

NHH



The value of expanding commercial rental space

A real options framework

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Abstract

In this thesis, we develop a real options framework to value options to expand a commercial property. We consider the investment in a development project under uncertainty in future rental rates and in the presence of regulatory uncertainty. We have consulted with industry experts to model the framework as realistic as possible without losing analytical tractability. Our framework contributes to the accurate valuation of real estate as well as the real options literature related to real estate, by adjusting and combining existing frameworks and applying them to the real estate industry. First, we consider a simple investment in the expansion of a property under uncertainty in future rental rates, but in the absence of regulatory uncertainty. We proceed to introduce the presence of a regulatory process in the form of a mandatory fixed approval lag, prior to the firm receiving the investment opportunity. Finally, we introduce uncertainty in the length of the regulatory process. For each model, we are able to obtain optimal investment thresholds and option values, and in most cases these results are closed form analytical solutions. We find that an option to expand an existing property has significant value under the right conditions, even when the immediate expansion is not profitable. We also show that the presence of a regulatory process alters the option value, and that a stochastic regulatory process implies a higher cost to the firm than a fixed regulatory process.

Keywords: Real options analysis, commercial real estate, geometric Brownian motion, regulatory uncertainty

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1. Introduction

Over the last years, commercial real estate transactions in Norway have steadily increased in both size and scope, and 2014 became a record year in terms of transaction volume for the Norwegian commercial real estate market (Fearnley, 2015). While low interest rates and volatile stock markets have dominated the global economy, commercial real estate has become an attractive investment, both as a safe haven for large institutional investors, and as a source of good returns for opportunistic investors. With increased competition, a thorough understanding of the commercial real estate industry and accurate valuations of real estate assets become increasingly important for investors. Current valuation techniques used in the commercial real estate industry have proved to be insufficient for several investment opportunities, particularly for development projects, and recent studies as well as industry experts indicate that more sophisticated valuation tools could improve the current situation (Leishman et al., 2000). In this thesis we introduce a framework that combines several strings of real options literature, and which can be applied for a more accurate and sophisticated valuation of commercial real estate. We begin by examining the dynamics of the commercial real estate industry, as it pertains to the real estate market in general, as well as to the valuation of commercial real estate.

1.1 Dynamics in the Commercial Real Estate Market

The commercial real estate market can be divided into the space market, the asset market, and the development market (Geltner & Miller, 2007). The space market is where the right to use rental space is traded, and it is often referred to as the rental market. On the demand side of the space market are businesses and individuals who require space for a specific purpose. On the supply side of the space market are property owners, who supply tenants with rental space. From a valuation perspective, the space market is where property owners lock in cash flow by charging tenants rent for the right to use their space. It is important to notice that the rental rates determined in the space market are highly segmented and largely dependent on the location of the rental space. For example, the rental rate in the central business district (CBD) of a large city is generally much higher than the rental rate in the fringe area of a small city. Similarly, the market value of real estate assets determined in the asset market also depend on the location of the property. The asset market is where ownership of real estate is traded, and is comprised of businesses and investors who buy and

sell commercial space for their own use or for investment purposes. From a valuation standpoint, the ownership of a property represents a claim to the cash flow that the property can generate from leasing out space to tenants. Hence, the space market and the asset market are intricately intertwined and the rental rates an owner can charge as well as the value of a property are determined by supply and demand for both real estate space and real estate assets. Lastly, to guide supply and demand in the space market and the asset market, the real estate development market exists to increase supply of rental space when market conditions are favorable, thus balancing the rental rates in the market.

Important to notice is that the commercial real estate market is just one of many platforms in which investors may choose to allocate their funds. Hence, the commercial real estate market is competing with the securities markets and other direct investment markets for the funds of investors. Because such alternative investment opportunities exist for investors, it is important to be aware of what differentiates the commercial real estate market from other markets. The commercial real estate market's biggest contender for funds is perhaps the public stock market. However, there are some fundamental differences between public stock markets and the commercial real estate market. Specifically, by contrast to stock exchanges, the commercial real estate market is a private market. Generally, private markets are less liquid and less transparent than public markets (Geltner & Miller, 2007), and closing a transaction in a private market can take up to several years, whereas closing a transaction on a stock exchange can be done in seconds. Furthermore, by contrast to public stock markets, where current stock prices are updated continuously, there is little public information available in the commercial real estate market, and the market price of a property is thus less transparent to investors. Consequently, the market value of a property is uncertain, and investors must spend time and resources in order to determine the appropriate value of a real estate asset before investing.

The inefficiency in the commercial real estate market, relative to the stock market, gives investors the opportunity to make good returns by taking advantage of asymmetric information in the market. However, because of the opportunity for great returns on investment, there is heavy competition in today's real estate market.

1.2 The Current Situation

In recent years, historically low interest rates have dominated global monetary policy and investors are looking to other asset classes than traditional stocks and bonds to fill their investment portfolio. Consequently, investment in commercial real estate has become an attractive harbor for investors looking for relatively safe investments that provide a decent current yield (DNB, 2015). For example, the Norwegian Government's Pension Fund Global announced in 2006 that it would allocate up to 3% of the funds capital to real estate investments, and in later years that target has increased to 5% (NBIM, 2007). Other large funds, such as the US pension funds CalPERS and CalSTRS, have also opted to allocate large holdings of their portfolio to real estate (OECD, 2014). Investment by such large institutions have increased over the past decade, and are predicted to continue to rise globally (Weil, 2015). In fact, the average target allocation in real estate assets for institutional investors is estimated at 9.62% globally by the end of 2015. That suggests an increase of 24 bps from last year, which implies that most developed real estate markets, including the Norwegian market, are likely to feel the impact of increased investment (Weil, 2015).

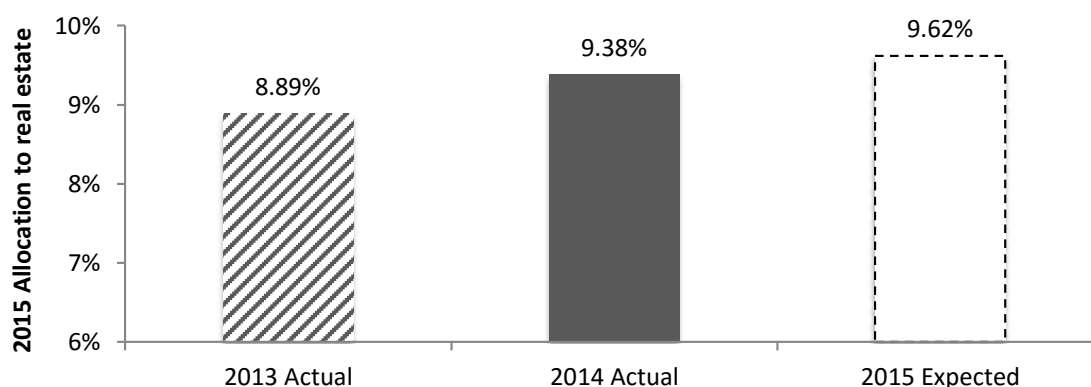


Figure 1: Expected allocation to real estate by large funds (Weil, 2015).

In fact, some of the largest acquisitions of real estate in Norway the past years, including Starwood Capital Group's acquisition of Statoil's headquarters (3,4 bn NOK) and Madison International Realty's acquisition of Fortin (4,65 bn NOK), have involved large international investors. While such investors continue to increase safe long-term investment in real estate assets in Norway, prices for such properties increase, and other real estate investors are pushed out of the market due to low returns on investment. As a result, smaller real estate

investors such as real estate investment companies, private investors, and family funds look to more opportunistic investment opportunities in properties with larger potential for capital gains (DNB, 2015). Such investments may include development properties or properties with potential for increased occupancy. The recent flow of international investors has thus caused increased competition in commercial real estate investment in prime properties as well as properties in the fringe areas. Together, all real estate investors have contributed to the recent increase in transaction volumes in Norway (Basale, 2015). As illustrated in Figure 2, transaction volumes the past 10 years in Norway have generally followed the cycles of the overall economy, but reached a record high 87 billion NOK in 2014. The trend with increased transaction volumes is expected to continue, and is estimated to reach 100 billion NOK in 2015 (CBRE, 2015).

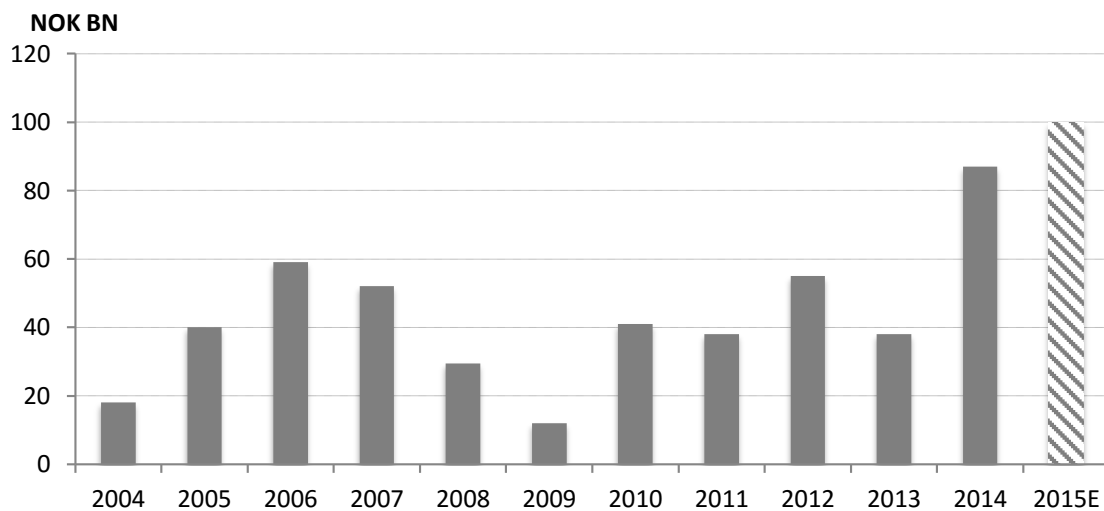


Figure 2: Historical transaction volume in Norway (Basale, 2015)

While the Norwegian asset market is experiencing increased competition from international investors and record high transaction volumes, the space market has experienced a similar trend. In the past 3-4 years, rental rates in Norway have been steadily increasing, however, experts now suggest that rental rates may fall in the near future due to the decline in activity in the oil industry (Fearnley, 2015). The same expectation is observed in occupancy rates across the country as several companies in the oil-industry are downsizing due to the decline in oil prices (Basale, 2015).

1.3 Real Estate Valuation

The demand for holding Norwegian commercial real estate is steadily increasing, and transaction volumes are consequently historically high (Fearnley, 2015). As competition for acquiring commercial real estate increases, the valuation of real estate assets becomes increasingly important. While an overvaluation of a property may have adverse effects on the profitability of an investment, an undervaluation may cause the investor to miss out on a profitable investment. Hence, knowing the price at which one should be willing to invest in a property is crucial for a successful investment in commercial real estate. However, determining the correct value of commercial real estate is a challenging task, and there are numerous uncertain factors that have to be considered in an accurate valuation.

The most dominant uncertainty related to investments in real estate is arguably the instability in the rental rates at which a property owner can lease out his space for in the future. Furthermore, a property, as it stands today, may be worth more if it was used for a different purpose, or had a few more stories, and there is uncertainty as to what degree a property may increase future cash flows through capital expenditures and investments. Therefore, it is common practice to consider the ‘highest and best use’ (HBU) principle when valuing a property. This principle states that each property should be valued as “the most probable use of land or improved property that is legally possible, physically possible, financially feasible and appropriately supportable from the market, and which results in maximum profitability” (Propex, 2003). While each of these considerations are driven by market forces, the practical use of a property is strictly regulated by local governments, and is thus a predetermined factor in the valuation of a property. However, if a property is currently not regulated to match the other conditions of the HBU principle, an investor may consider applying for changes in the property’s regulation and zoning in hopes to increase the future cash flow generated by the property. In such cases, the investor faces uncertainty as to how long the approval process will take, and to what extent their application will be approved.

When evaluating an investment in a development project, investors must consider the strict rules regarding a property’s zoning and regulation. In Norway, there are three levels of regulations that must be considered. Firstly, the municipal plan of a city outlines the allowed use of property. Investors who want to change the allowed use of a property must go through a lengthy process with local politicians. Secondly, the area plan is a more detailed plan that outlines concerns regarding the usage and infrastructure of a property. Lastly, the investor

must prepare a detailed plan for the actual development of a project. The expected length of the time a firm has to wait for approval of a real estate development project depends on the current status of a property, and at what level the investor must apply for changes in the existing plan.

In the current Norwegian real estate market, experts indicate that there is a significant inefficiency in market dynamics, and that different valuations of the same property often vary greatly, especially for projects that include the development of a property¹. The large variations in property valuations are mainly due to the lack of a generally accepted valuation process that accurately accounts for the uncertainty related to the potential improvement in a property's future cash flow (Leishman et al., 2000). In Norway, investors in the real estate development market are typically private owners of undeveloped land, or real estate investment companies with portfolios largely consisting of fully developed real estate assets, and only a few development projects. Hence, there is a lack of large institutional investors in the Norwegian real estate development market, as life insurance companies and pension funds generally look to real estate assets with safe long-term cash flows. From a valuation standpoint, the shortage of sophisticated investors willing to exhaust resources in the valuation of development projects in the early stages leads to less innovation in valuation techniques of real estate development projects, and thus less accurate valuations.

Today, the traditional discounted cash flow (DCF) method largely dominates all commercial real estate valuation. In a traditional DCF valuation, the value of a property is determined by estimating the expected future cash flow generated by a property, and discounting that cash flow back to present values. The challenge with such an approach is to accurately estimate the future cash flows from the property, especially if the investor believes the use of the property can be changed to increase future cash flows. In that case, the investor must make brave assumptions as to when the development project will be completed and the rental rate of the improved property at that time. In recent years, traditional DCF valuations have been criticized for failing to accurately consider such options to increase the future cash flow of a property. Critics argue that DCF valuations often disregard the flexibility of an investment, and concludes that pricing investments using a real options framework would add value to most types of investment valuations (Damodaran, 2005). Furthermore, Leishman et al.

¹ Experts we have been in contact with include Malling & Co, NewSec, and Entra.

(2000) show that a real options framework have provided more accurate valuations than traditional DCF valuations of real estate because the DCF method tends to overlook the potential value of further development of a property in the future, especially when it is not optimal to expand the property in the present. With the current valuation tools, investment in real estate development is depicted as a very risky affair due to the large differences in estimated value by different analysts. Consequently, banks are reluctant to finance such investments before the investor has secured approval of the project and has found tenants for the project, regardless of how promising the project looks. Thus, a more sophisticated valuation approach that provides more accuracy, less uncertainty, and a deeper insight to the value of real estate development projects may help increase the efficiency in the market by lowering the uncertainty surrounding the true value of real estate development projects.

2. Options Theory

In this section we will introduce financial options, and their basic characteristics. We will also discuss real options, and how real options differ from standardized financial options. Lastly, we explain how different mathematical techniques can be used to value real options in an uncertain environment.

2.1 Financial Options

An option is a derivative security, which represents a contract between a seller and a buyer, where the value of the security is derived from the value of an underlying asset, for example equity shares in a company. Trading of standardized options contracts on a national exchange started as early as in 1973 (Bodie, 2014). An option gives the buyer a right, but not the obligation, to execute the option on or before a specified expiration date. The buyer of the option pays a certain premium for this right, while the issuer receives this premium to commit to fulfill the option if the buyer wishes to exercise it. When purchasing an option, the investor does not necessarily believe the underlying asset will increase in value, and options are in fact often used as a way to hedge risk, and can also be used to leverage a position.

There are two types of options; call options and put options. A call option gives the buyer of the option a right to purchase the underlying asset for a specific amount, called the strike price, on or before a specified expiration date. This might be interesting if one believes the price of the underlying asset will rise above the strike price and thereby provide a payoff equal to the difference between the underlying asset value and the strike price. If the underlying asset value is not above the strike price at the time of expiration, the option will not be exercised because this will give a negative payoff. The payoff from a call option at the time of exercise can therefore be expressed as $\max(S_T - K, 0)$, where S_T is the value of the underlying asset at time of expiration, T , while K is the strike price of the option. The risk of this investment is that the price of the underlying asset will not rise above the strike price. However, notice that because an option provides the investor with the right, and not the obligation to exercise the contract, the investor will never lose more than the premium initially paid for the option. Figure 3 and Figure 4 shows the profit and loss for the buyer and seller of a call option, respectively.

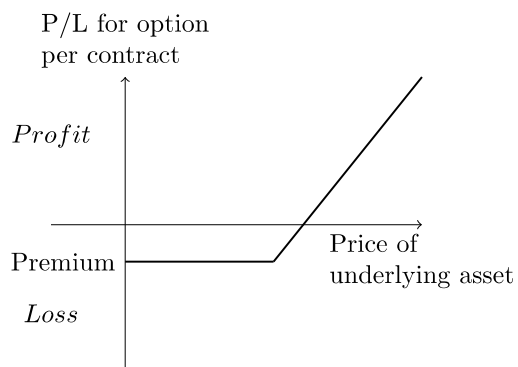


Figure 3: Profit/Loss for the buyer of a Call

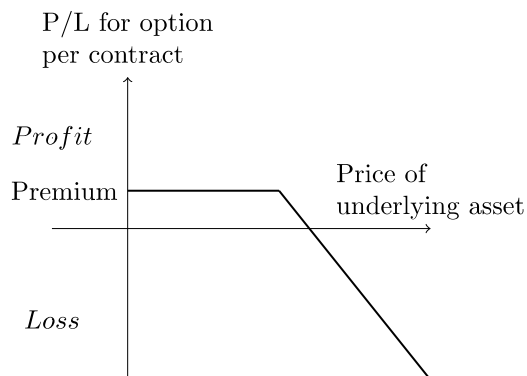


Figure 4: Profit/Loss for the seller of a Call

In contrast to the owner of a call option, the owner of a put option has the right to sell the underlying asset on or before the specified expiration date, at a specified price. When investing in a put option, the buyer believes the price of the underlying asset will fall below the strike price. As with the call option, a put option will not be exercised unless it is favorable to the investor, i.e. if the underlying asset's value is not beneath the strike price. Therefore, we get the following payoff function for the buyer of a put option at the time of exercise, $\max(K - S_T, 0)$. When buying a put option, the risk is that the underlying asset will not fall below the strike price. However, as with a call option, the holder of a put option cannot lose more than the premium initially paid. The profit and loss for a put option is shown in Figure 5 and Figure 6 for the buyer and the seller of a put option, respectively.

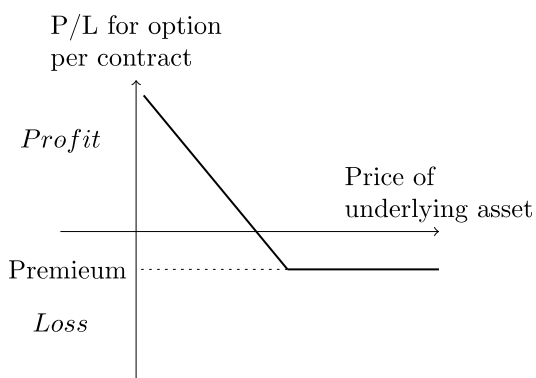


Figure 5: Profit/Loss for the buyer of a Put

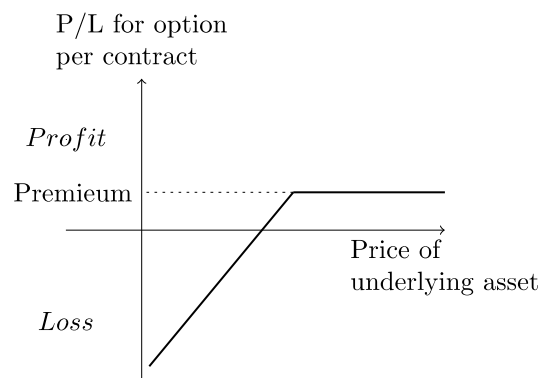


Figure 6: Profit/Loss for the seller of a Put

Generally, we use different expressions to describe the current state of an option. An option is defined as “in the money” if immediate exercise would produce a positive payoff to its holder. On the contrary, if immediate exercise would not produce a positive payoff, an

option is defined as “out of the money”. When the price of the underlying asset is equal to the predetermined strike price, the option is described as “at the money”.

