Strategic Insider Trading in Continuous Time: A New Approach

BY Knut K. Aase and Bernt Øksendal

DISCUSSION PAPER

Strategic Insider Trading in Continuous Time: A New Approach.

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Abstract

The continuous-time version of Kyle's (1985) model of asset pricing with asymmetric information is studied, and generalized by allowing time-varying noise trading. From rather simple assumptions we are able to derive the optimal trade for an insider; the trading intensity satisfies a deterministic integral equation, given perfect inside information, which we give a closed form solution to.

We use a new technique called forward integration in order to find the optimal trading strategy. This is an extension of the stochastic integral which takes account of the informational asymmetry inherent in this problem. The market makers' price response is found by the use of filtering theory. The novelty is our approach, which could be extended in scope.

KEYWORDS: Insider trading, asymmetric information, strategic trade, filtering theory, forward integration

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1 Introduction

We take as our stating point the seminal paper of Kyle (1985), where a model of asset pricing with asymmetric information is presented. Traders submit order quantities to risk-neutral market makers, who set prices competitively by taking the opposite position to clear the market. Excluding the market makers, the model has two kinds of traders: a single risk neutral informed trader and noise traders. The informed trader rationally anticipates the effects of his orders on the price, i.e., he acts non-competitively or strategically. In the presence of noise traders it is impossible for the market makers to exactly invert the price and infer the informed trader's signal. Thus markets are semi-strong, but not strong form efficient.

In this model the insider makes positive profits in equilibrium by exploiting his monopoly power optimally in a dynamic context. Noise trading provides camouflage which conceals his trading from market makers. An important issue is to demonstrate that this is possible in equilibrium without destabilizing prices.

Kyle's approach is to first study a one-period auction, then extend the analysis to a model in with auctions take place sequentially, and finally letting the time between the auctions go to zero, in which case a limiting model of continuous trading is obtained. Back (1992) formalize and extend the continuous-time version of the Kyle model, by i.a., the use of dynamic programming.

There is a rich literature on the one period model, as well as on discrete insider trading, e.g., Holden and Subrahmanyam (1992), Admati and Pfleiderer (1988), and others, all adding insights to this class of problems. Glosten and Milgrom (1985) present a different approach, containing similar results to Kyle. Before Kyle (1985) and Glosten and Milgrom (1985) there is also a huge literature on insider trading in which the insider acts competitively, e.g., Grossman and Stiglitz (1980).

The purpose of this article is to study the continuous-time model directly, not as a limiting model of a sequence of auctions, and use certain aspects of the modern methodological machinery in continuous-time modeling to resolve the problem of the informed trader, in a slightly more general setting with time-varying noise trading. The wealth of the insider can be represented as a stochastic integral of his orders with respect to the changes in the market price. This integral is not of a standard form, since the insider's order is not in the information set generated by the prices. This is precisely where a key

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part of the problem lies; the insider has more information then reflected in the market prices.

There is, however, an extension of the stochastic integral, called the *forward* integral, in which the usual information constraint of this type of analysis need not be satisfied. This is exactly what we need in the present context of asymmetric information.

The prices set by the market makers are in the form of a conditional expectation, which calls for the use of filtering theory. Combining these two methodologies, we are able to solve the insider's problem in a direct way, leading to a deterministic integral equation for the insider's trading intensity $\beta(t)$ at time t, given his information set with perfect forward information.

We solve the integral equation for the trading intensity $\beta(t)$ by by transforming this equation to a non-linear, separable differential equation, which calls for a simple solution. This we compare to the solution of Kyle (1985) (and also Back (1992)). In the special case of time homogeneous noise trading, we recover the Kyle-solution. For time-varying noise trading we get the result that the market depth is still a constant, and the expected (ex ante) profits of the insider depends on the average volatility process.

2 The Model

At date T there is to be a public release of information that will perfectly reveal the value of an asset; cf. fair value accounting. Trading in this asset and a risk-free asset with interest rate zero is assumed to occur continuously during the interval $[0, T]$. The information to be revealed at time T is represented as a signal $\tilde{v}$, a random variable which we interpret as the price at which the asset will trade after the release of information. This information is already possessed by a single insider at time zero. The unconditional distribution of $\tilde{v}$ is assumed to be normal with parameters $\mu_{\tilde{v}}$ and $\sigma_{\tilde{v}}$.

In addition to the insider, there are liquidity traders who have random, price-inelastic demands, and risk neutral market makers. All orders are market orders and the net order flow is observed by all market makers. We denote by z_t the cumulative orders of liquidity traders through time t. The process z is assumed to be a Brownian motion with mean zero and variance rate σ_t^2, i.e., $dz_t = \sigma_t dB_t$, where $\sigma_t > 0$ is a deterministic continuously differentiable function on $[0, T]$, for a standard Brownian motion B defined on a probability space (Ω, P). Note that we do not assume that z is independent

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of $\tilde{v}$. We let x_t be the cumulative orders of the informed trader, and define

$$(2.1) \qquad y_t = x_t + z_t \qquad \text{for all } t \in [0, T]$$

as the total orders accumulated by time t.

Market makers only observe the process y, so they cannot distinguish between informed and uninformed trades. Let $\mathcal{F}_t^y = \sigma(y_s; s \leq t)$ be the information filtration of this process. Since the market makers are assumed to be perfectly competitive and risk neutral, they will set the price p_t at time t as follows

$$(2.2) \qquad p_t = E(\tilde{v} | \mathcal{F}_t^y),$$

which we will call a *rational* pricing rule. We assume that the insider's portfolio is of the form

$$(2.3) \qquad dx_t = (\tilde{v} - p_t)\beta(t)dt, \qquad x(0) = 0,$$

where β is some deterministic function, both assumptions consistent with Kyle (1985).[1] The function β_t is the trading intensity on the insiders information surprise $(v - p_t)$.

Denote the insider's wealth by w and the investment in the risk-free asset by b. The budget constraint of the insider can best be understood by considering a discrete time model. At time t the agent submits a market order $x_t - x_{t-1}$ and the price changes from p_{t-1} to p_t. The order is executed at price p_t, in other words, x_t is submitted *before* p_t is set by the market makers. The investment in the risk-free asset changes by $b_t - b_{t-1} = -p_t(x_t - x_{t-1})$, i.e., buying stocks leads to reduced cash with exactly the same amount. Thus, the associated change in wealth is (which was pointed out by Back (1992))

$$(2.4) \qquad b_t - b_{t-1} + x_t p_t - x_{t-1}p_{t-1} = x_{t-1}(p_t - p_{t-1}).$$

In other words, the usual accounting identity for the wealth dynamics is of the same type as in the standard price-taking model, except for one important difference; while, in the rational expectations model, the number of stocks in the risky asset at time t is depending only on the information available at this time, so that both the processes x and p are adapted processes with respect

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[1]The finite variation property of x is assumed by Kyle (1985), and an equilibrium where this is the case is found by Back (1992).

