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Research Article

A Maximum Principle Approach to Risk Indifference Pricing with Partial Information

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We consider the problem of risk indifference pricing on an incomplete market, namely on a jump diffusion market where the controller has limited access to market information. We use the maximum principle for stochastic differential games to derive a formula for the risk indifference price $p_{\text{risk}}^{\text{seller}}(G, \mathcal{E})$ of a European-type claim G.

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1. Introduction

Suppose the value of a portfolio $(\pi(t), S_0(t))$ is given by

$$X_x^{(\pi)}(t) = x + \pi(t)S(t) + S_0(t), \tag{1.1}$$

where x is the initial capital, $S(t)$ is a semimartingle price process of a risky asset, $\pi(t)$ is the number of risky assets held at time t, and $S_0(t)$ is the amount invested in the risk-free asset at time t. Then, the cumulative cost at time t is given by

$$P(t) = X_x^{(\pi)}(t) - \int_0^t \pi(u^-)dS(u). \tag{1.2}$$

If $P(t) = p$-constant for all t, then the portfolio strategies $(\pi(t), S_0(t))$ is called self-financing. A *contingent claim* with expiration date T is a nonnegative $\mathcal{F}_T$-measurable random variable G

that represents the time T payoff from seller to buyer. Suppose that for a contingent claim G there exists a self-financing strategy such that $X_x^{(\pi)}(T) = G$, that is,

$$p + \int_0^T \pi(u)dS(u) = G. \tag{1.3}$$

Then, p is the price of G in the complete market, that is,

$$p = E_Q[G], \tag{1.4}$$

where Q is any martingale measure equivalent to P on the probability space $(\Omega, \mathcal{F}_t, P)$.

In an incomplete market, an exact replication of a contingent claim is not always possible. One of the approaches to solve the replicating problems in an incomplete market is the utility indifference pricing. See, for example, Grasselli and Hurd [1] for the case of stochastic volatility model, Hodges and Neuberger [2] for the financial model with constraints, and Takino [3] for model with incomplete information. The utility indifference price p of a claim G is the initial payment that makes the seller of the contract utility indifferent to the two following alternatives: either selling the contract with initial payment p and with the obligation to pay out G at time T or not selling the contract and hence receiving no initial payment.

Recently, several papers discuss risk measure pricing rather than utility pricing in incomplete markets. Some papers related to risk measure pricing are the following: Xu [4] propose risk measure pricing and hedging in incomplete markets; Barrieu and El Karoui [5] study a minimization problem for risk measures subject to dynamic hedging; Klöppel and Schweizer [6] study the indifference pricing of a payoff with a minus dynamic convex risk measure. See also the references in these papers.

In our paper, we study a pricing formula based on the risk indifference principle in a jump-diffusion market. The same problem was studied by Øksendal and Sulem [7] with the restriction to Markov controls. So the problem is solved by using the Hamilton-Jacobi-Bellman equation. In our paper, the control process is required to be adapted to a given subfiltration of the filtration generated by the underlying Lévy processes. This makes the control problem non-Markovian. Within the non-Markovian setting, the dynamic programming cannot be used. Here we use the maximum principle approach to find the solution for our problem.

The paper is organized as follows. In Section 2, we will implement the option pricing method in an incomplete market. In Section 3, we present our problem in a jump-diffusion market. In Section 4, we use a maximum principle for a stochastic differential game to find the relation between the optimal controls of the stochastic differential game and of a corresponding stochastic control problem. Using this result, we derive the relationship between the two value functions of the two problems above, and then find the formulas for the risk indifference prices for the seller and the buyer.

2. Statement of the problem

Assume that a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ is given.

Definition 2.1. A nonnegative random variable G on $(\Omega, \mathcal{F}_t, P)$ is called a *European contingent claim.*

From now on, we consider a European-type option whose payoff at time t is some nonnegative random variable $G = g(S(t))$. In the rest of the paper, we will identify a contingent claim with its payoff function g.

Let $\mathbb{F}$ be the space of all equivalence classes of real-valued random variables defined on Ω.

Definition 2.2 (see [8, 9]). *A convex risk measure* $\rho : \mathbb{F} \to \mathbb{R} \cup \{\infty\}$ is a mapping satisfying the following properties, for $X, Y \in \mathbb{F}$:

(i) (convexity)

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \ \lambda \in (0,1); \tag{2.1}$$

(ii) (monotonicity) if $X \leq Y$, then $\rho(X) \geq \rho(Y)$.

If an investor sells a liability to pay out the amount $g(S(T))$ at time T and receives an initial payment p for such a contract, then the minimal risk involved for the seller is

$$\Phi_G(x+p) = \inf_{\pi \in \mathcal{P}} \rho\left(X_{x+p}^{(\pi)}(T) - g(S(T))\right), \tag{2.2}$$

where $\mathcal{P}$ is the set of self-financing strategies such that $X_x^{(\pi)}(t) \geq c$, for some finite constant c and for $0 \leq t \leq T$.

If the investor has not issued a claim (and hence no initial payment is received), then the minimal risk for the investor is

$$\Phi_0(x) = \inf_{\pi \in \mathcal{P}} \rho\left(X_x^{(\pi)}(T)\right). \tag{2.3}$$

Definition 2.3. The seller's *risk indifference price*, $p = p_{\text{risk}}^{\text{seller}}$, of the claim G is the solution p of the equation

$$\Phi_G(x+p) = \Phi_0(x). \tag{2.4}$$

Thus, $p_{\text{risk}}^{\text{seller}}$ is the initial payment p that makes an investor risk indifferent between selling the contract with liability payoff G and not selling the contract.

In view of the general representation formula for convex risk measures (see [10]), we will assume that the risk measure ρ, which we consider, is of the following type.

