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Channel Coordination in a Multi-period Newsvendor Model with Dynamic, Price-dependent Stochastic Demand

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Abstract

In this study, we extend the single-period newsvendor problem with stochastic demand into a multi-period and time-dependent one and find a solution for it. We analyze the multi-period newsvendor problem with stochastic demand in a Stackelberg framework where the wholesaler is the leader and the retailer (newsvendor) is the follower. We use an additive-multiplicative structure for the demand so that both its mean and variance are considered as functions of the current retail price. Moreover, in our model, the demand mean and variance can be either time-dependent or autonomous with respect to time. A price-dependent memory function is also embedded in this representation of demand that carries the effect of demand level at present over to the demand in future. This leads to a strategic game in which the players must balance immediate high profits with reduced future earnings. We propose a complete solution to this stochastic multi-period Stackelberg game, covering cases with finite and infinite horizons. The theory is illustrated by using Cobb-Douglas demand functions as the deterministic part, while a random variable brings in stochasticity into the model. Because our theory is very flexible with respect to the assumptions made about the demand, market memory, and the behavior of the two agents, it is applicable to a wide variety of functional forms and capable to model many different economic contexts.

Keywords: stochastic games, multi-period newsvendor problem, stochastic demand, price-dependent demand, time-dependent demand, pricing theory, market engineering

1 Introduction

The newsvendor problem deals with a situation in which a vendor intends to obtain maximum profit by meeting an uncertain demand D for a product that has to be sold within a limited time span. This problem has been studied in different literatures with the decision variables being either merely the optimal quantity of the product to be supplied q, or the optimal supply quantity as well as the optimal retail price to be charged per unit (q and R). The solution to this problem (especially the simple case which offers only the optimal q for a time-independent demand) is classical. In this paper, we present a solution for the general price-setting problem which finds the optimal quantities and prices, at any specific time in the optimization horizon (q_ts and R_ts) to meet a time-dependent stochastic demand D_t. We solve the problem in a Stackelberg structure in which the upstream vendor (the wholesaler), as the leader, has to find an optimal vector of wholesale prices for different times (W_ts) to ensure her maximum profit.[1] The downstream vendor (the retailer), who is the follower, then faces the wholesale price and accordingly decides on the number of the products to be ordered to the wholesaler (and supplied to the market) and the retail price to be charged from the customers. To make our solution more comprehensive, we consider the time-dependent demand to be a function of the retail price as well. We solve the problem in a discrete time framework, with the time horizon divided into intervals referred to as periods.

This paper circles around the classical balance between price and demand. The simplest possible case occurs when demand is a deterministic function of price. Total revenue is then trivially maximized at the combination of price and demand where the elasticity of demand is unitary. If we consider demand within a time frame, the situation may be different. The current price is obviously still important, but the general level of demand may depend on previous prices in a critical way. A well proven line of approach is to increase demand in the future by low prices in the present. This is likely to affect demand since low initial prices lead to a situation where a larger fraction of the potential customer base takes interest in the product. If sufficiently many of these are return customers, prices can be increased later without lowering demand too much, leading to increased revenue over time. In marketing, this is a common strategy for market penetration, which is considered to be the least risky entry in the Ansoff matrix, Ansoff (1957).

A deterministic function of price is computationally convenient, but not realistic. The basic idea in the present paper is to model demand by a random variable where the mean and variance may vary with time, and the current retail price in a systematic way. Moreover, we will take the above point of view from marketing, assuming that previous prices scale demand by affecting the number of customers taking interest in the product. This is particularly important when a company wants to sell high-tech products with a possibly short lifespan. An optimal pricing scheme is critical. At the end of the timespan, the product will be outdated and replaced by more advanced products.

Trade-ins and introductory offers are more common than ever before, in particular due to web-based shopping. Market penetration strategies such as providing the potential customers with free trial versions of a software or distribution of a small number of a newly introduced

[1] In this paper, following the convention adopted by Cachon (2003), we assume the upstream agent (i.e. the wholesaler) to be female and the downstream agent (i.e. the retailer) to be male.

cell-phone model for free are frequently employed. This marketing approach may incur huge initial losses, and succeeds only if demand is enhanced to a level that outweighs the initial costs. The main issue for such schemes is to obtain a proper balance between present revenue and revenue in the remaining lifespan of the product. The length of the introductory period is for obvious reasons a crucial factor for success.

In this paper, we assume that a good is produced by a manufacturer and sold to a retailer. We will assume that the manufacturer and the retailer are risk neutral in the sense that they want to maximize expected, discounted total profit. The manufacturer faces a downstream coordination problem. As coordination affects performance over time, coordination is quite delicate. If the wholesale price is too high, this may imply a retail price harmful to future demand.

We consider a multi-period Stackelberg game between a manufacturer and a retailer where the actions of the two parties affect the actions of a third party, the customers. When the wholesale price W and the retail price R are settled, the retailer faces a classical newsvendor problem, i.e., he should choose an order quantity q to maximize expected profit. The distribution of demand is known once prices are settled, in which case the solution to our newsvendor problem is well known. Our central problem is hence to compute equilibrium prices. We ignore inventory, as this would add another level of complexity to our problem.

In general, multi-period Stackelberg games of this type are difficult to solve, even in cases that ignore inventory. The main technical obstacle is that the problem quickly branches into a huge number of subproblems. Problems with a very small number (3 or 4) of periods are computationally intensive, and general cases where the number of periods is large are more or less impossible to compute. To circumvent these difficulties we found a quite ingenious twist that separates some special multi-period games into a sequence of dependent 1-period games. Single-period games of this type are classical, and are focusing cases with additive or multiplicative demand (see Section 3 for precise definitions). Our multi-period framework is a natural extension of the single-period additive-multiplicative model, and our method puts very few restrictions on the functions that can be used. The model provides a functional flexibility where it is possible to discuss a variety of economic contexts.

The main result in the paper is Theorem 4.1. This theorem offers an explicit solution to the multi-period Stackelberg game. The construction may take some effort to understand, but the final solution is surprisingly easy to implement. To solve this problem numerically, we will need to solve a sequence of coupled one-variable problems, and the suggested method produces unique equilibria for reasonable specifications of mean, variance and scaling. Counterexamples do exist, but all the counterexamples we know of involve artificial constructions that are unlikely to occur in real world applications.

A point of interest is that the paper establishes a link between the newsvendor problem, game theory and marketing. Under certain conditions, our model suggests a pre-sales period where the product is given away for free. Our model determines the length of this period and the subsequent pricing schemes. See Section 5.3.

The paper is organized as follows. In Section 2, we review some central literature related to this paper. The one-period problem is classical, but to the best of our knowledge, the multi-period game we discuss is new. Our paper relates to several important branches of economic literature, in particular the theory of pricing. The theory of pricing is among the most heavily studied topics in economics, and certainly there must exist important contributions of which we are not aware. Among the many papers on price-dependent demand in a newsvendor context, we put emphasis on papers discussing Stackelberg issues. In the literature on stochastic games and pricing our paper relates in particular to those papers dealing with Markov perfect equilibria, and we review some of the most relevant references.

In Section 3, we introduce basic notation and review classical formulas for the one-period case. In Section 4, we formulate Stackelberg games for the two-period case. If demand in the next period is scaled by a factor that depends on the current demand and retail price, the system decouples into two separate cases. This decoupling carries over to the multi-period case, and we can obtain a complete solution by backward induction; i.e., we first solve the problem for the final period, feed the solution into a similar problem for the previous period, and continue backwards until we reach the first period. Our main result is stated in Theorem 4.1. In Section 5, we demonstrate how the theory in Section 4 can be implemented in a variety of different explicit examples. In Section 6, we offer concluding remarks.

2 The newsvendor problem with price dependent demand

In the classical newsvendor problem, a retailer wants to order a quantity q from a manufacturer. Demand D is a random variable, and the retailer wishes to select an order quantity q to maximize his expected profit $E[\Pi^r[q, D]]$. When the distribution of D is known, this problem is easily solved. The basic problem is simple, but appears to have a never-ending number of variations. There is now a large literature on such problems, which is surveyed in an excellent way by Cachon (2003) and Qin et al. (2011). (See also the numerous references therein.)

The one-period newsvendor problem with price-dependent demand is classical (see Whitin 1955). Mills (1959) refined the construction by considering the case in which demand uncertainty is added to the price-demand curve, and Karlin and Carr (1962) considered the case in which demand uncertainty is multiplied by the price-demand curve. Young (1978) covering both the additive and the multiplicative case within the same framework, generalized the results in these early papers. For a useful review of the problem with significant extensions, see Petruzzi and Dada (1999).

The by now huge body of literature dealing with price sensitive demand and inventory decisions is excellently surveyed in Chan et al. (2004). The papers they review are certainly important, but as far as we could see, none of these consider Stackelberg issues. The same remark applies to the deep survey in Elmaghraby and Keskinocak (2003), who are also focusing on inventory decisions. Cachon and Zipkin (1999) discuss Stackelberg equilibria in a multiperiod setting with stationary demand. We do not know of any scientific research combining inventory and Stackelberg decisions in cases with price dependent demand. Both topics add considerable complexity to the pricing problem, and in general, a combination of the two appears to be difficult. However, some low dimensional cases can be handled numerically based on the methods we outline in this paper. (See the concluding remarks.)

Petruzzi and Dada (1999) consider multi-period cases with price-dependent demand,

and show how to adapt such models to include backorders. However, they do not discuss Stackelberg competition. See also Kocabiyikoglu and Popescu (2011), Xu et al. (2010) and Xu et al. (2011) for some recent contributions to the price-dependent case. Pricing strategies for retailers have been discussed intensively in the marketing literature, and we mention Rao (1984) and Fassnacht and Husseini (2013).

Stackelberg competition between manufacturers and retailers can be traced back to Marshall (1920) who remarks that while retailers are forced to sell popular brands at prices barely covering their expenses, wholesale prices are relatively high. The inverse association between retailer's and manufacturer's margins has been studied in several papers by R. Steiner, see, e.g., Steiner (1985, 1993). Steiner points to extensive empirical evidence, and introduces a dual stage model which can be used to study the dynamics of the inverse association. Lynch (2004) reviews Steiner's works focusing numerous empirical cases in support of Steiner's theory. Cohen (2013) enters into a deep analysis of vertical and horizontal price competition using Bayesian methods to analyze empirical data. Rey and Vergé (2010) study how resale price maintenance limits competition in a Stackelberg game between two manufacturers and two retailers and prove the existence of an equilibrium for such games. The dynamics of prices in game theoretical settings have been discussed in several publications by K. Bagwell, we mention Bagwell (1987, 2007).

In our Stackelberg game, the manufacturer is the leader and offers the retailer a wholesale price W for items delivered in the next period. The retailer is the follower and tries to select an order quantity q and a retail price R to maximize future expected profits. Stackelberg games for the one-period case with fixed R have been studied extensively by Lariviere and Porteus (2001), who provide quite general conditions under which unique equilibria can be found. Song et al. (2008) study a single-period buyback contract in a Stackelberg framework of a manufacturer and a price-setting retailer. They introduce a new transformation technique establishing unimodality of the profit functions, and identify necessary and sufficient conditions under which the optimal contract is independent of the shape of the demand distribution. In this paper, we extend the single-period theory to Stackelberg equilibria in multi-period cases in which demand in the future is a function of prices and demand in the past. We assume that unmet demand is lost, and hence ignore cases with backordering. Such an assumption can be made in important cases such as electricity markets and markets for fresh foods.

General theory for stochastic games dates back to the seminal works of Shapley (1953). A particular line of approach with relevance to the problem we discuss in this paper is the theory of Markov perfect equilibria (MPE). This line of research was initiated by E. Maskin and J. Tirole in the late 80s, see Maskin and Tirole (1988) and Tirole (1988). The concept has found important applications in the analysis of industrial organization, in macroeconomics and in political economy.

