Markets With Memory: Dynamic Channel Optimization Models With Price-Dependent Stochastic Demand

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Markets With Memory: Dynamic Channel Optimization Models With Price-Dependent Stochastic Demand

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Abstract

Almost every vendor faces uncertain and time-varying demand. Inventory level and price optimization while catering to stochastic demand are conventionally formulated as variants of newsvendor problem. Despite its ubiquity in potential applications, the time-dependent (multi-period) newsvendor problem in its general form has received limited attention in the literature due to its complexity and the highly nested structure of its ensuing optimization problems. The complexity level rises even more when there are more than one decision maker in a supply channel, trying to reach an equilibrium. The purpose of this paper is to construct an explicit and efficient solution procedure for multi-period price-setting newsvendor problems in a Stackelberg framework. In particular, we show that our recursive solution algorithm can be applied to standard contracts such as buy back contracts, revenue sharing contracts, and their generalizations.

Keywords: stochastic demand; time-dependent demand; price-dependent demand; memory functions, market engineering; demand manipulation; prescriptive analytics; pricing theory

JEL Classification: C61, C73, D81, D47

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1 Introduction

Almost every vendor faces uncertain demand. The uncertainty of demand may be of different natures and varying levels of tractability for statistical modeling. The demand uncertainty for a specific commodity may stem from consumer behavior or the economic development condition for that commodity. For instance, the stochasticity of demand for commodities such as sports apparel may arise from changing trends of fashion; while for electronic devices or computer software, it may be caused by better products being rolled out.

Anticipating future trends of market and satisfying stochastic demand remains a challenge for manufacturers and vendors. In general, the uncertain demand for a specific product is price-dependent, and dynamic in the sense that it evolves through time.

The main goal of this paper is to demonstrate that a general structure for stochastic dynamic demand can be utilized for decision-making purposes. This general structure, as we will see, is such that many other demand models turn into sub-classes of our formulation.

The single-period newsvendor problem in the face of price-dependent uncertain demand has long been studied in the literature (see, for example, Whitin 1955, and Salinger and Ampodia 2011).

However, there are a variety of situations in which decision makers need a multi-period perspective and analysis of stochastic demand for a commodity as its price (and price-elasticity of demand) changes. For example, in market penetration scenarios where the optimal strategy of an entrant supplier in the beginning may be very different from what is optimal later, a multi-period analysis of demand is necessary.

The current work should be considered as a generalization and systematization of such models of demand, as it analyses the stochastic price-dependent demand of arbitrary distribution in a multi-period structure. This comprehensive model of demand can then be implemented in different economic and management contexts to cover and analyse multi-period changes of demand as the price elasticity of demand evolves through time.

Another feature of our model of dynamic price-dependent demand is that it can be embedded in a game theoretic setting where two vendors cater to the demand within a vertical supply channel. The general solution algorithm presented in the constructive Theorem 5.1 thus provides the optimal level of inventory and optimal prices for both of the channel members at each period.

This paper circles around the classical balance between price and demand. In a dynamic setting, where we consider demand within a time frame, the current price is obviously still important, but the general level of demand may depend on previous prices in a critical way.

Moreover, we take this point of view from marketing and behavioral economics that previous prices scale demand, for example by affecting the number of customers taking interest in the product. This is particularly important when a company wants to sell high-tech products with a possibly short lifespan. An optimal pricing scheme is critical. At the end of the timespan, the product will be outdated and replaced by more advanced products.

Trade-ins and introductory offers are more common than ever before, in particular due to web-based shopping. Market penetration strategies such as providing the potential customers with free trial versions of a software (freemium) or free distribution of a small number of a newly introduced cell-phone model are frequently employed.

For example, as reported by *Forbes* on August 2015, in an attempt to gain a greater share of the market, Samsung announced its promotion called "Ultimate Test Drive," which allowed Apple iPhone users an opportunity to test out the Galaxy Note 5, Galaxy S6 edge+ or Galaxy S6 edge cellphone and tablet models for a price of just $1 for 30 days.

According to an article published in *Harvard Business Review* on May 2014, similar marketing strategies have become the dominant business model employed by internet start-ups trying to secure a niche in the market by gradually converting the freemium users into paying customers.

In general, according to an analysis published on March 2013 in *The Wall Street Journal*, the freemium business model has proven to be the fastest way for a company to grow and create massive value.

Such marketing approaches, however, may incur huge initial losses, and succeed only if demand is enhanced to a level that outweighs the initial costs. The main issue for such schemes is to obtain a proper balance between present revenue and revenue in the remaining lifespan of the product. The length of the introductory free distribution period is, for obvious reasons, a crucial factor for success.

Our analysis of the long-term revenue optimization problem addresses this issue by providing the optimal length of the free distribution time interval.

The main challenge in a multi-period discrete-time model of demand in which demand at the present may be affected by that of previous periods is the implicit interdependence of values of all periods. In order to emphasise the nestedness caused by this interdependence, we introduce the notion of memory functions. These memory functions carry effects of present demand onto the future. They are generally price and time dependent and can be adjusted to model markets with stronger or weaker memories.

By embedding memory effects in the model, not only do we emphasize the nestedness mentioned above, but also cover the downstream (customer-side) effects of pricing. Thus, our model will be able to systematically cover price elasticity of demand as it changes through periods. Otherwise, by solving the same problem for demand distribution at every period, we cannot see the after effects of the pricing scheme on demand.

The paper is organized as follows. In section 2, we outline our uncertain demand structure and how it is affected by market memory. Next, in section 3, we embed this memory-based demand structure in a general time-dependent profit and inventory optimization problem. In this section, we briefly discuss how this (single-vendor) optimization problem can be solved using backward induction method.

In section 5, we extend the solution procedure to a supply channel composed of two vendors competing in a Stackelberg framework. The direction of generalization in this section is based on the number of periods: first, in section 5.1, the static single-period equilibrium problem is solved, and, finally, in section 5.3, the general solution procedure for the dynamic game is presented.

The final theoretical results for equilibria problems are stated in Theorem 5.1 and its Corollary 5.2. The final solution procedure yields the numerical values for optimal decision variables at different times while considering all the model parameters to be also time-dependent, thereby ensuring full non-autonomy of the model.

While this article should be regarded as a methodology paper, in section 6 we provide examples of decision-making problems using our model. It should be noted that these examples are provided to familiarize the reader (user of the algorithm) with the solution procedure as well as its diverse scope of applicability. Thus, it is imperative to emphasize that the representative functionals offered in this section are merely speculative and not, for instance, the results of an empirical study.

Through these examples, we will see how the model can be implemented in strategic games where the parties must balance immediate profits with future earnings. Among the scenarios in this section, we analyze a case wherein the two vendors coordinate in an integrated (centralized) channel.

In the appendices, we first demonstrate that in order for our proposed memory-based demand structure to decouple the nested multi-period optimization or equilibrium problem, the expected profit expression is required to be of a specific mathematical structure. Then we prove that such a specific structure indeed appears in many conventional supply chain optimization and inventory management problems, thus making our memory based solution procedure applicable to a wide variety of contracts.

1.1 Literature Review

In this section, we survey different models of stochastic demand employed by researchers in various disciplines, including microeconomics, inventory management, and stochastic programming. The papers mentioned in this section address various issues when facing stochastic demand. These issues include maximizing objective functions, minimizing cost functions, obtaining equilibria, or analysis of the very nature of uncertain demand. It should be noted that we are primarily interested in the model of stochastic demand in each paper, not the specific objective functions within which these demand expressions are embedded.

The body of research on stochastic demand can be categorized into two main subgroups. The first class of papers deem the distribution of stochastic demand to be uncertain. The second group of researchers consider certain characteristics of the demand distribution to be part of the a priori knowledge of the decision makers about the market.

In the first group, Azoury (1985) formulates the problem of stochastic demand in a Bayesian setting. Assuming a known prior distribution for the uncertain demand, the newly gained information is incorporated into the posterior distribution with unknown parameters. These unknown parameters constitute a multi dimensional state space. The dimensionality of the resulting problem is then reduced such that the solution of the Bayesian model can be obtained by solving another dynamic program with a one dimensional state space. She also assumes the unmet demand to be backlogged and observable. Lariviere and Porteus (1999) provide a similar Bayesian inventory management analysis in which demand distribution belongs to a parametric family of distributions. However, they assume that the unmet demand is lost and unobserved. Bensoussan et al. (2007) in their analysis of the multi-period newsvendor model, also consider demand distribution to be the state of their stochastic programming problem. In their model, demand is a stationary Markov process with a known transition probability. Using the unnormalized probabilities, they convert the state transition equation to a linear one. Levi et al. (2007) use a non-parametric Monte Carlo sampling algorithm to garner information about the underlying distribution of demand. Assuming that that the demands in all periods are independent and identically distributed (i.i.d) random variables, they do not consider the inter-dependence of demand level in the current period to the future ones. All of these papers, addressing only the inventory management problem, consider the demand to be independent of the price. Therefore, they do not address the optimal pricing strategy problem.

Kim et al. (2015) in their analysis of a multi-period newvendor model, discretize the problem as a multi-stage stochastic programming problem. The stochasticity of demand in their model is formulated as a set of a finite number of scenarios and the occurrence of each scenario is associated with a probability. Moreover, they use a discrete probability distribution (Poisson) to represent the demand. The ensuing scenario-based stochastic problems can then be treated as discrete deterministic optimization problems. In their model, the demand is not price dependent.

Pasternack (1985, 2008) in his analysis of the single-period newsvendor problem, considers uncertain price-dependent demand for a perishable good to be of a general probability density function. The decision variables are the order quantity and the pricing policy which are obtained by solving the maximization problem for the expected profit.

Gümüş et al. (2013) study a dyadic channel coordination problem in which two channel partners supply used goods to a peer-to-peer Web-based market. They construct a Stackelberg structure between the manufacture (leader) and the retailer (follower), and analyse the necessary conditions for return policies to constitute the equilibrium strategy. In their model, the time frame is comprised of two periods. In the first period, the degree of uncertainty in customer valuation for a specific product is assumed to be heterogenous and follow a uniform distribution. Trying to formulate the inter-dependence of demands at the two periods, they assume the demand potentials for the second period to be positively correlated to the realization of demand in the first period, which becomes known history when the second period starts.

In our research, we also consider the mean and variance of demand to be arbitrary functions of time and price history. The dependence of demand mean and variance to the current price can be obtained from microeconomics theory or empirical data, while the inter-dependence of current demand to the prices of the past periods are represented by memory functions. The solution scheme that we propose is, however, independent of the functions representing the demand distribution. We consider the current demand to be a function of current price, pricing strategy in the past, and, demand history. Thereby, we avoid neglecting the downstream effect of pricing strategies. That is, our model also considers the fact that the pricing scheme set by the decision maker(s) will affect the availability of the commodity to the costumers, which in turn, will affect their purchase decision, future demand, and, the expected profit for the vendor(s). As a result of such a nested, price-dependent demand structure the decision makers will be able to engineer the demand through devising the optimal pricing strategy.

