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Uncertainty in Real Estate Development

A Real Options Framework

Kevin Alexander Nilssen Blytt

Håvard Vabø

Supervisor: Astrid Kunze

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Abstract

We develop a real options framework to facilitate optimal decision making and valuation for local real estate development projects in Bergen. With uncertain time to completion, the investor must continuously trade off the potential benefits from continuing investment versus the benefits from being revealed of the remaining investment costs. To depict the development process in Bergen, we allow for the investor to temporarily abandon or to abandon for salvage value to decide an optimal investment strategy and to obtain an accurate valuation of a real estate development project.

Preface

This thesis is written as a concluding part of the Master of Science in Economics and Business Administration at the Norwegian School of Economics.

We would like to thank our supervisor Astrid Kunze for pushing us to do our best. We would also like to thank Thor Erik Blytt of Synapsit AS and Nyskapingsparken for providing us with a great working environment. Additionally, we would like to direct gratitude to Per Jæger of Boligprodusentenes Forening for introducing us to Anita Nysæther Kristiansen. And of course to Anita Nysæther Kristiansen at Backer Bolig AS for helpful insights into the problems and drivers in the regulatory process and for numerical input data to our case. At last, Eivind Gamst must be mentioned for his excellent assistance on Matlab programming.

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1. Introduction

The delivery of a finished housing project to market is a complicated and risky task. Idea generation, finding of a suitable property, and output prices driven by a complex set of factors are amongst the challenges a developer encounters. On top of that, the owner of a property must account for uncertainties regarding the permitted utilization rate, the time it takes to grant an approval to develop – and how much it will cost to get her there. These latter aspects are all related to uncertainty in the introductory stages of a housing project, where the public – and private sector work side by side to draw up a viable plan for the best utilization of the property.

Uncertain planning processes is a much-debated theme in Norwegian real estate development literature and national newspapers alike. The academic literature focuses on the processual challenges of development[1], the co-operation between the private- and public sector when objectives can differ[2], and outlines risk factors in real estate development[3].

The Norwegian planning system is in large part driven by private initiatives, where developers take into account the needs and trends of the market, procure property, do the detailed zoning, build – and sell the finished project. This is the underlying system in which a developer operates, which has been baptized *"market housing"* (Nordahl, 2011). Having procured a property, a profit maximizing developer must maneuverer through these uncertain waters and develop a strategy that will maximize the potential of the investment. During the approval process, the developer must continuously trade off the potential benefits from having the opportunity to reach approval versus the benefits from being revealed from the remaining ongoing investment costs. The many uncertainties working simultaneously makes it difficult to accurately manage the trade-off without reaching suboptimal decisions.

To accurately value a development project and determine value-maximizing behaviour, an analyst must develop a framework where relevant uncertainties and characteristics of the investment decision are taken into account. A popular method to determine investment

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[1] (Nordahl, Barlindhaug, & Ruud, 2007); (Barlindhaug, Holm, & Nordahl, 2014) (Nordveit, 2015)

[2] (Nordahl, Barlindhaug, & Ruud, 2007)

[3] (Nordahl, 2012); (Barlindhaug & Nordahl, 2005)

opportunities is to discount expected net benefits and invest immediately if expected benefits exceed costs. Despite intuitive traits, several scholars point to the shortcomings of the static net present value approach to guide decision making under uncertainty[4]. An emerging strand of research has borrowed from derivatives theory to view real investments similar as financial options[5]. By altering the view on the dynamics of real investments, the approach incorporates managerial flexibilities under uncertainty.

One of the main uncertainties in this thesis is the expected time to completion of the regulatory process and whether the event of approval will happen or not. To gain insight, we have collected data from previous planning decisions to enhance the understanding of the flow in the decision making process. We apply these findings to obtain accurate input measures on expected time to completion and to determine if two managerial real options can be of value in the regulatory process. We find that the options to temporarily abandon, and to abandon for salvage value, to be valuable and should be incorporated into an optimal decision making strategy in this context.

Miltersen and Schwartz (2007) proposes a framework for valuation and optimal decision making when expected time to completion is uncertain. Originally developed for R&D investments, we apply and adjust their framework to determine optimal decision making under uncertain time to regulatory approval. Under a stochastically evolving price process and uncertain ongoing investment costs, our aim is to develop a framework for optimal decision making and valuation for real estate development projects located in Bergen. This can add to the existing Norwegian real estate literature and enhance the financial aspects of this strand of research. By presenting an understandable and accessible framework, an additional aim of this thesis is to provide practitioners with a helpful tool to optimize decision making. By changing critical input parameters, we test the implications of recent policy suggestions that has the potential to change operating conditions.

The rest of this thesis is structured as follows; Chapter two introduces the investment problem from the developer's perspective in the housing construction industry. That includes both a

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[4] McDonald and Siegel (1986), Pindyck (1991), Dixit and Pindyck (1994) were amongst the first to acknowledge this

[5] The first approach was performed by Myers (1977). Since then, seminal approaches such as Brennan and Schwartz (1985), McDonald and Siegel (1986), and Dixit and Pindyck (1994) have brought the theory forward.

description of the real estate market and the decision process in real estate development. Chapter three presents the theory behind the investment framework. Chapter four presents survival analysis data from previous planning decisions. Chapters five gives a quick presentation of the assumptions and notations of the models, while chapter six presents the switch and abandonment models. Chapter seven presents the numerical case before the simulation analyses takes place in chapter eight based on our Matlab outputs. In chapter nine we conclude.

2. Norwegian Real Estate Market

The role of real estate as an asset class has changed dramatically in the post WW2-period. Coming out of the world war, the "freehold democracy" was championed politically. Through means such as subsidized mortgages and beneficiary tax schemes for housing, the goal was that Norwegian households should own their own home (Lundesgaard & Røisland, 2012). Regulating sales prices for cooperative housing through a full-cost recovery scheme ensured affordability and accessibility for first time buyers (Sørvoll, 2010).

Coming into the 1980s, the market was split in two; a deregulated market with strong price appreciation, and a regulated market where prices moved slowly (Sørvoll, 2010). Additionally, a growing economy increased welfare and inflation, hence adding to the gap between willingness to pay and prices regulated through a historical full-cost principle (Kiøsterød, 2005). Throughout the 1980s, politicians acknowledge the need to bring the markets together to reduce the gap. After the deregulation, housing prices rose significantly (Nordahl, 2012). This again led to high inflation and high interest rates, which was the start of a economic recession.
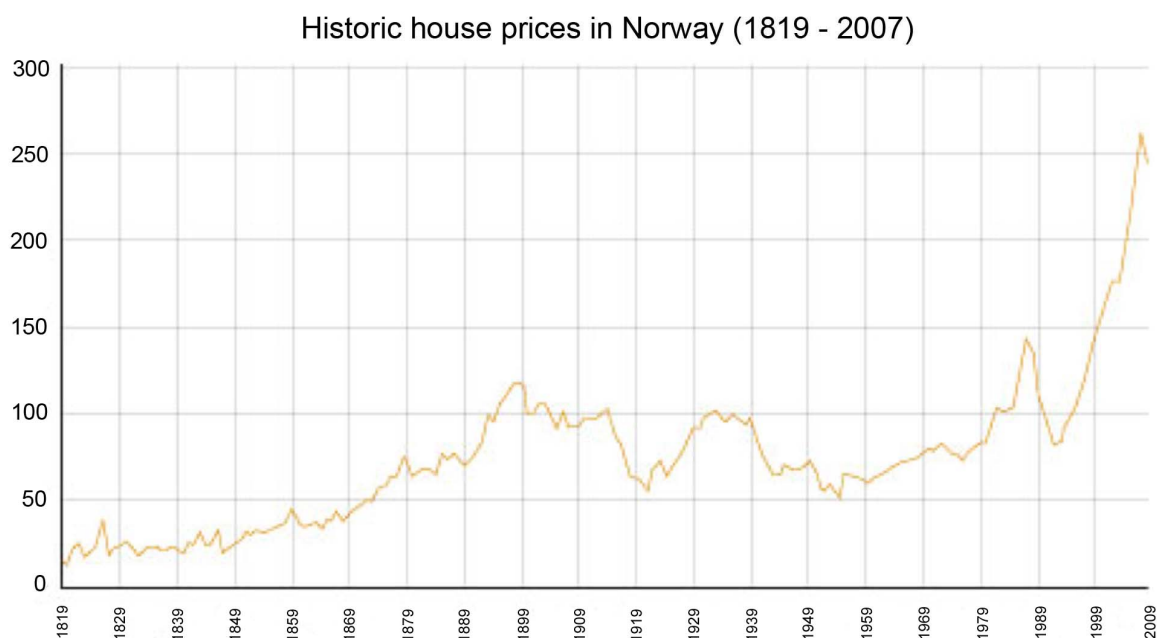
Figure 1: Real house price index Norway, 1819-2007 (Grytten, 2009)

By 1992, the return had flattened, and a new period of optimism was embarked upon as interest rates decreased (Evensen, et al., 1996). Thereafter, prices have with few exceptions risen

steadily. Coming into 1997, previous heights were reached, and in the period 1997-2005 prices increased by 95% (Lye & Nilsen, 2006). As can be seen from figure 2, the financial crisis hit the Norwegian real estate market relatively mildly and was short-lived compared to most countries.
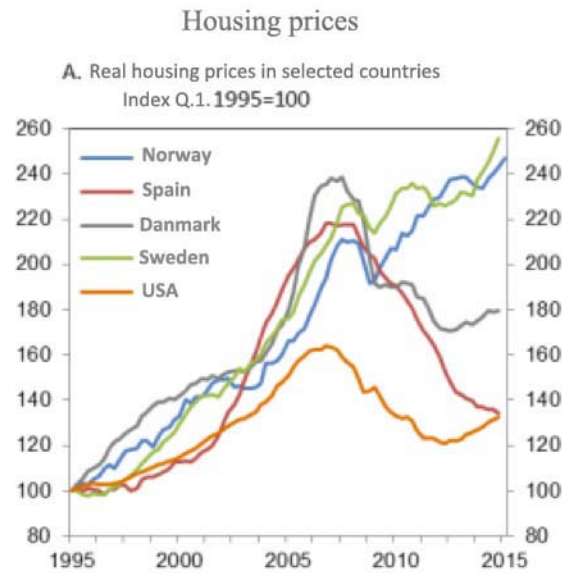
Figure 2: Real housing price index (1995-2015), selected countries
(Regjeringen, 2015)

The Norwegian market was however not sheltered during the financial crisis. The period 2007-2008 is one out of three periods where prices declined in real terms since 1980 (Barlindhaug, Holm, Nordahl, & Renå, 2014). By mid 2009, we see prices flattening, and a new period of strong price increases emerges. In the coming five-year period, all sampled cities grow steadily and at a high rate.

Growth in housing prices, 2007-2016 (in percentages).

% Change period	Fredrikstad	Oslo	Drammen	Kristiansand	Stavanger	Bergen	Ålesund	Trondheim	Tromsø
2007Q4-2008Q4	-9 %	-7 %	-6 %	-10 %	-11 %	-12 %	-2 %	-10 %	-2 %
2009Q1-2014Q1	29 %	38 %	50 %	14 %	39 %	44 %	31 %	44 %	44 %
2015Q1-2016Q1	10 %	10 %	10 %	4 %	-12 %	1 %	2 %	4 %	1 %

Table 1: Change in housing prices, time intervals (SSB, 2016)

Throughout 2015, we see the effect from the recession in the petroleum industry. Municipalities such as Stavanger, Bergen and Ålesund have seen either decreasing, or marginally increasing housing prices. High density areas surrounding Oslo have continued to grow steadily as can be seen from table 1.

2.1 Determinants of supply

Population growth and housing completions were strongly tied in the period 1985 to the early 2000s Barlindhaug et al. (2014). In the mid 2000s, driven by strong economic growth, the country attracted an increasing amount of immigrant labour and experienced increased centralization. Additionally, the downturn in housing prices during the financial crisis reduced supply of new housing projects. In the period after the financial crisis, demand continued to rise at a greater pace than the rate of new housing projects, which drove prices up Barlindhaug et al. (2014).

The supply deficit has attracted attention from scholars and politicians. To understand the reasoning behind a supply deficit in a period of price increase, attention has been directed to the system that approves new housing projects. Barlindhaug and Nordahl (2011) points out different reasons why supply is low, despite high prices. Amongst their main findings is stricter quality requirements, costly infrastructure requirements and complicated planning processes. They argue that municipal means to achieve their housing policy can affect both profitability and risk in development projects. Developers argue for this view, and desire that the municipality grants more approvals and open for development in new areas (Barlindhaug & Nordahl, 2011).

In an examination of local planning processes in the greater Bergen area, Nordtveit (2015) finds that municipalities surrounding Bergen in general are slower and less predictable than in comparable cities Stavanger and Oslo. In the greater Bergen area, it is Bergen municipality that scores the lowest on close to all parameters. Industry respondents explain their frustrations with lacking clerical capacity, poor communication throughout the process, and additional requirements that appear randomly and late in the process as their main concerns (Nordveit, 2015).

2.2 Local property development

The planning control system consists of three levels; national, regional and local. These levels represent the state, county and the municipality. At the national level, the stated goal is to facilitate for well-functioning real estate markets (Regjeringen, 2004). At higher levels, politicians draw up the framework for subordinate agencies via the tax system and the interest rate policy (Nordahl, 2011); (Nordahl, Barlindhaug, & Ruud, 2007). Municipalities draw the

terms locally with a municipal master plan every fifth year (Barlindhaug, Holm, & Nordahl, 2014). The plan consists of two levels; an "action"-, and an "aerial" plan. The role of the aerial plan is to provide a connection between future societal development and the use of land (Bergen Kommune, 2015). Its intention is to provide the greater guidelines for land use and to act as a framework for future planning decisions.

2.2.1 Sequential planning process in Bergen

The decision making process in Bergen is divided into two phases; "plan development" and "public processing and final decision". In the "plan development" phase, it is assumed that the process is driven by the developer, while the "public processing and final decision" is driven by municipal agencies and politicians (Nordveit, 2015). There are six steps within the two phases.
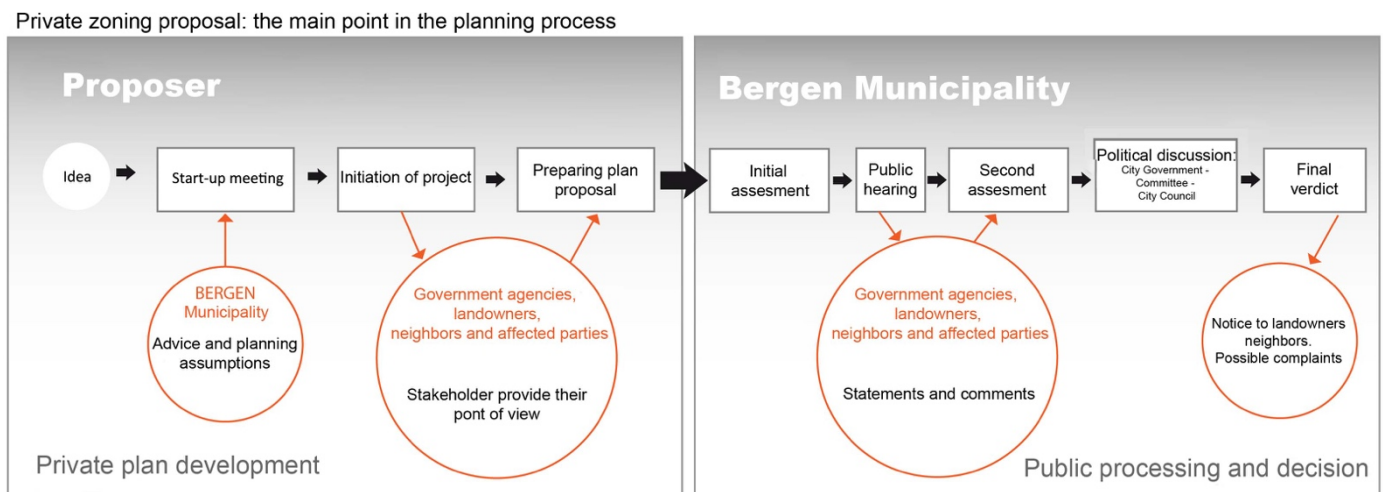
Figure 3: Main steps in the regulatory process (Bergen municipality, 2016)

1. **Start-up meeting**

The process starts with a "start-up meeting". Here, the developer and local government will meet to discuss ideas and possibilities, go through the general plan for the area and how the ideas functions within these limits. Municipal agencies must conclude in this phase whether further development is recommended and if an impact assessment must be performed.

