borrower risk attributes. It remains high (75%) even for very risky borrowers. Panel 2 of Figure 6 shows the maximum tradeoff benefit, as a percentage of unlevered bank asset value, which can be extracted by a bank that makes loans to sequential borrowers with constant risk attributes  $(\sigma, L)$ . It is shown that the benefit is decreasing in both dimensions of risk, since both the probability of default and the expected bankruptcy cost are increasing in risk. Panel 3 of Figure 6 shows the optimal number of borrower defaults after which the bank will default. One can see that for a borrower leverage below approximately 70% the bank defaults at exactly the same time as its first borrower. Figure 7 shows a slice from Panel 1 of Figure 6 for very high borrower leverage ( $L = 99.9999\%$ ), and compares the optimal bank leverage from our framework with the optimal leverage for unprotected debt from the Leland (1994) model. This shows that the Leland (1994) result is contained in our framework when the borrower leverage approaches 1.

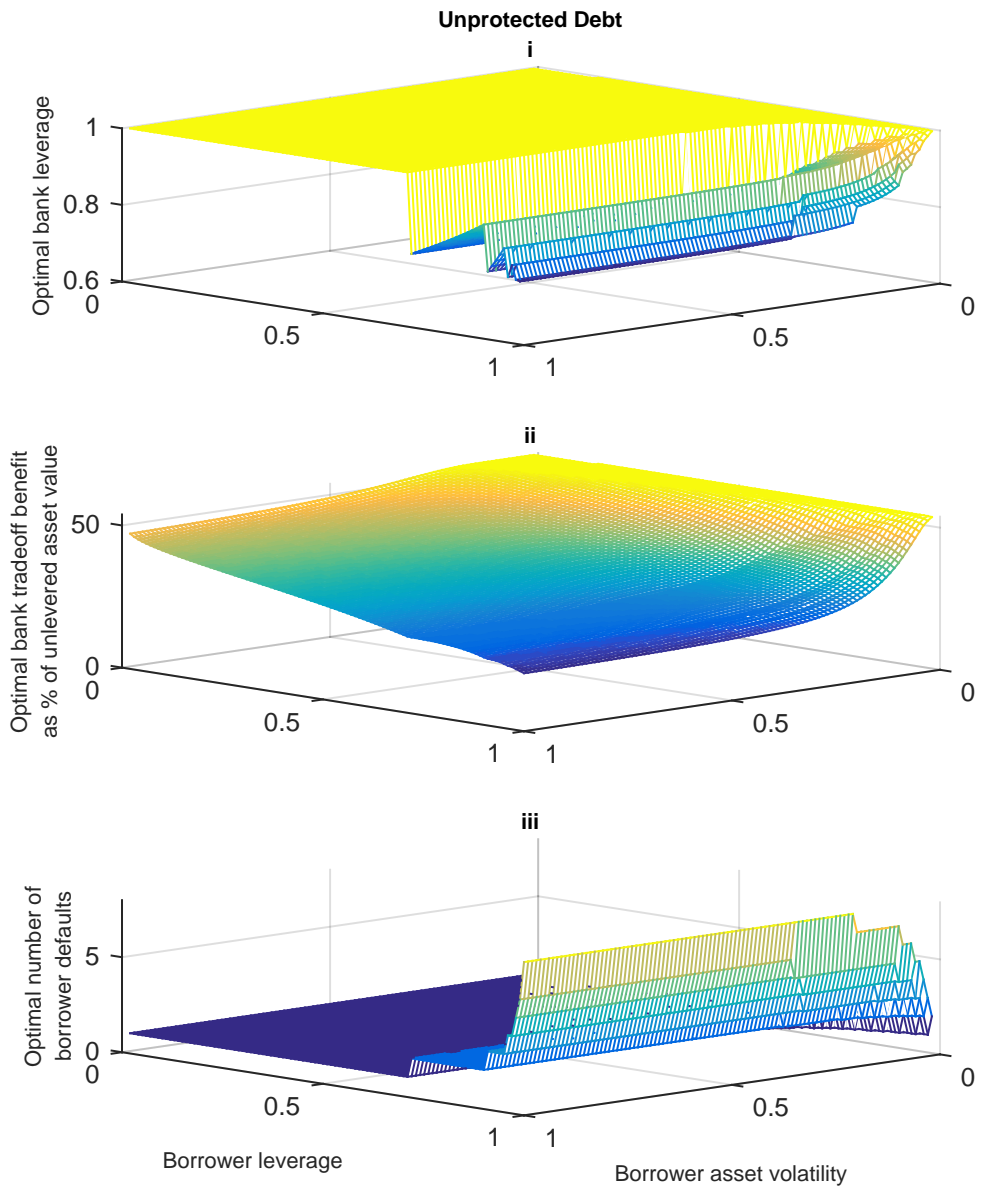
## 4 The optimal capital structure of a bank that issues protected debt