to the same filtration, here the order x depends on information available only at time T for the market makers (and the noise traders). As a consequence writing the dynamic equation for the insider' wealth as follows

$$(2.5) \qquad w_t = w_0 + \int_0^t x_s dp_s.$$

This is not well-defined as a stochastic integral in the traditional interpretation, since p_t is $\mathcal{F}_t^y$-adapted, and x_t is not. Thus it needs further explanation. However, since we assume that the strategy of the insider has the form (2.3) for some deterministic continuous function $\beta_t > 0$, then a natural interpretation of (2.6) is obtained by using *integration by parts*, as follows:

$$
\begin{aligned}
w_t^I &= w_0^I + x_t p_t - \int_0^t p_s dx_s \\
&= w_0^I + p_t \int_0^t (\tilde{v} - p_s)\beta_s ds - \int_0^t p_s(\tilde{v} - p_s)\beta_s ds \\
(2.6) \qquad &= w_0^I + \int_0^t (\tilde{v} - p_s)^2 \beta_s ds - \int_0^t (\tilde{v} - p_t)(\tilde{v} - p_s)\beta_s ds.
\end{aligned}
$$

Alternatively, one might obtain (2.6) by interpreting the stochastic integral in (2.5) as a *forward integral*. See Russo and Vallois (1993), Russo and Vallois (1995, 2000) for definitions and properties and Biagini and Øksendal (2005) for applications of forward integrals to finance.

Towards this end, let us define the information filtration of the informed trader as $\mathcal{G}_t = \mathcal{F}_t^y \vee \sigma(\tilde{v})$. Thus the informed trader knows $\tilde{v}$ at time zero and observes y_t at each time t. Obviously the filtration $\mathcal{G}_t \supset \mathcal{F}_t^y$ and this extension is not of a trivial, or technical type, but a significant one. For example, there is information in $\mathcal{G}_t$ for any $t \in [0, T)$ that will only be revealed to the market makers at the future time T. The key point here is that from (2.3) the order x_t depends on $\tilde{v}$ which is not in $\mathcal{F}_t^y$. Since the insider knows the realization of $\tilde{v}$ at time 0, he has long-lived forward-looking information. When z is not assumed to be independent of $\tilde{v}$, the extension of the ordinary stochastic integral to a semimartingale setting is not justified any longer.[2]

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[2]It does not help here to extend to a stochastic integral of a predictable process with respect to a semimartingale, as in Back (1992). In his case this procedure was valid, since z was explicitly assumed independent of $\tilde{v}$.

In the stochastic integral representing the budget constraints x_t is $\mathcal{G}_t$-measurable, and p_t is $\mathcal{F}_t$-measurable which is the violation of the standard, important requirement of any stochastic integral in the traditional interpretation.

There is, however, a stochastic integration theory based on the so-called *forward integral*, which turns out to be useful under the informational asymmetry that we have. It is a natural extension of the usual stochastic integral, with the informational constraints that we require of the dynamic wealth equation based on the above budget constraints. It is denoted by

$$(2.7) \qquad\qquad w_t = w_0 + \int_0^t x_s d^- p_s,$$

where $d^- p_s$ stands for forward integration. From its very definition, which is given by a limit (in probability) of the usual partial sums of the type appearing in (2.4), it follows that it will have the correct financial interpretation, given that the concept is meaningful. It turns out that it is, and naturally the forward integral will not possess many of the standard properties of the stochastic integral, but there is a version of Itô's formula that still is valid, and which we need in the following (see Appendix I for a definition, Itô's formula, and references).

We can now formulate the problem: The insider wants to solve, for each time point t

$$(2.8) \qquad\qquad \max_x E(w_T | \mathcal{G}_t)$$

subject to the price p satisfying the rational pricing rule (2.2), the insider's strategy x satisfying (2.3), and the dynamic forward stochastic differential equation (2.7) holding for all $t \in [0, T]$. Restricting the solution to (2.3) seems natural in a situation with $\tilde{v}$ normally distributed, since then the price p_t will be linear (see the next section), but we have not shown that this follows from (2.2).

Usually the assumption

$$(2.9) \qquad\qquad p_T = \tilde{v} \qquad \text{a.s.}$$

is made, but it can be demonstrated that this is a consequence of our other model assumptions (see Aase et. al (2012)). This result seems natural, ensuring that all information available has been incorporated in the price at

the time T of the public release of the information. However, in our present exposure we present a proof where this assumption is needed, which gives a much simpler, but more constructive proof, which can be extended i scope.

Since there is a tacit understanding that the price process p is continuous in this model, this result also means that the insider must trade continuously throughout the time interval $[0, T]$, and we can expect that the trading intensity β must be large as t approaches T in order for this condition to be satisfied. [3]

An *equilibrium* is a pair (p, x) such that p satisfies (2.2), given x, and x is an optimal trading strategy solving (2.8), given p. We now have the following result:

Theorem 2.1. *Given the linear trading strategy (2.3), the optimal trading intensity $\beta(t)$ is given by*

$$(2.10) \qquad \beta_t = \left(\frac{\int_0^T \sigma_s^2 ds}{S_0} \right)^{\frac{1}{2}} \frac{\sigma_t^2}{\int_t^T \sigma_s^2 ds}; \qquad 0 \le t \le T.$$

The corresponding price p_t set by the market makers is

$$(2.11) \qquad p_t = E(\tilde{v}) + \int_0^t \lambda_s dy_s,$$

where $\tilde{y}_t$ defined by $d\tilde{y}_t = \frac{1}{\sigma_t} dy_t$ is a Brownian motion with respect to the market makers' information, and the price sensitivity λ_t is given by

$$(2.12) \qquad \lambda_t \equiv \lambda = \frac{S_0^{\frac{1}{2}}}{\left(\int_0^T \sigma_s^2 ds \right)^{\frac{1}{2}}}; \qquad a \ constant \ over \ time.$$

In Section 4 we present a proof of this theorem. Here we discuss the properties of the solution.

3   Properties of the equilibrium.

The generalization relative to Kyle (1985) included in Theorem 2.1 allows for a time varying volatility parameter in the order process of the noise traders.

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[3] If the price $p_t \ne \tilde{v}$ for some $t < T$, and the agent did not trade in $[t, T)$, there would have to be a jump in the price at time T, which the results of our model rule out. This would not be rational for the insider to do, as he would miss some profit opportunities by not trading.