Theorem 2.4 (representation theorem [8, 9]). *A map* $\rho : \mathbb{F} \to \mathbb{R}$ *is a convex risk measure if and only if there exists a family* $\mathcal{L}$ *of measures* $Q \ll P$ *on* $\mathcal{F}_T$ *and a convex "penalty" function* $\zeta : \mathcal{L} \to (-\infty, +\infty)$ *with* $\inf_{Q \in \mathcal{L}} \zeta(Q) = 0$ *such that*

$$\rho(X) = \sup_{Q \in \mathcal{L}} \left\{ E_Q[-X] - \zeta(Q) \right\}, \quad X \in \mathbb{F}. \tag{2.5}$$

By this representation, we see that choosing a risk measure ρ is equivalent to choosing the family $\mathcal{L}$ of measures and the penalty function ζ.

Using the representation (2.5), we can write (2.2) and (2.3) as follows:

$$\Phi_G(x+p) = \inf_{\pi\in\mathcal{P}}\left(\sup_{Q\in\mathcal{L}}\left\{E_Q\left[-X_{x+p}^{(\pi)}(T)+g(S(t))\right]-\zeta(Q)\right\}\right), \tag{2.6}$$

$$\Phi_0(x) = \inf_{\pi\in\mathcal{P}}\left(\sup_{Q\in\mathcal{L}}\left\{E_Q\left[-X_{x}^{(\pi)}(T)\right]-\zeta(Q)\right\}\right), \tag{2.7}$$

for a given penalty function ζ.

Thus, the problem of finding the risk indifference price $p = p_{\text{risk}}^{\text{seller}}$ given by (2.4) has turned into two *stochastic differential game problems* (2.6) and (2.7). In the complete information, Markovian setting this problem was solved in [7] where the authors use Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations and PDEs to find the solution. In our paper, the corresponding partial information problem is considered by means of a maximum principle of differential games for SDEs.

3. The setup model

Suppose in a financial market, there are two investment possibilities:

(i) a bond with unit price $S_0(t) = 1$, $t \in [0,T]$;

(ii) a stock with price dynamics, for $t \in [0,T]$,

$$\begin{gathered} dS(t)S(t^-)\left[\alpha(t)dt+\sigma(t)dB_t+\int_{\mathbb{R}_0}\gamma(t,z)\widetilde{N}(dt,dz)\right], \\ S(0)=s>0. \end{gathered} \tag{3.1}$$

Here B_t is a Brownian motion and $\widetilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ is a compensated Poisson random measure with Lévy measure ν. The processes $\alpha(t)$, $\sigma(t)$, and $\gamma(t,z)$ are $\mathcal{F}_t$-predictable processes such that $\gamma(t,z) > -1$, for a.s. t, z, and

$$E\left[\int_0^T\left\{|\alpha(s)|+\sigma^2(s)+\int_{\mathbb{R}_0}|\log(1+\gamma(s,z))|^2\nu(dz)\right\}ds\right] < \infty \text{ a.s.}, \tag{3.2}$$

for all $T \geq 0$.

Let $\mathcal{E}_t \subseteq \mathcal{F}_t$ be a given subfiltration. Denote by $\pi(t)$, $t \geq 0$, the fraction of wealth invested in $S(t)$ based on the partial market information $\mathcal{E}_t \subseteq \mathcal{F}_t$ being available at time t. Thus, we impose on $\pi(t)$ to be $\mathcal{E}_t$-predictable. Then, the total wealth $X^{(\pi)}(t)$ with initial wealth x is given by the SDE

$$\begin{gathered} dX^{(\pi)}(t) = \pi(t^-)S(t^-)\left[\alpha(t)dt+\sigma(t)dB_t+\int_{\mathbb{R}_0}\gamma(t,z)\widetilde{N}(dt,dz)\right], \\ X^{(\pi)}(0)=x>0. \end{gathered} \tag{3.3}$$

In the sequel, we will call a portfolio $\pi \in \mathcal{P}$ *admissible* if π is $\mathcal{E}_t$-predictable, permits a strong solution of (3.3), and satisfies

$$\int_0^T \left\{ |\alpha(t)||\pi(t)|S(t) + \sigma^2(t)\pi^2(t)S^2(t) + \pi^2(t)S^2(t)\int_{\mathbb{R}_0} \gamma^2(t,z)\nu(dz) \right\} ds < \infty, \tag{3.4}$$

as well as

$$\pi(t)S(t)\gamma(t) > -1 \quad (\omega, t, z)\text{-a.s.} \tag{3.5}$$

The class of admissible portfolios is denoted by Π.

Now, we define the measures Q_θ parameterized by given $\mathcal{F}_t$-predictable processes $\theta = (\theta_0(t), \theta_1(t,z))$ such that

$$dQ_\theta(\omega) = K_\theta(T)dP(\omega) \quad \text{on } \mathcal{F}_T, \tag{3.6}$$

where

$$\begin{gathered} dK_\theta(t) = K_\theta(t^-)\left[\theta_0(t)dB(t) + \int_{\mathbb{R}_0} \theta_1(t,z)\widetilde{N}(dt,dz)\right], \quad t \in [0,T], \\ K_\theta(0) = k > 0, \end{gathered} \tag{3.7}$$

We assume that

$$\begin{gathered} \theta_1(t,z) \geq -1 \quad \text{for a.a. } t, z, \\ \int_0^T \left\{ \theta_0^2(s) + \int_{\mathbb{R}_0} (\log(1+\theta_1(s,z)))^2 \nu(dz) \right\} ds < \infty \text{ a.s.} \end{gathered} \tag{3.8}$$

Then, by the Itô formula, the solution of (3.7) is given by

$$\begin{aligned} K_\theta(t) = k \exp\Bigg[&-\int_0^t \theta_0(s)dB(s) - \frac{1}{2}\int_0^t \theta_0^2(s)ds \\ &+ \int_0^t\int_{\mathbb{R}_0} \ln(1-\theta_1(s,z))\widetilde{N}(dt,dz) \\ &+ \int_0^t\int_{\mathbb{R}_0} \{\ln(1-\theta_1(s,z)) + \theta_1(s,z)\}\nu(dz)ds \Bigg]. \end{aligned} \tag{3.9}$$

