The optimal extraction rate versus the expected real return of a sovereign wealth fund: Some simulations

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The optimal extraction rate versus the expected real return of a sovereign wealth fund: Some simulations

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Abstract

With reference to funds established for the benefits of the public at large, a university endowment, or other similar sovereign wealth fund, we demonstrate that the *optimal extraction rate* from the fund is significantly smaller than the *expected real rate of return* on the underlying fund. We consider the situation where the influx to the fund has stopped, it is in a steady state, and is invested broadly in the international financial markets. The optimal spending rate secures that the fund is a perpetuity, i.e., it will last 'forever', where the real value of the fund after payments is stationary, while spending according to the expected rate of return will deplete the fund with probability 1. Optimal portfolio choice and spending are then inconsistent. Our conclusions are contrary to the recommendations of an expert panel to the Norwegian Government Pension Fund Global, as well as at odds with part of the extant literature on the management of endowments of universities.

KEYWORDS: Optimal extraction rate, endowment funds, expected utility, recursive utility
JEL-Code: G10, G12, D9, D51, D53, D90, E21.

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1 The basic model

We consider the optimal consumption and portfolio selection problem using the life cycle model. We have an agent represented by the pair (U, e), where $U(c)$ is the agent's utility function over consumption processes c, and e is the agent's endowment process. The problem consists maximizing utility subject to the agent's budget constraint

$$\sup_{c,\varphi} U(c) \quad \text{subject to} \quad E\Big(\int_0^T \pi_t c_t dt\Big) \leq E\Big(\int_0^T \pi_t e_t dt\Big) := w, \qquad (1)$$

where φ are the optimal fractions of wealth in the various risky investment possibilities facing the agent, and w is the current value of the agent's wealth. The quantity π_t is the state price deflator at each time t, i.e., the Arrow-Debreu state prices in units of probability. The horizon $T \leq \infty$.

The consumer takes as given a dynamic financial market, consisting of N risky securities and one riskless asset, the latter with rate of return r. The agent's actions does not affect market prices of the risky assets, nor the risk-free rate of return r.

2 Optimal consumption and portfolio choice

In the paper we consider two different specifications of utility, (i) the standard model with separable and additive expected utility, and (ii) recursive utility of the Duffie-Epstein type with a Kreps-Porteus specification of the associated certainty equivalent, the latter derived from expected utility.

First we consider a continuous-time framework, where the agent's preferences are represented by standard expected additive and separable utility of the form

$$U(c) = E\Big(\int_0^T u(c_t, t)dt\Big). \qquad (2)$$

Here $u(c, t)$ is the agent's felicity index, which we assume to be of the CRRA-type, meaning that the real function $u(x, t) = \frac{1}{1-\gamma} x^{1-\gamma} e^{-\delta t}$, where γ is the agent's relative risk aversion and δ is the agent's impatience rate (the utility discount rate).

It follows from optimal consumption and portfolio choice theory that the optimal consumption per time unit, c_t^*, and the optimal wealth at time t, W_t^*, are connected. The starting point for this derivation is the following formula for the market value of current wealth W_t

$$W_t^* = \frac{1}{\pi_t} E_t\Big\{\int_t^T \pi_s c_s^* ds\Big\}. \qquad (3)$$

Here $E_t(X) = E(X|\mathcal{F}_t)$ is the conditional expectation of any random variable X given the information by time t, where $\mathcal{F}_t$, is the information filtration, $0 \leq t \leq T$, and π_t is the state price deflator. Under the assumption of no arbitrage possibilities, it is given by

$$\pi_t = e^{-\int_0^t (r_u + \frac{1}{2}\eta_u'\eta_u)du - \int_0^t \eta_u dB_u} \tag{4}$$

where r is the risk free rate of return, η is the market-price-of-risk and B is a standard d-dimensional Brownian motion.

We consider a financial market consisting of N risky assets, where $\eta_t'\eta_t = \lambda_t'(\sigma_t\sigma_t')^{-1}\lambda_t$, and the vector $\lambda_t = (\mu_1(t) - r, \mu_2(t) - r, \cdots, \mu_N(t) - r)'$ consists of the risk premiums of the risky assets, i.e., the excess expected returns of the risky assets over the riskless one at any time $t \in [0, T]$. The quantity $\mu_n(t)$ is the rate of return on asset n at time t, $n = 1, 2, \cdots, N$, and prime signifies transpose of a vector (or matrix). The matrix $\sigma_t\sigma_t'$ is the instantaneous variance/covariance matrix of the risky assets in units of prices.

For simplicity of exposition we assume that $d = N$.

2.1 Optimal consumption and extraction with expected utility

The agent's optimal consumption and portfolio choice is determined next. First we give a representation of the optimal consumption c_t^* at any time $t \in [0, T]$. By employing Kuhn-Tucker and the Saddle Point Theorem, we find the optimal consumption is given by

$$c_t^* = \pi_t^{-\frac{1}{\gamma}}(\mu e^{\delta t})^{-\frac{1}{\gamma}}, \tag{5}$$

where μ is the Lagrange multiplier, ultimately determined by equality in the budget constraint. Let $Y_t = (r_t, \eta_t, \lambda_t)$ signify the investment opportunity set. We can write the optimal wealth of the agent in terms of the optimal consumption as follows

$$W_t^* = c_t^* E_t\Big\{ \int_t^T e^{\frac{1-\gamma}{\gamma}[\int_t^s (r_u + \frac{1}{2}\eta_u'\eta_u - \frac{\delta}{1-\gamma})du + \int_t^s \eta_u\, dB_u]} ds \Big\}. \tag{6}$$

In this expression the conditional expectation is a random process, in which case the volatility of W_t^* is not the same as the volatility of c^*, and the instantaneous correlation coefficient between these two processes is not unity. We want an estimate of the conditional expectation. Towards this end, we make the simplifying assumption that Y_t is stationary and also deterministic, clearly with some loss of generality regarding the last point. This we refer to

as a *deterministic* investment opportunity set. Then, by the Fubini Theorem and the moment generating function of the normal distribution, we can write the above equation as follows

$$W_t^* = c_t^* \int_t^T e^{[\frac{1-\gamma}{\gamma}(r+\frac{1}{2}\eta'\eta)-\frac{\delta}{\gamma}+\frac{1}{2}(\frac{1-\gamma}{\gamma})^2\eta'\eta](s-t)}\, ds. \tag{7}$$

The optimal consumption to wealth ratio is then

$$\frac{c_t^*}{W_t^*} = k_T(t) \quad a.s. \tag{8}$$

where $k_T(t)$ is an estimate of the optimal extraction rate at the present time t, when the horizon is $T > t$. The expression for $k_T(\cdot)$ can be written

$$k_T(t) = \frac{k}{1 - e^{-k(T-t)}}, \tag{9}$$

where the k becomes a constant for all t by our above assumption, and is given by

$$k = r - \frac{r}{\gamma} + \frac{\delta}{\gamma} - \frac{1-\gamma}{2\gamma^2}\lambda'(\sigma\sigma')^{-1}\lambda. \tag{10}$$

Provided $k > 0$, the function $k_T(t) \to k$ as $T \to \infty$ for any fixed value of t. [1]

The assumption that k is non-random and time invariant is of course special. For example, it has as a consequence that the volatility of W is the same as the volatility of c. If the investment opportunity set is stochastic, naturally this is no longer be true and better in agreement with observations. However, in order to focus on the essential questions raised in this paper, we make this simplification here. We investigate separately the situation with a stochastic investment opportunity set in Section 2.7 below.

With a very long horizon T, it is optimal for the agent to consume approximately a fraction of the remaining wealth at any time t. In reality this fraction is a stochastic process. Here it is a deterministic function slowly increasing in t, and when the horizon approaches, it increases sharply (see e.g., Figure 1 below). If the horizon is unbounded at the outset, the fraction k is consumed forever. We may consider the factor $k_T(t)$ as an estimate as of time t.

[1] The result in (10) can alternatively be derived by dynamic programming, assuming that the horizon is infinite at the outset. A transversality condition must then be satisfied, which holds if $k > 0$ (see Merton (1971) for this approach).

2.2 The real rate of return versus the optimal extraction rate

Recall the dynamics of the wealth portfolio W_t. It is given by the following equation

$$dW_t = [W_t(\varphi_t'\lambda + r_t) - c_t]dt + W_t\varphi_t'\sigma dB_t, \quad W_0 = w. \tag{11}$$

Here φ_t is the vector of the portfolio fractions of the N risky securities at time t, and B is a Brownian motion of dimension N.

The problem (1) of maximizing utility subject to the agent's budget constraint results in both the optimal fractions in the various securities, and the associated optimal consumption (see Mossin (1968), Samuelson (1969), and Merton (1969-71) for the earliest treatments of this joint problem). With a deterministic investment opportunity set, the optimal portfolio weights at any time t are given by

$$\varphi_t = \frac{1}{\gamma}(\sigma\sigma')^{-1}\lambda \quad \text{for all } t. \tag{12}$$

We want to compare the optimal extraction rate k given equation (10) with the (conditional) expected real rate of return on the optimal wealth portfolio W_t^*, which is the solution to the stochastic differential equation (11) with $c_t = c_t^*$, the optimal consumption, and with the portfolio fractions given in equation (12). The (simple) return in the time interval dt is dR_t, where

$$dR_t = \frac{dW_t^* + c_t^* dt}{W_t^*}. \tag{13}$$

With this interpretation equation (13) is a standard expression for the real return with dividends.

Accordingly, from (13), equation (11) and the optimal portfolio rule in equation (12), the t-conditional expected real rate of return of the wealth portfolio is given by the following expression

$$E_t\big(dR_t\big)/dt = r + \frac{1}{\gamma}\lambda'(\sigma\sigma')^{-1}\lambda. \tag{14}$$

The optimal extraction rate k may be rewritten as follows

$$k = \frac{\delta}{\gamma} + \big(1 - \frac{1}{\gamma}\big)\big(r + \frac{\lambda'(\sigma\sigma')^{-1}\lambda}{2\gamma}\big). \tag{15}$$

We then have the following result

Proposition 1 *Assuming a deterministic investment opportunity set, the optimal extraction rate k is a constant and it depends on the return from the fund only via the certainty equivalent rate of return, and can be written*

$$k = \frac{\delta}{\gamma} + \big(1 - \frac{1}{\gamma}\big)\big(r + \frac{1}{2}\gamma\varphi'(\sigma\sigma')\varphi\big). \tag{16}$$

Proof. Starting with the risk premium

$$\frac{1}{\gamma}\lambda'(\sigma\sigma')^{-1}\lambda = \frac{1}{\gamma}\lambda'(\sigma\sigma')^{-1}(\sigma\sigma')(\sigma\sigma')^{-1}\lambda =$$
$$\gamma\frac{1}{\gamma}\lambda'(\sigma\sigma')^{-1}(\sigma\sigma')\frac{1}{\gamma}(\sigma\sigma')^{-1}\lambda =$$
$$\gamma(\frac{1}{\gamma}(\sigma\sigma')^{-1}\lambda)'(\sigma\sigma')\frac{1}{\gamma}(\sigma\sigma')^{-1}\lambda) = \gamma\varphi'(\sigma\sigma')\varphi,$$

where we have used (12). From this result it follows that the quantity

$$-\frac{1}{2}\gamma\varphi'(\sigma\sigma')\varphi$$

can be recognized as relative certainty equivalent for 'proportional risks', since $\varphi'\sigma$ is the volatility of the wealth portfolio (see equation (11))[2]. This means that certainty equivalent to the real rate of return in equation (13) is given by

$$r + \frac{\lambda'(\sigma\sigma')^{-1}\lambda}{\gamma} - \frac{1}{2}\gamma\varphi'(\sigma\sigma')\varphi = r + \frac{1}{2}\gamma\varphi'(\sigma\sigma')\varphi,$$

which is recognized to be that part of the optimal extraction rate in equation (15) that depends on the return of the fund. □

One comparison of interest is now between the expected real rate of return on the wealth portfolio given in (14) and the optimal extraction rate $k_T(t)$ given in (8). Assuming an infinite horizon, the inequality

$$k \leq r + \frac{1}{\gamma}\lambda'(\sigma\sigma')^{-1}\lambda \tag{17}$$

holds if and only if

$$\frac{r}{\gamma} \geq \frac{\delta}{\gamma} - \frac{1}{\gamma}\lambda'(\sigma\sigma')^{-1}\lambda\Big(\frac{1+\gamma}{2\gamma}\Big). \tag{18}$$

[2] It is really the Arrow-Pratt approximation to this quantity. In continuous-time models with Brownian-driven uncertainty, this approximation is in fact exact.

Since the second term on the right-hand side is negative, this inequality is true for reasonable values of the parameters of this problem.[3]

Alternatively, using the certainty equivalent and the representation for k given in equation (15), the inequality (17) is equivalent to

$$\frac{1}{2}\gamma\varphi'(\sigma\sigma')\varphi \geq \frac{(\delta - r)}{1+\gamma}. \tag{19}$$

That is, when half the expected excess return on the fund over the risk-free rate is larger than the right-hand side of (19), then the extraction rate is lower than the expected rate of return on the wealth portfolio.

Again, for reasonable values of the parameters of the problem, this can be seen to hold true. A very simple case occurs when $\delta \leq r$, in which case the inequality is obviously true, a fact which can be recognized from the inequality (18) as well.

Typically, the real risk-free rate close to 1% is consistent with US-data (see Table 1 below). Also, a reasonable value for the impatience rate is typically around 1%. In this case the risk premium of the fund is certainly positive, about 6% for the data of Table 1, so the inequality (19) is certainly true.

We notice that for plausible values of the parameters, the optimal extraction rate is strictly smaller than the expected real rate of return on the wealth portfolio.

It can be seen that when the extraction rate k equals the expected rate of return on the fund W, then the expected value $E(W_t) = W_0$ for any horizon t, and W_t can be shown to be a martingale. Seen from time 0,the end wealth of the agent corresponds to the random variable W_t, not the sure amount W_0. Considered from the beginning of the period, a risk averse agent would prefer the W_0 to the random wealth W_t. A claim that the agent considers the random future value W_t as equivalent to the expected value W_0 thus rests on an implicit assumption that the agent is risk-neutral.

To use the expected return on the endowment fund as the extraction rate, is on the other hand consistent with investing everything in the single risky asset, or group of assets, with the largest expected return(s) one can find, and completely ignore risk. Few responsible agents would recommend this 'optimum portfolio selection strategy' for an endowment fund.