Whether a put option or a call option, all options can generally be divided into two categories; American options and European options. An American option gives its buyer the right to exercise the option on, or before, the predetermined expiration date. A European option, on the other hand, can only be exercised at the expiration date, and not at any other time. The distinction between the two types of options is important as they may differ significantly in value. The value of an American option can never be less than that of a European option, if all other features are alike. Intuitively, that is because an American option has all the same features as a European option, but additionally, it can also be exercised before the expiration date. The additional flexibility of the American option will never be negative for the holder of the option.

2.2 Real Options

Real option theory, as its name implies, uses options theory to evaluate investments in physical or real assets, such as a development project, in a dynamic and uncertain business environment (Mun, 2006). Real options are different from financial options in that they are not traded on exchanges as securities, and that they do not normally involve decisions on an underlying asset that is traded as a financial security. Another distinction between real options and financial options is that the holder of the real option, which is typically the management of a firm, can influence the value of the project, as opposed to the holder of a financial option, which generally cannot influence the value of the underlying asset. Real options often represent certain types of management decisions. These decisions can typically be an option to expand a project, option to postpone an investment, option to abandon a project, or option to suspend a project. Hence, real option models provide an alternative approach to a traditional DCF to value projects with flexibility.

While the DCF approach assumes one pathway with fixed amounts, and all decisions are made at the start of the project, the real options approach considers several different pathways, where decisions are made during the project as a consequence of the uncertain environment. In other words, in a real options approach, management makes their strategic decisions along the way as new information becomes available. Thus, when applying real options theory to value a project, the project tends to get a higher value than when applying a

traditional DCF approach, because the real options approach better captures the value of management flexibility.

In real options theory, different managerial decisions can be interpreted as put or call options. For example, an option to expand a project at a convenient time in the future has the same features as an American call option. The present value of the future cash flows gained by the expansion can be thought of as the underlying asset, and the expansion cost can be thought of as the strike price of the option. If the present value of the cash flow is above the investment cost, the option is “in the money”. If, on the other hand, the present value of the cash flow is below the investment cost, the option is “out of the money”. Depending on whether the expansion has to be made at a specified time, or if it can be made at any time before a specified date, we have a European or an American call option. Figure 7 shows an example of the payoff of an option to expand. Notice that the payoff in Figure 7 is the same as for the call option illustrated in Figure 3.

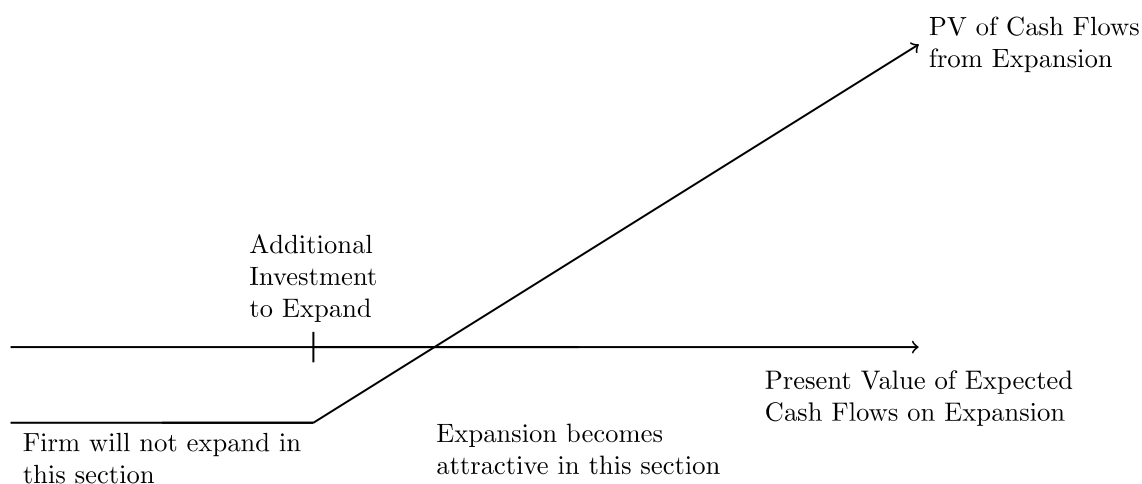


Figure 7: The Option to Expand

Contrary to an option to expand, an option to abandon a project can be seen as a put option. Such an option is valuable if the project starts losing money, because the managers does not have to continue with the business plan if it becomes unprofitable. If the project becomes unprofitable, management can abandon the project and the loss from the investment will not be as severe. As with the option to expand, the time at which the abandonment is possible to exercise dictates whether the abandonment option can be seen as an American or a European put option. Figure 8 outlines the payoff of an abandonment option. Notice that the payoff in Figure 8 is similar to the payoff from the put option in Figure 5.

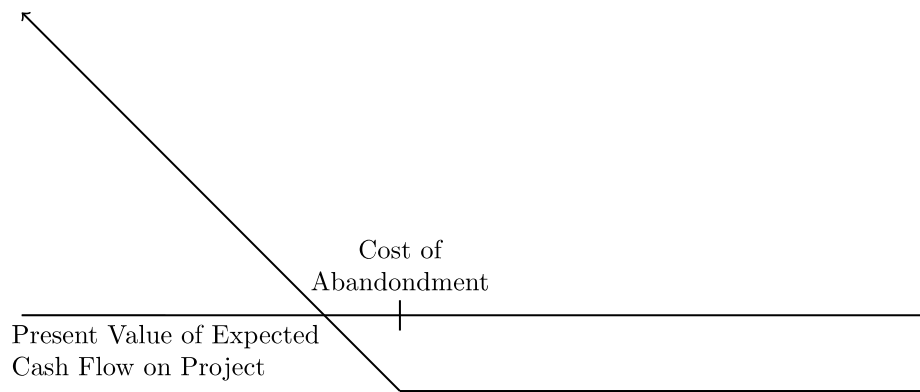


Figure 8: The Option to Abandon

2.3 Dynamic Optimization

The payoff from a firm's investment made today accumulates over time and are affected by uncertainty in the business environment, as well as decisions made by management and the firm's competitors. In real options theory, there are mainly two mathematical techniques to solve decision-making problems with these kinds of features. These methods are called dynamic programming and contingent claims analysis, and are used to obtain the value of real options. The two techniques are closely related, and may lead to the same results in many applications. However, the two approaches impose different assumptions about financial markets, as well as the appropriate discount rates (Dixit & Pindyck, 1994).

In real options valuation, contingent claims analysis builds on the idea that a project's stream of costs and benefits can be seen as a specific asset. In today's economy, there exist markets for a vast amount of assets, and if our asset happens to trade in one of these markets, we can use contingent claims analysis to solve our problem. More accurately, our asset doesn't explicitly have to be traded in a market, but contingent claims analysis requires that the financial markets are sufficiently complete in that it exists some traded assets that can track the value of our underlying project. All that is required is some combination of assets that will exactly replicate the pattern of returns from our investment project. This requirement is often referred to as the spanning assumption (Dixit & Pindyck, 1994). The value of the project must be equal to the value of the replicating portfolio. If it is not, we would have an arbitrage opportunity. In cases where the spanning assumption does not hold it is more suitable to use the dynamic programming approach.

Dynamic programming is a general tool for solving dynamic optimization problems. This approach breaks the optimization problem into two parts. On the one hand is the immediate investment decision, and on the other hand, is a valuation function that takes into consideration all future decisions. If the project is finite, the approach finds the value at the very last decision point of the project using standard static optimization. Then it integrates this solution into the second to last decision point, and in this way works its way back to the present. If the problem is infinite, the problem can be solved by finding the project value at the point where all subsequent events are recursive, i.e. each new problem looks exactly like the one before, and thereafter using the same approach as with a finite project, working from this point and back to the present. This approach does not only facilitate numerical results, but sometimes also analytical solutions (Dixit & Pindyck, 1994). Dynamic programming is especially useful when dealing with uncertainty, and the approach can deal with complex decision structures as well as complex relationships between the option value and the underlying asset (Rogers, 2013).

3. Related Literature

To this date, several research papers and articles have described valuation models and optimal investment decisions under uncertain conditions, and some also consider the case as it pertains to development projects in the real estate industry.

In early real options theory, Titman (1985) considered the case of underutilized urban land, such as parking lots, that may be converted to residential or commercial space. He assumed that landowners are wealth maximizers, and that a better utilization of property may significantly increase the owner's wealth. He goes on to develop a simple binomial real options model to value undeveloped land. The model provides intuition about when it is optimal to invest and when it is optimal to postpone investment in the development of land. The paper shows that because underutilized land can be developed to become several different asset classes, the option to develop the land has significant value. However, since development of urban land is an irreversible decision, Titman (1985) reasons that it may be rational to leave the underutilized land undeveloped in hopes of realizing greater payoffs in the future. A clear limitation to Titman's paper is that it only considers a simple two stage binomial model in which the underlying rental rates are estimated based on projections, rather than probability distributions.

McDonald and Siegel (1986) developed a more complex model in which both the benefit and the cost of a project follows a geometric Brownian Motion (GBM). The model also provides optimal investment thresholds as it compares the value of the immediate investment in a synthetic fuel plant to the value of the same investment at any time in the future. Thus, the model is able to isolate the value lost when investing at a suboptimal time, and the authors make the argument that it is in fact not optimal to invest until the project provides a value twice that of the investment cost. McDonald and Siegel (1986) focus on the value of flexibility in timing rather than the value of the isolated option to invest.

Capozza and Helsley (1989) did the opposite when they developed a real options model to value undeveloped urban land. They consider the investment case in which agricultural land can be developed into to urban land. The model assumes that future rental rates follow a GBM, and points out that while uncertainty in land prices and the rate of growth of a city affect the value of urban land, rental rates are yet the most dominant parameter to value land. Capozza and Helsley (1989) show that the irreversibility of the decision to develop a tract of

land, as well as the uncertainty in future rental rates, favors a delay in the development of raw land. Furthermore, they argue that the option value of undeveloped land may explain why undeveloped land close to urban areas sells at a premium over the value of the rental income from the land. One interesting extension to Capozza and Hesley (1989) would be to include risk related to regulatory issues in the development of urban land. Such regulatory issues may impose time lags in the investment of a development project.

Bar-Ilan and Strange (1996) introduced a framework which included such issues in the form of a time lag between the investment decision and the collection of cash flow. They used dynamic programming to develop a model that considers the effect of investment lags on an optimal decision under uncertainty. The framework outlines a simple model that emphasizes the effect investment lags have on capital budgeting decisions. The paper shows that conventional effects of price uncertainty in a real options problem may be reversed when investment lags are introduced. Moreover, they argue that investment lags have an offsetting effect on uncertainty, and that investors facing long lags should be less concerned with uncertainty. Since Bar-Ilan and Strange (1996) apply their model to fit a generic investment, and does not take industry specific details into consideration, their model is not very applicable in practice. However, the investment lag introduced in their framework could be used in practice as a very simplified model.

To model risks more specific to particular investment opportunities, Riddiough (1997) focused on the value of land where there is a risk of regulatory takings, i.e. the government confiscation of properties in exchange for a certain compensation. The underlying property value in Riddough's (1997) model follows a GBM and the probability of a property being confiscated is modeled via a Poisson process. By introducing the Poisson process, Riddiough (1997) finds that regulatory risk can significantly decrease land value. However, the article largely discusses the economic cost of regulations rather than viewing the problem from an investor's point of view. For example, an investor in real estate is more concerned with the value of the property itself than the loss of value of a property due to potential confiscation by the government.

Leishman et al. (2000) takes the investors point of view in their research paper when they argue that option pricing models have a greater potential for valuing land development than traditional valuation models such as residual valuation and DCF. They argue that because such traditional models do not consider the 'hope value' of a property, a real options

approach to the valuation of a development project in real estate provides a more accurate estimate of value. The ‘hope value’ of a property is described in the article as the value of the possibility that a development that is not profitable today may become very profitable at some point in the future if market conditions change. The paper proves empirically that option-pricing models in real estate valuation have proved to be more accurate than other traditional models. However, Leishman et al. (2000) also points out that option-pricing models are ‘time-series in construction’, with predictions depending on key financial variables such as interest rates, capitalization rates and volatility, and that traditional valuation methods often have the ability to remain more detailed and case specific.

One framework that attempts to include a realistic feature related to uncertainty in timing is developed by Miltersen and Schwartz (2007). They outline a model that considers a real option problem with uncertain maturity. The framework is modeled to match investment opportunities such as an exploration project that requires on-going investment costs, however, it can quite easily be applied to other similar scenarios. In this model, the time to completion of a real investment is uncertain, but the model is still able to obtain analytical solutions as to when to abandon the project or to switch the level of investment activity. The article extends the problem from a monopoly market to a duopoly, and argues that the model also can be used for markets with several market participants. Similarly to Riddough (1997), Miltersen and Schwartz (2007) includes a Poisson process to model the uncertainty in the time processes, and the framework proves to be quite flexible in terms of obtaining analytical solutions to other variables than option value. However, the framework does not specifically consider the opportunity cost of investing in a project.

Heydari and Siddiqui (2009), on the other hand, consider the opportunity cost of exercising an option, in a sequential decision making problem with fixed investment lags, to suggest that the optimal time of interrupting an interruptible load contract in the electricity industry is far out of the money. The model assumes that the underlying electricity price follows a GBM and the investment lags are of a fixed length. The model takes into consideration the cash flow gained from the interruption in excess of the opportunity cost of forgone income. Thus this framework considers the opportunity cost of investing when determining optimal investment thresholds. Heydari and Siddiqui (2009) show that, in this case, greater uncertainty results in greater option values, and on the other hand that greater interruption lags results in a lower threshold price of interruption. Their model assumes a fixed

opportunity cost, and can thus not be applied if the opportunity cost of the project is not constant.

Choi (2011) investigated the usefulness of real options analysis with his case study of real options applications in the real estate industry. His study involves the usefulness of real options when it comes to presale contracts of condominiums, valuation of raw urban land in the perspective of land developers, and valuation of an opportunity to install solar panels on properties. In the case of valuing raw urban land with opportunity to develop it into developed lots, Choi introduces a permit approval lag based on a gamma distribution and he also includes a sequential decision making problem in which the investor can make two different decisions at two different points in time. The article outlines a quite realistic framework for valuation of raw urban land, however, it does not provide analytical solutions for option values or optimal investment thresholds. Therefore, Choi (2011) relies on numerical analysis to obtain his results.

The model developed in this paper is a combination of several strings of the mentioned economic literature. Our analysis is particularly inspired by the works of Heydari and Siddiqui (2009), and Bar-Ilan and Strange (1996) as it considers irreversible investments in the presence of one or multiple lags. Furthermore, this paper is similar to those of Riddough (1997) and Capozza (2011) as it pertains to the real estate industry, and we have included several industry specific features in our model to reflect realistic assumptions. This paper thus covers a gap in the literature related to investment in real estate under uncertainty in both future rental rates and regulatory approval of development projects. Our most basic model first builds on the existing framework of Heydari and Siddiqui (2009) for a model in the absence of an approval lag. Thereafter, we use the extended framework of Heydari and Siddiqui (2009) which draws on the optimal investment threshold of an investment in the presence of a lag of fixed length. Lastly, we combine this framework to that of Miltersen and Schwarts (2007) to add uncertainty to the previously fixed lag. Although our paper outlines a framework that combines several strings of literature, we are able to provide closed form analytical solutions in most cases, and to shed light on some interesting implications about investment in development projects in the real estate industry.

4. Assumptions and Notation

In the following framework, we take the perspective of a price-taking firm with an opportunity to invest in the expansion of a building that is subject to regulatory approval. Thus, the firm has a potential option to incur a fixed investment cost to increase the rental space of a building, and thereby increase the cash flow of the property. We denote the total amount of rental space before and after the expansion by D_1 and D_2 respectively, where $D_1 < D_2$. The average annual rental rate for newly constructed buildings is denoted by P_t , and is modeled via a geometric Brownian motion (GBM),

$$dP_t = \mu P_t dt + \sigma P_t dZ_t, \quad P_0 = P \quad (1)$$

where μ is the annual growth rate of rental rates, σ is the annual volatility in rental rates, and dZ is the increment of a standard Brownian motion. For simplicity, we assume that operating costs remain a constant ratio of rental rates, and hence that P_t represents rental rates net of operating costs. Because only a negligible portion of operating expenses cannot be covered directly by tenants, this is a reasonable assumption. Furthermore, we assume that the investment is irreversible and that the project remains active forever once undertaken. The time at which the firm exercises its option to expand is denoted by τ , and the corresponding optimal investment threshold is given by P_τ . We assume that when the firm exercises the option to expand at $t = \tau$, the length of the construction period is known, and will end at $t = T$. The expected value of the project is denoted by $V(\cdot)$, and becomes active when the firm exercises the option at $t = \tau$. The value of the option to invest in the expansion is denoted by $F(\cdot)$. Finally, h is the duration of the application period, ρ is the exogenous discount rate, and T is the time at which construction is completed.

Generally, property owners typically accept some capital expenditures in order to maintain high rental rates over time. However, Crosby (2012) shows that rental rates still tend to depreciate over time as buildings get older. Therefore, we include the parameter, r , to account for the relative difference between rental rates for newly constructed buildings and those of older buildings. Thus, r is a decay factor that decreases rental rates over time. The rate at which rental income decays over time depends on building-specific features, and thus differs between buildings. Crosby (2012) outlines one way to accurately obtain the long run

average annual rate of depreciation in rental rates by observing the current rental rate of an existing building and comparing it to the market rental rate for a new building in the same area². We apply the same approach to obtain the average long term decay factor in rental rates. In (2), r is as function of the rental rate of a current building, C_0 , the age of the current building, M , and the rental rate for a new building, P .

$$C_t = P_t e^{-r(M+t)} \quad \text{where } r = -\frac{\ln(\frac{C_0}{P})}{M} \quad (2)$$

Throughout this paper, r will remain a variable of P to reflect that the decay factor in rental rates is a case specific measure that can vary for different locations, and different rental rates. In Appendix A we show the isolated effect of the decay factor on rental rates of a building for different values of a buildings age, M . All notations used in this paper are summarized in Table 1.

Notation	Description
P_t	Net rental rate at time, t
μ	Drift parameter
σ	Volatility
ρ	Discount rate
τ	Time of optimal investment
T	Time of finished construction
P_τ	Net rental rate at $t = \tau$
$V(\cdot)$	Expected value of active project
$F(\cdot)$	Expected value of option
I	Investment cost
h	Length of approval time
M	Age of existing building
C_t	Net rental rate from existing building
r	Decay factor of rental rates
$\frac{1}{\lambda}$	Expected approval time

Table 1: Summary of notations

One assumption that is important to emphasize is that once construction has started, the firm will complete the construction at $t = T$, regardless of the movement of rental rates during

² Crosby (2012) also adjusts the average long term depreciation in rental rates for a specific area by including a benchmark average. Creating such a benchmark requires a great deal of data. Therefore, we have excluded the benchmark adjustment in (2).

the construction period. Since the investment cost, I , is paid at the beginning of construction, and because rental rates cannot be negative, a delay or abandonment of the project during construction will only cause the firm to miss out on rental income it otherwise could have collected upon completion of the construction. In other words, a building will always have a higher value when rented out to tenants than when empty, regardless of the rental rate, because the tenants themselves pay the operating expenses associated with their use of the building³. Thus, since $P_t \geq 0$, abandoning a project before construction is finished, or opting not to lease out space after construction is complete, will never be optimal. In other words, even if management had the flexibility to abandon the project during the construction period, they would never do so, because the decision to abandon would never be optimal in our model. This assumption holds for each of the three models we develop.