We say that the control $\theta = (\theta_0, \theta_1)$ is admissible and write $\theta \in \Theta$ if θ is adapted to the subfiltration $\mathcal{E}_t$ and satisfies (3.8) and

$$E[K_\theta(T)] = K_\theta(0) = k > 0. \tag{3.10}$$

We set

$$
\begin{aligned}
dY(t) &= \begin{bmatrix} dY_1(t) \\ dY_2(t) \\ dY_3(t) \end{bmatrix} = \begin{bmatrix} dK_\theta(t) \\ dS(t) \\ dX^{(\pi)}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ S(t^-)\alpha(t) \\ S(t^-)\pi(t)\alpha(t) \end{bmatrix} dt \\
&\quad + \begin{bmatrix} K_\theta(t^-)\theta_0(t) \\ S(t^-)\sigma(t) \\ S(t^-)\pi(t)\sigma(t) \end{bmatrix} dB(t) + \int_{\mathbb{R}_0} \begin{bmatrix} K_\theta(t^-)\theta_1(t,z) \\ S(t^-)\gamma(t,z) \\ S(t^-)\pi(t)\gamma(t) \end{bmatrix} \widetilde{N}(dt,dz), \\
Y(0) &= y = (y_1, y_2, y_3) = (k, s, x), \\
d\widetilde{Y}(t) &= \begin{bmatrix} dY_1(t) \\ dY_2(t) \end{bmatrix} = \begin{bmatrix} dK_\theta(t) \\ dS(t) \end{bmatrix}, \\
\widetilde{Y}(0) &= \widetilde{y} = (y_1, y_2) = (k, s).
\end{aligned}
\tag{3.11}
$$

We now define two sets $\mathcal{L}$, $\mathcal{M}$ of measures as follows:

$$
\begin{aligned}
\mathcal{L} &= \{Q_\theta;\ \theta \in \Theta\}, \\
\mathcal{M} &= \{Q_\theta;\ \theta \in \mathbb{M}\},
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
\mathbb{M} &= \{\theta \in \Theta;\ E[M\theta(t,\widetilde{y}) \mid \mathcal{E}_t] = 0\ \forall t, \widetilde{y}\}, \\
M\theta(t,\widetilde{y}) &= M\theta(t,k,s) = \alpha(t) + \sigma(t)\theta_0(t) + \int_{\mathbb{R}_0} \gamma(t,z)\theta_1(t,z)\nu(dz).
\end{aligned}
\tag{3.13}
$$

In particular, by the Girsanov theorem, all the measures $Q_\theta \in \mathcal{M}$ with $E[K_\theta(T)] = 1$ are equivalent martingale measures for the $\mathcal{E}_t$-conditioned market $(S_0(t), S_1(t))$, where

$$
\begin{aligned}
dS_1(t) &= S_1(t^-)\left[E[\alpha(t) \mid \mathcal{E}_t]dt + E[\sigma(t) \mid \mathcal{E}_t]dB_t + \int_{\mathbb{R}_0} E[\gamma(t,z) \mid \mathcal{E}_t]\widetilde{N}(dt,dz)\right] \\
S_1(0) &= s > 0
\end{aligned}
\tag{3.14}
$$

(see, e.g., [11, Chapter 1]).

We assume that the penalty function ζ has the form

$$
\zeta(Q_\theta) = E\left[\int_0^T \int_{\mathbb{R}_0} \lambda(t, \theta_0(t,\widetilde{Y}(t)), \theta_1(t,\widetilde{Y}(t),z), \widetilde{Y}(t), z)\nu(dz)dt + h(\widetilde{Y}(T))\right],
\tag{3.15}
$$

for some convex functions $\lambda \in C^1(\mathbb{R}^2 \times \mathbb{R}_0)$, $h \in C^1(\mathbb{R})$, such that

$$
E\left[\int_0^T \int_{\mathbb{R}_0} |\lambda(t, \theta_0(t,\widetilde{Y}(t)), \theta_1(t,\widetilde{Y}(t),z), \widetilde{Y}(t), z)|\nu(dz)dt + |h(\widetilde{Y}(T))|\right] < \infty,
\tag{3.16}
$$

for all $(\theta, \pi) \in \Theta \times \Pi$.

Using the $Y(t)$-notation, problem (2.6) can be written as follows:

Problem A. Find $\Phi_G^{\mathcal{E}}(t,y)$ and $(\theta^*,\pi^*) \in \Theta \times \Pi$ such that

$$\Phi_G^{\mathcal{E}}(t,y) := \inf_{\pi\in\Pi}\left(\sup_{\theta\in\Theta} J^{\theta,\pi}(t,y)\right) = J^{\theta^*,\pi^*}(t,y), \tag{3.17}$$

where

$$\begin{aligned} J^{\theta,\pi}(t,y) &= J(\theta,\pi) \\ &= E^y\left[-\int_t^T \Lambda(\theta(u,\widetilde{Y}(u)))du - h(\widetilde{Y}(T)) + K_\theta(T)g(S(T)) - K_\theta(T)X^{(\pi)}(T)\right], \end{aligned} \tag{3.18}$$

$$\Lambda(\theta) = \Lambda(\theta(t,\widetilde{y})) = \int_{\mathbb{R}_0} \lambda(t,\theta_0(t,\widetilde{y}),\theta_1(t,\widetilde{y},z),\widetilde{y},z)\nu(dz). \tag{3.19}$$

We will relate Problem A to the following stochastic control problem:

$$\Psi_G^{\mathcal{E}} = \sup_{Q\in\mathcal{M}}\{E_Q[G] - \zeta(Q)\}. \tag{3.20}$$

Using the $\widetilde{Y}(t)$-notation, the problem gets the following form.