One would, perhaps, expect that as a consequence the market liquidity function λ_t would depend on time, suggested by the expression (5.26) in the next section. The result of Theorem 2.1 is that it does not. The intuition for this can be explained as follows:

The trading intensity β_t will typically increase as t approaches T, since the insider becomes increasingly desperate to utilize his residual information advantage. In particular, from expression (2.10) in Theorem 2.1 we see that β_t/σ_t^2 increases as t increases. It follows from the proof in the next section, equations (5.25) and (5.26), that the price sensitivity λ_t can be written

$$\lambda_t = \frac{\beta_t S_t}{\sigma_t^2}.$$

Here

$$S_t := E[(\tilde{v} - p_t)^2] \qquad \text{and} \qquad S_0 = E[(\tilde{v} - E[\tilde{v}])^2].$$

Furthermore S_t can be shown to have the form

$$S_t = \frac{S_0}{1 + S_0 \int_0^t \tilde{\beta}_s^2 ds}; \qquad t \in [0, T],$$

(see equation (5.10)) where

$$\tilde{\beta}_t = \frac{\beta_t}{\sigma_t}; \qquad 0 \le t \le T.$$

The quantity $\int_0^t \tilde{\beta}_s^2 ds$ measures the the "amount" of insider trading to liquidity trading by time t. As this quantity increases over time, the amount of private information S_t remaining at time t is seen, from the above expression, to decrease, where S_t is the (mean square) distance between $\tilde{v}$ and p_t. The function λ_t is seen to depend on two effects:

(i) The quantity β_t/σ_t^2 increases over time, which tends to increase λ_t as time t increases.

(ii) The quantity S_t decreases over time, suggesting that the insider's information advantage is deteriorating, which tends to decrease λ_t as t increases. In equilibrium (i) is offset by (ii) and $\lambda_t = \lambda$ is constant over time.

Notice that the important quantities are β_t/σ_t^2 and $\beta_t/\sigma_t = \tilde{\beta}_t$ in the above arguments. The mere fact that the amount of insider trading represented by $\int_0^t \beta_s^2 ds$ is large, is no guarantee that the market price p_t is close to the fundamental value $\tilde{v}$, i.e., that S_t is small. It could be that the amount

of noise trading $\int_0^t \sigma_s ds$ is also large, in which case the insider could hide his trade, and less information about the true value would be revealed to the market makers. Similarly, we do not know that β_t is *monotonically* increasing over time, only that β_t / σ_t^2 is. Notice that the equilibrium value of the price sensitivity λ can be interpreted as the square root of a ratio, where the numerator is the amount of private information, ex ante, and the denominator is the amount of liquidity trading.

From the expressions in Theorem 2.1 we notice that

$$\beta_t = \frac{1}{\lambda} \frac{\sigma_t^2}{\int_t^T \sigma_s^2 ds}$$

so β_t is inversley related to λ for each t. Since the quantity $1/\lambda$ measures the market depth, the insider will naturally trade more intensely, ceteris paribus, when this quantity is large.

From the general discussion in Kyle (1985) it is indicated that if the slope of the residual supply curve λ_t ever decreases (i.e., if the market depth ever increases), then unbounded profits can be generated. This is inconsistent with an equilibrium, so λ_t must be monotonically non-decreasing in any equilibrium. It is argued that this follows since in continuous time, the informed trader can act as a perfectly discriminating monopsonist, moving up or down the residual supply curve (i.e., the market is infinitely tight). Hence, he could exploit predictable shifts in the supply curve. From the analysis of Back (1992) it is known that, more generally, this slope must be a martingale given the market makers' information. Our result that λ_t is indeed a constant is, accordingly, consistent with the literature.

One would, perhaps, expect that the insider, since he can be assumed to know the function σ_t, may use it to further conceal his trade in that he will use a high β_t at a time when σ_t is large. This impression is confirmed by investigating the optimal trading intensity β appearing in expression (2.10) of Theorem 2.1.

However, when σ_t is low the insider must apply a correspondingly lower trading intensity, and it turns out that the expected (ex ante) profits average out. This can be demonstrated as follows: Consider the expected wealth of the insider given in (5.12)

$$E[w_T] = w_0 + S_0 \int_0^T \frac{\beta_t dt}{1 + S_0 \int_0^t \tilde{\beta}_s^2 ds}.$$

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Here the last term is the expected (ex ante) profits, which can be shown to be $\sqrt{S_0 \int_0^T \sigma_t^2 dt}$.[4] Thus, trading at a time-varying volatility σ_t corresponds exactly, when it comes to expected profits, to trading at a constant volatility σ determined by $\sigma^2 = \frac{1}{T} \int_0^T \sigma_t^2 dt$, the right comparison in this regard.

When the amount of liquidity trading $\int_0^t \sigma_s^2 ds$ is large, we noticed above that λ is small, in which case the insider's profit is large. However, a small value of λ is, in isolation, no guarantee for a large ex ante profit of the insider, since a large value of S_0 also makes the profit of the insider large, and λ large as well.

This points in one possible direction for extending the present model. Suppose that the private information is connected to quaterly accounting data for the firm, so T stands for one quarter, and let us extend the model beyond T to $2T$, $3T$, $\cdots$, etc. Let us, as in Admati and Pfleiderer (1988), imagine two types of liquidity traders, discretionary and non-discretionary. Just after each disclosure period of length T, the level of private information relative to the uninformed is at its minimum. It seems reasonable, from the above formula for the ex ante profits of the insider, that the discretionary traders, acting strategically to time their trades, should concentrate their trade to these times in order to loose less to the insider. That this kind behavior is optimal is expected from the conclusions of Admati and Pfleiderer (1988), who noticed that λ is a constant is not in accordance with empirical findings; the bid ask spread 2λ is varying over time.

We also have the following corollary:

Corollary 1. *Suppose $\sigma_t = \sigma > 0$ is a constant. Then the optimal trading intensity for the insider is*

$$(3.1) \qquad \beta_t = \frac{\sigma\sqrt{T}}{\sqrt{S_0}(T-t)}; \qquad 0 \le t < T.$$

The corresponding price p_t set by the market makers is given by

$$(3.2) \qquad dp_t = \lambda_t dy_t,$$

where

$$(3.3) \qquad \lambda_t \equiv \lambda = \frac{\sqrt{S_0}}{\sigma} \frac{1}{\sqrt{T}}; \qquad a\ constant\ for\ all\ t \in [0, T).$$

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[4]In the case when $\sigma_t = \sigma$ is a constant, we get that the expected profits equal $\sigma\sqrt{S_0 T}$, consistent with Kyle (1985).

This result follows from Theorem 2.1 by setting $\sigma_s \equiv \sigma$ in (5.27). The results of Corollary 1 are in agreement with Kyle (1985) and Back (1992) (when we set $T = 1$).

Eide (2007) focuses on the situation when the price process $\tilde{v}_t$ of the stock is assumed to have a specific dynamics (an Itô diffusion and a martingale with respect to an independent Brownian motion), and its current value $\tilde{v}_t$ (not $\tilde{v}_T$) is known to the insider at time t for all $t \in [0, T]$. She avoids the use of forward integrals by assuming a priori that the processes are semimartingales with respect to the relevant filtrations. Like Back she then assumes that the market makers set the price equal to $p_t = H(t, y_t)$ for some function H and that $H(t, y_t) = E(\tilde{v}_T | \mathcal{F}_t^y)$. These assumptions put the problem of finding a corresponding equilibrium into a Markovian context, which allows her to solve the problem by using dynamic programming. In conclusion, her a priori assumptions are stronger than ours, but they enable her to solve other problems than we do. In particular, the final stock value $\tilde{v} = \tilde{v}_T$ need not be normally distributed in her case.