This is, however, what Campbell (2012) seems to claim, where the author recommends that k is set equal to the real expected rate return. In the author's own words.

[3]Based on about 100 years of US-data, an estimate of the real short rate r is around 1 per cent, which is also the usual suggestion for the impatience rate δ.

"The sustainable spending rate of an endowment, which is the amount spent as a fraction of the market value of the endowment, must equal the expected return in order to achieve immortality."
This is called "vigorous immortality" by the author. As we have just demonstrated, this policy is a little bit too vigorous to be rational and consistent, and unfortunately implies the above mentioned contradiction. This policy will eventually deplete the fund with probability 1. This is shown in the next section.

2.3 An example of a typical fund

Let us illustrate the above theory by an example. We assume that the agent takes the US-market as given, where we let the risky part of our fund be represented by the $S\&P$-500 index. This corresponds to one of the best functioning securities market in the World, and should be representative in construction of the underlying market quantities. The relevant data are given as follows.

Table 1 represents the summary statistics of the data used by Mehra and Prescott (1985)[4]. By $\sigma_{cM}(t)$ we mean the instantaneous covariance rate between the return on the index S&P-500 and the consumption growth rate. Similarly, $\sigma_{Mb}(t)$ and $\sigma_{cb}(t)$ are the corresponding covariance rates between the index M and government bills b and between aggregate consumption c and Government bills, respectively[5].

	Expectation	Standard dev.	covariances
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{Mc} = 0.002268$

Table 1: Key US-data for the time period 1889-1978. Continuous-time compounding.

[4]The data is adjusted from discrete-time to continuous-time compounding.

[5]These quantities are "estimated" directly from the original data obtained from R. Mehra, where we use an assumption about ergodicity, and estimates are denoted by $\hat{\sigma}_{M,c}$, etc.

2.4 Illustrations based on expected additive and separable utility

As an example, consider a wealth fund described by the three upper rows of Table 1. The consumption data in Table 1, the fourth row, has to do with society at large, which is not under consideration here.

Let us assume a relative risk aversion of $\gamma = 2.5$, and an impatience rate $\delta = 0.01$. For the market structure of Table 1, we obtain that the expected rate of return on the wealth portfolio is 0.065 and the certainty equivalent rate of return is 0.037, corresponding to the optimal portfolio rule $\varphi = 0.95$. The optimal extraction rate under our assumptions is $k = 0.026$, corresponding to $T = \infty$. The drawdown rate is seen to be lower than the expected rate of return on the portfolio for these rather reasonable parameters of the preferences of the agent.

In Figure 1 we present graphs with a finite time horizon of $T = 300$ years using the expected utility model explained above, with the parameters of this example. The optimal long run extraction rate is $k = 0.026$ is the lower horizontal (blue) line in Figure 1. The expected return on the wealth portfolio is the upper horizontal (green) line in the figure. As the horizon approaches, there is a sharp increase in the rate of consumption. After about 200 years, the rate $k_T(200) = 0.028$, a modest increase from the steady state value of 0.026.

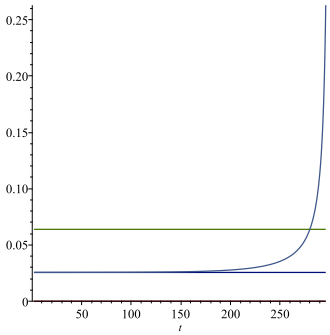

Fig. 1: The optimal drawdown rate vs expected return. $T = 300$.

The optimal consumption in this case has the expected growth rate

$$\mu_c = \frac{1}{\gamma}(r - \delta) + \frac{1}{2}\frac{1}{\gamma}(1 + \frac{1}{\gamma})\lambda'(\sigma\sigma')^{-1}\lambda$$

estimated to 0.039 and the estimate of the volatility σ_c is 0.1510, which equals the estimate of $\sigma_W = \varphi\sigma$. According to our assumption about a constant extraction rate, this implies that these two volatilities must be equal, i.e., $\varphi\sigma = 0.1510$.

As noticed, the optimal extraction rate k can be written as an arithmetic mean of the impatience rate and the certainty equivalent return rate, with weight $1/\gamma$. We may calculate how large the impatience rate must be in order to have an extraction rate equal to the expected return. The answer is $\delta = 0.108$. This level is rather unrealistic as an impatience rate.

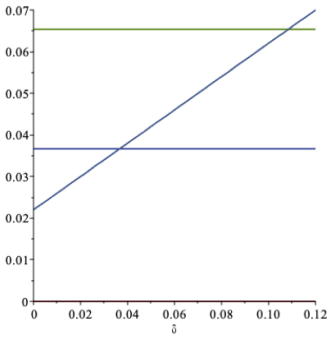

Fig. 2: The drawdown rate $k(\delta, \gamma)$ as a function of δ when $\gamma = 2.5$.

In Figure 2 these aspects are illustrated. The increasing curve is the drawdown rate $k(\delta, 2.5)$, the lower horizontal line is the certainty equivalent, $ce(\gamma)$, at $\gamma = 2.5$, and the upper horizontal line is the expected return, $er(\gamma)$, at $\gamma = 2.5$. As we see, the drawdown rate may exceed the expected return, but at a rather unrealistically high value of the impatience rate. For this data set, when the impatience is 0.0367, then $k(0.0367) = 0.0367 = ce(2.5)$. An impatience rate above this level is hardly sustainable. At this level of spending, the optimal spending rate 0.0367 should be compared to the expected rate of return 0.065.

From the inequality (18) we notice that when the impatience rate δ is large enough, the extraction rate may become larger than the expected rate of return. A high enough degree of impatience may then deplete the fund at a finite point in time in the future. This is usually not what politicians, or owners of colleges and universities have in mind when deciding on an optimal drawdown rate from a fund or an endowment.

Failure to realize this may have *negative* consequences for the beneficiaries of the fund. If k is set too large, equal to the expected rate of return from the fund for example, [6] then the fund will not last 'forever'.

[6] This is the value that is recommended by an expert panel for the Norwegian Government Pension Fund Global.

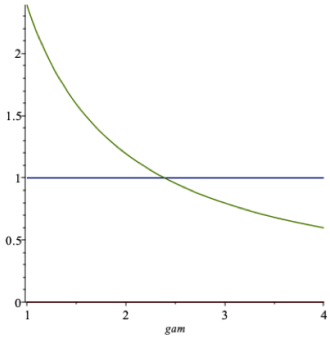

Fig. 3: φ as a function of γ.

In Figure 3 we show a graph of the fraction φ in the risky asset as a function of γ (the falling curve) for the data in Table 1. In this situation the S&P-500 is a proxy for the risky asset, so here is $N = 1$ with one risk-free asset, so φ is one-dimensional. When γ is larger than about 2.4 in this example, the agent does not borrow risk-free, since φ is then smaller than 1.

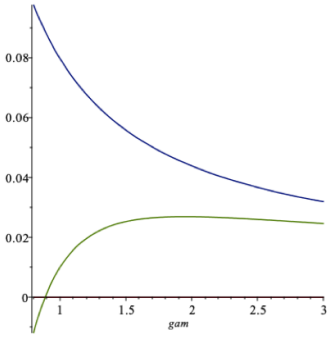

Fig. 4: $k(\gamma)$ and $ce(\gamma)$ as functions of γ; $\delta = 0.01$.

Figure 4 shows a graph of of the optimal extraction rate $k(\gamma)$ as a function of γ (the lowest curve) for the data of Table 1. The upper curve is a graph of the certainty equivalent return $ce(\gamma) = r + \frac{1}{2}\gamma\varphi'(\sigma\sigma')\varphi$. We notice that $k(\gamma) < ce(\gamma) < r + \gamma\varphi'(\sigma\sigma')\varphi$, where the latter quantity is the expected return, not shown in the figure.

The function $k(\gamma)$ is falling in γ when the risk aversion is larger than about 1.4 in the figure. It may be surprising that it does not decrease over the whole range of γ-values, but this can be attributed to the two, sometimes conflicting, roles that this parameter plays. In this model the elasticity of intertemporal substitution (EIS) in consumption $\psi = 1/\gamma$. Later we will separate these to properties of an individual using recursive utility, in which case we shall denote $\rho = 1/\psi$. The parameter ρ, called the marginal utility flexibility parameter by R. Frisch, is a measure of the individual's resistance against

substituting consumption across time (in a deterministic world). When this parameter increases, the agent will be inclined to extract more from the fund. Since $\rho = \gamma$ here, this explains the shape of the left part of the graph of k. We demonstrate later that with recursive utility, where the parameters ρ and γ are separated, the function $k(\rho)$ can be strictly increasing in the parameter ρ, under certain conditions.

A few other scenarios will be discussed next. When $\gamma = 2.0$, and $\delta = 0.01$, then the optimal extraction rate is $k = 0.027$, the expected rate of return on the wealth portfolio is 0.079 and the certainty equivalent rate of return is 0.044, corresponding to an optimal portfolio strategy of $\varphi = 1.19$. Furthermore $\sigma_c = 0.19$, $\mu_c = 0.05$. Now the agent takes on more portfolio risk, since the risk aversion has decreased.

When $\gamma = 2.0$, and $\delta = 0.03$, then the optimal extraction rate is $k = 0.037$, the expected rate of return on the wealth portfolio is 0.079 and the certainty equivalent rate of return is 0.044, corresponding to an optimal portfolio strategy of $\varphi = 1.19$. Furthermore $\sigma_c = 0.19$, $\mu_c = 0.04$. Now the agent takes on about the same portfolio risk, but the extraction rate has increased because of increased impatience, however the consumption growth rate has decreased.

From the expression (15) we notice that when $\gamma = 1$, then the optimal extraction rate equals δ, the impatience rate of the agent.

As a numerical example, when $\gamma = 1$, and $\delta = 0.02$, then $k = 0.02$, the expected rate of return on the wealth portfolio is 0.15 and the certainty equivalent rate of return is 0.079, corresponding to an optimal portfolio strategy of $\varphi = 2.38$. Furthermore and $\sigma_c = 0.38$, $\mu_c = 0.13$.

In theory reported in textbooks, we often see examples where gamma is both 1/2 (square root utility), and 1 (the Kelly Criterion), but it seems like such values are a bit too low in the present context, since this leads to positions that appear to be risky, and sometimes rather odd.

We formulate our main findings related to the theme of the paper. Under our assumptions about the investment opportunity set Y, the following holds

Proposition 2 *When (i) the objective is to maximize utility and, (ii) we consider a particular fund in isolation, the optimal drawdown rate will be lower than the expected real rate of return on the fund, for any reasonable levels of the impatience rate and the relative risk aversion.*

For an endowment fund with a well-defined owner, this analysis may be general enough to answer the question of optimal extraction from an endowment. The situation where consumption in society at large is considered as well, is treated in the last section of the paper.

2.5 The asymptotic behavior of a sovereign wealth fund

When the spending rate k is a constant, as in the above model, the wealth W_t is a geometric Brownian motion with dynamics

$$W_t = W_0 e^{\int_0^t [\mu_W - \frac{1}{2}\varphi'(\sigma\sigma')\varphi]ds + \int_0^t \varphi'\sigma dB_s}, \tag{20}$$

where

$$\mu_W = \begin{cases} 0, & \text{if } k = r + \gamma\varphi'(\sigma\sigma')\varphi; \\ \frac{1}{2}(1+\gamma)\varphi'(\sigma\sigma')\varphi + \frac{1}{\gamma}(r-\delta), & \text{if } k \text{ is optimal.} \end{cases} \tag{21}$$

In other words, when the spending rate k is equal to the expected rate of return, then $\mu_W = 0$ and W_t is a martingale. When k is optimal, given in (16), then either W_t is a submartingale or a supermartingale depending on the size of the impatience rate δ. In general, when $W_0 > 0$ then $W_t \in (0, \infty)$ for all t.

If $\mu_W > 0$ the process W_t is a submartingale, in which case $E_t(W_s) \geq W_t$ for all $s \geq t$; if $\mu_W < 0$ the process W_t is a supermartingale, in which case $E_t(W_s) \leq W_t$ for all $s \geq t$. We have the former, $\mu_W > 0$, if $\delta < \frac{1}{2}(1+\gamma)\gamma\varphi'(\sigma\sigma')\varphi + r$, and the latter, $\mu_W < 0$, if $\delta > \frac{1}{2}(1+\gamma)\gamma\varphi'(\sigma\sigma')\varphi + r$.

Of some interest here, we can also conclude about the asymptotic behavior of the wealth process from the sign of $\mu_W - \frac{1}{2}\sigma_W'\sigma_W$. Since here $\sigma_W'\sigma_W = \varphi'(\sigma\sigma')\varphi$, by the law of the iterated logarithm for Brownian motion and Feller's test for explosions the following results hold (see e.g., Karatzas and Schreve (1987), Feller (1952)):

$$(i)\,\text{If } \mu_W < \frac{1}{2}\varphi'(\sigma\sigma')\varphi, \text{then } \lim_{t\to\infty} W_t = 0, \text{and } \sup_{0\leq t<\infty} W_t < \infty \text{ a.s.} \tag{22}$$

$$(ii)\,\text{If } \mu_W > \frac{1}{2}\varphi'(\sigma\sigma')\varphi, \text{then } \lim_{t\to\infty} W_t = \infty, \text{and } \inf_{0\leq t<\infty} W_t > 0 \text{ a.s.} \tag{23}$$

Thus, when $\mu_W = 0$, i.e., when spending equals the expected return as advocated by e.g., Campbell (2012), the martingale property gives that $E(W_t) = W_0$ for all $t \geq 0$, but despite of this the wealth eventually converges to zero with probability 1, by the above result.

Moreover, using (21) when k is optimal and given in (16), we see that (22) is satisfied when $\delta > r + \frac{1}{2}\gamma^2\varphi'(\sigma\sigma')\varphi$, and (23) materializes when $\delta < r + \frac{1}{2}\gamma^2\varphi'(\sigma\sigma')\varphi$. The right-hand side of this inequality is larger than or equal to the certainty equivalent rate of return when $\gamma \geq 1$. So, for example, when δ is smaller than or equal to the certainty equivalent rate of return, then W_t converges to infinity as time $t \to \infty$, provided $\gamma \geq 1$, and the wealth never hits zero with probability 1.

