Explicit Solution Algorithms for Order and Price Postponement in Multi-periodic Channel Optimization

BY Reza Azad Gholami, Leif K. Sandal and Jan Ubøe

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Reza Azad Gholami* Leif Kristoffer Sandal† Jan Ubøe†

Department of Business and Management Science, NHH Norwegian School of Economics

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Abstract

Supply channels typically face uncertain and time-varying demand. Nonetheless, time-dependent channel optimization while addressing uncertain demand has received limited attention due to the high level of complexity of the ensuing nested equilibrium problems. The level of complexity rises when demand is dependent on current and previous prices. We consider a decentralized supply channel whose two members, a manufacturer and a retailer, must address the demand for a perishable commodity within a multi-period time horizon. Using a general (additive-multiplicative) stochastic model for the price-dependent demand, the purpose of this paper is to provide the channel members with analytic tools to devise optimal pricing and supply strategies at different times. In the first part of the paper, we propose a constructive theorem providing an explicit solution algorithm to obtain equilibrium states for bilevel optimization in decentralized supply channels. We also prove that the resulting equilibria are subgame perfect. In the second part, we allow the retailer to postpone her supply and pricing decisions until demand uncertainty is resolved at each period. Using subgame perfectness of the equilibria, we propose solution algorithms that use the extra information obtained by postponement. Finally, in a number of comparison theorems, we show that postponement strategies are always beneficial for a centralized channel (whose revenue structure is identical to that of a retailer). Whereas for a decentralized channel, due to vertical competitions, there may be scenarios wherein postponement strategies, i.e. access to extra information, turn out to be detrimental to the manufacturer and even to the whole channel.

Keywords: stochastic optimization; bilevel programming; game theory; pricing theory; stochastic demand; time-dependent demand; price-dependent demand

*Department of Business and Management Science, NHH Norwegian School of Economics, Helleveien 30, 5045 Bergen, Norway, reza.gholami@nhh.no

†Department of Business and Management Science, NHH Norwegian School of Economics

1 Introduction

Demand for almost every commodity is typically uncertain and time-varying. Better products being rolled out by competitors may reduce or eliminate demand for a certain commodity. Seasonal changes in demand trends for commodities such as apparel may rapidly render a fashionable product outdated. Thus gaining information about uncertain and time-varying demand for a commodity is vital for every vendor facing it.

With the advent and growth of online shopping, supply channels have become able to obtain reliable signals from the uncertain demand. Electronic-commerce retailers such as Amazon, Wal-Mart and eBay provide their customers with “wish lists” where potential buyers can suggest or pre-sale an item before its “future release”, thus reducing demand uncertainty for the retailers.

The main goal of this paper is to provide the supply channel members with analytic tools to use the uncertain demand data when devising long-term (multi-period) pricing and supply strategies.

The problem of finding equilibrium state for a supply channel facing uncertain demand in a single-period time setting has been long studied. Petruzzi and Dada (1999) solve the single-period newsvendor problem for both (purely) additive and multiplicative price-dependent uncertain demand models and compare the results with those of the benchmark deterministic model. Pasternack (2008) analyzes the static (single-period) problem of finding optimal pricing strategies and buy back contracts (return policies) for a retailer and a manufacturer facing uncertain demand for a perishable commodity. Gümüş et al. (2013) extend the study of inventory management for suppliers facing uncertain demand into a double-period time setting. Keren (2009) solves the single-period inventory problem for a specific demand distribution and two types of yield risks, with the decision variable being the order quantity.

However, there are many scenarios in which a multi-period analysis of pricing and demand is necessary. Market-penetration scenarios in which entrant suppliers try to manipulate demand by offering lower prices in the beginning are among such cases. Incurring initial losses that may manipulate the demand and cause higher profits in the future are not prescribed by single-period solutions that do not consider future effects of pricing on demand and profit.

Considering the effect of the pricing history on future demand and, as a result, on future profits in multi-period supply chain coordination is a challenging task. In many studies, the random demand in different periods are considered to be Markovian and independent from each other across time (Aviv and Federgruen 2001).

In Section 3, we embed the uncertain demand structure introduced in Section 2 in a dynamic (multi-period) bilevel profit optimization problem where two competing suppliers, a manufacturer and a retailer, try to maximize their respective revenues through addressing the demand. We analyze the problem in a Stackelberg framework where the manufacturer is the leader and the retailer is the follower. We assume both the agents to be risk-neutral so each one them tries to maximize her respective expected profit while being subjected to the optimality of the other player's solution. The decision variables to be determined are the wholesale price, retail price, and the order quantity which are set at the beginning of each period. We analyze the equilibrium problems within the scope of multiple periods and with a general contract where the manufacturer may or may not offer buy back prices to the retailer.

The analysis in that section results in Theorem 3.1 where we state the necessary conditions for the existence of equilibria in different periods. Moreover, we propose a solution algorithm to obtain the numerical variables constituting the equilibria. Moreover, in Proposition 3.2, we prove that the obtained equilibria are subgame perfect—a property we will use later when analyzing price postponement strategies.