Before we present the proof of Theorem 2.1, we will also need the dynamics of the profit of the insider for illustrations in the next section. This we first provide.

3.1 The dynamics of the profit of the insider

Later we will need the dynamics of the profits of the insider. As before let

$$(3.4) \qquad S_t = S_t^{(\beta)} := E[(\tilde{v} - p_t)^2]$$

be the mean square error process and define

$$(3.5) \qquad S_{s,t} = S_{s,t}^{(\beta)} := E[(\tilde{v} - p_s)(\tilde{v} - p_t)]; \qquad 0 \le s \le t \le T.$$

Then, taking expectation in (2.7), the insiders expected profit at any time $t \in [0, T]$ can be written

$$(3.6) \qquad E[w_t] = w_0 + \int_0^t S_t^{(\beta)} \beta_s ds - \int_0^t S_{s,t}^{(\beta)} \beta_s ds.$$

We need to compute $S_{s,t}^{(\beta)} = E[(\tilde{v} - p_t)(\tilde{v} - p_s)]$: We have

$$\begin{aligned} E[(\tilde{v} - p_t)(\tilde{v} - p_s)] &= E[(\tilde{v}^2) - E[(\tilde{v}p_s) - E(\tilde{v}p_t) + E(p_t p_s)] \\ &= E(\tilde{v}^2) - E(p_s^2) - E(p_t^2) + E(p_t p_s). \end{aligned}$$

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We first compute $E(p_t p_s)$. By (4.4) we have that p_t is a square-integrable martingale. Hence

$$E[p_s p_t] \quad = \quad E[p_s^2],$$

and consequently

$$
\begin{aligned}
E[(\tilde{v} - p_t)(\tilde{v} - p_s)] \quad &= \quad E(\tilde{v}^2) - E(p_s^2) - E(p_t^2) + E(p_t p_s) \\
&= \quad E(\tilde{v}^2) - E(p_s^2) - E(p_t^2) + E(p_s^2) \\
&= \quad E(\tilde{v}^2) - E(p_t^2) \,.
\end{aligned}
$$

But

$$E(p_t^2) = E(\tilde{v}^2) - E(\tilde{v} - p_t)^2 = E(\tilde{v}^2) - S(t) \,,$$

and hence

(3.7) $$S_{s,t}^{(\beta)} \quad = E[(\tilde{v} - p_t)(\tilde{v} - p_s] \quad = S_t(\beta) \,.$$

In particular, note that

(3.8) $$S_{s,t}^{(\beta)} \geq 0 \qquad \text{for all } s \in [0, t]$$

and

(3.9) $$S_{s,T}^{(\beta)} = 0 \qquad \text{if } p_T = \tilde{v}.$$

We then have shown the following:

Theorem 3.1. *The profit of the insider is given by*

$$E[w_t] = w_0 + \int_0^t S_s^\beta \beta_s ds - S_t^\beta \int_0^t \beta_s ds$$

for any $t \in [0, T]$.

4   Illustrations

In this section we provide some illustrations of the results of the paper. First we consider the situation where the volatility σ_t is constant through time, and address the situation with a time varying volatility below.

4.1 Constant volatilities

We start with some illustrations of the trading intensity β_t for various choices of the parameters.

We let the time horizon $T = 12$, and consider three different scenarios, where in 1) $\sigma = 0.20$, $\sigma_{\tilde{v}} = .30$, 2) $\sigma = 0.50$, $\sigma_{\tilde{v}} = .20$, and 3) $\sigma = 0.50$, $\sigma_{\tilde{v}} = .40$.
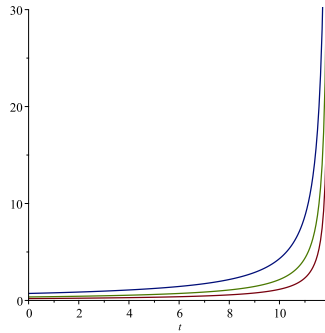

Fig. 1: The trading intensities of the insider as functions of t.

In Figure 1 we illustrate the three $\beta_i(t)$'s for each of the above scenarios $i = 1, 2$ and 3. Here $\beta_1(t)$ is the lowest graph, $\beta_2(t)$ is the highest graph and the one in the middel is $\beta_3(t)$. Thus, when the ratio of $\sigma/\sigma_{\tilde{v}}$ is largest, the trading intensity is the largest, as we know from Corollary 1.
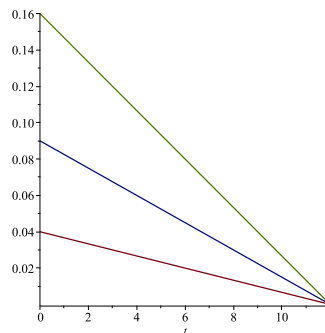

Fig. 2: The square deviations S_t as functions of t.

In Figure 2 we illustrate the time developments of the functions S_t in these three scenarios. Here the two lowest graph is $S_2(t)$, the next lowest is $S_1(t)$, while the largest one corresponds to $S_3(t)$. Since $S_0 = \sigma_{\tilde{v}}^2$, it is natural that $S_3(t)$ starts out at the highest level, and thjs gives the ranking of these curves, since they are all linear and end up in the same point $(12, 0)$. At the

horizon, when the true value of the asset is known in the market, naturally all these expected square deviations between the true value and the market price p_T must then be zero, since $p_T = \tilde{v}$ (a.s.).
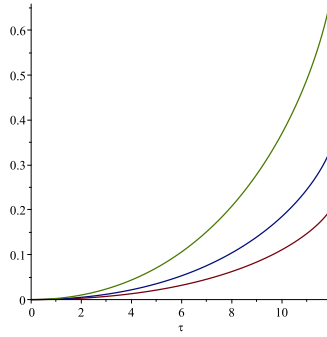

Fig. 3: The profits of the insider as functions of t.

Moving to the profit functions of the insider for these three scenarios, they are illustrated in Figure 3. We consider $(E[w_i(t)] - w_0)$ as functions of $t \in [0, 12]$ for scenario $i = 1, 2$ and 3. The lowest profit curve corresponds to scenario 1, the next lowest to scenario 2, and the highest profit curve corresponds to scenario 3. Naturally when the volatility of the true price is largest, this gives the insider an informational advantage, which she uses to obtain a larger profit. In this situation the volatility of the noise traders is also the highest, which allows the insider to better camouflage her actions from the market maker. In the situation where the volatilities of the true price are the same, the insider obtains the highest profit function when the volatility of the noise trade is the largest, again for the same reason.