These results are is not so surprising as they may seem at first sight, since it is well known that neither convergence in L^1, nor almost sure convergence implies the other. When W_t is not uniformly integrable, as here, this may typically be the case.

As we have argued above, it is reasonable that δ is smaller than, or at the most equal to, the certainty equivalent rate of return. It follows that the impatience rate will satisfy this requirement provided $\gamma \geq 1$. Hence, the prospects for a long term sustainable management of a sovereign wealth fund are really promising using the optimal spending rate k as outlined above.

Finally, if $\delta = r + \frac{1}{2}\gamma^2\varphi'(\sigma\sigma')\varphi$ when k is optimal, then

$$W_t = W_0 e^{\int_0^t \varphi'\sigma dB_s},$$

in which case

$$E(W_t) = W_0 e^{\frac{1}{2}\int_0^t \varphi'(\sigma\sigma')\varphi ds} \ \to \infty \text{ as } t \to \infty. \tag{24}$$

In this situation $\inf_{0\leq t<\infty} W_t = 0$, and $\sup_{0\leq t<\infty} W_t = \infty$, a.s.

We summarize the most essential findings as follows

Theorem 1 *(i) With the optimal spending rate k, the fund value W_t goes to infinity as $t \to \infty$ as long as the impatience rate δ is smaller than or equal to the certainty equivalent rate of return on the fund, assuming $\gamma \geq 1$.*

(ii) If the spending rate is set equal to the expected rate of the return on the fund, then the fund value goes to 0 with probability 1 as time goes to infinity.

We also have the following corollary:

Corollary 1 *With the optimal spending rate k we have the following:*

(i) $W_t \to \infty$ almost surely as $t \to \infty$ provided $\delta < r + \frac{1}{2}\gamma^2\varphi'(\sigma\sigma')\varphi$, in which case W_t is also a submartingale.

(ii) $W_t \to 0$ almost surely as $t \to \infty$ provided $\delta > r + \frac{1}{2}(1+\gamma)\gamma\varphi'(\sigma\sigma')\varphi$, in which case W_t is also a supermartingale.

We can also say something about the expected time to the wealth process W_t reaches a certain value, or more precisely, if the wealth process today satisfies $a < W_0 < b$, we can calculate the conditional expected time to the process W reaches a for the first time, say, given that a is reached before b. This is of course a topic of interest in the present model, and is what we consider next.

2.6 A conditional first exit expectation result

Consider a Feller process $X(t)$ on an interval F in the real line, and let $\tau^*(J) = \inf\{t : X(t) \notin J\}$, $J = (a, b), [a, b] \in F$. Suppose $P_x[\tau^*(J) < \infty] = 1$, $x \in J$ and let $p^+(x, J) = P_x[X(\tau^*(J)) = b]$, and $p^-(x, J) = 1 - p^+(x, J)$. Then the following result holds (Aase (1977)):

$$E_x\{\tau^*(J)|X(\tau^*(J)) = b\} = \frac{1}{p^+(x,J)} E_x\Big\{\int_0^{\tau^*(J)} p^+(X(t), J)dt\Big\}.$$

In the same paper we find the following application of this result to a geometric Brownian motion: For a diffusion where $F = (0, \infty)$, $\mu(x) = \mu \cdot x$, $\sigma^2(x) = \sigma^2 \cdot x^2$, where μ, σ^2 are two constants, and $J = (a, b)$, $0 < a < b < \infty$. Let $c = 1 - (2\mu/\sigma^2)$, it follows that

$$E_x\{\tau^*(J)|X(\tau^*(J)) = b\} = \begin{cases} \frac{2}{\sigma^2 c}\Big\{\Big(\ln\frac{b}{a}\Big)\frac{b^c+a^c}{b^c-a^c} - \Big(\ln\frac{x}{a}\Big)\frac{x^c+a^c}{x^c-a^c}\Big\}, & c \neq 0; \\ \frac{1}{3\sigma^2}\Big\{\Big(\ln\frac{b}{a}\Big)^2 - \Big(\ln\frac{x}{a}\Big)^2\Big\}, & c = 0. \end{cases}$$

A similar result holds for the boundary a by use of $p^-(x, J) = 1 - p^+(x, J)$. Notice that here $p^+(x, J) = \frac{x^c-a^c}{b^c-a^c}$.

Since we have a geometric Brownian motion process, where $X(t) = W_t$, these results are immediately applicable to our situation, which we explore below.

In the example related to Figure 1 above, we calculate the conditional expected time to the fund leaves a given interval. Consider the interval (a, b) where $a = (1/10)W_0$ and $b = 1.5W_0$. In this scenario and with the *optimal* spending rate, the parameters are $\mu_W = 0.03881$, $\sigma_W = 0.1584$ and the constant $c = -2.09$. The first exit probabilities are $p^+(W_0, J) = 0.995$ and $p^-(W_0, J) = 0.005$, so it is much more likely that the first exit takes place at upper level b than at the lower a. We obtain that $E_{W_0}\{\tau^*(J)|X(\tau^*(J)) = b\} = 14$ years while $E_{W_0}\{\tau^*(J)|X(\tau^*(J)) = a\} = 65$ years.

In the situation where the spending rate is the *expected rate of return*, $\mu_W = 0$ and $c = 1$ while $\sigma_W = 0.1584$ remains the same. The first exit probabilities have changed to $p^+(W_0, J) = 0.64$ and $p^-(W_0, J) = 0.36$, so it is still more likely that the first exit takes place at upper level b than at the lower a, but much less likely than above. Here $E_{W_0}\{\tau^*(J)|X(\tau^*(J)) = b\} = 22$ years while $E_{W_0}\{\tau^*(J)|X(\tau^*(J)) = a\} = 85$ years. Yet we know that in this situation W_t will eventually end up in zero, although it may take a long time, while in the former case with optimal extraction in place this does not ever happen with probability 1.

2.7 A stochastic investment opportunity set.

A more realistic situation arises when we also allow the investment opportunity set to be stochastic. In particular, this will have as a consequence that the volatility of of the consumption growth rate is different from the volatility of the growth rate of the agent's wealth[7].

For example, looking at Table 1 we notice that the short rate has a volatility of $\sigma_r = 5.74\%$ and a covariance rate with the $S\&P$-500 index of 0.0015, with an associated correlation rate $\kappa_2 = 0.16$, implying that the short rate is indeed stochastic. We now allow for this extension by letting the short term interest rate be subject to the following dynamics

$$dr_t = q(m - r_t)dt + \sigma_r dB_t.$$

This represents an Ornstein-Uhlenbeck process (see Breiman (1986), Vasicek (1977), Merton (1971)), where the interest rate is mean reverting around the level m, with a constant force represented by the parameter q[8].

Here the drift represents this force that keeps pulling the interest rate towards its long-term mean m with magnitude proportional to the the deviation of the interest rate from this mean. This process possesses a stationary distribution, and will also itself be a stationary process if it is started according to this distribution.

When the interest rate is low, demand for capital to start up projects will increase, which in its turn contributes to increasing the interest rate. When the interest rate is high, demand for capital goes down, which decreases the "price" of capital. Together, this is consistent with a mean reverting short rate.

The 'solution' to the above stochastic differential equation is given by

$$r_u = r_t e^{-q(u-t)} + m(1 - e^{-q(u-t)}) + \sigma_r \int_t^u e^{-q(u-v)} dB_v, \tag{25}$$

for $0 \leq t \leq u \leq T$. From this it follows that

$$E_t(r_u) = r_t e^{-q(u-t)} + m(1 - e^{-q(u-t)}) \ \to m \text{ as } u \to \infty, \text{ and}$$

$$\text{var}_t(r_u) = \frac{\sigma_r^2}{2q}(1 - e^{-2q(u-t)}) \ \to 0 \text{ as } q \to \infty.$$

This shows that the parameter m is the long-term mean of the short term interest rate, and the variance decreases as the parameter q increases. Also notice that $\mathrm{E}_t(r_u) \to m$ as $q \to \infty$, for any fixed $u > t$.

[7]When we introduce recursive utility, this fact will be an important one.

[8]In its original application this was a viscous resistance force acting on a particle in liquid suspension.

The-market-price-of-risk η_t and the excess rate of return vector λ_t could also be allowed to be stochastic, but it is more unclear what forms this should take. Sometimes is η_t assumed to be a deterministic process in t, but we shall assume that both these parameter processes are constants in what follows.

Under these conditions we now derive the optimal extraction rate from an endowment fund. With the same agent as in the previous section, we obtain the following

$$W_t = \frac{1}{\pi_t} E_t\Big\{ \int_t^T \pi_s c_s ds \Big\} = \frac{1}{\pi_t} E_t\Big\{ \int_t^T \pi_s^{(1-\frac{1}{\gamma})} (\mu e^{\delta s})^{-\frac{1}{\gamma}} ds \Big\} =$$

$$\pi_t^{-\frac{1}{\gamma}} \int_t^T E_t\Big\{ e^{(1-\frac{1}{\gamma})[-\int_t^s r_u\, du - \frac{1}{2}\eta'\eta(s-t) - \eta' \int_t^s dB_u]} (\mu e^{\delta s})^{-\frac{1}{\gamma}} \Big\} ds =$$

$$c_t^* \int_t^T E_t\Big\{ e^{-(1-\frac{1}{\gamma})\int_t^s \left(r_t e^{-q(u-t)} + m(1-e^{-q(u-t)}) + \sigma_r' \int_t^u e^{-q(u-v)} dB_v \right) du} \cdot$$

$$e^{-(1-\frac{1}{\gamma})\frac{1}{2}\eta'\eta(s-t) - \frac{\delta}{\gamma}(s-t) - (1-\frac{1}{\gamma})\eta' \int_t^s dB_u} \Big\} ds.$$

Here

$$c_t^* = \pi_t^{-\frac{1}{\gamma}} \mu^{-\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma} t}.$$

Moreover, the double integral in the exponent

$$\int_s^t \int_t^u e^{-q(u-v)} dB_v du = \int_s^t \int_v^s e^{-q(u-v)} du\, dB_v = \int_t^s H(s-v) dB_v,$$

where

$$H(x) = \frac{1}{q}(1 - e^{-qx}).$$

Thus, in the above exponent there are two correlated stochastic integrals $X_t = \sigma_r' \int_t^s H(s-u) dB_u$ and $Y_t = \eta' \int_t^s dB_u$, both normally distributed with means zero. X_t is associated with the stochastic interest rate as of time t, Y_t with the rest of the economy, including the financial market, as of time t. These must naturally be interrelated, which is the case since the same Brownian motion vector B is in both X_t and Y_t. Moreover, since $X_t + Y_t$ can really be written as one single stochastic integral, namely

$$X_t + Y_t = \sigma_r' \int_t^s H(s-u) dB_u + \eta' \int_t^s dB_u = \int_t^s (\sigma_r' H(s-u) + \eta') dB_u,$$

their sum is also normally distributed with zero mean and variance[9]

$$\mathrm{var}_t(X_t + Y_t) = \int_t^s (\sigma_r' H(s-u) + \eta')(\sigma_r H(s-u) + \eta) du.$$

[9]This expression can alternatively be written using the standard formula $\mathrm{var}_t(X_t+Y_t) = \mathrm{var}_t(X_t) + \mathrm{var}_t(Y_t) + 2\mathrm{cov}_t(X_t, Y_t)$.

Using the moment generating function of the normal variate, we then obtain

$$W_t = c_t^* \int_t^T \Big(e^{(1-\frac{1}{\gamma})^2 \frac{1}{2}\sigma_r'\sigma_r \int_t^s H^2(u-t)du - (1-\frac{1}{\gamma})qm\int_t^s H(u-t)du - (1-\frac{1}{\gamma})H(s-t)r_t} \cdot$$

$$e^{-(1-\frac{1}{\gamma})\frac{1}{2}\eta'\eta(s-t) - \frac{\delta}{\gamma}(s-t) + \frac{1}{2}(1-\frac{1}{\gamma})^2\eta'\eta(s-t) + 2(1-\frac{1}{\gamma})^2\sigma_r'\eta\int_t^s H(u-t)du} \Big) ds. \qquad (26)$$

Notice in particular that the right-hand-side in this equality depends or r_t which is a random variable, implying that the consumption to wealth ratio c_t^*/W_t is random. Hence the extraction rate is not a constant as in the previous section (as $T \to \infty$). As a consequence $\sigma_c(t) \neq \sigma_W(t)$.

Unlike the corresponding integral (7) in the previous section, this does not lead to a simple closed form expression for the extraction rate as a function of time, but the rate can be readily analyzed by numerical methods as a function of r_t, t and T, and the rest of the parameters of the problem.

Example. Let us consider the same situation as in Section 2.4, where we have an agent with relative risk aversion $\gamma = 2.5$ and an impatience rate $\delta = 0.01$ for the market data of Table 1. In addition we use the following interest parameters: $r_0 = 0.01$, $m = 0.0080$, $\sigma_r = 0.0574$, $\kappa_2 = 0.16$, all consistent with the data of Table 1. Finally $T = 200$ and $t = 0$.

The parameter q:	0.50	1.00	2.00	20.0
The optimal extraction rate:	0.019	0.023	0.025	0.026

Table 2: Optimal extraction rate as a function of parameter q.

In Table 2 we have computed the optimal extraction rate for 4 different values of the 'force' parameter q in the interest rate model. This can be compared to the optimal extraction rate of $k = .026$ in the example of Section 2.4, with a deterministic investment opportunity set. Recall, in this situation the expected real rate of return on the wealth portfolio is still (close to) 0.065 and the corresponding certainty equivalent rate of return is still 0.037 (approximately).

We notice that a stochastic interest rate has the effect of lowering the optimal extraction rate, which seems reasonable, since there is now more uncertainty in this economy, and the agent *is* risk averse. When the parameter q increases, as we have seen, the interest rate converges to the deterministic rate $m = 0.008$, which is the spot interest rate in Section 2.4, in which case the optimal extraction rate converges to $k = 0.026$ *from below*.

3 Stochastic extraction rate with expected utility

In this section we attack the problem of the previous section in a little different way. First we recognize that it is an empirical fact that the volatility of aggregate consumption in society is different from the volatility of the securities market. This can be seen from Table 1, and is true also for newer data sets, and for a variety of countries. Second, we bring this knowledge to bear also on an endowment fund. In principle we should then start with a stochastic investment opportunity set, which would imply that the volatility of consumption is different from the volatility of the agent's wealth. Here we just assume that this is the case at the outset, and investigate what implications this has on the optimal extraction rate, when the fund is part of the society as a whole (more on this later).

