A bioeconomic analysis of an age-structured fish stock continuous in time and age

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Abstract

An age-structured bioeconomic model, that is completely continuous in state and time, is developed. To make the model more realistic, a nonlinear recruitment function is introduced and steady state solutions are studied analytically. The objective is to maximize discounted net revenue. As the combination of continuous state and time requires solving partial differential equations, the analytical study has been complemented with numerical results. The main objective of numerical simulations is to investigate how optimal harvesting patterns vary with different assumptions and parameter values. In particular the purpose is to find out how persistent pulse fishing patterns are when we use this kind of modeling framework. Main emphasis is put on cost and demand parameters. The main results from the analysis are that pulse fishing patterns are highly sensitive both to cost and demand parameters, and pulse fishing tend to disappear completely when there is a significant relationship between price and quantity.

Keywords: Continuous Age-structured Model, Optimal Resource Management.

1   Introduction

Age-structured models have been applied in the literature on fisheries economics for decades, and they are more and more replacing aggregated biomass models as a tool for bioeconomic analysis. The structure of such models usually follow the Leslie matrix (Leslie 1945, 1948) which represents discrete updating of the stock. Combining the Leslie matrix structure with a discrete time version of the Baranov equation (Baranov 1938), we get the classical age-structured Beverton-Holt model (Beverton and Holt 1957). Also when age-structured models are applied for actual management of real world fish-stocks, they are usually in discrete time, although the underlying equations in these models are continuous; see Tahvonen (2010) and the references therein.

In ecology, biology and medicine on the other hand, models that are completely continuous both in state and time are often analyzed, usually for the purpose of studying the population dynamics only (Kapur 1979; Li and Brauer 2008; Ellner 2009). These models usually do not contain harvesting or maximization of net revenue. If they contain harvesting, it is at most maximization of sustainable yield (Murphy and Smith 1991).

The purpose of the present article is to formulate an age-structured model that is completely continuous both in state and time, like the ecological models, but in addition can be used for bioeconomic analysis and optimization including non-linearities in the economic model. In other words, the purpose is to introduce an economic objective in the model to be maximized subject the underlying dynamic constraints representing the biology. It is recognized that optimization is very difficult even within the framework of discrete time age-structured bioeconomic models (Clark 2010), and it is an even more ambitious task when the model in addition is non-linear and completely continuous in state and time. Combined continuous state and time requires partial differential equations, and partial differential equations are inherently more difficult to deal with both analytically and numerically than ordinary differential equations. In the optimization, therefore, numerical methods have been resorted to.

One motivation for putting emphasis on perfectly continuous models is that both biological processes, like birth, growth and death, as well as human activity, like commercial fishing, are taking place continuously and not at discrete time steps. Continuity typically also allows deeper

analytical study. In addition, the continuous case may serve as a conceptual framework and guideline for discrete cases.

To make the model more realistic, we introduce a nonlinear recruitment function, which describes how the number of survived recruits depends on the number of produced eggs. The original linear model (in which recruitment is, in a sense, proportional to spawning stock size) does not have nontrivial structurally stable steady states and usually leads to unrealistic exponentially growing solutions. The modified model with a concave recruitment function has a structurally stable steady state solution for any constant harvesting effort. Within that nonlinear model we analytically study the long term outcome of constant harvesting effort. It is shown that if the harvesting effort is above a certain critical level then the stock will collapse. Otherwise, if it is below the critical level, the stock will approach a certain steady state distribution, which is given by an explicit expression. Results of that analysis are used in the long term constant effort harvesting optimization. In the economic submodel, non-linearities are introduced through a downward sloping demand curve.

To study optimal harvesting with variable harvesting effort we use numeric optimization for the discretized version of our model. The objective is then to investigate how optimal harvesting patterns vary with different assumptions and parameter values. In particular to find how persistent pulse fishing patterns, as found first by Hannesson (1975) and later by numerous others using discrete models (Tahvonen 2009; Steinshamn 2010), are when we use a model that is completely continuous along both the state and time dimension. The model parameters that will be investigated are first and foremost demand- and cost parameters such as the slope of the demand curve and the cost per unit effort.

The outline of the article is as follows. First a fairly detailed description of the conceptual model will be given as this kind of model is quite new in the bioeconomic literature. Along with this an outline of the optimization method will be given. Then the numerical specification of the model will be explained, and the results from the optimization runs presented. Finally some conclusions and policy implications will be made.

2 The Model

2.1 Stock Representation and Dynamics

Let the age-structured fish stock be represented by a function $u(a,t)$, where $a \in [0, \infty)$ is age. For any time t, $u(a,t)$ is the stock density at age a. The total stock in the $[a_1, a_2]$ range at time t is $\int_{a_1}^{a_2} u(a,t)\,da$. The whole stock, measured in number of fish, at time t is then given by

$$N(t) \rightleftharpoons \int_0^\infty u(a,t)\,da.$$

One can consider a finite age range $[0, a_{max}]$ instead of $[0, \infty)$ if the maximum age, a_{max}, above which no fish of that species can survive, is known. That will not introduce any noticeable changes in the analysis.

It is straightforward to calculate total biomass in this model. Let $w(a) > 0$ be the weight of a fish at age a. Then the total biomass at time t is

$$B(t) \rightleftharpoons \int_0^\infty u(a,t)\,w(a)\,da.$$

It is natural to assume that $w(a)$ is a continuous, monotonically increasing function.

To describe the dynamics of the stock we will assume that its mortality is proportional to $u(a,t)$. The proportionality factor $m(a,t)$ may include natural mortality and fishing mortality, where each of them typically depends on age a and may also depend on time. The rate of change of $u(a,t)$, in the absence of harvesting, will be determined by natural mortality only. The following dynamics equation can be obtained formally in many ways (Ellner 2009):

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial a} - m(a,t)\,u.$$

This first order linear partial derivative equation is often written as

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -m(a,t)\,u. \tag{1}$$

It is a particular case of the so-called "transport" equation, which describes dynamics of

substance suspended in a flowing liquid. Without mortality (i.e. when $m(a, t) \equiv 0$) its general solution has the form $u(a, t) = f(a - t)$ (where f is an arbitrary function), i.e. a wave moving to the right (along the a-axis) with the speed 1. With positive mortality one can expect that the solution will again be a wave, but decreasing in amplitude while it travels.

It is worth noticing that natural mortality may also depend on u itself due to e.g. competition for food or cannibalism, but this will complicate the model unnecessarily and is ignored here.