2. **Initiation of project**

In the next step, the developer must announce that he is initiating the project. The initiation must be made to government and other affected stakeholders. This phase entails that a full proposal will be developed in accordance with formal structures.

3. Initial assessment

The "public processing and final decision" phase starts when the developer has delivered his proposal and awaits the first responses to his case. The plan is forwarded to relevant municipal agencies to give an assessment.

4. Public hearing

After the initial treatment, the plan is presented to relevant stakeholders for comments. Neighbours, local interest groups, and others, forward their comments and statements regarding the plan.

5. Second assessment

Having gathered the opinion of relevant stakeholders, the developer submits an updated proposal where he argues for how comments/statements has been taken into account. Municipal agencies can come with additional remarks for the developer to internalize into his proposal. The municipality conclude this section by writing a memorandum that is forwarded to politicians.

6. Political discussions and final verdict

The memorandum from municipal agencies are considered first by the committee for environmental – and urban development. This group of politicians give their opinion to the city council who gives the final verdict.

2.3 Regulatory risk

Regulatory risk can be defined as deviations in profitability due to municipal demands and restrictions (Nordahl, 2014). In addition to market- and financial risk, regulatory risk can be of equal significance since this defines the framework and possibilities for a development project Nordahl et al. (2007). Nordahl (2012) argues that since municipal agencies are exempt from considering the financial consequences of their decision making, the developer bears a disproportionately large part of the financial risk in this structure. ECON (2005) lists it as comprising the following:

1. The utilization rate allowed
2. The time frame of a final verdict
3. Procedural order rules

The first point reflects the favourability of the outcome of municipal decision making. All else equal, the developer will prefer a utilization rate that maximizes profits. Jaeger and Plantinga (2007) argues for restriction effects, which is the case when regulatory restrictions preclude the *"highest and best use"* of the property.

The second part of regulatory risk is the uncertain expected time until approval. This class of risk is of severe importance as the developer have running expenses, but often no income in the period (Nordahl, 2012). In addition to running expenses, costs following "loops" in the process, and/ or improvements that must be made with the planning proposal can be assumed to be increasing with time (Nordahl, 2012). Additionally, uncertainty in time to approval have an important implication for when the finished product can be offered to the market (Nordahl, 2014). Since the developer will outline project characteristics years before sales can happen, uncertainty in time to completion will increase the risk for low demand when the project is ready for the sales stage.

The final part of regulatory risk is a much-debated part of development in Norway. Procedural order rules are the contribution a project has to make to surrounding infrastructure. Typically, this is a means to ensure that an increasing population in the area will maintain or better the local infrastructural level, or ensure the maintenance of public services Barlindhaug et al. (2014). These are measures that must be paid for in order to obtain the final approval.

3. Investment decision making frameworks

In this chapter we direct our focus towards the underlying theory which is used to understand optimal decision making under uncertainty. We start briefly by introducing some characteristic issues often encountered in uncertain investment decisions, and go on to provide some alternative views on optimal decision making criteria. The approaches mentioned as "traditional" includes static discounted cash flow approaches where the decision to invest is satisfied as long as future net benefits are positive.

In section 3.2 and 3.3 the reader is introduced to basic financial – and real options theory. The section concludes with the description of dynamic programming and stochastic behaviour.

3.1 Irreversible investment, uncertainty and strategic decision making

To maximize the potential of irreversible investments, one must apply a framework that captures the dynamics of the underlying asset. A natural starting point in neoclassical economic theory would be to make strong assumptions ex-ante about the potential of the investment, the expected sales price, and project cost of capital. Once calculated, the values are discounted back at present value and, dependent on the framework applied, a decision is made based on some pre-determined decision rule. These approaches share appeal through strong intuition and mathematical simplicity. The major weakness drawn from that simplicity is the fact that we are applying current information to investments taking place in the future, and assume that we cannot react to changing states of nature.

Several scholars[6] argues that under given circumstances, the traditional net present value approach fail to incorporate the behavioural traits of an investment by ignoring the option properties. The first to acknowledge the dynamism of real investments was Mossin (1968), who argued that once a ship was laid up, the owner foregoes the opportunity to do so, which is valuable. The main takeaway from his article is that when investments are irreversible and

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[6] (Brennan & Schwartz, 1985), (Bjerksund & Ekern, 1990) and (Capozza & Li, 2002) are amongst others who propose an altered decision making rule applied to cases in different industries.

future states of the world uncertain, the option to postpone the decision has value (Tvedt, 2000).

McDonald and Siegel goes to such lengths as to argue that the standard net present value decision rule *"is only valid if the variance of the present value of future benefits and costs is zero"* (McDonald & Siegel, 1986, s. 708). When investments are irreversible and future cash flows evolve stochastically, they find it optimal to wait until benefits are twice the investment costs. Closely related to this argument is the analysis from Pindyck (1991), which states that if investments are irreversible and has the ability to be postponed, the classical decision making criteria is obsolete.

In a paper on real options in real estate development, Lucius (2001) argues that the traditional understanding of real estate as an immobile, inflexible, and deterministic investment must be altered. He claims that the greater the emphasis on uncertainty, the less adequate is traditional approaches to real estate investment. When taking into account entrepreneurial flexibilities, standard methods ignore alternative decisions and undervalue projects. Bulan et al. (2002) describes real estate development as *"essentially irreversible",* arguing that this complicates shifting to alternative uses. This reduces the value of the managerial flexibility to sell a project once construction is initiated. Cunningham (2006) points to the durability and inseparability of built structures and property to argue for real estate as an irreversible investment, making techniques closely related to financial options compatible to real estate development (Cunningham, 2006).

3.2 Financial options theory

Black and Scholes (1973), with the help of Merton (1973) revolutionized valuation of financial instruments that is dependent on an underlying asset. Their seminal work in the field lead to a large increase in liquidity of such products as practitioners were able to more accurately price the products in real time and hence trust in the products increased substantially.

An option contract is an agreement between two financial entities which gives the holder the right, but not the obligation, to buy or sell an underlying asset at a future date for a predetermined price, the "strike price" (Mun, 2002).

A call option is the right to buy an underlying asset at the predetermined price at some time in the future (Mun, 2002). Call options has value if the price of the underlying asset is above the predetermined strike price at termination (Mun, 2002). The option is "in the money" in this scenario. The opposite is true if the asset price moves below the strike price. In this case, the option is "out of the money". If the option is out of the money at maturity, the holder will forego his opportunity to execute the option, leaving the option worthless.

The value of a call option can be expressed as:

(3.1) $$Max\,[\,S - K, 0\,]$$

Where S is the value of the underlying asset at maturity and K is the contracted strike price. The expression can be depicted as:
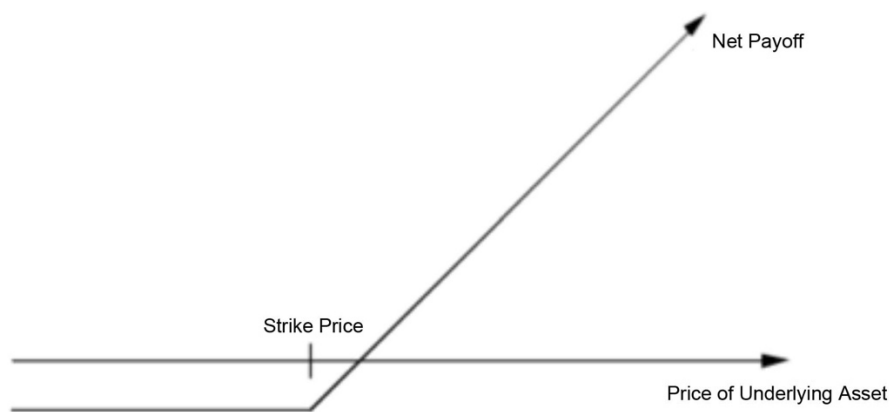
Figure 4: Payoff, call option (Damodaran, 2005)

Put options gives the right to sell the underlying asset at an agreed upon strike price at termination (Mun, 2002). For the holder, a put option has value (i.e. is "in the money") if the underlying asset's price is below the strike price. In such a situation, the put holder has the right to sell the underlying asset above the going market price.

The value of a put option for the holder can be expressed as:

(3.2) $$Max\,[\,K - S, 0\,]$$

Where K is the strike price and S is the value at which the underlying asset is currently trading. The relationship can be depicted as:
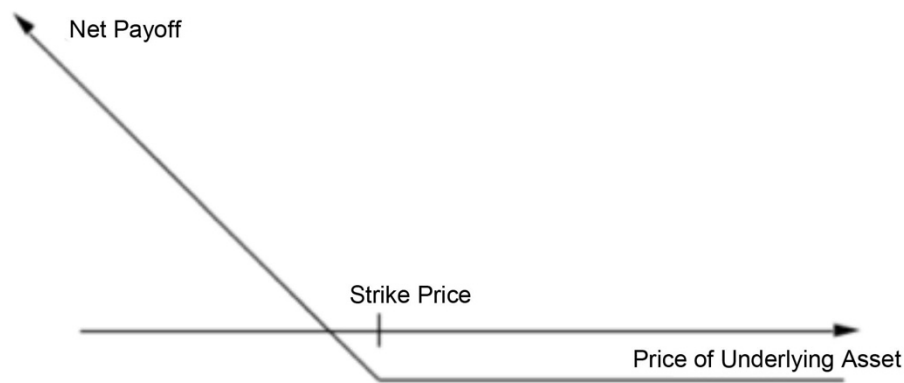
Figure 5: Payoff, Put option (Damodaran, 2005)

An option consists of two classes of value – intrinsic value and time value. Intrinsic value is the monetary amount the option is above – or below the pre-determined price, whilst time value is based on the fact that option value is driven by volatility (Damodaran, 2005). As the holder has an option to buy or sell, the holder also has the option not to. This makes the option's payoff asymmetric. Hence, greater volatility increases the upside potential, while the downside potential is the same (Damodaran, 2005)

The seller, or writer, of the option is said to have a "short" position in the instrument and must adhere to the decision of the option holder. That is to either buy – or sell the underlying asset if the holder executes the option (Berk & DeMarzo, 2014). In order to hold this risk, the writer is rewarded with an option premium, which is his maximum payoff from the arrangement.

We separate between American and European options on the ability of the holder to execute before, or strictly at, maturity. That is, the holder of an American option can execute at any given time until, and at maturity, while the European holder can only execute the option at maturity (Damodaran, 2005). Hence, an American option can never be worth less than a European option with identical option characteristics (Mun, 2002). It is however exceptional that the holder of an American option will execute the option early, as the holder will lose time value of the option by doing so (Damodaran, 2005). One exception is for American-style call options where early exercise can be beneficial if the underlying stock will go ex-dividend the day after and the option itself is deep into the money (Mun, 2002).

Another exception is for deep into the money American put options. It can be valuable with early exercise because it means that the holder will receive the intrinsic value earlier so that it can start to earn interest quicker (Damodaran, 2005).

3.3 Real options

"To create a good analogy of real options, visualize it as a strategic road map of long and winding roads with multiple perilous turns and forks along the way. Imagine the intrinsic and extrinsic value of having such a strategic road map when navigating through unfamiliar territory, as well as having road signs at every turn to guide you in making the best and most informed driving decisions. This is the essence of real options." (Mun, 2002, s. 10)

From financial options, Myers (1977) brought derivatives theory over to irreversible investments to value non-financial – or real assets. The first distinction between financial – and real options is that a financial option gives the holder the right, but not the obligation to buy or sell an underlying asset, while real options gives the holder the right, but not the obligation, to make a business decision (Berk & DeMarzo, 2014).

In a typical capital investment decision, the real options view is to regard the investment decision as a call option, where the present value of future benefits is the price of the underlying asset and the investment cost the strike price (Pindyck, 1991). The applicability is however more widespread than the investment decision itself. As Damodaran (2005) puts is; real options are "ubiquitous" in business decisions. He does however emphasize that despite the vast amount of options available to managers, only under certain conditions will they have value.

Akin to financial options, the real options approach allows for future states of the world to be revealed before investment decisions are reached. The approach permits for the incorporation of management's ability to alter the course of action for investment opportunities that develops contrary to expectations (Mun, 2002).

Elnan et al. (2007) points out that since real options seldom are traded, it becomes an important managerial exercise to identify valuable real options. For the same reason, Amram and Kulatilaka (1999) proposes a four-step solution process on how to apply real options successfully. The first step involves framing of the decision, which means to identify available options, concretize relevant sources of uncertainty and to create a decision rule for optimal decision making. Step two includes the implementation of the model that is now framed. This involves projecting relevant input parameters accurately and to decide the "options calculator" to be used, i.e. the mathematical approach. Steps three and four focuses on the output provided

for the user, and how to make best use of them. They emphasize the large potential for output generation using this approach, and at the same time argue that the approach can have many viable uses dependent on the preferred application. Amram and Kulatilaka (1999) categorizes four different types of outputs that can be valuable; valuation results, critical values for strategic decision making, the strategy space, and the investment risk profile.

Valuation results entails a performance comparison between traditional discounted cash flow approaches and the real options approach, where the implicit option value is found as a subtraction between the static net present value and the adjusted net present value. Output generation can also consider strategic considerations and allow for strategy optimization by calculating, reviewing and taking into account threshold levels for investment, abandonment and other embedded real options. The "strategy space" further allows for reviewing optimal decision making within a range of values for two input factors in an X-Y plane. The strategy space can also be useful when considering the levels of critical input factors. If there is large uncertainty regarding future levels of inputs, for instance projected costs, building a strategy space where correct strategies are identified within given ranges can help management to capitalize on forthcoming strategic challenges (Amram & Kulatilaka, 1999).

3.4 Stochastic Dynamic Programing

"Dynamic programming solves the problem of how to make optimal decisions when the current decision influences future payoffs" (Amram & Kulatilaka, 1999, s. 110)

The two main approaches to solve sequential investment problems is dynamic programming and contingent claims analysis. To have a contingent claim means that the value of a derivative is contingent on the value of other financial instruments. To obtain accurate results using this approach, the asset in question ideally has to be perfectly correlated with another traded asset to accurately replicate the payoffs of the derivative and thus apply the law of one price.

Dynamic programming on the other hand is a mathematical optimization method based on the theory of sequential decision making. Dynamic programming focuses on future decisions, where the value of the project is a result of decisions made throughout project

lifetime. Central to dynamic programming is Richard Bellman's theory of the principle of optimality:

"An optimal policy has the property that, whatever the initial action, the remaining choices constitutes an optimal policy with respect to the sub problems starting at the state that results from the initial actions." (Dixit & Pindyck, 1994, s. 100)

Breaking down these decisions into two components, the immediate decision and a valuation function which takes into account the consequences of all sequential decisions after the initial decision is made, one can create sub-problems that is easier to calculate. Finally, backward induction is applied to find the value of a project (Dixit & Pindyck, 1994).