2 Demand Model Framework

2.1 Problem Description

A commodity is to be supplied to a specific market with uncertain demand. The vendor(s) are risk-neutral as they seek to maximize their expected discounted total profit within a timespan. We consider a discrete time structure for our model, in which the timespan is divided into intervals called *periods*, which may be of different lengths and hence differently value-discounted. We assume all model parameters and variables to be constant within each period. First, we introduce the demand model, and then substituting it in a generic inventory model, solve the profit optimization problem. It should be noted that, for the sake of generality, we consider an inventory model applicable to commodities facing uncertain demand.

In addition, we assume that when decisions are being made for the kth period, the prices of the past denoted as $\{R_i \mid 1 \leq i < k\}$ are common knowledge. Another important assumption is that demand at the $k-1$th period, D_{k-1}, becomes known, i.e. observable at the beginning of period k. This is especially true when analysing internet-based demand, where online shopping enables potential customers to report their "wishlists."

2.2 Demand Structure

We assume the uncertainty of demand at each period to be of a general i.e. additive-multiplicative form. Such a demand structure is comprised of two deterministic and uncertain parts each a function of time and price history.

We consider demand at period k, D_k to be of the following general form.

$$D_1 = \widetilde{\mu}_1(R_1) + \widetilde{\sigma}_1(R_1)\varepsilon_1 \tag{1}$$

$$D_k = \widetilde{\mu}_k(R_1, \ldots, R_k) + \widetilde{\sigma}_k(R_1, \ldots, R_k)\varepsilon_k \tag{2}$$

where

R_k = price per unit at period k

$\widetilde{\mu}_k(R_1, \ldots, R_k)$ = given function, representing mean of demand at period k

$\widetilde{\sigma}_k(R_1, \ldots, R_k)$ = given function, representing the standard deviation of demand at period k

ε_k = stochastic variable of arbitrary distribution, continuously distributed, normalized such that:

$\mathrm{E}[\varepsilon_k] = 0$ and $\mathrm{Var}[\varepsilon_k] = 1$ for all k, and supported on intervals with density function, $f_{\varepsilon_k} > 0$, a.e. on its support.

2.3 Normalization of The Stochastic Variables

It is obvious that with the normalization described earlier, we have:

$$\widetilde{\mu}_k = \mathrm{E}[D_k] \quad \widetilde{\sigma}_k = \mathrm{Var}[D_k]$$

Also, note that this normalization enables our demand model to cover both the additive case (Within (1955)) and the multiplicative case (Karlin and Carr (1962)). Without loss of generality, we can summarize all single-period multiplicative models of stochastic demand as below.

$$D(R) = \widetilde{\sigma}(R)\xi \quad \mathrm{E}[\xi] = 1 \qquad \text{where } \xi \text{ is a stochastic variable.} \tag{3}$$

Such a demand expression is equivalent to a special case in our model in which $\widetilde{\mu}$ and $\widetilde{\sigma}$ are equal: $D(R) = \widetilde{\sigma}(R)(1 + \varepsilon)$. It is readily observable that in the multiplicative model the coefficient of variation of demand is equal to 1 and, a fortiori, independent of price.

3 The Single-vendor Profit and Inventory Optimization Problem

In this section, we use our proposed demand structure in a quite general inventory management problem. Substituting the demand structure in the expected profit expression, we illustrate how our model of demand can be solved to obtain vectors of optimal decision variables at any specific period.

For simplicity, we analyse the case with only one vendor here, and in section 5.1, we delineate how the same structure can solve cases where two vendors try to maximize their respected profits.

We denote the net running value of the profit obtained at period k, by Π_k, and its expected value with respect to f_{ε_k} by $\overline{\Pi}_k$. For the sake of generality, we assume the expected profit at period k to be of the following form.

$$\mathrm{E}[\Pi_k] = \overline{\Pi}_k \left(D_k, R_k, q_k, M_k, s_k, g_k, c_{r_k}\right) \tag{4}$$

where

$D_k =$ actual uncertain demand at period k

$R_k =$ price per unit at period k, (decision variable)

$q_k =$ inventory level: quantity of items to be supplied at period k, (decision variable)

$M_k =$ manufacturing cost per unit at period k, given parameter

$s_k =$ salvage price per unit at period k, given parameter

$g_k =$ goodwill cost per unit incurred due to stockout at period k, given parameter

$c_{r_k} =$ marginal cost incurred upon procuring a unit at period k, given parameter.

It should be noted that in addition to the decision variables in this single-vendor model, i.e. R_k and q_k, all of the other parameters are considered as time-dependent entities. This full-blown non-autonomy with respect to both the variables and the parameters is kept throughout the entire analysis and is present in the expression of the final results in Theorem 5.1 and its Corollary 5.2 for general double-vendor models.

The supply quantity q_k is to address the stochastic demand D_k. Thus, without loss of generality, we can consider q_k to follow a structure similar to that of D_k as described in (1) and (2), i.e. a function of mean and standard deviation of demand at k ($\widetilde{\mu}_k$ and $\widetilde{\sigma}_k$).

$$q_k = q_k\Bigg(\widetilde{\mu}_k(\mathbf{R}_k), \widetilde{\sigma}_k(\mathbf{R}_k)\Bigg) \tag{5}$$

Assuming the total number of periods to be n, we have the expected discounted total profit of the whole n periods as below.

$$J = \mathrm{E}[\Pi_1] + \alpha_2 \mathrm{E}[\Pi_2 \mid D_1] + \cdots + \alpha_n \mathrm{E}[\Pi_n \mid D_1, \cdots, D_{n-1}] \tag{6}$$

where $0 < \alpha_k \leq 1$ are given discount factors.

Now we have to solve the following nested n-variable optimization problem.

$$\max_{\mathbf{R}_n} J = \sum_{k=1}^{n} \alpha_k \overline{\Pi}_k(\mathbf{R}_k) \quad \text{The barred symbol indicate expected value.} \tag{7}$$

Solving (7), we want to obtain the vector of optimal prices $\hat{\mathbf{R}}_k$.

3.1 A General Solution Scheme Based on Backward Induction

In the final period, there is no need to worry about future demand. Moreover, by the time the decisions are being made for the last period, all the previous decision variables and the demands themselves are common knowledge. In addition, because no term other than $\overline{\Pi}_n$ in J as expressed in (7) depends on R_n, the problem of finding the nth argmax of J boils down to the single-variable problem of finding the R_n which maximizes $\overline{\Pi}_n$.

$$\max_{R_n} J \equiv \max_{R_n} \overline{\Pi}_n \tag{8}$$

Hence, given $\mathbf{R}_{n-1}$ and $\mathbf{D}_{n-1}$, and assuming that the mapping $R_n \mapsto \mathrm{E}[\Pi_n \mid \mathbf{D}_{n-1}]$ has a global maximum, we can construct a *function* $\hat{R}_n = \hat{R}_n(\mathbf{R}_{n-1}, \mathbf{D}_{n-1})$ that maximizes this conditional expected value. Inserting this function in (7) and iterating the same procedure for $\max_{R_{n-1}} J, \cdots, \max_{R_1} J$, we obtain the vector $\hat{\mathbf{R}}_n$ with each $\hat{R}_k$ being the global argmax to the corresponding $\overline{\Pi}_k$.

Later, in section 5, we extend the procedure to solve the equilibrium problem when there are two suppliers (a leader and a follower) in the supply channel. The ensuing bilevel optimization will be of the following general form.

$$\begin{aligned} &\max_{\{l_i\}} \left(J^{\text{ Leader}} \mid f_j^* \right) \\ s.t. \quad &\{f_j^*\} = \left\{ Argmax \left(J^{\text{ Follower}} \mid l_i^* \right) \right\} \end{aligned} \tag{9}$$

Where l_i and f_j denote the decision variables for the leader and the follower, respectively. Moreover, throughout this paper, an asterisk superscript denotes an optimal decision variable.

4 Market Memory

It is classical to assume that the demand at present depends on the current price: $D_k = \psi(R_k)$. However, not all markets behave as simply as this. Many markets have some kind of memory, in the sense that pricing in the past may affect demand at present. In a dynamic market, the customers may become anchored to past prices, and this may affect their purchasing behavior. Besides, the functional form ψ may vary with time; thus making $D_k = \psi_k(R_k)$.

Consider a market in which a commodity with a limited lifespan is supplied to a base of potential customers. It is natural to assume that strategic customers are sensitive to previous prices when comparing them to the current price. Thus, one can conclude that, in general, in addition to the current price, previous prices may have a bearing on the current customer base by scaling the demand. For example, in a specific scenario, a price increase by 20% may reduce the customer base by, for example, 10%. We argue that a general time-dependent model of supply and price optimization should also consider the effect of anchoring to the past prices on current demand.

We build our time-dependent model of uncertain demand on the simple premise that the probability of an item being sold at time k for the price of R_k depends on the customers' interest, which in its own right, in general, may depend on the past prices.

$$D_k \propto \mathcal{P}(\text{purchase}_k) = \mathcal{P}(\text{purchase}_k \mid \text{interested}_k) \cdot \mathcal{P}(\text{interested}_k) \tag{10}$$

$$D_k = \psi_k(R_k) \cdot \mathcal{H}_k(R_{k-1}, \cdots, R_1) \tag{11}$$

The dependence of the current demand on current price, $\psi_k(R_k)$, has been a subject of classical microeconomic study. Whereas, obtaining a functional format for $\mathcal{H}_k(\mathbf{R}_{k-1})$ may fall into the domain of behavioral economics.[1]

4.1 Direction of Generalization: TENTATIVE TITILE

When embedding D_k into the optimization procedure outlined in (9), depending on the demand structure, we will encounter different game theoretic scenarios.

[1]Should I remove this (minimal) reference to Behavioral Economics?

In the memory-less market, where $\mathcal{H}_k = 1$, the demand at the current period is assumed not to be affected by past prices. If the demand functional format remains identical (as is the case in some microeconomic analyses), i.e. $\psi_k(R_k) = \psi(R_k)$, the procedure outlined in (9) turns into a *repeated game*.

In contrast, a fully dynamic game emerges where the functional formats for $\psi_k(R_k)$s vary with time.

In addition, assuming demand's dependence on past prices, i.e. having $\mathcal{H}_k(R_{k-1}, \cdots, R_1)$ covers the effects of price anchoring on demand and adds up to the level of non-autonomy in the ensuing equilibrium problem.