2.2   Initial Condition: Given Stock Distribution

For many purposes it is useful also to know the initial stock distribution at time $t = 0$, for example in order to find the complete solution with the path from the initial state to optimum. Here it is assumed that the initial stock has a given distribution:

$$u(a, 0) = u_0(a), \qquad a \geq 0. \tag{2}$$

This serves as an initial condition for the above transport equation (1).

2.3   Integral Boundary Condition: Recruitment

Information about recruitment, defined as the stock of age zero $u(0, t)$, is also needed. This is supposed to stem from a biological reproduction function, or so-called stock-recruitment function. Suppose that $r(a) \geq 0$ represents average fertility of individuals at age a. Then recruitment at time t is

$$u(0, t) = \int_0^\infty u(a, t) \, r(a) \, da, \qquad t \geq 0. \tag{3}$$

This integral serves as boundary condition.

2.3.1   Concave Recruitment

The above model for recruitment does not take into account the possible influence of the stock itself on the survival of eggs. Later we will see that such simple model, which is linear in u

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results in completely unrealistic long-term dynamics (see, e.g., Sections 3.2 and 3.3).

In this section a simple model that takes this effect into account, is introduced. In Eq. (3) the number of recruits $u(0, t)$ equals, or is proportional to, the number of produced eggs $\int_0^\infty u(a, t)\, r(a)\, da$. Now it is instead assumed that

$$u(0, t) = \sigma\left(\int_0^\infty u(a, t)\, r(a)\, da\right), \qquad t \geq 0, \tag{4}$$

i.e.,the number of survived recruits is a function $\sigma(\cdot)$ of the number of produced eggs. The function $\sigma(x)$ is continuous positive monotonically increasing and concave for all $x > 0$, $\sigma(0) = 0$, $\sigma'(0) = 1$ (i.e. for small number of eggs there is no effect of overpopulation, almost all the recruits survive) and $\sigma(+\infty) = R_{\max}$, where $R_{\max}$ is the upper limit on recruitment. An example of such function is shown on the Fig. 4. As a result in this modified model production of recruits is a two-stage process. Thee first step it is production of eggs (which is determined by the function $r(a)$)and is linear in u. The second stage is survival of recruits (determined by the function $\sigma(x)$) which is nonlinear in u.

In fact, one could consider a much wider class of functions σ, not necessarily everywhere concave and not necessarily approaching a finite limit (and we will briefly discuss it below). However, in this study we intentionally want to make this function as "close" to the identity function as possible, but introduce a "saturation" effect to the dynamic system. Below we will see that it makes the model much more realistic. We will start with a model where all eggs survive and show that its behavior is essentially unrealistic, which makes such model inappropriate for describing long-term dynamics and optimization. Then, starting from Section 3.4, we will add the concave recruitment function to the model, which will make its behavior more regular.

3 Basic Analysis

3.1 Initial Boundary Problem

Combining equation (1) for the stock dynamics with the condition for the initial stock (2) and the recruitment equation (3), an initial-boundary problem for the transport equation is

obtained:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -m(a, t)\, u, \qquad a > 0, \qquad t > 0.$$

$$u(a, 0) = u_0(a), \qquad a \geq 0.$$

$$u(0, t) = \int_0^\infty u(a, t)\, r(a)\, da, \qquad t \geq 0.$$

Consider first the simplest case without harvesting where natural mortality is age- and time-independent, i.e., $m(a, t) = m = \text{const}$:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -m\, u(a, t), \qquad a > 0, \quad t > 0.$$

$$u(a, 0) = u_0(a), \qquad a \geq 0.$$

$$u(0, t) = \int_0^\infty u(a, t)\, r(a)\, da, \qquad t \geq 0.$$

This problem can be partially solved

$$u(a, t) = \begin{cases} \exp(-mt) \cdot u_0(a - t), & a \geq t \\ \exp(-ma) \cdot u(0, t - a), & a \leq t \end{cases}.$$

Here the stock distribution function $u(a, t)$ exponentially decreases along the characteristics of the partial derivatives equation (see Fig. 1).

For the general case of mortality changing in age and time the corresponding relation is

$$u(a, t) = \begin{cases} \exp\left(-\int_0^t m(a - \tau, t - \tau)\, d\tau\right) \cdot u_0(a - t), & a \geq t \\ \exp\left(-\int_0^a m(a - \alpha, t - \alpha)\, d\alpha\right) \cdot u(0, t - a), & a \leq t \end{cases}.$$

If the boundary values had been exogenously given, this would give the complete solution. In our case, however, it does not give the complete solution since the boundary value at any
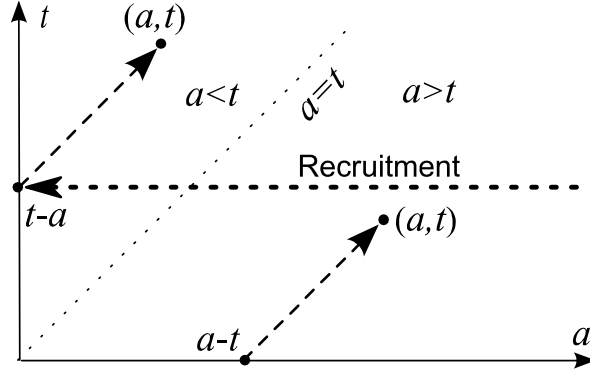
Figure 1: Expressing solution at any age-time point through the initial or boundary values.

time t is expressed through the values of $u(a, t)$ for all $a \in (0, \infty)$.

3.2   Age-Dependent Mortality

Consider the problem where natural mortality, $m(a)$, is age-dependent but not time-dependent. In order to study the solution, $u(a, t)$, we start with the dynamics equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -m(a)\, u, \qquad a > 0, \quad t > 0. \tag{5}$$

and the recruitment equation (3), but without initial condition.