In Model 1, a firm holds an option to expand the rental space of a building, and does not have to get approval from regulatory authorities prior to construction. Therefore, the firm can begin construction at any time, without having to wait for approval of the expansion. However, the firm will not invest unless market conditions are favorable. As mentioned above, this model includes a construction period of fixed length. During this period, the firm receives no cash flow, but rental rates continue to follow a GBM. Furthermore, since the construction period is fixed and known prior to investment, the new building will be completed at $t = T$, regardless of the change in rental rates during the construction period. Hence, this model begins with an uncertain waiting period in which the firm waits for market conditions to become favorable, and is followed by a fixed construction period in which the firm receives no cash flow. Once construction is completed, the firm starts receiving cash flow from the expanded building.

In Model 2, we assume that the firm holds a potential option to expand the rental space of a building, but that prior to receiving the rights to begin construction, they have to wait a period of fixed length for regulatory authorities to approve the project. The length of the approval period is determined by the parameter h . Once approval is granted, the firm receives the option to expand. However, even if market conditions are favorable at the time the firm submits the application for approval to expand the building, the firm has to wait out the full length of the approval period before they receive the option to begin construction.

³ Leases where tenants pays operating expenses are commonly referred to as triple net leases, or NNN leases.

During this period, rental rates continue to move stochastically. Thus, while market conditions may be favorable when the firm first applies for the expansion, they may become unfavorable during the approval period. If so, the firm will have to wait for an uncertain period of time before it becomes optimal to begin construction. As in Model 1, the construction period is fixed, and once construction has started, the time at which the expansion is completed is known. Hence, this model starts with a fixed approval lag, which is followed by an uncertain waiting period, which again is followed by a fixed construction lag before the firm starts receiving cash flow.

In Model 3, we assume that the firm holds a potential option to expand the rental space of a building, but that prior to receiving the rights to begin construction they have to wait an uncertain amount of time for regulatory authorities to approve the project. The length of the approval period is determined by a Poisson process, where the parameter λ denotes the mean arrival rate of approval. Once approval is granted, the firm receives the option to expand. However, as in the previous models, the firm in Model 3 will wait for market conditions to become optimal before they begin construction. When construction starts, the firm will wait out the fixed construction period before they start receiving cash flow. Thus, this model begins with a stochastic approval lag, which is followed by an uncertain waiting period, which again is followed by a fixed construction lag.

5. Analytical Formulation

In this section, we will lay out a simple analytical framework for valuing an option to expand a building's rental space under uncertain conditions. While we consider an option to expand a property with a current cash flow from an existing building, we will focus on the isolated option value, rather than the value of the existing building in addition to the option to expand. We begin with the simplest model without an approval lag, and go on to extend the model to include both a fixed and a stochastic approval lag.

5.1 Model 1: Investment in the absence of an approval lag

A firm has the option to invest in an expansion of the rental space of a building either immediately or at a random time in the future, τ , by incurring a fixed investment cost, I . By owning a property with an option to expand, the firm will hold the value of the cash flow from the existing property, and they will also hold the value of the option to expand, $F(P)$. At $t = \tau$ the firm exercises the option to invest in the expansion, and receives the value of the active project, $V(P)$. In this model, the value of the active project, $V(P)$, includes the opportunity cost of not investing in the expansion, and rather keeping the cash flow from the existing building. Hence, the value of the active project, $V(P)$, is the value that the firm will receive from investing in the expansion in excess of the value it would receive if it kept the existing building "as-is". In this simple model, no regulatory authorities exist, and the firm does not have to apply for approval of the project prior to expanding. Consequently, the firm can invest immediately if market conditions are favorable. When the firm invests, they will demolish the existing building and construct a new building with an increased rental space. Since the length of the construction period is fixed and known prior to investment, the expansion will be completed at $t = T$, regardless of the movement in rental rates during the construction period. As shown in Figure 9, the project becomes active once construction begins, rather than when construction is finished, and the value of the active project, $V(P)$, thus includes the foregone cash flow from the existing building during the construction period. The firm has no flexibility during the construction period, and is therefore only concerned with when it is optimal to invest in the expansion, and the value of the option to expand.

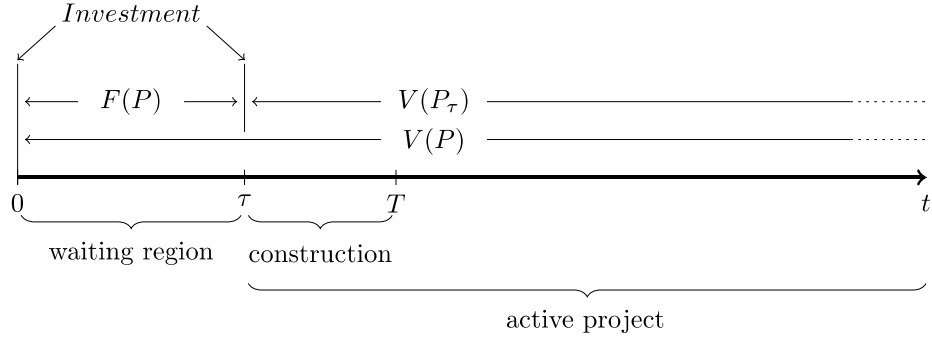


Figure 9: Investment under uncertainty in the absence of approval lag

In the waiting region in Figure 9 the firm has the option to incur a fixed cost, I , to receive the uncertain payoff, $V(P)$. The value of that option is denoted $F(P)$. The construction period in Figure 9 represents the time at which the expansion is under construction, and is known before investment. For simplicity, we assume that the construction cost, I , occurs at $t = \tau$, and that while construction is on-going, the firm will receive no cash flow from the property. Furthermore, we assume that the construction period is of a fixed length, $T - \tau$, which is known to the firm prior to investment. This is a reasonable assumption, as developers generally have a good estimate of the length of the construction period prior to the start of the project. Thus, at $t = \tau$, the firm invests in the project, and the project becomes active. At $t = T$, the expansion is completed, and the firm starts receiving cash flow from the expanded building.

We use standard dynamic programming and work backwards by first determining the value of the active project, $V(P)$. As illustrated in Figure 9, if market conditions are favorable at $t = 0$, the waiting region is effectively eliminated, and the value of the option is equal to the value of immediate exercise. Therefore, we start by finding the value of exercising the option immediately, i.e. $t = \tau = 0$, as indicated in (3),

$$V(P) = \mathbf{E}_P \left[\int_T^\infty D_2 P_t e^{-r(t-T)} e^{-\rho t} dt - \int_0^\infty D_1 C_t e^{-\rho t} dt \right] - I \quad (3)$$

where \mathbf{E}_P is the expectation, which is conditional on P . The first integral on the right hand side of (3) represents the present value of the cash flow from the new building after the expansion is completed. Since the building will not be ready to lease out until construction is completed, this cash flow begins at $t = T$, and because we assume an infinite horizon the cash flow continues forever. Moreover, since the rental rate from the new building decreases

as the building gets older, P_t is adjusted to reflect that the decrease starts when the construction is completed at $t = T$, rather than at the beginning of construction, when $t = \tau$. Hence the cash flow from the new building, D_2P_t , is adjusted with $e^{-r(t-T)}$ to reflect the decay in rental rates over time. For example, one year after the construction is finished, $t - T$ will equal 1 and the rental rate will have decayed by the factor, r , for 1 year. The last term in the first integral, $e^{-\rho t}$, is included to discount every cash flow from the building back to $t = 0$.

The second integral on the right hand side of (3) represents the opportunity cost of the expansion, which includes the loss of rental income from the existing building during both the construction period, and in the time after construction. Since the firm receives no cash flow during the construction period, and because (3) denotes the value of immediate exercise, this opportunity cost begins at $t = 0$. The cash flow from the existing building will effectively cease once construction starts, and will never reoccur thereafter. Therefore, the second integral goes to infinity. The opportunity cost, i.e. the cash flow that the firm would have received from the existing building, D_1C_t , is discounted by $e^{-\rho t}$ to obtain the present value of the cash flow at $t = 0$. The last term on the right hand side of (3), I , represents the investment cost of the project.

We can simplify (3), as denoted in (4).

$$\begin{aligned}
V(P) &= \mathbf{E}_P \left[\int_T^\infty D_2P_t e^{-r(t-T)} e^{-\rho t} dt - \int_0^\infty D_1P_t e^{-r(M+t)} e^{-\rho t} dt \right] - I \\
&= \int_T^\infty D_2P e^{\mu t} e^{-r(t-T)} e^{-\rho t} dt - \int_0^\infty D_1P e^{\mu t} e^{-r(M+t)} e^{-\rho t} dt - I \\
&= \int_T^\infty D_2P e^{(\mu-r-\rho)t+rT} dt - \int_0^\infty D_1P e^{(\mu-r-\rho)t-rM} dt - I \quad (4)
\end{aligned}$$

In the top equality in (4) we insert the function for C_t , as denoted in (2), where we have that $C_t = P_t e^{-r(M+t)}$. Hence, C_t , is a function of the rental rate at time t , P_t , and it has decayed for $M + t$ years. In the middle equation, we implement the expectation to P_t , which is given by $P e^{\mu t}$, for each term. Lastly, we simplify the expression and get the bottom equation in (4). By integrating (4) we obtain the expression indicated in (5).

$$V(P) = \left[\frac{D_2P e^{(\mu-r-\rho)t+rT}}{\mu-r-\rho} \right]_T^\infty - \left[\frac{D_1P e^{(\mu-r-\rho)t-rM}}{\mu-r-\rho} \right]_0^\infty - I \quad (5)$$

By simplifying (5) we obtain the expression for the value of immediate exercise, $V(P)$, indicated in (6).

$$V(P) = \frac{D_2 P e^{(\mu-\rho)T} - D_1 P e^{-rM}}{\rho + r - \mu} - I \quad (6)$$

In equation (6), the first term in the numerator can be interpreted as the present value of the annual rental income from the newly constructed building. The expected rental rate at $t = T$ is given by $P e^{\mu T}$. To obtain the present value of this rental rate, we discount it back by T years with the discount rate, ρ , and obtain the term that represents the present value of the rental rate per square meter of the new building, $P e^{(\mu-\rho)T}$. By simply multiplying this rental rate by the size of the new building, D_2 , we obtain the expected present value of the rental income from the new building after construction. The second term in the numerator in (6) represents the current income from the existing building. The rental rate of the existing building has decayed for M years with the factor r . Hence, the rental income per square meter of the existing building at $t = 0$ is given by $P e^{-rM}$, and is the same as the rental rate in equation (2) when $t = 0$. By simply multiplying the rental rate per square meter of the existing building, $P e^{-rM}$, with the size of the existing building, D_1 , gives us the current annual rental income from the existing building. We combine the first and second term in the numerator to get the expected annual present value of the cash flow from the expanded building, in excess of the current cash flow from the existing building. This cash flow is a perpetuity which is discounted by the discount factor, ρ , and adjusted by the decay factor, r , and the drift parameter, μ . Quite intuitively, the formula in (6) is similar in structure to a perpetual Gordon growth model, and provides the value of the perpetual excess cash flow from the new building with a growth rate, $\mu - r$, and the discount rate, ρ .

By simplifying equation (6), we obtain the much simpler expression denoted in (7), which we use to denote the value of immediate exercise in further calculations.

$$V(P) = aP - I \quad \text{where} \quad a = \frac{D_2 e^{(\mu-\rho)T} - D_1 e^{-rM}}{\rho + r - \mu} \quad (7)$$

Thus, aP represents the expected present value of the cash flow from the expanded building, in excess of the cash flow from the existing building. The parameter, I , is the investment cost of the expansion, which subtracted from aP provides the value of immediate exercise, $V(P)$.

Next, we move further back in time and include the value of the option to invest if market conditions are not favorable to obtain the expression denoted in (8). From (8), we see that the value of the option to invest, $F(P)$, depends on whether or not the rental rate, P , has reached the optimal investment threshold, P_τ . If $P < P_\tau$, then the value of the option to invest is given by the top branch of the right hand side of (8). According to Dixit and Pindyck (1994), the solution to the value of the option when $P < P_\tau$ must take the form $A_1 P^{\beta_1}$, where A_1 is a constant yet to be determined, and $\beta_1 > 1$ is the positive root of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - \rho = 0$. See Appendix B for details. If, on the other hand, $P \geq P_\tau$, it is optimal to invest in the expansion immediately, and the value of the option to expand is equal to the value of immediate investment, $V(P)$, as shown by the bottom branch on the right hand side of (8).

$$F(P) = \begin{cases} A_1 P^{\beta_1}, & \text{if } P < P_\tau \\ aP - I, & \text{if } P \geq P_\tau \end{cases} \quad (8)$$

The optimal investment threshold, P_τ , and the endogenous constant, A_1 , denoted in (8) are obtained by applying the value matching and smooth pasting conditions between the two branches in (8), as shown in (9).

$$\left. \begin{array}{l} \text{Value matching } A_1 P_\tau^{\beta_1} = aP_\tau - I \\ \text{Smooth pasting } \beta_1 A_1 P_\tau^{\beta_1} = aP_\tau \end{array} \right\} \Rightarrow A_1 = \frac{aP_\tau^{1-\beta_1}}{\beta_1} \text{ and } P_\tau = \frac{\beta_1}{(\beta_1 - 1)} \frac{I}{a} \quad (9)$$

The value matching and smooth-pasting conditions come from the consideration of the value function, $F(P)$, at the time of optimal to investment, τ . The value-matching condition indicates that at the optimal investment threshold, the value of waiting, and the value of immediate investment are equal. The smooth-pasting condition says that at the optimal investment threshold, the slopes of the value functions for immediate exercise, and waiting to expand, are equal and continuous. This condition can be expressed by differentiating the value matching condition, as shown in the bottom branch of (9). The two conditions must both hold because if $F(P)$ were not value matching and smooth at the investment threshold, the firm could do better by investing in the expansion at a different point in time (Dixit & Pindyck, 1994). Intuitively, if the firm invests in the expansion at a rental rate other than the optimal investment threshold, they will receive a lower return on investment.

Interestingly, notice that the optimal investment threshold, and thereby the option value, can also be found by denoting the option value as indicated in (10),

$$F(P) = \max_{P_\tau \geq P} \left(\frac{P}{P_\tau} \right)^{\beta_1} V(P_\tau) \quad (10)$$

where the expected stochastic discount factor as explained by Dixit and Pindyck (1994) is denoted in (11). We will use this expected discount factor when we extend our model to include a fixed approval lag in section 5.2.

$$\mathbf{E}_P[e^{-\rho\tau}] = \left(\frac{P}{P_\tau} \right)^{\beta_1} \quad (11)$$

By applying the first order necessary condition to (10), we can obtain the same optimal investment threshold, P_τ , as in (9). This derivation is shown in Appendix C where we also show that the second order necessary condition is satisfied. Since $\beta_1 > 1$, we see in (12) that it is optimal to invest in the expansion when the expected present value of the new rental income, in excess of the foregone rental income from the current property, is greater than the present value of the investment cost.

$$aP_\tau = \left(\frac{\beta_1}{\beta_1 - 1} \right) I \quad (12)$$

In Appendix D we show that the optimal investment threshold, as well as the value of the option, both increase with volatility. Thus, as with options in general, uncertainty supports a delay in decision-making, because the value of the option, i.e. the value of waiting, increases with volatility.

5.2 Model 2: Investment under a fixed approval lag

Now, we extend the framework of section 5.1 to allow for a fixed approval lag. Before the firm can invest in the expansion, they have to wait for approval of the project by the local government for a fixed amount of time, h . As indicated in Figure 10, upon approval the firm will either invest in the expansion immediately if $P_h \geq P_\tau$, or they will delay the expansion, if $P_h < P_\tau$. If it is optimal to exercise the option immediately, the firm will receive the value of the active project, $V(P_h)$, and if it is optimal to wait, they will receive the value of the

option to invest, $F(P_h)$. Notice that the fixed approval lag introduced in this model occurs at a time before the uncertain waiting region, while the construction lag occurs at a time after the uncertain region. From a modeling point of view, the timing of these lags relative to the uncertain waiting region requires that we treat the approval lag and the construction lag differently. Standard dynamic programming dictates that we begin our analysis at the end period, and work our way backwards. Therefore, from a modeling standpoint, the first lag we encounter when solving the investment problem is the construction lag, which is included in the value function of the active project, $V(P_h)$. The approval lag, on the other hand, must be considered in the last step because the fluctuation in rental rates causes an uncertain length of the waiting region. Therefore the expected value of the project depends on whether or not the market rental rate has reached the optimal investment threshold at the time of approval, $t = h$.