Here one should notice that the profit $E[w_t]$ of the insider can be written at each $t \in [0, T]$ as follows

$$E[w_t] - w_0 = \int_0^t S_s \beta_s \, ds - S_t(\beta) \int_0^t \beta_s ds, \quad t \in [0, T],$$

where $E[w_T] - w_0 = \int_0^T S_s \beta_s \, ds$, since at the horizon $S_T = 0$, see Theorem 3.1.

4.2   Time varying volatilities

Our analysis also allows the volatility of the noise traders to vary through time, which is an extension of the situation considered by Kyle (1985).

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Below we consider three scenarios.

The first is a cyclical volatility. Many economic phenomena display some degree of cyclical behavior, for various reasons, one being that that supply of certain goods may be seasonally affected. Here we simply assume that there is a deterministic cycle that lasts for 12 time units i.e., months), with dynamics

$$\sigma_1(t) = 0.6 \sin\left(\frac{\pi}{6} t\right) + 0.6, \quad t \in [0, T].$$

The second one gives a lower volatility of the noise traders as time progresses; $\sigma_2(t) = e^{-0.1t}$, $t \in [0, T]$. This could indicate som increasing degree of 'rationality' on behalf of the noise traders as time goes, as they more and more come to the realization that they are loosing, and consequently trade less and less.

The third case is $\sigma_2(t) = 0.6e^{0.1t}$, $t \in [0, T]$. Here the noise traders trade more and more as time goes. Figure 4 illustrate these three situations, where the graphs are self eksplanatory. In all three cases $\sigma_{\tilde{v}}^2 = 0.09$.
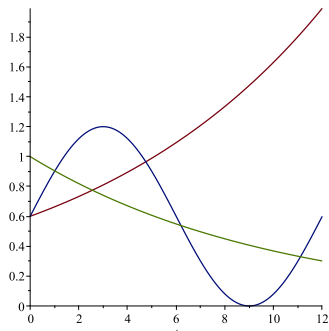

Fig. 4: The volatilities $\sigma_i(t)$ as functions of t.

For these three types of volatilites we next illustrate the trading intensities $\beta_i(t)$ of the insider as a function of time, $i = 1, 2, 3$. It is given in Figure 5.
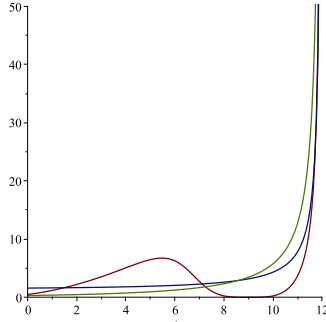
Fig. 5: The insider's trading intensities $\beta_i(t)$ as a functions of t.

The behavior reflected by these graphs would, perhaps, not be readily foreseen without some serious calculations. Starting with $\beta_1(t)$, when the volatility of the noise traders is very low, here zero at one point in time, the insider reduces trade to zero in order not to loose her informational advantage to the market maker. Towards the end, when the volatility of the noise traders increase, we observe some of the same trading intensity increase as in Figure 1. However, the rather high volatility around $t = 3$, comes to early for the insider to really increase trade, since there is stil a fairly long time to the horizon. By trading too much at this early stage, would reveal too much information to the market maker, making it more difficult to increase profits later.

The intensity $\beta_2(t)$ starts out highest of the three, but ends up lower than $\beta_3(t)$. This is natural, since the insider's trade intensity decreases relative to the case with increasing noise volatility. Because of the increasing volatility of noise trade, it is reasonable that the insider trades much towards the end in scenario 3, and it is here that the intensity is highest.

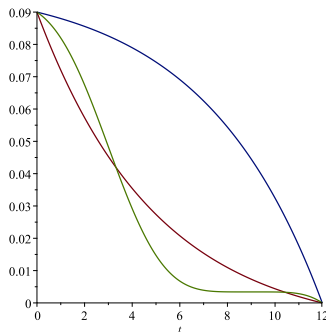The square deviation functions $S_i(t)$, $i = 1, 2, 3$ are displayed in Figure 6.

Fig. 6: The square deviations $S_i(t)$ as functions of t.

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Here $S_2(t)$ is convex, $S_3(t)$ is concave, and $S_1(t)$ is varying with time, and at its lowest around time $t = 7$.

At time zero all graphs start at $\sigma_{\tilde{v}}^2 = 0.09$ as they should according to theory, and then decrease with time to zero when t approaches T. For $S_1(t)$, in contrast to the situation with a constant volatility, the main decrease comes before $t = 6$, after which the curve flattens out. By this time a fair amount of the information has already been resolved by a combination of the insider's trade and the deterministic cyclicality of the trade by the noise traders, which the insider takes into account in her trade.

The functions $S_2(t)$ and $S_3(t)$ are symmetrically situated around a hypothetical straight line (Figure 2), which would have been the case with constant volatilities. Here $S_3(t)$ is uniformly the largest for all t, which is reasonable, because of the increasing variance of the noise traders in this scenario.

Finally, we consider the developments of the profit functions of the insider. The graphs are given in Figure 7.
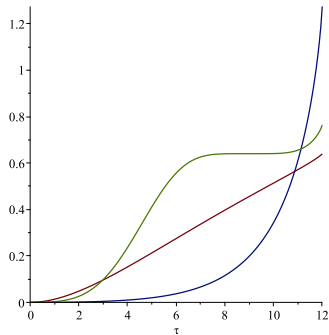

Fig. 7: The profit functions of the insider as functions of time.

The insider in scenario 1 is seen to make most of her profits before $t = 6$, which is consistent with the previous figures. Despite of the intense trading activity towards the end, the profit does not increase much later. For scenario 2 the profit ends up lowest of all at the end, and in scenario 3 the insiders intensive trade works out, and the final profit here ends up as the highest of the three. But notice that if trade were interrupted at time points 6, 7 or 8, the ranking of the profits would be quite different. Also notice that all the profits start out low, caused by the negative second term in the dynamic version of the profit function.

As can bee seen, it is an advantage to have a solution for the possibility of a time-varying volatility of the noise traders, since it can be used to throw

some more light on both the role of the noise traders, as well as on this interesting model of insider trading.

We now present the proof of Theorem 2.1 It can be noted to be rather different from the corresponding development in Kyle (1985).

5   The solution of the problem

From the requirement that the market makers are able to calculate the correct conditional expectation of $\tilde{v}$ at all times, we are led to consider filtering theory, which involves the following system of equations:

(5.1) $d\tilde{v}_t = 0,$ $\tilde{v}_0 = \tilde{v},$ (system equation)

and

(5.2) $d\hat{y}_t = \tilde{v}\beta_t dt + dz_t,$ (observation equation).