In an MPE each player's mixed strategy can be conditioned only on the state of the game. Fudenberg and Tirole (1991) provide a short proof that MPEs always exist in stochastic games with a finite number of states and actions. Doraszelski and Escobar (2010) discuss generic properties of MPEs in stochastic dynamic games, and show that almost all such games have a finite number of locally isolated MPEs. Haller and Lagunoff (2000) have obtained similar results. MPEs lead in general to computable models. A central paper is Ericson and Pakes (1995), who introduce a computable model of dynamic competition.

For recent contributions to this theory, see Doraszelski and Satterthwaite (2010) and the references therein.

In dynamic limit pricing, Gaskins (1971) and Judd and Petersen (1986), a dominant firm will use its market power to slow fringe growth, and the crucial issue is to find the correct balance between present and future earnings. Although this setting is principally different from ours, these pricing schemes have similarities with our approach. In Section 5.3, we consider a multi-period case where prices are initially very small, followed by an aggressive phase in the latter stages. The principal issue is again the balance between present and future earnings. Similar aspects arise in the theory of network externalities, Katz and Shapiro (1985). Øksendal et al. (2013) consider continuous time Stackelberg games for Itô-Lévy processes with price-dependent demand. They prove that equilibria can be found by solving a coupled system of stochastic differential equations. In principle, such systems can be solved, but even simple cases lead to equations that cannot be solved by any conventional means. Solutions appear to require mathematical optimization techniques not yet discussed in the literature.

The discussion in Øksendal et al. (2013) partly explains why general multi-period problems are difficult to solve. Some types may admit numerical solutions, but the general problem is difficult to compute or analyze even in the two-period case. By comparison, the discrete version we consider in this paper is transparent. Our scaling approach decouples a multi-period problem into a sequence of one-period problems, each of which is fairly easy to solve. Our model retains the main essence of the problem itself, while simultaneously providing a solution that can be analyzed without the need for advanced optimizing techniques.

3 The basic model: the single-period newsvendor problem with price and time dependent demand

The solution to the single-period newsvendor problem with stochastic demand forms the basic building block of our model in this research. Therefore, in this section, we review some properties of the single-period model and in the next sections, we propose our multi-period model based on it.

Main symbols:

W = wholesale price per unit (chosen by the manufacturer)

q = order quantity (chosen by the retailer)

R = retail price per unit (chosen by the retailer)

D = demand (random)

M = production cost per unit (fixed)

S = salvage price per unit (fixed)

Π^r = profit for the retailer

Π^m = profit for the manufacturer

In the classical newsvendor model, the manufacturer sets the wholesale price W for one unit of a certain commodity that needs to be sold within a short timespan. The retailer orders a quantity q units of the commodity to the manufacturer and plans to sell them for the price R (per unit) in a market with stochastic demand D. Any unsold items can be salvaged at the price S. The retailer's profit Π^r is calculated as below.

$$\begin{aligned}\Pi^r &= R\,min[D,q] + S(q-D)^+ - Wq \\ &= R\,min[D,q] + S(q - min[D,q]) = Wq \\ &= (R-S)\,min[D,q] - (W-S)q.\end{aligned} \tag{1}$$

From this expression, we obtain the expected profit for the retailer:

$$E[\Pi^r] = (R-S)E[min[D,q]] - (W-S)q. \tag{2}$$

In our model, we consider the additive-multiplicative model for the demand as given below.

$$D[R,k] = \mu[R,k] + \sigma[R,k]\varepsilon \tag{3}$$

The demand in (3) is both price-dependent and time-dependent as $\mu[R,k]$ and $\sigma[R,k]$ are given deterministic functions of R (the retail price) and k (the number of the current period in the multi-period framework), and ε is a random variable with an arbitrary distribution, satisfying $E[\varepsilon] = 0$ and $Var[\varepsilon] = 1$. This model formulation is equivalent to the one used by Young (1978), the only difference being that Young assumes a different normalization where $E[\varepsilon] = 1$. The format in (3) covers both the additive (Within (1955)) and the multiplicative cases (Karlin and Carr (1962)). Note that in (3) and using our normalization, the mean and standard deviation of the demand are $\mu[R,k]$ and $\sigma[R,k]$ respectively, and the coefficient of variation (CV) is therefore $\sigma[R,k]/\mu[R,k]$. Thus, in our additive-multiplicative model, we have an eye on the mean and variance of the demand at every stage of the model that makes it easier to impose economic constraints such as

$$\lim_{R\to\infty} E[D] = 0$$

on our mathematical expressions. In addition, our additive-multiplicative model allows the coefficient of variation to be a function of price and time, as opposed to the multiplicative demand model, Carlin and Carr (1962), in which the coefficient of variation becomes a constant. In section 4.3.2, we will discuss that a constant coefficient of variation does not happen in general, and thus, a model resulting in a constant CV may be unrealistic in some cases.

For a given R, it is well known that the maximum expected profit is obtained when:

$$P(D \leq q) = \frac{R-W}{R-S} \tag{4}$$

Inserting the expression for the demand in (3) into (2) and using (4), we can prove the following proposition where F_ε denotes the cumulative distribution of ε.

Proposition 3.1.
Assume that ε is a continuous distribution, supported on an interval, with density $f_\varepsilon > 0$ a.e. on its support. Given R and W, $R \geq W > S$, the retailer will make an order

$$q = \mu[R,k] + \sigma[R,k] \cdot F_\varepsilon^{-1}\left[\frac{R-W}{R-S}\right]. \tag{5}$$

in which case, he obtains the expected profit

$$\overline{\Pi}^r = E[\Pi^r] = (R-W)\mu[R,k] + L_\varepsilon[R,W]\sigma[R,k] \tag{6}$$

where L_ε is defined by

$$L_\varepsilon[R,W] = (R-S)\int_{-\infty}^{z} x f_\varepsilon[x]dx \quad z = F_\varepsilon^{-1}\left[\frac{R-W}{R-S}\right] \tag{7}$$

Proof
See the Appendix.

Our setup is slightly non-standard since we use a different normalization than Young (1978). Nonetheless, the result in Proposition 3.1 is more or less well known within the literature. In our normalization, we assume that $E[\varepsilon] = \int_{-\infty}^{\infty} x f_\varepsilon[x]dx = 0$, and hence, $L_\varepsilon[R,W] \leq 0$. In the literature, the term $L_\varepsilon \,.\, \sigma$ is often referred to as *loss due to randomness.*

Note that L_ε does not depend on the choice of the function $\sigma[R,k]$. For the construction used to solve multi-period Stackelberg games in this paper, it is important that the deterministic function $\sigma[R,k]$ enters as a multiplicative factor in (6). Thus, it is essential that the σ dependence is handled through the format we use in (3).

4 Multi-period vertical contracting

Having discussed the solution to the single-period problem, we are now ready to provide a theoretical analysis of the multi-period Stackelberg game. In particular, we focus on the case in which demand in the next period is scaled by a factor that depend on price and demand in the current period. This is a type of Markovian assumption in that it only requires knowledge of the current state, not of how prices and demand arrived at that state.

In the multi-period game, we assume that the parties are risk neutral and try to maximize their discounted expected profits:

$$J_r = \overline{\Pi}_1^r + \alpha\overline{\Pi}_2^r + +\alpha^2\overline{\Pi}_3^r + ... + \alpha^{N-1}\overline{\Pi}_N^r \tag{8}$$

$$J_m = \overline{\Pi}_1^m + \alpha\overline{\Pi}_2^m + +\alpha^2\overline{\Pi}_3^m + ... + \alpha^{N-1}\overline{\Pi}_N^m \tag{9}$$

where N is the number of periods, α is the discounting factor, and barred symbols indicate expected values.

4.1 The profit optimization algorithm for the single-period Stackelberg game

In the one-period newsvendor model, to formulate a Stackelberg game, we assume that both parties are risk neutral. The manufacturer (leader) offers a wholesale price W. We ignore the possibility that the retailer can negotiate this wholesale price. Given W, the retailer (follower) then solves (6) to find the R^* which maximizes $\overline{\Pi}^r$, and then, substituting this R^* into (5) to find out the optimum order quantity q^*. The manufacturer also knows that the retailer is going to choose q^* to maximize the expected profit. Therefore, given each possible value of W, the manufacturer can anticipate the resulting order quantity $q^* = q^*[W]$, and so chooses W to maximize expected profit (which happens be to be deterministic in this case). The manufacturer's profit is given by:

$$\Pi^m = (W - M)q \tag{10}$$

where M is the production cost per unit.

The maxima-finding algorithm for the two agents' expected profits is illustrated in Algorithm 1.

Algorithm 1: Optimization algorithm for the single-period Stackelberg game

```
input      : μ[R,k], σ[R,k]          // Describing mean and variance of the demand
                                      // Since this is a single-period game, k = 1
output     : R*, q*, W* ; Π̄^{r*}, Π̄^{m*}   // The three optimal decision
                                      // variables and corresponding maximum
                                      // expected profits for the two agents
parameters: α, M, S, W_max, R_max    // Predetermined constants
                                      // W_max < R_max
for ∀ W_i ∈ {M, W_max} do
                                      // The outer loop for the leader (wholesaler)
                                      // Generating a grid of W_i s
    for ∀ R_j ∈ {W_i, R_max} do
                                      // The inner loop for the follower (retailer)
                                      // Generating a grid of R_j s
        Set Π̄^r_j := Π̄^r_j[R_j, W_i]      // Using equation (6).
                                      // Within the entire inner loop on j, W_i is constant.
    end
    Find the maximum value of Π̄^r_j s and the corresponding argmax
                                      // Using Newton-Raphson method or a heuristic
    Set Π̄^{r̂}_i := max(Π̄^r_j)
    Set R̂_i := argmax(Π̄^r_j)
    Set q̂_i := q[R̂_i]                   // Using equation (5)
    Set Π̄^m_i := Π̄^m_i[W_i, q̂_i]        // Using equation (10). Because for every value of
                                      // W_i, the corresponding q̂_i has already been calculated,
                                      // Π̄^m_i becomes a function of only W_i.
end
Find the maximum value of Π̄^m_i s and the corresponding argmax
Set Π̄^{m*} := max(Π̄^m_i)
Set W* := argmax(Π̄^m_i) Set I := i*   // With i* representing the location of W* on
                                      // W_i grid
Set R* := R̂_I
Set q* := q̂_I
Set Π̄^{r*} = Π̄^{r̂}_I
Return R*, q*, W* ; Π̄^{r*}, Π̄^{m*}
```

4.2 General two-period games

We now consider the general two-period game. The crucial point in this two-stage game is to take into account the level of information at each level, and the rest is just a matter of book-keeping making sure that the problem is properly posed as an optimization problem.

For a two-period Stackelberg game, demand in the first period is given by:

$$D_1 = \mu_1[R_1, k] + \sigma_1[R_1, k]\varepsilon_1. \tag{11}$$

In the second period, we have:

$$D_2 = \mu_2[R_1, R_2, D_1] + \sigma_2[R_1, R_2, D_2]\varepsilon_2. \tag{12}$$

We assume that μ_1, μ_2, σ_1, and σ_2 are deterministic functions, and that the random variables ε_1 and ε_2 are centered and normalized. We should also note that, as this is a discrete-time model, the term k, representing the period number, remains constant within the duration of each period.

The idea is that the level of demand in the first period can (to some extent) carry over to the second period. Moreover, a high price in the first period can lead to reduced demand in the second period, whereas a low initial price can have the opposite effect by stimulating demand.