Granot and Yin (2008) solve the single-period problem of price and order postponement in a decentralized newsvendor model. The demand in their model is price-dependent and purely multiplicative. They analyze and compare the effect of different demand mean functions on the profit obtained by the whole channel and each individual supplier. Lenk (2008) extends the single-period study of the effect of price postponement on supply chain coordination into a two-stage newsvendor problem. Xu and Bisi (2011) study a price postponement scenario in a single-period newsvendor model with wholesale price-only contract. They, too, consider purely multiplicative or additive structures for the uncertain demand and make a series of assumptions about demand distribution which assure the unimodality of ensuing profit functions for both the manufacturer and the retailer.

Having solved the multi-period equilibria for no-postponement scenarios in Section 3, in Section 4 we propose and analyze the order postponement feedback policy in a multi-period setting. In this scenario, at each period, the retailer postpones sending her order quantity to the manufacturer until she observes the demand uncertainty in that period. We solve the problem of finding the optimal feedback policy and in Theorem 4.1 and its Corollary 4.3 show how the results of postponement equilibria outperform those of the non-postponing strategy adopted in Section 3.

In Section 5, we analyze another feedback policy in which the retailer postpones her retail pricing decision until after demand uncertainty is resolved. Using the subgame perfect property of the original equilibria found in Section 3, we solve the bilevel multi-period optimization problem with the additional information obtained after postponement. The structure of the ensuing equilibria and the solution algorithm are offered in Theorem 5.1. In Section 5.3, we compare the results of price postponement equilibria with those of the non-postponing strategy.

Finally, in Section 6, we provide a few number of examples containing simulated realizations of the random scenarios described in the previous sections. These examples have been provided to merely familiarize the reader with the implementation of the solution algorithms. They illustrate our theoretical model's scope of applicability and its flexility in performing prescriptive analyses accordingly. It is imperative to note that these examples and the mathematical features of the scenarios simulated therein are merely speculative and not the results of empirical studies.

2 Preliminary Model Description

In a dynamic setting and time-dependent structure, first we propose a general model for stochastic demand at each point in time. Then, in sections 3, embedding this demand structure into various profit-optimization games, we arrive at equilibria solutions for each scenario. We divide the time scope into n discrete intervals referred to as periods. All the model variables and parameters are assumed to remain constant within each period.

In general, we consider demand at each period k to be a function of the entire retail price history, and time.

$$D_k = \tilde{\mu}_k(\mathbf{r}_k) + \tilde{\sigma}_k(\mathbf{r}_k)\,\epsilon_k \tag{1}$$

where r_k is the retail price at k, $\mathbf{r}_k = [r_1, \cdots, r_k]$ is the vector of the entire retail price history up to period k. Moreover, $\tilde{\mu}_k(\cdot)$ and $\tilde{\sigma}_k(\cdot)$ are deterministic functions of $\mathbf{r}_k$ and time (period k), and ϵ_k is the stochastic variable at k.

The stochastic variable ϵ_k is normalized such that $\mathrm{E}[\epsilon_k] = 0$ and $\mathrm{Var}[\epsilon_\mathrm{k}] = 1$. We also assume that the density function for ϵ_k and its cumulative distribution function, $f_{\epsilon_k}(\cdot)$ and $F_{\epsilon_k}(\cdot)$ respectively, are known over its support $[\underline{\epsilon}_k, \overline{\epsilon}_k]$. Furthermore, we assume $F_{\epsilon_k}(\underline{\epsilon}) = 0$ and $F_{\epsilon_k}(\overline{\epsilon}_k) = 1$. Plus, we assume that F_{ϵ_k} is invertible on the support interval and denote the resulting inverse cumulative distribution function (quantile function) by $F^{-1}_{\epsilon_k}(\cdot)$.

In a purely additive model for the uncertain demand, the volatility of demand is considered to be constant and in a purely multiplicative model, the mean and standard deviation of demand are assumed to be equal, thus making the coefficient of variation of demand a constant (i.e. 1). Both assumptions, as we will see in the next section, are restrictive and undesirable (Young 1978).[1]

[1] In the multiplicative demand model, $D = \mu(r)\zeta$, $\mathrm{E}[\zeta] = 1$. This structure is a special case of our model with the assumption

An additive-multiplicative model, on the other hand, allows us to cover cases with coefficient of variation of demand being affected by the retail price.

2.1 Open-loop and Closed-loop Equilibria problems

Having outlined our general demand structure in section 2, we embed it in a class of channel optimization problems where the suppliers of a perishable good face the uncertain demand described earlier. The supply chain is comprised of a manufacturer and a retailer. We consider a Stackelberg competition framework in which the manufacturer is the leader and the retailer acts as the follower. Considering the uncertain demand for the product, at the beginning of each period k, the manufacturer sets the optimal wholesale price w_k, and the retailer has to find the optimal retail price r_k, and order quantity q_k accordingly. We denote the equilibrium values of the wholesale and retail prices and order quantity by w_k^*, r_k^*, and q_k^* respectively.