Let $\mathcal{F}_t^{\hat{y}} = \sigma(\hat{y}_s; s \leq t)$ be the information filtration of the process $\hat{y}$. The *innovation process* y is defined by

(5.3) $dy_t = (\tilde{v} - E(\tilde{v}|\mathcal{F}_t^{\hat{y}}))\beta_t dt + dz_t$

From filtering theory (see Allinger and Mitter (1981)) we then know that y generates the same filtration as $\hat{y}$, i.e., $\mathcal{F}_t^{\hat{y}} = \mathcal{F}_t^{y}$, and that $\tilde{y}$ defined by $d\tilde{y}_t := \frac{1}{\sigma_t}dy_t$ is a Brownian motion with respect to the information filtration $\mathcal{F}_t^{y}$. [5]

Using (2.2), (2.3) and the definition $y = x + z$, we see that what we have called the innovation process y in the above is equal to the total accumulated order process of the previous section. Returning to the equation (2.7), there is a analog of Itô's formula for forward integration, which says that

(5.4) $d^-(x_t p_t) = x_t d^- p_t + p_t d^- x_t + dp_t dx_t,$

(see formula (7.8) of Appendix I). Since x has finite variation, $dp_t dx_t = 0$ and we get

(5.5) $w_T = w_0 + x_T p_T - x_0 p_0 - \int_0^T p_t d^- x_t.$

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[5]The result that $\frac{1}{\sigma}y$ is a Brownian motion with respect to the market makers' information was observed by Back (1992), using a different type of argument. The result by Allinger and Mitter proved a long-standing conjecture by Kailath.

Since $(\tilde{v} - p_t) \perp p_t$ in $L^2(P)$, i.e., $E[(\tilde{v} - p_t)p_t] = 0$, we see that

(5.6)
$$E[\int_0^T p_t d^- x_t] = \int_0^T E[p_t(\tilde{v} - p_t)]\beta_t dt = 0.$$

Therefore, using the consistency requirement (2.9), we get that

$$E[w_T] = w_0 + E[x_T p_T] = w_0 + E[p_T \int_0^T (\tilde{v} - p_t)\beta_t dt] =$$

$$w_0 + E[\int_0^T \tilde{v}(\tilde{v} - p_t)\beta_t dt] =$$

(5.7)
$$w_0 + E[\int_0^T (\tilde{v} - p_t)^2 \beta_t dt] = w_0 + \int_0^T S_t \beta_t dt,$$

where

(5.8)
$$S_t := E[(\tilde{v} - p_t)^2]$$

satisfies the Riccati equation

(5.9)
$$S_t' := \frac{dS_t}{dt} = -\frac{\beta_t^2}{\sigma_t^2} S_t^2; \qquad S_0 = E[(\tilde{v} - E[\tilde{v}])^2].$$

The solution of this equation is

(5.10)
$$S_t = \frac{S_0}{1 + S_0 \int_0^t \tilde{\beta}_s^2 ds}; \qquad t \in [0, T],$$

where

(5.11)
$$\tilde{\beta}_t = \frac{\beta_t}{\sigma_t}; \qquad 0 \le t \le T.$$

Hence, by combining (5.7) and (5.10), we get

(5.12)
$$E[w_T] = w_0 + S_0 \int_0^T \frac{\beta_t dt}{1 + S_0 \int_0^t \tilde{\beta}_s^2 ds}.$$

Returning to our problem formulation in (2.8), the problem is now reduced to maximizing the above integral in the function β. The first order condition

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for this problem consists in equating the relevant directional derivative to zero, which is equivalent to use a perturbation method, or the calculus of variations, to maximize this integral over all functions β.

To this end let $\mathcal{A}$ be the family of all continuously differentiable functions $\beta : [0, T) \to R$ such that

(5.13)
$$\int_0^t \tilde{\beta}_s^2 < \infty \qquad \text{for all} \qquad t < T.$$

We use a perturbation argument to find the function $\beta \in \mathcal{A}$ which maximizes $E[w_T]$: Suppose $\beta \in \mathcal{A}$ maximizes

$$J(\beta) := S_0 \int_0^T \left(1 + S_0 \int_0^t \tilde{\beta}_s^2 ds\right)^{-1} \beta_t dt.$$

Choose an arbitrary function $\xi \in \mathcal{A}$ and define the real function g by

(5.14)
$$g(y) = J(\beta + y\xi); \qquad y \in R.$$

Then g is maximal at $y = 0$ and hence

$$0 = g'(0) = \frac{d}{dy} J(\beta + y\xi)|_{y=0} =$$

$$\frac{d}{dy} \left(S_0 \int_0^T \left(1 + S_0 \int_0^t \frac{(\beta_s + y\xi_s)^2}{\sigma_s^2} ds\right)^{-1} (\beta_t + y\xi_t) dt \right)\Big|_{y=0} =$$

$$S_0 \int_0^T \left(1 + S_0 \int_0^t \tilde{\beta}_s^2 ds\right)^{-1} \xi_t dt - S_0^2 \int_0^T \left(1 + S_0 \int_0^t \tilde{\beta}_s^2 ds\right)^{-2} \left(\int_0^t \frac{2\beta_s \xi_s}{\sigma_s^2} ds\right) \beta_t dt$$

$$= \int_0^T S_t \xi_t dt - 2 \int_0^T S_t^2 \left(\int_0^t \frac{\beta_s \xi_s}{\sigma_s^2} ds\right) \beta_t dt.$$

Changing the order of integration in the last term we get

$$\int_0^T S_t \xi_t dt - 2 \int_0^T \left(\int_s^T S_t^2 \beta_t dt\right) \frac{\beta_s \xi_s}{\sigma_s^2} ds = 0,$$

or

$$\int_0^T \{S_t - 2(\int_t^T S_s^2 \beta_s ds) \frac{\beta_t}{\sigma_t^2}\} \xi_t dt = 0.$$

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Since $\xi \in \mathcal{A}$ was arbitrary, we conclude that an optimal β_t must satisfy the equation

(5.15)
$$\sigma_t^2 S_t = 2\beta_t \int_t^T S_s^2 \beta_s ds$$

where, as before, S_t is given by equation (5.10). This is an integral equation in the unknown function β. Differentiating (5.15) with respect to t we get

$$2\sigma_t \sigma_t' S_t + \sigma_t^2 S_t' = 2\beta_t' \int_t^T S_s^2 \beta_s ds - 2\beta_t^2 S_t^2.$$

Combining this with (5.9) we obtain

(5.16)
$$2\sigma_t \sigma_t' S_t + \beta_t^2 S_t^2 = 2\beta_t' \int_t^T S_s^2 \beta_s ds.$$

We now combine (5.15) and (5.16) to get

$$2\sigma_t \sigma_t' S_t + \beta_t^2 S_t^2 = \frac{\beta_t'}{\beta_t} \sigma_t^2 S_t$$

or

$$\frac{\beta_t'}{\beta_t} = \frac{2\sigma_t'}{\sigma_t} + \frac{\beta_t^2}{\sigma_t^2} \frac{S_0}{\left(1 + S_0 \int_0^t \frac{\beta_s^2}{\sigma_s^2} ds\right)}.$$