As in Leland (1994), we introduce an assumption that the bank's debt is protected by a covenant that sets the default threshold equal to the face value of the bank's debt. The bank's debt would be protected e.g. if the bank's supervisors instruct the bank to raise additional equity capital before the market value of the bank's funding falls below its face value. We maintain the capital market frictions from Section 3. If this protected debt is issued at par then its value is

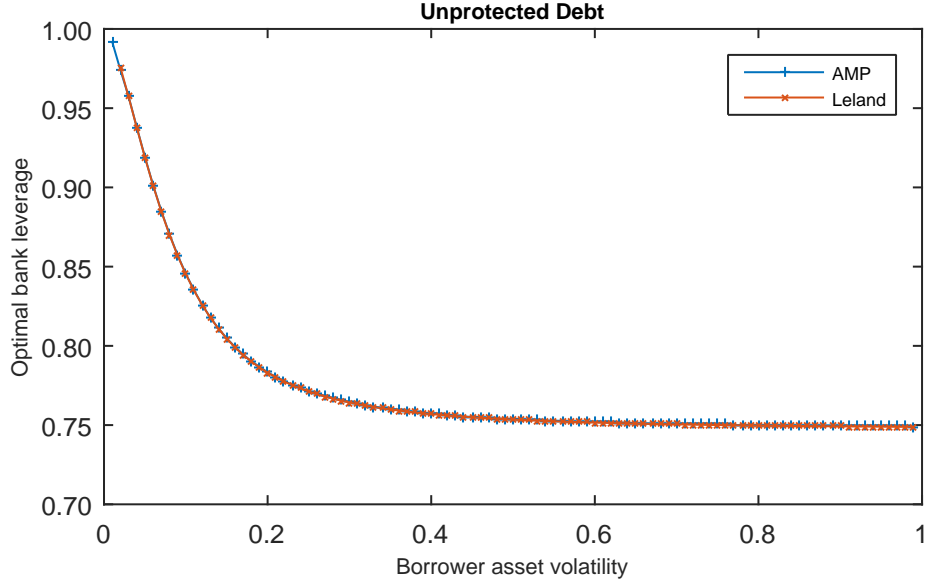
$$F = \frac{iF}{r} - \left( \frac{iF}{r} - (1 - \alpha)F \right) G^{n_F},$$

where  $n_F$  is the distance to the bank's default threshold ( $F$ ) measured in number of borrower defaults. The right hand side of the above expression equals Expression 26 if  $\bar{B}_f^* = F$  and  $n_f^* = n_F$ .





**Figure 6.** Optimal bank leverage, tradeoff benefit and number of borrower defaults assuming unprotected bank debt for different borrower sequences defined by volatility and leverage.  $\alpha = 5\%$ .  $\theta = 35\%$ .  $r = 2\%$ .



**Figure 7.** Optimal bank leverage under Leland (1994) and our model.  $\theta = 35\%$ .  $r = 2\%$ .  $\alpha = 5\%$ .  $L = 99.9999\%$  in our model (AMP).