Since the whole problem is linear with respect to u, one could expect that the solution would exponentially grow (or decline) in time. We will search for a solution in separated form:

$$u(a, t) = v(a)\, y(t).$$

Substituting this form into the problem statement (5) and (3) yields

$$v'y + vy' = -m(a)\, vy \qquad a > 0, \qquad t > 0. \tag{6}$$

$$v(0) = \int_0^\infty r(a)\, v(a)\, da. \tag{7}$$

Eq. (6) can be rewritten

$$\frac{y'(t)}{y(t)} = -\left(\frac{v'(a)}{v(a)} + m(a)\right) \qquad a > 0, \quad t > 0.$$

Since the left side of this equation depends on t only, while the right side depends on a only, the equation can hold for all $a > 0$ and $t > 0$ only if both sides, in fact, do not depend on t and a and are equal to one and the same constant, which we will call μ, i.e.,

$$\frac{y'(t)}{y(t)} = \mu \qquad t > 0.$$

$$\frac{v'(a)}{v(a)} + m(a) = -\mu \qquad a > 0.$$

Thus, for y we get an equation

$$y'(t) = \mu\, y(t),$$

and its general solution is

$$y(t) = Ce^{\mu t}.$$

It means that the constant μ represents the growth constant. If it is positive the stock exponentially grows and if μ is negative the stock exponentially decays.

Similarly, for $v(a)$ we have

$$v'(a) = -[m(a) + \mu]\, v(a),$$

and its general solution is

$$v(a) = De^{-\int_0^a [m(\alpha)+\mu]\, d\alpha} = De^{-\int_0^a m(\alpha)\, d\alpha} e^{-\mu a} = De^{-M(a)} e^{-\mu a}, \qquad (8)$$

where $M(a) = \int_0^a m(\alpha)\, d\alpha$ is accumulated natural mortality. Since $m(\alpha) \geq 0$, $M(a)$ is a monotone non-decreasing function, and $M(0) = 0$. In particular, if $m(a) = m = \text{const}$, $M(a) = ma$ and

$$v(a) = De^{-(m+\mu)a}.$$

Substituting the general solution (8) for $v(a)$ into equation (7) gives

$$\int_0^\infty r(a)\, De^{-M(a)}e^{-\mu a}\, da = De^{-M(0)}e^{-\mu 0} = D.$$

which yields the following condition for μ:

$$\int_0^\infty e^{-M(a)}\, e^{-\mu a}\, r(a)\, da = 1, \tag{9}$$

known as Euler-Lotka equation (Lotka 1922, 1939; Ellner 2009).

It is easy to see that this condition uniquely determines μ. Indeed, since $e^{-M(a)}\, r(a)$ is positive for every a and $e^{-\mu a}$ is a continuous monotone decreasing function of μ, the whole left hand side is a continuous monotone decreasing function of μ as well. Besides it approaches 0 and $+\infty$ when $\mu \to +\infty$ and $\mu \to -\infty$, respectively. Thus, there exists a unique μ that satisfies the Euler-Lotka equation.

The complete quasi-stationary solution is now given by

$$u(a,t) = C\, e^{-M(a)-\mu a+\mu t}. \tag{10}$$

When $\mu > 0$ the age profile

$$u(a,0) = C\, e^{-M(a)-\mu a}$$

exponentially grows in time. Conversely, if $\mu < 0$ the age profile exponentially decays. Only if $\mu = 0$ the age profile remains unchanged (stationary). However, this steady state $u(a,t) = C\, e^{-M(a)}$ (in fact, infinitely many steady states, since C can be any non negative value) is not structurally stable. Whatever small variations of functions $m(a)$ and $r(a)$ would immediately move μ away from 0 and result in a solution, which exponentially grows or decays in time.

Eq. (10) can be rewritten as

$$u(a,t) = C\, e^{-M(a)}\, e^{-\mu(a-t)}. \tag{11}$$

The term $e^{-\mu(a-t)}$ is an (exponentially shaped) wave running to the right along the a-axis. The first term $e^{-M(a)}$ monotonically decreases in a and is responsible for the attenuation of the

wave.

Let us note that unlimited exponential growth if $\mu > 0$ and the absence of nontrivial (structurally stable) steady states indicate that the corresponding linear model is completely unrealistic and inappropriate for analysis of long term stock dynamics.

3.3 Harvesting and Fishing Mortality

Consider the problem with stationary natural mortality factor $m(a, t) = m(a)$ (like in Section 3.2) but with the added fishing mortality caused by fishing effort rate $E(t)$. The effective fishing mortality affecting $u(a, t)$ is now $F(a, t) = k(a)E(t)$, where $k(a) \geq 0$ is a continuous fishing selectivity function. Then stock dynamics is determined by the equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -\left[m(a) + E(t)k(a)\right] u(a, t), \qquad a > 0, \quad t > 0. \tag{12}$$

First let us consider stationary harvesting $E(t) = E = $ const. Then Eq. (12) becomes the same as (5) in which $m(a)$ is replaced with $m(a) + Ek(a)$. Thus, using results of Section 3.2 the quasi-stationary solution for constant harvesting effort E is

$$u(a, t) = C\, e^{-M(a) - EK(a) - \mu_E(a-t)}, \tag{13}$$

where $M(a) = \int_0^a m(\alpha)\, d\alpha$ and $K(a) = \int_0^a k(\alpha)\, d\alpha$ are the accumulated mortality and selectivity functions, respectively, and μ_E is a unique solution of the equation

$$\int_0^\infty e^{-M(a) - EK(a)}\, e^{-\mu a}\, r(a)\, da = 1. \tag{14}$$

Since $m(a) \geq 0$ and $r(a) \geq 0$, $M(a)$ and $K(a)$ are monotone non-decreasing and $M(0) = K(0) = 0$. It is obvious that when E increases, the first exponent under the integral decreases, so the solution μ_E of (14) should decrease. Besides, it can be shown that $\mu_E \to -\infty$ as $E \to +\infty$. Let μ_0 be a unique solution of (14) for $E = 0$, i.e., corresponds to zero harvesting. If $\mu_0 > 0$ (which implies that in the absence of harvesting the stock exponentially grows in t) there exists a unique $E_0 > 0$ such that $\mu_{E_0} = 0$. Harvesting at rate E_0 would keep the stock constant in time:

$$u^0(a) = C\, e^{-M(a) - E_0 K(a)}.$$

In other words, E_0 is the sustainable level of fishing effort in this case. If $E < E_0$, the stock will grow, and if $E > E_0$ it will decline. Here (as always before) $C > 0$ is an arbitrary constant. The only possible steady state solution, corresponding to $E = E_0$, is not unique and structurally unstable.

So here our model again leads to unrealistic unlimited exponential growth if $E < E_0$ and and thus is not appropriate for long term stock optimization. To make the model more realistic, in the next section we will add to the model a nonlinear recruitment function, which was described in Section 2.3.1.