In a non-postponement analysis, both the agents are risk-neutral and their optimization problem is based on maximizing their respective expected profits within the n-period time scale. In such scenarios, after w_k^* is announced, the retailer announces her r_k^* and q_k^* without postponement.

Whereas in an order quantity or retail price postponement scenario, the retailer postpones declaration of one of her decision variables (either q_k or r_k) until she has observed demand uncertainty ϵ_k. At each period k the retailer uses this extra delayed information in order to incorporate the real value of her period (i.e. local-in-time) profit in her optimization problem. In Theorem 4.1, its Corollary 4.4, and in Sections 5.3 and 5.3.1 we discuss how different postponement strategies, allowing for post-observation optimization, will affect the profits for the two decision makers and for the whole channel.

We refer to the post-observation equilibrium variables as $\hat{w}_k$, $\hat{r}_k$, and $\hat{q}_k$. In the subsequent sections, we refer to the non-postponement optimization procedures as the open-loop, ex-ante, or pre-observation analyses. We also use the terms post-observation, closed-loop, and ex-post analysis, interchangeably to refer to the postponement analysis.

$\mu(r) = \sigma(r)$, where demand will be $D = \mu(r) + \mu(r)\epsilon$. Despite its computational tractability, we find the assumption that the mean and standard deviation of demand are necessarily equal quite strong and not always justifiable.

3 Pre-observation Equilibrium: An Open-loop Model Without Postponement

At the beginning of each period, the manufacturer offers a wholesale price. Then the retailer sends her order quantity (which may be zero) to the manufacturer and declares her retail price to the market. At the end of the period, if the retailer is left with a surplus of items, which means her order quantity was larger than the actual demand, she will sell them for a salvage price. She may or may not receive a buy back offer from the manufacturer for the surplus items. Because the commodity is perishable, she will not be able to store the unsold items to be offered to the market in the next periods.

In this section we solve the problem of maximizing the expected profits within the whole timescale encompassing all the periods. Thus, for instance, a pricing strategy that is optimal for a single period problem may be found out to be suboptimal within the multi-period setting. Thereby, the prescribed pricing and order quantity for the manufacturer and the retailer will enable then to make strategic sacrifices in order to boost the demand and rip the highest expected profits within the multi-period timescale. The decision variables to be determined are the wholesale price, retail price, and order quantity in each period, and the objective functions to be maximized are the holistic discounted expected profit for each decision maker.

3.1 The Static (Single-period) Equilibrium Problem

The final model in section 3.5, its equilibrium structure, and our proposed algorithm for its numerical solution presented in theorem 3.1, will include the general multi-period problem. However, for illustration purposes we start out with a single-period Stackelberg equilibrium problem. Later we expand the scheme to solve the generalized equilibrium problem in a multi-period (dynamic) setting.

Model Variables and Parameters

$w =$ wholesale price per unit, (decision variable)

$r =$ retail price per unit, $r > w$ (decision variable)

$q =$ quantity of items to be supplied to the market, (decision variable)

$D =$ actual uncertain demand

$c_m =$ manufacturing cost per unit, $c_m < w$ (given parameter)

$c_r =$ retailer's marginal cost per unit, $c_r < r - w$ (given parameter)

$s =$ salvage price per unit

$b =$ buy back price per unit

$\pi^m =$ manufacturer's profit

$\pi^r =$ retailer's profit

Note that because this is a single-period analysis, we have suppressed the subscripts k. In such a single-period setting the general demand expression in (1) will turn into a specific simplified form described below.

$$D = \mu(r) + \sigma(r)\epsilon \tag{2}$$

In the multi-period analysis, however, all the decision variables and parameters may vary with time. This feature adds up to the level of non-autonomy the model can cover.

In general the single-period equilibrium is obtained by solving the following bi-level maximization problem.

$$\begin{aligned} \max_q \mathrm{E}[\pi^r(r,w,q)] &\quad \text{to obtain} \quad q^*(r,w) \\ \max_r \mathrm{E}[\pi^r(r,w)] &\quad \text{to obtain} \quad r^*(w) \\ \max_w \mathrm{E}[\pi^m(w)] &\quad \text{to obtain} \quad w^* \to r^*, q^* \end{aligned} \tag{3}$$

Note that in (3), optimization procedures are applied on expected values of the players' profits. The retailer's profit, π^r is calculated as below.

$$\begin{aligned} \pi^r(r,q,w) &= r\min(D,q) + s(q-D)^+ - c_r q - wq + b(q-D)^+ \\ &= (r-s-b)\min(D,q) + (s+b-c_r-w)q \end{aligned} \tag{4}$$

In (4), for the sake of generality, we have considered buy back contracts represented by b. In a buy back contract the manufacturer pays the retailer $b < w$ per unit unsold. It should be noted that a buy back contract does not necessarily mean that the unsold items will be physically sent back to the manufacturer (Chacon 2003). In order to share the risks stemming from market uncertainty and incentivize a larger order quantity, the manufacturer credits the retailer for each unsold item. Obviously $r > b + s$.