The basic difference from our analysis in the previous section is that the volatility of the part of the consumption that originates from the fund, σ_c^* is different from $\sigma_{W^*}(t)$, the volatility of the wealth of the agent. As a consequence the extraction rate will not be a constant, but rather a stochastic process, which we denote X_t, i.e., $X_t = \frac{c_t^*}{W_t^*}$.

We are then left with two stochastic differential equations, one for the wealth W_t^* and one for the optimal consumption c_t^* given by

$$dW_t^* = [W_t^*(\varphi_t'\lambda + r) - c_t]dt + W_t^*\varphi_t'\sigma dB_t, \quad W_0^* = w, \tag{27}$$

and

$$dc_t^* = c_t^*(\mu_{c^*}(t)dt + \sigma_{c^*}(t)dB_t) \tag{28}$$

respectively, where

$$\mu_{c^*}(t) = \frac{1}{\gamma}(r_t - \delta) + \frac{1}{2}\frac{1}{\gamma}(1 + \frac{1}{\gamma})\lambda_t'(\sigma\sigma')_t^{-1}\lambda_t \tag{29}$$

and

$$\sigma_{c^*}(t) = \frac{1}{\gamma}\,\eta_t. \tag{30}$$

We seek the dynamics of $X_t = \frac{c_t^*}{W_t^*}$.

3.1 The stochastic differential equation for the extraction rate X

Towards this end we use the the multidimensional version of Itô's lemma, to obtain the following dynamics for X_t (for simplicity of notation we omit the

star superscripts from now on):

$$dX_t = \frac{1}{W_t^2}\Big(W_t\, dc_t - c_t\, dW_t - dc_t\, dW_t + X_t\, dW_t^2\Big).$$

This can be written

$$dX_t = X_t(X_t - \alpha) + X_t\theta dB_t \tag{31}$$

where

$$\alpha_t = r_t + \varphi'\lambda_t - \mu_c(t) + \sigma_W(t)(\sigma_c(t) - \sigma_W(t))$$

and

$$\theta_t = \sigma_c(t) - \sigma_W(t).$$

In our case $\theta_t \leq 0$ and $\alpha_t \geq 0$ for all t.

We now make the simplifying assumption that both these parameters are constants. This may seem at odds with our two previous analyses, and is admittedly made here in order to simplify the analysis. It enables us to observe the effects of a stochastic investment opportunity set without having to keep track of the detailed analyses of what and where this stochasticity enters. We believe that results based on this assumption will still be of value in discussion of our main theme.

When $\theta = 0$, then $\alpha = k$, where k is the optimal extraction rate given in Section 2.1. In this case the stochastic increments of dX_t are all zero when $X_0 = k$, so the optimal extraction rate becomes the constant k, consistent with the results of Section 2.

The stochastic differential equation in (31) is *non-linear*, with values in $(0, \infty)$. From standard classification theory the boundary $+\infty$ is *attracting*, which means that the process X arrives at this border sooner or later with strictly positive probability, but the time that this takes could be infinite. Similarly, the boundary 0 is also attracting. This is based on inspection of the integrability of the function

$$\varphi(x) = e^{-\frac{2}{\theta^2}(x - x_0)}\Big(\frac{x}{x_0}\Big)^{\alpha}$$

near the relevant boundaries, where $x_0 \in (0, \infty)$ is arbitrary. Since the exponential function dominates the power function when $x \to \infty$, the result for the upper boundary follows. For the lower boundary there is no integrability problem when $\alpha > -1$. Here $\alpha \geq 0$.

More generally, if a diffusion process is given on an interval (a, b) as

$$dX_t = f(X_t)dt + g(X_t)dB_t,$$

the test function is

$$\varphi(x) = \exp\Big\{ -\int_{x_0}^{x} \frac{2f(y)}{g^2(y)} dy \Big\}.$$

When, on the other hand, the function φ is not integrable near the relevant boundary, it is called repelling, or natural. This means that the probability is zero that the boundary is ever reached.[10]

As far as we know, the solution to this particular stochastic differential equation has not been analyzed before. It is related to a logistic type model used in population dynamics (see Polansky (1979)), but then with a different, mean reverting drift term, in which case both the boundaries 0 and $+\infty$ are repelling, and the resulting process has a stationary probability distribution. This model would not seem reasonable in the present case.

We can, for example, compute the conditional expected time to the first exit from an interval given that a specified boundary will be the first exit point (see Aase (1977)). This requires that the process X has a scale function $u(x)$ and a *speed measure* $\mathrm{spm}(dx)$. The scale function for X is the incomplete gamma function, where $u'(x) = \exp(-\frac{2}{\theta^2}(x - x_0))(\frac{x}{x_0})^\alpha$ so the process $\tilde{X}_t = u(X_t)$ is of zero drift with speed measure $\mathrm{spm}(dx) = (1/\theta^2 x^2 u'(x)^2)dx$ (see Feller (1954), Breiman (1968)). These facts can be used to find the above mentioned conditional expected exit times, which also means that the process X is a reasonably well-behaved Feller-process. The speed measure can also be used to further characterize the boundaries. For example, a regular boundary b is called *absorbing* if $\mathrm{spm}(\{b\}) = \infty$. The two boundary points of X, 0 and ∞, are exit boundaries.

According to the classifications given in e.g., Gard (1988), the stochastic differential equation for X is *reducible* to a linear stochastic differential equation. By this is meant that we can find a transformation h satisfying the requirements of Itô's lemma, such that the process $Y(t) = h(t, X_t)$ satisfies a *linear* stochastic differential equation. By guessing the form of h to be $h(t, x) = x^n$, we find that for $n = -1$, then the process Y satisfies the following stochastic differential equation

$$dY_t = \big((\alpha + \theta^2)Y_t - 1\big)dt - \theta Y_t dB_t,$$

where $\theta \leq 0$ and $\alpha \geq 0$. This is a *linear* stochastic differential equation, and we can use the associated theory for such equations to find the 'solution' as

[10]Comparing with the standard geometric Brownian motion process with drift $f(x) = x\mu$ and volatility $g(x) = x\sigma$ where μ and σ are constants, when $\mu < (1/2)\sigma^2$, then 0 is attracting and $+\infty$ is repelling (natural), but when $\mu > (1/2)\sigma^2$ then $+\infty$ is attracting and 0 is repelling, which is most often met in standard applications to finance.

a function of the d-dimensional Brownian motion B and time t. It follows that

$$Y_t = \Phi(t)\big(Y_0 - \int_0^t \Phi(s)^{-1} ds\big),$$

where

$$\Phi(t) = \exp\{(\alpha + \frac{1}{2}\theta^2)t - \theta B_t\},$$

i.e., a geometric Brownian motion. Since our process of interest, the optimal extraction rate $X_t = 1/Y_t$, we have the solution of the stochastic differential equation (31) as follows

$$X_t = \frac{e^{-(\alpha+\frac{1}{2}\theta^2)t+\theta B_t}}{X_0^{-1} - \int_0^t e^{-(\alpha+\frac{1}{2}\theta^2)s+\theta B_s} ds}. \tag{32}$$

This shows that the process X_t can be written as a quotient, where the numerator is a geometric Brownian motion, and the denominator is a constant minus a time (Lebesgue) integral of the same lognormal process. The numerator converges to 0 with probability 1 as $t \to \infty$ by the law of the iterated logarithm for Brownian motion. The integral in the denominator therefore converges a.s. according to standard integrability tests. For the parameters that we have, the denominator is larger than 0 with probability 1 for all t, since X can not become negative if $X_0 > 0$; if X reaches 0 it stays there. The denominator is seen to decrease with t, and in isolation this causes X to increase, but on the other hand the numerator decreases as noticed above. Which effect is the strongest we can not say without further analysis (see below).

The expected value of the denominator is $1/X_0 - (1/\alpha)(1 - e^{-\alpha t})$. As $t \to \infty$ this value converges to $(1/X_0 - 1/\alpha)$, which must be strictly positive. It is therefore natural to search for values of X_0 that are lower than α in order to obtain reasonably stable results.

As for the geometric Brownian motion, the process X does not have a stationary distribution. Therefore we shall be content in describing its behavior for a limited period of time $0 \leq t \leq \tau$ for some finite τ. The idea is that when the economy reaches time τ, a revised analysis is called for based on the additional information acquired in this time interval. This is a reasonable strategy in most cases.

Unlike for the linear process Y, where we can find deterministic differential equations for both the expectation and variance of Y_t for any time t, we have no such general methods when it comes to the process X. The differential equations for any moment around zero will depend on the next order such moment, and so on. However, these moments can still be of some

use in obtaining more information about the properties of X, in particular in the short run, as will be demonstrated below.

3.2 Moments for the process X

Since we have an explicit 'solution' of X_t in terms of the driving Brownian motion, we can in principle find the moments by direct methods, but this is rather cumbersome, and will in the end require numerical analysis. As an alternative we proceed as follows: Starting with the stochastic differential equation given in (31), equivalent to the following stochastic integral equation

$$X_t = X_0 + \int_0^t X_s(X_s - \alpha)ds + \theta \int_0^t X_s dB_s, \tag{33}$$

and assuming that X is well enough behaved for the last stochastic integral to be a zero mean martingale, by taking expectation in (33) we obtain

$$m_t = X_0 + \int_0^t (M_s - \alpha\, m_s)ds,$$

where $M_t = E(X_t^2)$. This gives the following ordinary, first order inhomogeneous differential equation

$$\frac{dm_t}{dt} = M_t - \alpha\, m_t. \tag{34}$$

Once M_t is known, this equation can be solved by standard methods (i.e., by quadrature).

When X is small in the region of interest, say of order 0.01-0.03, then X^2 is of course much smaller, and so is its expectation. However, despite of this we have no plans to ignore M_t, and an equation for this quantity can be obtained as follows: Using Ito-calculus on the function $f(x) = x^2$, we obtain

$$X_t^2 = X_0^2 + 2\int_0^t (X_s^3 - \alpha X_s^2 + \frac{1}{2}\theta^2 X_s^2)ds + 2\theta \int_0^t X_s 2dB_s.$$

As above this leads to the following ordinary differential equation

$$\frac{dM_t}{dt} = 2N_t - (2\alpha - \theta^2)M_t,$$

where $N_t = E(X_t^3)$. Approximating, for the moment, this third moment around 0 by X_0^3, we have the following ordinary, first order inhomogeneous differential equation

$$\frac{dM_t}{dt} + (2\alpha - \theta^2)M_t = 2N_t,$$

which can be solved by quadrature. The solution is

$$M_t = e^{(\theta^2-2\alpha)t}\big(X_0^2 + \frac{2N_t}{\theta^2-2\alpha}(1-e^{-(\theta^2-2\alpha)t})\big). \tag{35}$$

By inserting this into the above equation (34) for the mean, we obtain the following solution:

$$\begin{aligned} m_t = X_0 e^{-\alpha t} &+ \frac{X_0^2}{\theta^2-\alpha}\big(e^{(\theta^2-2\alpha)t} - e^{-\alpha t}\big) \\ &+ \frac{2N_t}{\theta^2-2\alpha}\Big(\frac{1}{\theta^2-\alpha}\big(e^{(\theta^2-2\alpha)t} - e^{-\alpha t}\big) - \frac{1}{\alpha}\big(1-e^{-\alpha t}\big)\Big). \end{aligned} \tag{36}$$

We proceed to find N_t in terms of the fourth moment of X around zero. Using Ito-calculus on the function $f(x) = x^3$, we obtain

$$X_t^3 = X_0^3 + 3\int_0^t (X_s^4 - \alpha X_s^3 + \theta^2 X_s^3)ds + 3\theta\int_0^t X_s^3 dB_s,$$

which gives

$$E(X_t^3) = X_0^3 e^{3(\theta^2-\alpha)t} + \frac{X_0^4}{\theta^2-\alpha}(e^{3(\theta^2-\alpha)t} - 1) \tag{37}$$

where we have approximated $E(X_t^4)$ by X_0^4. We now substitute this expression for $E(X_t^3) = N_t$ into (35) and (36), and the resulting expressions will be utilized in what follows.

Notice that when $t \to \infty$, then $E(X_t^3) \to \frac{X_0^4}{\alpha-\theta^2}$ provided $(\alpha - \theta^2) > 0$. This we can make use of to find a reasonable value for X_0 for this range of the parameters α and θ.

3.3 Numerical results

As a supplement to the above analysis, we use simulations directly on the equation (31). The results of this are represented in the figures 5-7 below.

With reference to the US economy, recent data (see Asghar and Mortensen (2017)) indicates a growth rate of the wealth portfolio around 2% with a volatility around 3%. With a short rate of around 0.0090, this gives a market price of risk $\eta = 0.37$. The corresponding quantity of the fund of Table 1 is actually close to this value, although the individual parameters are different. In this situation, and with $\sigma_c = 0.035$ and $\sigma_W = 0.1584$ (S&P-500), the parameter $\alpha = k + (\sigma_c - \sigma_W)\sigma_W = 0.0064$ which is smaller than the optimal extraction rate k in our example of Section 2.4, since $\sigma_c < \sigma_W$. In the illustrations below, we use the following parameters $\delta = 0.01$ and $\gamma = 2.5$,

for the data in Table 1.

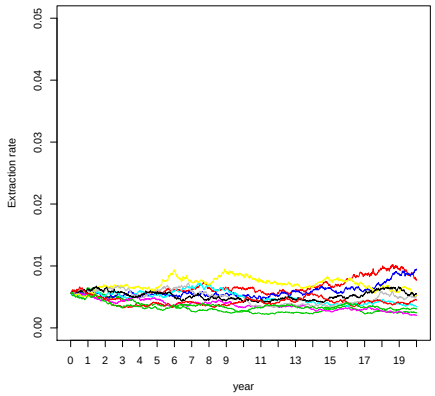

Fig. 5: Some simulations of the sample paths of X.

Figure 5 graphs 10 sample paths of the process X. We use step-lengths 1/100. With respect to the yearly data in Table 1, this represents 20 years. With a stochastic extraction rate, we notice that there is a some variability in these sample paths due to the volatility of the process X. Here $\alpha = 0.0064$, $X_0 = 0.0056$ and $\theta = \sigma_c - \sigma_W = -0.1229$. X_0 is determined such that $m(t)$ is approximately constant in an appropriate time interval $[0, \tau)$.