In the case of constant (uniform) selectivity $k(a) = k = \text{const}$, $K(a) = ka$ and (14) can be rewritten

$$\int_0^\infty e^{-M(a)}\, e^{-(\mu + Ek)a}\, r(a)\, da = 1. \tag{15}$$

In this case the relation between E and μ_E is quite simple: $\mu_E = \mu_0 - Ek$. So $E_0 = \mu_0/k$, where μ_0 is a unique solution of (15) for $E = 0$.

3.4 Steady State Harvesting

The previous section shows that with linear recruitment (3) a steady state exists only when $E = E_0$. For all other values of E the stock will either exponentially grow or decline. We would argue that the biological model is only realistic (or, at least, irrelevant for studying long-term effects) if a steady state exists even in the case with zero harvesting, and therefore linear recruitment must be ruled out. However, introducing concave recruitment function and, thus, making recruitment nonlinear as in (4) completely changes the situation. In this more realistic model steady states will become possible for a whole range of harvesting efforts. Starting from this section we will always assume that the number of survived recruits is a function $\sigma(\cdot)$ of the number of produced eggs as described in Section 2.3.1.

In this section we will consider stationary fishing effort: $E(t) = E = \text{const}$, and focus on a

steady state, i.e. when stock age-distribution does not depend on t. Thus

$$u(a, t) = u(a),$$

and Eq. (12) transforms into

$$\frac{du}{da} = -\left[m(a) + Ek(a)\right] u(a), \qquad a > 0. \tag{16}$$

Its general solution is

$$u(a) = C \, e^{-\int_0^a [m(a) + Ek(a)] \, d\alpha},$$

which can be written in the form

$$u(a) = C \, e^{-[M(a) + EK(a)]} = C \, l_E(a), \tag{17}$$

where $M(a) = \int_0^a m(\alpha) \, d\alpha$ and $K(a) = \int_0^a k(\alpha) \, d\alpha$. Here the function

$$l_E(a) \rightleftharpoons e^{-[M(a) + EK(a)]}$$

represents the survival curve (Ellner 2009), i.e. the fraction of all recruits survived from birth (at age 0) to age a. This function which depends on E as a parameter.

Since $u(0) = C$ Eq. (4) immediately implies that the constant C should satisfy the stationarity equation for recruitment

$$C = \sigma \left(C \cdot R(E) \right), \tag{18}$$

where

$$R(E) \rightleftharpoons \int_0^\infty e^{-[M(a) + EK(a)]} \, r(a) \, da = \int_0^\infty l_E(a) \, r(a) \, da. \tag{19}$$

So, $R(E)$ if a "reproduction factor", i.e., the number of produced eggs per each survived recruit.

Note that since $\sigma(0) = 0$, Eq. (18) always has a trivial solution $C = 0$, which corresponds to the trivial steady state $u(a, t) = 0$.

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Proposition. For a fixed E a nontrivial steady state exists (and is structurally stable) if and only if $R(E) > 1$. If $R(E) \leq 1$ there is no nontrivial steady state and the stock would collapse.

Proof. Consider a function

$$\varphi(C) \rightleftharpoons \sigma\left(C \cdot R(E)\right) - C$$

for $C \geq 0$. Than Eq. (18) can be written as $\varphi(C) = 0$. Since $\sigma(\cdot)$ is continuous, concave, and $\sigma(0) = 0$, $\sigma'(0) = 1$, it implies that $\varphi(C)$ is also continuous, concave, $\varphi(0) = 0$, and $\varphi'(0) = R(E) - 1$.

Now, if $R(E) \leq 1$, $\varphi(C) < 0$ for all $C > 0$. So the equation $\varphi(C) = 0$ has only one (trivial) solution.

Conversely, if $R(E) > 1$, then $\varphi'(0) > 0$. So, for small enough C the values $\varphi(C)$ are positive. However, since $\sigma(+\infty) = R_{\max}$, $\varphi(+\infty) = -\infty$, so, for large enough C the values $\varphi(C)$ are negative. As a result, there exists a solution of the equation $\varphi(C) = 0$. This solution is unique since $\varphi(C)$ is concave and thus can cross a strait line (zero-line) in no more than two points, one of which is $C = 0$. Finally, according to Eq. (19), $R(E)$ depends continuously on all the model parameters. Thus, small enough variations of parameters would not violate the inequality $R(E) > 1$, which implies that the corresponding steady state is structurally stable. ▲

Obviously, $R(E)$ is monotone decreasing continuous and $R(+\infty) = 0$. This observation and the above proposition immediately imply:

Corollary. If $R(0) \leq 1$, the stock will collapse for any harvesting level E. If $R(0) > 1$, there exists a unique critical harvesting level E_0 such that $R(E_0) = 1$. In that case a nontrivial steady age distribution exists if and only if $E < E_0$ (harvesting level is below critical), it is structurally stable, and the steady state age-distribution for the constant harvesting E is

$$u_E(a) = C_E \, e^{-[M(a) + EK(a)]}. \tag{20}$$

The above proof shows that we could consider a much wider class of functions σ, providing steady states for the whole interval $0 < E < E_0$. Specifically, it is sufficient for σ to satisfy the following condition: there exists a constant c_0, $0 < c_0$, such that if $0 < c < c_0$ there exists a

unique $x > 0$ for which $\sigma(x) = cx$; and if $c_0 \leq c$ this equation holds only at $x = 0$. Here c_0 is some fixed positive constant, which can be equal to $+\infty$. Or, in other words, the graph of $\sigma(x)$ has a unique nontrivial intersection with the straight line cx if $0 < c < c_0$ and there is only trivial intersection if $c_0 \leq c$.

That would include the Ricker function $\sigma(x) = x \exp(r(1 - x/k))$ (which is not everywhere concave); the Beverton-Holt stock-recruitment relation $\sigma(x) = ax/(1 + bx)$; the upside-down parabola $\sigma(x) = cx(1 - x/b)$; the power function $\sigma(x) = cx^{\alpha}$, where $0 < \alpha < 1$; the constant function $\sigma(x) = C > 0$ and many more. In the latter example (and always when $\sigma(0) > 0$) a nontrivial solution for $\sigma(x) = cx$ exists for any $0 < c$ (which means that $c_0 = +\infty$) and, as a result, a nontrivial steady state will exist for any harvesting level E.