We can calculate the interest rate<sup>2</sup> that the bank pays on this debt as

$$i = \frac{r}{1 - G^{n_F}} [1 - (1 - \alpha)G^{n_F}]. \quad (32)$$

From Expression 32, we see that the bank's protected debt is risk free if there is no bankruptcy cost ( $\alpha = 0$ ). The bankruptcy cost  $\alpha$  is the only source of default loss for the bank's creditors.

#### 4.1 Optimal default threshold

Unlike the case of unprotected bank debt, there is no possibility for the bank's shareholders to maximize the value of their claim by picking the optimum default threshold, for a given capital structure. The default threshold is exogenously determined by the protective covenant and the existing capital structure. The existing capital structure, however, can be optimized to produce the maximum enterprise value for the bank, which indirectly maximizes the value of the bank's equity.

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<sup>2</sup>Note that  $r \leq i \leq \frac{r}{1 - G^{n_F}}$  when  $\alpha = 0$ .

## 4.2 Optimal capital structure

From Expression 32 and Proposition 3, both the interest rate paid by the bank  $i$  and the bank's default threshold  $F$  can be expressed as functions of some number of borrower defaults  $n$ . Using these functional forms of  $i$  and  $F$  in the tradeoff value, Expression 21, we get

$$\begin{aligned} TO(n) &= \frac{\theta}{r} i(n) F(n) (1 - G^n) - \alpha F(n) G^n \\ &= F(n) [G^n (\alpha \theta - \alpha - \theta) + \theta] \end{aligned} \quad (33)$$

Observe that  $n$  is not necessarily an integer in this case since the bank's default event may be triggered before a borrower default event, depending on the market value of the bank's assets.

**Proposition 9.** *Maximizing the tradeoff with respect to  $n$  gives the optimum*

$$0 \leq n^* = \begin{cases} n_1^* & \text{if } TO(n_1^*) > TO(n_2^*) \\ n_2^* & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} n_1^* &= \lfloor x \rfloor + \{x\}/2 \\ n_2^* &= (\lfloor x \rfloor - 1) + (\{x\} + 1)/2, \end{aligned}$$

and

$$x = \frac{\log \left[ \frac{-\theta}{(\gamma+1)(\alpha\theta - \alpha - \theta)} \right]}{\log G} + 1. \quad (34)$$