We let Π_1^r and Π_2^r denote the profits for the retailer in the two periods. Π_1^m and Π_2^m denote the corresponding profits for the manufacturer. We assume that both parties try to maximize discounted expected profits:

$$J_r = E[\Pi_1^r + \alpha.\Pi_2^r] \tag{13}$$

$$J_m = E[\Pi_1^m + \alpha.\Pi_2^m] \tag{14}$$

where $0 \leq \alpha \leq 1$ is a fixed discounting factor. When decisions are taken for the second period, we assume that the values R_1, W_1, and D_1 are common knowledge. Conditional on D_1, and given values for R_1, R_2, and W_2, it follows from (5) and (6) that:

$$q_2 = \mu_2[R_1, R_2, D_1] + \sigma_2[R_1, R_2, D_1].F_{\varepsilon_2}^{-1}\left[\frac{R_2 - W_2}{R_2 - S}\right] \tag{15}$$

$$E[\Pi_2^r|D_1] = (R_2 - W_2)\mu_2[R_1, R_2, D_1] + L_{\varepsilon_2}[R_2, W_2]\sigma_2[R_1, R_2, D_1]. \tag{16}$$

In the second and final period, there is no need to worry about future demand. Hence, given R_1, D_1, and W_2, the retailer chooses R_2 to maximize $E[\Pi_2^r|D_1]$. By assuming that the mapping $R_2 \mapsto E[\Pi_2^r|D_1]$ has a unique maximum, we can hence construct a function $R_2 = R_2[R, D_1, W_2]$ that maximizes this conditional expected value. At the time when the manufacturer chooses W_2, the values of R_1 and D_1 are common knowledge. Hence, the manufacturer chooses W_2 to maximize conditional profit:

$$E[\Pi_2^m|D_1] = (W_2 - M_2)q_2 \tag{17}$$

where q_2 is given by (15) and $R_2 = R_2[R_1, D_1, W_2]$. Given values for R_1 and D_1, it follows that $E[\Pi_2^m|D_1]$ is a function of only W_2. Assuming that this function has a unique maximum, we can then construct a function $W_2 = W_2[R_1, D_1]$ that maximizes the manufacturer's conditional expected profit.

By the law of double expectation, we have:

$$J_r = E[\Pi_1^r] + \alpha.E[E[\Pi_2^r|D_1]] \tag{18}$$

$$J_r = E[\Pi_1^m] + \alpha.E[E[\Pi_2^m|D_1]] \tag{19}$$

Given a value for W_1, the retailer, knowing that the manufacturer is a Stackelberg optimizer, can anticipate that the manufacturer will offer the price $W_2 = W_2[R_1, D_1]$ in the second period. By (6), we have:

$$E[\Pi_1^m] = (R_1 - W_1)\mu_1[R_1] + L_{\varepsilon_1}[R_1, W_1]\sigma_1[R_1]. \tag{20}$$

Given R_1, the distribution of D_1 is known. Equation (18), together with (16) and (20), enables us to compute the final value of J_r given this particular choice of R_1. The retailer chooses R_1 to maximize this value. From this choice, the manufacturer obtains the (deterministic) profit of:

$$\Pi_1^m = (W_1 - M_1)\left(\mu_1[R_1] + \sigma_1[R_1].F_{\varepsilon_1}^{-1}\left[\frac{R_1 - W_1}{R_1 - S}\right]\right) \tag{21}$$

The manufacturer's (possibly) random profit is:

$$\Pi_2^m = (W_2 - M_2)q_2[R_1, D_1, W_2]. \tag{22}$$

Knowing that the retailer will choose R_1 as above, the manufacturer can hence choose W_1 to maximize her total expected profit. This shows that both the manufacturer and the retailer face a well-posed optimization problem.

4.3 Two-period games with memory function

4.3.1 Memory function

The general construction in Section 4.2 is sufficiently explicit to enable solutions of the problem for most choices of functions μ_1, μ_2, σ_1, and σ_2. However, the problem is so deeply nested that one cannot expect to find an analytical solution. Extending (12) and in order to consider the effect of the demand, stochasticity, and retail price in the previous period on demand in the current period, we consider the following important case:

$$\mu_2[R_2, R_1, D_1, \varepsilon_1] = \tilde{\mu}_2[R_2] \cdot g_2[R_1, D_1, \varepsilon_1],\ \sigma_2[R_2, R_1, D_1, \varepsilon_1] = \tilde{\sigma}_2[R_2] \cdot g_2[R_1, D_1, \varepsilon_1] \tag{23}$$

with g_1 being a common scaling factor. In this case, (16) takes the form:

$$E[\Pi_2^r|D_1] = (R_2 - W_2)\mu_2[R_2, R_1, D_1, \varepsilon_1] + \sigma_2[R_2, R_1, D_2, \varepsilon_1](R_2 - S)\int_{-\infty}^{F_{\varepsilon_2}^{-1}\left[\frac{R_2 - W_2}{R_2 - S}\right]} x f_{\varepsilon_2}[x]dx \tag{24}$$

$$= g_2[R_1, D_1, \varepsilon_1]\left((R_2 - W_2)\tilde{\mu}_2[R_2] + \tilde{\sigma}_2[R_2](R_2 - S)\int_{-\infty}^{F_{\varepsilon_2}^{-1}\left[\frac{R_2 - W_2}{R_2 - S}\right]} x f_{\varepsilon_2}[x]dx\right) \tag{25}$$

and the optimal values of R_2 and W_2 are then independent of R_1, D_1 and ε_1. The multiplier effect in (25) is the crucial observation in this paper; it reduces the retailer's optimization problem to a problem of maximizing:

$$J_r[R_1] = (R_1 - W_1)\mu_1[R_1] + L_{\varepsilon_1}[R_1, W_1]\sigma_1[R_1] + E[\alpha \cdot g_2[R_1, D_1, \varepsilon_1] \cdot \overline{\Pi}_2] \tag{26}$$

where $\overline{\Pi}_2$ is the expected profit the retailer would have obtained in the final period had the scaling factor been 1. This simplification separates our original problem into two separate subproblems, which are both easily solved. The problem for the final period is a standard one-period problem with price-dependent demand. The second problem is quite similar, the only difference being the extra term $E[\alpha \cdot g_2[R_1, D_1, \varepsilon_1] \cdot \overline{\Pi}_2]$.

4.3.2 Coefficient of variation and scaling of demand

Randomness in demand is primarily driven by two effects. The first effect is caused by individual randomness in preferences. If agents act independently, such randomness causes the coefficient of variation, σ_D/μ_D, to increase as the pool of interested agents decrease. In the additive model, Mills (1959), $\mu_D[R]$ is decreasing while $\sigma_D[R]$ is constant, and the coefficient of variation will increase with R. If the number of agents is very large, however, we can expect that this effect is largely diversified, i.e., the resulting variance is relatively small.

The second type of randomness is driven by events, e.g., weather conditions, introduction of competing products etc. This type of randomness leads to highly correlated agents. The effect does not vanish when the number of agents is very large, and the high level of correlation leads to a fairly constant coefficient of variation.

The scaling structure in (23) can alternatively be written as follows:

$$D_2[R_1, D_1, R_2] = g_2[R_1, D_1, \varepsilon_1] \cdot \tilde{D}[R_2] \quad \text{where} \quad \tilde{D}[R_2] = \mu_2[R_2] + \sigma_2[R_2]\varepsilon_2 \tag{27}$$

Demand in period 2 is hence primarily a function of price in period 2, while the previous price scales the overall level of demand. As a consequence of the proposed scaling structure, mean and variance are scaled proportionally. Unless the randomness in the individual preference structure is very large, it appears reasonable that randomness due to events is dominant, leading to a constant coefficient of variation. The multiplicative scaling structure we propose in (27) hence appears reasonable under fairly general conditions.

4.4 Multi-period games with memory function

Whereas it is straightforward to formulate an n-period game in the general case, numerical solutions are difficult to obtain even if n is moderately large. The nonlinear structure of the problem branching into separate cases for each particular choice made on every level quickly renders the problem computationally intractable.

In this section, we show how to generalize the scaling approach described in the previous section to multi-period problems. First, we discuss an important technical issue. Consider the three-period problem:

$$D_1 = \mu_1[R_1] + \sigma_1[R_1]\varepsilon_1 \tag{28}$$

$$D_2 = g_2[R_1, D_1, \varepsilon_1](\mu_2[R_2] + \sigma_2[R_2]\varepsilon_2) \tag{29}$$

$$D_3 = g_2[R_1, D_1, \varepsilon_1]g_3[R_2, D_2, \varepsilon_2](\mu_3[R_3] + \sigma_3[R_3]\varepsilon_3) \tag{30}$$

where g_2 and g_3 are scaling factors. For generality, we assume that ε_1, ε_2, ε_3 are not necessarily independent. Moreover, the scaling factors $g_2[R_1, D_1, \varepsilon_1]$ and $g_3[R_2, D_2, \varepsilon_2]$ are not typically independent. Our task in this section is to determine restrictions to be imposed on g_2 and g_3 in order to make the optimization problems in different periods independent from each other, so that the global optimization problem can be decomposed into independent subproblems. In the following analysis, we consider only the retailer's profit optimization procedure. The same arguments also hold true for the wholesaler's.

Starting the backward induction process from the final period, for the retailer we have:

$$J_3^r = g_2[R_1, D_1, \varepsilon_1] g_3[R_2, D_2, \varepsilon_2] \overline{\Pi}_3[R_3] \tag{31}$$

where according to (6),

$$\overline{\Pi}_k[R_k] = (R_k - W_k)\mu_k[R_k] + L_{\varepsilon k}[R_k, W_k]\sigma_k[R_k].$$

Thus,

$$J_3^r = g_2[R_1, D_1, \varepsilon_1]\, g_3[R_2, D_2, \varepsilon_2]) \Big[(R_3 - W_3)[\mu_3[R_3] + L_{\varepsilon 3}[R_3, W_3]\sigma_3[R_3] \Big]. \tag{31$'$}$$

Because period 3 is the final period, there is no need to worry about future demand, and therefore, given W_3, the retailer chooses R_3 to maximize J_3^r. Note that because R_i, D_i and ε_i $(i = 1, 2)$ have happened in the past, they are not considered as decision variables at period 3 and the optimal values of R_3 and W_3 are independent of them. Thus, the optimization problem reduces to the single-variable problem of maximizing $\overline{\Pi}_3[R_3]$.

Assuming that $\widehat{\Pi}_3$ is the (unique) maximum value of $\overline{\Pi}_3[R_3]$, the backward induction method proceeds to the next subproblem, i.e., the problem of maximizing the expected profit in the second period.

$$\begin{aligned} J_2^r &= g_2[R_1, D_1, \varepsilon_1] \Big[\overline{\Pi}_2[R_2] + E\Big[\alpha g_2[R_2, D_2, \varepsilon_2] \widehat{\Pi}_3\Big]\Big] \\ &= g_2[R_1, D_1, \varepsilon_1] \Big[(R_2 - W_2)\mu_2[R_2] + L_{\varepsilon 2}[R_2, W_2]\sigma_2[R_2] + \alpha \widehat{\Pi}_3 E\Big[g_3[R_2, D_2, \varepsilon_2]\Big]\Big] \end{aligned} \tag{32}$$

In (32), unlike the case in (31$'$), and because of the term $E\Big[g_3[R_2, D_2, \varepsilon_2]\Big]$, the optimal value for R_2 does not become independent of R_1 and D_1, as according to (29), $D_2 = g_2[R_1, D_1, \varepsilon_1](\mu_2[R_2] + \sigma_2[R_2]\varepsilon_2)$. However, in the backward induction process, the optimal values for R_1 and D_1 are not obtained yet. Therefore, such a system of implicit optimization problems does not decompose into separate single-variable optimization subproblems.

This issue does not arise if $g_3 = g_3[R_2, \varepsilon_2]$ or, in general, if the memory function g_{k+1} depends only on R_k and ε_k. When every scaling factor is independent of the previous demand, the system can be solved using backward induction. As, we have shown in (26), this always applies if $n = 2$ because there is only one scaling factor involved.