A wide class of such functions can be specified by the following conditions:

(a) $\sigma(x)$ is continuous positive on an interval $0 < x < a$, where $a \leq +\infty$ is some constant;

(b) $\sigma(x)$ is concave for all $x > 0$ and $\lim_{x \to \infty} \sigma(x)/x \leq 0$, or $\sigma(x)$ is concave on an interval $[0, b]$ for some $b \geq 0$ and for $x > b$ $\sigma(x)$ is non-increasing.

3.5 The Optimization Problem

3.5.1 Variable Fishing

Suppose that harvesting is performed with the fishing effort rate $E(t)$ for a certain period of time $t \in [0, T]$. The resulting stock dynamics $u_E(a, t)$ for the given harvesting profile $E(\cdot)$ is a solution of the following initial problem, which depends on a (functional) parameter $E(\cdot)$:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -\left[m(a) + E(t)k(a)\right] u(a, t), \qquad a > 0, \quad t > 0,$$

$$u(a, t) = u(a),$$

$$u(0, t) = \sigma \left(\int_0^A u(a, t)\, r(a)\, da \right), \qquad t \geq 0.$$

At each time t the harvested biomass (or supply) rate

$$h_E(t) \rightleftharpoons E(t) \int_0^A u_E(a, t)\, w(a)\, k(a)\, da$$

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is proportional to the harvesting effort E at time t and also depends on $E(t')$ at any preceding moment $t' < t$ indirectly since $u_E(a, t)$ depends on the profile of $E(t')$ on the interval $t' \in [0, t)$.

We will assume payoff rate $\pi_E(t)$ at time t is determined by the following factors: costs are proportional to fishing effort, i.e., linear in E, but the revenue is proportional to the sold biomass $h_E(t)$ with the price coefficient, which by itself may depend on $h_E(t)$:

$$\pi_E(t) \rightleftharpoons p\left(h_E(t)\right) h_E(t) - c\, E(t).$$

Here $p(h)$ is the landing price function, which may depend on the supply rate (e.g., drop with the increase of supply), and c is the cost parameter. We will study cases of constant price or downward sloping demand.

Now consider the following optimization problem: find an optimal rate of fishing effort that maximizes the discounted payoff

$$W \rightleftharpoons \int_0^T e^{-\delta t} \pi(t)\, dt \sim \max_{E(\cdot)},$$

where δ represents discounting.

In numerical simulations presented below (see Sec. 4.2.1.) for any given profile $E(\cdot)$ we compute a solution of the above problem on a discrete age-time grid as described in Sec. 6. The age-time step is 0.5 years, so there are 25 age nodes and 101 time nodes. Thus, in the optimization procedure $E(\cdot)$ is treated as a 101-dimentional parameter.

3.5.2 Constant Fishing

The problem of optimizing a harvesting profile $E(t)$ (as stated in Sec. 3.5.1) involves solving a complete initial problem for any given profile $E(t)$. However, if focus on the long-term effects of fishing with constant effort E we will arrive to a much simpler optimization problem.

Suppose that harvesting is performed with the constant fishing effort rate E and $u_E(a)$ is the corresponding stationary age distribution (see Sec. 3.4). The stationary harvested biomass rate corresponding to the effort rate E is given by the expression

$$h_E \rightleftharpoons E \int_0^A u_E(a)\, w(a)\, k(a)\, da.$$

Consider the following optimization problem: find a stationary effort rate E that maximizes the steady-state payoff rate

$$\pi_E \rightleftharpoons p(h_E)\, h_E - c\, E \sim \max_E.$$

Perhaps the main advantage of the steady state case is that for each value of E it can be solved analytically (reduces to a simple equation) and does not require solving complete initial problem. Besides, it gives an adequate description of the long-term harvesting with constant E. The steady state corresponding to the stationary harvesting with the effort rate E is analyzed in Sec. 3.4 and can be found from Eqs. (17) and (18).

4 Numerical Analysis

4.1 Numerical specification

In order to perform optimization a number of relationships have to be numerically determined and specified. In particular information about initial distribution, recruitment, growth, natural mortality and age selectivity. The numbers chosen here are based on observations of the Arcto-Norwegian cod stock, but they are not supposed to represent this stock exactly.

Regarding initial stock distribution, this is in most cases found as a two-stage process where the model is run twice and the final distribution from the first run is used as initial distribution in the second and final run. This is done in order to mimic a sort of equilibrium situation. The input in the first run is the following:

$$u_0(a) = 800 \cdot \mathrm{e}^{\frac{2.7}{a+1} - 1.5\ln(a+1)}.$$

Subsequent recruitment is based on Eq. (4) with the function

$$r(a) = 25 \cdot \mathrm{e}^{-\frac{6}{a+1}}$$

and recruitment survival function

$$\sigma(x) = \overline{x} \cdot \sigma_0 \left(\frac{x}{\overline{x}} \right), \qquad \text{where } \sigma_0 (x) = 1 - e^{-x}, \qquad \overline{x} = 20000.$$

The individual growth function applied is

$$w(a) = 90 \cdot e^{-\frac{24}{a+1}}.$$

Natural mortality is stationary over time and represented by the function

$$m(a) = 0.2 + 16 \cdot e^{-0.67a}.$$

Age selectivity of the fishing mortality is given by

$$k(a) = \frac{1}{a_{\max}} a$$

and we have specified $a_{\max} = 12$.

The above relationships are associated with biological or technological aspects of the model and are all functions of a only. They are used in all runs. Figures 2–7 illustrate the corresponding functions.
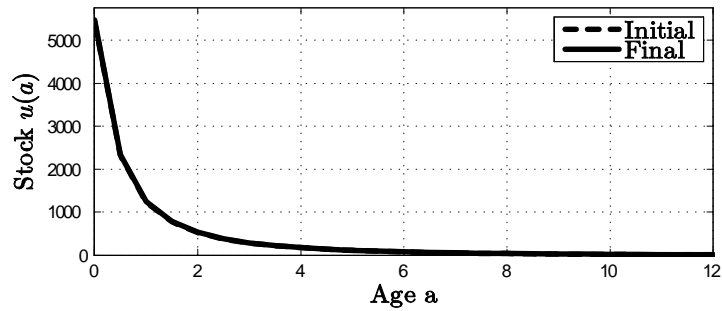
Figure 2: Initial age distribution $u_0(a)$.