In order to obtain the expected value of the retailer's profit, we need to calculate $\mathrm{E}\big[\min(D, q)\big]$. Given f_ϵ, F_ϵ, and $\underline{\epsilon}$ we define and calculate the expected sales, $\mathcal{S}$, as follows.

$$\begin{aligned}
\mathcal{S}(q) \coloneqq \mathrm{E}\big[\min(D, q)\big] &= \int_{\underline{\epsilon}}^{\overline{\epsilon}} \min(\mu + \sigma t, q)\, f_\epsilon(t)\, dt \\
&= \int_{\underline{\epsilon}}^{\frac{q-\mu}{\sigma}} (\mu + \sigma t)\, f_\epsilon(t)\, dt + \int_{\frac{q-\mu}{\sigma}}^{\overline{\epsilon}} q\, f_\epsilon(t)\, dt \\
&= q - (q-\mu) F_\epsilon\Big(\frac{q-\mu}{\sigma}\Big) + \sigma \int_{\underline{\epsilon}}^{\frac{q-\mu}{\sigma}} t f_\epsilon(t) dt
\end{aligned} \tag{5}$$

$$\frac{\partial\, \mathcal{S}(q)}{\partial q} = 1 - F_\epsilon\Big(\frac{q-\mu}{\sigma}\Big) \tag{6}$$

From (4) and (5), we obtain the expected value of the retailer's profit $\overline{\pi}^r$.

$$\overline{\pi}^r(r, w, q) \coloneqq \mathrm{E}[\pi^r(r, w, q)] = (r - s - b)\, \mathcal{S}(q) + (b + s - c_r - w)\, q \tag{7}$$

Following the outline in (3), now the retailer can calculate her optimal order quantity, q^* as a function of r and w.

$$\frac{\partial \overline{\pi}^r}{\partial q} = (r - s - b)\left(1 - F_\epsilon\Big(\frac{q-\mu}{\sigma}\Big)\right) + (b + s - c_r - w) = 0 \tag{8}$$

From the expressions in (6) and (7) it is readily observable that $\mathrm{E}[\pi^r(r, w, q)]$ is convex with respect to q; therefore, solving (8) yields $q^*(r, w)$ as the argmax of the retailer's expected profit.

$$q^*(r, w) = \mu(r) + \sigma(r)\, F_\epsilon^{-1}\left(\frac{r - w - c_r}{r - s - b}\right) \tag{9}$$

Substituting (9) in (5) and the result in (7), we obtain the following.

$$\begin{aligned}
\overline{\pi}^r(r, w) &= (r - w - c_r)\, \mu(r) + (r - s - b)\, \sigma(r) \int_{\underline{\epsilon}}^{z} t f_\epsilon(t) dt \\
\text{where} \quad z(r, w) &= F_\epsilon^{-1}\Big(\frac{r - w - c_r}{r - s - b}\Big)
\end{aligned} \tag{10}$$

Note that because $\underline{\epsilon} < z < \delta$, the term $\int_{\underline{\epsilon}}^{z} t f_\epsilon(t) dt$ is always negative, which in turn makes $(r - s - b)\sigma(r) \int_{\underline{\epsilon}}^{z} t f_\epsilon(t) dt$ also negative. This means that stochasticity in demand always reduces the expected profit for the retailer.

Following the procedure outlined in (3) a numerical solution to $\max_r \overline{\pi}^r(r, w)$ in (10) yields $r^*(w)$ which is in turn substituted in the expression for the manufacturer's expected profit (12).

$$\pi^m = (w - c_m)q - b(q - D)^+ = (w - cm - b)q + b\min(D, q) \tag{11}$$

$$\begin{aligned}\overline{\pi}^m(w) &= \mu\left(r^*(w)\right)\left(w - c_m\right) + \sigma\left(r^*(w)\right)\Big[(z^*(w)\Big(w - c_m - \frac{r^* - w - c_r}{r^* - s - b}\Big) \\ &\quad + b\int_{\underline{\epsilon}}^{z^*} t f_\epsilon(t)dt\Big] \qquad (12)\\ &\text{where}\quad z^*(w) = F_\epsilon^{-1}\Big(\frac{r^* - w - c_r}{r^* - s - b}\Big)\end{aligned}$$

A numerical solution to $\max\limits_{w} \overline{\pi}^m$ will complete the procedure in (3) and yield the equilibrium values of w^*, r^*, and q^*.

3.2 The Dynamic (Multi-period) Equilibrium Problems

Having solved the open-loop equilibrium problem in a single-period setting, we now proceed to the general open-loop problem in a multi-period time frame. In a multi-period setting, both the manufacturer and the retailer try to maximize their total expected profit over the whole duration of n periods. We start with analyzing the retailer's optimization problem. The manufacturer will face an structurally identical problem.

$$\max_{\mathbf{r}_k} \overline{\Pi}^r = \sum_{k=1}^{n} \alpha_k \mathrm{E}[\pi_k^r | D_1, \cdots, D_{k-1}] \tag{13}$$

where α_k is the given discount factor at period k, ($\alpha_1 = 1$).[2]

From the structure of the expected profit at a single-period in (10) and without loss of generality we can conclude that $\mathrm{E}[\pi_k^r]$ is a function of the mean and variance of the demand, which in turn may depend on the entire price history. The dependence of $\mu(\mathbf{r}_k)$ and $\sigma(\mathbf{r}_k)$ on the vector of the whole retail prices in the past makes the optimization problem (13) highly nested.