Problem B. Find $\Psi_G^{\mathcal{E}}(t,\widetilde{y})$ and $\check{\theta} \in \mathbb{M}$ such that

$$\Psi_G^{\mathcal{E}}(t,\widetilde{y}) := \sup_{\theta\in\mathbb{M}} J_0^\theta(t,\widetilde{y}) = J_0^{\check{\theta}}(t,\widetilde{y}), \tag{3.21}$$

where

$$J_0^\theta(t,\widetilde{y}) = E^y\left[-\int_t^T \Lambda(\theta(u,\widetilde{Y}(u)))du - h(\widetilde{Y}(T)) + K_\theta(T)g(S(T))\right]. \tag{3.22}$$

Define the *Hamiltonian* $H : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\Theta\times\Pi\times\mathbb{R}\times\mathbb{R}\times\mathcal{R} \to \mathbb{R}$ for Problem A by

$$\begin{aligned} &H(t,k,s,x,\theta,\pi,p,q,r(\cdot,z)) \\ &\quad = -\Lambda(t,\widetilde{Y}(t)) + s\alpha p_2 + s\alpha\pi p_3 + k\theta_0 q_1 + s\sigma q_2 + s\sigma\pi q_3 \\ &\qquad + \int_{\mathbb{R}_0}\{k\theta_1 r_1(\cdot,z) + s\gamma(t,z)r_2(\cdot,z) + s\pi\gamma(t,z)r_3(\cdot,z)\}\nu(dz), \end{aligned} \tag{3.23}$$

and the Hamiltonian $\widetilde{H} : [0,T]\times\mathbb{R}\times\mathbb{R}\times\Theta\times\mathbb{R}\times\mathbb{R}\times\mathcal{R} \to \mathbb{R}$ for Problem B by

$$\begin{aligned} &\widetilde{H}(t,k,s,\theta,p,q,r(\cdot,z)) \\ &= -\Lambda(t,\widetilde{Y}(t)) + s\alpha p_2 + k\theta_0 q_1 + s\sigma q_2 + \int_{\mathbb{R}_0}\{k\theta_1(t,z)r_1(\cdot,z) + s\gamma(t,z)r_2(\cdot,z)\}\nu(dz). \end{aligned} \tag{3.24}$$

Here $\mathcal{R}$ is the set of functions $r : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that the integrals in (3.23) and (3.24) converge. We assume that H and $\widetilde{H}$ are differentiable with respect to k, s, and x. The adjoint equations (corresponding to θ, π, and $Y(t)$) in the unknown adapted processes $p(t)$, $q(t)$, $r(t,z)$ are the backward stochastic differential equations (BSDEs)

$$\begin{aligned} dp_1(t) &= \left(\frac{\partial \Lambda}{\partial k}(t,\widetilde{Y}(t)) - \theta_0(t)q_1(t) - \int_{\mathbb{R}_0} \theta_1(t,z)r_1(t,z)\nu(dz)\right)dt \\ &\quad + q_1(t)dB(t) + \int_{\mathbb{R}_0} r_1(t,z)\widetilde{N}(dt,dz), \end{aligned} \tag{3.25}$$

$$\begin{aligned} p_1(T) &= -\frac{\partial h}{\partial k}(\widetilde{Y}(T)) + g(S(T)) - X^{(\pi)}(T), \\ dp_2(t) &= \left(\frac{\partial \Lambda}{\partial s}(t,\widetilde{Y}(t)) - \alpha(t)p_2(t) - \sigma(t)q_2(t) - \int_{\mathbb{R}_0} \gamma(t,z)r_2(t,z)\nu(dz)\right)dt \\ &\quad + q_2(t)dB(t) + \int_{\mathbb{R}_0} r_2(t,z)\widetilde{N}(dt,dz), \end{aligned} \tag{3.26}$$

$$\begin{aligned} p_2(T) &= -\frac{\partial h}{\partial s}(\widetilde{Y}(T)) + K_\theta(T)g'(S(T)), \\ dp_3(t) &= \left(-\alpha(t)p_3(t) - \sigma(t)q_3(t) - \int_{\mathbb{R}_0} \gamma(t,z)r_3(t,z)\nu(dz)\right)dt \\ &\quad + q_3(t)dB(t) + \int_{\mathbb{R}_0} r_3(t,z)\widetilde{N}(dt,dz), \\ p_3(T) &= -K_\theta(T). \end{aligned} \tag{3.27}$$

Similarly, the adjoint equations (corresponding to θ and $\widetilde{Y}(t)$) in the unknown processes $\widetilde{p}(t)$, $\widetilde{q}(t)$, $\widetilde{r}(t,z)$ are given by

$$\begin{aligned} d\widetilde{p}_1(t) &= \left(\frac{\partial \Lambda}{\partial k}(t,\widetilde{Y}(t)) - \theta_0(t)\widetilde{q}_1(t) - \int_{\mathbb{R}_0} \theta_1(t,z)\widetilde{r}_1(t,z)\nu(dz)\right)dt \\ &\quad + \widetilde{q}_1(t)dB(t) + \int_{\mathbb{R}_0} \widetilde{r}_1(t,z)\widetilde{N}(dt,dz), \\ \widetilde{p}_1(T) &= -\frac{\partial h}{\partial k}(\widetilde{Y}(T)) + g(S(T)), \\ d\widetilde{p}_2(t) &= \left(\frac{\partial \Lambda}{\partial s}(t,\widetilde{Y}(t)) - \alpha(t)\widetilde{p}_2(t) - \sigma(t)\widetilde{q}_2(t) - \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{r}_2(t,z)\nu(dz)\right)dt \\ &\quad + \widetilde{q}_2(t)dB(t) + \int_{\mathbb{R}_0} \widetilde{r}_2(t,z)\widetilde{N}(dt,dz), \\ \widetilde{p}_2(T) &= -\frac{\partial h}{\partial s}(\widetilde{Y}(T)) + K_\theta(T)g'(S(T)). \end{aligned} \tag{3.28}$$

Lemma 3.1. *Let $\theta \in \Theta$ and suppose that $\tilde{p}(t) = (\tilde{p}_1(t), \tilde{p}_2(t))$ is a solution of the corresponding adjoint equations* (3.28). *For all $\pi \in \mathbb{R}$, define*

$$p_1(t) = \tilde{p}_1(t) - X^{(\pi)}(t), \tag{3.29}$$

$$p_2(t) = \tilde{p}_2(t), \tag{3.30}$$

$$p_3(t) = -K_\theta(t). \tag{3.31}$$

If $\theta \in \mathbb{M}$, then $p(t) = (p_1(t), p_2(t),$ and $p_3(t))$ is a solution of the adjoint equations (3.25), (3.26), *and* (3.27). *Then, the following relation holds:*

$$\begin{aligned} &H(t, Y(t), \theta, \pi, p(t), q(t), r(t,z)) \\ &\quad = \widetilde{H}(t, \tilde{Y}(t), \theta, \tilde{p}(t), \tilde{q}(t), \tilde{r}(t,z)) - S(t)\pi K_\theta(t)\left(\alpha(t) + 2\theta_0(t)\sigma(t) + 2\int_{\mathbb{R}_0} \theta_1(t,z)\gamma(t,z)\nu(dz)\right). \end{aligned} \tag{3.32}$$