Figure 6 illustrates a graph of the expected value $m(t)$ of the process X, along with an approximate 95% confidence band based on the standard deviation of X_t. Since the process X is not Gaussian, the confidence limits are not symmetrically located around the mean. It turns out that an approximate 'stationary' graph is obtained for these parameter values when $X_0 = 0.0056$. It follows that $X_0 = m(0) \approx m(t)$ for $0 < t \leq \tau$, for some suitable value of τ (in Figure 6, $\tau = 20$). Provided $X_0 \leq \alpha$, we then observe that:

$$X_0 \approx m(t) \leq \alpha \leq k \leq \text{expected rate of return on the fund, for } t \in [0, \tau].$$

Interpreting $m(t)$ as close to the optimal extraction rate at time t in the short term perspective, we notice that with a stochastic real extraction rate X, this seems to call for additional caution when it comes to optimal extraction from a fund which is part of society. When compared to the optimal extraction rate 0.026 in Section 2.2, a more conservative, policy seems reasonable under the above assumptions, in this case around 0.0056.[11]

[11] In this example it is sufficient to used X_0^3 as an approximation for N_t.

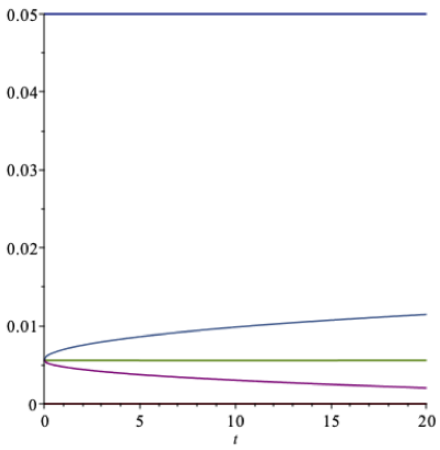

Fig. 6: Expected drawdown rate with confidence bands.

In Figure 7 we present the statistics based on 10.000 such sample paths. Again the parameter $\alpha = 0.0064$, and $\theta = -0.1229$. The horizontal curve in the middle of the graph represents the *median* in this sample, with confidence limits on each side computed directly from the generated sample paths. The median is located around 0.0056, at $t = 0$, and then slightly decreases with time. Since the distribution of X_t is skewed to the right for any t, the median and the mean do not coincide exactly. [12].

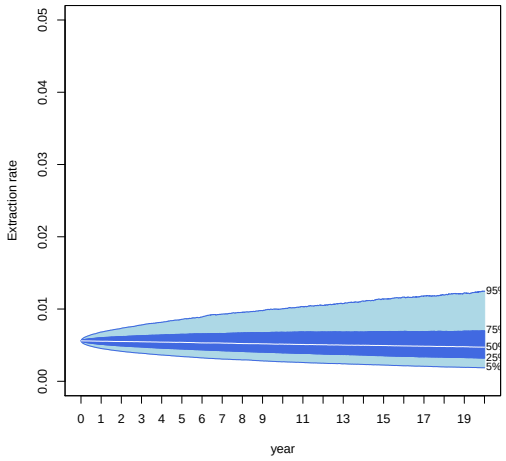

Fig. 7: Median drawdown rate based on simulations.

The expected rate of return on the wealth portfolio is 0.065 and the cer-

[12]The upper 95% confidence curve in Figure 2 is about 2 standard deviations from the mean, while the lower curve is about 1.2 standard deviations from the mean.

tainty equivalent rate of return is 0.049, following from an optimal portfolio strategy of $\varphi = 1.19$ in the fund.

When the rest of the economy is taken into account, the optimal extraction rate will reflect this. In the present example this rate is slightly decreasing with time, and well below the constant rate of Section 2.2, due to the volatility of of the consumption to wealth ratio X.

Although we have not so far established a formal proof that the mean of the optimal extraction rate in the stochastic case is smaller than the expected real rate of return of the fund, we have demonstrated this for the numerical example given above. By definition the parameter α is strictly smaller than k when $\sigma_c < \sigma_W$, and with $(m(t) - X_0)/X_0 \approx -\alpha$, the leading term in $m(t)$ at time $t = 1$, $m(t) \leq k$, provided $k \geq (1 - (\sigma_c - \sigma_W)\sigma_W)/(1 + 1/X_0)$. For the example above, this inequality holds.

We formulate our findings of this section, loosely, as the following meta result:

In the model of the present section, when the fund is part of society as a whole and the objective is to maximize utility of total consumption, the optimal extraction rate is randomly fluctuating across time, with a mean $m(t)$ that is smaller than α for t in the short run from time 0. The parameter α is smaller than the corresponding constant optimal extraction rate k (of Section 2). It follows that the optimal extraction policy gives a lower average drawdown rate than the expected real rate of return on the fund provided $\sigma_c < \sigma_W$.

In the example given above there is a problem not addressed so far, with origin in the preferences. With a relative risk aversion of 2, the volatility of consumption becomes 0.19. In order to reach the value 0.0355, which is an implicit assumption following from the equations for the optimal consumption in (28)-(30), γ has to be unrealistically large. This is actually related to a consumption puzzle (see e.g., Aase (2016)), which has its origin in the problems that models based on the expected, additive and separable utility representation has in rationalizing data.

To be more precise, both the value 0.0355 of the volatility *and* the value of 0.0181 of the expected consumption growth rate of Table 1 should match the formulas (29) and (30). This gives two equations in two unknowns, and the unique answer is $\delta = -0.10$ and $\gamma = 10.6$, with an associated fraction φ in the risky asset of about 2%. This low fraction is a consequence of the very high risk aversion. In its turn this leads $\alpha = -0.9495$, which calls for a policy of constantly injecting capital into the fund, rather than extracting from it, at a meaningless rate. These values for the preference parameters are clearly rather unrealistic, and consequently, so is the resulting policy recommendation.

Finally it should be noticed that even if the optimal spending rate X_t is negative, with a finite horizon this does not mean no spending at all. For details, see section 4.6 below.

Recursive utility is known to give better results when confronted with real market and consumption data than the standard model based on expected utility, a fact we will illustrate in the next section, where we consider the continuous-time life-cycle model based on recursive utility. First we give the theoretical details, and then we turn to numerical examples like the ones just presented and based on the same data, in order to be able to compare the results. This theory will not, as we shall demonstrate, leave us with the puzzle just explained.

4 Recursive utility

This preference structure is known to give much more reasonable results than the expected utility model when it comes to calibrating to real data, see e.g., Aase (2016a,b), where the celebrated Equity Premium Puzzle is solved using recursive utility, among other things.

We use the framework established by Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994) which elaborate the foundational work by Kreps and Porteus (1978) of recursive utility in dynamic models. Recursive utility leads to the separation of risk aversion from the elasticity of intertemporal substitution in consumption, within a time-consistent model framework.

The recursive utility $U : L \to \mathbb{R}$ is defined by two primitive functions: $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}$. The function $f(c_t, V_t)$ corresponds to a felicity index, and A corresponds to a measure of absolute risk aversion of the Arrow-Pratt type for the agent. In addition to current consumption c_t, the function f also depends on future utility V_t at time t, a stochastic process with volatility $\tilde{\sigma}_V(t) := Z_t$ at each time t.

The utility process V for a given consumption process c, satisfying $V_T = 0$, is given by the representation

$$V_t = E_t\Big\{ \int_t^T \big(f(c_s, V_s) - \frac{1}{2}A(V_s)\,\tilde{\sigma}_V(s)'\tilde{\sigma}_V(s)\big)ds\Big\}, \quad t \in [0, T]. \tag{38}$$

If, for each consumption process c_t, there is a well-defined utility process V, the stochastic differential utility U is defined by $U(c) = V_0$, the initial utility. The pair (f, A) generating V is called an aggregator.

The utility function U is monotonic and risk averse if $A(\cdot) \geq 0$ and f is jointly concave and increasing in consumption.

As for the last term in (38), recall the Arrow-Pratt approximation to the certainty equivalent of a mean zero risk X. It is $-\frac{1}{2}A(\cdot)\sigma^2$, where σ^2 is the variance of X, and $A(\cdot)$ is the absolute risk aversion function.

In the discrete time world the starting point for recursive utility is that future utility at time t is given by $V_t = g(c_t, m(V_{t+1}))$ for some function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where m is a certainty equivalent at time t (see e.g, Epstein and Zin (1989)). If h is a von Neumann-Morgenstern index, then $m(V) = h^{-1}(E[h(V)])$. The passage to the continuous-time version in (38) is explained in Duffie and Epstein (1992b), and in a direct form from the discrete time analog, by Svensson (1989).

Unlike expected utility theory in a *timeless* situation, i.e., when consumption only takes place at the end, in a *temporal* setting where the agent consumes in every period, derived preferences do not satisfy the substitution axiom (e.g., Mossin (1969), Kreps (1988)). Thus additive Eu-theory in a dynamic context, i.e., in situations where a financial market is utilized by the agents to smooth consumption across time and states of the world, has a weak axiomatic underpinning, unlike recursive utility (Kreps and Porteus (1978)).

4.1 The specification

We work with the Kreps-Porteus utility, where the aggregator has the following CES specification

$$f(c, v) = \frac{\delta}{1-\rho} \frac{c^{(1-\rho)} - v^{(1-\rho)}}{v^{-\rho}} \quad \text{and} \quad A(v) = \frac{\gamma}{v}. \tag{39}$$

The parameter $\delta \geq 0$ is the agent's impatience rate, $\rho \geq 0$, $\rho \neq 1$ is what R. Frisch called the marginal utility flexibility parameter, and $\gamma \geq 0$, $\gamma \neq 1$, is the relative risk aversion. The parameter $\psi = 1/\rho$ is the elasticity of intertemporal substitution in consumption, referred to as the EIS-parameter. The higher the value of the parameter ρ is, the more aversion the agent has towards consumption fluctuations *across time* in a deterministic world. The higher the value of γ, the more aversion the agent has to consumption fluctuations, due to the different states of the world that can occur. Clearly these two properties of an individual's preferences are different. In the conventional Eu-model, however, $\rho = \gamma$.

It can be shown that this specification is the continuous-time analogue of the one used by Epstein and Zin (1989-91) in discrete time.

4.2 The optimal consumption and portfolio rules

As with the standard EU-model we will need the optimal consumption of an agent, here one with recursive utility (U, e), who takes the market as given, and shifts her endowment e in each period from the given e_t to the optimal one c_t^* using the financial markets. In each period the agent decides how much to consume, and how much to invest in the given opportunity set for future consumption. Thus these two problems are intimately connected.

4.2.1 The first order conditions

Using Pontryagin's maximum principle, properly extended to a stochastic environment, we can solve for the basic version of recursive utility as follows [13]. The first order conditions can be written

$$\alpha\, \pi_t = Y(t) \frac{\partial f}{\partial c}(c_t^*, V_t) \quad \text{a.s. for all } t \in [0, T]. \tag{40}$$

Notice that the first order condition depends on the *future* utility V_t. This means, among other things, that the agent is in general not myopic (in the sense of Mossin (1968)).

4.2.2 The optimal consumption

It has been shown in Aase (2016b) that the answers to these two problems are given as follows: The stochastic representation for the optimal consumption growth rate is given by

$$\frac{dc_t^*}{c_t^*} = \mu_c(t)\, dt + \sigma_c(t)\, dB_t. \tag{41}$$

where

$$\mu_c(t) = \frac{1}{\rho}\,(r_t - \delta) + \frac{1}{2}\,\frac{1}{\rho}\,(1 + \frac{1}{\rho})\,\eta_t'\eta_t - \frac{(\gamma - \rho)}{\rho^2}\,\eta_t'\,\sigma_V(t) + \frac{1}{2}\frac{(\gamma - \rho)\gamma(1 - \rho)}{\rho^2}\sigma_V'(t)\sigma_V(t) \tag{42}$$

and

$$\sigma_c(t) = \frac{1}{\rho}\Big(\eta_t + (\rho - \gamma)\sigma_V(t)\Big). \tag{43}$$

[13] With this method we do not need to go via the ordinally equivalent version with corresponding $A = 0$.

Here $V_t\sigma_V(t) = \tilde{\sigma}_V(t)$, the latter appearing in the definition (38) of recursive utility. Both σ_V and V_t exist as a solution to a backward stochastic differential equation for V. The quantity η_t is the market-price-of-risk vector, and $\eta_t'\eta_t$ corresponds to our previous $\lambda_t'(\sigma_t\sigma_t')^{-1}\lambda_t$.

For recursive utility in *discrete* time it is known that the consumption to wealth ratio is equal to

$$\frac{c_t}{W_t} = \frac{1-\beta}{(\frac{V_t}{c_t})^{1-\rho}},$$

where $\beta = e^{-\delta}$. It is seen that this ratio is a constant only when $\rho = 1$, in which case our model is not valid. Thus the consumption to wealth ratio is in general a stochastic process. However, in the continuous-time model this is a bit different, as a constant consumption to wealth ratio is possible without requiring that $\rho = 1$. With a stochastic investment opportunity set, however, this ratio is not constant. This means that the volatility of consumption σ_c is not equal to the volatility of wealth $\sigma_W = \varphi\sigma$. In order to secure this, below we make the same assumption about the interest rate as in Section 2.7.

We can in general express the volatility of the utility growth rate as follows

$$\sigma_V(t) = \frac{1}{1-\rho}\big(\sigma_W(t) - \rho\sigma_c(t)\big), \tag{44}$$

where the $\sigma_V(t)$, one of the primitives of the model, is connected to 'observable' quantities via this relationship. Combining (44) with equation (43), we find that the volatility of the optimal consumption is given by

$$\sigma_c(t) = \frac{1-\rho}{\rho(1-\gamma)}\eta_t - \frac{\gamma-\rho}{\rho(1-\gamma)}\sigma_W(t), \tag{45}$$

(see Aase (2016b)). With expected utility $\gamma = \rho$, and $\sigma_c(t) = \frac{1}{\gamma}\eta_t$, but this formula is not possible to reconcile with data, unless γ is disproportionately large. The result (45) on the other hand, can be used to explain market and consumption data with reasonable values for the two preference parameters ρ and γ.