Additionally, it should be noted that as was the case in the single-period problem, the retailer's problem must be solved while considering retail price at k as a function of the manufacturing price at that period; $r_k = r_k(w_k)$. In other words, in the order presented in (3) the retailer solves her optimization problem for any feasible value of w_k offered by the manufacturer. Doing so she obtains her optimal decision variable as a function of the manufacturer's decision variable; $r_k^* = r_k^*(w_k)$. Only when the third step in the bilevel optimization problem (3) is accomplished, i.e. when the manufacturer's problem to find the numerical value of w^* is solved, can the retailer substitute this value in the functional format of her optimal decision variable. This will yield the numerical value of $r_k^*(w_k^*)$. We assume that both the players are rational and each one can solve both her own and

[2]This allows for time-dependent discounting which in turn allows for different length of periods.

the other's optimization problem. That is, essentially in the bilevel optimization problem, each agent when solving her own optimization problem is simultaneously constrained to the optimality of the other player's solution. The bilevel nature of the optimization algorithm also adds up to level of inter-dependence between decision variables and to the complexity of the ensuing equilibrium problem.

3.3 A General Solution Procedure

Using backward induction method, we begin the solution of the multi-variable nested optimization problem by analyzing the final period. It is readily observable that the only profit expression in (13) which depends on r_n is $\mathrm{E}[\pi_n^r]$. Thus maximization of $\overline{\Pi}^r$ with respect to r_n is equivalent to maximization of $\mathrm{E}[\pi_n^r]$ with respect to r_n.

$$\max_{r_n} \overline{\Pi}^r \equiv \max_{r_n} \mathrm{E}[\pi_n^r] \tag{14}$$

Moreover, at period n all of the previous decision variables and demands have become common knowledge. Therefore, given $\mathbf{r}_{n-1}^*$ and $\mathbf{D}_{n-1} = [D_1, \cdots, D_{n-1}]$ and assuming that the mapping $r_n \mapsto \mathrm{E}[\pi_n^r|\mathbf{D}_{n-1}]$ has a global maximum, this global maximum can be expressed as a function of the previous retail prices and demand history.

$$r_n^* = r_n^*(\mathbf{r}_{n-1}, \mathbf{D}_{n-1}) \tag{15}$$

Now the backward induction method proceeds to the period $n-1$ where having r_n^* as expressed in (15) enables us to conclude that maximization of $\overline{\Pi}^r$ with respect to r_{n-1} is equivalent to maximization of $\alpha_{n-1}\mathrm{E}[\pi_{n-1}^r] + \alpha_n \mathrm{E}[\pi_n^r]$ with respect to r_{n-1}. The resulting r_{n-1}^* will be a function of $(\mathbf{r}_{n-2}^*, \mathbf{D}_{n-2})$. Inserting this new function into (13) and iterating the same procedure backward in time, we obtain the vector $\mathbf{r}_n^*$.

3.4 Generalizing Demand's Dependence on Time and Prices

The microeconomic relationship between an elastic demand structure and the current price is classically portrayed as $D_k = \psi(r_k)$, where k denotes the current period.

However, not every market behaves in such a simple manner, as strategic buyers base their purchase on the (possibly repetitive) trends of previous prices to which they have become *anchored*.

In general, potential buyer's valuation of a commodity and, in turn, their purchase decision may become biased by their comparison of the current price and those of the past. For example, in a specific scenario, a price increase by 20% may reduce the customer base by, for example, 10%.

Thus, a general time-dependent model of supply and price optimization should also consider the effect of anchoring to the past prices on current demand.

We base our time-dependent model of uncertain demand on the simple premise that the probability of an item being sold at time k for the price of r_k depends on the customers' interest, which in its own right, in general, may depend on the past prices,

$$D_k \propto \mathcal{P}(\text{purchase}_k) = \mathcal{P}(\text{purchase}_k \mid \text{interested}_k) \cdot \mathcal{P}(\text{interested}_k) \tag{16}$$

$$D_k = \psi_k(R_k) \cdot \mathcal{H}_k(R_{k-1}, \cdots, R_1) \tag{17}$$

where the functional form $\mathcal{H}$ represents price history. Obtaining such a functional form may fall into the domain of behavioral economics.

Obviously, such a general demand model, which considers the effects of anchoring to the past prices, also covers the classical memoryless demand case where $H_k = 1$.

If the demand functional format remains identical (as is the case in some microeconomic analyses), i.e. $\psi_k(R_k) = \psi(R_k)$, the procedure outlined in Section 3.3 turns into a *repeated game*.

In contrast, a fully dynamic game emerges when the functional formats for $\psi_k(R_k)$s vary with time, adding to the level of non-autonomy in the ensuing equilibrium problems. In addition, assuming demand's dependence on past prices, i.e. $\mathcal{H}_k = \mathcal{H}_k(R_{k-1}, \cdots, R_1)$, makes the equilibrium problems highly nested.