In addition to this, the following two parameters are varied in order to see how they affect optimal effort and harvest patterns. These are related to the economic submodel and are the cost- and demand parameters respectively. The cost parameter applied is a constant cost per unit effort. The demand parameter that is varied in this model is the parameter determining the slope of a linear demand curve. In other words, the price of the catch varies with the size
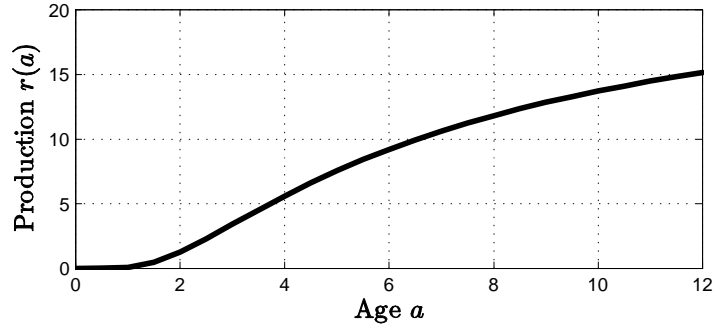
Figure 3: Reproduction function $r(a)$.
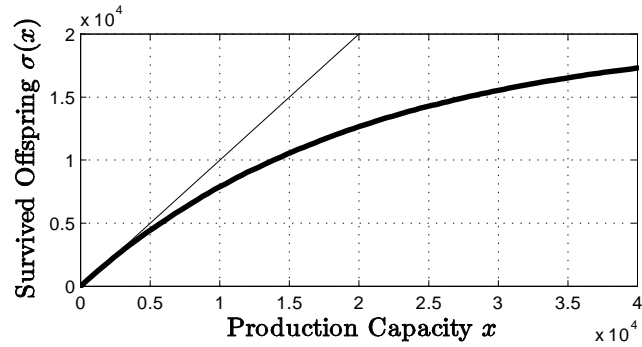
Figure 4: Recruitment survival function $\sigma(x)$

of the catch.

Cost parameters applied here are 5,000 (low cost), 10,000 (medium cost) and 20,000 (high cost). Regarding demand parameter two linear demand functions are used in addition to constant price. The parameters are as follows where $p(\cdot)$ is the demand function and h is harvest.

Constant $p(h) = p = 10$

slight $p(h) = 10.1 - 0.0005 \cdot h$ (ranges from 9.9 to around 10.1)

steep $p(h) = 10 - 0.005 \cdot h$ (ranges from 9 to around 11)

4.2 Results of Optimization

4.2.1 Optimal Paths

We have performed over 100 runs of the model under various assumptions. Based on these we decided to concentrate on variations in cost and demand parameters, as indicated in the previous section, with other assumptions constant. The three values of the cost parameter combined with three values of the demand slope parameter (including zero) give us nine possible combinations
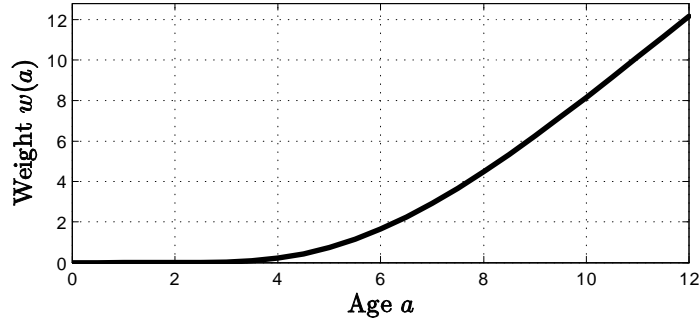
Figure 5: Weight function $w(a)$.
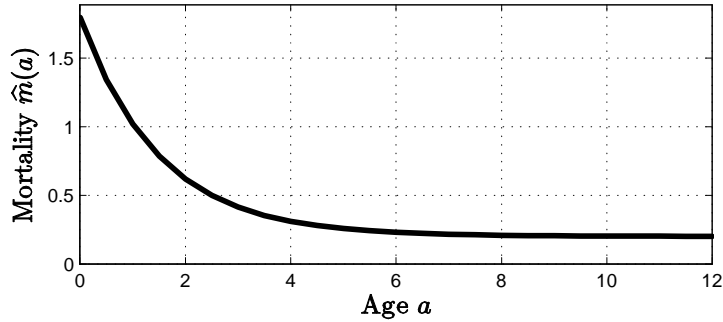
Figure 6: Natural mortality profile $m(a)$.

that we analyze in more detail. In addition a maximum limit on fishing effort, $E_{\max} = 0.5$ (which represents a limited capacity of the fishing fleet), is applied and a discount rate of 3 percent. First we look at the optimal time path for the fishing effort, $E(t)$, given an initial stock profile and the corresponding harvest, stock development and pay-off. As an alternative to this we also consider the constant effort level that maximizes the steady state pay-off as a simplified optimization routine.

In order to mimic long-term equilibrium all runs are performed as a two-stage process described above. The results are illustrated in the panel in Figure 8: Going from left to right in the panel shows the effect of increasing the cost parameter. Going from top to bottom in the panel illustrates the effect of increasing the slope of the demand curve.

From the upper row of figures it is seen that increasing the cost parameter decreases the amplitude of the pulses. The amplitude, that is the level of fishing effort, is a direct effect of the cost parameter. And it is a usual result that higher costs have a conservating effect on the stock. Higher costs implies a larger stock and, at the same time, less pressure on the stock in terms of effort.