3.5 Geometric Brownian Motion

Geometric Brownian motion is a continuous-time stochastic process often used in options pricing to describe the uncertain development in the value process of the underlying asset (Baxter & Rennie, 2001). Geometric Brownian Motion with drift can be described as follows:

$$(3.3.3) \qquad dX_t = X_0 exp\left[\left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma_t dW_t\right]$$

Where μ is the drift, σ the volatility and W the Wiener process. The Wiener process is a continuous stochastic process where each increment is normally distributed with expected value of zero and variance dt. Differing from a Brownian Motion, the Geometric Brownian motion is lognormally distributed (Dixit & Pindyck, 1994). This distribution is popular to describe processes where the values tend to be positively skewed (Mun, 2002). For simulations of values such as prices, this property is intuitive as a lognormal distribution, equal to prices, can never take negative values, thus skewing positively. The drift parameter μ typically represents the instantaneous increase of the underlying price process, while the volatility, σ, represents the volatility of the price process (Brewer, Feng, & Kwan, 2012). Both parameters are measured in annual terms.
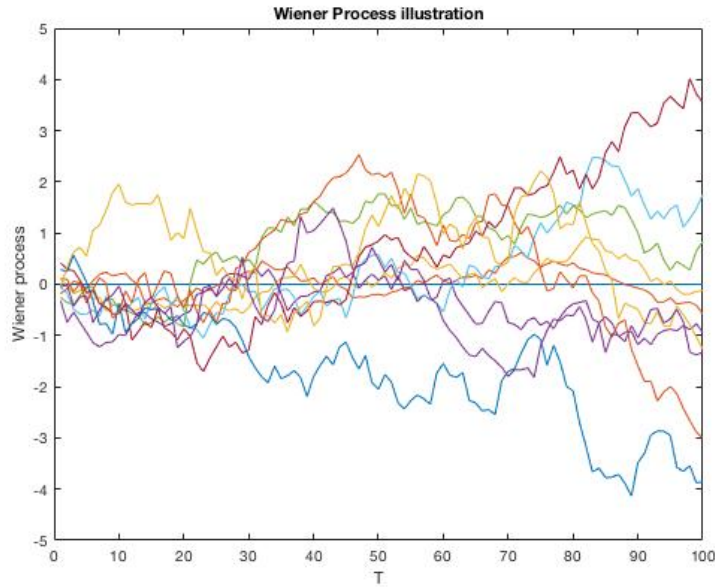
Figure 6: Illustration of Wiener processes (Matlab)

3.6 Poisson process

A process that makes infrequent but discrete jumps, where the jumps can be of random or fixed size (Dixit & Pindyck, 1994). These jumps are called "events", where λ is mean arrival rate for the event to occur, within the time step dt. The poisson process can be written mathematically as follows: (the process is denoted q)

$$(3.3.4) \qquad dq = \begin{cases} 1, & probability \ \ \lambda_n dt \\ 0, & probability \ \ 1 - \lambda_n dt \end{cases}$$

The first equation is the probability that the event will occur, and the last equation is the probability for the event not to occur. In the equation above the jump is 1. This can be changed to a random variable (Dixit & Pindyck, 1994).

4. Data collection

In this chapter we present data from previous planning decisions. As uncertainty in time to completion is at the core of this thesis, we have gathered data to investigate the flow of decision making. By doing so, our aim is to provide the forthcoming models with accurate input data. Additionally, we apply the data to perform the first step in real options analysis; identify real options that can be valuable.

Data is presented by the statistical approach survival analysis using the Stata software. By using the survival analysis framework, we can get further insights in historical expected time to completion and potential behaviours in the regulatory process. The event we are looking for is defined as time to approval, the finishing step in obtaining regulatory approval.

4.1 Data collection of flow in public processes

The source of our data is Bergen municipality's database "BraPlan". This is a public database containing previous – and current planning decisions for detailed zoning. Each planning proposal is designated with dates for ruling in each sequence of the planning process.

The dataset consists of residential-, commercial-, industry- and recreational projects. After filtering out unwanted subjects, we are left with a total of 648 cases dispersed over all seven districts from 1990-2011. We choose 2011 as an end-date to avoid selection bias. That is, if we had included for instance an additional two-three years of data, only cases with short time to completion could by nature have been included.

4.1.1 Survival analysis

Survival analysis is a statistical approach for analysing positive-valued random variables, such as time to a given event (Miller, 1998). With the passage of time, one can analyse the behaviour of life courses, and the occurrence of events in the period. Each subject needs a description of time spent in each state or step, with the date of each transition or action. The different states are mutually exclusive at each point in time (Jenkins, 2005). Survival analysis can also be called transition data or duration analysis in economics. The event that one wants to check for can be of all sorts. Some examples can be time to failure, death, success.

What distinguishes survival analysis from other statistical techniques is censoring of data. Censoring can be explained as incomplete information, when there is only partial information on the subjects. The information that is given can be either left – or right censoring. Left censoring is apparent when information about the project start is missing, while right censoring appears when the relevant event has not yet occurred (Jenkins, 2005). Censored data can be caused by drop-out, discontinuation, loss to follow-up or missing information. An additional reason can be censoring as a result of ending the study (Miller, 1998). The duration of the process or the time to event, can then be measured using non-negative real numbers, often derived from start dates, exit dates for complete cases or last observation for censored cases (Jenkins, 2005)

To give a small introduction to the calculus, the dependent variable is assumed to have a continuous probability distribution $f(t)$. The first function, $F(t)$, is the probability that the duration is less than t.

(4.1) $$F(t) = Prob(T \leq t) = \int_0^t f(s)ds$$

The survival function $S(t)$ is the probability that the duration will be at least t. This can be written as a function where $T \geq 0$, and the function of distribution is given as $dF(t)$.

(4.2) $$S(t) = 1 - F(t) = P\{T > t\}$$

The hazard rate is a conditional probability that the duration will end after time t, given that the project has lasted until time t, or in other words still remain in the sample.

(4.3) $$\lambda(t) = \frac{f(t)}{1-F(t)} = \frac{f(t)}{S(t)}$$

4.1.2 Dataset

To summarize the dataset, the two following tables gives a quick introduction to the inputs and subjects in the dataset. Table 2 presents the main information about the subjects.

```
        failure _d:  Event == 1
  analysis time _t:  Duration
```

Category	total	mean	min	median	max
		per subject			
no. of subjects	648				
no. of records	648	1	1	1	1
(first) entry time		0	0	0	0
(final) exit time		3.165883	.0038462	2.776923	16.74615
subjects with gap	0				
time on gap if gap	0				
time at risk	2051.4923	3.165883	.0038462	2.776923	16.74615
failures	413	.6373457	0	1	1

Table 2: Description of survival-time data (Stata).

Information per subject includes both the cases of approval and censored subjects. The total amount of cases is 648 where there are 413 cases left after censoring. The difference of 235 cases are lost due to early termination or incomplete information. The total mean exit time is 3.17 years for all subjects, with minimum 1 day and maximum 16.75 years. It is important to note that the minimum value does not describe that the approval process has been completed after one day, but describes the fact that the case is censored in the first step of development. The total time at risk is found to be 2051.49 years, the total timeframe for every project that is included in the set.

```
        failure _d:  Event == 1
  analysis time _t:  Duration
```

	time at risk	incidence rate	no. of subjects	Survival time		
				25%	50%	75%
total	2051.492308	.2013169	648	2.526923	3.930769	5.765385

Table 3: Summary of survival-time data (Stata).

The incidence rate is in total 20.13% and is the likelihood for approval over the total time at risk. The 25th percentile is below 2.5 years, half the observations are below 3.93 years and the 75 percent are below 5.77 years.
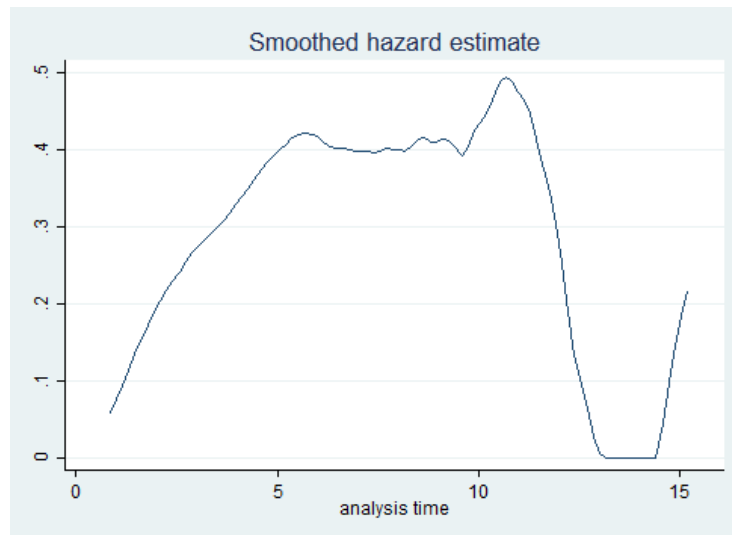
Figure 7: Smoothed hazard estimate (Stata).

The hazard rate graph depicts the conditional probability of having an event or in this case get the approval, at each time step. This illustrates that likelihood is strictly increasing in the start, with highest probability from around 5 to 11 years. This gives a good introduction to the next section, where we try to find an expected time to completion from the dataset.

4.1.3 Time to completion

If a case has missing value in the start-up meeting cell, we choose to calculate from the initiation of the project as the start-up point for the project. This is a conservative approach, but will at the same time ensure that the approximation is not overstated. This problem can be categorized as left censoring, one of two types of data censoring in survival analysis. Left censoring is where there is no observed start date of the project, preventing an exact duration for the analysis (Jenkins, 2005). We then combine the start-up meeting and initiation to give a best approximation of the observed start date for each case. In some cases, two start-up meetings are held, or as much as three initiations of the project is listed. This can be the case if a developer has restarted the process, or if the feedback from the start-up meeting required a larger change in the plan. Consistent with a conservative measure, we consistently choose the higher of the alternative values.

Table 4 shows the restricted mean survival time

```
       failure _d:  Event == 1
  analysis time _t:  Duration

              │  no. of  restricted
              │ subjects      mean    Std. Err.   [95% Conf. Interval]
      ────────┼──────────────────────────────────────────────────────
         total │     648   4.524934(*)  .1341584     4.26199    4.78788

(*) largest observed analysis time is censored, mean is underestimated
```

Table 4: Survival analysis, restricted mean (Stata)

For the sample, we extract an average of 4.52 years throughout the 21-year period. This mean is slightly underestimated, as the notation in Table 4 describes. To better understand why the mean is underestimated, we present the Kaplan-Meier survival estimate.
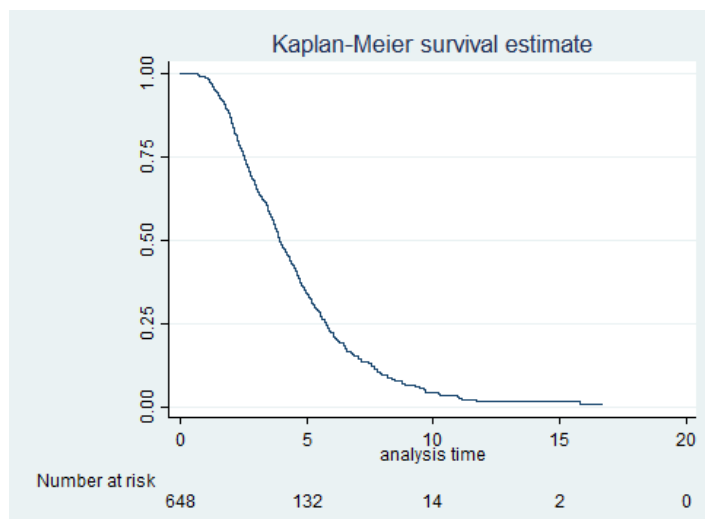
Figure 8: Kaplan-Meier survival estimate (Stata).

The restricted mean is in the area beneath the Kaplan-Meier survival curve, which is the survival probability over time. This mean restricts to the longest follow-up time, and since the largest time is a censored case, the function does not reach zero completely. This results in an underestimated mean, which we will adjust for in the extended mean. For more detailed information about the survival data and table describing the numbers behind, see appendix C. The extended mean calculates survival time by exponentially extending the survival curve to zero, this is shown in figure 9 below.
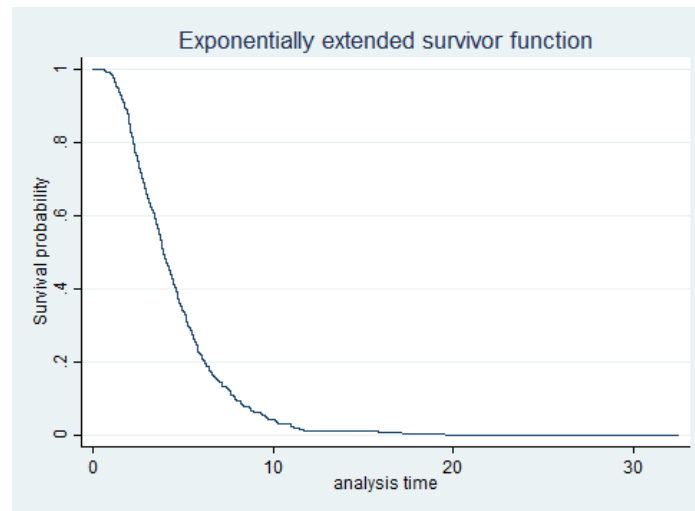
Figure 9: Exponentially extended survivor function (Stata).

From the exponential extension done above, we find the table 5 is the extended mean of the survival time, which can be translated to duration until approval.

```
          failure _d:  Event == 1
    analysis time _t:  Duration

                    no. of      extended
                  subjects          mean
          _____

          total        648      4.550572
```

Table 5: Survival analysis, extended mean (Stata).

The mean is at last calculated to be 4.55 years.

4.1.4 Strategic behavioural patterns in the development process

In addition to act as precise input data, we can take learning from analysing the flow of the sequential process. That will allow us to depict the process as it actually occurs and to identify which options that can be valuable in this context.

From our data we find two options that can potentially be valuable; the option to temporarily shut down and re-start (switch state) and the option to abandon the project.

The option to scale up/down depending on going market prices can be found in many industries, but will be restricted by the ability of decision makers to act upon it. For instance, if a manufacturing plant has contractual agreements to deliver a certain amount of output in each time period, the option to shut down can be either non-existent or very expensive. Further,

the costs of shutting down and re-start operations will affect optimal decision making in addition to price movements and contractual arrangements. The option to temporarily shut down and re-start will be a trade-off between keeping operations going and receive the proceeds of production/progress, and the value of being relieved of ongoing costs.

The option to abandon has the characteristics of a put option. Dependent on the criteria for disinvestment, the abandonment option can be both of European – and American nature, and can be valuable in circumstances characterised by large capital outlays and high uncertainty (Trigeorgis, 2002). If project value moves in adverse directions, the option to abandon and recover some of the capital outlays can be worth more than the proceeds of further investment.

4.1.5 Temporary abandonment

From our data we see that several cases have multiple decision dates in the same step, indicating that the project has been restarted. This is most common in the first phase of development.

To determine if the option to temporarily abandon can be valuable we use Statistics Norway's house price index. Within this period, the local market has seen the end of a strong cycle, disrupted by a short decline, before it started a new period of increasing prices.
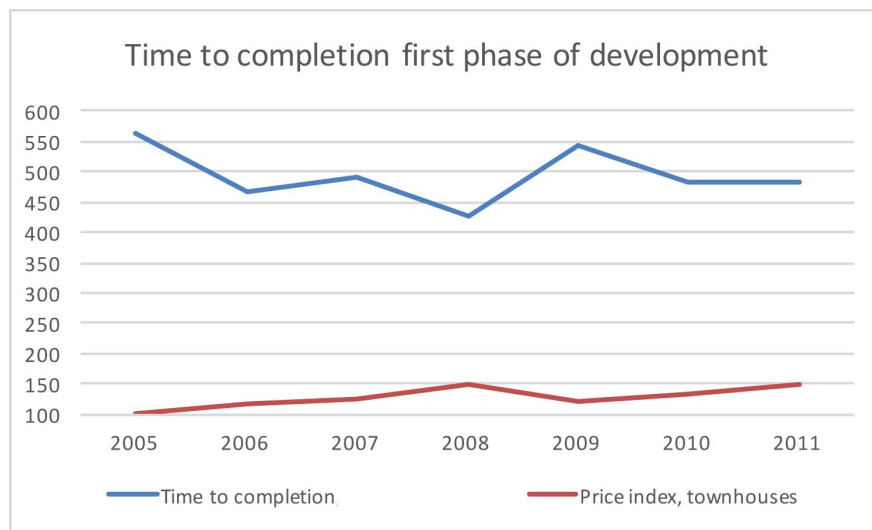
Figure 10: Time to completion in the first phase of development (in days) versus house price index for townhouses

From the graph above we see two periods where prices are increasing and a year where prices are declining. In the same periods, we see tendencies of an inverse relationship between

expected time to completion and the expected value of the outcome. Running a correlation-test on the data, we find the values to have a correlation coefficient of -0,77. Even though a strong correlation coefficient does not necessarily entail causality, we can put the numbers into its context and attempt to interpret.

We interpret the increased time to completion as a timing feature of development; in times where the value of the outcome is decreasing (increasing), expected time to completion is increasing (decreasing).