## 4.3 Results

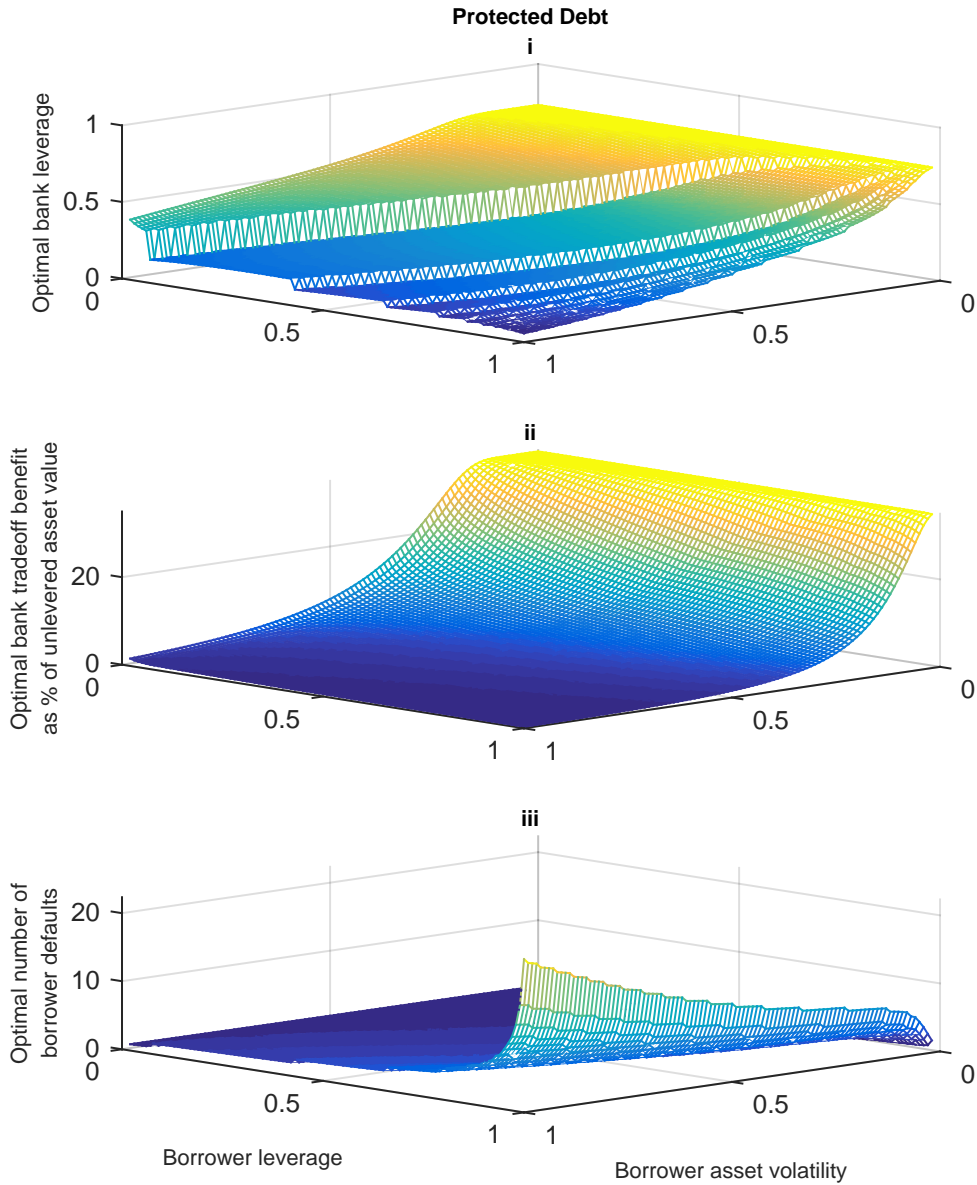
Panel 1 of Figure 8 shows the optimal bank leverage in this scenario, which is lower than its unprotected debt counterpart. Panel 2 of Figure 8 shows the maximum tradeoff benefit that the bank can extract when its debt is protected by covenant. This benefit is a lot lower than in the case of unprotected debt because bank default is more likely to occur now, which reduces the expected tax benefit and increases the expected bankruptcy cost. Panel 3 of Figure 8 shows the optimal number of borrower defaults after which the bank will default. Figure 9 shows the slice from

Panel 1 of Figure 8 for borrower leverage very close to 1, and compares our result with that from Leland (1994). The results are the same when there is no bankruptcy cost ( $\alpha = 0$ ). We use this setting because Leland (1994) only includes a closed-form solution for protected debt when there is no bankruptcy cost. We also include the result from our framework for high borrower leverage and a bankruptcy cost of 5%, to show the large effect of this cost in the case of protected debt. Comparing these results to the results from the unprotected debt case (Section 3), the optimal bank leverage is much lower for any combination of borrower asset volatility and leverage. In addition, the sensitivity of the optimal leverage to borrower asset volatility is much higher for protected debt in the presence of bankruptcy costs.