Integrating this we obtain, with c_i integration constant, $i = 1, 2, \cdots$

$$\log \beta_t = 2 \log \sigma_t + \log\left(1 + S_0 \int_0^t \frac{\beta_s^2}{\sigma_s^2} ds\right) + c_1$$

or

(5.17)
$$\beta_t = c_2 \sigma_t^2 \left(1 + S_0 \int_0^t \frac{\beta_s^2}{\sigma_s^2} ds\right).$$

Define

(5.18)
$$\alpha_t = \frac{\beta_t}{\sigma_t^2}.$$

Then equation (5.17) gives the non-linear, separable differential equation

$$\alpha_t' = c_2 S_0 \sigma_t^2 \alpha_t^2,$$

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which has the general solution

$$\alpha_t = \left(c_3 - c_2 S_0 \int_0^t \sigma_s^2 ds\right)^{-1}$$

or

(5.19)
$$\beta_t = \sigma_t^2 \left(c_3 - c_2 S_0 \int_0^t \sigma_s^2 ds\right)^{-1}.$$

Substituting (5.19) into the right hand side (RHS) of (5.17) we get

$$RHS = c_2 \sigma_t^2 \left(1 + S_0 \int_0^t \sigma_s^2 (c_3 - c_2 S_0 \int_0^s \sigma_u^2 du)^{-2} ds\right)$$

$$= c_2 \sigma_t^2 \left(1 - \frac{1}{c_2} \Big|_0^t \frac{1}{c_3 - c_2 S_0 \int_0^s \sigma_u^2 du}\right)$$

$$= \sigma_t^2 \left[c_2 - \left(\frac{1}{c_3 - c_2 S_0 \int_0^t \sigma_u^2 du} - \frac{1}{c_3}\right)\right]$$

$$= \frac{\sigma_t^2 \left((\int_0^t \sigma_u^2 du)(c_2 S_0 - c_2^2 c_3 S_0) + c_2 c_3^2\right)}{c_3 (c_3 - c_2 S_0 \int_0^t \sigma_u^2 du)}.$$

Therefore, (5.17) holds if and only if

$$c_2 S_0 - c_2^2 c_3 S_0 = 0,$$

i.e.,

(5.20)
$$c_2 c_3 = 1.$$

Substituting this into (5.19) we get

(5.21)
$$\beta_t = \frac{\sigma_t^2 c_2}{1 - c_2^2 S_0 \int_0^t \sigma_s^2 ds}.$$

Since by the consistency requirement the relation (2.9) holds, we must have $S_T = 0$ and hence

(5.22)
$$\lim_{t \to T^-} \beta_t = \infty.$$

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Using this in (5.21) we deduce that

$$(5.23) \qquad c_2^2 S_0 \int_0^T \sigma_s^2 ds = 1$$

which gives

$$(5.24) \qquad \beta_t = \frac{\sigma_t^2 \left(\int_0^T \sigma_s^2 ds \right)^{\frac{1}{2}}}{S_0^{\frac{1}{2}} \int_t^T \sigma_s^2 ds}.$$

By the Kalman filter theory (see e.g., Kalman (1960), Davis (1977-84), Kallianpur (1980) or Øksendal (2003), Ch. 6) we know that the corresponding conditional expected value $p_t = E(\tilde{v}|\mathcal{F}_t^y)$ is given by

$$(5.25) \qquad dp_t = \frac{\beta_t S_t}{\sigma_t^2} dy_t = \lambda_t dy_t,$$

with

$$(5.26) \qquad \lambda_t = \frac{S_t \left(\int_0^T \sigma_s^2 ds \right)^{\frac{1}{2}}}{S_0^{\frac{1}{2}} \int_t^T \sigma_s^2 ds}; \qquad 0 \le t < T.$$

Now recall from equation (5.10) that

$$S_t = E[(\tilde{v} - p_t)^2] = \frac{S_0}{1 + S_0 \int_0^t \left(\frac{\beta_s}{\sigma_s} \right)^2 ds}; \qquad S_0 = \text{var}(\tilde{v}) = \sigma_{\tilde{v}}^2.$$

By the use of (5.24) we find that

$$S_t = \frac{S_0}{1 + \left(\int_0^T \sigma_s^2 ds \right) \int_0^t \frac{\sigma_u^2}{(\int_u^T \sigma_s^2 ds)^2} du} = \frac{S_0 \int_t^T \sigma_s^2 ds}{\int_0^T \sigma_s^2 ds}.$$

Inserting this expression for S_t into the expression for λ_t in (5.26), we obtain

$$(5.27) \qquad \lambda_t \equiv \lambda = \frac{\sqrt{S_0}}{\sqrt{\int_0^T \sigma_s^2 ds}}; \qquad \text{a constant.}$$

This completes the proof of Theorem 2.1.

6   A short discussion

Under a set of rather natural assumptions we have formulated an insider's problem as maximizing the expected value of future wealth subject to the price of the stock satisfying the rational pricing rule (2.2) and the strategy satisfying (2.3). This latter constraint seems reasonable, since from (5.5) we see that the insiders wealth can be written ($x_0 = 0$)

$$(6.1) \qquad w_T = w_0 + \tilde{v}x_T - \int_0^T p_t d^- x_t = w_0 + \tilde{v}x_T - \int_0^T p_t dx_t,$$

where the equality follows since x has finite variation. As a consequence the final net wealth equals the value of the final position less the cost of acquiring it. The cost formula is analogous to the usual one for the cost of a discriminating monopsonist. It also follows that this final wealth can be written

$$(6.2) \qquad w_T = w_0 + \int_0^T (\tilde{v} - p_t)d^- x_t = w_0 + \int_0^T (\tilde{v} - p_t)dx_t,$$

(assumption (4.1) on p. 1326 in Kyle (1985)).

From our assumptions we derive that the rational pricing rule has the form

$$(6.3) \qquad p_t = E(\tilde{v}) + \int_0^t \lambda_s dy_s$$

(assumption (4.3) p. 1326 of Kyle (1985)). Even in the case of time-varying noise trading we obtain that the price response function $\lambda_t = \lambda$ for all t, a constant.[6]

Conceptually it was an advantage to use an extended stochastic integral to achieve our goal, and given this new concept our approach was rather direct and gave a unique solution to the problem, provided our assumptions.

7   Conclusions

The continuous-time version of Kyle's (1985) model of asset pricing with asymmetric information has been studied, and generalized by allowing time-varying noise trading. From rather simple assumptions we are able to derive

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[6]The results (6.1)-(6.3) follow from our assumptions, which are the same as the ones that Kyle employ, even if he chooses to call them assumptions (Kyle (1985) (4.1)-(4-3) p. 1236).

the optimal trade for an insider; the trading intensity satisfies a deterministic integral equation, given perfect inside information, which we give a closed form solution to. We also have a dynamic relation for the profit of the insider.