To simplify the notation, we define:

$$\overline{g}[R] = E[g[R, \varepsilon]]. \tag{33}$$

First, we solve for the final period to obtain expected profits $\overline{\Pi}_n^r$ and $\overline{\Pi}_n^m$. Once these values are known, the previous level can be computed as shown in Section 4.3. That produces numerical values of $\overline{\Pi}^r_{n-1,n}$ and $\overline{\Pi}^m_{n-1,n}$ (total discounted expected profits in the two periods). To determine the strategy for $(n-2)$nd level, we consider the problem:

$$\begin{aligned} J_r[R_{n-2}] = &(R_{n-2} - W_{n-2})\mu_{n-2}[R_{n-2}] + L_{\varepsilon_{n-2}}[R_{n-2}, W_{n-2}]\sigma_{n-2}[W_{n-2}] \\ &+ \alpha \cdot \overline{\Pi}^r_{n-1,n} \cdot \overline{g}_{n-1}[R_{n-2}] \end{aligned} \tag{34}$$

$$\begin{aligned} J_m[W_{n-2}] = &(W_{n-2} - Mn - 2)\Big(\mu_{n-2}[R_{n-2}] + \sigma_{n-2}[R_{n-2}] \cdot F^{-1}_{\varepsilon_{n-2}}\Big[\frac{R_{n-2} - W_{n-2}}{R_{n-2} - S}\Big]\Big) \\ &+ \alpha \cdot \overline{\Pi}^m_{n-1,n} \cdot \overline{g}_{n-1}[R_{n-2}] \end{aligned} \tag{35}$$

From 34 and 35, we see that the problem for period $n-2$ is reduced to a 1-period problem that only involves R_{n-2} and W_{n-2}. The only difference from the problem for period $n-1$, is that the values of $(\overline{\Pi}^r_{n-1,n}, \overline{\Pi}^m_{n-1,n})$ are different from the values $(\overline{\Pi}^r_n, \overline{\Pi}^m_n)$. Hence all we have to do to solve this problem is repeat the previous step with updated values for $(\overline{\Pi}^r, \overline{\Pi}^m)$.

To simplify notation, we have supressed dependence on arguments that are not yet active; μ_{n-2} and σ_{n-2} are in general functions of $(R_{n-3}, \varepsilon_{n-3})$ but according to our assumptions, this dependence enters as an independent multiplicative factor and can hence be factored out of the optimization problem. (See equations $31'$ and 32 for example.)

By using the argument above repeatedly, it is clear that we can solve this problem for any value of n. Starting with the values $(\overline{\Pi}^r, \overline{\Pi}^m) = 0$ in the final period, we solve essentially the same problem in all periods. The values of $(\overline{\Pi}^r, \overline{\Pi}^m)$ are updated as the construction progresses backwards, but those updated values come for free from the solution of the previous step. We state the final result as follows.

Theorem 4.1.
Let n be the number of periods and assume that demand in period k is given by:

$$D_k = (\mu_k[R_k] + \sigma_k[R_k]\varepsilon_k) \cdot \prod_{i=2}^{k} g_i[R_{i-1}, \varepsilon_{i-1}] \tag{36}$$

where $\varepsilon_1, ..., \varepsilon_n$ are continuously distributed with $\mathrm{E}[\varepsilon_k] = 0$ and $\mathrm{Var}[\varepsilon_k] = 1$ for all k, and supported on intervals with $f_{\varepsilon_k} > 0$ a.e. on their supports. If, for each k, the one-period Stackelberg problem below has a unique equilibrium at $R_k = \hat{R}_k, W_k = \hat{W}_k$

$$\begin{aligned} J_r^{(k)}[R_k] &= (R_k - W_k)\mu_k[R_k] + L_{\varepsilon_k}[R_k, W_k]\sigma_k[R_k] + \alpha \cdot \overline{\Pi}^r_k \cdot \overline{g}_{k+1}[R_k] \\ J_m^{(k)}[W_k] &= (W_k - M_k)\Big(\mu_k[R_k] + \sigma_k[R_k] \cdot F^{-1}_{\varepsilon_k}\Big[\frac{R_k - W_k}{R_k - S}\Big]\Big) + \alpha \cdot \overline{\Pi}^m_k \cdot \overline{g}_{k+1}[R_k] \end{aligned} \tag{37}$$

where $\overline{\Pi}^r_k$ and $\overline{\Pi}^m_k$ are found recursively from:

$$\overline{\Pi}^r_n = 0 \quad \overline{\Pi}^m_n = 0 \tag{38}$$

$$\overline{\Pi}^r_k = J_r^{k+1}[\hat{R}_{k+1}] \quad \overline{\Pi}^m_k = J_m^{k+1}[\hat{W}_{k+1}] \quad k = 1, 2, ..., n-1, \tag{39}$$

then the problem of maximizing

$$J_r = \overline{\Pi}^r_1 + \alpha\overline{\Pi}^r_2 + \alpha^2\overline{\Pi}^r_3 + ...\alpha^{n-1}\overline{\Pi}^r_n \tag{40}$$

$$J_m = \overline{\Pi}^m_1 + \alpha\overline{\Pi}^m_2 + \alpha^2\overline{\Pi}^m_3 + ...\alpha^{n-1}\overline{\Pi}^m_n \tag{41}$$

has a unique equilibrium at $\hat{\boldsymbol{R}} = (\hat{R}_1, \hat{R}_2, ..., \hat{R}_n)$, $\hat{\boldsymbol{W}} = (\hat{W}_1, \hat{W}_2, ..., \hat{W}_n)$.

Remarks

The multiplicative factor $\prod_{i=2}^{k} g_i[R_{i-1}, \varepsilon_{i-1}]$ controls the memory of the process. If all the scaling factors are equal to 1, there is no memory, and the problem decouples into independent one-period problems. Note that given R_i, the value of ε_i is known if and only if the value of D_i is known.

Comparing the case where scaling is given by $\prod_{i=2}^{k} g_i[R_{i-1}, \varepsilon_{i-1}]$ with the corresponding general case where the scaling factor is $\prod_{i=2}^{k} g_i[R_{i-1}, \varepsilon_{i-1}, D_{i-1}]$, we note that the latter specification leads to dependence in the forward phase of the system. Both versions carry a Markov type of information, i.e., we only need to know the value of the scaling at time t to progress forward.

The condition that ε_k is supported on an interval with $f_{\varepsilon_k} > 0$ a.e. on its support is needed to ensure that F_{ε_k} is invertible. If F_{ε_k} is not invertible, it is possible that the retailer's expected profit is maximized at several order quantities between which the retailer is indifferent. Different order quantities lead to different profits for the manufacturer, but the manufacturer lacks an instrument to ensure that the retailer chooses order quantities that are optimal for the manufacturer.

As we can see from the assumptions in the theorem, the scaling constants and the shape of the demand distributions are allowed to change from period to period. This offers a functional flexibility where several kinds of economic contexts can be built into the model. In this setting, the function driving the model can change systematically over time. It is possible to model seasonal trends, but it is also possible to model, e.g., stochastic games where strategic customers postpone purchases with the hope of lower prices in the future.

Theorem 4.1 raises several questions related to uniqueness. In a multi-variable problem such as this, local maxima and/or non-unique maxima are detrimental to computational performance. The strength of Theorem 4.1, however, is that it reduces the dimension of the search space to one, and maxima for functions of one variable can always be handled by an exhaustive search.

There exist cases leading to non-unique equilibria. Such cases occur when two or more choices lead to the same expected profit for either one of the agents.In our model, such cases do not cause degeneracy because in the backward induction process, any of these multiple choices will lead to the same profit in the current period and when the backward induction proceeds to the previous period, the state space will not be affected by non-uniqueness of the equilibrium state in the future. This is due to the fact that the memory functions are functions of previous prices, not the current or future ones. This can be observed in (31′) where having multiple cases of R_3 as argmaxes of J_3^r will not change the equlibium state in the previous previod, as stated in (32).

4.5 The infinite-period case

For given values of $\overline{\Pi}_k^r$ and $\overline{\Pi}_k^m$, the parties try to maximize:

$$J_r^{(k)}[R_k] = (R_k - W_k)\mu_k[R_k] + L_{\varepsilon_k}[R_k, W_k]\sigma_k[R_k] + \alpha \cdot \overline{\Pi}_k^r \cdot \overline{g}_{k+1}[R_k] \tag{42}$$

$$\begin{aligned} J_m^{(k)}[W_k] = (W_k - M_k)\Big(\mu_k[R_k] + \sigma_k[R_k] \cdot F_{\varepsilon_k}^{-1}\Big[\frac{R_k - W_k}{R_k - S}\Big]\Big) \\ + \alpha \cdot \overline{\Pi}_k^m \cdot \overline{g}_{k+1}[R_k]. \end{aligned} \tag{43}$$

The first-order conditions for this problem yield two equations for the two unknowns R_k and W_k. In the multi-period case, we start by using $\overline{\Pi}_n^r = 0$ and $\overline{\Pi}_n^m = 0$ and iterate backwards until we reach the starting period. However, if the horizon is infinite, this approach fails because an infinite number of iterations is needed to reach the start.

If $\mu[R]$, $\sigma[R]$, $g[R,\varepsilon]$, and ε do not depend on k, or

$$\lim_{k\to\infty}\big(\mu[R,k],\sigma[R,k],g[R,k,\varepsilon_k],\varepsilon_k\big) = \big(\mu[R],\sigma[R],g[R,\varepsilon],\varepsilon\big),$$

i.e., the same functions are used for any k, then cases with an infinite horizon can be solved. To do so, one needs a steady state for the system; i.e., we must find $\overline{\Pi}^r$ and $\overline{\Pi}_m^r$ such that:

$$\overline{\Pi}^r = (R-W)\mu[R] + L_\varepsilon[R,W] + \alpha\cdot\overline{\Pi}^r .\overline{g}[R] \tag{44}$$

$$\overline{\Pi}^m = (W-M)\Big(\mu[R] + \sigma[R]\cdot F_\varepsilon^{-1}\Big[\frac{R-W}{R-S}\Big]\Big) + \alpha\cdot\overline{\Pi}^m\cdot\overline{g}[R]. \tag{45}$$

The first-order conditions from (42)-(43), together with (44)-(45), yield four equations in the four unknowns, $R, W, \overline{\Pi}^r$, and $\overline{\Pi}^m$. If this system has a unique solution, we have a unique candidate for the infinite-horizon case.

5 Practical implementation of the model

In this section, we illustrate the theory in section 4 by explicit examples. In these examples, we use a Cobb-Douglas demand function with a normally distributed random term. The problem is as easily solved when using other functional forms. The problem (given W) is reduced to finding maxima for a function of one variable, which is straightforward for almost any $\mu_k[R], \sigma_k[R], \varepsilon_k$, and $g_k[R,\varepsilon_k]$.

We start with the one-period case, and gradually increase the number of periods, n, until we reach the infinite horizon case.

We remark that the purpose of this section is to offer practical advice on how our theory can be implemented in some special cases. To take full advantage of the model, one should try to vary scaling factors and functional forms in a systematic way. This makes it possible to model a wide range of economic contexts. A full discussion of the model and all the variations it has to offer, is, however, beyond the scope of this paper.

5.1 The one-period case

We consider the initial demand function:

$$\widetilde{D_1}[R,k] = \frac{1000\,e^{-0.1k}}{R^2} + 10\,e^{-0.01k}\cdot\varepsilon \tag{46}$$

where ε is $\mathcal{N}(0,1)$. Because a normally distributed variable can take negative values, we must impose restrictions to exclude artificial cases. If q, as given by (5), is negative, we set $q=0$. Moreover, if the expected profit in (6) is negative, we also assume $q=0$. Note that in (46), we have considered the demand to monotonously decrease with time. This feature can be the result of competition or other exogenous factors. In section 5.3.3, we will modify the expression for the demand to analyze the cases in which demand does not necessarily decrease with time. We choose

$$M = 2 \quad S = 1 \tag{47}$$

By using the formula in (5) and (6) we can compute the manufacturer's profit as a function of W. This function is illustrated on the left side of Figure 1.