The effect of a steeper demand curve is shown when we go from top to bottom in the panel,
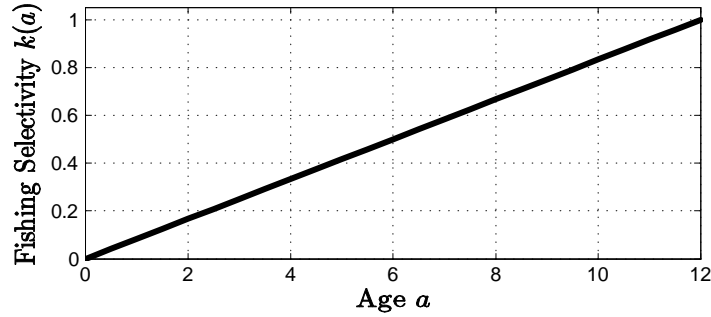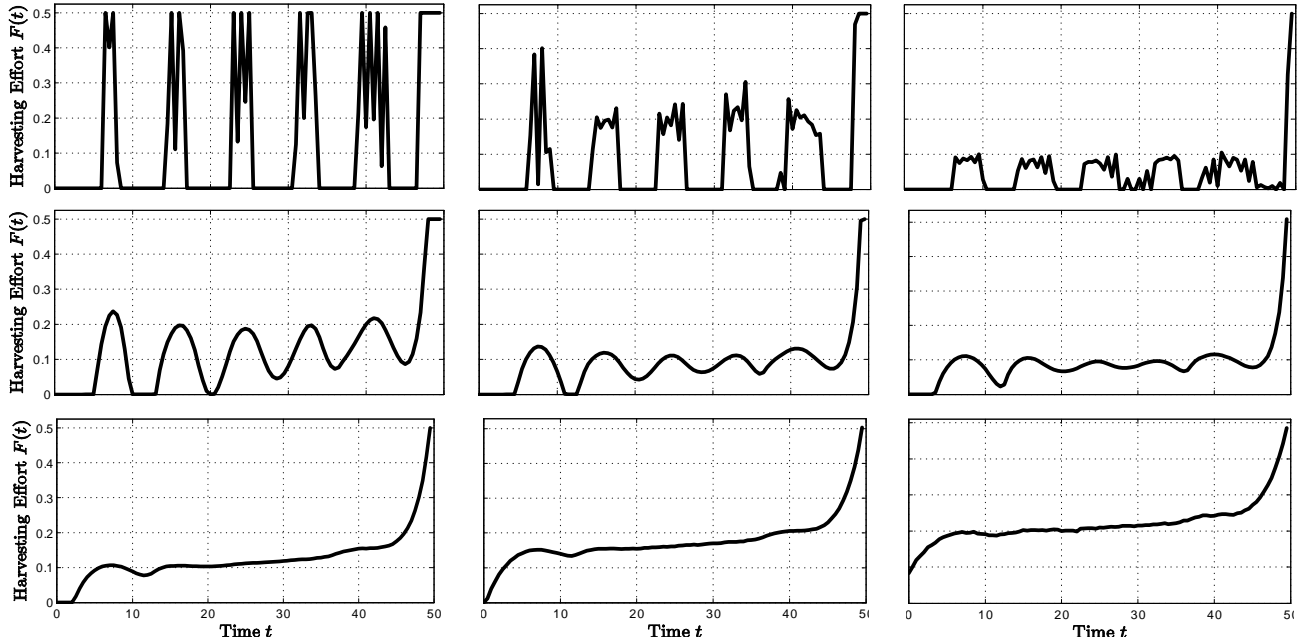
Figure 7: Fishing selectivity function $k(a)$.

Figure 8: Effect of increasing the cost parameter (left to right) the effect of increasing the slope of the demand curve (top to bottom).

and this is even more interesting. It is seen that even a very modest slope, namely 0.0005, attenuates the pulses and replace them by a cyclical fishing pattern instead of pulse fishing, as seen in the middle row of figures. Again it is seen, by going from left to right, that increased cost parameter implies lower amplitude. In the bottom row of figures, it is seen that increasing the slope of the demand curve further, to 0.005, removes both pulses and cycles more or less completely from the optimal pattern.

This comes out as the most robust and interesting result from this analysis: even very modest downward sloping demand curves tend to eliminate pulses and cycles from the optimal fishing pattern. In other words, constant price seems to be a necessary condition for optimal pulse fishing, and, although there is an extensive literature on pulse fishing, this is an aspect that has been paid very little attention earlier. And although the relationship between demand

and pulse fishing may not be very surprising, with the suggested model we can calculate the exact relationship. And the relationship between costs and pulse fishing are not that obvious as the cost function still is linear in E.

4.2.2   Steady State Optimum

Consider the following optimization problem: find a stationary effort rate E that maximizes the steady-state payoff rate

$$\pi_E \rightleftharpoons E\left[p\int_0^A u_E(a)\,w(a)\,k(a)\,da - c\right] \sim \max_E,$$

where the steady state corresponding to the stationary harvesting with the effort rate E is analyzed in Sec. 3.4.

We will study the case where the stock does not collapse without harvesting i.e., $R(0) > 1$, where $R(E)$ is defined by (19).

As it was already mentioned in Sec. 3.5.2 u_E is determined by Eqs. (17) and (18) and thus, computing π_E for each E reduces to the solution of a simple equation (18).

The following figures 9–11 illustrate steady state age-distribution, payoff and biomass for various values of E, computed using Eqs. (20) and (18). The parameters of the model are shown in Sec. 4.1.

5   Policy Implications and Conclusions

Pulse fishing as an optimal strategy in age-structured models was first discovered by Hannesson (1975). Such policies are optimal in linear discrete-time age-structured models of the Beverton-Holt type as long as there is less than perfect selectivity. And it seems reasonable to claim that selectivity is less than perfect in virtually all real-world fisheries. On the other hand, from a practical point of view, pulse fishing would be difficult to implement by any responsible managing authority as it would imply that fishers would be without income for long periods. Rotating fishing might be an alternative, where fishing fleets target one species after the other,
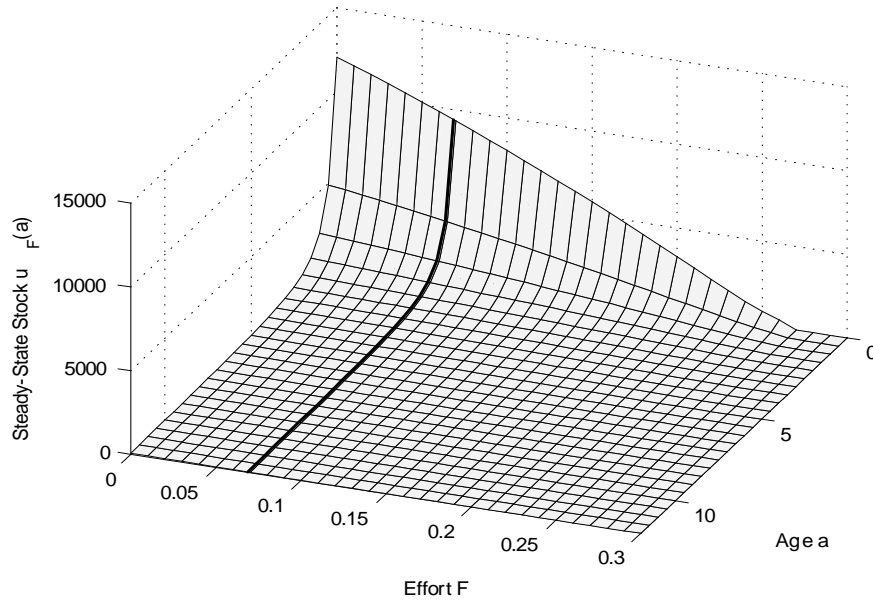
Figure 9: Steady state age-distributions for various values of F. Max payoff profile is indicated with a thick line.
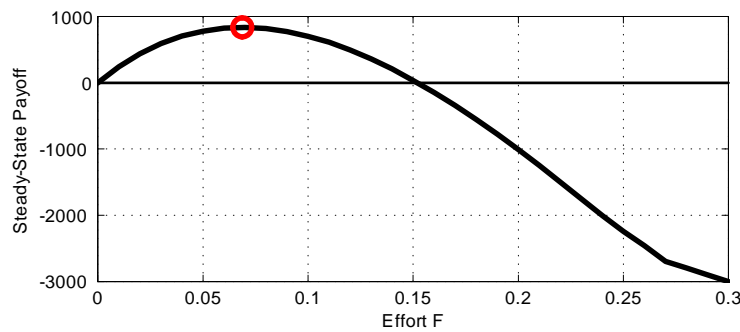