4.2.3 The optimal portfolio selection strategy

Turning to optimal investments for the future consumption, the optimal portfolio fractions in the risky assets are given at each point in time by

$$\varphi(t) = \frac{1-\rho}{\gamma-\rho}(\sigma_t\sigma_t')^{-1}\lambda_t - \frac{\rho(1-\gamma)}{\gamma-\rho}(\sigma_t\sigma_t')^{-1}(\sigma_t\sigma_{c^*}(t)), \tag{46}$$

assuming $\gamma \neq \rho$. In this formula, and otherwise throughout, a term like $(\sigma_t\sigma_{c^*}(t))$ is to be interpreted as the covariance rate between the market for

risky securities and the optimal consumption (and not as a mere multiplication of volatilities, which implies an instantaneous correlation coefficient of 1).

In Aase (2016b) it is demonstrated how these results give a more reasonable fit to real market data than the corresponding results based on the standard model.

For both the results (45) and (46) there is an underlying assumption of a market structure along the lines of Section 3, with a stochastic investment opportunity set.

In the next section we investigate this model with a stochastic interest rate of the Vasicek type.

4.3 A stochastic investment opportunity set with general recursive utility

We now derive the optimal extraction rate from an endowment fund when the agent has recursive utility. Towards this end, we start with the following expression for the wealth of the agent

$$W_t = \frac{1}{\pi_t} E_t\Big\{ \int_t^T \pi_s c_s ds \Big\}.$$

First observe that the optimal consumption can be represented as

$$c_t = c_0 \exp\Big\{ \int_0^t (\mu_c(s) - \frac{1}{2}\sigma_c'(s)\sigma_c(s))ds + \int_0^t \sigma_c(s) dB_s \Big\}$$

where μ_c is given in (42) and σ_c in (43). With the expression for the state price deflator π given in (4) we can write π_s and c_s in terms of π_t and c_t for any $t \leq s \leq T$ as follows

$$\pi_s = \pi_t \exp\{-(\int_t^s (r_u + \frac{1}{2}\eta_u'\eta_u)du + \int_t^s \eta_u dB_u)\}$$

and

$$c_s = c_t \exp\Big\{ -\frac{\delta}{\rho}(s-t) + \frac{1}{2\rho}(\gamma-\rho)(1-\gamma) \int_t^s \sigma_V'(u)\sigma_V(u))du$$

$$+\frac{1}{\rho}(\int_t^s (r_u + \frac{1}{2}\eta_u'\eta_u)du) + \frac{1}{\rho}\int_t^s ((\rho-\gamma)\sigma_V(u) + \eta_u)dB_u \Big\}.$$

In order to obtain testable results, we now assume that σ_c, σ_W and η are all non-random and constant through time, but the short rate r_u is assumed to

be of the Vasicek type discussed before. By carrying out the same type of calculations as in Section 2.7, we arrive at the following wealth to consumption ratio

$$\frac{W_t}{c_t^*} = \int_t^T \exp\Big\{\frac{1}{2}(\frac{1}{\rho}-1)\eta'\eta(s-t) - \frac{\delta}{\rho}(s-t) + \frac{1}{2\rho}(\gamma-\rho)(1-\gamma)\sigma_V'\sigma_V(s-t)$$

$$+\frac{1}{2}(1-\frac{1}{\rho})^2\sigma_r'\sigma_r\int_t^s H^2(u-t)du - (1-\frac{1}{\rho})qm\int_t^s H(u-t)du - (1-\frac{1}{\rho})H(s-t)r_t$$

$$+\frac{1}{2}(\sigma_c-\eta)'(\sigma_c-\eta)(s-t) + 2(\frac{1}{\rho}-1)(\sigma_c-\eta)'\sigma_r\int_t^s H(u-t)du\Big\}ds. \quad (47)$$

Comparing to the corresponding wealth to consumption ratio in equation (26) for expected utility, we notice that when $\rho = \gamma$, then $\sigma_c = \frac{1}{\gamma}\eta$ and this expression reduces to the one with expected utility, as we know it should.

Notice again that the right-hand side in this equality depends or r_t which is a random variable, implying that the consumption to wealth ratio c_t^*/W_t is random. The extraction rate is therefore not a constant, and $\sigma_c(t) \neq \sigma_W(t)$ consistent with the above theory.

As for the corresponding expression in Section 2.7 with a stochastic interest rate, the above integral does not lead to a simple closed form expression for the extraction rate, but the rate can again be analyzed by numerical methods as a function of r_t, t and T, and the rest of the parameters of the problem.

Example. Let us consider an example where the relative risk aversion $\gamma = 2.0$, the marginal utility flexibility parameter $\rho = 0.95$ and the impatience rate $\delta = 0.02$ for the market data in Table 1. In addition we use the following interest parameters: $r_0 = 0.01$, $m = 0.0080$, $\sigma_r = 0.0574$, $\kappa_2 = 0.16$, all consistent with the data of Table 1. Here $T = 200$ and $t = 0$.

The parameter q:	0.02	0.05	0.125	20.0
The optimal extraction rate:	0.013	0.017	0.018	0.019

Table 3: Optimal extraction rate as a function of parameter q.

When $q = 20$, this corresponds to a constant extraction rate of $k = 0.019$, since then the interest rate is deterministic. At this point we start with $\sigma_c = \sigma_W = 0.19$ and set κ close to 1 ($\kappa = 0.9981$). This gives $\varphi = 0.56$. The optimal extraction rate is calculated in Table 3 for some values of the force parameter q in the interest rate model. As q decreases, the variance of the interest rate increases, the volatility σ_W becomes different from σ_c, while κ decreases from 1. This is consistent with an example to be presented in the

next section, with the same preference parameters, but with a deterministic investment opportunity set.

In the present situation the expected real rate of return on the wealth portfolio is 0.0796 and the corresponding certainty equivalent rate of return is 0.044. With recursive utility the optimal extraction rate is typically smaller than the expected rate of return.

We notice that randomness in the interest rate has also here the effect of lowering the optimal extraction rate, which is reasonable since there is more uncertainty in this economy the lower the value of q is, and the agent *is* risk averse. When the parameter q increases, the interest rate converges to the deterministic rate $m = 0.008$, which is the spot interest rate, in which case the optimal extraction rate here tends to $k = 0.019$ from below. This is the rate with a deterministic investment opportunity set (see below)[14].

For this reason we next present a simpler version of recursive utility, which is relevant for the present problem, with a deterministic investment opportunity set. This model also has the advantage that it gives the clear answer that the optimal extraction rate under risk aversion is smaller than the expected rate of return on the endowment, for any reasonable set of parameters of the preferences.

4.4 Recursive utility: A deterministic investment opportunity set

In this section we make the same assumptions as in Section 2 except that we now consider recursive utility. In this situation we assume a deterministic investment opportunity set, in which case $\sigma_c(t) = \sigma_W(t)$ for all t, and moreover, these are assumed constant in t.

Based on the above results we first find the optimal extraction rate corresponding to the constant k in Section 2. It is a routine matter to verify that when $\sigma_c = \sigma_W$, it follows that the volatility of utility $\sigma_V = \sigma_c$ as well. Furthermore, the optimal fractions in the risky assets are then the same as in the expected utility model and given by

$$\varphi(t) = \frac{1}{\gamma}(\sigma_t \sigma_t')^{-1}\lambda. \tag{48}$$

[14]The exact nature of the various changes in the parameters is difficult to calculate exactly. This situation is a bit more involved than the previous one with expected utility.

The expected growth rate of the optimal consumption is now given by

$$\mu_c(t) = \frac{1}{\rho}(r_t - \delta) + \frac{1}{2}\frac{1}{\rho}(1+\frac{1}{\rho})\,\eta_t'\eta_t - \frac{(\gamma-\rho)}{\rho^2}\,\eta_t'\,\sigma_c(t) + \frac{1}{2}\frac{(\gamma-\rho)\gamma(1-\rho)}{\rho^2}\sigma_c'(t)\sigma_c(t) \quad (49)$$

and the volatility of the optimal consumption growth rate is

$$\sigma_c(t) = \frac{1}{\gamma}\eta_t, \quad (50)$$

which is the same expression as we have in the expected utility model. Furthermore, from the expression (47) we can deduce that the optimal extraction rate k now reduces to the following [15]

$$k = \frac{\delta}{\rho} - \frac{1-\rho}{\rho}\big(r + \frac{\lambda'(\sigma\sigma')^{-1}\lambda}{2\gamma}\big), \quad (51)$$

when $T = \infty$. This is also consistent with the more general theory outlined above, under the special assumptions of this section, where the correlation rate κ is set equal to 1. From this expression it follows that the expected rate of return is larger than or equal to the extraction rate whenever

$$\frac{r}{\rho} \geq \frac{\delta}{\rho} - \frac{1+\rho}{2\rho\gamma}\lambda'(\sigma\sigma')^{-1}\lambda. \quad (52)$$

Since the second term on the right-hand side is negative, this inequality holds true for all reasonable values of the parameters, just as in the case of expected utility. Under the assumptions of this section, we have the following

Proposition 3 *With recursive utility, assuming a deterministic investment opportunity set, the optimal extraction rate k is a constant and depends on the return from the fund only via the certainty equivalent rate of return. It is given by*

$$k = \frac{\delta}{\rho} + \big(1 - \frac{1}{\rho}\big)\big(r + \frac{1}{2}\gamma\varphi'(\sigma\sigma')\varphi\big). \quad (53)$$

The expected real rate of return on the fund is larger than or equal to the optimal extraction rate if and only if the inequality (52) holds. For any reasonable set of parameters of this problem, this inequality is true.

[15] This formula was first derived by Svensson (1989) in his special model of recursive utility in continuous time.

Proof: The fact that the term $\lambda'(\sigma\sigma')^{-1}\lambda/\gamma$ equals $\gamma\varphi'(\sigma\sigma')\varphi$ follows as in the proof of Proposition 1, since the optimal portfolio rule φ is given by expression in (48) in this model as well. Again $\frac{1}{2}\gamma\varphi'(\sigma\sigma')\varphi$ is the certainty equivalent to the stochastic part of the rate of return of the fund. The rest of the argument follows as in the proof of Proposition 1. The second assessment of the proposition was explained above. □

Again, the logic of extracting the expected real rate of return rests on an implicit assumption that the agent is risk neutral. In the above derivation, on the other hand, the agent is risk averse with relative risk aversion $\gamma > 0$, so this would again imply a contradiction, as explained in Section 2.4.

We notice from the representation of k given in (53) that the difference from the corresponding result with expected utility is that the 'weight' factor $1/\gamma$ is replaced by $1/\rho$, where ρ is the marginal utility flexibility parameter, the reciprocal of the EIS parameter. This has several consequences, to be discussed now.

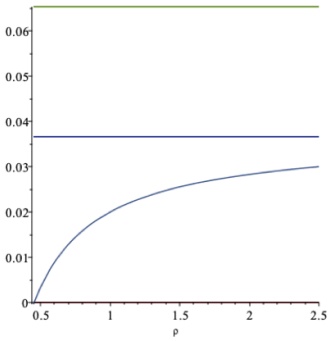

Fig. 8: k as a function of ρ.

In Figure 8 we illustrate how k vary with ρ. The increasing curve is $k(\rho;\gamma,\delta)$ as a function of ρ when $\gamma = 2.5$ and $\delta = 0.02$, the lowest horizontal line is the certainty equivalent $ce(\gamma)$ and the upper line is the expected return $er(\gamma)$ for this value of the relative risk aversion (both these are constant as functions of ρ). When ρ increases, the extraction rate is seen to increase to the $ce(\gamma)$ as long as $\delta \leq ce(\gamma)$, and decrease to $ce(\gamma)$ when $\delta > ce(\gamma)$.

The extraction rate is a decreasing function of γ for given ρ and δ provided $\rho > 1$ for this model. For general recursive utility, this is no longer so, and this function may be decreasing as γ increases also when $0 \leq \rho \leq 1$.

As a function of δ the extraction rate is again a straight line that crosses the certainty equivalent at $\delta = ce(\gamma)$.

Let us illustrate by a numerical example, corresponding to the one of the previous section. We let $\gamma = 2.0$, $\rho = 0.95$ and $\delta = 0.02$, the same parameters as in the example of Section 4.3. The optimal extraction rate is then

$k = 0.019$. The expected rate of return on the wealth portfolio is 0.078 and the certainty equivalent rate of return is 0.043, following from an optimal portfolio strategy of $\varphi = 1.13$. Furthermore, $\sigma_c = \sigma_W = 0.19$. Now the agent takes on about the same portfolio risk as before, and the extraction rate is about the same as when $q = 20$ in the previous section. This example is illustrated in the Figure 9 below.

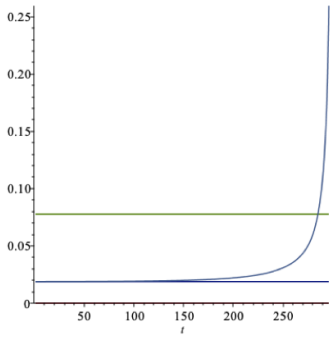

Fig. 9: Recursive utility. Drawdown rate vs return; $\gamma > \rho$.

We now relate this model to the examples in Section 2.4. First we notice that our earlier results for $\gamma = 1$, are translated to $\rho = 1$ in the present model. Thus, in the bigger picture, it is really the condition that $EIS = 1$ that yields the optimal extraction rate $k = \delta$. Accordingly, this result is not a 'risk aversion type result', but rather a result where consumption substitution plays the main role.

Consider the following example. Suppose $\rho = 1/EIS = 1$, $\gamma = 2.5$ and $\delta = 0.02$. This gives the optimal extraction rate $k = 0.02$. The expected rate of return on the wealth portfolio is now 0.065 and the certainty equivalent rate of return is 0.037. Furthermore $\sigma_c = 0.15$, $\mu_c = 0.02$ and $\varphi = 0.95$. These results indicate a less risky strategy than in the corresponding example of Section 2.4 where $\gamma = 1$. Part of the explanation is that the agent is now more risk averse. Still the optimal extraction rates are the same and equal to δ.