In Theorems 3.1, 4.1, and 5.1 we propose solution algorithms for the general non-autonomous dynamic games. Obviously, the proposed solution algorithms are significant generalizations which among others, cover the trivial n-periodic repeated games as well as the non-trivial fully non-autonomous memory-less cases.

3.4.1 Memory-based Uncertain Demand

In our expression for memory-based demand, we embed a class of functional forms within the uncertain demand structure such that the demand at each period be not only a function of price at that period, but also carry the effects of pricing policies and the demand in the previous periods. We will refer to these functional forms as memory functions and denote them by $\Phi_k(\mathbf{r}_{k-1})$.

As discussed earlier, the additive-multiplicative structure of demand in (1) enables us to cover general demand expressions with non-constant coefficient of variation. Here, for the sake of greater generality, we consider the coefficient of variation of demand to be a function of the retail price as well.

$$CV_{D_k} = CV_{D_k}(r_k) \tag{18}$$

In this paper we limit our analysis to the reasonable case where previous prices scale the level of the current demand.

$$\begin{aligned} D_k(\mathbf{r}_k) &= \Phi_k(\mathbf{r}_{k-1}) d_k(r_k) \\ \text{where } d_k(r_k) &= \mu_k(r_k) + \sigma_k(r_k)\epsilon_k \end{aligned} \tag{19}$$

Comparing (19) with (1) we observe that

$$\begin{aligned} \widetilde{\mu}_k(\mathbf{r}_k) &= \Phi_k(\mathbf{r}_{k-1})\mu_k(r_k) \\ \widetilde{\sigma}_k(\mathbf{r}_k) &= \Phi_k(\mathbf{r}_{k-1})\sigma_k(r_k). \end{aligned} \tag{20}$$

The memory functions embedded within the uncertain demand $D_k(\mathbf{r}_k)$ must be such that at the $k+1$st period, $\Phi_{k+1}(\mathbf{r}_k)$ retains the information from the entire previous periods' memories while being affected by the last piece of information that has becomes available, i.e. r_k. This feature can be obtained by the following expression.

$$\frac{\Phi_{k+1}}{\Phi_k} = \phi_k(r_k) \tag{21}$$

We call these $\phi_k(r_k)$s the memory elements. Notice that the possibility of having different functional forms for ϕ_ks in different periods enables our demand structure to cover more non-autonomy. With the memory structure in (21), we will have:

$$\begin{aligned} \Phi_k(\mathbf{r}_{k-1}) &= \prod_{i=1}^{k} \phi_i(r_{i-1}) \\ \Phi_1(\cdot) &= \phi_1(\cdot) = 1 \end{aligned} \tag{22}$$

3.5 Embedding the Demand Structure in the Equilibrium Problems

The general construction outlined in Section 3.3 is sufficiently explicit to enable solutions of the problem for most choices of functions $\widetilde{\mu}$ and $\widetilde{\sigma}$. However, as discussed in section 3.2 the resulting bilevel optimization problem in its multi-period setting is so deeply nested that one cannot expect to find an analytical solution.

The importance of our memory-based demand scheme lies in the structure it will create when embedded inside the expressions for the channel members' expected profits. At each period k, we denote the local-in-time profit for the retailer and the manufacturer by $\widetilde{\pi}_k^r$ and $\widetilde{\pi}_k^m$, respectively.

The memory-based expression for demand at each period D_k is given in (19). Due to linearity of the expressions for $\widetilde{\pi}_k^r$ and $\widetilde{\pi}_k^m$ with respect to D in the single-period case, it is straightforward to see that for the kth period, the resulting expressions for the order quantity and the expected values of the profits will be as below.

$$D_k(k,\mathbf{r}_k)=\tilde{\mu}_k(\mathbf{r}_k)+\tilde{\sigma}_k(\mathbf{r}_k)\,\epsilon_k=\Phi_k(\mathbf{r}_{k-1})\,\Big[\mu_k(r_k,k)+\sigma_k(r_k,k)\,\epsilon_k\Big] \tag{23}$$

$$\begin{aligned}\mathrm{E}[\widetilde{\pi}_k^r]&=(r_k-w_k-c_{r_k})\,\widetilde{\mu}_k(\mathbf{r}_k)+(r_k-s_k-b_k)\,\widetilde{\sigma}_k(\mathbf{r}_k)\int_{\underline{\epsilon}_k}^{z_k}tf_\epsilon(t)dt\\&=\overbrace{\Big[(r_k-w_k-c_{r_k})\mu_k(r_k)+(r_k-s_k-b_k)\,\sigma_k(r_k)\int_{\underline{\epsilon}_k}^{z_k}tf_\epsilon(t)dt\Big]}^{:=\overline{\pi}_k^r}\cdot\Phi_k(\mathbf{r}_{k-1})\\&\text{where}\quad z_k(r_k,w_k)=F_\epsilon^{-1}\Big(\frac{r_k-w_k-c_{r_k}}{r_k-s_k-b_k}\Big)\end{aligned} \tag{24}$$