Figure 10: Steady state payoff for various values of F. Circle corresponds to maximum payoff.

but this too is improbable from a practical viewpoint these days, although it was applied by the Japanese and Soviet fishing fleet back in the 60s and 70s before extended economic zones.

It is therefore of some relief to see that the conclusion that pulse fishing is optimal hinges heavily on an assumption about linearity in the economic model. As soon as only slightly downward sloping demand is introduced, pulse fishing patterns start disappearing as optimal policies. Another way to put this is to say that by adding more realistic assumptions to the model, the resulting policies become more realistic as well, namely in the form of more even harvesting patterns. In other words, by making age-structured models more realistic from an economic point of view, and not only from a biological point of view, it is possible to implement more realistic optimal management policies.
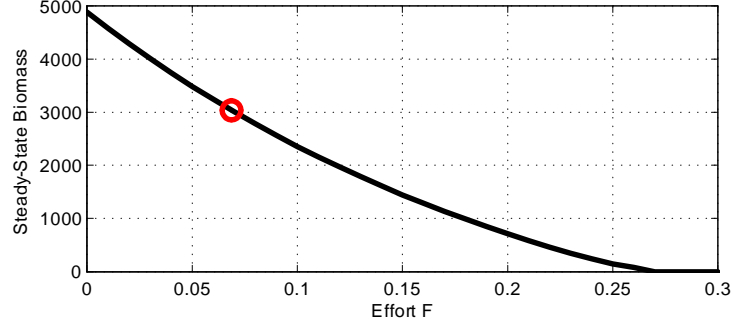
Figure 11: Steady state biomass for various values of F. Circle corresponds to maximum payoff.

6 APPENDIX

Discrete Age and Time. Equal Steps

In all our simulations we used a discrete age-time analogue of the continuous model with equal steps in age and time. Such discrete model seems to be the most natural and intuitive.

Specifically, suppose that the time and age ranges are discrete and all the discretization steps are equal $\Delta a = \Delta t = \Delta$. So, the age and time grids are $a \in \{0, \Delta, 2\Delta, \ldots, K\Delta\}$ and $t \in 0, \Delta, 2\Delta, \ldots$, and the complete stock dynamics is represented by a set of values u_{kn} where $k = 0, 1, \ldots, K$ and $n = 0, 1, \ldots$. Then all the functions of the model which depend on a, i.e., $u_0(a)$, $r(a)$, $w(a)$, $m(a)$, $k(a)$ are represented by the corresponding discrete functions u_k^0, r_k, w_k, m_k, k_k, and the discrete model is specified by the following ingredients:

Initial condition:

$$u_{k0} = u_k^0, \qquad k = 0, 1 \ldots, K.$$

Recruitment (boundary) condition:

$$u_{0n} = \sigma \left(\sum_{k=1}^{K} u_{kn} \, r_k \, \Delta \right), \qquad n = 0, 1 \ldots.$$

Transition equation:

$$u_{kn} = u_{k-1,n-1} \cdot e^{-\bar{m}_{k,n-1} \cdot \Delta},$$

where

$$\bar{m}_{kn} = m(a_k) + E(t_n)k(a_k) = m_k + E_n k_k,$$
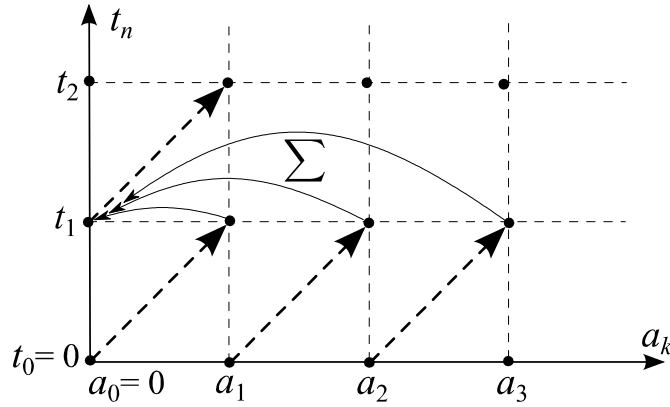
See Fig. 12 for an illustration of this scheme.

Figure 12: Discrete age-time scheme with equal steps.

For very small steps $e^{-\bar{m}_{kn}\cdot\Delta} \approx 1 - \bar{m}_{kn} \cdot \Delta$. Thus,

$$u_{kn} = u_{k-1,n-1} \cdot (1 - \bar{m}_{k,n-1} \cdot \Delta).$$

7   Acknowledgement

8   References

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An age-structured bioeconomic model, that is completely continuous in state and time, is developed. To make the model more realistic, a nonlinear recruitment function is introduced and steady state solutions are studied analytically. The objective is to maximize discounted net revenue. As the combination of continuous state and time requires solving partial differential equations, the analytical study has been complemented with numerical results. The main objective of numerical simulations is to investigate how optimal harvesting patterns vary with different assumptions and parameter values. In particular the purpose is to find out how persistent pulse fishing patterns are when we use this kind of modeling framework. Main emphasis is put on cost and demand parameters. The main results from the analysis are that pulse fishing patterns are highly sensitive both to cost and demand parameters, and pulse fishing tend to disappear completely when there is a significant relationship between price and quantity.