Proof. Differentiating both sides of (3.29), we get

$$\begin{aligned} dp_1(t) &= d\tilde{p}_1(t) - dX^{(\pi)}(t) \\ &= \left(\frac{\partial \Lambda}{\partial k}(t, \tilde{Y}(t)) - \theta_0(t)\tilde{q}_1(t) - \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{r}_1(t,z)\nu(dz) - S(t)\alpha(t)\pi(t)\right)dt \\ &\quad + (\tilde{q}_1(t) - S(t)\sigma(t)\pi(t))dB(t) + \int_{\mathbb{R}_0} (\tilde{r}_1(t,z) - S(t)\pi(t)\gamma(t,z))\widetilde{N}(dt,dz). \end{aligned} \tag{3.33}$$

Comparing this with (3.25) by equating the dt, $dB(t)$, $\widetilde{N}(dt,dz)$ coefficients, respectively, we get

$$\begin{aligned} &\frac{\partial \Lambda}{\partial k}(t, \tilde{Y}(t)) - \theta_0(t)q_1(t) - \int_{\mathbb{R}_0} \theta_1(t,z)r_1(t,z)\nu(dz) \\ &\quad = \frac{\partial \Lambda}{\partial k}(t, \tilde{Y}(t)) - \theta_0(t)\tilde{q}_1(t) - \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{r}_1(t,z)\nu(dz) - S(t)\alpha(t)\pi(t), \end{aligned} \tag{3.34}$$

$$q_1(t) = \tilde{q}_1(t) - S(t)\sigma(t)\pi(t), \tag{3.35}$$

$$r_1(t,z) = \tilde{r}_1(t,z) - S(t)\gamma(t,z)\pi(t). \tag{3.36}$$

Substituting (3.35) and (3.36) into (3.34), we get

$$S(t)\pi(t)\left(\alpha(t) + \theta_0(t)\sigma(t) + \int_{\mathbb{R}_0} \theta_1(t,z)\gamma(t,z)\nu(dz)\right) = 0. \tag{3.37}$$

Since $\theta \in \mathbb{M}$, (3.37) is satisfied, and hence $p_1(t)$ is a solution of (3.25).

Proceeding as above with the processes $p_2(t)$ and $p_3(t)$, we get

$$q_2(t) = \tilde{q}_2(t), \qquad r_2(t) = \tilde{r}_2, \tag{3.38}$$

$$-\alpha(t)p_3(t) - \sigma(t)q_3(t) - \int_{\mathbb{R}_0} \gamma(t,z)r_3(t,z)\nu(dz) = 0, \tag{3.39}$$

$$q_3(t) = -K_\theta(t)\theta_0(t), \qquad r_3(t,z) = -K_\theta(t)\theta_1(t,z). \tag{3.40}$$

With the values $p_3(t)$, $q_3(t)$, and $r_3(t,z)$ defined as above, relation (3.39) is satisfied if $\theta \in \mathbb{M}$. Hence, $p_1(t)$, $p_2(t)$, and $p_3(t)$ are solutions of (3.29), (3.30), and (3.31), respectively.

Equations (3.23) and (3.24) give the following relation between H and $\widetilde{H}$:

$$H(t,y,\theta,\pi,p,q,r(\cdot,z)) = \widetilde{H}(t,\tilde{y},\theta,p,q,r(\cdot,z)) + s\pi\left(\alpha p_3 + \sigma q_3 + \int_{\mathbb{R}_0} \gamma(t,z)r_3(\cdot,z)\nu(dz)\right). \tag{3.41}$$

Hence,

$$\begin{aligned}
&H(t,Y(t),\theta,\pi,p(t),q(t),r(t,z)) \\
&\quad = \widetilde{H}(t,\tilde{Y}(t),\theta,p_1(t),p_2(t),q_1(t),q_2(t),r_1(t,z),r_2(t,z)) \\
&\qquad - S(t)\pi(t)\left(\alpha(t)p_3(t) + \sigma q_3(t) + \int_{\mathbb{R}_0} \gamma(t,z)r_3(t,z)\nu(dz)\right) \\
&\quad = \widetilde{H}(t,\tilde{Y}(t),\theta,\tilde{p}_1(t),\tilde{p}_2(t),\tilde{q}_1(t),\tilde{q}_2(t),\tilde{r}_1(t,z),\tilde{r}_2(t,z)) \\
&\qquad - S(t)\sigma(t)\pi(t)K_\theta(t)\theta_0(t) - \int_{\mathbb{R}_0} S(t)\gamma(t,z)\pi(t)K_\theta(t)\theta_1(t,z)\nu(dz) \\
&\qquad - S(t)\pi(t)K_\theta(t)\left(\alpha(t) + \sigma(t)\theta_0(t) + \int_{\mathbb{R}_0} \gamma(t,z)\theta_1(t)\nu(dz)\right) \\
&\quad = \widetilde{H}(t,\tilde{Y}(t),\theta,\tilde{p}(t),\tilde{q}(t),\tilde{r}(t,z)) \\
&\qquad - s\pi K_\theta(t)\left(\alpha(t) + 2\sigma(t)\theta_0(t) + 2\int_{\mathbb{R}_0} \gamma(t,z)\theta_1(t,z)\nu(dz)\right).
\end{aligned} \tag{3.42}$$