4.1.6 Abandon development for salvage value

In our dataset, approximately 36% of the cases were abandoned during the process. In table 6, we see the amount of censored and approved cases over time. From the amount of lost cases it seems obvious that the option to abandon for salvage value can be valuable. This is even more

Interval		Beg. Total	Deaths	Lost	Survival	Std. Error	[95% Conf. Int.]	
0	1	648	8	136	0.9862	0.0048	0.9726	0.9931
1	2	504	58	30	0.8692	0.0150	0.8365	0.8958
2	3	416	97	15	0.6628	0.0216	0.6186	0.7032
3	4	304	77	17	0.4901	0.0233	0.4437	0.5348
4	5	210	59	19	0.3459	0.0228	0.3015	0.3906
5	6	132	44	8	0.2270	0.0208	0.1875	0.2689
6	7	80	25	5	0.1538	0.0186	0.1195	0.1920
7	8	50	18	0	0.0984	0.0158	0.0702	0.1321
8	9	32	10	1	0.0672	0.0135	0.0439	0.0970
9	10	21	7	0	0.0448	0.0114	0.0261	0.0709
10	11	14	6	0	0.0256	0.0088	0.0122	0.0473
11	12	8	3	1	0.0154	0.0070	0.0057	0.0343
12	13	4	0	1	0.0154	0.0070	0.0057	0.0343
14	15	3	0	1	0.0154	0.0070	0.0057	0.0343
15	16	2	1	0	0.0077	0.0065	0.0011	0.0311
16	17	1	0	1	0.0077	0.0065	0.0011	0.0311

Table 6: List of survival estimates (Stata).

We see that subjects are lost during the entire lifetime of the study. From table 7, we see that subjects are lost even in the later stages of the process, indicating that they find this decision to be optimal. Even though the final verdict may be in reach.
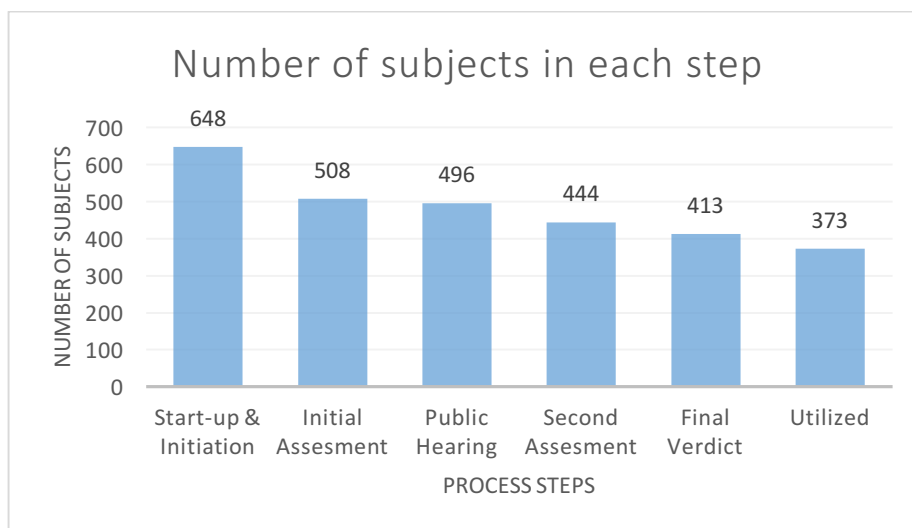
Table 7: Frequency table, number of survival in each process step.

We can see that there it can be valuable to abandon even after receiving approval, since there are 40 cases that do not utilize the approval or in some sort forfeit their right to build.

5. Assumptions and notations

To develop an accurate valuation model for property development, we draw on the work of Miltersen and Eduardo Schwartz. In their paper, "Real options with uncertain maturity and competition" (2007), they develop several models with the purpose of fully capturing uncertainty at the early stages of different kinds of projects. They propose the models to be used in R&D, mine – or oil exploration projects. We see great resemblance between the characteristics of these investment decisions and those of real estate development. First of all, like with most projects, there is uncertainty about the value of the outcome at completion. Secondly, the total investment costs are uncertain. Adding to this, the user cannot be certain on an accurate time frame at which the project will be completed.

Even though the models proposed by Miltersen and Schwartz (2007) include advanced American – and European option characteristics, they are able to draw closed form solutions. Their main simplification in order to obtain closed form solutions is that project completion is governed by an independent exponentially distributed random variable. This simplification makes the assumption that the probability of completion is equal at every small time-increment. This Poisson jump process has the probability λ per unit of time to reach a conclusion of the development process. Miltersen and Schwartz (2007) admits this to be a bold simplification, but it is necessary to avoid the complexity of partial differential equations. The expected remaining time to completion can be written as a function of time, $T = \frac{1}{\lambda}$. The time distribution and expected time to completion do not depend on calendar time. Date and time to completion will use the same distribution to simplify the model.

The investor pays an ongoing investment cost per unit of time until the project is completed. At completion, the investor must choose if the project is worth pursuing. If so, he must pay a final investment cost that will eventually allow him to claim the value of the outcome. The value of the outcome is the expected net present value generated by the investment project. As project value is dependent on stochastically evolving future cash flows, the project owner will at each time-instant re-adjust his approximation of the value of the outcome to consider if the project is worth further investment. The underlying idea from Miltersen and Schwartz (2007) is that, as future cash flows are uncertain, revealed information about uncertain cash flows and updated projections on the value of the outcome will provide the investor with information that will affect his/her investment behaviour.

In their paper, they propose both monopoly – and duopoly models. The duopoly models have the interesting trait that they add the characteristics of game theory into the models. As we suppose that the owner of the development project has already bought the property in question, game theoretic questions are not very relevant. Hence, our focus is solely on monopoly models. They propose a total of four monopoly models; a model with an abandonment option, a switching option model, a combined abandonment and switch option model, and lastly they propose a model where the project owner can choose to abandon a project during development and postpone payment on the final investment decision.

For all of the models, it is assumed that the project requires ongoing investment costs, k per unit of time. The uncertain time to project completion is denoted T, and the final investment cost K. The value of the outcome is denoted V. At the date of completion of the project, the project can expire worthless. Hence, the value of the outcome at the date of decision is $\text{MAX}(V_T - K, 0)$. In a real estate development context, the ongoing investment cost, k, considers the cost of moving the project forward to the next small time-interval and thus progress development. T, is the date at which the holder expects the development project to grant regulatory approval. K represents the the costs of preparing the site and construct the project the owner intends to build at the property. The value of the outcome, V, is the sales price for the residential real estate project.

In the model including an abandonment option, there will be some changes from the model of Miltersen and Schwartz (2007). They assume that if the project is abandoned, project value drops to zero. Our model will take inspiration from Teisberg (1994) and Bar-Ilan & Strange (1996), who include salvage value when exercising the option to abandon. The compensation from abandonment will be calculated as a percentage of the total sales value. The starting point for the salvage value will be the fraction $\frac{Property\ price\ pre-development}{Sales\ price\ of\ project} = a$. Further, we must adjust for the fact that the proposed project is not worth keeping alive, additional costs that occur from walking away from contracts, and other costs that can be recovered.

Inspired by Bar-Ilan and Strange (1996), we separate the investment in two periods. The first is the planning process, where the owner applies for an approval to further develop. The second is construction – and sales after the permit is granted. If the owner of the project chooses to exercise the option on the value of the outcome, access to period two is granted. Thereafter, construction and sales will happen simultaneously over a two-year period.

Overview of mathematical notations	
Sales value per square meter	V
Notation Model 1	$N(V)$
Notation Model 2	$M(V)$
NPV threshold	N
Value of regulation	$\Phi(V)$
Switch Option	S
Abandonment Option	A
Volatility	σ
Drift	μ
Subjective discount rate	ρ
Poisson death parameter, stage 1	λ_1
Poisson death parameter, stage 2	λ_2
Expected time to completion; Zoning	T_1
Expected time to completion; Construction and sales	T_2
On-going investment costs	k
Final investment costs	K
Value ratio, vacant land to square meter value	a
Expected annual square meter sold in stage 2	Q_1
Expected total square meters	Q_2

Table 8: Mathematical Notations

6. Real options models

In our thesis, we assume two different scenarios, both of which are based on analysis and assumptions from the panel data, observations from mass media, and other academic papers.

In accordance with the dynamic programming approach, the developer will continuously trade off the expected future benefits of keeping the project alive versus the value of ceasing operations. Depending on the model, the owner will have different options as opposed to progress the project full speed. In model one, an option to switch between an active – and passive state will allow the holder to postpone progress to learn more about market conditions. In model two, the developer can abandon the project for salvage value.

6.1 Model 1: Option to temporary abandonment

Miltersen and Schwartz (2007) propose a model where the owner of the project can switch costlessly between an active – and passive state for the development project if the value of the outcome drops below some threshold level. As time goes by, the stochastic value of the outcome will be revealed, and the investor has the option to re-start the project if the value of the outcome evolves above the switching threshold level.

In mathematical terms this will translate into a change in on-going investment costs in the active state, k, to drop to $k = 0$, in the passive state. This will have an impact on the intensity of completion λ_1. The intensity level is $\lambda_1 > 0$ in the active state, but is changed in the passive state to $\lambda_1 = 0$. This reflects that there is no progress in the passive state, and therefore no possibility to reach completion. Since the owner can at any given time instant determine the expected net present value of the outcome, the option to switch operating mode adds a timing feature of development, since by mothballing the investment project the owner ensures that it is never completed out of the money.

6.1.1 Framework and Calculation

To be able to differentiate the two states, one need to construct two separate valuation functions. The first describes the project value when the project is passive. In this state, the project derives value from being passive and the opportunity to switch into an active project.

The other function describes the value of the active project, including the effect from being able to switch to temporary abandonment.

Value function for the passive project:

(6.1)
$$\phi(V) = (1 - \rho dt)E_P[\phi(V + dV)]$$

The function for the passive project state includes only the expected future values if the project changes into an active state. These expected future values are discounted by ρ.

When constructing the Bellman equation for the active project state, this has to be done in two steps using backward induction. The first equation describes the value from stage two, construction and sales. The second equation is the value of the active state, including both the development stage and construction and sales.

(6.2)
$$N(V) = VQ_1 dt - \rho K dt + (1 - \lambda_2 dt)(1 - \rho dt)E_V[N(V + dV)]$$

(6.3)
$$\phi(V) = (1 - \rho dt)\lambda_1 dt\, E_V[N(V + dV)] + (1 - \lambda_1 dt)(1 - \rho dt)E_V[\phi(V + dV)] - kdt.$$

The first two terms on the right hand side of equation N(V) describes the immediate profit from construction and sales in the time interval, dt. V equals the sales price per square meter, while the quantity Q_1 is the annual sales of the constructed units, in square meters.

The last term on the right hand side of N(V) is the expected profits from future time intervals multiplied by the probability of construction not to end in the second stage, λ_2. The values are discounted by the subjective discount rate ρ.

The first term on the right hand side of $\phi(V)$ includes the expected future values of the active project if the zoning proposal is approved, discounted by the subjective discount rate ρ. The second term gives the continuation value of the ongoing zoning approval process. Since the first stage is yet to finish, we multiply by the probability of the project not to reach the final verdict, $1 - \lambda_1$, and subtract the ongoing investment costs from zoning, k.

To take into account the continuous value changes, we use Ito's Lemma of the Bellman equations, often called the Fundamental Theorem of Stochastic Calculus or a Taylor series expansion (Dixit & Pindyck, 1994). By breaking down the multi-period decision problems into smaller steps, we find a solution to the Ordinary differential equations (ODE):

(6.4)   *Passive:* $\frac{1}{2}\sigma^2 V^2 \Phi''(V) + \mu V \Phi'(V) - \rho \Phi(V) = 0,$ $V < S_n$

(6.5) Active: $\frac{1}{2}\sigma^2 V^2 \Phi''(V) + \mu V \Phi'(V) - (\rho + \lambda_1)\Phi(V) - k + \lambda_1 N(V) = 0,$
$$V > S_n$$

The first two terms of the ODEs describe the uncertain development in the underlying price process. The last term in equation $V < S_n$ is the discounted value of future expected benefits from shifting to an active state.

The third term in $V > S_n$ describes the disadvantage of ending the project and thus be unable to enjoy future price increases. Since the zoning approval is never obtained unless the project is in the money, the term $\lambda_1 N(V)$ describes the expected future benefits of paying the final investment cost to claim the value of the outcome.

The net present value of the active project, discounted by the subjective discount rate, less the drift of the value process, and adding the probability of completion in the second stage:

(6.6) $$N(V) = \frac{VQ_1}{(\rho - \mu + \lambda_2)} - K$$

To solve the differential equation, we start off by making more understandable equations that are easier to solve. This is done by using homogeneous substitution, taking parts of the term of the ODEs and introducing new coefficients. In this step we can also substitute the Poisson parameter as a function of time, so that $\lambda = \frac{1}{T}$. By making these changes, we are able to create a more intuitive equation, that makes it easier to understand the underlying mechanisms of the equations. Ito's Lemma can be re-written as:

(6.7) $$\phi_1(V) = A_1 V^{x_1} + A_2 V^{x_2}$$

(6.8) $$\phi_2(V) = B_1 V^{y_1} + B_2 V^{y_2} + \frac{VQ_1}{(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)} - \frac{kT_1+K}{1+\rho T_1}$$

$\phi_1(V)$ describes the value of the project when it is passive, i.e. out of the money. The first term includes the probability of the price to go up so that the value becomes $\phi_2(V)$. The second term includes the effect from the value to drop below another threshold, which would be the case if we had a third state, for example if we had included the option to abandon. This is described in more detail below.

The first term in the second function, $\phi_2(V)$, includes the effect from the price to reach another threshold level. This is the opposite effect from that of the second term in $\phi_1(V)$. This is also described further below. The second term in $\phi_2(V)$ includes the probability and value-effect from the price to drop below the passive threshold level and go to $\phi_1(V)$. The last part of $\phi_2(V)$ is the discounted net future benefits from obtaining zoning approval and sell the project.

The power functions used to solve the ODEs are shown as follows, where y, is the notations for active state powers and x, is passive state powers:

(6.9)
$$y_1 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2(\rho + \lambda_1)\sigma^2}}{\sigma^2} > 1$$

(6.10)
$$y_2 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) - \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2(\rho + \lambda_1)\sigma^2}}{\sigma^2} < 0$$

(6.11)
$$x_1 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}}{\sigma^2} > 1$$

(6.12)
$$x_2 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}}{\sigma^2} < 0$$

Since $y_1 > 1$ and $y_2 < 0$, and knowing that the value of the project can never exceed the value of the outcome V, $\lim_{V \to \infty}\left(\frac{\Phi(V)}{V}\right)$ must be a finite value. This results in the values V^{x_2} and V^{y_1} becoming zero, and the constants A_2 and B_1 to be excluded. That gives:

(6.13)
$$\phi_1(V) = A_1 V^{x_1}$$

(6.14)
$$\phi_2(V) = B_2 V^{y_2} + \frac{V Q_1}{(1 + (\rho - \mu)T_1)\left(\rho - \mu + \frac{1}{T_2}\right)} - \frac{kT + K}{1 + \rho T_1}$$

Because V^* is endogenous, we need two boundary conditions to solve this differential equation. The unknown coefficients are found by applying smooth pasting and value matching conditions and solve a set of boundary constraints in respect to the unknown variables. The following boundary conditions is needed to solve the set of ODEs:

(6.15)   *Value Matching condition:* $\Phi_1(S_n) = \Phi_2(S_n)$

(6.16)   *Smooth Pasting condition:* $\Phi_1'(S_n) = \Phi_2'(S_n)$

(6.17)   *Switch threshold:* $\lambda(S_n - K) = \lambda\Phi_2(S_n) + k$

The value matching condition describes the fact that at the switching threshold level, the value function should be continuous at the switching point. The smooth pasting condition tells us that at the switching point, the value function should be differentiable. The coefficients A_1, B_2 and S_n can be found analytically by using the boundary constraints in 7.15 and 7.16:

(6.18)
$$A_1 = \frac{(1-y_2)Q_1 S_n (1+\rho T_1) + y_2 (1+(\rho-\mu)T_1)(kT_1+K)\left(\rho-\mu+\frac{1}{T_2}\right)}{(x_1-y_2)(1+(\rho-\mu)T_1)(1+\rho T_1)\left(\rho-\mu+\frac{1}{T_2}\right)S_n^{x_1}}$$

(6.19)
$$B_2 = \frac{(1-x_1)Q_1 S_n (1+\rho T_1) + x_1 (1+(\rho-\mu)T_1)(kT_1+K)\left(\rho-\mu+\frac{1}{T_2}\right)}{(x_1-y_2)(1+(\rho-\mu)T_1)(1+\rho T_1)\left(\rho-\mu+\frac{1}{T_2}\right)S_n^{y_2}}$$

(6.20)   *Switch threshold*: $\lambda(S_n - K) = \lambda\Phi_2(S_n) + k$

The threshold for the switching point is based on an instantaneous trade-off argument where the owner studies the instantaneous costs and benefits from switching from one state to the other. The benefits are given on the left hand side of the equation, by $\lambda(S_n - K)$. The instantaneous costs to be considered are the increased probability of losing the investment project at completion given by $\lambda\Phi_2(S_n)$ and the increased ongoing investment cost, k, per unit of time.