We refer to $\overline{\pi}_k^r$ as scaled expected profit for the retailer at k. Thus (24) can be simplified as below.

$$\mathrm{E}[\widetilde{\pi}_k^r]=\overline{\pi}_k^r\cdot\Phi_k(\mathbf{r}_{k-1}) \tag{25}$$

The manufacturer's local-in-time expected profit is calculated as below.

$$\begin{aligned}\mathrm{E}[\widetilde{\pi}_k^m]=\Bigg\{&\mu_k\left(r_k^*(w_k)\right)\big(w_k-c_{m_k}\big)+\sigma_k\left(r_k^*(w_k)\right)\Big[(z_k^*(w_k)\Big(w_k-c_{m_k}-\frac{r_k^*-w_k-c_{r_k}}{r_k^*-s_k-b_k}\Big)\\&+b_k\int_{\underline{\epsilon}_k}^{z_k^*}tf_\epsilon(t)dt\Big]\Bigg\}\cdot\Phi_k(\mathbf{r}_{k-1}^*)\\&\text{where}\quad z_k^*(w)=F_\epsilon^{-1}\Big(\frac{r_k^*-w_k-c_{r_k}}{r_k^*-s_k-b_k}\Big)\end{aligned} \tag{26}$$

Analogous to the single-period case, the numerical value for the optimal order quantity is then obtained from the following expression.

$$q_k^*=\Phi_k(\mathbf{r}_{k-1}^*)\,\Big[\mu_k(r_k^*)+\sigma_k(r_k^*)\,F_{\epsilon_k}^{-1}\Big(\frac{r_k^*-w_k^*-c_{r_k}}{r_k^*}\Big)\Big] \tag{27}$$

Similarly we refer to the term inside the curly brackets in (26) as the scaled expected profit for the manufacturer at k and denote it by $\overline{\pi}_m^r$. Whence (26) is simplified as below.

$$\mathrm{E}[\widetilde{\pi}_k^m]=\overline{\pi}_k^m\cdot\Phi_k(\mathbf{r}_{k-1}^*) \tag{28}$$

It is important to note that in general, the argmax of the expected profit in a specific period k for either supplier, i.e. the result of $\max\limits_{r_k,m_k}\mathrm{E}[\widetilde{\pi}_k^{r,m}]$ is not equal to the value of the kth optimal decision variable for that supplier when the objective function is the whole expected profit within the periods 1 to n. In other words, in general

$$\max_{r_k,m_k}\mathrm{E}[\widetilde{\pi}_k^{r,m}]\not\equiv\max_{r_k,m_k}\overline{\Pi}^{r,m}. \tag{29}$$

We refer to the results of the RHS of (29) as myopic solutions and to those of its LHS as the holistic ones. Our objective is to find the vectors of the latter – those decision variables which considering the effect of the pricing in the past on current and future demand, manipulate the demand such that they yield highest amounts of expected profits for each decision maker over the time interval between 1 and n.

To that end, we begin by analyzing the retailer's optimization problem and re-write the general optimization problem in (13) using the results of (25).

$$\begin{aligned} \max_{\mathbf{r}_n} \overline{\Pi}^r &= \overline{\pi}_1^r(r_1, w_1, q_1) + \cdots + \alpha_k \Phi_k(\mathbf{r}_{k-1})\, \overline{\pi}_k^r(r_k, w_k, q_k) \\ &\quad + \cdots + \alpha_n \Phi_n(\mathbf{r}_{n-1})\, \overline{\pi}_n^r(r_n, w_n, q_n) \end{aligned} \tag{30}$$

Analogous to the approach adopted in Section 3.3, we observe that the variable r_n appears only in the final discounted profit term–more precisely in $\overline{\pi}_n^r$. Thus following the backward induction process, we begin the optimization from the final period.

$$\max_{r_n} \overline{\Pi}^r(\mathbf{r}_n) \equiv \max_{r_n} \overline{\pi}_n^r(r_n) \tag{31}$$

At each period k we define J_k^r as the discounted expected value of the profit obtained from that period onward, i.e. within the time interval $\{k, \cdots, n\}$.

$$J_k^r = \alpha_k \Phi_k(\mathbf{r}_{k-1}) \overline{\pi}_k^r(r_k) + \cdots + \alpha_n \Phi_n(\mathbf{r}_{n-1}) \overline{\pi}_n^r(r_n) \tag{32}$$

Notice that $J_1^r = \overline{\Pi}^r$. We also observe that in this structure, beginning from the last period, the variable r_k in Π^r appears for the first time in the expression for J_k^r. Having solved the RHS of (31) we obtain r_n^* and proceed to the previous period $n-1$. Knowing r_n^* means that in the holistic optimization problem (30) the unknown variable r_{n-1} appears only in the two final terms for the expected profit. This is stated below.