□

Lemma 3.2. *Let $p_1(t)$, $p_2(t)$, and $p_3(t)$ be as in Lemma 3.1. Suppose that, for all $\pi \in \mathbb{R}$, the function*

$$\theta \longrightarrow E[H(t,Y(t),\theta,\pi(t),p(t),q(t),r(t,z)) \mid \mathcal{E}_t], \quad \theta \in \Theta, \tag{3.43}$$

has a maximum point at $\widehat{\theta} = \widehat{\theta}(\pi)$. Moreover, suppose that the function

$$\pi \longrightarrow E[H(t,Y(t),\widehat{\theta}(\pi),\pi,p(t),q(t),r(t,z)) \mid \mathcal{E}_t], \quad \pi \in \mathbb{R}, \tag{3.44}$$

has a minimum point at $\widehat{\pi} \in \mathbb{R}$. Then,

$$M\widehat{\theta}(\widehat{\pi}) = 0. \tag{3.45}$$

Proof. The first-order conditions for a maximum point $\widehat{\theta} = \widehat{\theta}(\pi)$ of the function $E[H(t, Y(t), \theta, \pi(t), p(t), q(t), r(t,z)) \mid \mathcal{E}_t]$ is

$$E\left[\nabla_\theta (H(t, Y(t), \theta, \pi(t), p(t), q(t), r(t,z)))_{\theta=\widehat{\theta}(\pi)} \mid \mathcal{E}_t\right] = 0, \tag{3.46}$$

where $\nabla_\theta = (\partial/\partial\theta_0, \partial/\partial\theta_1)$ is the gradient operator. The first-order condition for a minimum point $\widehat{\pi}$ of the function $E[H(t, Y(t), \widehat{\theta}(\pi), \pi, p(t), q(t), r(t,z)) \mid \mathcal{E}_t]$ is

$$E\left[\nabla_\pi (H(t, Y(t), \widehat{\theta}(\pi), \pi(t), p(t), q(t), r(t,z)))_{\pi=\widehat{\pi}} \mid \mathcal{E}_t\right] = 0, \tag{3.47}$$

that is,

$$\begin{aligned} E\Bigg[&\nabla_\theta (H(t, Y(t), \theta, \widehat{\pi}, p(t), q(t), r(t,z)))_{\theta=\widehat{\theta}(\widehat{\pi})} \left(\frac{d\widehat{\theta}(\pi)}{d\pi}\right)_{\pi=\widehat{\pi}} \\ &+ \nabla_\pi (H(t, Y(t), \theta, \pi, p(t), q(t), r(t,z)))_{\pi=\widehat{\pi}, \theta=\widehat{\theta}(\widehat{\pi})} \mid \mathcal{E}_t\Bigg] = 0. \end{aligned} \tag{3.48}$$

Choose $\pi = \widehat{\pi}$. Then, by (3.46) and (3.48), we have

$$E\left[\nabla_\pi (H(t, Y(t), \theta, \pi, p(t), q(t), r(t,z)))_{\pi=\widehat{\pi}, \theta=\widehat{\theta}(\widehat{\pi})} \mid \mathcal{E}_t\right] = 0, \tag{3.49}$$

that is,

$$E\left[S(t)\alpha(t)p_3(t) + S(t)\sigma(t)q_3(t) + \int_{\mathbb{R}_0} S(t)\gamma(t,z)r_3(t,z)\nu(dz) \mid \mathcal{E}_t\right] = 0. \tag{3.50}$$

Substituting the values $p_3(t)$, $q_3(t)$, and $r_3(t,z)$ as in Lemma 3.1 into (3.50), we get

$$E\left[S(t)K_\theta(t)\left\{\alpha(t) + \sigma(t)\theta_0(t) + \int_{\mathbb{R}_0} \gamma(t,z)\theta_1(t,z)\nu(dz)\right\} \mid \mathcal{E}_t\right] = 0. \tag{3.51}$$

This gives,

$$M\widehat{\theta}(\widehat{\pi}) = 0. \tag{3.52}$$

□

4. Maximum principle for stochastic differential games

Problem A is related to what is known as stochastic games studied in [12]. Applying in [[12], Theorem 2.1] to our setting we get the following jump-diffusion version of the maximum principle (of Ferris and Mangasarian type [13]).

Theorem 4.1 (maximum principle for stochastic differential games [12]). *Let* $(\widehat{\theta},\widehat{\pi}) \in \Theta \times \Pi$ *and suppose that the adjoint equations* (3.25), (3.26), *and* (3.27) *admit solutions* $(\widehat{p}_1(t),\widehat{q}_1(t),\widehat{r}_1(t,z))$, $(\widehat{p}_2(t),\widehat{q}_2(t),\widehat{r}_2(t,z))$, *and* $(\widehat{p}_3(t),\widehat{q}_3(t),\widehat{r}_3(t,z))$, *respectively. Moreover, suppose that, for all* $t \in [0,T]$, *the following partial information maximum principle holds:*

$$\begin{aligned}
&\sup_{\theta\in\Theta} E[H(t,Y(t),\theta,\widehat{\pi}(t),\widehat{p}(t),\widehat{q}(t),\widehat{r}(t,z)) \mid \mathcal{E}_t] \\
&\quad = E[H(t,Y(t),\widehat{\theta}(t),\widehat{\pi}(t),\widehat{p}(t),\widehat{q}(t),\widehat{r}(t,z)) \mid \mathcal{E}_t] \\
&\quad = \inf_{\pi\in\Pi} E[H(t,Y(t),\widehat{\theta}(t),\pi,\widehat{p}(t),\widehat{q}(t),\widehat{r}(t,z)) \mid \mathcal{E}_t].
\end{aligned} \tag{4.1}$$

Suppose

$$\begin{aligned}
\theta &\longrightarrow J(\theta,\widehat{\pi}) \text{ is concave,} \\
\pi &\longrightarrow J(\widehat{\theta},\pi) \text{ is convex.}
\end{aligned} \tag{4.2}$$

Then $(\theta^*,\pi^*) := (\widehat{\theta},\widehat{\pi})$ *is an optimal control and*

$$\begin{aligned}
\Phi_G^{\mathcal{E}}(x) &= \inf_{\pi\in\Pi}\left(\sup_{\theta\in\Theta} J(\theta,\pi)\right) \\
&= \sup_{\theta\in\Theta}\left(\inf_{\pi\in\Pi} J(\theta,\pi)\right) \\
&= \sup_{\theta\in\Theta} J(\theta,\widehat{\pi}) \\
&= \inf_{\pi\in\Pi} J(\widehat{\theta},\pi) \\
&= J(\widehat{\theta},\widehat{\pi}).
\end{aligned} \tag{4.3}$$