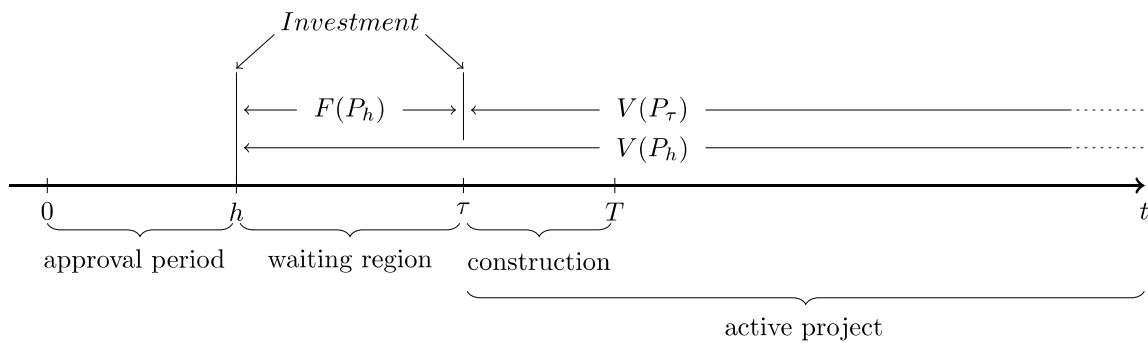


Figure 10: Investment under uncertainty in the presence of an approval lag of fixed duration

Notice that if $P_h \geq P_\tau$, it is optimal to invest in the expansion immediately after the time of approval, and the waiting region is effectively eliminated. On the other hand, if $P_h < P_\tau$, the value of holding the option to expand and receive the cash flow from the existing property is greater than the value of expanding. Thus the firm will wait until $P \geq P_\tau$ at the optimal investment threshold, $t = \tau$, before they expand. The top branch on the right hand side of (13) denotes the value of the option to expand the rental space of the property at $t = h$ when $P_h < P_\tau$, while the bottom branch denotes the value of exercising the option immediately at $t = h$, and thus represents the expected NPV of the project if $P_h \geq P_\tau$.

$$F(P_h) = \begin{cases} \left(\frac{P_h}{P_\tau}\right)^{\beta_1} V(P_\tau), & \text{if } P_h < P_\tau \\ V(P_h), & \text{if } P_h \geq P_\tau \end{cases} \quad (13)$$

In equation (14) we show that the expected value of the option to expand at $t = h$, conditional on the market rental rate, P , is determined by the probability that at $t = h$ the rental rate, P_h , will have reached the optimal investment threshold, P_τ . Thus equation (14) represents a combination of the two branches in (13) where the weights of the two branches are determined by the probability that the rental rate will reach the optimal investment threshold at the time of approval.

$$E[F(P_h)] = \mathbf{E}_P \left[\left(\frac{P_h}{P_\tau} \right)^{\beta_1} [aP_\tau - I] \right] \times \mathbf{P}_P[P_h < P_\tau] \\ + \mathbf{E}_P[aP_h - I] \times \mathbf{P}_P[P_h \geq P_\tau] \quad (14)$$

The first term on the right hand side of (14) is the probability that the rental rate will not have reached the optimal investment threshold by the time of approval, times the top branch of the right hand side of (13). The second term in (14) is the probability that the rental rate will have reached the investment threshold by the time of approval, times the bottom branch of the right hand side of (13). By the characteristics of a GBM and the definition of conditional probability and optimal thresholds, the probability that $P_h \geq P_\tau$ at $t = h$ is described in (15), where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard normal distribution, and $R(h, P, P_\tau)$ is indicated in (16) (Etheridge, 2002).

$$\mathbf{P}_P[P_h \geq P_\tau] = \Phi(R(h, P, P_\tau)) \quad (15)$$

$$R(h, P, P_\tau) = \frac{(\mu - \frac{1}{2}\sigma^2)h - \ln(\frac{P_\tau}{P})}{\sigma\sqrt{h}} \quad (16)$$

Returning to equation (14), we can now calculate the conditional expectation of the value of the project at $t = h$ as indicated in (17),

$$E[F(P_h)] = \left(\frac{P}{P_\tau} \right)^{\beta_1} e^{\gamma h} [aP_\tau - I] \times [1 - \mathbf{P}_P[P_h \geq P_\tau]] \\ + [aPe^{\mu h} - I] \times \mathbf{P}_P[P_h \geq P_\tau] \quad (17)$$

where $\gamma = \beta_1\mu + \frac{1}{2}\beta_1(\beta_1 - 1)\sigma^2$, as explained by Siddiqui and Heydari (2010). Ultimately, in order to determine the optimal investment rule for the firm, we must optimize equation (17) with respect to P_τ , as indicated in (18).

$$\max_{P_\tau \geq P} \mathbf{E}_P[F(P_h)] \quad (18)$$

By the first order necessary condition of (18), we get the non-linear equation shown in (19),

$$\begin{aligned} 0 = & \beta_1 \left(\frac{P}{P_\tau} \right)^{\beta_1} \left(-\frac{1}{P_\tau} \right) e^{\gamma h} [aP_\tau - I] \times \mathbf{P}_P[P_h < P_\tau] \\ & + \left(\frac{P}{P_\tau} \right)^{\beta_1} a e^{\gamma h} \times \mathbf{P}_P[P_h < P_\tau] \\ & + \left(\frac{P}{P_\tau} \right)^{\beta_1} e^{\gamma h} [aP_\tau - I] \times \phi(R(h, P, P_\tau)) \frac{1}{P_\tau \sigma \sqrt{h}} \\ & - [aP e^{\mu h} - I] \times \phi(R(h, P, P_\tau)) \frac{1}{P_\tau \sigma \sqrt{h}} \end{aligned} \quad (19)$$

where $\phi(R(h, P, P_\tau))$ is the probability distribution function (pdf) given the time to approval, h , the rental rate, P , and the optimal investment threshold, P_τ . Since (19) cannot be solved analytically, we must solve it numerically to obtain the threshold for when the firm should invest in the expansion of the property, P_τ . By substituting P_τ from (19) into (17) we obtain the expected option value at $t = h$. The present value is found by simply discounting (17) as shown in (20) below.

$$F(P) = \mathbf{E}[F(P_h)] e^{-\rho h} \quad (20)$$

We see from (20) that the present value of the option depends on the length of the approval lag, h . Thus, by setting $h = 0$, we obtain the expression of the optimal investment threshold in the absence of an approval lag in (19), and thus the value of the option without an approval lag in (20). In Appendix E we show that when $h \rightarrow 0$, then $\mathbf{E}[F(P_h)] \rightarrow F(P)$.

5.3 Model 3: Investment under a stochastic approval lag

So far, we have introduced a model with a fixed investment lag. Now, we extend the framework from section 5.2 by assuming that the time until approval of the project, h , is no longer fixed, but rather determined by an exponentially distributed stochastic variable. The arrival time of the approval is thus stochastic and follows a Poisson process where the parameter λ denotes the mean arrival rate of approval during an infinitesimal period of time, dt . The value of the option to expand prior to approval is denoted by $G(P)$ and depends on the probability of approval, which is determined by the input parameter, λ . The probability

that the application is approved during the short time interval, dt , is given by λdt , and the probability that the application is not approved is $(1 - \lambda dt)$. Hence, if the application has not been approved for t years, then the probability that it will be approved within the next short time period is λdt , as shown in (21),

$$dq = \begin{cases} 0, & \text{with probability } (1 - \lambda dt) \\ 1, & \text{with probability } \lambda dt \end{cases} \quad (21)$$

where q denotes a Poisson process. The expected duration of the approval period is $\mathbf{E}(h) = \frac{1}{\lambda}$. Therefore, a higher λ implies a shorter expected approval period. Notice that in the short time period, dt , the application will either be approved, and the firm will receive the value of the option to invest in the expansion, $F(P)$, or the application will not be approved and the firm will continue to hold the value of the function $G(P)$, as illustrated in Figure 11.

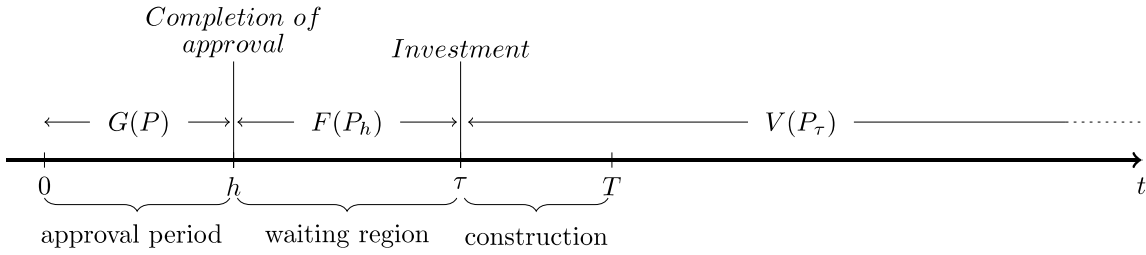


Figure 11: Investment under uncertainty in the presence of an approval lag of uncertain duration

The dynamics of the expected value of the project during the approval period are found following a similar approach as Miltersen and Schwartz (2007). Thus, the expected value of the project during the approval period, $G(P)$, is denoted in (22).

$$G(P) = (1 - \rho dt)\lambda dt \mathbf{E}_P[F(P + dP)] + (1 - \rho dt)(1 - \lambda dt) \mathbf{E}_P[G(P + dP)] \quad (22)$$

The first term on the right hand side of (22) represents the expected present value of the option to expand, multiplied by the probability that the expansion will be approved within the next short time period. The second term represents the possibility that the expansion will not be approved in the next short time period, and thus that they will continue to hold the value of the unapproved option. We expand the right hand side of (22) using Itô's Lemma, as shown in equation (23).

$$G(P) = (1 - (\rho + \lambda)) \left[G(P) + \frac{1}{2} \sigma^2 P^2 G''(P) dt + \mu P G'(P) dt \right] \\ + (1 - \rho dt) \lambda dt \left[F(P) + \frac{1}{2} \sigma^2 P^2 F''(P) dt + \mu P F'(P) dt \right] \quad (23)$$

We simplify equation (23) and obtain the differential equation indicated in (24).

$$\frac{1}{2} \sigma^2 P^2 G''(P) + \mu P G'(P) - (\lambda + \rho) G(P) + \lambda F(P) = 0 \quad (24)$$

Notice that (24) provides two expressions, one for each of the two branches of $F(P)$ in equation (8) in Section 5.1. Therefore, each expression must be solved separately as indicated in (25) and (26).

$$\frac{1}{2} \sigma^2 P^2 G''(P) + \mu P G'(P) - (\lambda + \rho) G(P) + \lambda A_1 P^{\beta_1} = 0 \quad \text{if } P < P_\tau \quad (25)$$

$$\frac{1}{2} \sigma^2 P^2 G''(P) + \mu P G'(P) - (\lambda + \rho) G(P) + \lambda [aP - I] = 0 \quad \text{if } P \geq P_\tau \quad (26)$$

The value of the function $G(P)$, as denoted in (27), is determined by solving (25) and (26), and thus depends on whether or not $P < P_\tau$ at $t = 0$.

$$G(P) = \begin{cases} A_1 P^{\beta_1} + A_0 P^{\omega_1}, & \text{if } P < P_\tau \\ \frac{\lambda a P}{\rho + \lambda - \mu} - \frac{\lambda I}{\rho + \lambda} + B P^{\omega_2}, & \text{if } P \geq P_\tau \end{cases} \quad (27)$$

The first term on the right hand side of the top branch in (27) is the value of the option to expand should the project be approved, and is comparable to the same term in (8). However, since the project is yet to be approved, the value of the option is adjusted via the second term on the right hand side of the top branch in (27). Intuitively, the second term reduces the value of the option due to the possibility that that the firm may have to wait for a long time for the project to be approved. On the other hand, the first term on the right hand side of the bottom branch in (27) represents the expected present value of the excess future cash flow from the rental income of the finished project, and the second term represents the investment cost. Lastly, the third term on the right hand side in the bottom branch of (27) adjusts the expected net present value of the project by considering the probability that the rental rate will drop below the optimal investment threshold by the time the application is approved. Intuitively, the last term in the bottom branch in (27) should be interpreted as the value of the

firm's flexibility to not have to decide today, but rather wait with the decision. Hence, the term BP^{ω_2} is positive.

In equation (27), the input parameters ω_1 and ω_2 are the positive and negative roots of the quadratic equation $\frac{1}{2}\sigma^2\omega(\omega - 1) + \mu\omega - (\rho + \lambda) = 0$, respectively. Notice that when $\lambda = 0$, then $\omega_1 = \beta_1$ and $\omega_2 = \beta_2$. However, when there is a positive possibility that the application will be approved, i.e. $\lambda > 0$, then $\omega_1 > \beta_1 > 1$ and $\omega_2 < \beta_2 < 0$. The implication is that when $\lambda = 0$, the value of $G(P)$ is eliminated. Intuitively, when $\lambda = 0$ the probability of approval is equal to 0. On the other hand, as $\lambda \rightarrow \infty$, the loss in value due to the possibility of a long approval period is eliminated, and $G(P) \rightarrow F(P)$.

To solve for $G(P)$ we must obtain the endogenous constants A_0 and B , by applying the value matching and smooth pasting conditions to (25) and (26) as shown in (28) and (29).

$$A_1P^{\beta_1} + A_0P^{\omega_1} = \frac{\lambda aP}{\rho + \lambda - \mu} - \frac{\lambda I}{\rho + \lambda} + BP^{\omega_2} \quad \text{value matching} \quad (28)$$

$$A_1\beta_1P^{\beta_1-1} + A_0\omega_1P^{\omega_1-1} = \frac{\lambda a}{\rho + \lambda - \mu} + B\omega_2P^{\omega_2-1} \quad \text{smooth pasting} \quad (29)$$

By solving for A_0 and B and substituting into (27) we obtain the value of $G(P)$. See Appendix F for details.

6. Numerical Results

In our numerical presentation we have developed a base case with estimated input parameters that reflect current market conditions in Oslo. We have been in contact with experts in the Norwegian commercial real estate industry to inquire about realistic expectations for input parameters in the model⁴. Consequently, all numbers are presented in NOK unless otherwise stated.

We begin our presentation of numerical results with a discussion of the input parameters in our base case. We continue by presenting the results for each of the three models. Lastly we compare the results from the three models, with an emphasis on the effect of the approval lag and uncertainty regarding future rental rates.

6.1 Data

While some parameters in our base case are observable in the market, others must be estimated based on historical data or economic intuition. As previously mentioned, we assume that operating expenses remain a fixed percentage of gross rental income, and thus that net rental rates are observed in the market for both the existing property and newly constructed properties. Experts have indicated that operating expenses are typically 10% of gross rental income. With that in mind, we determine that $C_0 = 1\,350$ and $P = 3\,100$, $M = 85$, $D_1 = 15\,000$, $D_2 = 37\,000$, and $T - \tau = 2$. Furthermore, investment cost is observed to be 20 000 per square meter of new construction, thus, $I = 740\,000\,000$. While we classify the mentioned parameters as observable in the market, every property has individual features when it comes to age, current rental space, potential for expansion, and construction specifics. Therefore, expertise and diligence is required not only when calculating the estimated parameters, but also when determining the observable input parameters. Table 2 summarizes all base case input parameters.

⁴ We have been in contact with Malling & Co, Entra, NewSec, Pangea Property Partners, DTZ Realkapital, NAI First Partners. Since most commercial real estate brokers publish market reports every quarter, these data are recently updated.

Parameter	Value
C_0	1 350
P_0	3 100
M	85
D_1	15 000
D_2	37 000
I	740 000 000
$T - \tau$	2
σ	10.58%
μ	2.5%
ρ	10%
h	4
λ	0.25

Table 2: Base case input parameters

In the following sections, we will briefly discuss the most important input parameters in the model.

6.1.1 Standard Deviation

To calculate the volatility of office rental rates in Oslo, we have used a dataset of gross rental rates, comprised of between 697 and 93 properties for each observation, for Class A properties in Oslo⁵. The dataset trails back to the beginning of year 2000 and includes observations for every 6 months up to the current period. Based on these data, we estimate the historical volatility to $\sigma = 10.54\%$. We use this measure in our base case⁶.

6.1.2 Permit Approval Time

Obtaining approval from the local government to expand a rental space is often a lengthy political and bureaucratic process. Although developers have a good estimate of time to approval, the duration of such a process is ultimately uncertain. In our model, we use a Poisson process to model this uncertainty. According to developers and real estate experts in Oslo, the average expected time interval for approval of a project with the size and scope of our base case, is around 4 years. Hence, $h = 4$ in our base case. However, depending on case

⁵ The dataset is from Arealstatistikk and covers Class A properties in all of Oslo. More geographically detailed data may be preferred if available for a property's area. Class A properties are considered high-end properties.

⁶ Other volatility measures such as an exponentially weighted moving average (EWMA) or generalized autoregressive conditional heteroskedasticity (GARCH) may provide different results.

to case specifics, an approval process can be shorter, sometimes less than one year, and it can be longer, sometimes more than 10 years. We found that using a $\lambda = 0.25$ gave us a distribution for the time of arrival that fitted best with the information we got from local experts. The Poisson distribution with $\lambda = 0.25$ is shown in Figure 12.

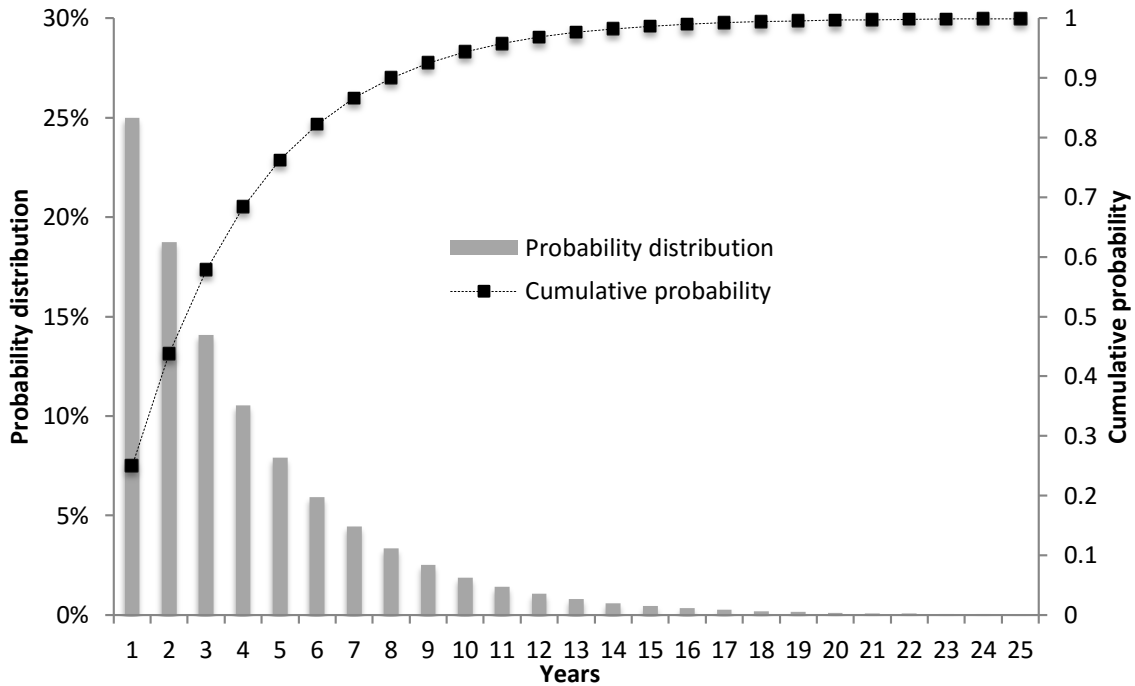


Figure 12: Poisson distribution with $\lambda = 0.25$

6.1.3 Development Costs

We have determined that investment costs for a project like our base case would be approximately 20 000 per square meter. Therefore, $I = 740\,000\,000$ in our base case. These costs include all costs associated with the demolition of the existing building and the construction of the finished project. Hence, cash outflows such application costs, demolition costs, construction costs, and compensation paid to current tenants for temporary termination of lease contracts must all be included in I . Certain tax considerations such as value-added tax must also be considered.

6.1.4 Discount Rate

Determining an appropriate discount rate can be a challenging task. Firms often use the Weighted Average Cost of Capital (WACC) when selecting a discount rate for their investments. Since most firms are mainly financed by equity and debt, the WACC is simply

the weighted average cost of these two sources of financing. However, since discount rates differ between investors, we have spoken to local real estate companies in Oslo and found that an appropriate discount factor would be around 10%. Generally, commercial real estate investors consider development projects to be more risky than investment in existing assets. Thus, while the market yield for office buildings in Oslo roughly range from 4.5% to 6.5% for finished properties, development projects tend to be discounted at a higher rate. Thus, it is reasonable to add a risk premium to current market yields to account for the added risk of development projects. We will therefore use a discount factor of 10% in our base case.

6.1.5 Drift

The drift parameter represents the expected annual increase in rental rates over time. As shown in Figure 13, the average annual increase in office rental rates in Oslo from the past 15 years has been 3.58%⁷. However, in the long run, rental rates should increase at the same rate as the rest of the economy on average. If rental rates continued to increase at a higher rate than the rest of the economy, then the value of holding real estate would increase disproportionately to the rest of the economy. Therefore, we assume a drift parameter equal to the inflation target set by the Norwegian Central Bank at $\mu = 2.5\%$.

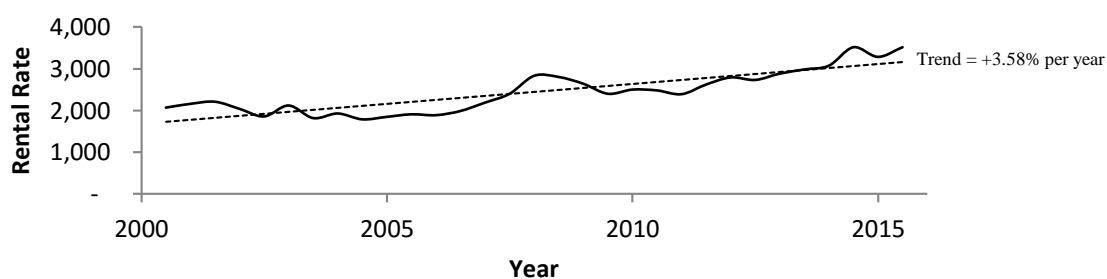


Figure 13: Historical drift in rental rates

⁷ The dataset is from Arealstatistikk and covers Class A properties in all of Oslo.