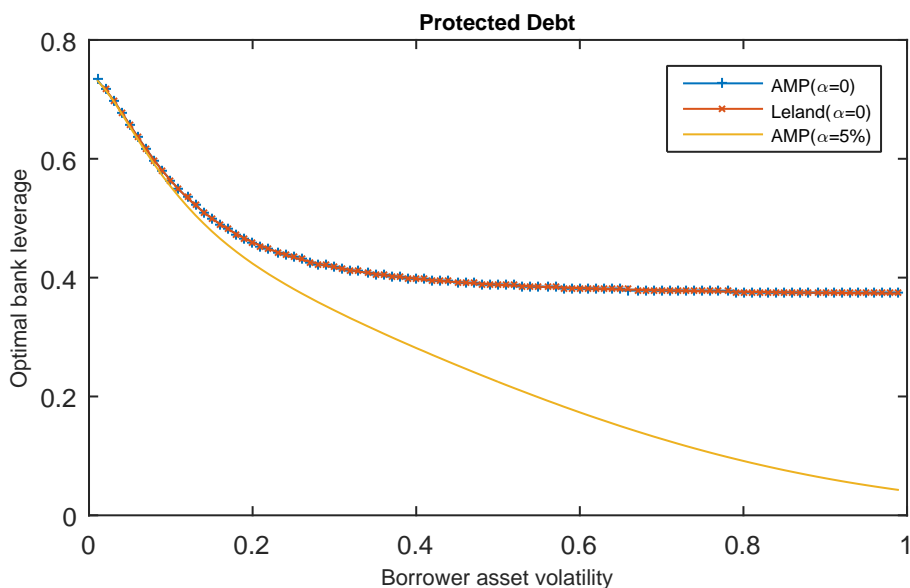
## 5 Conclusion

Banks are highly leveraged relative to other types of firms. Gornall and Strebulaev (2013) estimate the average leverage of U.S. banks in the range of 87%-95% over the last 80 years. This empirical fact has drawn many financial researchers to offer potential explanations. Many of these explanations focus on how the government's regulation of the banking sector introduces distortions, like the moral hazard from having an implicit government guarantee. Such a distortion would induce the bank's shareholders to choose a higher leverage for the bank, as the bank would pay a lower cost for its debt relative to the risk in its assets. We model the risk in a bank's assets anew, by treating the bank as a pure lender, offering asset backed loans, and valuing its assets as such. In the unconstrained scenario, our results show that it is optimal for the bank's shareholders to leverage the bank close to 100% even in the absence of regulatory distortions. The result is driven by the lower risk, as measured by the state price of default, in the bank's asset value process relative to a geometric Brownian motion based asset value process. Even though the bank's shareholders pay a fair cost for their debt, they find it optimal to select a bank leverage close to 100%.

Although our model doesn't explicitly capture the benefits of diversification in a bank's loan portfolio, simulation results show that this benefit could be represented by using a lower borrower asset volatility. Our result for the bank's optimal leverage is discontinuous at certain points on the borrower volatility/leverage plane, but this is due to the discontinuous changes in the bank's interest income, due to borrower defaults. In the absence of regulatory intervention, the bank continues to



**Figure 8.** Optimal bank leverage, tradeoff benefit and number of borrower defaults for different borrower sequences defined by volatility and leverage.  $\alpha = 5\%$ .  $\theta = 35\%$ .  $r = 2\%$ .



**Figure 9.** Optimal bank leverage under Leland (1994) and our model.  $\theta = 35\%$ .  $r = 2\%$ .  $L = 99.9999\%$  in our model (AMP).

service its debt as long as it has the cash flow to do so. Multiple researchers (Dermine and Lajeri, 2001; Chen et al., 2006; Nagel and Purnanandam, 2015) share the idea that a bank’s assets should not be modeled as a standard geometric Brownian motion. Our model is different in the following ways. First, we build the bank’s asset value process on standard (Black and Cox, 1976) blocks. Second, we explicitly add borrower leverage as a dimension of risk for the bank. Third, our results converge to those from Black and Cox (1976) and Leland (1994) as we allow borrower leverage to approach 1, which connects us with the established literature on structural credit models and optimal capital structure for regular firms. Last, we model all assets and liabilities as perpetual instruments, and thus avoid having to interpret the meaning of an arbitrary event horizon. After all, most firms don’t cease to be at time  $T$ .

## Appendix

### A Proof of Proposition 3

We generalize the single borrower default time  $\tau(1)$  described in Section 2.1 to the default time of any borrower  $j$  as

$$\tau(j) = \inf\{t > \tau(j-1) : A_t^j = \bar{A}^j\}, j \geq 1. \quad (35)$$

Here,

$$dA_t^j = rA_t^j dt + \sigma A_t^j dW_t,$$

and, extending Expression 2,  $A_{\tau(j-1)}^j = \frac{B\Psi^{j-1}}{L}$ .