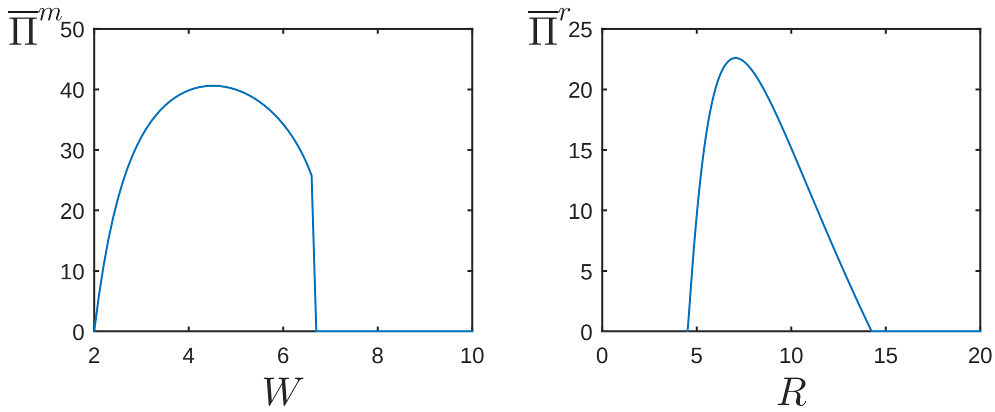

Figure 1: Expected profit for the manufacturer (left) and the retailer (right)

The manufacturer obtains maximum profit at the unique value $W = 4.51$. Given $W = 4.51$, the retailer's profit in 6 is a function of R only. This function is shown on the right side of Figure 1. The retailer's best response in to choose $R = 7.05$, which, according to 5, leads to an order quantity $q = 16.18$. To summarize, our particular Stackelberg game has a unique equilibrium at:

$$(W, \overline{\Pi}^m) = (4.51, 40.60) \tag{48}$$

$$(R, q, \overline{\Pi}^r) = (7.05, 16.18, 22.59) \tag{49}$$

These values will be important as they will be used as input for the two-period case analyzed in section 5.2.

5.2 The two-period case

In this section, we extend the discussion in section 5.1 to a two-period Stackelberg game. We assume that:

$$D_1 = \mu_1[R_1] + 10 \cdot \varepsilon_1 \tag{50}$$

where $\mu_1[R] = 1000 \cdot R^{-2} \cdot e^{-0.1}$ and ε_1 is $\mathcal{N}(0, 1)$; i.e., we use the same demand function used in section 5.1. We further assume that $M = 2$ and $S = 1$ (as before). Now, let:

$$D_2 = g_2[R_1, \varepsilon_1]\big(\mu_2[R_2] + 10 \cdot \varepsilon_2\big) \tag{51}$$

where $g_2[R_1, \varepsilon_1$ is a scaling factor. Regardless of the choice of $g_2[R_1, \varepsilon_1]$, it follows from (25) and the results in Section 4 that the second stage of the game will have a unique equilibrium at:

$$(W_2, \overline{\Pi}_2^m) = (4.51, 40.60 \cdot E[g[R_1, \varepsilon_1]]) \tag{52}$$

$$(R_2, q_2, \overline{\Pi}_2^r) = (7.05, 16.18, 22.59 \cdot E[g[R_1, \varepsilon_1]]) \tag{53}$$

Hence, the arguments W_2, R_2, and q_2 are independent of the scaling factor. However, the maximal values depend on the scaling factor, and both parties must take this into account when considering their first-period strategies.

5.2.1 Choosing appropriate scaling factors

We now consider the scaling factors:

$$\overline{g}[R_1] = E[g[R_1, \varepsilon_1]] = 1 + \gamma(K - R_1) \tag{54}$$

where $\gamma \geq 0$ and $K \geq 0$ are given constants. The constant k can be interpreted as a price cap; i.e., any initial price above K reduces demand, whereas demand is more likely to increase if $R_1 < K$. If the scaling factor is negative, maxima are turned into minima. Hence, if $E[g[R_1, \varepsilon_1]] \leq 0$, the optimal order q_2 is zero. To avoid this problem, we consider cases where:

$$\overline{g}[R_1] = max[1 + \gamma(K - R_1), 0] \tag{55}$$

Typically, $M \leq W_1 \leq R_1$ is expected. Ruling out short selling implies that $W_1 \geq 0$ and $R_1 \geq 0$. If $R_1 < W_1$, the optimal order quantity is q_1 zero. However, $R_1 < W_1$ might represent an optimal strategy. If $R_1 < W_1$, the retailer orders nothing in the first period. Then he might just as well choose $R_1 = 0$ because this is the most efficient way to increase demand in period 2. A strategy of this type makes good sense economically; it corresponds to a situation in which a small number of items ($q \approx 0$) are given away for free ($R_1 = 0$) in the first period to increase interest for the product in the second period. In our optimization problem, given W_1, the retailer should find the maximum over all R_1 with $R_1 \geq W_1$. The retailer should then compare this maximum value with the value he could obtain by using the alternative $R_1 = 0$, $q_1 = 0$, and then choose the best alternative.

Values for the parameters γ and K are specified below. We investigate how different values of these parameters affect the solutions. Given the choices described above, the strategies and expected profits in the second period are given by (52) and (53). Hence, the retailer's total expected profit, given W_1, is:

$$\begin{aligned} J_r[R_1] = {} & (R_1 - W_1)\mu_1[R_1] + L_{\varepsilon_1}[R_1, W_1]\sigma_1[R_1] \\ & + \alpha \cdot 22.59 \cdot max[(1 + \gamma(K - R_1), 0]. \end{aligned} \tag{56}$$

The manufacturer's total expected profit is:

$$\begin{aligned} J_m[W_1] = {} & (W_1 - M_1)\Big(\mu_1[R_1] + \sigma_1[R_1] \cdot F_{\varepsilon_1}^{-1}\Big[\frac{R_1 - W_1}{R_1 - S}\Big]\Big) \\ & + \alpha \cdot 40.60 \cdot \max[(1 + \gamma(K - R_1), 0]. \end{aligned} \tag{57}$$

The manufacturer knows that, given W_1, the retailer will choose R_1 to maximize $J_r[R_1]$. Given $R_1 = Argmax[J_r[R_1]]$ in (57), $J_m[W_1]$ is a function of W_1 only.

5.2.2 Numerical results for some particular cases

We will now show how specific choices for the parameters α, γ, and K affect the equilibria from (56) and (57).

• Case 1: $\alpha = 1$ (no discounting), $\gamma = 0$ (no dependence on R_1).
In this particular case, there is no memory and the system is effectively decoupled into two identical one-period problems. Both problems coincide with the problem we solved in section 5.1, and the equilibrium states are:

$$W_1 = W_2 = 4.51 \qquad J_m = 81.20 \tag{58}$$
$$R_1 = R_2 = 7.05 \qquad q_1 = q_2 = 16.18 \qquad J_r = 45.18 \tag{59}$$

• Case 2: $\alpha = 1$ (no discounting), $\gamma = 0.1$, $K = 5$.
Now the pricing effect of R_1 is active. The system has a unique equilibrium state at:

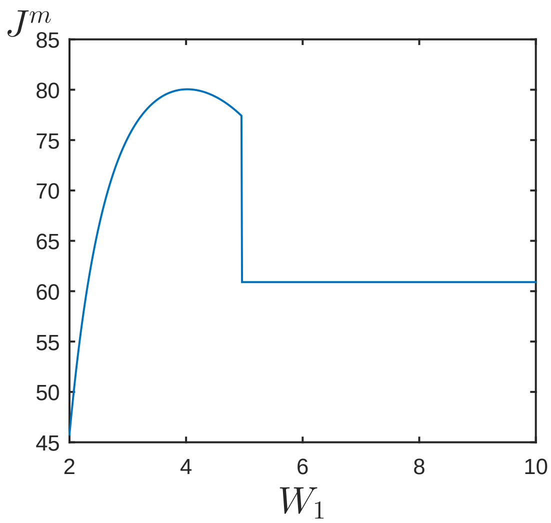

Figure 2: Case 2, Expected total profit for the manufacturer as a function of W_1

$$W_1 = W_2 = 4.51 \qquad J_m = 81.20 \tag{60}$$
$$R_1 = R_2 = 7.05 \qquad q_1 = q_2 = 16.18 \qquad J_r = 45.18 \tag{61}$$

The graph of the function $J_m[W_1]$ is shown in Figure 2. The section to the right is flat because when W_1 is too high, the retailer's best choice is to order $q_1 = 0$, in which case he chooses $R_1 = 0$ to get the benefit of higher demand in period 2. The graph reveals that this does not correspond to the manufacturer's best choice.

In equilibrium, the discounted scaling factor (the blow-up factor), $\eta = \alpha(1+\gamma(K-R_1)) < 1$. Hence, demand is lower than in Case 1. Although profits are higher than in Case 1, the profit margin is lower, but this is offset by a higher order quantity. The manufacturer has less control and makes a smaller expected profit.

• Case 3: $\alpha = 1$ (no discounting), $\gamma = 0.25,\ K = 5$.
The system has equilibrium states at:

$$W_1 \geq 3.93, W_2 = 4.51 \qquad J_m = 91.35 \tag{62}$$

$$R_1 = 0, R_2 = 7.05 \qquad q_1 = 0, q_2 = 16.18 \qquad J_r = 50.83 \tag{63}$$

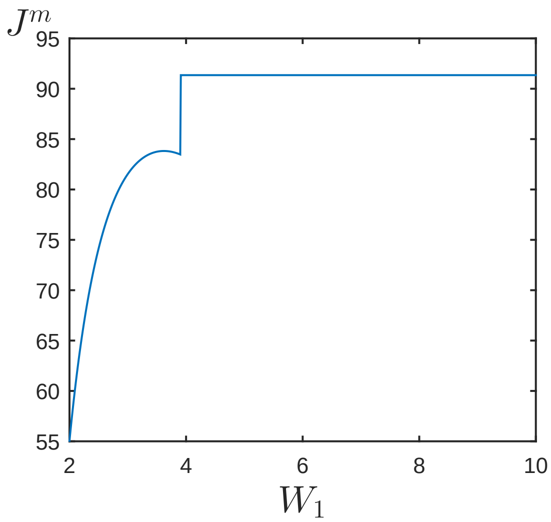

Figure 3: Case 3, Expected total profit for the manufacturer as a function of W_1

The graph of the function $J_m[W_1]$ is shown in Figure 3. The section to the right is flat because, when $W_1 \geq 3.93$, the retailer's best choice is to order $q = 0$, in which case he chooses $R_1 = 0$ to get the benefit of higher demand in period 2. Unlike in case 2, it is in the manufacturer's best interest to provoke a strategy of this sort because it maximizes expected profit.

• Case 4: $\alpha = 0.8,\ \gamma = 0.25,\ K = 5$.
The system has a unique equilibrium at:

$$W_1 \geq 3.73, W_2 = 4.51 \qquad J_m = 73.96 \tag{64}$$

$$R_1 = 5.57, R_2 = 7.05 \qquad q_1 = 26.67, q_2 = 16.18 \qquad J_r = 51.47 \tag{65}$$

The graph of the function $J_m[W_1]$ is shown in Figure 4. Because of the discounting, it no longer pays to have zero sales in the first period; profits in the second period are less valuable because of discounting.