6.2 Model 1: Investment in the absence of an approval lag

Using the input parameters determined in section 6.1, we begin by looking at numerical results from the simple model with an investment in the absence of an approval lag. Figure 14 illustrates the value of the option to expand the rental space of a building, $F(P)$, at different levels of volatility in rental rates. The Figure also illustrates the expected NPV of exercising the option immediately, $V(P)$.

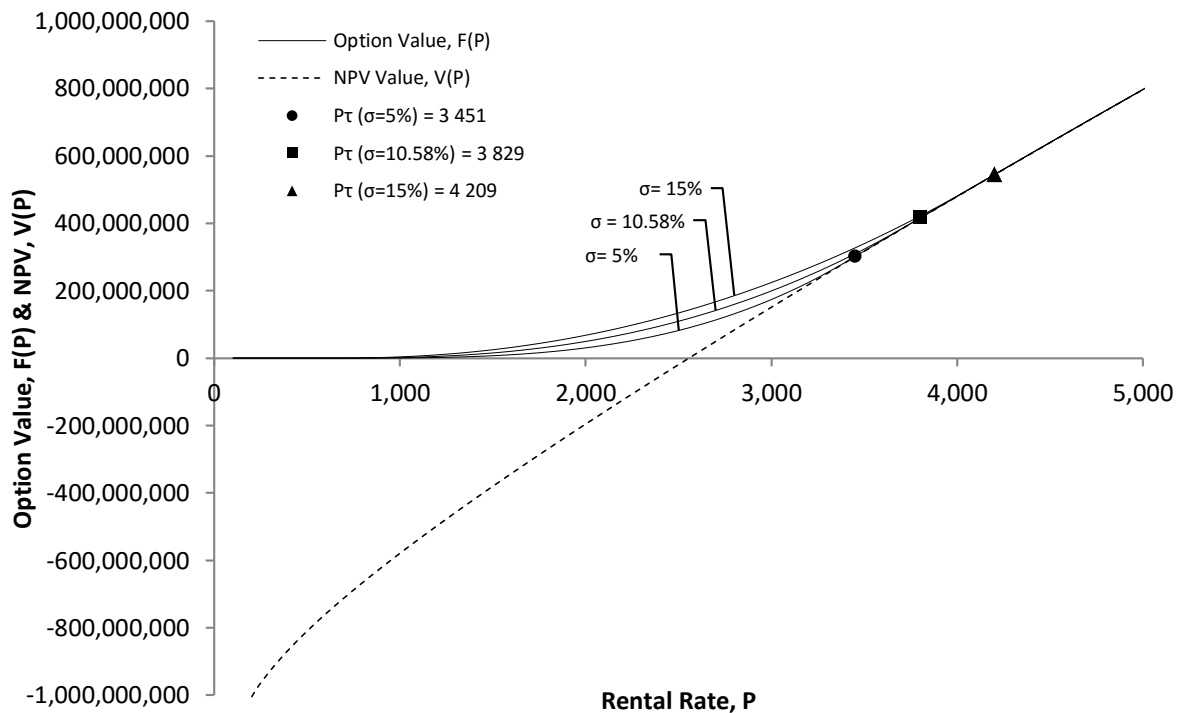


Figure 14: Option values for different levels of volatility using the framework in Model 1.

As expected, an increase in volatility leads to greater option values. Notice that the value of the option to expand, $F(P)$, is always greater than, or equal to, the value of immediate exercise, $V(P)$. At the optimal investment threshold, P_t , the value of the option to expand and the expected NPV of immediate exercise are equal. At any rental rate $P < P_t$, we observe that $F(P) > V(P)$, and similarly, when $P \geq P_t$, we observe that $F(P) = V(P)$. Intuitively, the optimal investment threshold represents the lowest market rental rate at which exercising the option to expand is the preferred decision. The implication is that as soon as the market rental rate for new buildings reaches the optimal investment threshold, the firm should immediately invest in the expansion. Notice that this holds for any level of volatility. However, we observe from Figure 14 that while an increase in volatility leads to

an increase in option value, the expected NPV of immediate exercise remains the same for any level of volatility. Consequently, the optimal investment threshold for each level of volatility can be found along the same NPV-line, and thus P_τ is also increasing in volatility. Notice that because the decay factor, r , depends on P , as denoted in equation (2), the NPV line is slightly curved. For very low values of P , the rental rate of the existing building becomes greater than the rental rate for new buildings, and thus r becomes negative. Therefore, the NPV of immediate exercise decreases at an increasing rate as P becomes very low, as shown in Figure 14. Figure 15 illustrates the optimal investment threshold as a function of volatility, and shows that an increase in volatility leads to an increase in the optimal investment threshold.

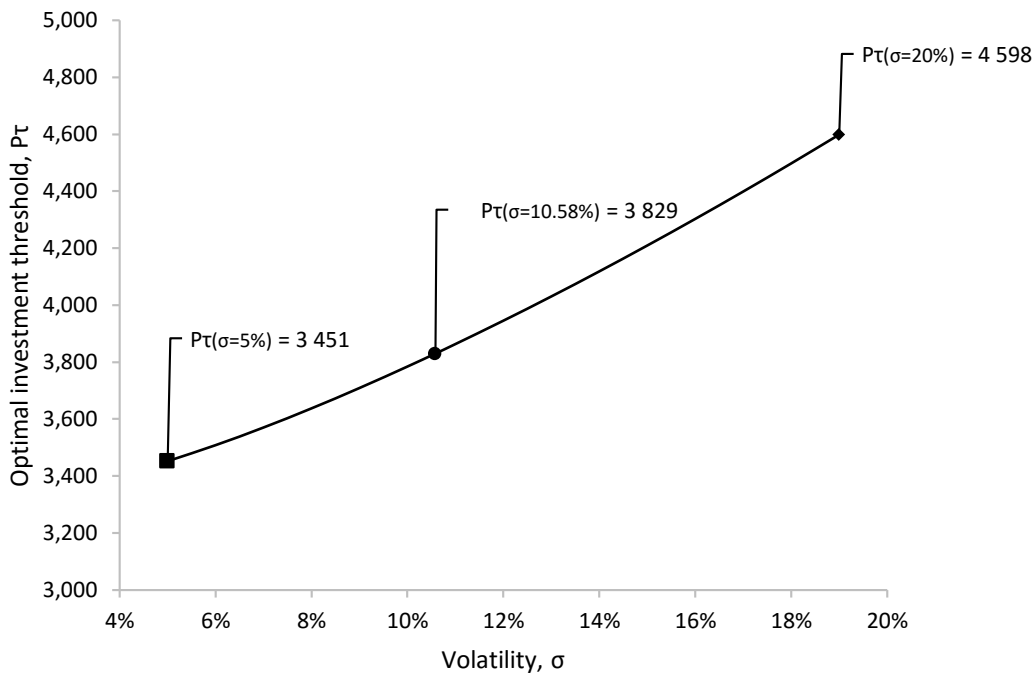


Figure 15: Optimal investment threshold as a function of volatility

Intuitively, Figure 15 tells us that an increase in the uncertainty in future rental rates favors a delay in the investment of an expansion. In practice this implies that investors in a volatile real estate market should delay their investment in development projects longer than investors operating in a less volatile market. In our base case, where $\sigma = 10.58\%$, the optimal investment threshold, $P_\tau = 3\,829$, is greater than the market rental rate, $P = 3\,100$. Consequently, the value of the option to expand is greater than the value of immediate exercise, and the firm's optimal decision will be to delay the investment, and allow the

market rental rate to reach the optimal investment threshold before they invest in the expansion. Remembering that the option to expand depends on the current market rental rate, P , we observe from Figure 14 that the firm's option is currently "in the money". The point where the NPV line in Figure 14 crosses the X-axis can be interpreted as the strike price of the option, X . At any rental rate above the strike price, immediate exercise of the option is profitable, and is thus "in the money". However, for any value of $P < P_t$ the value of the option to expand is greater than the value of expanding immediately. Therefore, although immediate exercise of the option would provide the firm with $NPV = 186 \text{ MNOK}$ in our base case, the firm will wait with the investment because the value of the option, $F(P) = 222 \text{ MNOK}$, is greater than the value of immediate exercise. In practice this impacts the decision making process of investors. Contrary to traditional beliefs, this model suggests that even though a project provides a positive NPV, real estate investors should consider the value of waiting when deciding whether to invest or not, because the value of waiting maybe greater than the value of investing immediately.

The difference between the value of the option to invest, and the value of immediate exercise at a given rental rate, P , is illustrated by the gap between the NPV-line and the option value line in Figure 14. Notice that the gap is larger for high levels of volatility than for low levels of volatility, and hence that option values for high levels of volatility are greater than option values for low levels of volatility. Quite intuitively, if there was no uncertainty in future rental rates, i.e. $\sigma = 0\%$, the value of the option would either be 0, if the rental rate was low, or it would be equal to the value of the NPV, if the rental rate was sufficiently high. Since the rental rate where the NPV is equal to the option value represents the point at which it is optimal to invest, the optimal investment threshold in the case of price certainty is equal the strike price of the option. Therefore, in the case of no uncertainty in rental rates, the value of an option to expand is equal to the NPV of the project, which is only a positive value if the project is profitable. In our base case $X = 2\,553$. This implies that property owners with the option to expand a building, would prefer more volatile rental rates. However, in practice volatile rental rates are also likely to increase discount rates which would decrease the value of the existing building as well as affect the value of the real option negatively.

Interestingly, notice that an increase in volatility has a greater impact on the size of the gap between the option value and the NPV when the rental rate, P , is close to the strike price than for rental rates that are far higher or far lower than the strike price. For example, we

observe from Figure 14 that for high values of P , the option value is equal to the value of immediate exercise, regardless of the volatility. Hence, the level of volatility becomes less significant as P becomes very high, and does in fact become completely insignificant when a change in volatility does not result in an optimal investment threshold, P_τ , that is greater than the market rental rate, P . Thus, for an investment in the absence of an approval lag, real estate investors should be most concerned with the volatility of the market rental rate when the NPV of immediate exercise of the option is close to 0.

6.3 Model 2: Investment under a fixed approval lag

In Model 2, we extended Model 1 to include a fixed approval lag, h . The framework used in Model 2 considers an optimal stopping time problem where the optimal investment threshold, P_τ , depends on P . While this model does not provide a closed form analytical solution for the optimal threshold, we are still able to solve for it numerically. Hence, we should be mindful when analyzing the optimal threshold in Model 2, and cautious to draw any strict conclusions from the optimal threshold. In this model, we obtain the expected value of the option at the time of approval, $t = h$, and simply discount that value back to the present to obtain the present value of the option to expand.

Figure 16 illustrates option values for different lengths of approval time, h . Notice that the present value of $F(P_h)$ represents a combination of the option value if $P_h < P_\tau$ and of the option value if $P_h \geq P_\tau$, as denoted in equation (14). Therefore, we cannot obtain an NPV line for any $h > 0$. On the other hand, when $h = 0$, we obtain the same option values in Model 2 as in Model 1, and also the same NPV. Thus, the option value line where $h = 0$ in Figure 16 can be interpreted as the value of the option to expand in the absence of an approval lag, and provides the same results as Model 1. Recall from the previous section that because r is dependent on P , the NPV line is slightly curved for low values of P , as illustrated in Figure 16.

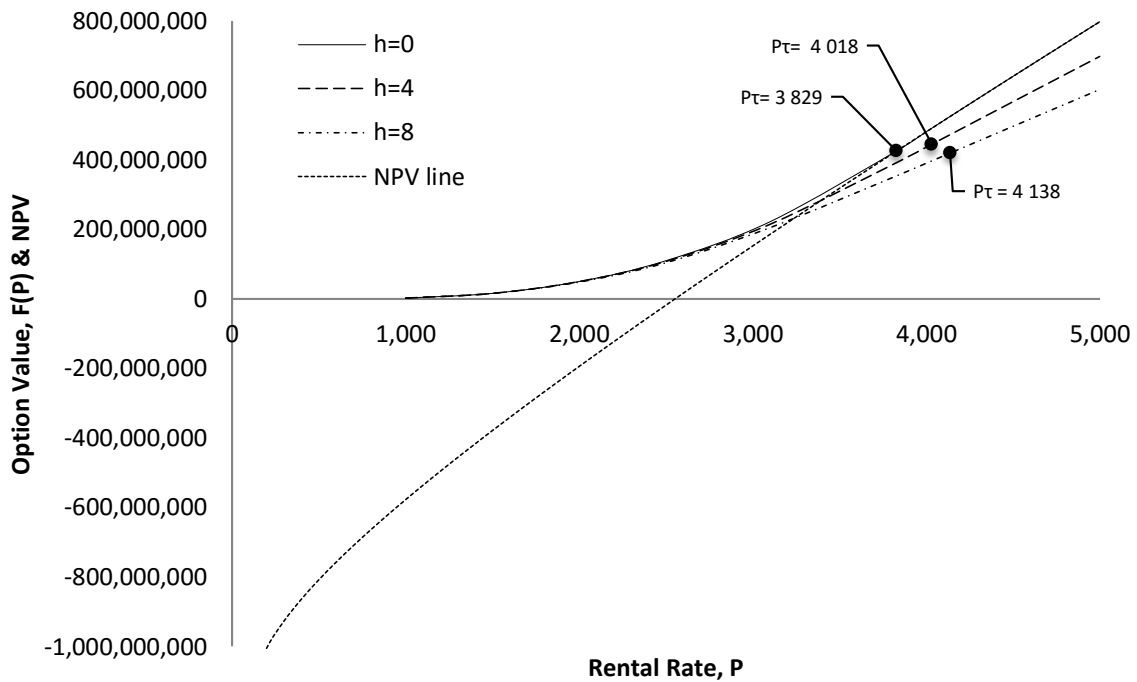


Figure 16: Option value for different values of P using the framework in Model 2.

In our base case, we include an approval lag of $h = 4$ years, and we obtain the optimal investment threshold, $P_\tau = 4\,018$, marked in Figure 16. Since $P < P_\tau$ in our base case, we obtain that $\Phi = 19,52\%$, which implies that there is a 19,52% probability that the market rental rate will reach the optimal investment threshold by the end of the approval period, in 4 years. If $P_h \geq P_\tau$ by the time of approval, the optimal decision for the firm would be to exercise the option immediately upon approval. On the other hand, we know that there is a $(1 - 19,52\%) = 80,48\%$ probability that the rental rate will not have reached the optimal threshold by the time the firm receives the approval. In that case, it would be optimal for the firm to delay the investment. Consequently, in our base case, the probability that the optimal decision at $t = h$, is to wait, is greater than the probability that the optimal decision upon approval is to exercise the option immediately. Notice that when the approval period becomes longer, i.e. h increases, the probability that $P_h \geq P_\tau$ at the end of the approval period increases. Intuitively, a longer approval period increases the probability of an increased rental rate because the expected drift in rental rates over time, μ , has a greater effect over long time periods. However, an increased approval period also suggests that the firm has to wait longer before they start receiving cash flow from the expanded building. Therefore, although increasing the time to approval increases the probability of the rental rate reaching the optimal investment threshold by the time of approval, an increased

approval period reflects negatively on the option value because the firm has to wait longer before they receive the cash flow. Intuitively, the firm does not mind waiting for approval as long as $P_h < P_\tau$. However, if P_t reaches P_τ before the project is approved, the firm would miss out on the cash flow it would have received in the absence of an approval lag. Thus the introduction of an approval lag causes a decrease in the value of the option to expand.

In our base case, we obtain the option value $F(P_h) = 320 \text{ MNOK}$ at $t = h$. By discounting this value back to $t = 0$, we obtain the present value of the option to expand the rental space of the building, $F(P) = 214 \text{ MNOK}$, which is 8 MNOK lower than the option value obtained for the same investment in the absence of an approval lag. We see that for any value of $h > 0$, the option value for investment with a fixed approval lag is lower than the option value in the absence of an approval lag. Notice also that option values decrease as h increases. There are several practical implications to this observation. Firstly, our model suggests that real estate investor should be willing to pay a premium for properties that are already regulated for expansion. Secondly, the result implies that the value of real estate development projects decreases as regulatory demands increases. Therefore, real estate investors should be concerned with the rate at which regulatory authority approve projects. Furthermore, our results show that policy makers should keep in mind the implications of longer regulatory processes.

Interestingly, notice that the difference in option value between Model 1 and Model 2 in our base case is less than 4%. Intuitively, the relatively small loss in option value in Model 2 due to the introduction of an approval lag can be explained by that as long as $P_h < P_\tau$, the firm will not exercise the option. Thus, when the firm receives the approval, they will not invest unless market conditions are favorable. Consequently, the loss in option value due to the approval lag is greater for high values of P than for low values of P because the probability that the market rental rate will reach the optimal investment threshold before the project is approved is high for high values of P . On the other hand, for low values of P , the probability that the rental rate will reach the optimal investment threshold before approval of the project is low.

Notice also that, whether or not the market rental rate will reach the investment threshold by the time of approval depends on the volatility in rental rates. Figure 17 below illustrates the optimal investment threshold as a function of volatility. As in Model 1, the optimal investment threshold in Model 2 increases with volatility.

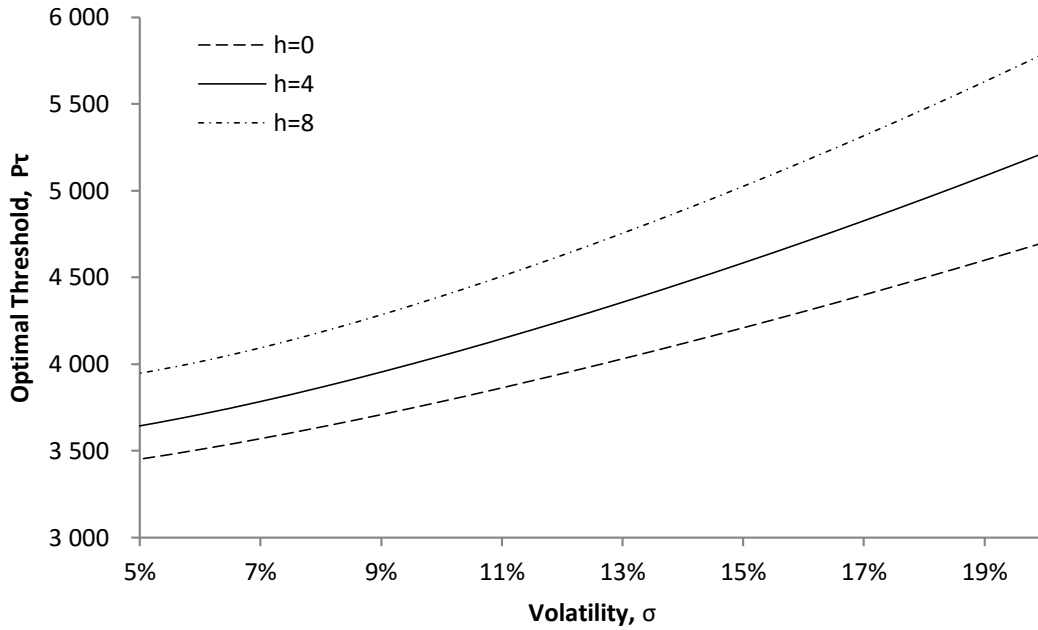


Figure 17: Threshold as a function of the volatility for different approval lags

Figure 17 illustrates that greater uncertainty in future rental rates increases optimal investment thresholds. Consequently, an increase in volatility favors a delay in investment in this model, and it also leads to a greater option value. In practice, this observation implies that real estate investors should be less willing to invest immediately in a development project when they are uncertain about future rental rates, than when they have good information.