In Theorem 5.1, we propose a solution algorithm for the general non-autonomous dynamic game. Obviously, the proposed solution algorithm can be applied to the repeated game and the memory-less cases.

4.2 Memory-based Uncertain Demand

We formulate the demand scaling factors described in the previous section within *memory functions*. These memory functions carry the effects of the past prices onto current demand. They are generally price and time dependent and can be adjusted to model markets with stronger or weaker memories. In our expression for memory-based demand, we devise memory functions such that the demand at each period be not only a function of price at that period, but also carry the effects of pricing policies and the demand in the previous periods.

To define the memory functions for the demand, we assume the demand structure to have the following two features.

Feature 1. At any period, the coefficient of variation of the demand, CV_{D_k}, may depend on the current retail price. In other words, we allow

$$CV_{D_k} = \frac{\widetilde{\sigma}_k(\mathbf{R}_k)}{\widetilde{\mu}_k(\mathbf{R}_k)} = CV_{D_k}(R_k) \tag{12}$$

where the price sequence at period i, $\mathbf{R}_i = \{R_j, j = 1, \cdots, i\}$.[2]

[2]Throughout this paper and for the sake of brevity, we have used boldface letters to denote sequences of variables like demand, retail and wholesale prices. Thus, for a variable Z, $\mathbf{Z}_i = \{Z_j, j = 1, \cdots, i\}$, where j is the number of the period.

Note that assuming this feature is a step towards generalization as, in the existing literature, in multiplicative demand models, where $D = \sigma\,\xi$, the coefficient of variations is assumed to be constant ($CV_D = 1$). This highly restrictive assumption is relaxed by assuming Feature 1, instead.

Now we recast the general expression for demand in (2), as below.

$$\begin{aligned} &D_k(\mathbf{R}_k) = \Phi_k(\mathbf{R}_{k-1})d_k(R_k) \\ &\text{where } d_k(R_k) = \mu_k(R_k) + \sigma_k(R_k)\varepsilon_k. \end{aligned} \tag{13}$$

and

$$\begin{aligned} \widetilde{\mu}_k(\mathbf{R}_k) &= \Phi_k(\mathbf{R}_{k-1})\mu_k(R_k) \\ \widetilde{\sigma}_k(\mathbf{R}_k) &= \Phi_k(\mathbf{R}_{k-1})\sigma_k(R_k). \end{aligned} \tag{14}$$

We call Φ_ks the memory functions as they bring the effects of pricing history into the current demand structure.

Moreover, we assume that the the memory function for the $k+1$st period, $\Phi_{k+1}(\mathbf{R}_k)$, retains the information from the previous period's memory, $\Phi_k(\mathbf{R}_{k-1})$, while being affected by the last piece of information that has become available, i.e. R_k.

The level of retainment of the information about the past prices (determining the potential buyers' *anchoring* to the past prices) may vary depending on the market and the behavior of strategic buyers. Feature 2 expresses this feature in a multiplicative format.

Feature 2.

$$\frac{\Phi_{k+1}}{\Phi_k} = \phi_k(R_k) \tag{15}$$

We call these $\phi_k(R_k)$s the memory elements. Notice that the possibility of having different functional forms for ϕ_ks in different periods enables our demand structure to cover more non-autonomy. With the general memory structure in (15), we will have:

$$\Phi_k(\mathbf{R}_{k-1}) = \prod_{i=2}^{k} \phi_i(R_{i-1}) \tag{16}$$

The memory structure satisfying assumptions 1 and 2 enables us to explicitly solve nested multi-period optimization/equilibria problems in a large variety of inventory management and game theoretic contexts.

As it can be seen in (13), the memory functions make D_k an implicit function of all the previous pricing schemes. Note that given R_i, the value of ε_i is known if and only if the value of d_i is known. The idea is that the level of demand in each period can (to some extent) carry over to the next period. This dependence is usually such that a high price in one period can lead to reduced demand in the next period, whereas a low initial price can have the opposite effect by stimulating demand. This nested demand expression for all periods can then be substituted in total profit maximization problem by the decision maker, enabling her to obtain the optimal set of decision variables.

These memory functions, as we will see later on, are adjustable such that they can enable the model to represent different levels of influence from the past. For example $\Phi_i = 1, \forall i \in \{2, \cdots, n\}$ represent a memory-less market in which D_is are decoupled from each other.

4.3 Embedding the Memory-based Demand in the Expected Profit Expression

The general construction outlined in (7) and (8) is sufficiently explicit to enable solutions of the problem for most choices of functions $\widetilde{\mu}_k$, and $\widetilde{\sigma}_k$. However, the problem is so deeply nested that one cannot expect to find an analytical solution. The importance of our memory-based structure of demand as described in section 4.2 is that in many classical supply chain optimization problems, as it has been shown in the Appendix 1, the running expected profit at each period, as outlined in (4), has the following form.

$$\overline{\Pi}_k(\mathbf{R}_k) = \Psi_k(R_k)\widetilde{\mu}^p_k(\mathbf{R}_k) + \Theta_k(R_k)\widetilde{\sigma}^p_k(\mathbf{R}_k) \tag{17}$$

The power p in most classical contracts is equal to 1. With the memory structure introduced in (13), we will have:

$$\widetilde{\mu}^p_k(\mathbf{R}_k) = \mu^p_k(R_k)\,\Phi^p_k(\mathbf{R}_{k-1}) \quad \text{and} \tag{18}$$

$$\widetilde{\sigma}^p_k(\mathbf{R}_k) = \sigma^p_k(R_k)\,\Phi^p_k(\mathbf{R}_{k-1}) \tag{19}$$

and can recast (17) as below.

$$\overline{\Pi}_k = \overbrace{\left(\Psi_k(R_k)\mu_k^p(R_k) + \Theta_k(R_k)\sigma_k^p(R_k)\right)}^{:=\widetilde{\Pi}_k(R_k)} \Phi_k^p(\mathbf{R}_{k-1}) = \widetilde{\Pi}_k(R_k)\,\Phi_k^p(\mathbf{R}_{k-1}) \tag{20}$$

Now the profit optimization problem in (7) can be simplified as follows.

$$J = \sum_{k=1}^{n} \alpha_k \Phi_k^p(\mathbf{R}_{k-1})\,\widetilde{\Pi}_k(R_k) \tag{21}$$

The multiplier effect in (21) is the crucial observation in this paper; as it reduces the nested n-variable optimization problem to n single-variable optimization problems.

This decoupling effect is shown (22). Again, starting from the final period, we observe that the only term in J containing R_n is $\widetilde{\Pi}_n$. Thus, we have:

$$\max_{R_n} J \equiv \max_{R_n} \widetilde{\Pi}_n \tag{22}$$

This is much more straightforward to solve compared to the general case in (8), as here we can immediately obtain the numerical value for $\hat{R}_n$. Substituting this value in (21), we solve the next single-variable maximization problem with respect to R_{n-1}. Continuing the same procedure backward in time, we can obtain all the optimal values of $\hat{R}_k$s.

5 Equilibria In Cases With Two Vendors

Having outlined our general model of memory-based stochastic demand, and embedding it in a single-vendor profit optimization problem, we now extend the scope of the analysis to problems where two vendors facing stochastic demand, try to maximize their own respective profits. For the analysis of our proposed demand structure, we begin by the newsvendor model as it epitomizes the problem of inventory management when the demand for a commodity with short lifespan is stochastic.

We assume that a good is produced by a manufacturer and sold to a retailer. We also assume that the manufacturer and the retailer are risk neutral in the sense that they try to maximize expected discounted total profit. We consider a multi-period Stackelberg game between the manufacturer and the retailer where the actions of the two parties affect the actions of a third party, the customers. In this Stackelberg structure, the upstream vendor (the manufacturer), as the leader, has to find a sequence of optimal wholesale prices at different periods (W_ks) to ensure her maximum profit.[3] The downstream vendor (the retailer), who is the follower, then faces the wholesale price and accordingly decides on the number of products to be ordered to the manufacturer (and supplied to the market) and the sequence of optimal retail prices (R_ks).

The dynamics of prices in game theoretical settings have been discussed in several publications by K. Bagwell, we mention Bagwell (1987, 2007). Petruzzi and Dada (1999) consider multi-period cases with price-dependent demand, and show how to adapt such models to include backorders. However, they do not discuss Stackelberg competition. Pricing strategies for retailers have been discussed intensively in the marketing literature, and we mention Rao (1984) and Fassnacht and Husseini (2013).

The discussion in Øksendal et al. (2013) partly explains why general multi-period problems are difficult to solve. Some types may admit numerical solutions, but the general problem is difficult to compute or analyze even in the two-period case. By comparison, the discrete version we consider in this paper is transparent. Our memory scheme decouples a multi-period problem into a sequence of one-period problems, each of which is fairly easy to solve. Our model retains the main essence of the problem itself, while simultaneously providing a solution that can be analyzed without the need for advanced optimizing techniques.

5.1 The basic model: the game in single-period

The solution to the single-period newsvendor problem epitomizes a supply chain coordination scenario while facing stochastic demand. Therefore, in this section, we review some properties of the single-period model and in the next sections, we propose our multi-period model based on it.

[3] In this paper, following the convention adopted by Cachon (2003), we assume the upstream agent (i.e. the manufacturer) to be female and the downstream agent (i.e. the retailer) to be male.

Main symbols:

W = wholesale price per unit (chosen by the manufacturer)

R = retail price per unit (chosen by the retailer)

q = order quantity (chosen by the retailer)

D = demand (random)

M = production cost per unit (fixed)

s = salvage price per unit (fixed)

Π^r = profit for the retailer

Π^m = profit for the manufacturer

In the classical newsvendor model, the manufacturer sets the wholesale price W for one unit of a certain commodity that needs to be sold within a short timespan. The retailer orders a quantity q units of the commodity to the manufacturer and plans to sell them for the price R (per unit) in a market with stochastic demand D. Any unsold item can be salvaged at the price $s < R$. The retailer's profit Π^r is calculated as below.

$$\begin{aligned}\Pi^r &= R\min(D,q) + s(q-D)^+ - Wq \\ &= R\min(D,q) + s(q-\min(D,q)) - Wq \\ &= (R-s)\min(D,q) - (W-s)q.\end{aligned} \tag{23}$$

From this expression, we obtain the expected profit for the retailer:

$$\mathrm{E}[\Pi^r] = (R-s)\mathrm{E}[\min(D,q)] - (W-s)q. \tag{24}$$

In our model, we consider the additive-multiplicative model for the demand as given in (1). For a given R, it is well known that the maximum expected profit is obtained when:

$$P(D \le q) = \frac{R-W}{R-s} \tag{25}$$

Inserting the general expression for the demand in (1) into (24) and using (25), we can prove the following proposition where F_ε denotes the cumulative distribution of ε.