5.3 The multi-period case

As explained in the theoretical section, once we have an algorithm that solves the two-period case, the same algorithm can be used repeatedly to solve $n-$period problems. We merely

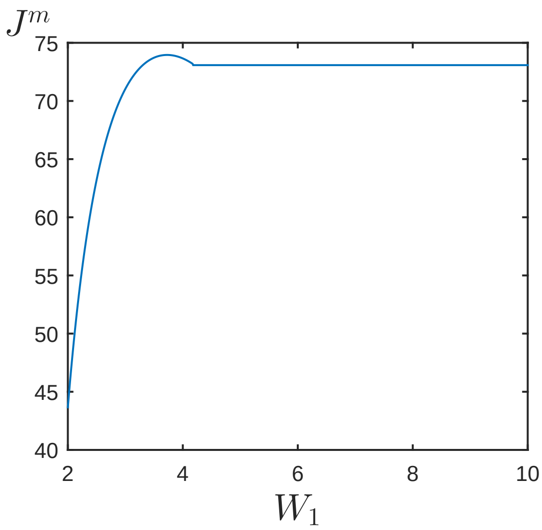

Figure 4: Case 4, Expected total profit for the manufacturer as a function of W_1

have to update the remaining profits as the construction progresses backward. Starting with a given demand distribution D_1 in period 1, we consider the case in which demand in period k is given by:

$$D_k = \widetilde{D_1}[R_k] \cdot \prod_{i=2}^{k} g_i[R_{i-1}, \varepsilon_{i-1}] \tag{66}$$

where the tilde signifies that, at each step, an independent draw is made from the original distribution D_1. When setting D_k, the values of R_1, R_2, ..., R_{k-1}, and D_1, D_2,...,D_{k-1} are all known, in which case the values ε_1, ε_2, ..., ε_{k-1} are also known. In principle, the scaling factor g_k can change with k. For illustration purposes, we only consider cases in which the expected scaling factors satisfy the following:

$$\overline{g}_k[R_k] = max[1 + \gamma(K - R_k), 0] \tag{67}$$

where $\gamma > 0$ and $K > 0$ are given constants. As mentioned above, more complicated expressions can be computed without problems.

• Case 1: $\alpha = 1$ (no discounting), $\gamma = 0.01$, $K = 7$, $n = 25$, $M = 2$, $S = 1$, $\mu_k[R] = \frac{1000}{R^{(2-\beta\frac{n-k}{n})}}$, $\beta = 0.8$ and $\sigma_k[R] = \frac{1}{R}\mu_k$, so that the coefficient of variation decreases as R increases.
The decision variables in each period (R_k, W_k, q_k) as well as the values for $D_k, \mu_k, \overline{\Pi}_k^r$, and $\overline{\Pi}_k^m$ are shown in Figure 5. Where $\overline{\Pi}_k^r$ and $\overline{\Pi}_k^m$ represent the expected present value of the discounted profit obtained at each period by the retailer and the manufacturer, respectively.[2]

The optimal strategy in this case is to increase demand by letting $R_1 = R_2 = R_3 = 0$, then start selling in period 4. In this case, $max[\eta] = \alpha \cdot max[g] = \alpha(1 + \gamma K) = 1.07$. To obtain increased profits from an initial strategy in which $R_1 = 0$, it is clearly necessary that $max[\eta] = \alpha \cdot max[g] > 1$. This requirement, however, is not sufficient as we will see in case 2. It is observable that, in the sales periods, $k \geq 4$, the profit margin $R_k - W_k$ remains fairly

[2] So J^r and J^m are the sums of $\overline{\Pi}_k^r$s and $\overline{\Pi}_k^m$s, respectively.

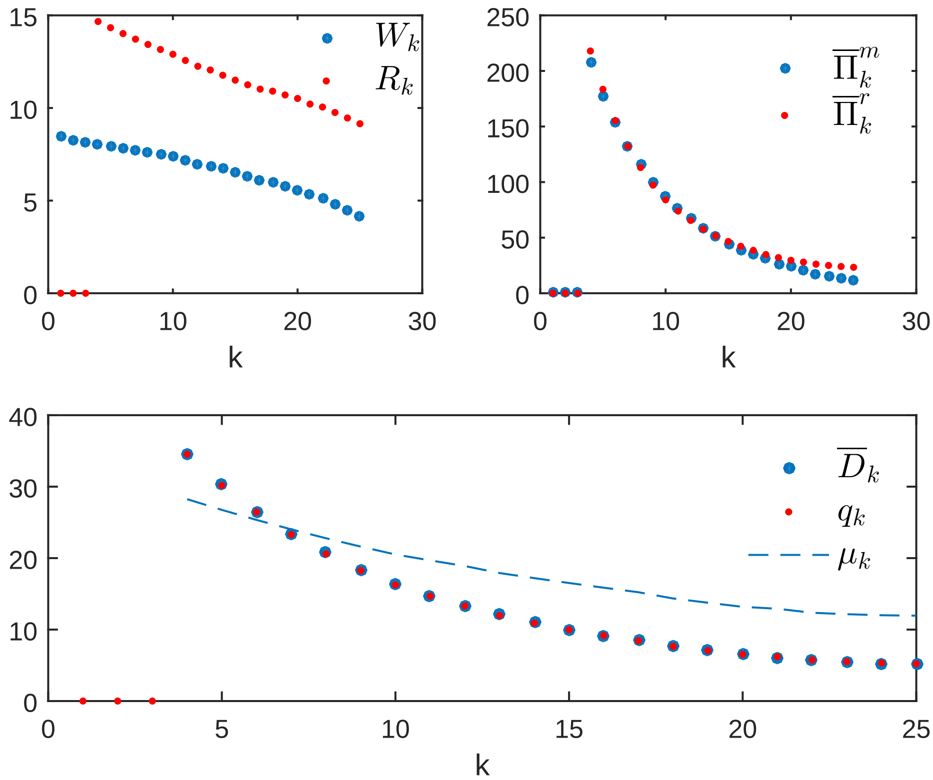

Figure 5: Stackelberg equilibrium state in period k

constant with time. Figure 5 also illustrates the expected values of demand at each period corresponding to the retail prices R_k. As shown in Figure 5, the expected values of demand, $\overline{D}_k$ and the optimal ordering quantity q_k are very close to each other, which means that in a situation where the stochasticity is decreasing as R increases, the retailer remains close to risk-aversion and adheres to the deterministic optimal solution. The expected value of demand in each period is calculated as below.

$$\begin{aligned} D_k &= (\mu_k[R_k] + \sigma_k[R_k]\varepsilon_k) \cdot \prod_{i=2}^{k} g_i[R_{i-1}, \varepsilon_{i-1}] \\ \overline{D}_k &= E[D_k] = \mu_k[R_k] \cdot E\left[\prod_{i=2}^{k} g_i[R_{i-1}, \varepsilon_{i-1}]\right] = \mu_k[R_k] \cdot \prod_{i=2}^{k} g_i[R_{i-1}, \varepsilon_{i-1}], \end{aligned} \tag{68}$$

A comparison between the values of $\overline{D}_k = \frac{\mu_k[Rk]}{\prod_{i=2}^{k} g_i[R_{i-1}, \varepsilon_{i-1}]}$, and μ_k, illustrated in Figure 5, reveals that the memory functions tend to keep the retailer from taking higher risks necessary for over-ordering.

• Case 2: $\alpha = 0.95$, $\gamma = 0.01$, $K = 7$, $n = 25$, $M = 2, S = 1$, with $\mu_k[R]$ and $\sigma_k[R]$ remaining the same as that of case 1.
In this case, $max[\eta] = \alpha \cdot max[g] = \alpha(1 + \gamma K) \approx 1.02$. Nonetheless, this blow-up factor

is not big enough to justify an initial retail price $R_1 = 0$. Therefore, sales take place in all periods. Figures 6 shows the equilibrium prices and the expected value of the demand for this case.

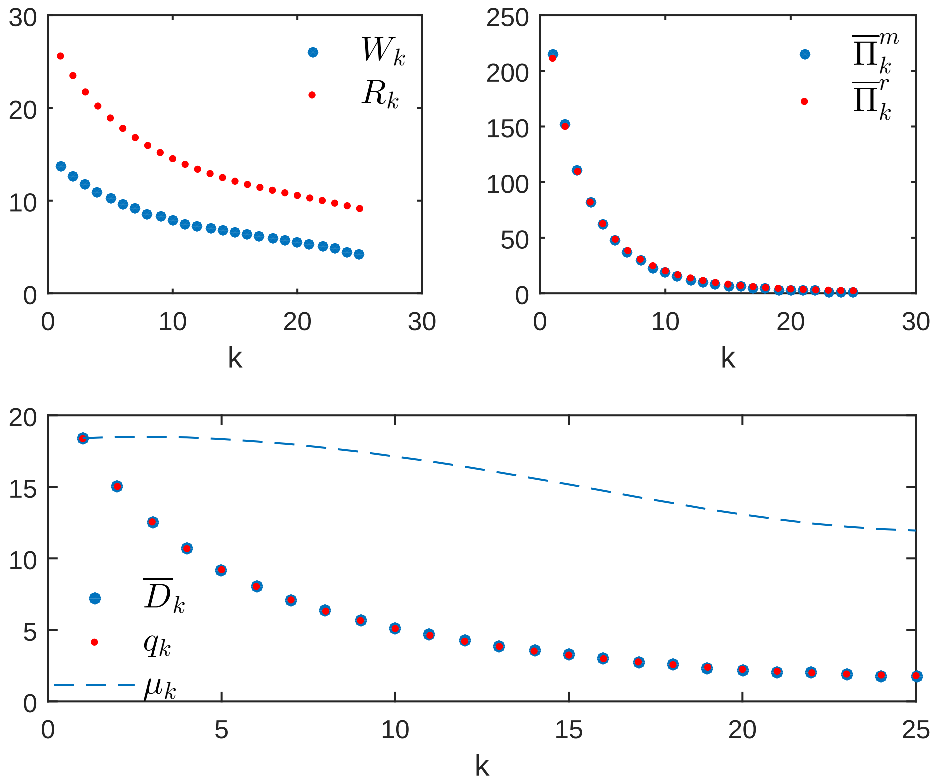

Figure 6: Stackelberg equilibrium state in period k

5.3.1 Constant coefficient of variation

As discussed in section 4.3.2, randomness in demand is usually such that it results in a constant coefficient of variation. In this section, we analyze such cases. For illustration purposes, we consider cases with relatively high coefficient of variation so that the effect of stochasticity on demand and optimal order quantity are easily observable.

• Case 1: $\alpha = 1$ (without discounting), $\gamma = 0.01$, $K = 7$, $n = 25$, $M = 2, S = 1$, with $\mu_k[R] = \frac{1000}{R^{(2-\beta\frac{n-k}{n})}}$, $\beta = 0.8$, and $\sigma_k[R] = 0.5 \cdot \mu_k[R]$.

• Case 2: $\alpha = 0.95$, $\gamma = 0.01$, $K = 7$, $n = 25$, $M = 2, S = 1$, with $\mu_k[R] = \frac{1000}{R^{(2-\beta\frac{n-k}{n})}}$, $\beta = 0.8$, and $\sigma_k[R] = 0.5 \cdot \mu_k[R]$.

Figures 7 and 8 illustrate the Stackelberg equilibrium results for cases 1 and 2, respectively. It can be seen in both figures that with the advent of higher volatility into the demand, the retailer becomes able to take a higher risk and over-order with respect to the expected value of demand ($q_k > \overline{D}_k, \forall k$). However, the effect of multiplicative memory functions on order quantity makes $q_k < \mu_k$ as stated in (68). Such an effect prevents the retailer from taking higher risks when over-ordering.

Figure 7 shows that without discounting (i.e. with a more promising future), the retailer can start with $R_1 = R_2 = R_3 = 0$. Such an optimal policy boosts the demand starting from $k = 4$. (Compare the values of $\overline{D}_{k>4}$ in Figures 7 and 8.)