At last we are left with the value of the investment project:

(6.21)
$$\phi(V) = \begin{cases} A_1 V^{x_1} & V < S_n, \\ B_2 V^{y_2} + \frac{VQ_1}{(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)} - \frac{kT+K}{1+\rho T_1} & V \geq S_n. \end{cases}$$

6.2 Model 2: Option to abandon

Miltersen and Schwartz (2007) propose a model where the owner of a project has the option to abandon development at any time-instant if the value of the outcome drops below an abandonment threshold level. They assume an instantaneous trade-off argument where the owner trades off the net benefits of further development against the value of being able to walk away from the project, assuming that the owner pays an ongoing investment cost per unit of time and thus have a contingent claim on the value of the outcome.

In our thesis, this model is based on the fact that several application cases contain dates of abandonment in our panel data. Hence, this option can be valuable. Additionally, we include a salvage value if the developer chooses to abandon the project.

6.2.1 Framework and Calculation

When constructing the Bellman equation for the active project, this has to be done in two steps using backward induction. The first equation describes the value from stage 2, construction and sales. The second equation is the value of the active state, including both the development stage and construction and sales.

(6.22) $$M(V) = VQ_1 dt - \rho K dt + (1 - \lambda_2 dt)(1 - \rho dt)E_V[M(V + dV)]$$

(6.23) $$\phi(V) = (1 - \rho dt)\lambda_1 dt \, E_V[M(V + dV)] + (1 - \lambda_1 dt)(1 - \rho dt)E_V[\phi(V + dV)]$$
$$-kdt.$$

The intuition behind these two functions are the same as in model 1, described in section 6.1.1.

In the event that it is optimal to sell the project rather than continue investing, it is realistic to assume that the project or land contains some value. The possible abandonment value of the project is determined by the equation, aVQ_2, where a represent the value ratio as a fraction of the value of the outcome. V is the price per square meter and Q_2 is the total amount of square meters that is expected to be sold.

We use Ito's Lemma of the Bellman equations. By breaking down the multi-period decision problems into smaller steps, we find a solution to the compensation from abandonment and the Ordinary differential equations (ODE):

(6.24) $$a(V) = aVQ_2, \qquad\qquad\qquad\qquad When: V < A_m$$

(6.25) $$\tfrac{1}{2}\sigma^2 P^2 \Phi''(V) + \mu P \Phi'(V) - (\rho + \lambda_1)\Phi(V) - k = 0, \qquad When: A_m \leq V < N$$

(6.26) $$\tfrac{1}{2}\sigma^2 P^2 \Phi''(V) + \mu P \Phi'(V) - (\rho + \lambda_1)\Phi(V) - k + \lambda_1 M(V) = 0, \quad When: V \geq N$$

Function (7.25) describes the value of the investment project when it is above the abandonment threshold level, but not in the money. The second function describes the project when it is above the investment threshold level, N.

The first two terms of the ODEs describe the uncertain development in the underlying price process. The third term in equation (7.25) describes the disadvantage of ending the project

and thus be unable to enjoy future price increases. The last term in (7.25) is the ongoing investment costs from obtaining an approval, k. The term $\lambda_1 M(V)$ from function (7.26) is only relevant when $V \geq N$, i.e. the project is in the money. If this is the case when the project is completed, the owner will invest the final investment cost, K, to claim the value of the active project.

The net present value of the active project, discounted by the subjective discount rate, less the drift of the value process, and adding the probability of completion in the second stage.

(6.27)
$$M(V) = \frac{VQ_1}{(\rho-\mu+\lambda_2)} - K$$

By re-arranging $M(V)$ and solving for V, we find the investment threshold level, N.

(6.28)
$$N = \frac{K(\rho-\mu+\lambda_2)}{Q_1}$$

The powers are the same as in model one, and the same conditions apply for removing D_1. Can then arrange the ODEs as follows:

(6.29)
$$\Phi_1(V) = aVQ_2$$

(6.30)
$$\Phi_2(V) = C_1 V^{y_1} + C_2 V^{y_2} - \frac{kT_1}{1+\rho T_1}$$

(6.31)
$$\Phi_3(V) = D_2 V^{y_2} + \frac{VQ_1}{(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)} - \frac{kT_1+K}{1+\rho T_1}$$

The criteria for the value matching – and smooth pasting conditions are the same as in model 1. In this model, we add a set of conditions since we have an additional state. The boundary constraints are given by (7.32-7.35):

(6.32)
$$Value\ matching\ condition: \Phi_1(A_m) = \Phi_2(A_m)$$

(6.33)
$$Smooth\ pasting\ condition: \Phi_1'(A_m) = \Phi_2'(A_{m,})$$

(6.34)
$$Value\ matching\ condition: \Phi_2(N) = \Phi_3(N)$$

(6.35)
$$Smooth\ pasting\ condition: \Phi_2'(N) = \Phi_3'(N)$$

Using the boundary constraints above, we find equations for the coefficients C_1, C_2, D_2 and the threshold A_m analytically. The calculation is in more detail in appendix B.

(6.36)
$$C_1 = -\frac{y_2 k T_1 - (1-y_2) a A_m Q_2 (1+\rho T_1)}{(y_1-y_2)(1+\rho T_1) A_m^{y_1}}$$

(6.37)
$$C_2 = \frac{y_1 k T_1 - (1-y_1) a A_m Q_2 (1+\rho T_1)}{(y_1-y_2)(1+\rho T_1) A_m^{y_2}}$$

(6.38)
$$D_2 = -\frac{A_m^{(1-y_2)} Q_1 Q_2 a (1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)}{y_2(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)}$$

(6.39) $\quad$ *Abandonment threshold*: $\quad \Phi_1(A_m) = \Phi_2(A_m)$

The value of the investment project is:

(6.40) $\quad \Phi(V) = \begin{cases} aVQ_2 & V < A_m, \\ C_1 V^{y_1} + C_2 V^{y_2} - \frac{kT_1}{1+\rho T_1} & A_m \le V < N, \\ D_2 V^{y_2} + \frac{VQ_1}{(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)} - \frac{kT_1+K}{1+\rho T_1} & V \ge N. \end{cases}$

This is true when $A_m < N$. In this case it can be optimal to continue investing when the project is out of the money, but above abandonment threshold. If this assumption changes to $A_m > N$, there is no longer optimal to invest in that state. New function for the value of the project would be simplified to:

(6.41) $\quad \Phi(V) = \begin{cases} aVQ_2 & V < A_m, \\ D_2 V^{y_2} + \frac{VQ_n}{(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)} - \frac{kT_1+K}{1+\rho T_1} & V \ge A_m. \end{cases}$

7. Numerical case

In this section we construct a numerical case based on qualitative interviews with industry professionals, project data from a development project in Bergen, and conclusions from own data gathering. We do so to implement realistic scenarios into our models and to provide a guiding example on how to optimize decision making in the given context. Under the assumptions we make, we can change critical input parameters to approximate how these changes impact investment behaviour. The underlying scenario covering the case is a real estate developer situated in Bergen. Their core business model is to buy raw property and convert it into residential real estate for sale. Thus, we assume that the project property is already bought for a fixed up-front fee. Differing between models, the owner of the project will have certain opportunities to alter the speed of development as information starts to uncover. If/when completed, construction and sales happens simultaneously over an expected period of two years.

7.1 Basis

Our case is based on a development project that is currently in the finishing stages of sales. Geographically, the project is placed in the intersection of the districts Fana and Årstad, and is initiated by a relatively small developer. According to interviews, the case in question is representative for their perception on how the development process normally occur.

Property size	Square meters
Total size	$6721m^2$
Gross area, building	$1853m^2$
Gross area, including garage	$2842m^2$

Table 9: Property information for base case

Cost approximation	Costs	Percentage of total
Contractor	NOK 64 500 000,00	60 %
Property	NOK 11 376 548,00	11 %
Procedural order rules	NOK 10 000 000,00	9 %

Table 10: Cost approximation from local developer

Fees approximation	Costs	Percentage of total
External planning	NOK 3 707 141,00	3 %
Internal planning	NOK 3 699 229,00	3 %
Sales/marketing	NOK 1 650 000,00	2 %
Other construction costs	NOK 6 513 344,00	6 %
Interest rates, pre-construction period (5 years)	NOK 1 650 413,00	2 %
Interest rates, construction period (2 years)	NOK 4 208 457,00	4 %
Total approximated	**NOK 114 305 132**	**100 %**

Table 11: Fees approximation from local developer

7.2 Inputs

7.2.1 Uncertainty

Expected time to completion: First we find the expected time for the development, T_1. As discussed earlier, we apply our dataset to obtain this input measure which gives 4.5 years as our base. The expected time for construction and sales, T_2, is assumed to be 2 years.

Drift parameter, μ: The drift represents the expected annual increase in the price process over time. To find this input figure, we annualize the percentage change in prices per square meter of houses in Bergen in the period 2005-2016. We find the drift parameter of the price process to equal 7.8%. This is however not durable in the long run, as prices on real estate have to

grow on average at the same pace as the rest of the economy. Knowing that the Norwegian government has an inflation target of 2.5% per year, that will be the long-run equilibrium growth rate of real estate prices.

Volatility of value process, σ: To find the volatility of the value process, we apply the Statistics Norway database for prices per square meter. Standard deviation of prices in the period 2005-2016 equals 23%.

Discount rate, ρ: Since we are considering a case where markets are not complete, we must make a subjective assessment about the required rate of return.

Barlindhaug and Nordahl (2011) find that uncertain future sales prices, construction costs, and the uncertain outcome of the regulatory process will increase risk and thus increase cost of capital. They comment that industry participants they have been in contact with outline cost of capital to be in the region 12-15% depending on project characteristics. Concluding on the discount rate, we find 15% to be a proper level.

7.2.2 Costs, price process and salvage value

Ongoing investment costs: We separate between ongoing investment costs and the final investment cost. Financial expenses related to property purchase, internal – and external planning and procedural order rules are assumed ongoing investment costs. The remaining costs constitutes the final investment cost.

Financial expenses: In our case, the purchasing fee amounted to approximately NOK 11 375 000. As it was financed through a mix of equity and debt, running interest rate expenses occurred throughout the development period. Without knowing the exact loan term agreement, we can work backwards to approximate an implicit annual interest rate in the waiting period.

Over a 5-year period, the company paid 1 650 000 in financial expenses. Assuming an equity ratio of 50%, amount borrowed constitutes 5 687 500. Hence, the simple five-year interest rate amount to 29%. Assuming yearly compounding of interest rates, we can find the effective annual interest rate using a simple formula:

(7.1) $$r = (1 + 0{,}29)^{\frac{1}{5}} - 1 = 5.22\%$$

To approximate yearly financial expenses in the planning process we have to find a figure to add to the ongoing investment costs. We are not able to do simulations in our models having financial expenses as a function of the loan, but have to land on an exact figure. Thus, we use the average yearly interest expenses, *NOK 330 000.*

Internal and external planning: From the table 11 we see that over a five-year period the internal planning is set to be NOK 3 699 229, and external planning is NOK 3 707 141. That equals a total of NOK 7 406 370. Annualized cash-flow assuming a 15% discount rate:

(7.2) $$\frac{x_1}{1.15} + \frac{x_2}{1.15^2} + \frac{x_3}{1.15^3} + \frac{x_4}{1.15^4} + \frac{x_5}{1.15^5} = \frac{7\ 406\ 370}{1.15^5}, \quad x = 1\ 098\ 480$$

Fees: Planning fees occur for the first phases of the process and must be paid in order to move to the next sequence. The fee structure is available at the municipal homepages[7]. We can estimate the expected costs for this project at Bergen municipality's fee-calculator.

(7.3) $$\frac{x_1}{1.15} + \frac{x_2}{1.15^2} + \frac{x_3}{1.15^3} + \frac{x_4}{1.15^4} + \frac{x_5}{1.15^5} = 409\ 860, \quad x = 122\ 267$$

Measures	Amount	Estimated price
Acreage: Fees on property acreage	$6721\ m^2$	NOK 211 510, 00
Start-up meeting – detailed zoning	1	NOK 24 810, 00
Process meeting – detailed zoning	2	NOK 29 740, 00
Proposed planning acc. to regulations on environmental impact assessment	1	NOK 143 800,00
Sum		NOK 409 860,00

Table 12: Expected zoning fees (Bergen kommune, 2016)

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[7] The fee-calculator can be accessed here: https://www3.bergen.kommune.no/gebyrkalkulator/#/menu1

Procedural order rules: From table 10 we see that procedural order rules adds to NOK 10 million. To incorporate as a part of the ongoing investment costs, we discount to present value and divide it into five equals amounts.

Discounted to present value: 10 000 000/1.15^5 = 4 971767

Annual cash flow equals: 994 353

Applying a subjective discount rate of 15%, we can summarize the ongoing investment costs for our project and conclude on the amount, k.

$$k = 2\,545\,100$$

Final investment cost, K: When the development project is completed, the developer will have an option to pay the final development cost, K, to obtain the net benefits of the active project. In this project, the final investment cost will include all relevant costs associated with construction of the intended project. As can be seen from the table above, the final investment cost includes financial expenses in the construction period, sales/marketing efforts to generate sales and the contractor-cost. Additionally, we add the full repayment of the property-loan as a lump sum to simplify.

Present value: $K = 82\,550\,500/1{,}15\text{^}5 = 41\,042\,000$

Salvage value: In model two we allow the owner of the project to sell it for a pre-determined fraction of the value of the outcome. This is an addition to Miltersen and Schwartz (2007) that seem proper when applied to real estate development. To find a fair estimate of salvage value, we take; $\frac{Property\ value\ pre\ development}{Sales\ price} = a = 15\%$.

Input Summary	
T_1	4,5 $years$
T_2	2 $years$
μ	2,5%
σ	23%
ρ	15%
k	NOK 2 545 000
K	NOK 41 042 000
Q_1	751,5m^2
Q_2	1503m^2
a	15%

Table 13: Numerical input summary

8. Using the models to evaluate alternatives: Simulation analyses

In this section we combine our analytical models to the numerical case presented previously to outline optimal investment behaviour in this context. We go on to compare our findings to those of static discounted cash flow approaches for both models and compare our results to other empirical papers. To prepare the decision maker for potential future operating changes, we change critical input parameters based on two recent governmental initiatives.

In 2015, government presented their new strategy for the housing market, which emphasized the importance of efficient and predictable planning processes in order to increase housing supply (Regjeringen, 2015). A vast amount of initiatives was proposed, and only minor proposals have been implemented as of 2016. Additional to measures meant to reduce processing time, political parties have also proposed to reduce the impact of costly quality restrictions related to construction. By removing restrictions that are costly to implement, but have very limited impact on the actual quality, the stated goal is to increase supply.

8.1 Model 1: Switch option

In the first model we have a situation where the developer has the option to halt investment if the value of the outcome drops below the investment threshold level, and vice versa re-start investment if the price moves above the same threshold level, in addition to the perpetual call option on the value of the outcome.

Remember that in the passive state, both ongoing investment costs and the probability that the project will end in the next time instant drops to zero. Oppositely, when price steps above the investment threshold level (i.e. is in the money), ongoing investment costs become $k,$ while the intensity of completion is λ_1. The intuition behind the switching threshold is that the owner of the project continuously trade-off the costs and benefits of holding the project active/passive. The increased benefits from going into an active state is given by $\lambda_1(S_n - K)$ per time unit, while costs increase by $\lambda_1 N_2(S_n) + $ k, where $\lambda_1 N_2(S_n)$ describes the increased intensity of losing the project, and k the ongoing investment costs.

In figure 11, the threshold level is depicted as a function of expected time to completion. As mentioned previously, the threshold level represents the price level per square meter in which

it is optimal for the developer to halt further investment if the value drops below, and alternatively re-start investment if the value climbs above. Put differently, figure 11 provides us with the value-maximizing behaviour when taking into account the price process, time to completion, ongoing investment costs, option of the value of the outcome and the fact that we have an option to switch between states.