Proposition 5.1.

Assume that ε is a continuous distribution, supported on an interval, with density $f_\varepsilon > 0$ a.e. on its support. Given R and W, $R \geq W > s$, the retailer will make an order

$$q = \mu(R) + \sigma(R)\, F_\varepsilon^{-1}\left(\frac{R-W}{R-s}\right). \tag{26}$$

in which case, he obtains the expected profit

$$\overline{\Pi}^r = E[\Pi^r] = (R-W)\mu(R,k) + L_\varepsilon(R,W)\sigma(R,k) \tag{27}$$

where L_ε is defined by

$$L_\varepsilon(R,W) = (R-s)\int_{-\infty}^{z} \zeta f_\varepsilon(\zeta) d\zeta \quad z = F_\varepsilon^{-1}\left(\frac{R-W}{R-s}\right) \tag{28}$$

Proof

See the Appendix 3.

Our setup is slightly non-standard since we use a different normalization than that of Young (1978). Nonetheless, the result in Proposition 5.1 is more or less well known within the literature. In our normalization, we assume that $\mathrm{E}[\varepsilon] = \int_{-\infty}^{\infty} \zeta f_\varepsilon(\zeta) d\zeta = 0$, and hence, $L_\varepsilon(R,W) \leq 0$. In the literature, the term $L_\varepsilon \,.\, \sigma$ is often referred to as *loss due to randomness.*

It should be noted that the channel under study is considered to be a segment of a more complete market, such that a segmentation of the pool of customers are addressed by it. The market demand structure, in general, is an aggregation of the individual demands from possibly heterogenous consumers who may be affected by the supply of competing products from other vendors. This feature is embedded in D through the choice of $\mu[R,k]$ and $\sigma[R,k]$. Therefore, although the manufacturer and the retailer in our model are basically monopolistic suppliers, the model considers competition via demand structure.

In the one-period newsvendor model, to formulate a Stackelberg game, we assume that both parties are risk neutral. The manufacturer (leader) offers a wholesale price W. We ignore the possibility that the retailer can negotiate this wholesale price. Given W, the retailer (follower) then solves (27) to find the $\hat{R}$ which maximizes $\overline{\Pi}^r$, and then, substituting this $\hat{R}$ into (26) to find out the optimum order quantity $\hat{q}$. The manufacturer also knows that the retailer is going to choose $\hat{q}$ to maximize the expected profit. Therefore, given each possible value of W, the manufacturer can anticipate the resulting order quantity $\hat{q} = \hat{q}(W)$, and so chooses W to maximize expected profit (which happens be to be deterministic in this case). The manufacturer's profit is given by:

$$\Pi^m = (W - M)q \tag{29}$$

5.2 Multi-period vertical contracting

Having discussed the solution to the single-period problem, we are now ready to provide a theoretical analysis of the multi-period Stackelberg game. In particular, we focus on the case in which demand in the next period is scaled by a factor that depend on price and demand in the current period. This is a type of Markovian assumption in that it only requires knowledge of the current state, not of how prices and demand arrived at that state.

In the multi-period game, we assume that the parties are risk neutral and try to maximize their discounted expected profits:

$$J^r = \overline{\Pi}_1^r + \alpha_2 \overline{\Pi}_2^r + \cdots + \alpha_n \overline{\Pi}_n^r \tag{30}$$

$$J^m = \overline{\Pi}_1^m + \alpha_2 \overline{\Pi}_2^m + \cdots + \alpha_n \overline{\Pi}_n^m \tag{31}$$

where n is the number of periods.

5.3 Multi-period games with memory functions

Whereas it is straightforward to formulate an n-period game in the general case, numerical solutions are difficult to obtain even if n is moderately large. The nonlinear structure of the problem branching into separate cases for each particular choice made on every level quickly renders the problem computationally intractable.

In this section, we show how to generalize the memory-based approach described in the previous section to multi-period problems. First, we discuss an important technical issue. Consider a general three-period problem. Substituting the memories from (16), we will have the following structure.

$$D_1 = \mu_1(R_1) + \sigma_1(R_1)\,\varepsilon_1 \tag{32}$$

$$D_2 = \phi_2(R_1)\,\big(\mu_2(R_2) + \sigma_2(R_2)\,\varepsilon_2\big) \tag{33}$$

$$D_3 = \phi_2(R_1)\,\phi_3(R_2)\,\big(\mu_3(R_3) + \sigma_3(R_3)\,\varepsilon_3\big) \tag{34}$$

In the following analysis, we consider only the retailer's profit optimization procedure. The same arguments also hold true for the manufacturer's. In the presence of the memory functions, the maximization problem expressed in (30), turns into the following.

$$\max_{R_1,R_2,R_3} J^r = \overline{\Pi}_1(R_1) + \alpha_2\,\phi_2(R_1)\,\overline{\Pi}_2(R_2) + \alpha_3\,\phi_2(R_1)\,\phi_3(R_2)\,\overline{\Pi}_3(R_3) \tag{35}$$

Starting the backward induction process from the final period, we define J_k^r as the expected discounted profit earned within the interval between period k and $n = 3$, inclusive.

$$J_3^r = \alpha_3\,\phi_2(R_1)\,\phi_3(R_2)\,\overline{\Pi}_3(R_3) \tag{36}$$

$$J_2^r = \alpha_2\,\phi_2(R_1)\,\overline{\Pi}_2(R_2) + J_3^r \tag{37}$$

$$J_1^r = J^r = \overline{\Pi}_1(R_1) + J_2^r \tag{38}$$

where according to (27), the running expected profit obtained at each period is

$$\overline{\Pi}_k(R_k) = (R_k - W_k)\mu_k(R_k) + L_{\varepsilon_k}(R_k, W_k)\sigma_k(R_k).$$

Thus,

$$J_3^r = \alpha_3\,\phi_2(R_1)\,\phi_3(R_2)\bigg((R_3 - W_3)\mu_3(R_3) + L_{\varepsilon_3}(R_3, W_3)\sigma_3(R_3)\bigg) \tag{39}$$

Because period 3 is the final period, there is no need to worry about future demand, and therefore, given W_3, the retailer chooses the optimal R_3 to maximize J_3^r. Note that because R_1 and R_2 have happened in the past, they are not considered as decision variables at period 3 and the optimal values of R_3 and W_3 are independent of them. Thus, the optimization problem reduces to the single-variable problem of maximizing $\overline{\Pi}_3(R_3)$.

Assuming that $\hat{R}_3$ is the (global) argmax value of $\overline{\Pi}_3(R_3)$, we set $\widehat{\Pi}_3 = \overline{\Pi}_3(\hat{R}_3)$. Then the backward induction proceeds to the next subproblem, i.e., the problem of maximizing the expected profit in the second period. From (37):

$$\begin{aligned}\max_{R_2} J_2^r &= \phi_2(R_1)\Big(\alpha_2\,\overline{\Pi}_2(R_2) + \alpha_3\phi_3(R_2)\widehat{\Pi}_3\Big)\\ &= \alpha_2\,\phi_2(R_1)\Big(\Big[(R_2 - W_2)\mu_2(R_2) + L_{\varepsilon_2}(R_2, W_2)\sigma_2(R_2)\Big] + \frac{\alpha_3}{\alpha_2}\,\phi_3(R_2)\widehat{\Pi}_3\Big)\end{aligned} \tag{40}$$

Notice that in (40) similar to the case in (39) the only decision variable for the retailer is R_2, as R_1 has happened in the past. Therefore the retailer faces another single-variable optimization problem.

The same procedure is applied backward until all the three optimal decision variables are found. Assuming that $\hat{R}_2$ is the global argmax of J_2^r as optimized with respect to R_2, from (40), we set

$$\widehat{\Pi}_2 = \overline{\Pi}_2(\hat{R}_2) + \frac{\alpha_3}{\alpha_2}\phi_3(\hat{R}_2)\widehat{\Pi}_3 = \frac{J_2^r(\hat{R}_2)}{\alpha_2\phi_2(R_1)} \tag{41}$$

Now the remaining single-variable optimization problem, is derived from (38) as below.

$$\max_{R_1} J_1^r = \overline{\Pi}_1(R_1) + \alpha_2\,\phi_2(R_1)\widehat{\Pi}_2 \tag{42}$$

Generalizing the same procedure for an n-period game ($n > 3$), we start by solving for the final period to obtain expected profits $\widehat{\Pi}_n^r$ and $\widehat{\Pi}_n^m$. Once these values are known, the profit values in the previous period can be computed through backward induction process. That produces numerical values of $\widehat{\Pi}_{n-1}^r$ and $\widehat{\Pi}_{n-1}^m$. To determine the strategy for $(n-2)$nd period, we consider the problem:

$$\begin{aligned}\max_{R_{n-2}} J_{n-2}^r =& \Big((R_{n-2} - W_{n-2})\mu_{n-2}(R_{n-2}) + L_{\varepsilon_{n-2}}(R_{n-2}, W_{n-2})\sigma_{n-2}(W_{n-2})\\ &+ \frac{\alpha_{n-1}}{\alpha_{n-2}}\,\phi_{n-1}(R_{n-2})\,\widehat{\Pi}_{n-1}^r\Big)\alpha_{n-2}\prod_{i=2}^{n-2}\phi_i(R_{i-1})\end{aligned} \tag{43}$$

$$\begin{aligned}\max_{W_{n-2}} J_{n-2}^m =& \Big((W_{n-2} - M_{n-2})\Big[\mu_{n-2}(R_{n-2}) + \sigma_{n-2}(R_{n-2})\,F_{\varepsilon_{n-2}}^{-1}\Big(\frac{R_{n-2} - W_{n-2}}{R_{n-2} - s}\Big)\Big]\\ &+ \frac{\alpha_{n-1}}{\alpha_{n-2}}\,\phi_{n-1}(R_{n-2})\,\widehat{\Pi}_{n-1}^m\,)\Big)\alpha_{n-2}\prod_{i=2}^{n-2}\phi_i(R_{i-1})\end{aligned} \tag{44}$$

We also set α_1 and ϕ_1 equal to 1. Note that in (43) and (44) the term $\prod_{i=2}^{n-2}\phi_i(R_{i-1}) = \Phi_n(\mathbf{R_{n\text{-}1}})$ represents the previous prices and has no bearing on the optimization problem. Thus the equilibrium problem for period $n-2$ is reduced to a single-period problem that only involves R_{n-2} and W_{n-2}. The only difference from the problem for period $n-1$, is that the values of $(\widehat{\Pi}^r_{n-1}, \widehat{\Pi}^m_{n-1})$ are different from the values $(\widehat{\Pi}^r_n, \widehat{\Pi}^m_n)$. Hence all we have to do to solve this problem is repeat the previous step with updated values for $(\widehat{\Pi}^r, \widehat{\Pi}^m)$.