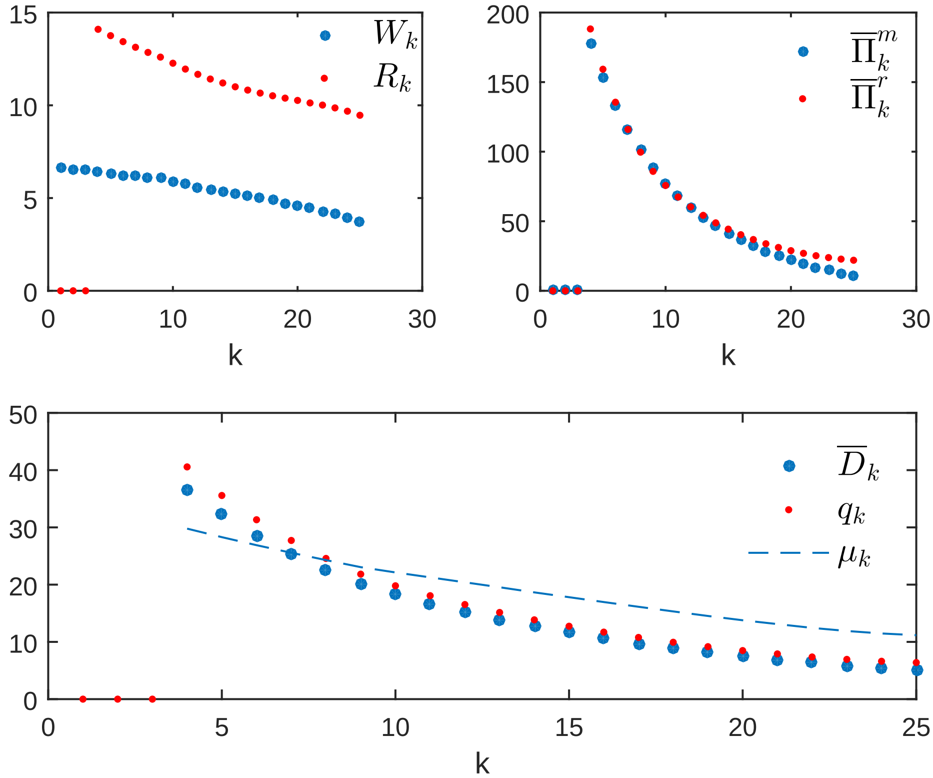

Figure 7: Stackelberg equilibrium state at k, constant coefficient of variation, $\alpha = 1$

5.3.2 Constrained optimization

So far, we have implemented Theorem 4.1 without considering optimization constraints. However, it is conceivable that market-penetration strategies as well as factors of exogenous nature impose constraints on the optimal policy.[3] The flexibility of Theorem 4.1 allows us to employ it in tandem with constraints.

Figure 9 shows that with a constraint imposed only on the maximum amount of R, the manufacturer is forced to reduce the wholesale price (compared to the unconstrained case) in order to keep the profit margin high enough to convince the retailer not to reject the proposed W by choosing $q = 0$.

Similarly, imposing a constraint on the maximum amount of W, will also lead to a reduction in retail prices.

[3]For instance, government regulations may impose such restrictions on final price.

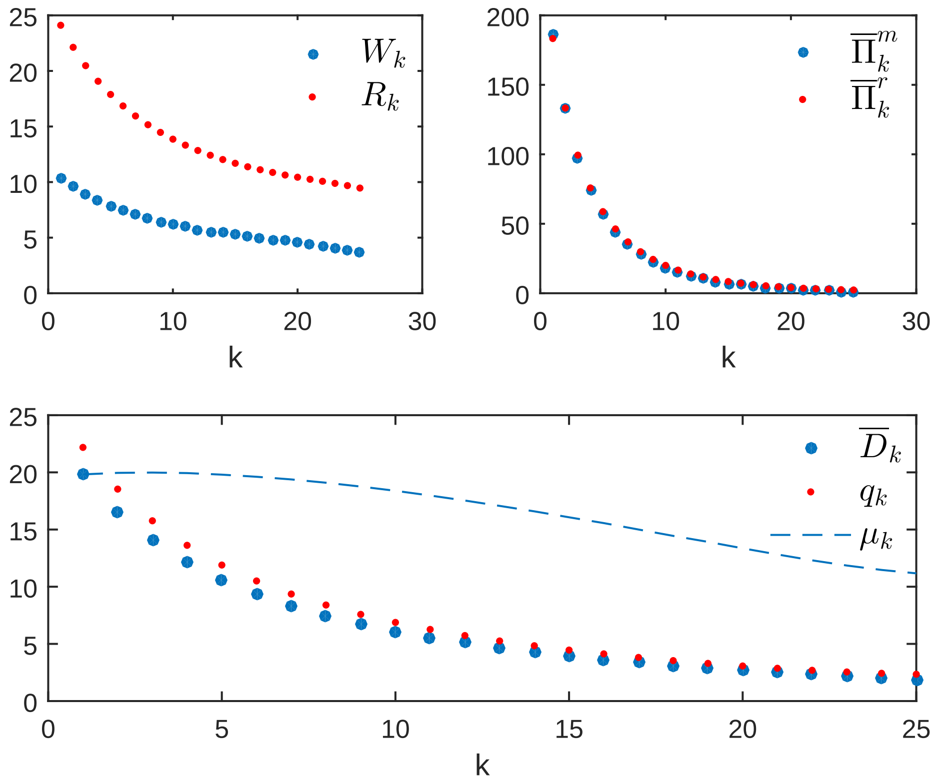

Figure 8: Stackelberg equilibrium state at k, constant coefficient of variation, $\alpha = 0.95$

5.3.3 Boosting the demand

As discussed earlier, we considered $\frac{\partial \overline{D}}{\partial k} < 0$, in general. However, there may be opportunities for the wholesaler and/or the manufacturer to boost the demand, sometimes in the process. This, for example, can be done through rolling out a new model of the product or implementing publicity campaigns.

To illustrate such situations, we consider a case in which the demand, as represented by μ_k, is amplified twice at $k = 8$ and $k = 16$ when $n = 25$.

$$\begin{aligned} \mu_k[R] &= \frac{1000}{R^{(2-\beta\frac{7-k}{25})}}, k < 8 \\ \mu_k[R] &= \frac{1000}{R^{(2-\beta\frac{15-k}{25})}}, 8 \leq k < 16 \\ \mu_k[R] &= \frac{1000}{R^{(2-\beta\frac{25-k}{25})}}, 16 \leq k \leq 25 \end{aligned} \tag{69}$$

$$\alpha = 1, \gamma = 0.01, K = 7, n = 25, M = 2, S = 1, \beta = 0.8, \text{and}\, \sigma_k[R] = 0.5 \cdot \mu_k[R]$$

Figure 10 shows that with such incremental increases in demand, the retailer can start with $R_1 = R_2 = ... = R_7 = 0$ and continue with jumps in retail prices after any time the demand is boosted. Such jumps in R will lead to higher profit margins. Compare the results shown in

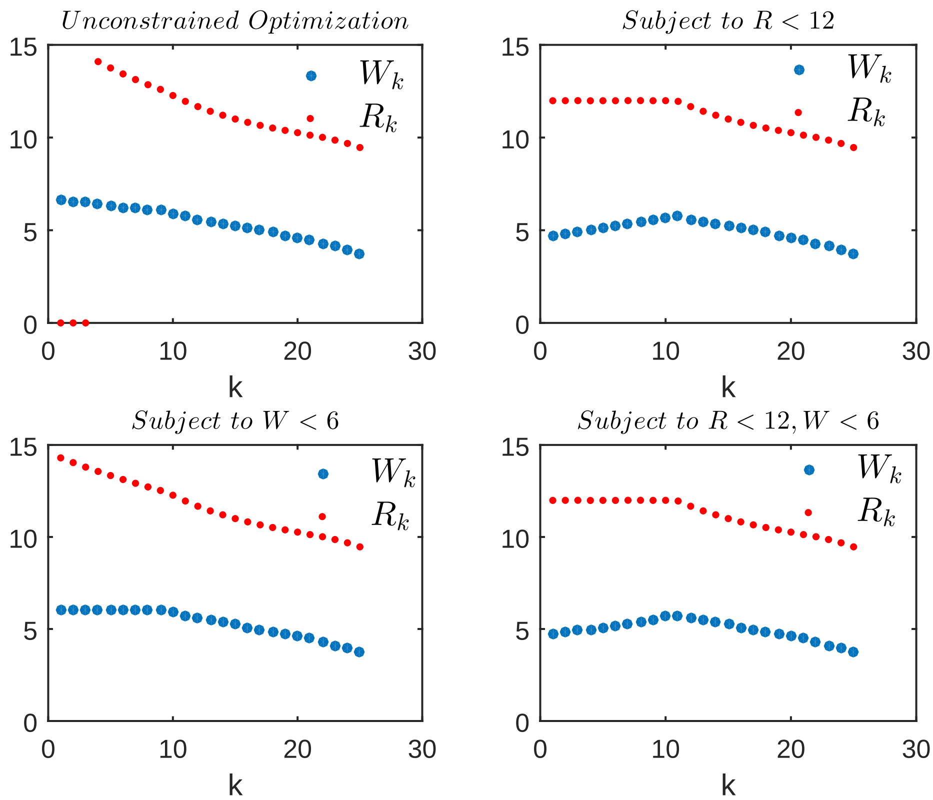

Figure 9: Stackelberg equilibrium states at k, constrained optimization

Figure 10 with the result of the similar case in which demand was not amplified, as depicted in Figure 7.

5.4 The infinite horizon case

In section 4.5, we saw that in order for the equilibrium state to exist in an infinite horizon, it is necessary that μ and σ be independent of time. Then, if we iterate (42) and (43), backwards, equilibrium prices will progressively stabilize. This is to be expected as we approach an infinite-horizon problem with a well-defined equilibrium state. If $\alpha(1 + \gamma K) > 1$, it is clear that an arbitrarily large total profit can be obtained, and that there is no nondegenerate equilibrium strategy. Hence, we need $\alpha(1 + \gamma K) \leq 1$, in which case we can try to solve a fixed-point problem by using the first order conditions from (42)-(43) together with (44)-(45).

By using $\mu[R] = \frac{1000}{R^2}$, $\sigma[R] = 0.5 \cdot \mu[R]$, and the values $\alpha = 0.9, \gamma = 0.01, K = 7, M = 2$, and $S = 1$, this fixed-point problem is straightforward to solve, and we find:

$$R = 7.61, q = 18.59, J^r = 391.38 \tag{70}$$

$$W = 3.90, J^m = 330.85 \tag{71}$$

The equilibrium state is shown in Figure 11.

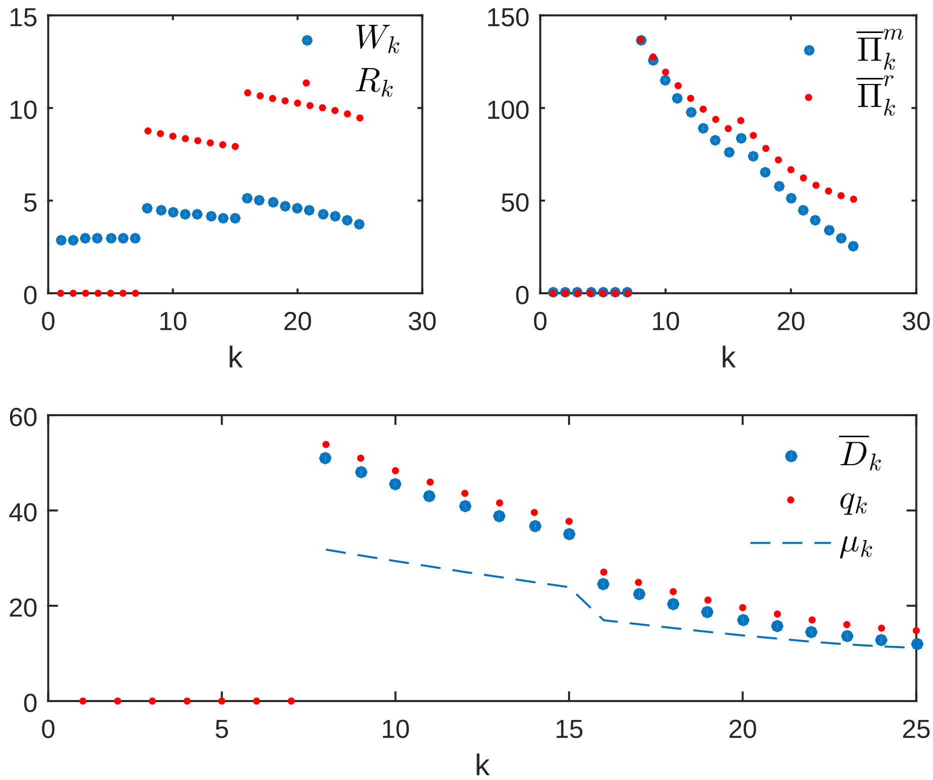

Figure 10: Stackelberg equilibrium states at k, incremental demand boosts

5.5 Cooperative behavior of the agents

So far, we have analyzed the equilibria in a Stackelberg framework. However, it is possible for the two parties to cooperate. Such a deviation from the Stackelberg game is implemented by considering the two agents as a single decision-maker, substitution of $W_k = M_k$ in Theorem 4.1, and then optimizing only J^r with respect to R.