On the y-axis, the reader can see the price (in millions) per square meter, while the x-axis gives different expected times to completion. Using our base case as an example, we see that the optimal switching point can be found at NOK 42 600 per square meter. If prices per square meter moves below this value at any time during the approval process, it is optimal for the decision maker to temporarily abandon further investment. Oppositely, if prices rise above this level, it is optimal to re-start the planning process with the aim to obtain approval as soon as possible.
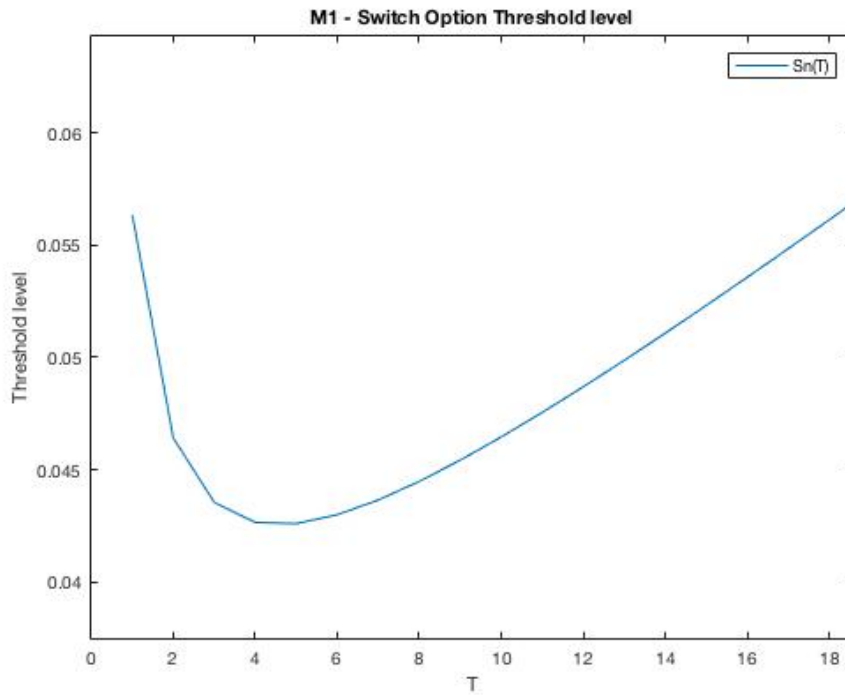
Figure 11: Switch threshold for different expected time to completion of stage 1 (Matlab)

As one can see from figure 11, the switching threshold level is high for short expected times to completion. Further, the threshold level decreases until $T = 5$. Intuitively, we would expect a positive relationship between expected time to completion and the respective threshold levels. To determine this strictly, we must look beyond our first intuition and revisit the forces

that are working in opposite directions. For very short times to completion, ongoing investment costs will be smaller as fewer periods are drawn. In isolation, this follows our first intuition covering the positive relationship. Another effect, the final investment cost, K, works in the opposite direction. As this value is discounted to present value, a shorter time to completion will in isolation increase this value and support a negative relationship between expected time to completion and the switching threshold level. The final effect that we must consider is the value of the outcome, V. All else equal, postponing a payment will support a positive relationship between threshold level and expected time to completion.
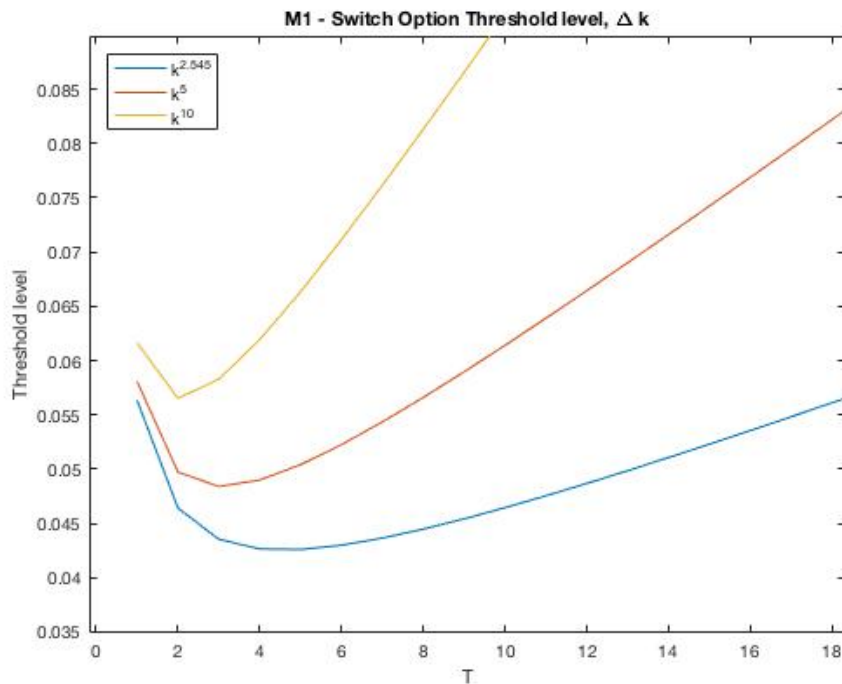
Figure 12: Switching threshold level with different ongoing investment costs (Matlab)

With this is mind, we can conclude that the postponement of K dominates the opposing effects until $T = 5$. Thereafter, the effect from postponing the payment of V and the relative increase in ongoing investment costs are the dominating factors. Miltersen and Schwartz (2007) argues for such a non-monotonic relationship to be reasonable for very short times to completion. Values below $T = 5$ can hardly be assumed as very short times to completion. One reason why our turning point is found at a relatively high value is our level of ongoing investment costs. Miltersen and Schwartz (2007) approximates that if ongoing investment costs as a percentage of the final investment cost are in the region 1-4%, the value of the investment project may be increasing with the expected time to completion, as the opposite effects

mentioned above work in favour of postponing K. As our ongoing investment costs are 6.2% of the final investment cost, one plausible explanation why the effect from increasing expected time to completion is negative until such high levels can be partly explained by this. In figure 12 we investigate some of this effect by changing the ongoing investment costs to 24% (yellow) – and 12% (red) of the final investment cost. The blue line is our base case. From figure 12 we see clearly that the negative relationship appears for the (much) shorter times to completion of 2,1 – and 3 years respectively, thus supporting this view.

Figure 13 shows the value of the investment project as a function of the value of the outcome, V. The red line represents our base case scenario with its expected time to completion of 4.5 years, while the yellow – and blue represent cases where expected time to completion is 6.75 (yellow) – and 3.15 (blue) years respectively.

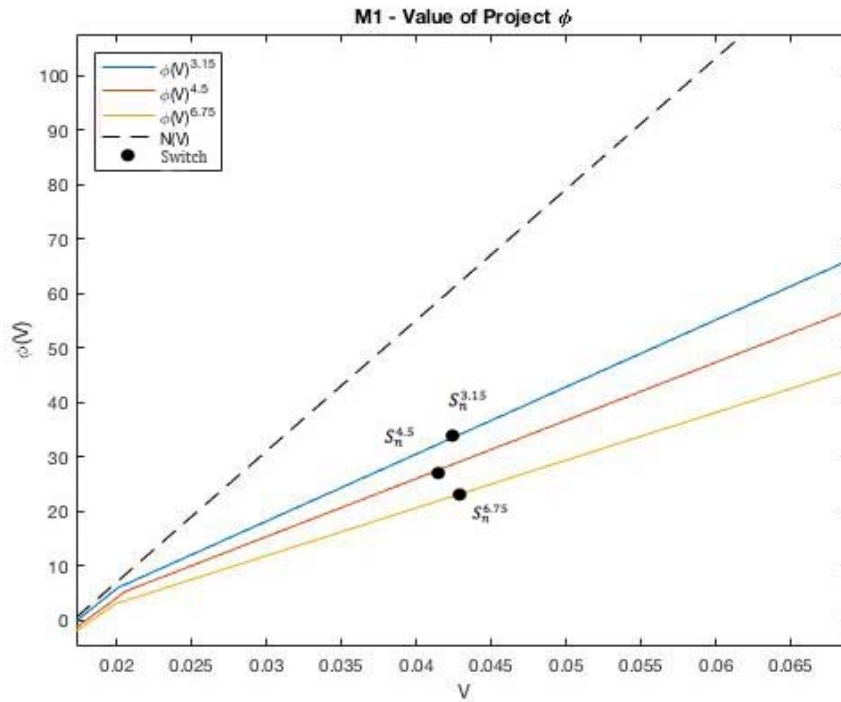*Figure 13: Value of investment project as a function of the value of the outcome (Matlab)*

We see the change in threshold levels $S_n^{3.15} = 0.0435$, $S_n^{4.5} = 0.0426$ and $S_n^{6.75} = 0.0433$, is consistent with the switch threshold curve in figure 11. The dashed line is the net present value had we removed the uncertain waiting period, and thus the ongoing investment costs and the project were completed immediately.

Comparison of decision making rules; discounted cash flow approach vs. switching model

As Miltersen and Schwartz (2007) points out, since switching between states is costless, the decision to abandon will never be optimal under these circumstances. On a similar note, they argue that since state-shifting is costless, the development phase will never be completed unless the project is in the money, denoted by $V \geq K$. From chapter 8, we remember that K equals NOK 41 020 000. For a project containing 9 units of 167 square meters per unit, the final investment cost per square meter equals NOK 27 292. Compared to our state-switching point of NOK 42 600, our base case is consistent with theory in the sense that the development phase will never be progressed unless into the money.

From a simple discounted cash-flow approach using the same inputs as in our model, we find that the investment threshold level, i.e. $NPV = 0$, to be at 32 738 per square meter. Using the classical decision making rule, all projects where the projected value of the outcome is less than 32 738 should be rejected. Comparing this figure to the NOK 42 600 threshold we found previously, we must remember that the model also consists of an American call option on the value of the outcome. Introducing an option to switch costlessly between states allows the developer a timing feature in the sense that he can to a greater extent decide under which circumstances the irreversible, final investment cost should be taken. Since expected time to completion of the first phase is driven by an exponential random variable, the project can be lost in the next time-instant. This effect is taken into account in the continuous cost-benefit trade-off a developer must make to optimize decision making, given by λ_1 in $\lambda_1 N_2(S_n)$.

Hence, our findings are consistent with the likes of McDonald and Siegel (1986), Titman (1985) and others, that find the investment threshold level to be higher than the traditional NPV-rule when future states of the world are uncertain and investments are irreversible. Common arguments in the literature is that when faced with an American call option, the investment threshold level should be adjusted to take into account the fact that by making the investment, one also foregoes the opportunity to wait for new information to be revealed.

In our context, this would relate to the price-risk, namely that by executing the option, one foregoes the opportunity to enjoy future price increases. From a strategic decision making point of view, this can help explain why attractive urban areas are left undeveloped.

Capozza and Li (1994) find that the option to vary capital intensity in real estate development will affect both timing of development and project values. They argue that this ability will add to the positive hurdle value above the traditional rule to accept projects with positive expected net cash flows. In a similar study, Capozza and Li (2002) confirm their own findings from 1994, but adds to theory by showing that even under certainty is it optimal to delay a project further than the classical net present value rule, as long as cash flows are growing. They add that the ability to vary capital intensity will increase the likelihood to delay even further.

By undertaking the final investment cost at maturity, we exercise both the option to switch between a passive – and active project, and the call option on the value of the outcome. Therefore, it seems intuitive that our investment threshold level takes both these options values into account when determining the optimal investment threshold to be NOK 42 600 as opposed to NOK 32 738.

Outlining potential effects from changes in inputs

We have seen from figures 11-12 that the sensitivity of the switching point is related to expected time to completion. For projects with relatively larger ongoing investment costs, such as the red case from figure 12 where the ongoing costs in relation to the final investment cost are 12%, we see clearly that reducing the expected time to completion from $T = 4.5$ to $T = 3$ to reduce the switching threshold level. Moving up to the yellow case, where ongoing investment costs are 24% of the final investment costs, we see a significant reduction of the switching threshold level of going from $T = 4.5$ to $T = 2$.

For the case with reduced construction costs due to the removal of costly quality requirements, that will affect the contractor costs which is part of the final investment costs, K. Since the stated goal is to reduce quality restrictions that are redundant, we can make the assumption that the decrease in construction costs will in large part be received by the developer. From figure 14 (below) we see the effect from reducing final investment costs, all else equal.
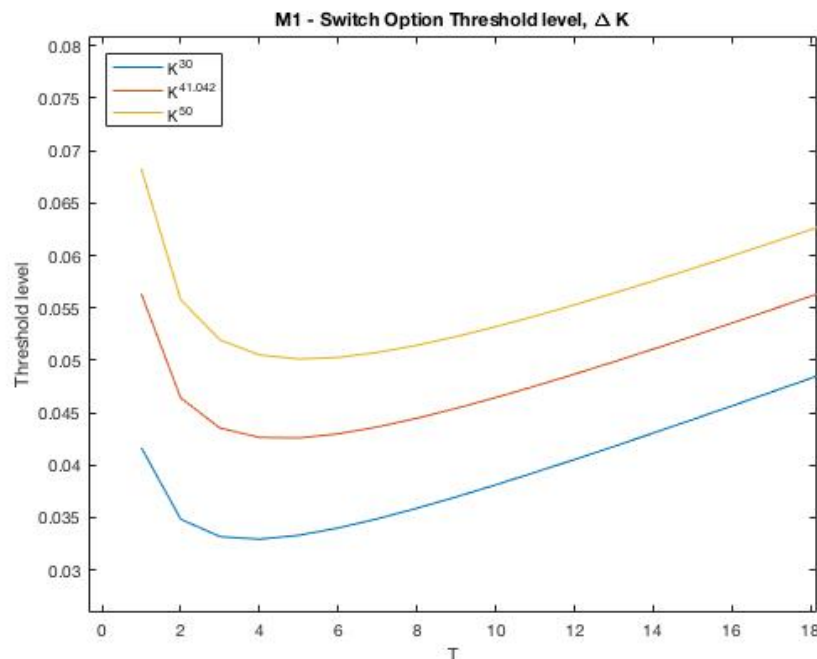
Figure 14: Switching thresholds for different expected times to completion
with different values for the final investment cost (Matlab)

The three lines in figure 14 represent cases where K = 30 (blue), 41.042 (red) and 50 (yellow). The examples above outlines significant changes in the final investment costs compared to the base case (red), both up and down. The blue case represents the argument above figure 14, where government removes quality restrictions, if everything else is held constant. On the contrary, the yellow line represents a case where the final investment costs have increased. This is unrelated to the removal of costly quality restrictions, but can become present if other initiatives are taken that will increase construction costs. We see clearly that as final investment costs increase, threshold levels for all expected times to completion also increase. This is equal to developers holding projects passive for larger values, thus increasing the actual time to completion, and reducing housing supply. For the blue case, where the final investment costs are reduced by 26%, the final investment cost per square meter equals 19 960 with the new assumption. At the same time, the threshold level is reduced to approximately NOK 33 000. Hence, a reduction of 26% of the final investment cost can potentially equal a 21% decrease in the investment threshold level.

8.2 Model 2: Abandonment Option

When the owner has an option to abandon the development project if the value of the outcome drops below some threshold level, he will continuously trade off the net benefits of keeping the project active versus liquidating for salvage value. One important distinction from the previous model is that when the project is no longer active, progress can never be re-started, making this decision irreversible. Hence, we are considering the put option of abandonment versus the embedded American call option on the value of the outcome. As opposed to the switching model, we will have two different threshold levels – one to describe the price at which it is more profitable to abandon the project for salvage value, and the other a point where it is optimal to invest if the project is completed. Throughout this section, we will also see a special case where the abandonment threshold level is above the final investment cost.