To simplify notation, we have suppressed dependence on arguments that are not yet active; μ_{n-2} and σ_{n-2} are in general functions of (R_{n-3}) but according to our assumptions, this dependence enters as an independent multiplicative factor and can hence be factored out of the optimization problem. (See equations 39 and 40 for example.)

By using the argument above repeatedly, it is clear that we can solve this problem for any value of n. Starting with the values $(\widehat{\Pi}^r_{n+1}, \widehat{\Pi}^m_{n+1}) = 0$ in the final period, we solve essentially the same problem in all periods. The values of $(\widehat{\Pi}^r, \widehat{\Pi}^m)$ are updated as the construction progresses backwards, but those updated values come for free from the solution of the previous step. We state the generalized result as follows.

Theorem 5.1.
Let n be the number of periods and assume that demand in period k is given by:

$$D_k = \left(\mu_k(R_k) + \sigma_k(R_k)\varepsilon_k\right)\Phi_k\big(\boldsymbol{R}_{k-1}\big) \tag{45}$$

where

$$\Phi_1 = 1, \Phi_k\big(\boldsymbol{R}_{k-1}\big) = \prod_{i=2}^{k} \phi_i(R_{i-1}) \quad k > 1$$

and $\varepsilon_1, \cdots, \varepsilon_n$ are continuously distributed with $\mathrm{E}[\varepsilon_k] = 0$ *and* $\mathrm{Var}[\varepsilon_k] = 1$ *for all k, with $f_{\varepsilon_k} > 0$ a.e. on their supports. If for each k, the one-period Stackelberg problem below has a unique equilibrium at $R_k = \hat{R}_k, W_k = \hat{W}_k$.*

$$\begin{aligned}
J^r_k(R_k) &= \Big((R_k - W_k)\mu_k(R_k) + L_{\varepsilon_k}(R_k, W_k)\sigma_k(R_k) + \frac{\alpha_{k+1}}{\alpha_k}\,\phi_{k+1}(R_k)\,\widehat{\Pi}^r_{k+1}\Big)\alpha_k\Phi_k(\boldsymbol{R}_{k-1}) \\
J^m_k(W_k) &= \Bigg((W_k - M_k)\Big(\mu_k(R_k) + \sigma_k(R_k)F^{-1}_{\varepsilon_k}\Big[\frac{R_k - W_k}{R_k - s_k}\Big]\Big) \\
&\quad + \frac{\alpha_{k+1}}{\alpha_k}\phi_{k+1}(R_k)\,\widehat{\Pi}^m_{k+1}\Bigg)\alpha_k\,\Phi_k(\boldsymbol{R}_{k-1})
\end{aligned} \tag{46}$$

where $\overline{\Pi}_k^r$ *and* $\overline{\Pi}_k^m$ *are found recursively from:*

$$\widehat{\Pi}_{n+1}^r = 0 \quad \widehat{\Pi}_{n+1}^m = 0 \tag{47}$$

$$\widehat{\Pi}_k^r = \frac{J_k^r(\hat{R}_k)}{\alpha_k\,\Phi_k(\boldsymbol{R}_{k-1})} \quad \widehat{\Pi}_k^m = \frac{J_k^m(\hat{W}_k)}{\alpha_k\,\Phi_k(\boldsymbol{R}_{k-1})} \quad , k = 1, 2, \cdots, n,\ \alpha_1 = \Phi_1 = 1 \tag{48}$$

then the problem of maximizing

$$J^r = \overline{\Pi}_1^r + \alpha_2\,\Phi_2(\boldsymbol{R}_1)\,\overline{\Pi}_2^r + \cdots + \alpha_n\,\Phi_n(\boldsymbol{R}_{n-1})\,\overline{\Pi}_n^r \tag{49}$$

$$J^m = \overline{\Pi}_1^m + \alpha_2\,\Phi_2(\boldsymbol{R}_1)\,\overline{\Pi}_2^m + \cdots + \alpha_n\,\Phi_n(\boldsymbol{R}_{n-1})\,\overline{\Pi}_n^m \tag{50}$$

has a unique equilibrium at $\hat{\boldsymbol{R}} = (\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_n)$, $\hat{\boldsymbol{W}} = (\hat{W}_1, \hat{W}_2, \cdots, \hat{W}_n)$.

Theorem 5.1 delineates how the memory structure can decouple nested equilibria problems. The result of the theorem can be generalized to all optimization or equilibrium problems in which the running (single period) profit expression is of the structure stated in (17). In the appendix B, we show that the expected profit expressions in all the classical coordination contracts (and their combinations), including the wholesale price contracts, the buy back contracts, and, the revenue sharing contracts are indeed of this structure.

Corollary 5.2.

Let n *be the number of periods and assume that demand in period* k *is given by:*

$$D_k = (\mu_k(R_k) + \sigma_k(R_k)\varepsilon_k)\,\Phi_k\big(\boldsymbol{R}_{k-1}\big) \tag{51}$$

where

$$\Phi_1 = 1, \quad \Phi_k\big(\boldsymbol{R}_{k-1}\big) = \prod_{i=2}^{k} \phi_i(R_{i-1}) \quad k > 1$$

and $\varepsilon_1, \cdots, \varepsilon_n$ *are continuously distributed with* $\mathrm{E}[\varepsilon_k] = 0$ *and* $\mathrm{Var}[\varepsilon_k] = 1$ *for all* k *with* $f_{\varepsilon_k} > 0$ *a.e. on their supports. Assuming that the running expected profit for the retailer and the manufacturer at period* k *can be written in the following formats,*[4]

$$\overline{\Pi}_k^r(\boldsymbol{R}_k) = \Psi_k^r(R_k)\,\mu_k^p(\boldsymbol{R}_k) + \Theta_k^r(R_k)\,\sigma_k^p(\boldsymbol{R}_k)$$

$$\overline{\Pi}_k^m(\boldsymbol{R}_k) = \Psi_k^m(R_k)\,\mu_k^p(\boldsymbol{R}_k) + \Theta_k^m(R_k)\,\sigma_k^p(\boldsymbol{R}_k)$$

[4]In classical supply-chain optimization contracts the power p is equal to 1.

if for each k, the one-period Stackelberg problem below has a unique equilibrium at $R_k = \hat{R}_k, W_k = \hat{W}_k$.

$$
\begin{aligned}
J_k^r(R_k) &= \Big(\Psi_k^r(R_k)\mu_k^p(R_k) + \Theta_k^r(R_k)\sigma_k^p(R_k) + \frac{\alpha_{k+1}}{\alpha_k}\,\phi_{k+1}^p(R_k)\,\widehat{\Pi}_{k+1}^r\Big)\alpha_k\,\Phi_k(\boldsymbol{R}_{k-1}) \\
J_k^m(W_k) &= \Big(\Psi_k^m(R_k)\mu_k^p(R_k) + \Theta_k^m(R_k)\sigma_k^p(R_k) + \frac{\alpha_{k+1}}{\alpha_k}\,\phi_{k+1}^p(R_k)\,\widehat{\Pi}_{k+1}^m\Big)\alpha_k\,\Phi_k(\boldsymbol{R}_{k-1})
\end{aligned}
\tag{52}
$$

where $\widehat{\Pi}_k^r$ *and* $\widehat{\Pi}_k^m$ *are found recursively from:*

$$\widehat{\Pi}_{n+1}^r = 0 \quad \widehat{\Pi}_{n+1}^m = 0 \tag{53}$$

$$\tag{54}$$

$$\widehat{\Pi}_k^r = \frac{J_k^r(\hat{R}_k)}{\alpha_k\,\Phi_k(\boldsymbol{R}_{k-1})} \quad \widehat{\Pi}_k^m = \frac{J_k^m(\hat{W}_k)}{\alpha_k\,\Phi_k(\boldsymbol{R}_{k-1})} \quad , k = 1, 2, \cdots, n,\ \alpha_1 = \Phi_1 = 1 \tag{55}$$

then the problem of maximizing

$$J^r = \overline{\Pi}_1^r + \alpha_2\,\Phi_2(\boldsymbol{R}_1)\,\overline{\Pi}_2^r + \cdots + \alpha_n\,\Phi_n(\boldsymbol{R}_{n-1})\,\overline{\Pi}_n^r \tag{56}$$

$$J^m = \overline{\Pi}_1^m + \alpha_2\,\Phi_2(\boldsymbol{R}_1)\,\overline{\Pi}_2^m + \cdots + \alpha_n\,\Phi_n(\boldsymbol{R}_{n-1})\,\overline{\Pi}_n^m \tag{57}$$

has a unique equilibrium at $\hat{\boldsymbol{R}} = (\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_n)$, $\hat{\boldsymbol{W}} = (\hat{W}_1, \hat{W}_2, \cdots, \hat{W}_n)$.

Remarks

The Corollary 5.2 is a generalization of Theorem 5.1 based on general functional structures introduced in Section 4.3. In appendices B.1, B.2, and B.3 we calculate the structures $\Psi_k^{r,m}$s and $\Theta_k^{r,m}$s for some of the conventional supply chain contracts. These functional structures substituted in the procedure outlined in Corollary 5.2 yield the optimal results for a supply channel bound to the associated contract.

In multi-variable problem such as the ones discussed here, multiple local maxima are detrimental to computational performance. The strength of Theorem 5.1, however, is that it reduces the dimension of the search space to one, and maxima for functions of one variable can always be handled by an exhaustive search.

Theorem 5.1 and its corollary, state that the uniqueness of the multi-period equilibria is contingent upon the uniqueness of the associated single-period equilibrium results. The unimodality of the multi-period equilibria solutions is determined by the unimodality of each of the decoupled single-period (i.e. single-variable) optimization problems. Finding necessary conditions to guarantee the unimodality of the single-period price-setting newsvendor problem has been exhaustively studied in the literature; see for example Xu et al. (2011) and Rubio-Herrero and Baykal-Gürsoy (2018).