The literature does not give clear answers regarding the EIS-parameter. In calibrations to market data, it has been observed that EIS is typically larger than one, and $\rho < \gamma$ and $\rho < 1$ (see e.g. Aase (2016a-b) or Bansal and Yaron (2004)), which indicates preference for early resolution of uncertainty ($\rho < \gamma$), or $\rho > 1$ and $\gamma < \rho$. When $\gamma < \rho$ the the agent has preference for late resolution of uncertainty, which is not irrational, but it typically calibrates to data when also $\gamma < 1$, which seems a bit too low for the relative risk aversion. Guvenen (2009) seems to think that $EIS < 1$ is the most

natural choice, although this is not a result, rather an assumption[16]

To take an example when $\rho > 1$ and $\gamma < \rho$, assume that $\gamma = 0.9$, $\rho = 1.03$ and $\delta = 0.02$. Then the optimal extraction rate is $k = 0.022$. The expected rate of return on the wealth portfolio is 0.17 and the certainty equivalent rate of return is 0.088 with an optimal portfolio strategy of $\varphi = 2.65$. Here $\sigma_c = 0.42$, $\mu_c = 0.13$. Now the agent takes on a much more risky portfolio strategy due to the rather low relative risk aversion. This gives a rather large discrepancy between the expected real rate of return from the fund and the optimal extraction rate (see Figure 10).

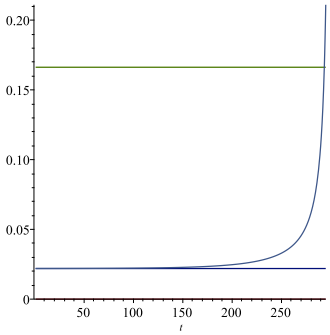

Fig. 10: Recursive utility. Drawdown rate vs return; $\gamma < \rho$.

When γ decreases in this example, ceteris paribus, the extraction rate increases. This is what seems natural, since then the agent becomes more risk neutral. However, these models do not converge well to risk neutrality as γ decreases to 0, as we have seen also in Section 2.2 in Figure 4.

Comparing these two last examples, we find the situation when $\gamma > \rho$ to the more plausible of the two, and to give the most reasonable portfolio investment strategy.

4.5 The asymptotic behavior of a sovereign wealth fund: Recursive utility

When the spending rate k is a constant, as in the model of the last section, the wealth W_t is a geometric Brownian motion as in Section 2.3 where we considered expected utility. The wealth dynamics is

$$W_t = W_0 e^{\int_0^t [\mu_W - \frac{1}{2}\varphi'(\sigma\sigma')\varphi]ds + \int_0^t \varphi'\sigma dB_s}, \tag{54}$$

[16]When discussing whether EIS is larger or smaller than 1, many economists implicitly seem to be taking the standard expected utility model as the 'thruth', in which case $EIS < 1$, since $\gamma > 1$ is considered most reasonable ($\gamma = 1/EIS$ for expected utility).

where

$$\mu_W = \begin{cases} 0, & \text{if } k = r + \gamma\varphi'(\sigma\sigma')\varphi; \\ \frac{1}{2}(1+\frac{1}{\rho})\gamma\varphi'(\sigma\sigma')\varphi + \frac{1}{\rho}(r-\delta), & \text{if } k \text{ is optimal.} \end{cases} \tag{55}$$

In other words, when the spending rate k is equal to the expected rate of return, then $\mu_W = 0$ and W_t is a martingale. When k is optimal, here given in equation (53), then either W_t is a submartingale or a supermartingale depending on the size of the impatience rate δ:

If $\mu_W > 0$ the process W_t is a submartingale if $\delta < \frac{1}{2}(1+\rho)\gamma\varphi'(\sigma\sigma')\varphi + r$, and if $\mu_W < 0$, the wealth process is a supermartingale provided $\delta > \frac{1}{2}(1+\rho)\gamma\varphi'(\sigma\sigma')\varphi + r$.

Next consider

$$\mu_W - \frac{1}{2}\sigma_W'\sigma_W = \begin{cases} -\frac{1}{2}\varphi'(\sigma\sigma')\varphi, & \text{if } k = r + \gamma\varphi'(\sigma\sigma')\varphi; \\ \frac{1}{2}((1+\frac{1}{\rho})\gamma - 1)\varphi'(\sigma\sigma')\varphi + \frac{1}{\rho}(r-\delta), & \text{if } k \text{ is optimal.} \end{cases} \tag{56}$$

Again we can conclude about the asymptotic behavior for a geometric Brownian motion from the sign of $\mu_W - \frac{1}{2}\sigma_W'\sigma_W$. Since here $\sigma_W'\sigma_W = \varphi'(\sigma\sigma')\varphi$, and by the law of the iterated logarithm for Brownian motion, the following results hold:

$$\text{If } \mu_W - \frac{1}{2}\varphi'(\sigma\sigma')\varphi < 0, \text{then } W_t \to 0 \text{ with prob. 1 as } t \to \infty; \tag{57}$$

$$\text{If } \mu_W - \frac{1}{2}\varphi'(\sigma\sigma')\varphi > 0, \text{then } W_t \to \infty \text{ with prob.1 as } t \to \infty. \tag{58}$$

Thus, when $\mu_W = 0$, where the spending rate is the expected rate of return on the fund, the martingale property then gives that $E(W_t) = W_0$ for all $t \geq 0$, but despite of this, by the above result eventually the wealth converges to zero with probability 1, and with good margin.

Moreover, using (55) when k is optimal and given in (53), we see that the situation in (57) happens when $\delta > r + \frac{1}{2}((\rho+1)\gamma - \rho)\varphi'(\sigma\sigma')\varphi$. The other case in (58) materializes when $\delta < r + \frac{1}{2}((\rho+1)\gamma - \rho)\varphi'(\sigma\sigma')\varphi$.

As we have argued above, it is reasonable that δ is smaller than or equal to the certainty equivalent (*ce*) rate of return, which holds for $\gamma = 1$. For (58) to be true, δ can be larger than the *ce* if $\gamma > 1$, but must be smaller than the *ce* when $\gamma < 1$.

Hence, the prospects for a long term sustainable management of a sovereign wealth fund are really promising using the optimal spending rate k as outlined above.

Finally, if $\delta = r + \frac{1}{2}((\rho+1)\gamma - \rho)\varphi'(\sigma\sigma')\varphi$ when k is optimal, then

$$W_t = W_0 e^{\int_0^t \varphi'\sigma dB_s},$$

in which case

$$E(W_t) = W_0 e^{\frac{1}{2}\int_0^t \varphi'(\sigma\sigma')\varphi ds} \to \infty \text{ as } t \to \infty. \tag{59}$$

We summarize as follows for recursive utility:

Theorem 2 *(i) With the optimal spending rate k, the fund value W_t goes to infinity as $t \to \infty$ as long as the impatience rate δ satisfies $\delta \leq r + \frac{1}{2}((\rho+1)\gamma - \rho)\varphi'(\sigma\sigma')\varphi$.*

(ii) If the spending rate is set equal to the expected rate of the return on the fund, then the fund value goes to 0 with probability 1 as time goes to infinity.

We also have the following corollary with recursive utility:

Corollary 2 *With the optimal spending rate k we have the following:*

(i) $W_t \to \infty$ almost surely as $t \to \infty$ provided $\delta < r + \frac{1}{2}((\rho+1)\gamma - \rho)\varphi'(\sigma\sigma')\varphi$, in which case W_t is also a submartingale.

(ii) $W_t \to 0$ almost surely as $t \to \infty$ provided $\delta > r + \frac{1}{2}((\rho+1)\gamma - \rho)\varphi'(\sigma\sigma')\varphi$, in which case W_t is also a supermartingale.

As with expected utility, we can also here say something about the expected time to the wealth process W_t reaches a certain value. This is of course a topic of interest in the present model as well, and is what we consider in the next section.

First we take closer look at a special sovereign fund, the Norwegian SWF Government Pension Fund Global, in daily language referred to as the "Oil Fund".

4.6 The Norwegian SWT Government Fund Global

For this sovereign fund the Norwegian Ministry of Finance set down a commission in 2016 to consider the asset allocation problem. Table 2 below reflects the commission's market view on equity and risky bonds[17]

[17]The report uses geometric returns. We translate this into expected continuously compounded arithmetic returns: Equity: $0.0472 = \ln(1 + 0.035) + 0.5 \cdot (0.16)^2$; and Bonds: $0.0068 = \ln(1 + 0.005) + 0.5 \cdot (0.06)^2$. The covariance reported in the table stems from the following calculation $0.16 \cdot 0.06 \cdot 0.4 = 0.00384$, where the intertemporal correlation coefficient is 0.4.

	Expectation	Standard dev.	Covariance
Equity	4.72%	16.00%	0.00384
Bonds	0.68%	6.00%	
Equity premium	4.04%	14.67%	

Table 4: The commission's market view, Norwegian Ministry of Finance (2016).

The commission recommends an equity share of $\varphi = 70\%$. Given a riskless rate of 0.68% and an equity premium with expectation 4.04% and standard deviation 14.67%, this translates into an implicit risk aversion of $\gamma = 2.68$. The expected return and standard deviation of the fund are then 3.75% and 11.56%, respectively.

The certainty equivalent fund return is $ce = 1.87\%$, which less than half the expected rate of return on the fund. Observe that the certainty equivalent fund return is substantially less than the current fiscal rule, which is 3%.

Suppose for the moment that the utility impatience rate $\delta = 1.87\%$. In this case, where the impatience rate and the certainty equivalent fund return are equal, the optimal spending rate $k = 1.87\%$ regardless of the elasticity of intertemporal substitution (EIS $= 1/\rho$).

Now, suppose instead that the utility impatience rate is $\delta = 1.5\%$. In fact, if the EIS is sufficiently large, the optimal consumption rate k might become zero or even negative, which clearly must be ruled out in the infinite horizon case but which still makes sense with a finite horizon. Say, for instance, that the fixed horizon is $T = 100$ years from now, and that EIS $= 5$. It then follows that the optimal spending rate $k \approx 0\%$ to three decimal places ($k = 0.00027$). Does this mean no spending at all? Clearly not. The optimal spending the first year is 0.0102 of the fund value, the optimal spending in year 2 is 0.0103, the optimal spending in year 50 is 0.0201, and in year 90 it is 0.1001 of the fund value, and so forth (recall equation (9)).

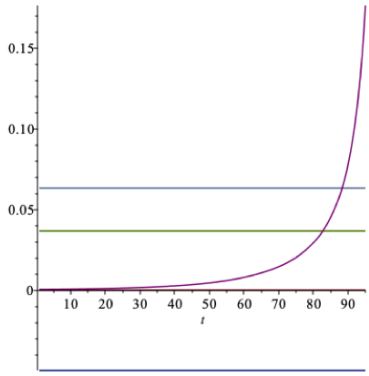

Fig. 11: Spending rate when k is zero; $\gamma > \rho$, $1/\rho = 5$.

In Figure 11 this situation is illustrated, where the upper horizontal line is the real expected rate of return, the next line is the certainty equivalent rate of return, the curve is the optimal spending rate with $T = 100$ as the horizon, and the horizontal line close to the origin is the value $k = 0.00027$.

If we increase the EIS further, the value of k becomes negative. Still the optimal spending with a finite horizon is strictly positive, and increasing as the horizon comes closer, as in Figure 11.

In this situation we can calculate the conditional expected time to the fund leaves a given interval at a specified level for the first time, treated in Section 2.6. Consider the interval (a, b) where $a = (1/10)W_0$ and $b = 2W_0$. In this scenario and with the *optimal* spending rate, the parameters are $\mu_W = 0.02315$, $\sigma_W = 0.1156$ and the constant $c = -2.46$. The first exit probabilities are $p^+(W_0, J) = 0.9972$ and $p^-(W_0, J) = 0.0028$, so it is much more likely that the first exit takes place at upper level b than at the lower a. From the results of Section 2.4 we obtain that $E_{W_0}\{\tau^*(J)|X(\tau^*(J)) = b\} = 41.35$ years while $E_{W_0}\{\tau^*(J)|X(\tau^*(J)) = a\} = 121.42$ years. Here $E_{W_0}\{\tau^*(J)\} = 41.58$ years.

In the situation where the spending rate is the *expected rate of return*, $\mu_W = 0$ and $c = 1$ while $\sigma_W = 0.1156$ remains the same. The first exit probabilities have changed to $p^+(W_0, J) = 0.86$ and $p^-(W_0, J) = 0.14$, so it is still more likely that the first exit takes place at upper level b than at the lower a, but less so than in the optimal case. Now we get $E_{W_0}\{\tau^*(J)|X(\tau^*(J)) = b\} = 74$ years while $E_{W_0}\{\tau^*(J)|X(\tau^*(J)) = a\} = 184$ years. Here $E_{W_0}\{\tau^*(J)\} = 132$ years. Yet we know that in this situation W_t will eventually end up in zero, although it may take a long time, while in the former case with optimal extraction in place this does not *ever* happen with probability 1.

There are several important lessons we can draw from this example. Firstly, for reasonable parameter values, it is optimal to consume consid-

erably less than the expected rate of return of the fund. Secondly, if the utility impatience rate and the certainty equivalent fund return are equal, the optimal consumption rate equals the two regardless of EIS. Thirdly, if the utility impatience rate is less than the certainty equivalent fund return, the latter is an upper bound for the optimal consumption rate.

We now move from analysis of these simple examples of recursive utility, in some sense a degenerate version of this preference relation, to the situation where the extraction rate is a random process, along the lines of Section 4.

5 Stochastic extraction rate with recursive utility

Paralleling the discussion in Section 3, we make the same assumptions here, and derive the optimal consumption to wealth ratio X_t when the consumption volatility is different from the wealth volatility. We now have $X_t = c_t/W_t$ where W_t has the same dynamics as before, but where the dynamics of the optimal consumption is given in the equations (41)-(43) of Section 4.2. The stochastic differential equation for X_t is again the following

$$dX_t = X_t(X_t - \alpha_t)dt + X_t\theta_t dB_t. \tag{60}$$

Here $\theta_t = \sigma_c(t) - \sigma_W(t)$ and

$$\alpha_t = -\mu_c(t) + \varphi_t\lambda_t + r_t + \varphi_t'\sigma_t\sigma_c(t) - \varphi_t'\sigma_t\sigma_t'\varphi_t, \tag{61}$$

where φ_t is given in equation (46). This stochastic differential equation is of the same class as the one we encountered in Section 3.

We illustrate by a numerical example. The figures below give simulations of the paths of X, the expected drawdown rate as a function of time, and statistics based on 10000 simulations of the path of X a situation with the following parameters $\gamma = 1.39$, $\rho = 0.76$ and $\delta = 0.04$, and $r = 0.008$ (which we consider as reasonable).

In Figure 12 we illustrate 10 sample paths of the process X_t for the fund illustrated in Table 1. Here the preference parameters used are as given above. This gives $\alpha = 0.026$ and $\theta = -0.1229$, corresponding to the data of Table 1, where $\sigma_c = 0.0355$, and $\sigma_W = 0.1584$.