The value of the bank's assets,  $B_t$ , is time continuous. If  $\tau(N_t)$  is the time of default number  $N_t$ , then

$$\lim_{t \nearrow \tau(N_t)} B_t = \lim_{\Pi_t \rightarrow 1} \frac{c}{r} B\Psi^{N_t} \left(1 - \frac{\Pi_t}{\gamma + 1}\right) = B\Psi^{N_t+1}$$

and, from the par condition (8),

$$\lim_{t \searrow \tau(N_t)} B_t = \lim_{\Pi_t \rightarrow G} \frac{c}{r} B\Psi^{N_t+1} \left(1 - \frac{\Pi_t}{\gamma + 1}\right) = B\Psi^{N_t+1}.$$

The bank's asset value is at its lowest level, to that point, when one of its borrowers defaults. So,

$$B_s = m_s \forall s \in \{\tau(j)\},$$

where  $\tau(j)$  is the default time of borrower number  $j$ .

From Propositions 1 and 2 we know that

$$B_t = \frac{c}{r} B\Psi^{N_t} \left(1 - \frac{Ge^{\sigma\gamma(Y_t - N_t \cdot d)}}{\gamma + 1}\right). \quad (36)$$

The bank's defaults the first time that its asset value ( $B_t$ ) touches the default threshold ( $\bar{B} < B$ ).

The bank's default time is defined as

$$\tau_{\bar{B}} = \inf\{t \geq 0 : B_t = \bar{B}\}. \quad (37)$$

Since  $M_{\bar{B}}$  is the maximum number of defaults that the bank can endure without its asset value falling below the default threshold  $\bar{B}$ ,

$$\begin{aligned} m_{\tau(M_{\bar{B}})} &= B\Psi^{M_{\bar{B}}} > \bar{B} \geq B\Psi^{M_{\bar{B}}+1} = m_{\tau(M_{\bar{B}}+1)} \\ \implies \tau(M_{\bar{B}}) &< \tau_{\bar{B}} \leq \tau(M_{\bar{B}} + 1), \end{aligned}$$

where  $\tau(M_{\bar{B}})$  is the time at which borrower number  $M_{\bar{B}}$  defaults. We rearrange Expression 36 for  $t \in [\tau(M_{\bar{B}}), \tau(M_{\bar{B}} + 1))$  to get

$$B_t = \frac{c}{r} B\Psi^{M_{\bar{B}}} \left( 1 - \frac{Ge^{\sigma\gamma(Y_t - M_{\bar{B}} \cdot d)}}{\gamma + 1} \right). \quad (38)$$

Combining Expressions 37 and 38 yields

$$\tau_{\bar{B}} = \inf\{t \geq 0 : B_t = \bar{B}\} = \inf\{t \geq 0 : Y_t = d \cdot n_{\bar{B}}\}, \quad (39)$$

where  $n_{\bar{B}}$  is given in Expression 15. Here,  $n_{\bar{B}}$  can be considered the bank's distance to default measured in number of borrower defaults. Observe that

$$\lfloor n_{\bar{B}} \rfloor = M_{\bar{B}} \quad (40)$$

is the integer part of  $n_{\bar{B}}$ , and

$$\{n_{\bar{B}}\} = 1 - \frac{\log \left[ (\gamma + 1) \left( 1 - \frac{r}{c} \frac{\bar{B}}{B\Psi^{\lfloor n_{\bar{B}} \rfloor}} \right) \right]}{\log G} \quad (41)$$

is the fractional part of  $n_{\bar{B}}$ . So, trivially,

$$n_{\bar{B}} = \lfloor n_{\bar{B}} \rfloor + \{n_{\bar{B}}\}.$$

Expression 39 implies that

$$Q(m_t < \bar{B}) = Q(\tau_{\bar{B}} < t) = Q(n_t > n_{\bar{B}} \cdot d).$$



## B Proof of Proposition 4

Since  $Y_t$  is an arithmetic Brownian motion with drift  $\nu$ ,  $\tau_{\bar{B}}$  has an inverse gaussian distribution that can be parameterized as

$$\tau_{\bar{B}} \sim \text{IG}(\mu, \lambda).$$

where

$$\begin{aligned} \mu &= \frac{n_{\bar{B}} \cdot d}{\nu}, \text{ and} \\ \lambda &= (n_{\bar{B}} \cdot d)^2. \end{aligned}$$