Theorem 4.2. *Let* $\widetilde{p}_1(t)$, $\widetilde{p}_2(t)$ *be, respectively, solutions of adjoint equations* (3.28), *and let* $p_1(t)$, $p_2(t)$, $p_3(t)$ *be defined as in Lemma 3.1. Suppose* $\theta \to \widetilde{H}(t,\widetilde{Y}(t),\theta,\widetilde{p}(t);\widetilde{q}(t),\widetilde{r}(t,\cdot))$ *is concave. Let* $(\widehat{\theta}(\widehat{\pi}),\widehat{\pi})$ *be an optimal pair for Problem A, as given in Lemma 3.2. Then,*

$$\breve{\theta} := \widehat{\theta}(\widehat{\pi}) \tag{4.4}$$

is optimal for Problem B.

Proof. By Theorem 4.1 for Problem B, $\breve{\theta}$ solves Problem B under partial information $\mathcal{E}_t$ if

$$\sup_{\theta\in\mathbb{M}} E[\widetilde{H}(t,\widetilde{Y}(t),\theta,\widetilde{p}(t),\widetilde{q}(t),\widetilde{r}(t,z)) \mid \mathcal{E}_t] = E[\widetilde{H}(t,\widetilde{Y}(t),\breve{\theta},\widetilde{p}(t),\widetilde{q}(t),\widetilde{r}(t,z)) \mid \mathcal{E}_t], \tag{4.5}$$

that is, if there exists $C = C(t)$ such that

$$E\left[\nabla_\theta(\widetilde{H}(t,\widetilde{Y}(t),\theta,\widetilde{p}(t),\widetilde{q}(t),\widetilde{r}(t,z))) - C(t)M(\theta))_{\theta=\breve{\theta}} \mid \mathcal{E}_t\right] = 0, \tag{4.6}$$

$$E[M\breve{\theta}(t) \mid \mathcal{E}_t] = 0. \tag{4.7}$$

Let $\widehat{\pi}$, $\widehat{\theta}(\widehat{\pi})$ be as in Lemma 3.2. Then,

$$E\left[\nabla_\theta(H(t,Y(t),\theta,\widehat{\pi}(t),p(t),q(t),r(t,z))_{\theta=\widehat{\theta}(\widehat{\pi}(t))} \mid \mathcal{E}_t\right] = 0, \tag{4.8}$$

$$E[M\widehat{\theta}(\widehat{\pi})(t) \mid \mathcal{E}_t] = 0. \tag{4.9}$$

Hence, by Lemma 3.1,

$$\begin{aligned} 0 &= E\Bigg[\nabla_\theta\Bigg\{\widetilde{H}(t,\widetilde{Y}(t),\theta,\widetilde{p}(t),\widetilde{q}(t),\widetilde{r}(t,z)) \\ &\qquad - S(t)\widehat{\pi}(t)K_\theta(t)\left(\alpha(t)+2\sigma(t)\theta_0+2\int_{\mathbb{R}_0}\gamma(t,z)\theta_1(z)\nu(dz)\right)\Bigg\}_{\theta=\widehat{\theta}(\widehat{\pi}(t))} \mid \mathcal{E}_t\Bigg] \\ &= E\left[\nabla_\theta(\widetilde{H}(t,\widetilde{Y}(t),\theta,\widetilde{p}(t),\widetilde{q}(t),\widetilde{r}(t,z)) - 2S(t)\widehat{\pi}(t)K_\theta(t)M\theta)_{\theta=\widehat{\theta}(\widehat{\pi}(t))} \mid \mathcal{E}_t\right]. \end{aligned} \tag{4.10}$$

Therefore, if we choose

$$C(t) = 2S(t)\widehat{\pi}(t)K_\theta(t), \tag{4.11}$$

we see that (4.6) holds with $\check{\theta} = \widehat{\theta}(\widehat{\pi})$, as claimed. □

5. Risk indifference pricing

Let $(\theta^*,\pi^*) = (\check{\theta},\widehat{\pi})$ be as in Theorem 4.2 with the corresponding state process $Y^* = Y^{\theta^*,\pi^*}$. Suppose that $Y = Y^{\widehat{\theta}(\widehat{\pi}),\pi}$ is the state process corresponding to an optimal control $(\widehat{\theta}(\widehat{\pi}),\pi)$. Then, the value function $\Phi_G^{\mathcal{E}}$, which is defined by (3.17) and (3.18), becomes

$$\begin{aligned} &\Phi_G^{\mathcal{E}}(t,y) \\ &= \inf_{\pi\in\Pi}\left(\sup_{\theta\in\Theta} J^{\theta,\pi}(t,y)\right) \\ &= \inf_{\pi\in\Pi}\left(\sup_{\theta\in\Theta} E^y\left[-\int_t^T \Lambda(\theta(u,\widetilde{Y}(u)))du - h(K_\theta(T),S(T)) + K_\theta(T)g(S(T)) - K_\theta(T)X^{(\pi)}(T)\right]\right) \\ &= \inf_{\pi\in\Pi}\left(E^y\left[-\int_t^T \Lambda(\theta^*(u,\widetilde{Y}^*(u)))du - h(K_{\theta^*}(T),S(T)) + K_{\theta^*}(T)g(S(T)) - K_{\theta^*}(T)X^{(\pi)}(T)\right]\right). \end{aligned} \tag{5.1}$$

We have that, for all $\pi \in \Pi$,

$$E^y\left[K_{\theta^*}(T)X^{(\pi)}(T)\right] = E^y\left[K_\theta(T)X^{(\pi)}(T)\right] = kE^{k,s,x}_{(1/k)Q_\theta}\left[X^{(\pi)}(T)\right] = kx, \tag{5.2}$$

since $(1/k)Q_{\check{\theta}}$ is an equivalent martingale measure for $\mathcal{E}_t$-conditioned market. On the other hand, the first part of (5.1) does not depend on the parameter π. Hence, (5.1) becomes

$$\begin{aligned}\Phi_G^{\mathcal{E}}(t,y) &= E^y\left[-\int_t^T \Lambda(\check{\theta}(u,\widetilde{Y}(u)))du - h(K_{\check{\theta}}(T),S(T)) + K_{\check{\theta}}(T)g(S(T))\right] - kx \\ &= \sup_{\theta\in\mathbb{M}} J_0^{\theta}(t,\widetilde{y}) - kx \\ &= \Psi_G^{\mathcal{E}}(t,\widetilde{y}) - kx.\end{aligned} \tag{5.3}$$

We have proved the following result for the relation between the value function for Problem A and the value function for Problem B in the partial information case that is the same as in Øksendal and Sulem [7] for the full information case.