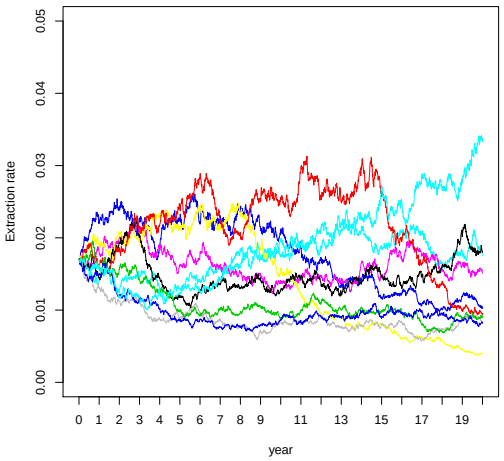

Fig. 12: Some simulations of the sample paths of X

In this illustration we use step-lengths 1/100, so with respect to the yearly data in Table 1, this represents 20 years. With a stochastic extraction rate, we notice some variations due to the volatility of the paths.

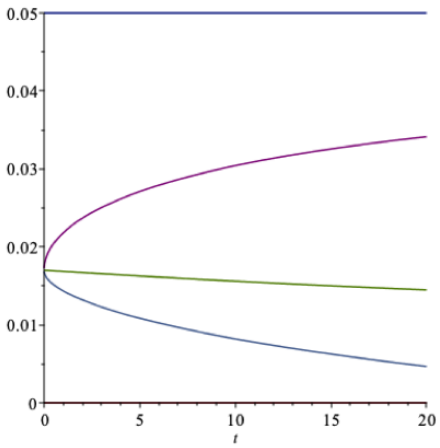

Fig. 13: Expected drawdown rate, $\alpha = 0.026$.

In Figure 13 we have a graph of the mean value function $m(t) = E(X_t)$ as a function of t. As can be seen, the mean is fairly stable in the interval considered ($\tau = 20$), and a 95% confidence band is provided. We notice that the expected drawdown rate is close to 0.017 for these parameters.

In Figure 14 we present the statistics based on 100 sample paths. The horizontal curve in the middle represents the median in this sample, with confidence limits on each side. The median is located around 0.017, and decreases slightly.

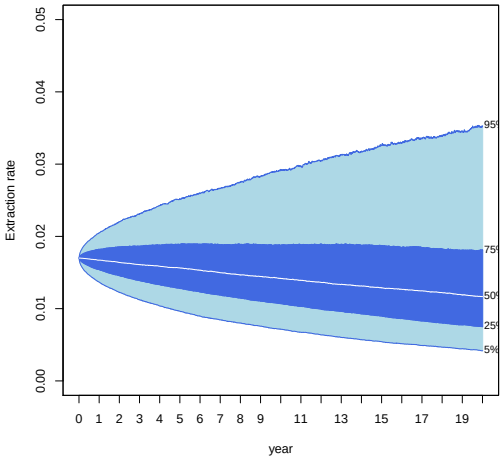

Fig. 14: Median drawdown rate of X.

For the above parameters and with the data of Table 1, the expected rate of return on the wealth portfolio is 0.065 and the certainty equivalent rate of return is 0.049, following from an optimal portfolio strategy of $\varphi = 0.95$ in the fund. Furthermore, for these values of the preference parameters it follows that $\sigma_c = 0.0355$, $\mu_c = 0.018$ both match the equations (42)-(43) exactly, unlike for the corresponding situation with expected utility treated in Section 3. Thus, with recursive utility, this puzzle disappears (see e.g., Aase (2016a) for more examples related to stock market and consumption data regarding, for example, the equity premium puzzle).

Now the optimal portfolio investment strategy given in equation (46) takes into account also current consumption, unlike for the expected utility model. Due to the somewhat low relative risk aversion, the fraction in the risky fund is fairly large.

In the present example this extraction rate (0.017) could be compared to the model of the previous section. With the same preference parameters as above applied to the model of the previous section, we obtain an optimal, constant extraction rate of $k = 0.034$. Again there is a drop due to the stochastic nature of X.

With regard to the specific preference parameters chosen, this choice is of course not to be taken literally. We have merely demonstrated that with recursive utility, *reasonable* parameters can be found which are internally consistent with the other model assumptions. For the expected utility model in Section 3.3, this was not possible.

So far we have not shown that the stochastic extraction rate of this sec-

tion is smaller than the real expected rate of return of the fund portfolio, for reasonable values of the preference parameters. This was not formally shown for the stochastic rate with expected utility in Section 3, although the numerical example indicated that this is the case. Suppose now that the parameters $\sigma_c(t)$ and $\sigma_W(t)$ are constants. What we can say about this is the following.

Since we know that the third order moment around zero is not really stationary, a better approximation is obtained if we make this assumption about the fourth order moment, and use the limiting result following from equation (37) for the quantity $E(X_t^3)$. In this situation we have by use of (37) in (36) that $m_t \to \frac{2X_0^4}{(\alpha-\theta^2)(2\alpha-\theta^2)\alpha}$ when $t \to \infty$, assuming $\alpha > \theta^2$. By using this limit as a guide towards the determination of initial value of the drawdown rate X_0, we obtain a fairly stable graph of $m(t)$ in the vicinity of $t = 0$. We then interpret this value as the expected drawdown rate in the short run. The solution for this example is $X_0 = 0.017$, which gives a reasonably values for $m(t)$, reflected in the figures 13 and 14.

One may ask when is X_0, so determined, smaller than k, the constant optimal drawdown rate in the parts of the paper where $\theta = 0$. An answer, based on the above discussion, is given by the following inequality.

$$X_0 \leq k \Longleftrightarrow \frac{1}{2}(\alpha - \theta^2)(2\alpha - \theta^2)\alpha \leq k^3. \tag{62}$$

In the above numerical example for the fund in Table 1, where $\sigma_c = 0.0355$ and $\sigma_W = 0.1584$, this equivalence is true for any relevant value of k, whenever $\alpha > \theta^2$.

Figures 13 and 14 illustrate with graphs of the mean value function $m(t)$ and the median, respectively, both starting at $X_0 = 0.017$. We see that this gives a fairly stationary mean $m(t)$, and median, at this level, with associated 95% confidence intervals. Also notice the difference between these two figures and the corresponding figures 6 and 7. For example are the confidence bands broader in the present model, and also command a higher extraction rate. Regarding the broader bands, this seems economically plausible, since both models refer to the same fund; when the extraction rate is higher, this comes at a price of more risk. A technical explanation can be seen from inspecting the explicit expression for X given in (32); variations will also depend upon the value of X_0 to some extent, and its relationship to the parameter α.

The choice of X_0 is made with a view towards the limiting value for $m(t)$, which exists in our present approximation for the moments. This we know is not correct for the process X, but it merely works as a guide to determine X_0 and hence $m(t)$ in the short run from $t = 0$ onwards, when $\alpha > \theta^2$. With this caveat in mind, we recapture the following meta result.

In the model of the present section where the fund is part of society as a whole, and the objective is to maximize recursive utility of total consumption, the optimal extraction rate X is randomly fluctuation with time having dynamics given in (60).

Suppose the parameters satisfy $\alpha > \theta^2$. Then the expected value $m(t)$ of the optimal extraction rate X is approximately stationary in the short run, with level X_0 strictly smaller than the corresponding constant optimal extraction rate k.

The intuition is again that the added uncertainty caused by having $\theta \neq 0$ implies that a more cautious, optimal extraction policy is called for, provided we interpret $m(t)$ as the optimal extraction rate at time t.

6 Additional Consumption in Society

The analysis in the preceding sections is under the assumption that the fund can be considered in isolation from consumption in the rest of society.

For a fund established by society for the benefits of its inhabitants, it may be of interest to investigate if the above analysis is general enough, since the ownership and purpose of the fund may be more complex. If the fund is owned by a state, the rest of the wealth in society may matter. Typically, for a sovereign wealth fund owned by the state, the government could, perhaps, be inclined to compare the extraction from the fund with consumption in society that originates from other, and more common sources. This we now address.

Let us assume that there is a consumption stream in society that does not originate from the fund, denoted c_t^S, while the consumption that originates from the fund is denoted c_t^F, so that total consumption $c_t = c_t^F + c_t^S$ at any time t. The objective is to maximize utility $U(c)$ subject to the relevant budget constraint. Here we assume $U(c) = E(\int_0^T u(c_t, t)dt)$ where $u(x, t)$ is power utility of the kind used in sections 1 and 2 of the paper.

In order to discuss this problem, let us return to equation (3) for the market value of the optimal wealth. This equation can be expressed as follows under our present model assumptions

$$W_t = \frac{1}{\pi_t} E_t\Big\{ \int_t^T \pi_s c_s^F ds \Big\} + \frac{1}{p_t} E_t\Big\{ \int_t^T p_s c_s^S ds \Big\}. \tag{63}$$

Here W_t is the total wealth in society at time t and p_t is the state price deflator related to the consumption c^S that stems from other sources than the fund, so we can write $W_t = W_t^F + W_t^S$ for all t, where W_t^F is the optimal

wealth from the fund, and where W_t^S is the wealth stemming from other sources than the fund, at any time $t \in [0, T]$.

The central planner's problem is then to solve the following

$$\sup_c U(c) \quad \text{subject to} \quad E\Big\{ \int_0^T (\pi_t c_t^F + p_t c_t^S) dt \Big\} \leq w,$$

where w is the present value of wealth in the society.

The Lagrangian of this problem is

$$\mathcal{L}(c^F, c^S; \mu) = E\big\{ \int_0^T u(c_t^F + c_t^S, t) dt - \mu\big(\int_0^T (\pi_t c_t^F + p_t c_t^S) dt - w\big) dt \big\}$$

where μ is the Lagrange multiplier. Using directional derivatives, the first order conditions are

$$(c_t^F + c_t^S)^{-\gamma} e^{-\delta t} = \mu \pi_t, \quad \text{and} \quad (c_t^F + c_t^S)^{-\gamma} e^{-\delta t} = \mu p_t, \ \forall t \in [0, T]$$

where γ is the relative risk aversion and δ is the impatience rate. As a direct consequence of this, $\pi_t = p_t$ for all t, so the two state price deflators must be identical (a.s.).

In the same vein we consider the two wealths. Here we make the somewhat heroic assumption that all assets in society are marketed, so that, for example, we can consider labor as a shadow asset contained in W^S. We then get

$$dW_t = dW_t^F + dW_t^S = \big(W_t^F(\varphi_t^F \lambda_t^F + r_t) - c_t^F\big) dt + W_t^F \varphi_t^F \sigma^F dB_t$$

$$+ \big(W_t^S(\varphi_t^S \lambda_t^S + r_t) - c_t^S\big) dt + W_t^S \varphi_t^S \sigma^S dB_t.$$

The first order conditions of optimal portfolio selection, using either dynamic programming or otherwise, leads in the same manner to the following

$$\varphi_t^F = \frac{1}{\gamma} (\sigma^F \sigma'^F)^{-1} \lambda^F \quad \text{for all } t, \text{ and}$$

$$\varphi_t^S = \frac{1}{\gamma} (\sigma^S \sigma'^S)^{-1} \lambda^S \quad \text{for all } t.$$

The conclusion of this is that an endowment fund, wether owned by the state, by a university or otherwise, should be managed optimally as a fund, separated from the rest of the consumption problem in society. This separation principle is also rather intuitive.

6.1 A real case

Let us discuss a concrete case, and consider again the Norwegian Government Pension Fund Global, formerly simply the Norwegian Oil Fund, from the perspective of the last section. The idea of the origins of this fund is that also future generations are supposed to benefit from the oil exploration of the present generation, not only those who live in Norway at the present.

Consider, for example, a situation where the pension liabilities increase in the future for some limited and transitory amount of time, and then returns to a more normal state after this period. If the fund is supposed to take care of this particular problem, one can simply use actuarial methods to calculate the relevant extraction rates in the future. This problem is not connected, or at the best, just vaguely related to the problem analyzed above. In principal, no utility function is needed for the actuarial calculations involved. Thus we must make assumptions about both ownership of the fund, as well as the intended purpose of the fund.

Despite of the change of the name of the former Norwegian Oil Fund, the actual daily use of this fund seems to be more in line with the description considered in this paper. The conclusion from the last section is then to use the separation principle and treat this fund in isolation, where an optimal extraction policy must be consistent with the portfolio selection strategy used. Since this is one of broadly diversifying over assets in international security markets, including various government bonds, and also real estate, it is clear that this implies risk aversion on the investment side. Consistent with this, the extraction rate should also take into account both risk aversion and consumption substitution, as explained in this paper.

This is contrary to the current state of affairs of the Norwegian Government Pension Fund Global, where the extraction from this fund is determined by a mandate from the Parliment (Stortinget) to be set equal to the expected real return on the fund. As we have shown, this is not the sustainable spending rate of this fund, and will deplete the fund in the future with probability one.

7 Conclusions

We have derived concrete formulas for optimal extraction from an endowment fund consistent with risk aversion, and demonstrated that the optimal extraction rate is strictly smaller than the expected rate of return. The difference is far from negligible, and amounts to several percentage points in most real situations.

The explanation given is simple: If the fund is managed by diversification, this means risk aversion in the optimal portfolio choice problem. Then, to be consistent, the spending rate must also reflect risk aversion, which is what we have pointed out.

We have taken a security market as given, assumed to be in equilibrium, and introduced a price taking agent in this market. In this setting we have reconsidered the problem of optimal consumption and portfolio selection. In the context of an endowment fund, the results from analyzing this more general problem can immediately be utilized in order to determine an optimal spending rate. We have considered both expected utility, in which case risk aversion plays a prominent role, and recursive utility where consumption substitution is separated from risk aversion.

When the investment opportunity set is deterministic, there exists an explicit and closed form solution for optimal extraction. First and foremost, this solution is demonstrated to be smaller than the expected real rate of return on the endowment fund, for plausible values of the preference parameters. The difference is far from negligible.

If the extraction rate is the one of expected return, this implies that the agent is *risk neutral* at the level of extraction, and must then, to be consistent, be risk neutral at the level of investments as well. But the consequence of the latter behavior is rarely advocated by anyone responsible for an endowment fund.

We demonstrate that a popular and much advertised extraction policy, the expected real rate, is not consistent with a sustainable spending rate, and will with probability one eventually deplete any fund that is managed by diversification.

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