6.4 Model 3: Investment under a stochastic approval lag

In Model 3, we extend the investment problem by assuming that the length of the approval period is no longer fixed, but rather determined by an exponentially distributed stochastic variable, as explained in section 5.3. Figure 18 illustrates option values and corresponding NPV's using our base care parameters for different levels of volatility in rental rates.

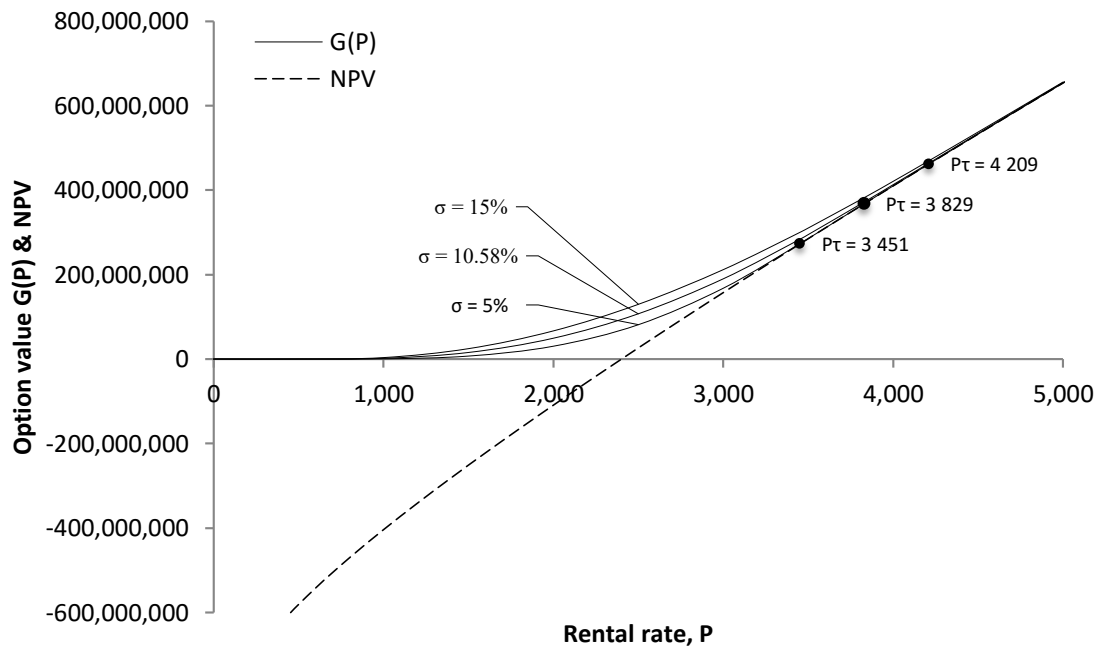


Figure 18: Option value for different values of σ using the framework in Model 3.

Notice that the base case optimal investment threshold in this model, $P_t = 3,829$ is equal to that of Model 1. That implies that the expected length of the approval period does not affect the optimal investment threshold or the firm's optimal decision. Intuitively, the optimal investment threshold represents the rental rate that maximizes the profitability of the project, and since the project itself is not affected by the approval period, the optimal investment threshold in the presence of a stochastic approval lag remains the same as in the absence of an approval lag. As expected, we see from Figure 18 that an increase in volatility leads to an increase in option values, and thus an increase in the optimal investment threshold.

Furthermore, notice that the interpretation of the NPV in this model is different from that of Model 1. In Model 1, the NPV is the value of the now-or-never exercise of the option, and since there is no approval lag in Model 1, the investment is available immediately. In Model 3 however, the firm cannot invest in the expansion until the project gets approved. Therefore, the NPV's plotted in Figure 18 should rather be interpreted as the value of having to commit to invest in the project immediately, before the project is approved. Upon the time of approval, the firm would then be obligated to pay the investment cost to receive the cash flow from the expanded property, regardless of the rental rate at that time. Hence, the NPV line illustrated in Figure 18 reflects the now-or-never value of committing immediately to invest at an unknown time in the future, in a project with an uncertain payoff. The notion

that the NPV line is slightly curved due to the varying decay factor, r , still holds in this model.

Notice that in our base case, the rental rate, $P = 3\,100$, is below the optimal investment threshold, $P_\tau = 3\,829$. Recall from equation (27) that when $P < P_\tau$ the value of the option to expand in the presence of a stochastic approval lag is equal to the value of the option to expand in the absence of an approval lag, adjusted for the possibility that the firm may have to wait for a long time before the project gets approved. With our base case input parameters, we obtain the option value, $F(P) = 209\text{ MNOK}$, which is lower than the option values for both Model 1 and Model 2. Intuitively, the implied cost to the firm of having to wait an uncertain amount of time for approval is $222\text{ MNOK} - 209\text{ MNOK} = 13\text{ MNOK}$. Recall that the implied cost to the firm when the approval period was fixed was 8 MNOK . Hence, the firm would prefer the case with a fixed approval lag to the case with a stochastic approval lag. In practice, this implies that real estate investors should be willing to pay a premium for information about the length of the approval period. Notice that this observation holds for any level of volatility, and any level of expected approval time. Figure 19 below illustrates option values and corresponding NPV's for different expected approval periods.

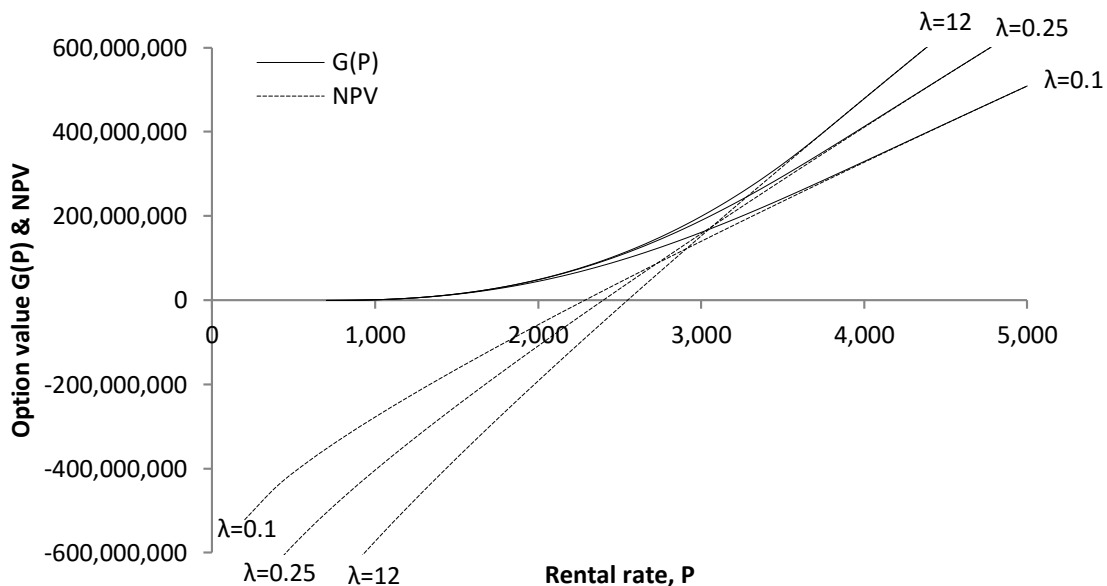


Figure 19: Option values and corresponding NPV for different values of λ .

Recall that the expected approval period is denoted, $\frac{1}{\lambda}$, and thus that the expected approval periods for $\lambda = 0.1$, $\lambda = 0.25$, and $\lambda = 12$ represent expected approval periods of 10 years, 4 years, and 1 month respectively. As expected, and increase in approval time, i.e. a decrease

in λ , leads to a lower option value. As $\lambda \rightarrow \infty$, the expected approval time goes towards 0, and the option values and NPV's in Figure 19 becomes equal to those of Model 1. Hence, the option values and NPV's for the case where $\lambda = 12$ in Figure 19 are very close to the values of Model 1.

The gap between the option value line and its corresponding NPV line in Figure 19 represents the difference between the value of holding the option and the value of committing to invest immediately. Hence, the gap can be interpreted as the value of the flexibility to delay the investment decision. If the firm had to make a decision immediately, the expected value of the investment would be given by the NPV line in Figure 19. However, when the firm has the flexibility to wait with the decision, the value of the option is illustrated by the option value line in Figure 19. Interestingly, the NPV line in Figure 19 will always be below the option value line. This implies that the optimal decision will never be to commit to the investment prior to approval of the project, if the firm has the option to wait with the decision.

6.4.1 Sensitivity to change in demand

When investing in real estate with potential options to expand, investors are concerned with the future demand for rental space at the market rental rate, P_t . Figure 20 shows option values for different levels of demand.

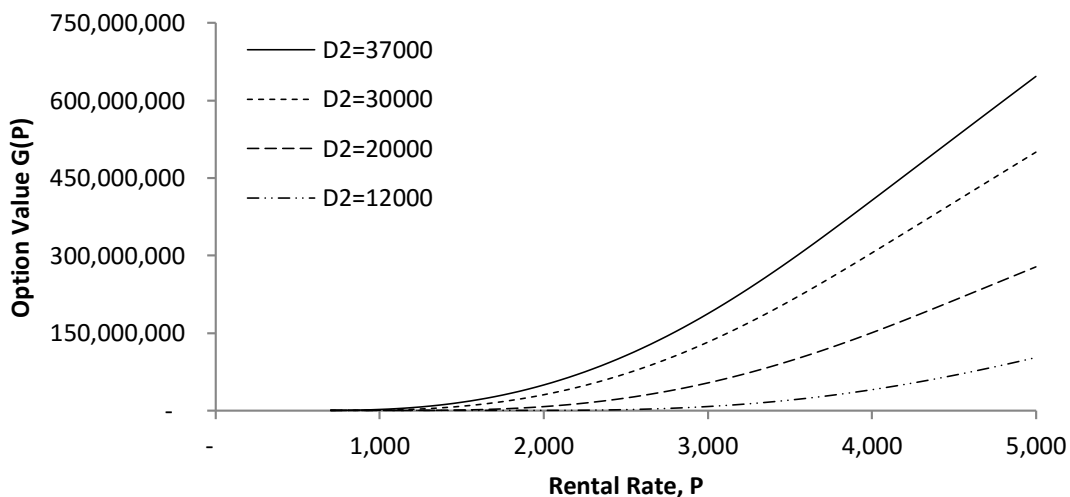


Figure 20: Option values at different levels of demand

As expected, an increase in demand for new rental space, leads to greater option values. Since the cash flow from the expanded property increases with D_2 , we also note that as long

as the investment cost does not increase disproportionately with the rental income, the optimal investment threshold, P_τ , decreases as D_2 increases. Interestingly, the option value stays significantly positive even, as $D_2 < D_1$. Intuitively, this can be explained by that the current property earns a rental income lower than the market rental rate for newly constructed properties. Therefore, by constructing a new building with a decreased rental space, but an increased rental rate, the firm can collect increased future cash flows. However, we see from Figure 20 that option values when $D_2 < D_1$, are much lower than when the firm has the option to expand, rather than simply increase rental rate per square meter. Similarly, when $D_2 = D_1$, the value of the option is also positive, and can be interpreted as the value of the option to renovate, rather than the option to expand. Notice however that the optimal investment threshold increases as D_2 , decreases, as shown in Figure 21.

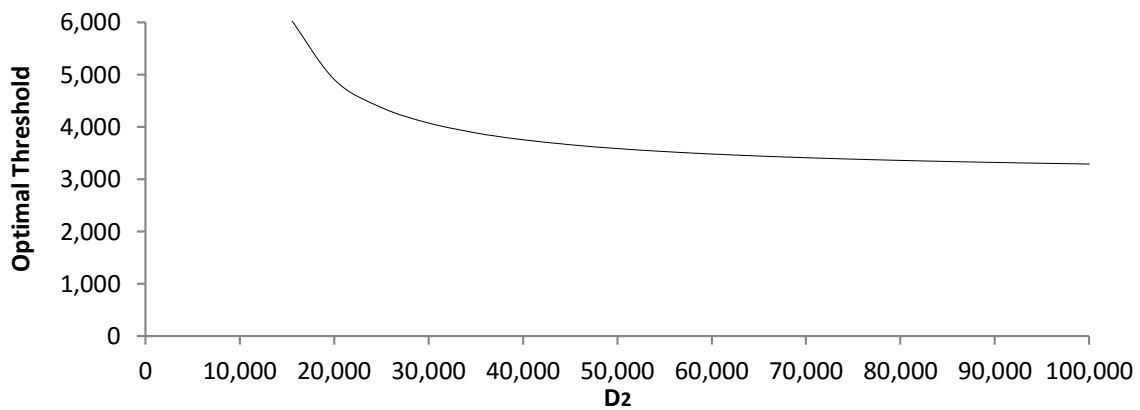


Figure 21: Optimal investment threshold at different levels of demand

We also note that the optimal investment threshold increases dramatically as D_2 approaches the existing rental space, $D_1 = 15\,000$. On the other hand, for higher values of D_2 , the optimal investment threshold flattens out, and goes towards, and eventually below, P . Thus, high demand in the space market of real estate speeds up investment in development projects, as reflected by a decrease in P_τ , while low levels of demand delays investment.

6.4.2 Sensitivity to change in investment cost

Figure 22 shows option values as a function of the rental rate for different values of the investment cost, I . The values of the investment cost is chosen around our base case scenario of $I = 740\,000\,000$. As expected, higher investment cost leads to lower option values. For our base case rental rate of $P = 3\,100$, a decrease in the investment cost down to $I = 540\,000\,000$ will increase the option value with 57%, from 209 MNOK to 327 MNOK.

On the other hand, an increase in investment costs to $I = 940\,000\,000$ will decrease the option value with 33%, from 209 MNOK to 139 MNOK.

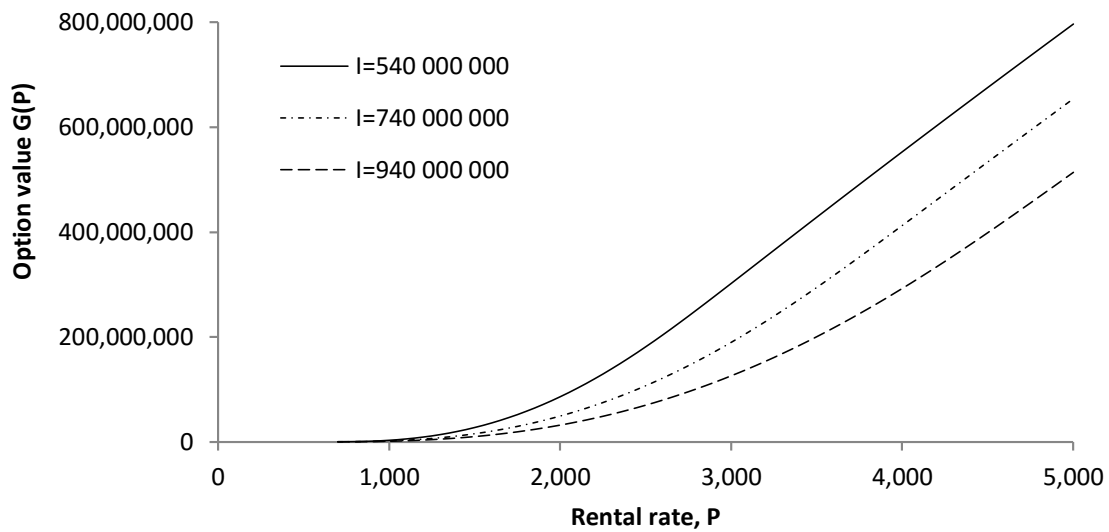


Figure 22: Option values for different investment costs

Interestingly, a decrease in investment cost leads to a greater change in option values than an increase in investment cost, and the relationship between the option value and the investment is thus nonlinear. Furthermore, notice that a 200 MNOK decrease in investment cost does not lead to an increase in the option value of the same amount, but rather a smaller amount. The same applies to a 200 MNOK increase in investment cost. This observation is mainly due to the discounting of the investment cost.

6.4.3 Sensitivity to change in construction time

Figure 23 illustrates option values as a function of the rental rate for different lengths of construction time, $(T - \tau)$. As expected, an increase in the construction time leads to a decrease in option values. There are mainly two reasons that can explain this decrease. Firstly, the future cash flows from the expansion will be discounted more when the construction time increases, as the cash flow will be received at a later time, thereby making the cash flow less valuable. Secondly, since the firm does not receive cash flow from the current property when construction is on-going, an increased construction period implies a longer period with no cash flow.

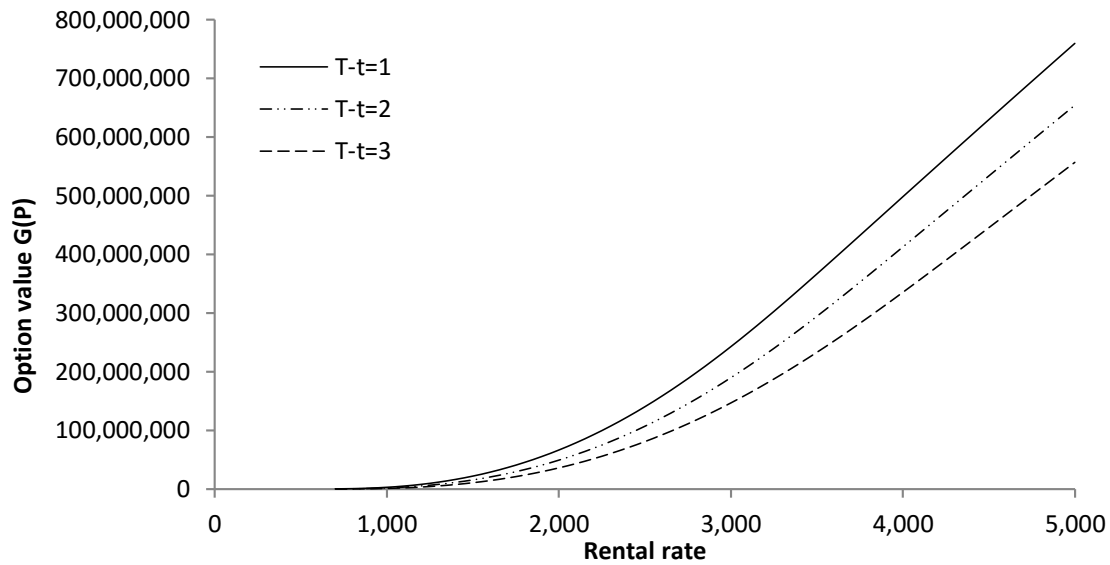


Figure 23: Option values for different construction times

Holding all other input parameters unchanged, a one-year decrease in construction time in our base case, will increase the option value by approximately 27%, to 266 MNOK, while a one-year increase in construction time from our base case will decrease the option value by approximately 22%, to 162 MNOK. Hence, an increase or a decrease in the length of the construction lag can severely impact the option price. Thus, the estimated length of the construction period is an important measure to determine accurately when investing in a development project.

Comparing the option value's sensitivity to the uncertain stochastic approval lag and the fixed construction lag, we observe from Figure 24 that the option value is considerably more sensitive to changes in the construction time.