Lemma 5.1. *The relationship between the value function $\Psi_G^{\mathcal{E}}(t,\widetilde{y})$ for Problem B and the value function $\Phi_G^{\mathcal{E}}(t,y)$ for Problem A is*

$$\Phi_G^{\mathcal{E}}(t,y) = \Psi_G^{\mathcal{E}}(t,\widetilde{y}) - kx. \tag{5.4}$$

We now apply Lemma 5.1 to find the risk indifference price $p = p_{\text{risk}}^{\text{seller}}$, given as a solution of the equation

$$\Phi_G^{\mathcal{E}}(t,k,s,x+p) = \Phi_0^{\mathcal{E}}(t,k,s,x). \tag{5.5}$$

By Lemma 5.1, this becomes

$$\Psi_G^{\mathcal{E}}(t,k,s) - k(x+p) = \Psi_0^{\mathcal{E}}(t,k,s) - kx, \tag{5.6}$$

which has the solution

$$p = p_{\text{risk}}^{\text{seller}} = k^{-1}\left(\Psi_G^{\mathcal{E}}(t,k,s) - \Psi_0^{\mathcal{E}}(t,k,s)\right). \tag{5.7}$$

In particular, choosing $k = 1$ (i.e., all measures $Q \in \mathcal{L}$ are probability measures), we get the following.

Theorem 5.2. *Suppose that the conditions of Theorem 4.2 hold. Then, the risk indifference price for the seller of claim G, $p_{risk}^{seller}(G,\mathcal{E})$, is given by*

$$p_{risk}^{seller}(G,\mathcal{E}) = \sup_{Q\in\mathcal{M}}\{E_Q[G] - \zeta(Q)\} - \sup_{Q\in\mathcal{M}}\{-\zeta(Q)\}. \tag{5.8}$$

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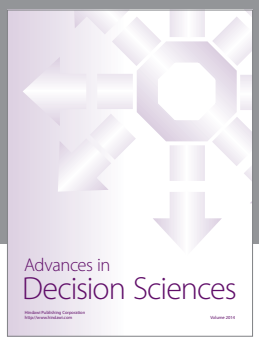

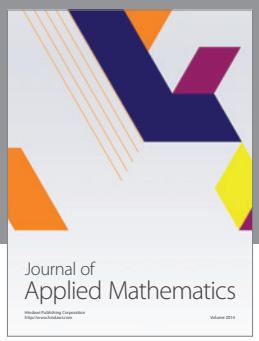

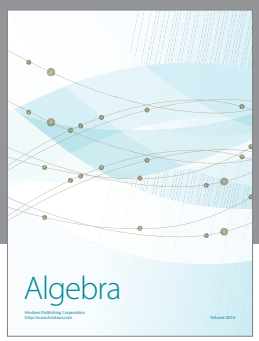

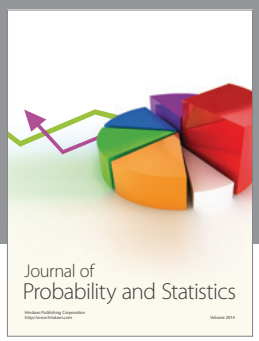

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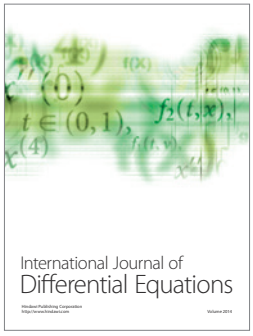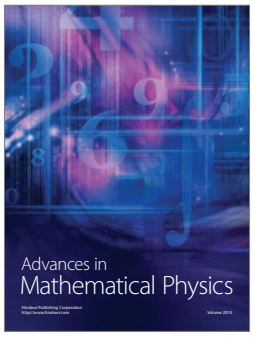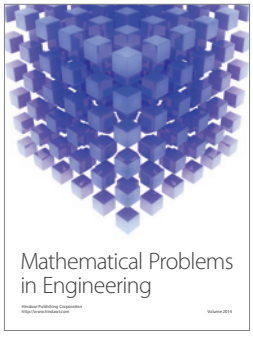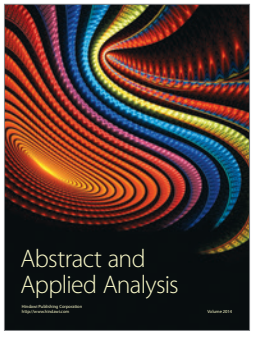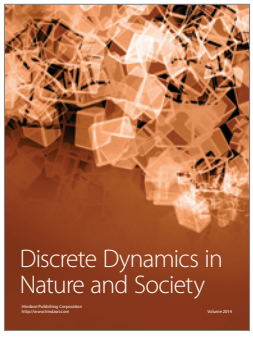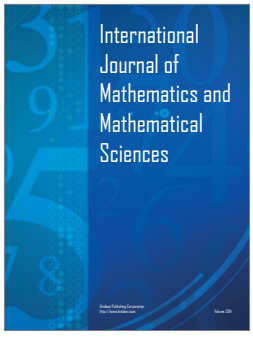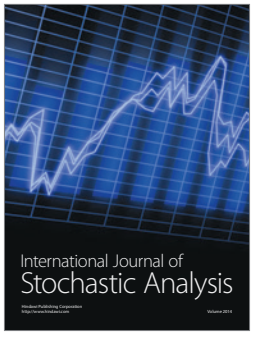
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