Figure 15 depicts the threshold level as a function of expected time to completion. As opposed to the switch option model, the abandonment threshold level is strictly increasing with expected time to completion. For our base case, we notice that the optimal value for abandonment is at a price per square meter of 0.0303, or NOK 30 300. For the events where the value of the outcome drops below this level, it is more valuable to abandon for salvage value than to continue development.
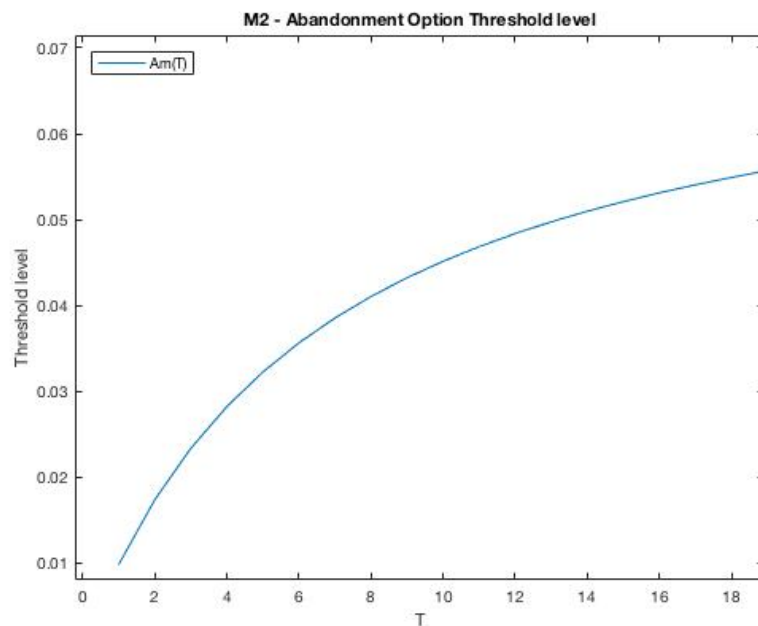
Figure 15: Abandonment threshold level as a function of expected time to completion (Matlab)

The strictly increasing change in the threshold values can be explained by the salvage value of the project at abandonment. By continue investing in the project, one postpones a future payment. Even more essential for the explanation of the threshold behaviour is the on-going costs, where duration is an important driver behind k. A longer expected duration will increase expected ongoing investment costs, essential to obtain the opportunity for approval. The trade-off between continued investments or abandon for salvage value will therefore have a strictly increasing relationship. If the value at abandonment was equal to zero, the graph would be more similar to the switching threshold shown in figure 11, which is shown in Miltersen and Schwartz (2007).

The value of the investment project is illustrated in figure 16 as a function of the value of the outcome, V. In addition, there are different scenarios of expected duration, illustrating its effects on project value over future price levels. The black line represents the net present value of the project if it was immediately active. That is, if construction and sales could be obtained immediately, denoted $M(V)$. The coloured lines display the value of the exploration stage in the different scenarios, denoted $\Phi(V)$. At last the dashed black line is the NPV threshold level, where the value of the active project is zero, denoted $N(V) = 0$.

Figure 16: Value of investment project as a function of the value of the outcome (Matlab)

The respective scenarios T = 3.15 (Blue), T = 4.5 (Red) and T = 6.75 (Yellow), shows the corresponding change in threshold levels $A_m^{3.15} = 0.0241$, $A_m^{4.5} = 0.0303$ and $A_m^{6.75} = 0.0379$. A steady increasing effect on the abandonment threshold with expectations o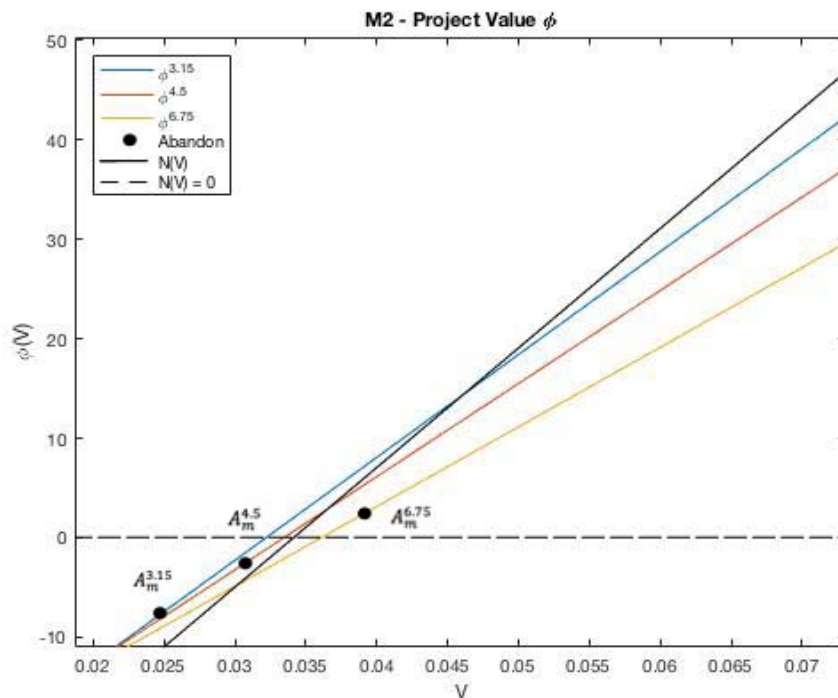f longer zoning duration. These thresholds can also be found in figure 15, reading off the threshold values for the respective time. The NPV threshold for the active project is $N = 0.0341$, the intersection where $N(V)$ crosses the NPV threshold line.

The first two thresholds, $A_m^{3.15} = 0.0241$ and $A_m^{4.5} = 0.030$, are beneath the NPV threshold, indicating that it is optimal to continue with development despite the project being out of the money. If completion is reached in this state, it would invest in a project that is out of the money, or not too far in the money, resulting in a negative jump in value, a state where it is no longer optimal to keep the project. An event where a manager has to decide whether to sell for salvage value or invest the fixed investment cost K, where in this model the manager is forced to sell. It would be optimal to include an option to wait with the decision to invest, letting the owner hold it passive until the price level reaches a value that is in the money, but this option is not included in the model. The project would in reality get a value increase when achieving a zoning approval, this is not considered either. In the case where we increase expected time to completion to T = 6.75 years while holding everything else constant, the threshold level $A_m^{6.75} = 0.0379$, is above the NPV threshold. This indicates that one would abandon the project even though the project is in the money. In this case we get $A_m > N$, causing the continue to invest state to lose its value, and the formula calculating the value to invest switches from equation 6.40 to 6.41.

Comparison of decision making rules; discounted cash flow approach vs. abandonment model

From section 9.1 we remember that applying a static discounted cash-flow approach would yield acceptance for projects if expected prices were above NOK 32 738. Below this point, the project will not be undertaken. From the base case we see that the abandonment threshold level is NOK 30 300 when salvage value is included. At this point, the developer will prefer the compensation received from salvaging the project rather than continuing and incur ongoing investment costs. Further, if completed, the owner of the project will prefer to abandon instead of undertaking the final investment cost at all prices below NOK 34 100. In the interval between NOK 30 300 and 34 100 it is optimal to continue development. Hence, the developer will prefer to continue investing in an area which would be regarded as value-decreasing had

we utilized a traditional discounted cash flow approach. Bar-Ilan and Strange (1996) argues that in the presence of an abandonment option, profit-maximizing developers can have an incentive to start development despite depressed markets to avoid being out of market when prices increase. However, if the project suddenly finishes whilst prices are in this region, it will become optimal to abandon for salvage value.

Outlining potential effects from changes in inputs

In this section our aim is similar as in section 8.1. Potentially changing circumstances due to governmental actions can be amongst many reasons that can potentially change operating conditions from a developer's point of view. Hence, to draw up scenarios that can be likely is an important exercise to obtain the realism of our framework. In addition to the changes proposed in section 8.1, we also add a scenario where the proposal is not granted fully. That is, the utilization rate is lowered from nine to seven units. Further, in this model the procedural order rules are a part of ongoing investment costs. Being a much-debated theme, and described as a reason in itself why several applications are abandoned, the effect of changes can be interesting from several points of view.

First we look at the effect from changes in ongoing investment costs. The impact of such changes is illustrated in figure 17, where the abandonment function is altered with three different values of on-going investment costs. The costs are from a very low level at k = 1 (Blue), base level k = 2.545 (Red), and a high level k = 5 (Yellow).
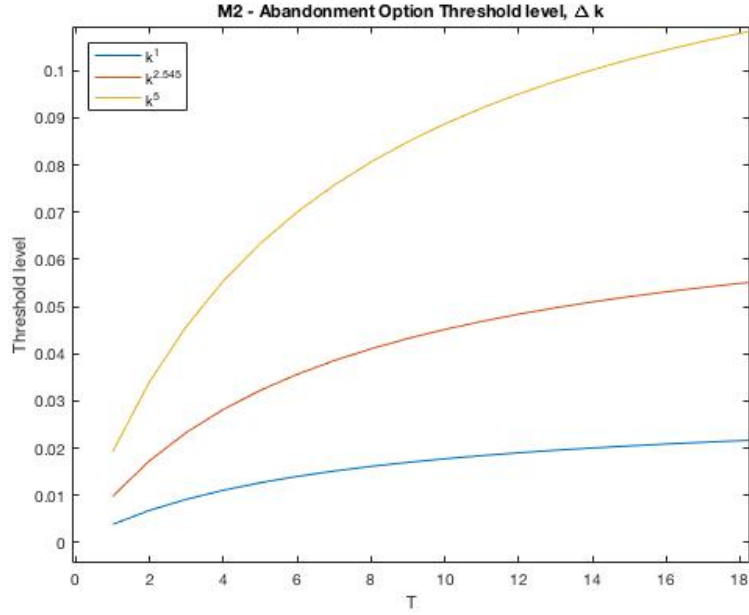
Figure 17: Abandonment threshold levels for different expected times to completion, for different ongoing investment costs (Matlab)

We see that the abandonment threshold level is higher for all expected times to completion when ongoing investment costs are increased. Higher ongoing investment costs can be due to higher expected procedural order rules, additional quality requirements, or other measures that will adversely affect ongoing investment costs relative to the sales price. On the contrary, reductions in ongoing investment costs can be due to reductions in the above-mentioned cost levels, or due to more efficient planning processes.

From figure 17 we see that governmental initiatives to reduce expected time to completion, or reduce ongoing investment costs will, all else equal, have the potential to increase housing supply as developer's incentives to abandon a project for salvage value is reduced.

Figure 18 presents a downward change in utility rate as a result of the outcome of the final verdict of the planning process. As mentioned in section 2.3, a large risk in real estate development is the inability to choose the optimal level of utilization of the property deterministically. Having planned for nine units, the developer will react negatively to a suboptimal utilization rate. In addition to reducing the utilization rate, Q_1 and Q_2, we reduce the contractor cost by 22% to account for the relatively less mass of building that must be constructed. However, ongoing investment costs and the remaining parts of the final investment cost is held constant as the developer planned his activities based on nine units.

From figure 18, we see the effect of lowered utilization rate. The three lines represents the base case (red) with its expected 4.5 years to completion, and the additional low estimate of 3.15 years (blue) and a high estimate of 6.75 years (yellow).
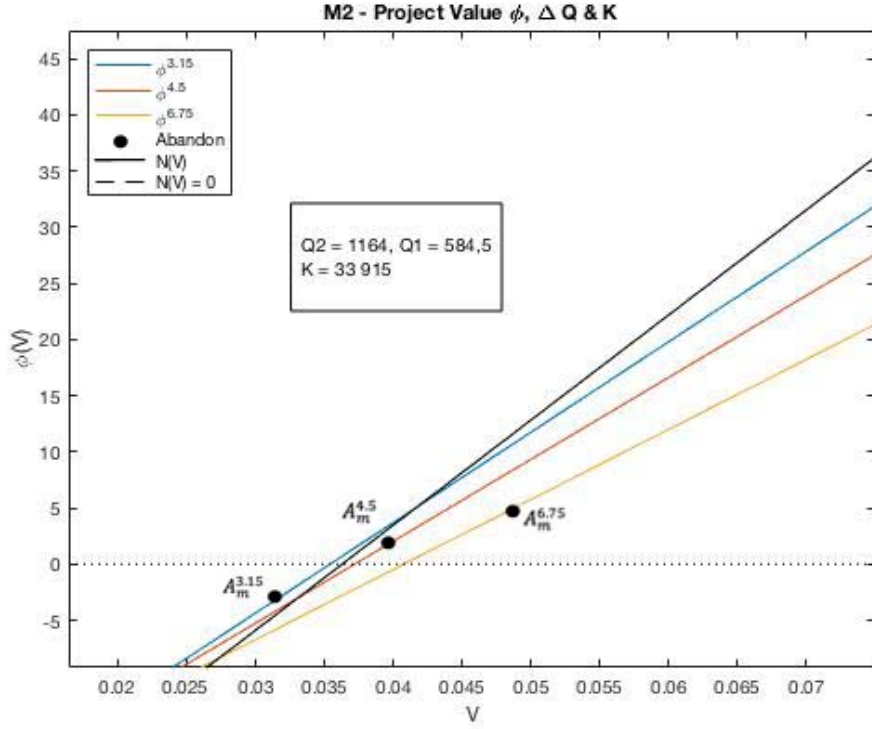
Figure 18: Value of investment project as a function of the value of the outcome (Matlab)

We see the change in threshold levels $A_m^{3.15} = 0.031$, $A_m^{4.5} = 0.039$ and $A_m^{6.75} = 0.0487$. The NPV threshold for the active project becomes $N = 0.0363$. Hence, the effect from a lowered utilization rate is an increase in both the abandonment – and investment thresholds for all expected times to completion. In previous discussions, we outlined the effects of reducing the expected time to completion by for instance 25% to T = 3.5. We see from the blue line the full effect of being able to reduce the expected time to completion for the planning process by one year, but at the same time delivering a negative shock to the developer in terms of a lower-than expected utilization rate. Of course, in many instances proposals can be unrealistic or necessary to scale down from a societal point of view. The scenario outlined above can serve as a typical scenario where government reduces one negative impact, but at the same time upholds another.

9. Conclusion

In this thesis we have proposed a framework that allows the user to take into account cost – and revenue uncertainty in real estate development projects with uncertain time to completion. We have applied two different models to account for managerial flexibilities in local real estate development projects. By allowing for these options in our framework, our aim was to develop a framework for optimal decision making and valuation for real estate development projects located in Bergen.

We drew on Miltersen and Schwartz (2007) for the main framework, but added a fixed time period to take into account the construction – and sales period, which was inspired by Bar-Ilan and Strange (1998). Additionally, we borrowed from Bar-Ilan and Strange (1998) and Teisberg (1994) to add the fractional salvage value that we deemed necessary when taking into account that a property generally has a re-sale value. Through the collection of data and performance of survival analysis, we identified two real options that could potentially be of value. Further, our data collection ensured accuracy for a key input parameter that was put in direct relation to ongoing investment costs.

Both our models allow us to obtain closed-form solutions to the value of the investment project and to decide optimal investment – and abandonment threshold levels. For the switch option model, we find the investment threshold to be significantly above what would be expected from a traditional discounted cash flow approach, which is consistent with existing theory. It is optimal to change intensity of development if the value of the outcome develops over/under NOK 42 600. The model is generally responding to what we would expect from theory, but has an inconsistency in terms of the relationship between the threshold level and expected time to completion. This is outlined by Miltersen and Schwartz (2007) to be plausible for low levels of on-going investment costs. When testing for this property, we see that the model responds by showing a more consistent relationship.

When allowing for salvage value in model 2, we find that marginally profitable projects can be more likely to be initiated compared to the static discounted cash flow approach. The model reacts equal to what theory suggests when we include salvage value. It is optimal to abandon the development project if the value of the outcome drops below NOK 30 300 and to invest if the value of the outcome moves above NOK 34 100.

For both models, we obtain results that can act as guidelines for optimal investment behavior for a profit-maximizing real estate developer in Bergen municipality. By changing critical input parameters based on plausible future policy changes, we have analyzed the effect from potential forthcoming policy changes to show the ability of the framework to consider future operating changes. This ability also tells us that the framework can be applied to different cases by fitting the input parameters to the relevant case.

Some interesting additions to the framework would be to combine the two models to one model to account for options to appear simultaneously. Making a switch model that uses different rates of intensity in the investment could also be interesting. Another idea could be to allow for real options behavior in the construction phase.