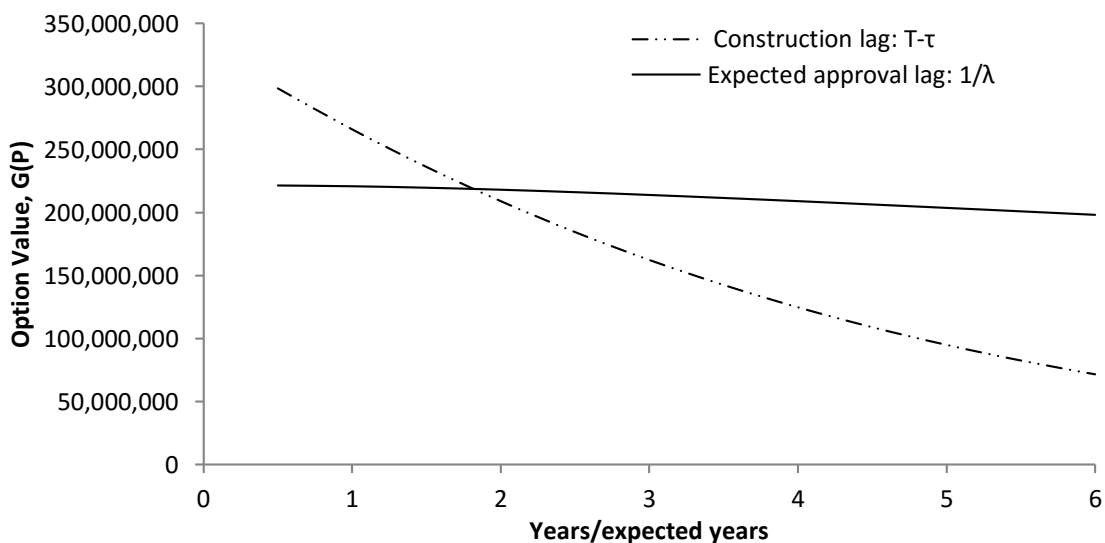


Figure 24: Option values for different construction times and expected approval lags

There are several reasons that explain why the option value is more sensitive to changes in the construction lag than to changes in the approval lag. Firstly, during the approval period the existing building still generates a cash flow, while during the construction period the existing building has been demolished and the cash flow during this period is therefore eliminated. Hence, the opportunity cost is greater with a long construction period than with a long approval period. Secondly, a longer approval period does not always intervene with the timing of the investment, since there is a probability that $P_h < P_\tau$ at the end of the approval period. If $P_h < P_\tau$ at the end of the approval period, it would be optimal to wait longer before investing in the expansion, even though the application is approved. On the other hand, the construction lag will always begin at the optimal time of investment, and a longer construction time will with certainty delay the cash flow from the expanded building.

6.4.4 Monte Carlo simulation of timeline

To estimate the amount of time it takes from first applying for approval of the project to the time of investment in the expansion in our base case, we have performed a Monte Carlo simulation with 10 000 simulations. The rental rate follows a Geometric Brownian Motion while the permit approval time follows a Poisson process, as stated in section 5. The simulation incorporates that if the rental rate reaches the optimal investment threshold, but the firm has not yet received approval of the project, the firm will have to wait for the approval before the expansion is made. On the other hand, if the firm has not yet received approval of the project, but the rental rate has reached the optimal investment threshold, the rental rate may still fall below the threshold during the approval period. Results of the simulation are shown in Figure 25.

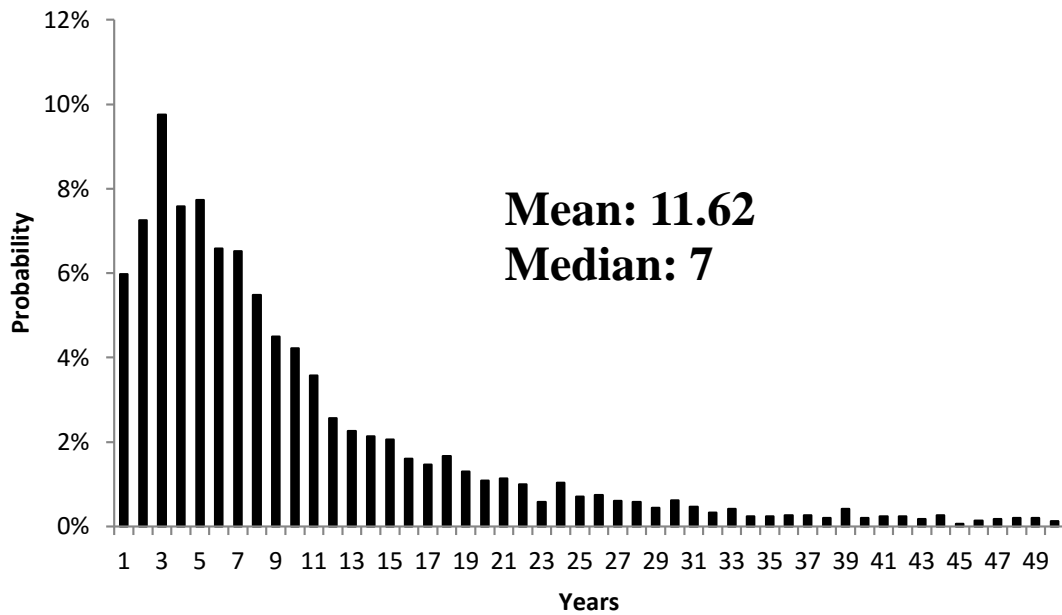


Figure 25: Probability distribution, 10 000 Monte Carlo simulations of time to investment

From the Monte Carlo simulation, we obtain a mean of 11.62 years and a median of 7 years. This tells us that the expected time from submitting the application of approval to the time when expansion is executed is almost twelve years. A median of 7 years tells us that the distribution is severely skewed, which we also observe from Figure 25. The simulation gave us several outliers and in some instances the time from applying for approval of the project to investment the expansion took up to a 100 years. However, the probability distribution illustrated in Figure 25 is in line with our expectations. Given the combination of the Poisson process with $\lambda = 0.25$ and the GBM with $\mu = 0.025$ and $\sigma = 0.1058$, we expect a probability distribution that illustrates a quite high probability that the firm should begin construction within 10 years, but there is also a probability that the firm has to wait for a longer time.

6.5 Comparison of Model 1, 2 and 3

In this section we compare the results from each of the three frameworks described in chapter 5. While each of the frameworks model slightly different scenarios, a comparison of the three models is helpful to understand the differences between them as well as to analyze some of the implications of the numerical results. Throughout this section, we use the base

case parameters, which includes $\lambda = 0.25$ and $h = 4$. Figure 26 illustrates base case option values for the three models and also marks the optimal investment threshold for each model.

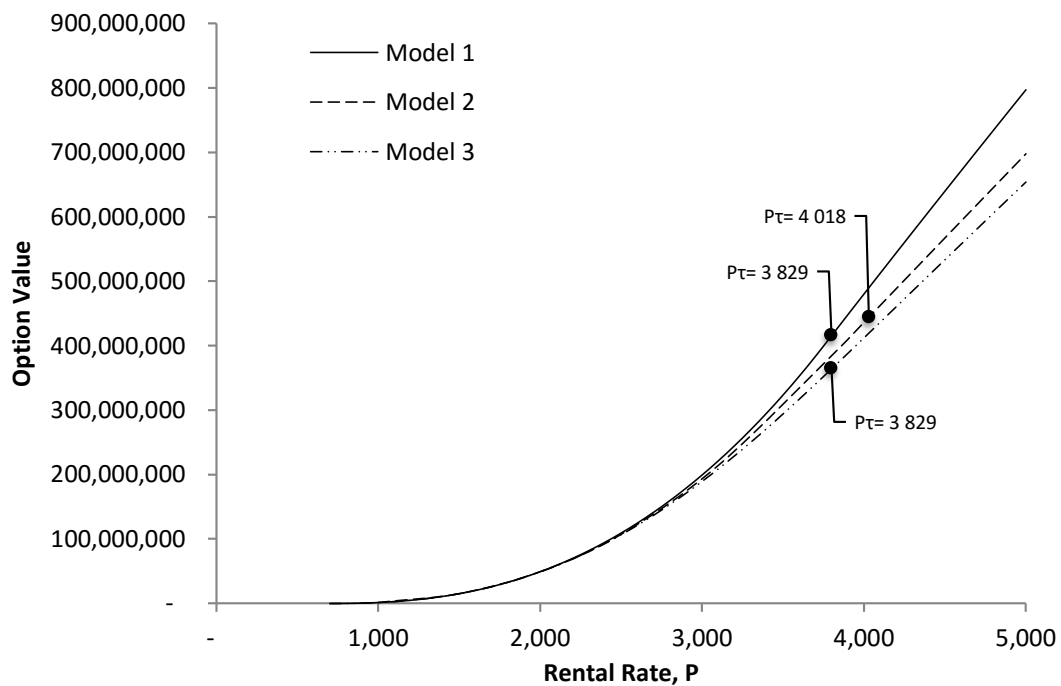


Figure 26: Comparison of the three models

Notice that for any rental rate, P , the option value from Model 1 is always greater than the option values from Model 2 and Model 3. Intuitively, the firm in Model 1 will always have the same opportunities to expand as the firms in Model 2 and Model 3. However, because the firm in Model 1 does not have to wait for approval of the project, they also have the opportunity to expand immediately. This additional flexibility adds to the value of the option, and thus the option value from Model 1 is always greater than that of Model 2 and Model 3 when $\frac{1}{\lambda} = h > 0$. However, since it is the absence of an approval period that causes the increased flexibility in Model 1, we observe that as $\lambda \rightarrow \infty$, and the expected approval time, $\frac{1}{\lambda}$, goes towards zero, the approval period in Model 3 will effectively be eliminated, and we obtain the same option values in Model 3 as in Model 1. Similarly, when $h \rightarrow 0$ in Model 2, the approval period is eliminated, and the option values from Model 1 and Model 2 will become the same. We can compare the three models to understand the implications of speedy regulatory processes, and the commercial value of information about the regulation process. In the absence of a regulatory authority, the value of real estate development

projects would be greater than in the presence of such authorities, and a long regulatory process reduces the value of an investment opportunity more than a short regulatory process.

Notice in Figure 26 that the optimal investment threshold for Model 3 is the same as that of Model 1, while the optimal investment threshold for Model 2 is different. Figure 27 illustrates optimal investment thresholds for the three models for different levels of volatility in rental rates.

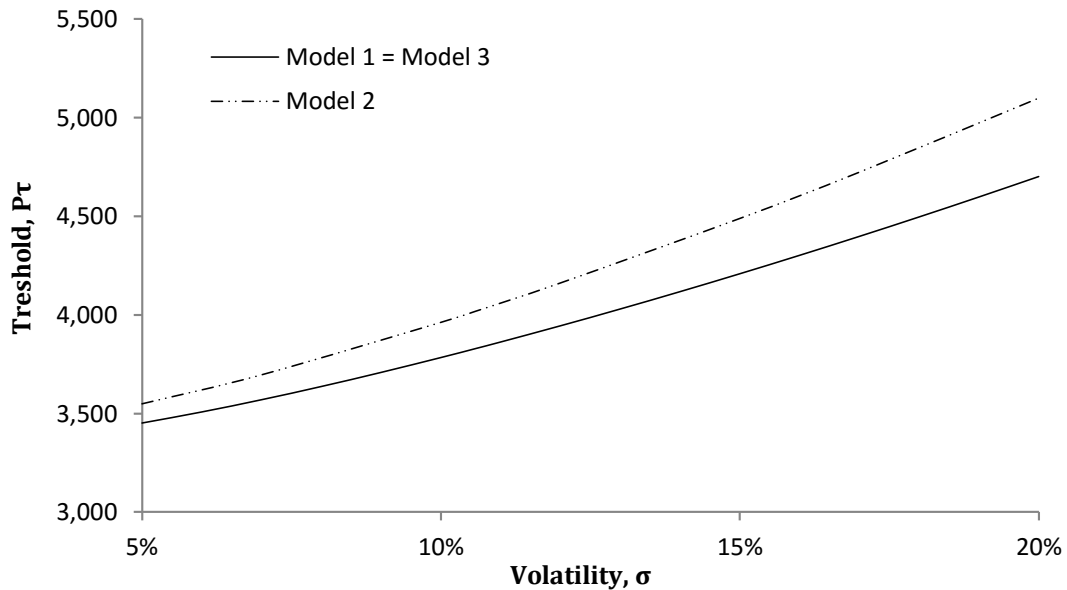


Figure 27: Optimal investment thresholds for different volatilities

Intuitively, the optimal investment threshold should be the same for each of the three models as the time to approval should not affect the optimal decision. In Model 1 and Model 3, we observe that the threshold remains constant, regardless of the expected approval time determined by λ . On the other hand, we observe that the optimal investment threshold in Model 2 is equal to that of Model 1 and Model 3 only when $h = 0$. When $h > 0$, the optimal investment threshold for Model 2 increases relative to the other models. The difference in optimal threshold between Model 2 and the other models can be explained by the analytical framework used in Model 2. Since the threshold in Model 2 is solved for numerically, rather than analytically, and the optimization problem includes the input parameter h , the optimal investment threshold depends on the length of the approval in Model 2. Hence, in contrast to in Model 1 and Model 3, the optimal investment threshold in Model 2 depends on the time to approval. Therefore, we should be mindful when interpreting the optimal threshold in Model 2, and cautious to draw any strict conclusions from the threshold. However, as Figure 27

illustrates, our numerical results show that the threshold in Model 2 is also increasing in volatility.

In our base case, Model 2 and Model 3 have the same expected approval lag, however, in Model 3, the length of the lag is uncertain. We observed in Figure 26 that Model 2 provides higher option values than Model 3 for any value of P . Intuitively, the uncertainty about the length of the approval time in Model 3 is not favorable for real estate investors, as they may have to wait longer than expected before receiving the option to expand. On the other hand, in Model 2, the property owner is certain that the approval period will last for $h = 4$ years, and does not face the risk that the lag could be longer. Therefore, uncertainty about the length of the approval lag negatively effects the option value and thus reduces the option value in Model 3. In practice, the implication is that information about the length of the regulatory process can increase the value of development projects in real estate. Figure 28 illustrates base case option values from Model 2 and Model 3 as a function of the expected length of the approval period. Base case option values for Model 1 is also included in the Figure as a reference point.

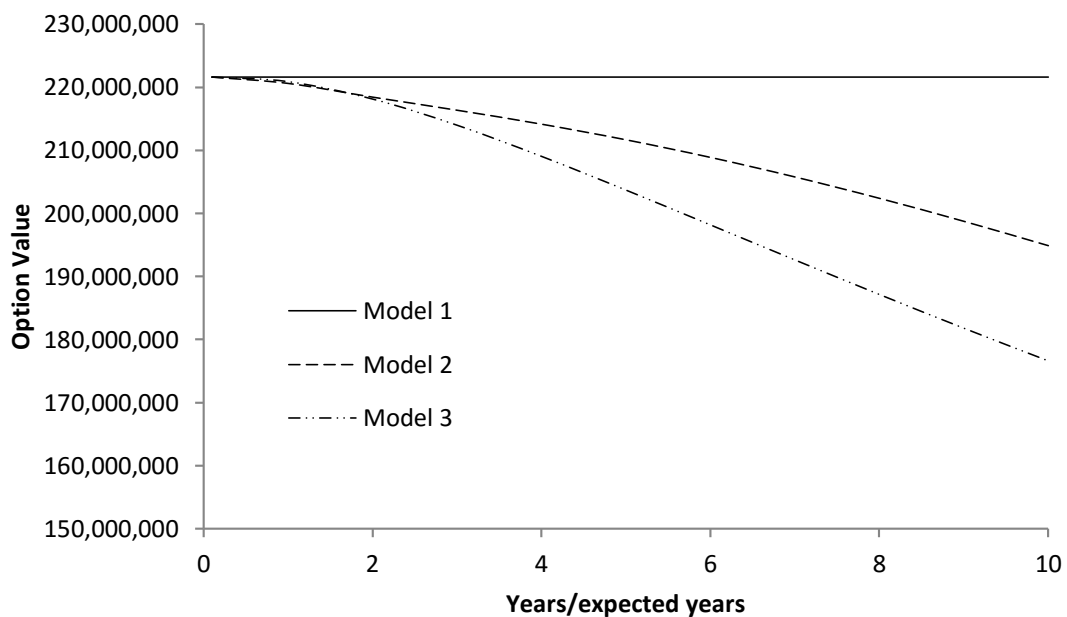


Figure 28: Option value as a function of approval period length/expected length

As expected, an increase in the length of the approval period leads to a decrease in option values for both Model 2 and Model 3. However, notice that Model 2 provides higher option values than Model 3 for any expected length of approval period. For low values of h , and

corresponding values of λ , the difference in option values between Model 2 and Model 3 are very small. Consequently, assuming large projects have longer expected approval periods than small projects, the value of information about the length of the approval period for small projects is very low compared to information about the length of the approval period for large projects. This observation has practical implications as the value of information about the length of the regulatory process becomes increasingly important as projects become larger. Hence, investors in large real estate projects should be willing to exhaust greater recourses in obtaining information about the regulatory process for a project than investors in smaller projects.

Interestingly, for our base case parameters, the introduction of a stochastic approval period decreases the option value by less than 6% compared to the model with investment in the absence of an approval lag. In other words, real estate development projects may have significant value even if investment in the project is not available for many years. Bahr-Ilan and Strange (1996) points out that even in a market with chronic excess capacity, such as the commercial real estate market, firms continue to invest in development projects. They suggest that because construction takes time, investors may choose to start development even under unfavorable conditions in order to avoid missing out if markets improve. A similar way of thought may be applied to our results. We observe that commercial real estate investors are willing to pay large premiums today for properties that may have potential for expansion or improvement in the future, even if such expansions are not currently profitable, or even available. Our results show that the introduction of an approval period does reduce option values, however, we observe that even projects with long expected approval periods have significant values if there is demand in the market. Hence our results support the claim that commercial real estate investors could be willing to pay a large premium today, for a project that may not be available for several years, in order to avoid missing out if the project should become available in the future. Additionally, because projects that are already approved are more valuable than projects that are subject to approval, speculative real estate investors may invest in non-approved projects today, in hopes of receiving approval of the project prior to the expected time of approval. In that way, such investors can acquire a building, receive approval of expansion before the expected approval time, and then sell the building before starting construction. They can thereby realize a profit, without actually investing in the expansion, but rather by simply obtaining the approval to expand.

7. Conclusion

We have developed an analytical framework for valuing options to expand the rental space of a commercial building, under uncertainty in both future rental rates and in the length of regulatory approval processes. Although accurate valuations are crucial for several participants in the commercial real estate industry, analytical formulations that consider potential options to improve the future cash flows of a property are limited both in theory and in practice. Therefore, to make the model as realistic as possible, we have put special emphasis on developing a practical model by considering input from experts in the industry. Our framework contributes to the accurate valuation of real estate as well as the real options literature related to real estate, by adjusting and combining existing frameworks, and applying them to the real estate industry.

For the model in the absence of an approval lag, we are able to obtain closed form analytical solutions for both the option value and the optimal investment threshold. We show that option values of a for can be significantly higher than the corresponding NPV, and that the flexibility to delay the investment decision is valuable. Extending the model to include a fixed approval lag, we show that the presence of a regulatory process prior to investment decreases option values, and that the loss in value increases with the length of the approval periods. By introducing uncertainty in the length of the approval period, we find that option values decrease further. Our analysis of the optimal investment threshold shows that uncertainty in future rental rates favors a delay in investment, regardless of the length of the approval period. We also find that when rental rates are low, the introduction of an approval period is less significant than for high rental rates.

There are several potential extensions to our framework that could be interesting for further research. The introduction of a stochastic investment cost could provide further insight to the dynamics of the industry, and also help the model become more realistic. One could argue that rental rates may be more predictable than a “random walk”. Therefore, an interesting alternation to our framework could be to replace the GBM with another stochastic process, such as a mean reverting process, and that way relax the limitations associated with the GBM. It would also be interesting to extend the model to include the risk preferences of the investor, and thus perhaps alter the preference for a fixed approval lag.