5.4 The infinite horizon case

According to Theorem 5.1, in the infinite horizon problem, for given values of $\widehat{\Pi}_k^r$ and $\widehat{\Pi}_k^m$, the parties try to optimize:

$$J_k^r(R_k) = \Big((R_k - W_k)\mu_k(R_k) + L_{\varepsilon_k}(R_k, W_k)\sigma_k(R_k) + \alpha\,\widehat{\Pi}_{k+1}^r\,\phi_{k+1}(R_k)\Big)\alpha\,\Phi_k(\mathbf{R}_{k-1}) \tag{58}$$

$$\begin{aligned} J_k^m(W_k) = \Bigg(&(W_k - M_k)\Big(\mu_k(R_k) + \sigma_k(R_k)\,F_{\varepsilon_k}^{-1}\Big[\frac{R_k - W_k}{R_k - s_k}\Big]\Big) \\ &+ \alpha\,\widehat{\Pi}_{k+1}^m\,\phi_{k+1}(R_k)\Bigg)\alpha\,\Phi_k(\mathbf{R}_{k-1}) \end{aligned} \tag{59}$$

Here, α is the fixed discount factor and remains constants for the whole duration of the problem from period 1 to n.[5]

The first-order conditions for this problem yield two equations for the two unknowns R_k and W_k. In the multi-period case, we start by using $\widehat{\Pi}_n^r = 0$ and $\widehat{\Pi}_n^m = 0$ and iterate backwards until we reach the starting period. However, if the horizon is infinite, this approach fails because an infinite number of iterations is needed to reach the start.

If $\mu(R)$, $\sigma(R)$, $\phi(R)$, and ε do not depend on k, or

$$\lim_{k\to\infty}\big(\mu(R,k), \sigma(R,k), \phi(R,k), \varepsilon_k\big) = \big(\mu(R), \sigma(R), \phi(R), \varepsilon\big),$$

i.e., the same functions are used for any k, then cases with an infinite horizon can be solved. To do so, one needs a steady state for the system; i.e., one must find $\widehat{\Pi}^r$ and $\widehat{\Pi}_m^r$ such that:

$$\widehat{\Pi}^r = (R - W)\mu(R) + L_\varepsilon(R, W) + \alpha\,\widehat{\Pi}^r\phi(R) \tag{60}$$

$$\widehat{\Pi}^m = (W - M)\Big(\mu(R) + \sigma(R)\,F_\varepsilon^{-1}\Big[\frac{R - W}{R - s}\Big]\Big) + \alpha\,\widehat{\Pi}^m\phi(R). \tag{61}$$

[5]The reason for this restrictions is that in the infinite-horizon case, a certain degree of autonomy is necessary for convergence to happen.

The first-order conditions from (58)-(59), together with (60)-(61), yield four equations in the four unknowns, $R, W, \widehat{\Pi}^r$, and $\widehat{\Pi}^m$.

6 Numerical implementation of the model

In this section, we illustrate the theory in section 5.2 by explicit examples. In these examples, we use a Cobb-Douglas demand function structure with a normally distributed random term. The problem is as easily solved when using other functional forms. The problem (given W) is reduced to finding maxima for a function of one variable, which is straightforward for almost any choice of μ_k, σ_k, and ϕ_k.

In section 6.1.1, we illustrate an important example in which the prescriptive analysis is employed to maximize the expected profits of the two vendors when the demand is boosted in between the n periods, e.g. through advertising campaigns. A numerical example of the cooperative behavior of the suppliers in which the two vendors form a centralized supply chain is illustrated in section 6.2.

We remark that the purpose of this entire section is to offer practical advice on how our theory can be implemented in some special cases to prescribe optimal decision variables. To take full advantage of the model, one should try to vary scaling factors and functional forms in a systematic way. This makes it possible to model a wide range of economic contexts. A full discussion of the model and all the variations it has to offer, is, however, beyond the scope of this paper.

6.1 Multi-period Buy Back Contracts

As evidenced in Theorem 5.1, once we have an algorithm that solves the two-period case, the same algorithm can be used repeatedly to solve $n-$period problems. We merely have to update the remaining profits as the construction progresses backward. We consider the case in which demand in period k is given by:

$$D_k = d_k(R_k) \prod_{i=2}^{k} \phi_i(R_{i-1}) \tag{62}$$

$$d_k(R_k) = \mu_k(R_k) + \sigma_k(R_k)\varepsilon_k \tag{63}$$

where ε_k is $\mathcal{N}(0,1)$. Because a normally distributed variable can take negative values, we must impose restrictions to exclude artificial cases. If q, as given by (26), is negative, we set $q = 0$. Moreover, if the expected profit in (27) is negative, we assume $q = 0$.

Setting the model parameters

We set the manufacturing cost at period k to be decreasing as time goes on, $M_k = 2 - 0.01k$, the buy back price $b_k = 0.3 \times M_k$, and the salvage price, $S_k = 0.2$, $\forall k$.

Note that in a buy back contract, as described by Cachon (2003), the remaining units at the end of each period need not be physically returned to the manufacturer. Instead, it may be such that the manufacturer credits the retailer with a price b_k (here equal to 30% of the manufacturing cost) for each unsold unit at the end of period k.

In principle, the scaling factors ϕ_ks can change with k. For illustration purposes, we consider only cases in which the expected scaling factors satisfy the following:

$$\phi_k(R_k) = \left[1 + \gamma_k(K_k - R_k)\right]^+ \approx e^{\gamma_k(K_k - R_k)} \text{ for small values of } \gamma_k \tag{64}$$

where $\gamma_k > 0$, the memory strength factor, and $K_k > 0$ are given parameters. The parameter K_k can be interpreted as a price cap; i.e., any price above K_k reduces demand, whereas demand is more likely to increase if $R_k < K_k$. More complicated expressions can be computed without problems. In the following examples, we set γ_k, K_k and α_k to remain constant through periods.

• Case 1: In addition to the parameters determined earlier, we set $\alpha = 1$ (no discounting), $\gamma = 0.01$, $K = 7$, $n = 25$, $\mu_k(R_k) = \frac{1000}{R_k^{(2-\beta\frac{n-k}{n})}}$, $\beta = 0.8$, and $\sigma_k(R_k) = \frac{1}{R_k}\mu_k$ so that the coefficient of variation decreases as R_k increases.

The optimal pricing variables in each period (R_k^*, W_k^*) as well as the values for $\overline{\Pi}_k^r$ and $\overline{\Pi}_k^m$ are shown in Figure 1. Where $\overline{\Pi}_k^r$ and $\overline{\Pi}_k^m$ represent the expected present value of the marginal profit obtained at each period by the retailer and the manufacturer, respectively.

The optimal strategy in this case is to increase demand by letting $R_1 = R_2 = R_3 = R_4 = 0$, then start selling in period 5.

Defining the blow-up factor as $\eta = \alpha \cdot \phi$, in this case, we have $\max(\eta) = \alpha \max(\phi) = \alpha(1 + \gamma K) = 1.07$. To obtain increased profits from an initial strategy in which $R_1 = 0$, it is clearly necessary that $\max(\eta) = \alpha \max(\phi) > 1$. This requirement, however, is not

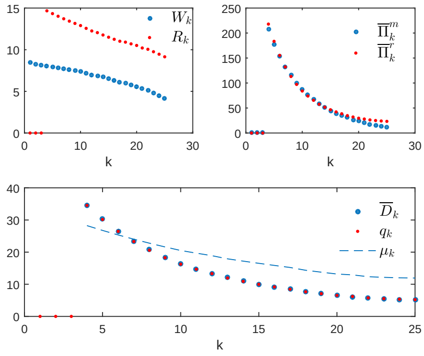

Figure 1: Stackelberg equilibrium state in period k

sufficient as we will see in Case 2. It is observable that, in the sales periods, $k \geq 5$, the profit margin $R_k - W_k$ remains fairly constant with time.

The total expected profits for the manufacturer and the retailer, J^m and J^r, respectively, are found as below.

$$J^m = 1547.35, \quad J^r = 1661.43$$

• Case 2: In this case, we analyze the previous example subject to discounting: $\alpha = 0.95$. The rest of the parameters and functional structures are the same as those of Case 1. In this case, $\max(\eta) = \alpha \max(\phi) = \alpha(1 + \gamma K) \approx 1.02$. Nonetheless, this blow-up factor is not big enough to justify an initial retail price $R_1 = 0$. Therefore, sales take place in all periods. Figures 2 shows the equilibrium prices and corresponding total expected profits for this case. For this case, we find $J^m = 1041.24$ and $J^r = 909.75$.

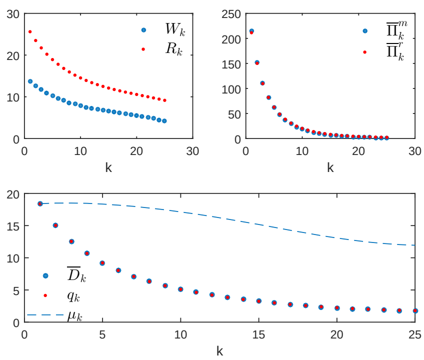

Figure 2: Stackelberg equilibrium state in period k

6.1.1 Boosting the demand

In general, we consider $\frac{\partial \overline{D}}{\partial k} < 0$. However, there may be opportunities for the manufacturer or the retailer to boost the demand, sometimes in the process. This, for example, can be done through rolling out a new model of the product or implementing advertising campaigns.

To cover such situations, we consider a case in which the demand, as represented by μ_k, is amplified twice at $k = 8$ and $k = 16$ when $n = 25$.

$$
\begin{aligned}
\mu_k(R) &= \frac{1000}{R^{(2-\beta\frac{7-k}{25})}}, k < 8 \\
\mu_k(R) &= \frac{1000}{R^{(2-\beta\frac{15-k}{25})}}, 8 \leq k < 16 \\
\mu_k(R) &= \frac{1000}{R^{(2-\beta\frac{25-k}{25})}}, 16 \leq k \leq 25
\end{aligned}
\tag{65}
$$

The rest of the parameters are: $\alpha = 1$, $\gamma = 0.01$, $K = 7$, $n = 25$, $M_k = 2 - 0.01k$, $b_k = 0.3\, M_k$, $S = 0.2$, $\beta = 0.8$, and $\sigma_k(R_k) = 0.5\, \mu_k(R_k)$. Here, we have suppressed the subscript k for parameters that are set to remain constant through periods.

Figure 3 shows that with such incremental increases in demand, the retailer's optimal strategy is to begin with $R_1 = R_2 = \cdots = R_7 = 0$ and continue with jumps in retail prices after any time the demand is boosted. Such jumps in R_k will lead to higher profit margins.

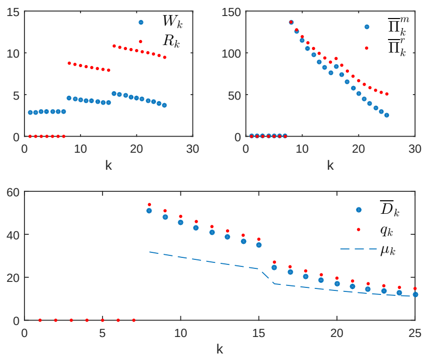

Figure 3: Stackelberg equilibrium states at k, incremental demand boosts

6.2 Cooperative Agents: Centralized Channel, No Double Marginalization

So far, we have analyzed the equilibria in a Stackelberg framework. However, it is possible for the two parties to cooperate. Note that, as outlined in (9), a Stackelberg equilibrium problem is essentially a bilevel optimization problem wherein the leader optimizes her objective function while being constrained by the optimality of the follower's solution.