The bank's state price of default ( $\Pi_{\bar{B}}$ ) is the expected discount factor at the time of default  $\tau_{\bar{B}}$ .

$$\Pi_{\bar{B}} = \mathbb{E}[e^{-r\tau_{\bar{B}}}] = \int_0^{\infty} e^{-rt} dF_{\tau_{\bar{B}}}(t),$$

which is the Laplace transform of the probability density function of  $\tau_{\bar{B}}$ , where  $s = r$ . Applying the Laplace transform (Seshadri, 1993) gives us

$$\begin{aligned} \Pi_{\bar{B}} &= \exp \left[ \frac{\lambda}{\mu} \left\{ 1 - \sqrt{1 + \frac{2\mu^2 r}{\lambda}} \right\} \right] \\ &= \exp \left[ n_{\bar{B}} \cdot d \cdot \nu \left\{ 1 - \sqrt{1 + \frac{2r}{\nu^2}} \right\} \right] \end{aligned} \quad (42)$$

We calculate the state price of default for a borrower (2.1) by a similar method. The borrower's asset value process (Expression 1) can be represented as

$$A_t = A_0 e^{\sigma Z_t},$$

where  $Z_t$  is the arithmetic Brownian motion ( $Z_0 = 0$ )

$$dZ_t = \xi dt + dW_t,$$

where

$$\xi = \frac{r}{\sigma} - \frac{\sigma}{2} = \frac{\sigma\gamma}{2} - \frac{r}{\sigma\gamma} = -\nu.$$

The borrower defaults the first time the process  $Z_t$  passes through the point

$$\frac{1}{\sigma} \log \left( \frac{\bar{A}}{A_0} \right) = \frac{1}{\sigma} \log (L\Psi) = -\frac{1}{\sigma\gamma} \log \left( \frac{1}{G} \right) = -d.$$

This implies that the borrower's default time

$$\tau(1) \sim \text{IG} \left( \frac{d}{\nu}, d^2 \right),$$

which makes the borrower's state price of default, as the expected discount factor at the time of default,

$$G = \exp \left[ d \cdot \nu \left\{ 1 - \sqrt{1 + \frac{2r}{\nu^2}} \right\} \right]. \quad (43)$$

Combining Expressions 42 and 43 gives us the result

$$\Pi_{\bar{B}} = G^{n_{\bar{B}}}.$$

Black and Cox (1976) derived the result  $G = (L\Psi)^\gamma$  using the calculus of variations, while we use the properties of an arithmetic Brownian motion to derive Expression 43. Both results are equivalent.

## C Proof of Proposition 6

Recall that  $n^* = \min\{j \geq 1 : cB\Psi^j < iF\}$ . We want to solve

$$\begin{aligned} & \sup_{\tau} \mathbb{E} \left[ \int_0^{\tau} (cB\Psi^{N_t} - iF)e^{-rt} dt \right] \\ &= \sup_{\tau} \mathbb{E} \left[ \int_0^{\tau(n^*)} (cB\Psi^{N_t} - iF)e^{-rt} dt + \underbrace{\int_{\tau(n^*)}^{\tau} (cB\Psi^{N_t} - iF)e^{-rt} dt}_{< 0 \text{ by definition of } n^*} \right] \\ &\implies \tau^* = \tau_{\bar{B}^*} = \tau(n^*) \end{aligned}$$

The initial value of the bank's equity is the maximum value of the objective function in the above optimization.