References

- Bar-Ilan, A. & Strange, W. C. (1996). Investment Lags. *American Economic Review*, 86(3), 610-622.
- Basale. (2015). Investing in Norway. *Handbook for the Norwegian commercial Real Estate Market*. Retrieved from <http://www.basale.no/images/docs/Handbook.pdf>
- Bodie, Z. (2014). *Investments* (10th global ed. utg.). Berkshire: McGraw-Hill Education.
- Capozza, D. R. & Helsley, R. W. (1989). The fundamentals of land prices and urban growth. *Journal of Urban Economics*, 26(3), 295-306.
- Choi, J. W. (2011). Real Options Models in Real Estate. *Electronic Thesis and Dissertation Repository. Paper 355*
- Crosby, N., Devaney, S., & Law, V. (2012). Rental Depreciation and Capital Expenditure in the UK Commercial Real Estate Market, 1993-2009. *Journal of Property Research*, 29(3), 227-246.
- Dixit, A. K. & Pindyck, R. S. (1994). *Investment under uncertainty*. Princeton, N.J: Princeton University Press.
- Damodaran, A. (2005). The Promise and Peril of Real Options. *NYU Working Paper No. S-DRP-05-02*
- Dunlap, N. A. C. P. M. (2012). Why invest in commercial real estate? *Journal of Property Management*, 77(1), 26.
- Akershus Eiendom. (2015). The Norwegian Commercial Property Market.
- Etheridge, A. (2002). *Course in Financial Calculus*. Cambridge: University Press.
- Fearnley, P. F. (2015). Eiendomsrapport 2015.
- Geltner, D. & Miller, N. (2007). *Commercial Real Estate Analysis and Investment*. I J. Clayton & P. Eicholtz (Red.).
- Henderson, V. (2007). Valuing the option to invest in an incomplete market. *Mathematics and Financial Economics*, 1, 2, July 2007, 103-128
- Heydari, S. & Siddiqui, A. (2009). Real Options Analysis of Multiple-Exercise Interruptible Load Contracts. *Conference on energy economics and technology*.
- Hudson-Wilson, S., Gordon, J. N., Fabozzi, F. J., Anson, M. J. P. & Giliberto, S. M. (2005). Why Real Estate? And How? Where? And When? *Journal of Portfolio Management*, 12-22.
- Leishman, C., Costello, G. (2000). Using Option Pricing Theory to Value Development Land
- McDonald, R. & Siegel, D. (1986). The value of waiting to invest. *Quarterly Journal of Economics*, 101, 707.
- Miltersen, K. R. & Schwartz, E. S. (2007). Real Options With Uncertain Maturity and Competition. *National Bureau of Economic Research*.
- Mun, J. (2006). A Comparison Between Real Options and Traditional Analysis in Layman's Terms
- NBIM. (2007). Inclusion of real estate investments in the Government Pension Fund - Global
- DNB, Næringsmegling (2015). Markedsappport (s. 4-5): DNB.
- OECD. (2014). Annual Survey of Large Pension Funds and Public Pension Reserve Funds (nr. 3). Retrieved from http://www.oecd.org/daf/fin/private-pensions/2014_Large_Pension_Funds_Survey.pdf:
- Propex, P. (2003). Highest and best use in Real Estate. Retrieved from http://www.propex.com/C_g_hbu0.htm

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- Riddiough, T. J. (1997). The Economic Consequences of Regulatory Taking Risk on Land Value and Development Activity. *Journal of Urban Economics*, 41(1), 56-77.
- Rogers, J. (2013). Strategy, Value and Risk: A Guide to Advanced Financial Management (3. p.202).
- Sunding, D., Zilberman, D., Liu, X. & Roland-Holst, D. (2004). The economics of climate change in agriculture. *An International Journal Devoted to Scientific, Engineering, Socio-Economic and Policy Responses to Environmental Change*, 9(4), 365-382
- Titman, S. (1985). Urban Land Prices Under Uncertainty. *The American Economic Review*, 75(3), 505-514.
- Weil, H. (2015). Institutional Real Estate Allocation Monitor. Retrieved from http://baker.realestate.cornell.edu/EE/images/uploads/publication/2014_Alloc_Monitor.pdf

8. Appendices

8.1 Appendix A:

As explained in chapter 4, the parameter, r , is a decay factor that captures the fact that rental rates tend to be higher for new buildings than for old buildings.

$$C_0 = P_0 e^{-rM}, \quad r = -\frac{\ln\left(\frac{P_c}{P_0}\right)}{M} \quad (\text{A-1})$$

Equation (A-1) above denotes r , and Figure A-1 below shows the isolated effect of the decay factor on the rental rate of buildings at different ages.

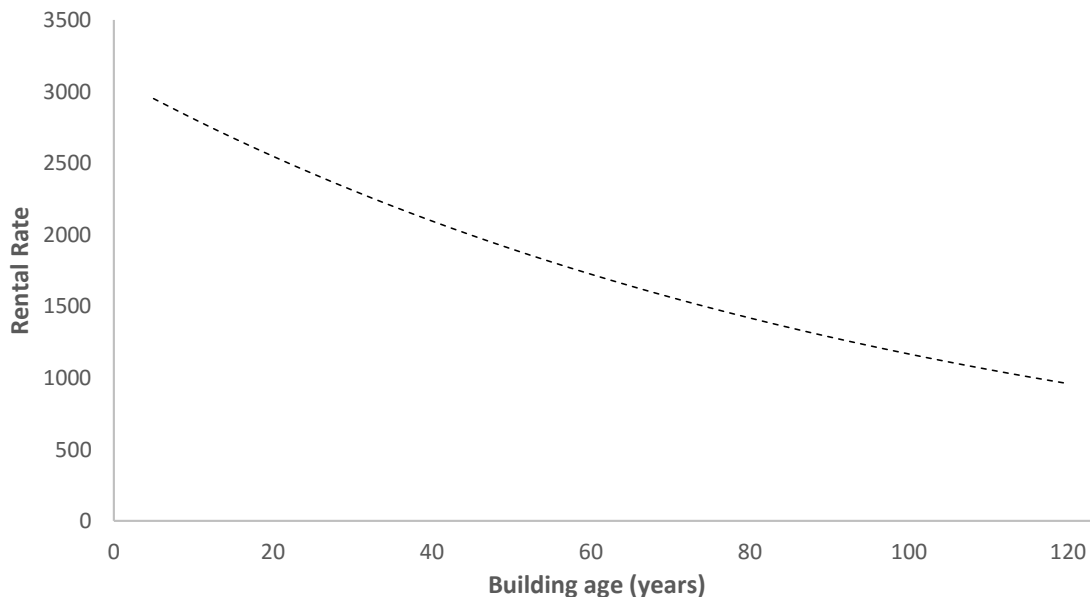


Figure A-1: Rental rate decay over time caused by parameter, r

Notice that since the decay factor, r , depends on the market rental rate for new rental space, P , the decay factor is highly sensitive to current market conditions which are reflected in P . Although equation (A-1) makes the realistic assumption that rental rates decay at an average rate per year, the actual amount by which rental rates decay are likely to fluctuate from year to year. Crosby (2012) suggests a similar approach as we do in (A-1) to estimate the decay factor, however, he compares the rental rate of the existing building to a benchmark portfolio rather than simply the market rate. This approach ensures that the existing buildings rental rate is compared to its most relevant peers, however, it requires large amounts of data that may be difficult to obtain.

8.2 Appendix B:

We have explained that the value of the option in section 5.1 is denoted by (B-1) as shown below.

$$F(P) = \begin{cases} A_1 P^{\beta_1}, & \text{if } P < P_\tau \\ aP - I, & \text{if } P \geq P_\tau \end{cases} \quad (\text{B-1})$$

So long as the rental rate, P , is greater than or equal to the optimal investment threshold, P_τ , the value of $F(P)$ is given by (B-2),

$$F(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2} \quad (\text{B-2})$$

where β_1 and β_2 denotes the positive and the negative roots of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - \rho = 0$ respectively. We know that as that as $P \rightarrow 0$ the value of the project should become very small. However, because $\beta_2 < 0$, we observe that $A_2 P^{\beta_2} \rightarrow \infty$ as $P \rightarrow 0$. Therefore, to make the function economically sound, we eliminate the second term of the right hand side as of (B-2), and obtain (B-1).

8.3 Appendix C:

To find the optimal investment threshold and prove that it is indeed optimal we show that the first and second order necessary condition is satisfied. We begin with equation (C-1), which is equation (10) from section 5.1.

$$\begin{aligned} F(P) &= \max_{P_\tau \geq P} \left(\frac{P}{P_\tau} \right)^{\beta_1} V(P_\tau) \\ F(P) &= \max_{P_\tau \geq P} \left(\frac{P}{P_\tau} \right)^{\beta_1} (aP_\tau - I) \end{aligned} \quad (\text{C-1})$$

To maximize equation (C-1) we apply the first order necessary condition i.e. $F'(P) = 0$. This is shown in equation (C-2) and as expected we get the same threshold as when applying the smooth pasting and value matching conditions.

$$\begin{aligned}
F'(P) &= \beta_1 \left(\frac{P}{P_\tau} \right)^{\beta_1-1} (-1) \left(\frac{P}{P_\tau^2} \right) (aP_\tau - I) + \left(\frac{P}{P_\tau} \right)^{\beta_1} a = 0 \\
F'(P) &= \beta_1 \left(\frac{P}{P_\tau} \right)^{\beta_1} \left(-\frac{1}{P_\tau} \right) (aP_\tau - I) + \left(\frac{P}{P_\tau} \right)^{\beta_1} a = 0 \\
&\implies P_\tau = \frac{\beta_1}{\beta_1 - 1} \frac{I}{a}
\end{aligned} \tag{C-2}$$

We continue with $F'(P)$ from the second line in (C-2) and obtain the simplified equation (C-3):

$$\begin{aligned}
F'(P) &= \beta_1 \left(\frac{P}{P_\tau} \right)^{\beta_1} \left(-\frac{1}{P_\tau} \right) (aP_\tau - I) + \left(\frac{P}{P_\tau} \right)^{\beta_1} a \\
&= \left(\frac{P}{P_\tau} \right)^{\beta_1} \left(\frac{\beta_1 I}{P_\tau} - a(\beta_1 - 1) \right)
\end{aligned} \tag{C-3}$$

We differentiate (C-3) further to observe if the second order necessary condition is negative, i.e. $F''(P) < 0$.

$$\begin{aligned}
F''(P) &= \beta_1 \left(\frac{P}{P_\tau} \right)^{\beta_1-1} \left(-\frac{P}{P_\tau^2} \right) \left(\frac{\beta_1 I}{P_\tau} - a(\beta_1 - 1) \right) + \left(\frac{P}{P_\tau} \right)^{\beta_1} \left(-\frac{\beta_1 I}{P_\tau^2} \right) \\
&= \beta_1 \left(\frac{P}{P_\tau} \right)^{\beta_1} \left(-\frac{\beta_1}{P_\tau^2} \right) (\beta_1 I - aP_\tau(\beta_1 - 1) + I)
\end{aligned} \tag{C-4}$$

By inserting $P_\tau = \frac{\beta}{\beta-1} \frac{I}{a}$ into the last bracket of (C-4) we obtain a simplified expression for $F''(P)$ as shown in (C-5)

$$\begin{aligned}
F''(P) &= \beta_1 \left(\frac{P}{P_\tau} \right)^{\beta_1} \left(-\frac{\beta_1}{P_\tau^2} \right) \left(\beta_1 I - a \frac{\beta_1}{\beta_1 - 1} \frac{I}{a} (\beta_1 - 1) + I \right) \\
&= \beta_1 \left(\frac{P}{P_\tau} \right)^{\beta_1} \left(-\frac{\beta_1}{P_\tau^2} \right) I
\end{aligned} \tag{C-5}$$

From equation (C-5) it is easy to observe that $F''(P) < 0$, and that the second order necessary condition therefore is satisfied.

8.4 Appendix D:

Optimal threshold:

We can rewrite P_τ from equation (9) in Section 5.1 as shows in (D-1) below.

$$\begin{aligned} P_\tau &= \frac{\beta}{(\beta - 1)} \frac{I}{a} \\ &= \beta(\beta - 1)^{-1} \frac{I}{a} \end{aligned} \quad (\text{D-1})$$

To find the derivative of P_τ with respect to σ we first find the derivative of β .

$$\begin{aligned} \frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - \rho &= 0 \\ \sigma\beta(\beta - 1) + \frac{1}{2}\sigma^2(\beta'(\beta - 1) + \beta\beta') + \mu\beta &= 0 \end{aligned} \quad (\text{D-2})$$

Equation (D-2) is the derivation of the quadratic equation applied in Section 5.1 to obtain β_1 .

In (D-3) below, we simplify and solve for β' . Since $\beta > 1$ we have that $\beta' < 0$.

$$\begin{aligned} -\sigma\beta(\beta - 1) &= \beta' \left(\frac{1}{2}\sigma^2(\beta - 1) + \frac{1}{2}\sigma^2\beta + \mu \right) \\ \beta' &= \frac{-\sigma\beta(\beta - 1)}{\frac{1}{2}\sigma^2(\beta - 1) + \frac{1}{2}\sigma^2\beta + \mu} \end{aligned} \quad (\text{D-3})$$

Next, we find the derivative of P_τ with respect to σ as indicated in (D-4).

$$\begin{aligned} \frac{\partial P_\tau}{\partial \sigma} &= (\beta'(\beta - 1)^{-1} + \beta(-1)(\beta - 1)^{-2}\beta') \frac{I}{a} \\ &= \frac{\beta'(\beta - 1) - \beta'\beta}{(\beta - 1)^2} \frac{I}{a} \\ &= \frac{-\beta'}{(\beta - 1)^2} \frac{I}{a} \end{aligned} \quad (\text{D-4})$$

Since β' is negative, we can observe from equation (D-4) that $\frac{\partial P_\tau}{\partial \sigma} > 0$

Optimal Value function

To show that the value function, F , from equation (10) in section 5.1 is strictly increasing with respect to the volatility, we differentiate the function with respect to the volatility. We will show that $\frac{\partial F}{\partial \sigma} > 0$. By differentiating equation (10) we obtain equation (D-5) below:

$$\begin{aligned}\frac{\partial F}{\partial \sigma} &= \beta \beta' \left(\frac{P}{P_\tau} \right)^{\beta-1} \left(-\frac{P}{P^2} \frac{\partial P_\tau}{\partial \sigma} \right) (aP_\tau - I) + \left(\frac{P}{P_\tau} \right)^\beta a \frac{\partial P_\tau}{\partial \sigma} \\ &= \left(\frac{P}{P_\tau} \right)^\beta \frac{\partial P_\tau}{\partial \sigma} \frac{1}{P_\tau} \left(P_\tau a (1 - \beta \beta') + \beta \beta' I \right)\end{aligned}\quad (\text{D-5})$$

By inserting $P_\tau = \frac{\beta}{\beta-1} \frac{I}{a}$ into the last bracket in equation (D-5) and simplifying, we obtain equation (D-6):

$$\begin{aligned}\frac{\partial F}{\partial \sigma} &= \left(\frac{P}{P_\tau} \right)^\beta \frac{\partial P_\tau}{\partial \sigma} \frac{1}{P_\tau} \left(\frac{\beta}{\beta-1} \frac{I}{a} a (1 - \beta \beta') + \beta \beta' I \right) \\ &= \left(\frac{P}{P_\tau} \right)^\beta \frac{\partial P_\tau}{\partial \sigma} \frac{1}{P_\tau} \frac{\beta}{\beta-1} I (1 - \beta')\end{aligned}\quad (\text{D-6})$$

Each term in equation (D-6) is strictly positive, and we have therefore shown that $\frac{\partial F}{\partial \sigma} > 0$.

8.5 Appendix E:

Equation (19) from section 5.2 is denoted below as equation (E-1). We will show that when the approval period, h , goes towards zero, we will get the same threshold as in Model 1.

$$\begin{aligned}0 &= \beta_1 \left(\frac{P}{P_\tau} \right)^{\beta_1} \left(-\frac{1}{P_\tau} \right) e^{\gamma h} [aP_\tau - I] \times \mathbf{P}_P[P_h < P_\tau] \\ &\quad + \left(\frac{P}{P_\tau} \right)^{\beta_1} a e^{\gamma h} \times \mathbf{P}_P[P_h < P_\tau] \\ &\quad + \left(\frac{P}{P_\tau} \right)^{\beta_1} e^{\gamma h} [aP_\tau - I] \times \phi(R(h, P, P_\tau)) \frac{1}{P_\tau \sigma \sqrt{h}} \\ &\quad - [aP e^{\mu h} - I] \times \phi(R(h, P, P_\tau)) \frac{1}{P_\tau \sigma \sqrt{h}}\end{aligned}\quad (\text{E-1})$$

The reason for why we get the same threshold in Model 2 as in Model 1 when h goes towards zero, can be explained by noticing that the expression for R , as denoted in (E-2), will go towards minus infinity when h goes towards 0.

$$R(h, P, P_\tau) = \left(\frac{(\mu - \frac{1}{2}\sigma^2)h - \ln(\frac{P_\tau}{P})}{\sigma\sqrt{h}} \right) \quad (\text{E-2})$$

Further we notice that the expression for ϕ , denoted in (E-3) goes towards 0, when R goes towards minus infinity.

$$\phi(R(h, P, P_\tau)) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{R^2}{2}} \right) \quad (\text{E-3})$$

Noticing that $P_p[P_h < P_\tau] = [1 - \Phi(R(h, P, P_\tau))]$ goes towards 1 when R goes towards minus infinity (because $\Phi(R(h, P, P_\tau))$ goes towards 0). We can substitute this into equation (E-1) and get to equation (E-4).

$$\begin{aligned} 0 = & \beta_1 \left(\frac{P}{P_\tau} \right)^{\beta_1} \left(-\frac{1}{P_\tau} \right) [aP_\tau - I] \times 1 \\ & + \left(\frac{P}{P_\tau} \right)^{\beta_1} a \times 1 \\ & + 0 \\ & - 0 \end{aligned} \quad (\text{E-4})$$

Simplifying (E-4) further we get equation (E-5) and as we can see, this will give us the same threshold as Model 1, since equation (E-5) is the same as equation (C-3) from Appendix C, which gave us the same threshold.

$$0 = \beta_1 \left(\frac{P}{P_\tau} \right)^{\beta_1} \left(-\frac{1}{P_\tau} \right) (aP_\tau - I) + \left(\frac{P}{P_\tau} \right)^{\beta_1} a \quad (\text{E-5})$$

8.6 Appendix F:

Equation (F-1) and (F-2) are determined from the value matching and smooth pasting conditions of equation (28) and (29) in section 5.3. Because $A_0P^{\omega_1}$ adjusts the option value, when $P < P_\tau$, for the possibility that the approval time might take longer than expected, $A_0 < 0$. On the other hand, BP^{ω_2} adjusts the option value, when $P \geq P_\tau$, for the possibility that the rental rate will drop below the optimal investment threshold. Thus $B > 0$.

$$A_0 = \frac{P_\tau^{-\omega_1}}{\omega_2 - \omega_1} \left[\frac{\lambda(\omega_2 - 1)aP_\tau}{\rho + \lambda - \mu} + (\beta_1 - \omega_2)A_1P_\tau^{\beta_1} - \frac{\omega_2\lambda I}{\rho + \lambda} \right] \quad (\text{F-1})$$

$$B = \frac{P_\tau^{-\omega_2}}{\omega_1 - \omega_2} \left[\frac{\lambda(1 - \omega_1)aP_\tau}{\rho + \lambda - \mu} + (\omega_1 - \beta_1)A_1P_\tau^{\beta_1} + \frac{\omega_1\lambda I}{\rho + \lambda} \right] \quad (\text{F-2})$$