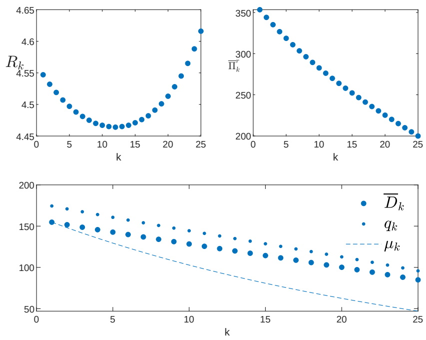

Figure 4: Centralized Channel

Thus, from a computational point of view, the multi-period single-vendor price-setting Newsvendor problem turns into an uncostrained special case of our general model. Such a deviation from the Stackelberg game is implemented by considering the two agents as a single decision-maker, substitution of $W_k = M_k$ in Theorem 5.1, and then optimizing only J^r with respect to R_ks.

Here, we consider $\alpha = 1$, $\gamma = 0.01$, $K = 7$, $n = 25$, $M_k = 2 - 0.01k, S_k = 0.2\,\forall k$, $\mu_k(R) = \frac{1000}{R^{(2-\beta\frac{n-k}{n})}}$, $\beta = 0.8$, and $\sigma_k(R) = \frac{\mu_k(R)}{R}$. Thus, the integrated channel in Case 4 faces the same market as the one in Case 1.

The results are illustrated in Figure 4. Comparing the results of the centralized channel with its decentralized counterpart, analyzed in Case (1), we observe that while the centralized channel charges comparatively lower prices in all periods, its overall expected profit is higher than the sum of expected profits for the two agents in a decentralized channel facing the same market. Denoting the total expected profit for the centralized channel by J^c and the expected profits for the members of the corresponding decentralized channel by J^r and J^m (obtained in Case 1), we have:

$$J^c = 6744.33 > J^m + J^r = 1547.35 + 1661.43 = 3208.78.$$

Due to vertical competition between its members, a decentralized channel suffers from double marginalization leading to its lower performance compared to a centralized supply channel.

Moreover, lower prices offered by the centralized channel leads to a lower level of demand suppression. Thus, unlike the decentralized channel in Case (1), the integrated channel does not need to resort to an early campaign of free distribution of products in order to boost the demand. According to Figure 4, in Case (4) sale takes place in all periods.

7 Concluding remarks

In this paper, within two constructive theorems, we present solution procedures for multi-period price-setting newsvendor problems in a Stackelberg framework. An important feature of the proposed recursive optimization algorithm is that it decouples the analysis of three highly nested $n-$dimensional problems to three sequences of n single-variable equations. In the appendices, the structural requirements for this memory-based scheme to work are shown to be present in many conventional supply channel contracts.

The assumptions made for constructing the memory functions are based on the simple fact that every market has some kind of a memory; that is, the potential buyers are anchored to past prices and their decision to purchase a product may to various degrees be affected by the history of pricing. This feature is embedded in the model by the introduction of γ_k, the memory strength factor. It is through this memory effect that a change in price may scale the pool of potential customers, thereby affecting the demand. Note that these assumptions are not needed for the model to work, yet they are computationally useful and not restrictive.

To demonstrate that such problems can be modeled and solved by the procedure outlined in Theorem 5.1 and its corollary, we provided numerical solutions to a variety of special cases. Note, however, that our framework is not limited to such special cases. The numerical illustrations raise questions of interest for future research.

In the numerical section, we have demonstrated an interesting link to marketing. Under certain conditions, an optimal strategy is to give away products in a pre-sales period. This stimulates demand, and the parties benefit from increased demand in the remaining time periods. Many high-tech products like mobile phones and computers have a very short lifespan. Our paper hence offers a new framework where the optimality of sales strategies for such products can be discussed and analyzed.

The parameters and functional structures that we have used to illustrate the scope of applicability of the solution procedures are merely speculative. To take full advantage of the model, one should try to vary scaling factors and functional forms in a systematic way, for example through the use of data obtained from empirical studies. Exploring the potential of our modeling approach is a topic for future research.

A Appendix 1 - The profit structures compatible with the use of multiplicative memory functions

In this section, we find the sufficient conditions for the profit structures enabling the memory-based algorithm to decouple nested optimization/equilibrium problems. In Appendix B, we show that many conventional supply channel contracts indeed follow these structures.

Let the following be the demand expression at period $k \in \{1, \cdots, n\}$, where n is the number of periods.

$$D_k(\mathbf{R_k}) = \widetilde{\mu}_k(\mathbf{R_k}) + \widetilde{\sigma}_k(\mathbf{R_k})\varepsilon_k \tag{66}$$

Now assume that the running profit expression for each period is as below.

$$\overline{\Pi}_k^r(\mathbf{R_k}) = \Psi_k(R_k)\, \widetilde{\mu}_k^p(\mathbf{R_k}) + \Theta_k(R_k)\, \widetilde{\sigma}_k^p(\mathbf{R_k}) \tag{67}$$

Using the memory function structure in (13) and (14), the expression in (66) turns into the following.

$$D_k(\mathbf{R_k}) = \big(\mu_k(R_k) + \sigma_k(R_k)\varepsilon_k\big)\, \Phi_k(\mathbf{R_k}) \tag{68}$$

Therefore, for (67), we will have:

$$\overline{\Pi}_k^r(\mathbf{R_{k\text{-}1}}) = \left(\Psi_k(R_k)\mu_k^p(R_k) + \Theta_k(R_k)\sigma_k^p(R_k)\right)\Phi_k^p(\mathbf{R_{k\text{-}1}}) \tag{69}$$

$$= \widetilde{\Pi}_k(R_k)\,\Phi_k^p(\mathbf{R_{k\text{-}1}}) \tag{70}$$

Thus, when solving the multi-variable optimization problem $\max\limits_{\mathbf{R_n}} J = \sum_{k=1}^n \alpha_k \overline{\Pi}_k^r$, finding the kth armgax, $\hat{R}_k$, will be equivalent to finding the argmax of the single-variable function $\widetilde{\Pi}_k(R_k)$. Whence decoupling becomes possible.

The same argument applies to the expected profit for the manufacturer. In section B, we will see that a special case of the structure in (67), where $p = 1$, indeed appears in many supply chain optimization contracts.

B Appendix 2 - The expected profit structure in supply chain coordination problems

In the appendix A, we realized that in order for the decoupling scheme (using memory functions) to work in multiple periods, the single-period profits (for either vendor) must be of specific structures. In other words, in such cases the multiplicative memory functions decouple the highly nested $n-$variable optimization (or equilibrium) problem, by turning it into n single-variable problems.

In the following sections, we will prove that the desired general structures indeed appear in many classical channel coordination contracts; hence making our theoretical formwork applicable to a large variety of multi-period optimization/coordinatin problems dealing with uncertain demand. This class of contracts include, e.g., *The Wholesale Price Contracts*, *The Buy Back Contracts*, and, *The Revenue Sharing Contracts*.[6]

[6] We have borrowed the nomenclature from Cachon (2003). Other sources may use different names; for example, Pasternack (1985) refers to buy back contracts as *return policies*.

Model Variables and Parameters

$D =$ uncertain demand

$\mu = \mathrm{E}[D]$ expected value of uncertain demand

$R =$ retail price per unit

$W =$ wholesale price per unit

$q =$ order quantity

$s =$ salvage price per unit

$g_r =$ retailer's goodwill penalty per unit, incurred for each unmet demand unit

$g_w =$ manufacturer's goodwill penalty per unit

$c_r =$ retailer's marginal cost per unit

$c_w =$ manufacturer's production cost per unit[7]

B.1 The wholesale price contract

We start with analysing the general newsvendor problem, also known as the wholesale price contract. In this contract, the retailer profit is obtained as below.

$$\begin{aligned}\Pi^r(R,q) &= R\min(D,q) + s(q-D)^+ - c_r q - g_r(D-q)^+ - Wq \\ &= R\min(D,q) + s(q-\min(D,q)) - c_r q - g_r(D-\min(D,q)) - Wq \\ &= (R-s+g_r)\min(D,q) - (c_r+W-s)q - g_r D\end{aligned} \tag{71}$$

In the analysis of this specific contract, and also in the subsequent sections for other contracts, our strategy is as follows.

1. Obtain the expression for the expected value of retailer's profit as a function of R and q.

2. Apply the F.O.C with respect to q on $\overline{\Pi}^r$, i.e. $\frac{\partial \overline{\Pi}^r}{\partial q} = 0$ to obtain q^* as a function of R.

[7] In order to adhere to the terminology suggested by Cachon (2003), here we have used c_w to denote the production cost incurred by the manufacturer. In the main body of the paper, however, we have denoted the manufacturer's production cost by M.

3. Check the concavity of $\overline{\Pi}^r$ with respect to $q^*(R)$.

4. Substitute the obtained q^* as a function of R in the expression for $\overline{\Pi}^r$ and see if it is of the general structure in (67).

5. Substitute the obtained q^* as a function of R in the expression for $\overline{\Pi}^m$ and see if it is of the general structure in (67).