Conceptually we use a new technique called forward integration in order to find the optimal trading strategy. This is an extension of the stochastic integral which takes account of the informational asymmetry inherent in this problem. The market makers' price response is found by the use of filtering theory. The novelty is our approach, which could be extended in scope.

It has been purpose of this article to study the continuous-time model directly, not as a limiting model of a sequence of auctions, and use certain aspects of the modern methodological machinery in continuous-time modeling to resolve the problem of the informed trader, in the more general setting with time-varying noise trading. The wealth of the insider can be represented as a stochastic integral of his orders with respect to the changes in the market price. This integral is not of a standard form, since the insider's order is not in the information set generated by the prices. This is precisely where a key part of the problem lies; the insider has more information then reflected in the market prices.

Wh illustrated by some examples the time developments of the various key quantities developed in the paper, like the trading intensity and the profit function of the insider, as well as the square deviation between the true value of the security and the price set by the market maker. In our illustrations we also included examples where the volatility of the noise traders were time dependent. It is evident that these examples would be very hard to analyze without our explicit results for time-varying volatilities.

Our line of attack is a natural framework to further investigate some of the problems underlying insider trading and differential information. In a companion paper we intend to analyze the situation when the market maker is not a fiduciary, unlike in the present model.

Appendix I: The forward integral

Consider a general information filtration $\mathcal{G}_t \supset \mathcal{F}_t$. If B_t is a Brownian motion with respect to $\mathcal{F}_t$, it need not be a semimartingale with respect to a bigger filtration $\mathcal{G}_t \supset \mathcal{F}_t$. A simple example is

$$\mathcal{G}_t = \mathcal{F}_{t+\delta}; \qquad t \geq 0$$

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where $\delta > 0$ is a constant.

First we ask the question what integrals of the form $\int_0^t x_s dB_s$ are supposed to mean when x_s is $\mathcal{G}_s$-adapted. In this paper $\mathcal{G}_t$ is the information filtration of the insider, while $\mathcal{F}_t$ is the corresponding information filtration generated by the order process y and thus possessed by the market makers. Below we consider forward integrals of processes driven by Brownian motion.

The *forward* integral $\int_0^t x_s d^- B_s$ is defined by

$$(7.1) \qquad \int_0^T x_t d^- B_t := \lim_{\Delta t_i \to 0} \sum_i x_{t_i}(B_{t_{i+1}} - B_{t_i}),$$

whenever the limit exists in probability, and $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ is a partition of $[0, T]$. Thus this integral is defined in the intuitive manner as a limit of sums, and it should be clear that when x_t is $\mathcal{F}_t$-adapted, this integral coincides with the ordinary Itô integral over non-anticipating functions. Viewed this way, the forward integral is a direct and very natural extension of the Itô integral to anticipating (non-adapted) functions.

More formally, suppose $x : [0, T] \to R$ is a measurable stochastic process adapted to the filtration $\mathcal{G}_t$ but not necessarily to the filtration $\mathcal{F}_t$. The forward integral of x with respect to B_t was first defined by Russo and Vallois (1993), and was applied to insider trading, in a framework different from the one in the present paper, in Biagini and Øksendal (2005). For our purpose, it is sufficient to consider the case when x is left continuous with right-sided limits (càglàd). Then the original definition simplifies to (7.1).

One can show that if x_t is adapted to some filtration $\mathcal{G}_t$ such that B_t is a $\mathcal{G}_t$-semimartingale, then the forward integral of x coincides with the semimartingale integral of x (if it exists). See Biagini and Øksendal (2005). Thus the forward integral is an extension of the semimartingale integral to (possibly) non-semimartingale contexts.

An Itô formula for the forward integrals was first obtained by Russo and Vallois (1995, 2000). It may be presented as follows: Let $X_t = X_t(\omega)$ be a stochastic process of the form

$$(7.2) \qquad X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s d^- B_s; \qquad X_0 \in R, \qquad \text{a constant,}$$

where α and β are measurable processes, such that

$$\int_0^t \{|\alpha_s| + \beta_s^2\} ds < \infty \qquad \text{a.s. for all } t,$$

and β is forward integrable. A short hand differential notation for (7.2) is

$$(7.3) \qquad\qquad d^- X_t = \alpha_t dt + \beta_t d^- B_t; \qquad X_0 \in R.$$

Such processes X_t are called forward processes.

Theorem 7.1. *(The one-dimensional Itô formula for the forward processes.)*
Let X_t be as above and let $f \in C^{1,2}(R \times R)$. Define

$$Y_t = f(t, X_t).$$

Then Y_t is again a forward process and

$$(7.4) \qquad d^- Y_t = \frac{\partial}{\partial t} f(t, X_t)\, dt + \frac{\partial}{\partial x} f(t, X_t)\, d^- X_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(t, X_t)\, \beta_t^2 dt.$$

Note the similarity between this and the classical Itô formula. We refer to Russo and Vallois (1995, 2000) for a proof.

The Itô formula extends to several dimensions, as follows:

Theorem 7.2. *(The multi-dimensional Itô formula for the forward processes.) Let*

$$(7.5) \qquad\qquad d^- X_t^{(i)} = \alpha_t^{(i)} dt + \sum_{k=1}^{m} \beta_t^{(i,k)} d^- B_t^{(k)}; \qquad 1 \leq i \leq n$$

be n forward processes, driven by m independent Brownian motions $(B_t^{(1)}, \cdots, B_t^{(m)})$. Let $f \in C^{1,2}(R \times R^n)$ and define

$$Y_t = f(t, X_t).$$

Then Y_t is again a forward process and

$$
\begin{aligned}
(7.6) \qquad d^- Y_t &= \frac{\partial}{\partial t} f(t, X_t)\, dt + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(t, X_t)\, d^- X_t^{(i)} \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(t, X_t)\, dX_t^{(i)} dX_t^{(j)},
\end{aligned}
$$

where

$$(7.7) \qquad\qquad dX_t^{(i)} dX_t^{(j)} = \sum_{k=1}^{m} \beta_t^{(i,k)} \beta_t^{(j,k)}\, dt.$$

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Example 5.3. Suppose $m = 1$ and $i = 2$, i.e.,

$$d^- X_t^{(i)} = \alpha_t^i \, dt + \beta_t^{(i)} \, d^- B_t; \qquad i = 1, 2.$$

Choose $f(t, x_1, x_2) = x_1 x_2$ and define

$$Y_t = f(t, X_t) = X_t^{(1)} X_t^{(2)}.$$

Then by (7.6) and (7.7) we get

(7.8) $\qquad d^-(X_t^{(1} X_t^{(2)}) = d^- Y_t = X_t^{(1)} d^- X_t^{(2)} + X_t^{(2)} d^- X_t^{(1)} + \beta_t^{(1)} \beta_t^{(2)} \, dt.$

This is the formula we use in (5.4), and later.

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