10. References

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Appendix - A

Underlying calculation of coefficients in model 1, using value matching and smooth pasting conditions to solve analytically. The first step is to find a function of A_1, by using the first boundary condition:

$$\text{Value Matching condition:} \qquad \Phi_1(S_n) = \Phi_2(S_n)$$

(10.1)
$$A_1 S_n^{x_1} = B_2 S_n^{y_2} + \frac{S_n Q_1}{(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)} - \frac{kT+K}{1+\rho T}$$

(10.2)
$$A_1 = \frac{B_2 S_n^{y_2}}{S_n^{x_1}} + \frac{S_n Q_1}{(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)S_n^{x_1}} - \frac{kT+K}{(1+\rho T)S_n^{x_1}}$$

Finding the smooth pasting condition to later insert the function of A_1, found above, to resolve for B_2.

$$\text{Smooth Pasting condition:} \qquad \Phi_1'(S_n) = \Phi_2'(S_n)$$

(10.3)
$$x_1 A_1 S_n^{x_1-1} = y_2 B_2 S_n^{y_2-1} + \frac{Q_1}{(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)}$$

$$y_2 B_2 S_n^{y_2-1} = \frac{Q_n}{(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)} + x_1 \left(\frac{B_2 S_n^{y_2}}{S_n^{x_1}} \right.$$

$$\left. + \frac{S_n Q_1}{(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)S_n^{x_1}} - \frac{kT+K}{(1+\rho T)S_n^{x_1}} \right) S_n^{x_1-1}$$

$$(x_1 - y_2)B_2 = \frac{Q_1}{(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)S_n^{y_2-1}} + \frac{x_1 Q_1}{(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)S_n^{y_2-1}} -$$
$$\frac{x_1(kT+K)S_n^{-1}}{(1+\rho T)S_n^{y_2-1}}$$

Common nominator: $(x_1 - y_2)(1 + (\rho - \mu)T)\left(\rho - \mu + \frac{1}{T_2}\right)(1 + \rho T)S_n^{y_2 - 1}$

(10.4) $\qquad B_2 = \dfrac{x_1(kT+K)(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)+(1-x_1)Q_n(1+\rho T)S_n}{(x_1-y_2)(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)(1+\rho T)S_n^{y_2}}$

At this stage, the function B_2, can be used and inserted to find the final function of A_1.

(10.5) $\quad A_1 = S_n^{y_2 - x_1}\left(\dfrac{x_1(kT+K)(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)+(1-x_1)Q_n(1+\rho T)S_n}{(x_1-y_2)(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)(1+\rho T)S_n^{y_2}}\right) +$

$\dfrac{S_nQ_n}{(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)S_n^{x_1}} - \dfrac{kT+K}{(1+\rho T)S_n^{x_1}}$

$$A_1 = \left(\frac{x_1(kT+K)(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)S_n^{-x_1} + (1-x_1)Q_n(1+\rho T)S_n^{1-x_1}}{(x_1 - y_2)(1 + (\rho-\mu)T)(1+\rho T)}\right)$$

$$+ \frac{S_n^{1-x_1}Q_n}{(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)} - \frac{(kT+K)S_n^{-x_1}}{(1+\rho T)}$$

Common nominator: $(x_1 - y_2)(1 + (\rho - \mu)T)\left(\rho - \mu + \frac{1}{T_2}\right)(1 + \rho T)$

(10.6) $\qquad A_1 = \dfrac{y_2(kT+K)(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)+(1-y_2)Q_n(1+\rho T)S_n}{(x_1-y_2)(1+(\rho-\mu)T)\left(\rho-\mu+\frac{1}{T_2}\right)(1+\rho T)S_n^{x_1}}$

Since the switching option is only available in the development process, the switch threshold will only consider the first stage. This makes it possible to get a analytical solution to the threshold between the active and passive state, we solve for S_n:

(10.7) $\qquad \lambda(S_n - K) = \lambda\Phi_2(S_n) + k$

$$\lambda(S_n - K) = \lambda\left(B_2V^{y_2} + \frac{S_nQ_1}{(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)} - \frac{kT+K}{1+\rho T_1}\right) + k$$

$$(S_n - K) = B_2V^{y_2} + \frac{S_nQ_1}{(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)} - \frac{kT+K}{1+\rho T_1} + \frac{k}{\lambda}$$

$$(S_n - K) = \left(\frac{(1 - x^1)Q_1 S_n(1 + \rho T_1) + x^1(1 + (\rho - \mu)T_1)(kT_1 + K)}{(x_1 - y_2)(1 + (\rho - \mu)T_1)(1 + \rho T_1)S_n^{y_2}} \right) S_n^{y_2}$$

$$+ \frac{S_n Q_1}{(1 + (\rho - \mu)T_1)} - \frac{kT + K}{1 + \rho T_1} + \frac{k}{\lambda}$$

$$(S_n - K) = \frac{(1 - x^1)Q_1 S_n(1 + \rho T_1) + x^1(1 + (\rho - \mu)T_1)(kT_1 + K)}{(x_1 - y_2)(1 + (\rho - \mu)T_1)(1 + \rho T_1)} + \frac{S_n Q_1}{(1 + (\rho - \mu)T_1)}$$

$$- \frac{kT + K}{1 + \rho T_1} + \frac{k}{\lambda}$$

Common nominator: $(x_1 - y_2)(1 + (\rho - \mu)T_1)(1 + \rho T_1)\lambda$

(10.8) $S_n = -\frac{kT_1 + K}{(1 + \rho T_1)} + \frac{K}{T_1} + \frac{x_1(kT_1 + K)(1 - (\rho - \mu)T_1)}{(x_1 - y_2)Q_1 T_1(1 + \rho T_1)} - \frac{1}{T_1} + \frac{(x_1 - 1)Q_1}{(x_1 - y_2)T_1(1 - (\rho - \mu)T_1)}$

Appendix - B

Underlying calculation of coefficients in model 2, using value matching and smooth pasting conditions to solve analytically. The first step is to find a function of C_1, by using the first boundary condition. Next we find the smooth pasting condition, and find an equation for C_2, before we again use the smooth pasting condition to find D_2.

Value matching condition: $\Phi_1(A_m) = \Phi_2(A_m)$

(10.9) $aA_m Q_2 = C_1 A_m^{y_1} + C_2 A_m^{y_2} - \frac{kT_1}{1 + \rho T_1}$

(10.10) $C_1 = \frac{aA_m Q_2}{A_m^{y_1}} - \frac{C_2 A_m^{y_2}}{A_m^{y_1}} + \frac{kT_1}{(1 + \rho T_1)A_m^{y_1}}$

Smooth pasting condition: $\Phi_1'(A_m) = \Phi_2'(A_{m,})$

(10.11) $aQ_2 = y_1 C_1 A_m^{y_1 - 1} + C_2 A_m^{y_2 - 1}$

(10.12) $C_2 = \frac{A_m^{1 - y_2}\left(aQ_2 - y_1 C_1 A_m^{y_1 - 1}\right)}{y_2}$

Smooth pasting condition: $\Phi_2'(N) = \Phi_3'(N)$

(10.13) $\qquad y_1 C_1 A_m^{y1-1} + y_2 C_2 A_m^{y2-1} = y_2 D_2 A_m^{y2-1} + \dfrac{Q_1}{(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)}$

(10.14) $\qquad D_2 = \dfrac{A_m^{1-y2}\left(y_1 C_1 A_m^{y1-1} - \frac{Q_1}{(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)} + y_2 C_2 A_m^{y2-1}\right)}{y_2}$

When we now have found a set of function for each coefficient, we substitute and find the algebraic functions for each coefficient. Starting with C_1:

(10.15) $\qquad C_1 = a A_m^{1-y1} Q_2 - A_m^{y2-y1}\left(\dfrac{A_m^{1-y2}\left(a Q_2 - y_1 C_1 A_m^{y1-1}\right)}{y_2}\right) + \dfrac{kT_1}{(1+\rho T_1)A_m^{y1}}$

$$-y_1 C_1 = a A_m^{1-y1} Q_2 - \dfrac{a A_m^{1-y1} Q_2}{y_2} + \dfrac{kT_1}{(1+\rho T_1)A_m^{y1}}$$

Common nominator: $y_2(1+\rho T_1)A_m^{y1}$

$$C_1 = -\dfrac{y_2 kT_1 - (1-y_2)a A_m Q_2(1+\rho T_1)}{(y_1-y_2)A_m^{y1}(1+\rho T_1)}$$

Inserting and finding the algebraic function of C_2.

(10.16) $\qquad C_2 = \dfrac{A_m^{1-y2}\left(a Q_2 - A_m^{y1-1}\left(\frac{a A_m Q_2}{A_m^{y1}} - \frac{C_2 A_m^{y2}}{A_m^{y1}} + \frac{kT_1}{(1+\rho T_1)A_m^{y1}}\right)y_1\right)}{y_2}$

$$-y_1 C_2 = \dfrac{(1-y_1)a A_m^{1-y2} Q_2 - \dfrac{y_1 kT_1 A_m^{-y2}}{(1+\rho T_1)}}{y_2}$$

Common nominator: $y_2(1+\rho T_1)A_m^{y2}$

(10.17) $\qquad C_2 = \dfrac{y_1 kT_1 - (1-y_1)a A_m Q_2(1+\rho T_1)}{(y_1-y_2)A_m^{y2}(1+\rho T_1)}$

Inserting and finding the algebraic function of D_2.

(10.18)

D_2

$$= \dfrac{A_m^{1-y2}\left(y_1\left(-\dfrac{y_2 kT_1-(1-y_2)a A_m Q_2(1+\rho T_1)}{(y_1-y_2)A_m^{y1}(1+\rho T_1)}\right)A_m^{y1-1} - \dfrac{Q_1}{(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)} + y_2\left(\dfrac{y_1 kT_1-(1-y_1)a A_m Q_2(1+\rho T_1)}{(y_1-y_2)A_m^{y2}(1+\rho T_1)}\right)A_m^{y2-1}\right)}{y_2}$$

$$D_2 = -\frac{y_1 y_2 k T_1 - y_1(1-y_2)a A_m^{1-y2} Q_2(1+\rho T_1)}{(y_1-y_2)(1+\rho T_1)} - \frac{Q_1 A_m^{1-y2}}{(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)} + \frac{y_2 y_1 k T_1 - y_2(1-y_1)a Q_2(1+\rho T_1)}{(y_1-y_2)(1+\rho T_1)}$$

common nominator: $(y_1-y_2)(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)(1+\rho T_1)$

(10.19)
$$D_2 = -\frac{A_m^{1-y2}\left(Q_1 - a Q_2(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)\right)}{y_2(1+(\rho-\mu)T_1)\left(\rho-\mu+\frac{1}{T_2}\right)}$$

We can also find a analytical solution to the threshold for the abandonment option, we solve for A_m:

Abandonment threshold: $\Phi_1(A_m) = \Phi_2(A_m)$

(10.20)
$$a A_m Q_2 = C_1 A_m^{y1} + C_2 A_m^{y2} - \frac{kT_1}{1+\rho T_1}$$

$$a A_m Q_2 = \left(-\frac{y_2 k T_1 - (1-y_2)a A_m Q_2(1+\rho T_1)}{(y_1-y_2)A_m^{y1}(1+\rho T_1)}\right) A_m^{y1} + \left(\frac{A_m^{1-y2}\left(a Q_2 - y_1 C_1 A_m^{y1-1}\right)}{y_2}\right) A_m^{y2}$$
$$-\frac{kT_1}{1+\rho T_1}$$

$$a A_m Q_2 = -\frac{y_2 k T_1 - (1-y_2)a A_m Q_2(1+\rho T_1)}{(y_1-y_2)(1+\rho T_1)} + \frac{A_m\left(a Q_2 - y_1 C_1 A_m^{y1-1}\right)}{y_2} - \frac{kT_1}{1+\rho T_1}$$

Common nominator: $(y_1-y_2)(1+\rho T_1)$

$$-\frac{y_2 k T_1 - (1-y_2)a A_m Q_2(1+\rho T_1)}{(y_1-y_2)(1+\rho T_1)} + \frac{A_m\left(a Q_2 - y_1 C_1 A_m^{y1-1}\right)}{y_2} - \frac{kT_1}{1+\rho T_1} - a A_m Q_2 = 0$$

(10.21)
$$A_m = -\frac{y_2 k T_1}{(y_1-1)a Q_2(1+\rho T_1)}$$

Appendix – C

Interval		Beg. Total	Deaths	Lost	Survival	Std. Error	[95% Conf. Int.]	
0	1	648	1	127	0.9983	0.0017	0.9879	0.9998
1	1	520	7	9	0.9847	0.0054	0.9697	0.9923
1	2	504	25	17	0.9350	0.0109	0.9098	0.9534
2	2	462	33	13	0.8673	0.0152	0.8342	0.8942
2	3	416	54	6	0.7539	0.0196	0.7131	0.7898
3	3	356	43	9	0.6617	0.0216	0.6174	0.7021
3	4	304	34	8	0.5867	0.0227	0.5408	0.6296
4	4	262	43	9	0.4887	0.0233	0.4423	0.5335
4	5	210	31	13	0.4143	0.0233	0.3684	0.4595
5	5	166	28	6	0.3431	0.0228	0.2987	0.3879
5	6	132	23	5	0.2822	0.0220	0.2398	0.3259
6	6	104	21	3	0.2244	0.0208	0.1849	0.2663
6	7	80	15	3	0.1815	0.0196	0.1449	0.2214
7	7	62	10	2	0.1517	0.0185	0.1176	0.1899
7	8	50	7	0	0.1305	0.0176	0.0985	0.1671
8	8	43	11	0	0.0971	0.0157	0.0692	0.1306
8	9	32	6	1	0.0786	0.0144	0.0535	0.1099
9	9	25	4	0	0.0660	0.0134	0.0430	0.0956
9	10	21	3	0	0.0566	0.0125	0.0354	0.0847
10	10	18	4	0	0.0440	0.0112	0.0256	0.0699
10	11	14	3	0	0.0346	0.0101	0.0186	0.0584
11	11	11	3	0	0.0252	0.0087	0.0120	0.0466
11	12	8	2	0	0.0189	0.0076	0.0079	0.0384
12	12	6	1	1	0.0154	0.0069	0.0058	0.0341
13	13	4	0	1	0.0154	0.0069	0.0058	0.0341
14	15	3	0	1	0.0154	0.0069	0.0058	0.0341
16	16	2	1	0	0.0077	0.0065	0.0011	0.0311
17	17	1	0	1	0.0077	0.0065	0.0011	0.0311

Table 14: List of survival estimate, semi-annually

Appendix - D

| | | Beg. | Cum. | Std. | | Std. | | |
Interval		Total	Failure	Error	Hazard	Error	[95% Conf. Int.]	
0	1	648	0.0017	0.0017	0.0034	0.0034	0.0000	0.0101
1	1	520	0.0153	0.0054	0.0273	0.0103	0.0071	0.0476
1	2	504	0.0650	0.0109	0.1035	0.0207	0.0630	0.1441
2	2	462	0.1327	0.0152	0.1503	0.0262	0.0991	0.2016
2	3	416	0.2461	0.0196	0.2798	0.0380	0.2053	0.3542
3	3	356	0.3383	0.0216	0.2606	0.0397	0.1829	0.3383
3	4	304	0.4133	0.0227	0.2403	0.0411	0.1597	0.3209
4	4	262	0.5113	0.0233	0.3644	0.0553	0.2559	0.4729
4	5	210	0.5857	0.0233	0.3298	0.0590	0.2141	0.4455
5	5	166	0.6569	0.0228	0.3758	0.0707	0.2372	0.5144
5	6	132	0.7178	0.0220	0.3898	0.0809	0.2313	0.5484
6	6	104	0.7756	0.0208	0.4565	0.0990	0.2625	0.6505
6	7	80	0.8185	0.0196	0.4225	0.1085	0.2099	0.6352
7	7	62	0.8483	0.0185	0.3571	0.1125	0.1367	0.5776
7	8	50	0.8695	0.0176	0.3011	0.1135	0.0787	0.5235
8	8	43	0.9029	0.0157	0.5867	0.1750	0.2437	0.9296
8	9	32	0.9214	0.0144	0.4211	0.1709	0.0860	0.7561
9	9	25	0.9340	0.0134	0.3478	0.1733	0.0083	0.6874
9	10	21	0.9434	0.0125	0.3077	0.1771	0.0000	0.6548
10	10	18	0.9560	0.0112	0.5000	0.2480	0.0139	0.9861
10	11	14	0.9654	0.0101	0.4800	0.2751	0.0000	1.0192
11	11	11	0.9748	0.0087	0.6316	0.3601	0.0000	1.3373
11	12	8	0.9811	0.0076	0.5714	0.3999	0.0000	1.3553
12	12	6	0.9846	0.0069	0.4000	0.3980	0.0000	1.1801
13	13	4	0.9846	0.0069	0.0000	.	.	.
14	15	3	0.9846	0.0069	0.0000	.	.	.
16	16	2	0.9923	0.0065	1.3333	1.2571	0.0000	3.7972
17	17	1	0.9923	0.0065	0.0000	.	.	.

Table 15: List of hazard rates, semi-annualy