In order to obtain the expected value of the retailer's profit as stated in (71), we need to obtain the expected sales, $\mathcal{S}(q)$, i.e. the expected value of $\min(D, q)$. For simplicity, we start with a distribution function, f_D, for D, instead of f_ε, bearing in mind that for $D = \mu + \sigma\,\varepsilon$, we have: $F_D^{-1}(\cdot) = \mu + \sigma\,F_\varepsilon^{-1}(\cdot)$.

$$\begin{aligned}\mathcal{S}(q) = \mathrm{E}[\min(D, q)] &= \int_0^\infty \zeta f_D(\zeta) d\zeta = \int_0^q \zeta f_D(\zeta) d\zeta + \int_q^\infty q f_D(\zeta) d\zeta \\ &= q - \int_0^q F_D(\zeta) d\zeta\end{aligned} \tag{72}$$

Alternatively, using our normalization based on f_ε instead of f_D, we obtain the following statement for the expected sales.

$$\mathcal{S}(q) = \mathrm{E}[\min(D, q)] = \mu + \sigma \int_{-\infty}^\infty \zeta f_\varepsilon(\zeta) d\zeta = q - (q-\mu) F_\varepsilon\left(\frac{q-\mu}{\sigma}\right) + \sigma \int_{-\infty}^{\frac{q-\mu}{\sigma}} \zeta f_\varepsilon(\zeta) d\zeta \tag{73}$$

We define $I(q) = \mathrm{E}[(q-D)^+] = q - \mathcal{S}(q)$ as the expected left over inventory. Similarly, let $L(q) = \mathrm{E}[(D-q)^+] = \mu - \mathcal{S}(q)$ be the expected lost sales function. Also observe that $\frac{d\mathcal{S}(q)}{dq} = 1 - F_D(q)$. Now we can calculate the expected value of retailer's profit, expressed in (71).

$$\overline{\Pi}^r(R, q) = (R - s + g_r)\mathcal{S}(q) - (c_r - s + W)q - g_r\mu \tag{74}$$

Strict concavity of $\overline{\Pi}^r(q)$ with respect to q is obvious:

$$\frac{\partial^2 \overline{\Pi}^r(q)}{\partial q^2} = -(R - s + g_r) \times f_D(q) \tag{75}$$

So we apply the F.O.C with respect to q and obtain $q^*(R)$ as below.

$$q^*(R) = F_D^{-1}\left(\frac{R - W + g_r - c_r}{R - s + g_r}\right) = \mu(R) + \sigma(R)\,F_\varepsilon^{-1}\left(\frac{R - W + g_r - c_r}{R - s + g_r}\right) \tag{76}$$

Substituting (76) in (74) and using (73) we have

$$\overline{\Pi}^r(R) = (R - W - c_r)\mu(R) + (R - s + g_r)\left(\int_{-\infty}^{F_\varepsilon^{-1}\left(\frac{R-W+g_r-c_r}{R-s+g_r}\right)} \zeta f_\varepsilon(\zeta) d\zeta \right) \sigma(R) \tag{77}$$

In a single-vendor system, F.O.C applied to 77 yields the R^*. However, in our Stackelberg framework, a numerical solution to $\frac{\partial \overline{\Pi}^r}{\partial R} = 0$ provides us with R^* as a function of W: $R^*(W)$. The optimal order quantity, q^* can then be obtained using R^*.

Let us now analyse the expected value of the manufacturer's profit.

$$\begin{aligned}
\overline{\Pi}^m\big(R^*(W), W\big) &= Wq^*(W) - c_w q^*(W) + g_w \mathcal{S}(q^*(W)) - g_w \mu\big(R^*(W)\big) \\
&= q^*\big(W - c_w + g_w\big) - g_w\left(\sigma F_\varepsilon^{-1}(y) \times y + \mu + \int_{-\infty}^{F_\varepsilon^{-1}(y)} \zeta f_\varepsilon(\zeta) d\zeta\right) \\
&= \mu(W)(W - c_w) + \sigma(W)\left(F_\varepsilon^{-1}(y)\Big(W - c_w - g_w(y-1)\Big)\right. \qquad (78) \\
&\quad \left. + \, g_w \int_{-\infty}^{F_\varepsilon^{-1}(y)} \zeta f_\varepsilon(\zeta) d\zeta \right) \\
&\quad \text{where } y = \frac{R - W + g_r - c_r}{R - s + g_r}
\end{aligned}$$

The structure of $\overline{\Pi}^m(W)$ as stated in (78) is also of the desired type.

B.2 The buy back contract

In a buy back contract, the manufacturer pays the retailer $b \leq W_b$ per unit remaining at the end of the selling season (period).

$$\overline{\Pi}^r(R, q) = (R - s + g_r - b)\mathcal{S}(q) - (W_b - b + c_r - s)q - g_r \mu \tag{79}$$

Again, assuming that $R - s + g_r - b > 0$, concavity of $\overline{\Pi}^r(q)$ with respect to q is obvious. Applying the F.O.C with respect to q, we obtain the following for $q^*(R)$.

$$q^*(R) = F_D^{-1}\left(\frac{R - W_b + g_r - c_r}{R - s + g_r - b}\right) = \mu(R) + \sigma(R)\, F_\varepsilon^{-1}\left(\frac{R - W_b + g_r - c_r}{R - s + g_r - b}\right) \tag{80}$$

Substituting (80) in (79), we obtain the following expression for the retailer's expected profit.

$$\overline{\Pi}^r(R) = (R - W_b - c_r)\mu(R) + (R - s + g_r - b)\left(\int_{-\infty}^{F_\varepsilon^{-1}\left(\frac{R-W_b+g_r-c_r}{R-s+g_r-b}\right)} \zeta f_\varepsilon(\zeta) d\zeta \right) \sigma(R) \tag{81}$$

Notice that, due to our normalization in which we set $\mathrm{E}[\varepsilon] = 0$, the integral term $\Gamma := \int_{-\infty}^{F_\varepsilon^{-1}\left(\frac{R+g_r-c_r-W_b}{R-s+g_r-b}\right)} \zeta f_\varepsilon(\zeta) d\zeta$ is always negative, making $(R-s+g_r-b)\times\Gamma\times\sigma(R)$ in (81) also negative. In the literature, the latter term is called the *loss due to stochasticity*. Similarly, we obtain the expected value of the manufacturer's profit as below.

$$\begin{aligned}\overline{\Pi}^m(W) = \mu(W)(W_b - c_w) + \sigma(W)\Bigg(&F_\varepsilon^{-1}(y)\Big(W - c_w + g_w(1-y) - by\Big)\\ &+ (g_w + b)\int_{-\infty}^{F_\varepsilon^{-1}(y)} \zeta f(\zeta) d\zeta\Bigg) \\ &\text{where } y = \left(\frac{R - W_b + g_r - c_r}{R - s + g_r - b}\right)\end{aligned} \tag{82}$$

B.3 The revenue sharing contract

With a revenue sharing contract the manufacturer charges W_r per unit purchased and the retailer gives the manufacturer a percentage of his revenue. Let θ be the fraction of supply chain revenue the retailer keeps, so $(1-\theta)$ is the fraction given to the manufacturer. The retailer's expected profit function is

$$\overline{\Pi}^r(R, q) = (\theta(R-s) + g_r)\mathcal{S}(q) - (W_r + c_r - \theta s)q - g_r\mu. \tag{83}$$

We observe that $\overline{\Pi}^r$ is concave with respect to q because $\theta(R-s)+g_r > 0$. Applying F.O.C. with respect to q yields:

$$q^*(R) = F_D^{-1}\left(\frac{\theta R - W_r + g_r - c_r}{\theta(R-s) + g_r}\right) = \mu(R) + \sigma(R)\, F_\varepsilon^{-1}\left(\frac{\theta R - W_r + g_r - c_r}{\theta(R-s) + g_r}\right) \tag{84}$$

Substituting (84) into (83), we have:

$$\overline{\Pi}^r(R) = (\theta R - W_r - c_r)\mu(R) + \left((\theta(R-s) + g_r)\int_{-\infty}^{F_\varepsilon^{-1}\left(\frac{\theta R - W_r + g_r - c_r}{\theta(R-s)+g_r}\right)} \zeta f_\varepsilon(\zeta) d\zeta\right)\sigma(R). \tag{85}$$

Also notice how $\theta = 1$ turns (85) into (77).

Next, we obtain the manufacturer's expected profit as below.

$$\begin{aligned}
\overline{\Pi}^m &= \Big(g_w + (1-\theta)(R-s)\Big)\mathcal{S}(q) + \Big(W_r + (1-\theta)s - c_w\Big)q - g_w\mu \\
&= \mu\left((1-\theta)R + W_r - c_w\right) \\
&\quad + \sigma\left(F_\varepsilon^{-1}(y)\Big(g_w + (1-\theta)R + W_r - c_w - y\big(g_w + (R-s)(1-\theta)\big)\Big)\right. \\
&\quad \left. + g_w \int_{-\infty}^{F_\varepsilon^{-1}(y)} \zeta f_\varepsilon(\zeta) d\zeta\right) \\
&\quad \text{where } y = \frac{\theta R - W_r + g_r + c_r}{\theta(R-s) + g_r}.
\end{aligned} \tag{86}$$

Notice how $\theta = 1$ turns (86) into (78).

C Appendix 3 - Proof of Proposition 5.1

Let F_ε denote the cumulative distribution of ε. Since ε is continuous and supported on an interval, with density $f_\varepsilon > 0$ a.e. on its support, the expected profit $\overline{\Pi}^r$ is strictly concave in q on the support of D, and the order quantity q from (25) is unique. It is clear that

$$q = \mu(R,k) + \sigma(R,k)\, F_\varepsilon^{-1}\left(\frac{R-W}{R-s}\right). \tag{87}$$

By using 1 and 25, we obtain:

$$\begin{aligned}
\mathrm{E}[\Pi^r] &= (R-s)\mathrm{E}[\min(D,q)] - (W-s)q \\
&= (R-s)\Big(\mu(R,k) + \mathrm{E}\Big[\min\left(\sigma(R,k)\varepsilon, \sigma(R,k)F_\varepsilon^{-1}\left(\frac{R-W}{R-s}\right)\right)\Big]\Big) \\
&\quad - (W-s)\Big(\mu(R,k) + \sigma(R,k)F_\varepsilon^{-1}\left(\frac{R-W}{R-s}\right)\Big).
\end{aligned} \tag{88}$$

Equations (1) and(87) indicate that

$$\mathrm{E}\Big[\min\Big(\varepsilon, F_\varepsilon^{-1}\left(\frac{R-W}{R-s}\right)\Big)\Big] \tag{89}$$

$$= \int_{-\infty}^{F_\varepsilon^{-1}\left(\frac{R-W}{R-s}\right)} \zeta f_\varepsilon(\zeta) d\zeta + F_\varepsilon^{-1}\Big(\frac{R-W}{R-s}\Big)\, \mathcal{P}\Big(\varepsilon \geq F_\varepsilon^{-1}\Big(\frac{R-W}{R-s}\Big)\Big) \tag{90}$$

$$= \int_{-\infty}^{F_\varepsilon^{-1}\left(\frac{R-W}{R-s}\right)} \zeta f_\varepsilon(\zeta) d\zeta + F_\varepsilon^{-1}\Big(\frac{R-W}{R-s}\Big)\left(1 - \frac{R-W}{R-s}\right). \tag{91}$$

Inserting (91) into (88) and simplifying the resulting expression yields:

$$\overline{\Pi}^r = \mathrm{E}[\Pi^r] = (R-W)\mu(R,k) + L_\varepsilon(R,W)\sigma(R,k) \tag{92}$$

where L_ε is defined as:

$$L_\varepsilon(R,W) = (R-s)\int_{-\infty}^{z} \zeta f_\varepsilon(\zeta)d\zeta \quad z = F_\varepsilon^{-1}\left(\frac{R-W}{R-s}\right). \tag{93}$$

□

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