Here, we consider $\alpha = 0.95$, $\gamma = 0.01$, $K = 7$, $n = 25$, $M = 2, S = 1$, $\mu_k[R] = \frac{1000}{R^{(2-\beta\frac{n-k}{n})}}$, $\beta = 0.8$, and $\sigma_k[R] = 0.5 \cdot \mu_k[R]$.

Comparing the results of this cooperative game as shown in Figure 12 with those of the corresponding Stackelberg game depcited in Figure 8, we see that the single agent in the cooperative game, can take a higher risk when over-ordering, charge a lower price (which in turn increases the demand relatively), and ensure a profit higher than the sum of the profit obtained by the two agents in the Stackelberg game.

6 Concluding remarks

In this paper, we considered multi-period newsvendor problems with price-dependent demand. In particular, we studied the case in which demand in one period is a function of prices and demand in the previous period. A problem of this type leads to time dependent

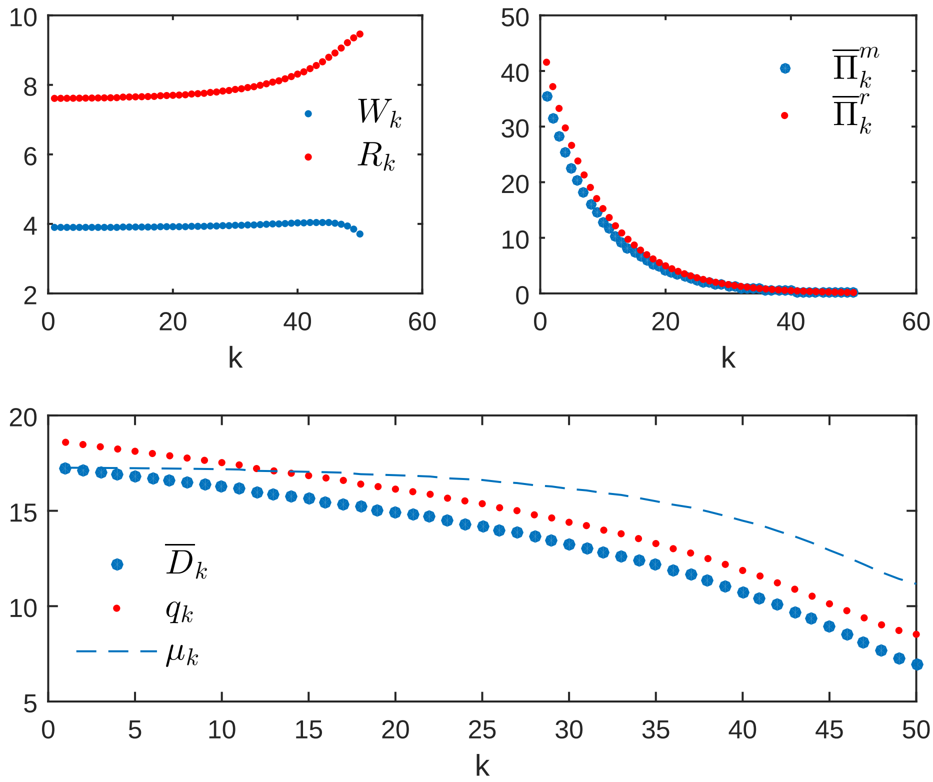

Figure 11: The equilibrium state as $n \to \infty$

pricing strategies. Increasing prices in one period can lead to short-term improvements, but as a consequence of the coupling, long-term demand can be reduced, as, thereby, can overall profits. The parties must then find an optimal balance between current profits and discounted future profits.

We showed how to obtain a complete solution to such problems when demand in one period is scaled by a multiplicative memory function that depends on prices in the previous period. The multi-period problem can then be separated into a sequence of one-period problems, and the solution can be found by backward induction starting from the final period. The problems are linked in that each stage needs the total profit from the previous stage as an input, but otherwise there is no coupling between the stages. When the scaling function is fixed, it is possible to consider infinite-horizon problems of this type, and Stackelberg equilibria can then be found solving an explicit fixed-point problem.

To demonstrate that such problems can be modeled and solved by the procedure outlined in Theorem 4.1, we provided numerical solutions to a variety of special cases. Note, however, that our framework is not limited to such special cases. The numerical illustrations raise questions of interest for future research.

In the numerical section, we have demonstrated an interesting link to marketing. Under certain conditions, an optimal strategy is to give away products in a pre-sales period. This stimulates demand, and the parties benefit from increased demand in the remaining time

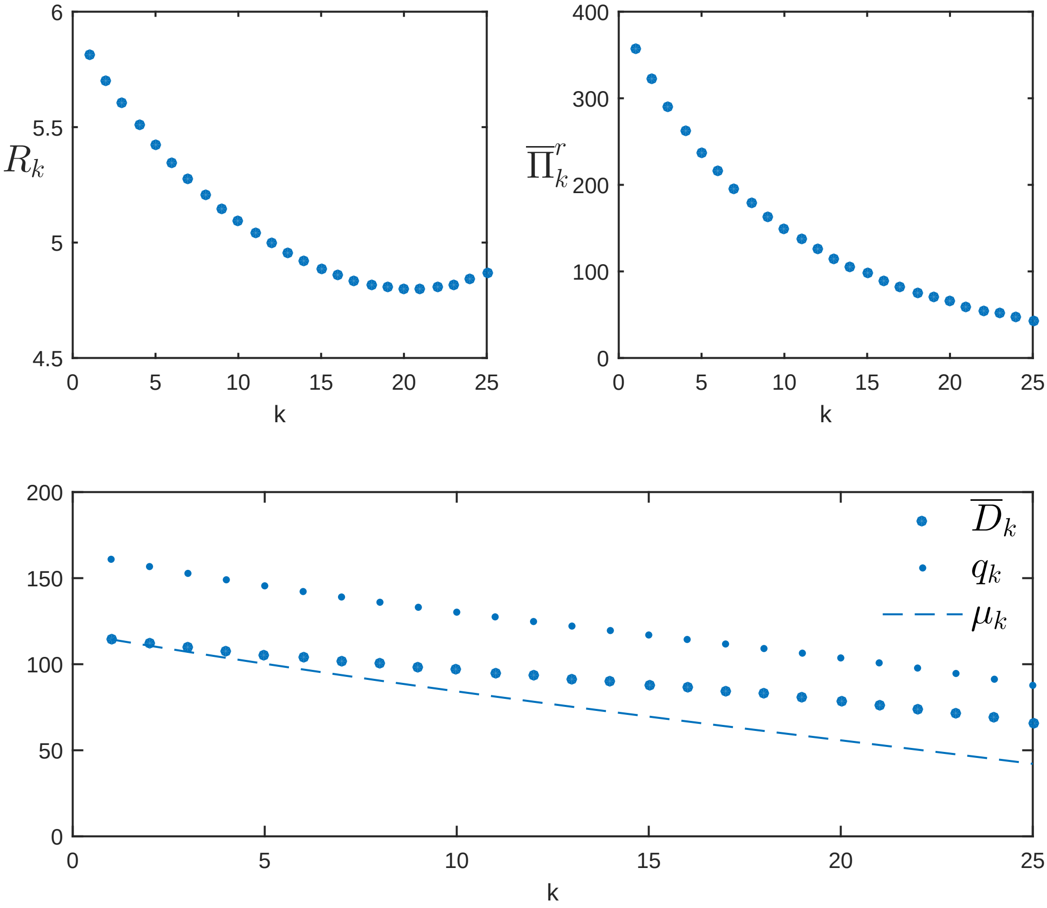

Figure 12: Expected demand in period k

periods. Many high-tech products like mobile phones and computers have a very short lifespan. Our paper hence offers a new framework where sales strategies for such products can be discussed and analyzed.

To take full advantage of the model, one should try to vary scaling factors and functional forms in a systematic way. Exploring the potential of our modeling approach is a topic for future research.

An interesting next step would be to extend our construction to include inventory. We have obtained some partial results using multiplicative scaling in cases with backlog. In summary multiplicative scaling can be used to reduce the complexity of Stackelberg games also in cases with inventory. The details are, however, quite technical and outside the scope of the present paper. Further discussion of this problem is hence left for future work.

7 Appendix: Proof of Proposition 3.1

Let F_ε denote the cumulative distribution of ε. Since ε is continuous, supported on an interval, with density $f_\varepsilon > 0$ a.e. on its support, the expected profit $\overline{\Pi}^r$ is strictly concave in q on the support of D, and the order quantity q from (4) is unique. It is clear that

$$q = \mu[R, k] + \sigma[R, k] \cdot F_\varepsilon^{-1}\left[\frac{R-W}{R-S}\right]. \tag{72}$$

By using 3 and 4, we obtain:

$$\begin{aligned} E[\Pi^r] &= (R-S)E[min[D, q]] - (W-S)q \\ &= (R-S)\left(\mu[R,k] + E\left[min\left[\sigma[R,k]\varepsilon, \sigma[R,k]F_\varepsilon^{-1}\left[\frac{R-W}{R-S}\right]\right]\right]\right) \\ &- (W-S)\left(\mu[R,k] + \sigma[R,k]F_\varepsilon^{-1}\left[\frac{R-W}{R-S}\right]\right). \end{aligned} \tag{73}$$

Equations (3) and(72) indicate that

$$E\left[min\left[\varepsilon, F_\varepsilon^{-1}\left[\frac{R-W}{R-S}\right]\right]\right] \tag{74}$$

$$= \int_{-\infty}^{F_\varepsilon^{-1}[\frac{R-W}{R-S}]} x f_{\varepsilon[x]dx} + F_\varepsilon^{-1}\left[\frac{R-W}{R-S}\right] \cdot P\left(\varepsilon \geq F_\varepsilon^{-1}\left[\frac{R-W}{R-S}\right]\right) \tag{75}$$

$$= \int_{-\infty}^{F_\varepsilon^{-1}[\frac{R-W}{R-S}]} x f_{\varepsilon[x]dx} + F_\varepsilon^{-1}\left[\frac{R-W}{R-S}\right] \cdot \left(1 - \frac{R-W}{R-S}\right). \tag{76}$$

Inserting the expression in (76) into (73) and simplifying the resulting expression yields:

$$\overline{\Pi}^r = E[\Pi^r] = (R-W)\mu[R,k] + L_\varepsilon[R,W]\sigma[R,k] \tag{77}$$

where L_ε is defined as:

$$L_\varepsilon[R, W] = (R-S)\int_{-\infty}^{z} x f_\varepsilon[x]dx \quad z = F_\varepsilon^{-1}\left[\frac{R-W}{R-S}\right]. \tag{78}$$

$\square$

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