$$\begin{aligned}
E(B) &= \mathbb{E} \left[ \int_0^{\tau(n^*)} (cB\Psi^{N_t} - iF)e^{-rt} dt \right] \\
&= \mathbb{E} \left[ \sum_{j=1}^{n^*} \left\{ (cB\Psi^{j-1} - iF) \int_{\tau(j-1)}^{\tau(j)} e^{-rt} dt \right\} \right] \\
&= \sum_{j=1}^{n^*} (cB\Psi^{j-1} - iF) \underbrace{G^{j-1} \frac{1-G}{r}}_{\text{Since } \mathbb{E}[e^{-r\tau(j)}] = G^j} \\
&= B[1 - (\Psi G)^{n^*}] - \frac{iF}{r}(1 - G^{n^*}) \\
&= B - \bar{B}^* G^{n^*} - \frac{iF}{r}(1 - G^{n^*}) \\
&= B - \left\{ \frac{iF}{r} - \left( \frac{iF}{r} - \bar{B}^* \right) G^{n^*} \right\}
\end{aligned}$$

## D Proof of Proposition 8

We derive Expression 28 in this section. We begin with the tradeoff value

$$TO(n_{\bar{B}}) = \frac{\theta}{r} \left[ \frac{cB\Psi^{(n_{\bar{B}}-1)}}{(1-\theta)} \right] (1 - G^{n_{\bar{B}}}) - \alpha B (\Psi G)^{n_{\bar{B}}}.$$

Differentiating this value with respect to  $n_{\bar{B}}$  and setting the result equal to zero gives us

$$\begin{aligned}
\frac{dT O}{dn_{\bar{B}}} &= \frac{\theta}{r} \left[ \frac{cB\Psi^{(n_{\bar{B}}-1)}}{(1-\theta)} \right] [\log \Psi - G^{n_{\bar{B}}}(\log \Psi + \log G)] - \alpha B (\Psi G)^{n_{\bar{B}}} (\log \Psi + \log G) = 0 \\
\implies \frac{\log \Psi}{G^{n_{\bar{B}}^*}} &= \left( \frac{r(1-\theta)\alpha\Psi}{\theta c} + 1 \right) (\log \Psi + \log G) \\
\implies n_{\bar{B}}^* &= \frac{\log \left[ \frac{\log \Psi}{\left( \frac{r(1-\theta)\alpha\Psi}{\theta c} + 1 \right) (\log \Psi + \log G)} \right]}{\log G}.
\end{aligned}$$

## E Proof of Proposition 9

$$TO(n) = F(n)[G^n(\alpha\theta - \alpha - \theta) + \theta]$$

Differentiating this value with respect to  $n$  and setting the result equal to zero gives us

$$\frac{dT_O}{dn} = F(n) [(\alpha\theta - \alpha - \theta)G^n \log G] + [G^n(\alpha\theta - \alpha - \theta) + \theta] \frac{d}{dn}F(n) = 0,$$

which implies that

$$\begin{aligned} \frac{c}{r} B\Psi^{[n]} \left(1 - \frac{G^{1-\{n\}}}{\gamma+1}\right) [(\alpha\theta - \alpha - \theta)G^n \log G] &= -[G^n(\alpha\theta - \alpha - \theta) + \theta] \log G \frac{c}{r} \frac{B\Psi^{[n]}}{\gamma+1} G^{1-\{n\}} \\ G^n(\alpha\theta - \alpha - \theta) &= -\frac{G^{1-\{n\}}}{\gamma+1} \theta \\ G^{n+\{n\}-1} &= -\frac{\theta}{(\gamma+1)(\alpha\theta - \alpha - \theta)} \\ n + \{n\} = [n] + 2\{n\} &= \frac{\log \left[-\frac{\theta}{(\gamma+1)(\alpha\theta - \alpha - \theta)}\right]}{\log G} = x, \end{aligned}$$

which has two possible solutions

$$n_1^* = [x] + \{x\}/2, \text{ or}$$

$$n_2^* = ([x] - 1) + (\{x\} + 1)/2.$$

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