$$\begin{aligned} J_{n-1}^r(\mathbf{r}_{n-1}) &= \alpha_{n-1} \Phi_{n-1}(\mathbf{r}_{n-2}) \overline{\pi}_{n-1}^r(r_{n-1}) + \alpha_n \Phi_n(\mathbf{r}_{n-1}) \overline{\pi}_n^r(r_n^*) \\ &= \Phi_{n-1}(\mathbf{r}_{n-2}) \overbrace{\left[\overline{\pi}_{n-1}^r(r_{n-1}) + \frac{\alpha_n}{\alpha_{n-1}} \phi_{n-2}(r_{n-1}) \underbrace{\overline{\pi}_n^r(r_n^*)}_{\text{given}} \right]}^{:= \mathfrak{J}_{n-1}^r \text{: a function of } r_{n-1} \text{ only}} \end{aligned} \tag{33}$$

Thus the problem of finding the optimal r_{n-1}^* boils down to the following single-variable optimization problem.

$$\max_{r_{n-1}} \overline{\Pi}^r(\mathbf{r}_{n-1}) \equiv \max_{r_{n-1}} J_{n-1}^r(\mathbf{r}_{n-1}) \equiv \max_{r_{n-1}} \mathfrak{J}_{n-1}^r(r_{n-1}) \tag{34}$$

Going backward in time, we can generalize this procedure as shown in (35), given that $\alpha_1 = 1$ and

$\Phi_1(\cdot) = 1$.

$$J_k^r \coloneqq \alpha_k \underbrace{\Phi_k(\mathbf{r}_{k-1})}_{\text{price history}} \overbrace{\left(\overline{\pi}_k^r(r_k) + \phi_{k+1}(r_k) \underbrace{\left[\frac{\alpha_{k+1}}{\alpha_k}\,\overline{\pi}_{k+1}^r(r_{k+1}^*) + \cdots + \frac{\alpha_n}{\alpha_k}\,\overline{\pi}_n^r(r_n^*) \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*)\right]}_{\coloneqq \mathcal{F}_k^r = \text{ expected (future) values, given at } k\text{th period}}\right)}^{\coloneqq \mathfrak{J}_k^r(r_k)}$$

$$\max_{r_k} J_k^r \tag{35}$$

In general, we define $\mathcal{F}_k^r$, the scaled expected future profit within $\{k+1, \cdots, n\}$ and $\mathfrak{J}_k^r$, the scaled expected profit within $\{k, \cdots, n\}$, as below.

$$\mathcal{F}_k^r \coloneqq \frac{1}{\alpha_k} \sum_{j=k+1}^{n} \prod_{i=k+2}^{j} \phi_i(r_{i-1}^*) \cdot \alpha_j \overline{\pi}_j^r(r_j^*) \tag{36}$$

$$\mathfrak{J}_k^r(r_k) \coloneqq \overline{\pi}_k^r + \phi_{k+1}(r_k)\mathcal{F}_k^r \tag{37}$$

As it is demonstrated in (35), when the backward induction process reaches the kth period, the scaled profit expected to gain in the future denoted by $\mathcal{F}_k^r$ has been determined and is treated as a constant. We also observe the following relationship between $\mathfrak{J}_{k+1}^r$ and $\mathcal{F}_k^r$.

$$\mathfrak{J}_{k+1}^r(r_{k+1}^*) = \frac{\alpha_k}{\alpha_{k+1}} \mathcal{F}_k^r \quad 1 \le k < n \tag{38}$$

Note that, unlike $\mathcal{F}_k^r$ and $\mathfrak{J}_{k+1}^r$, J_{k+1}^r includes the entire pricing history $\Phi_k(\mathbf{r}_{k-1})$ and hence is not known at k. In fact, J_k^rs are not resolved until the backward induction reaches $k = 1$. The effect of the past represented by $\Phi_k(\mathbf{r}_{k-1})$, though not yet determined by backward induction, is factorized in (35) such that it only scales the expected profit from k onward. Therefore, we will have:

$$\max_{r_k} \overline{\Pi}^r(\mathbf{r}_n) \equiv \max_{r_k} J_k^r(\mathbf{r}_k) \equiv \max_{r_k} \mathfrak{J}_k^r(r_k) \tag{39}$$

Combining (35) and (38) we can summarize the retailer's part of the multi-period bilevel optimization in the following recursive procedure.

$$\begin{gathered} \mathcal{F}_n^r = 0 \quad \text{no future earning after } n \\ \max_{r_k} \mathfrak{J}_k^r(r_k) = \max_{r_k} \left[\overline{\pi}_k^r(r_k) + \phi_{k+1}(r_k)\mathcal{F}_k^r\right] \quad k = n, \cdots, 1 \text{ (backward) } \to \text{ yields}\, r_k^* \\ \mathcal{F}_{k-1}^r = \frac{\alpha_{k-1}}{\alpha_k}\, \mathfrak{J}_k^r(r_k^*) \quad k = n, \cdots, 2 \text{ (backward)} \end{gathered} \tag{40}$$

From the procedure outlined in (40) it is readily observable that, in general, the holistic optimal retail prices (r_k^*s) are not the optimizers of individual $\overline{\pi}_k^r$s. The only situation where $r_k = argmax(\overline{\pi}_k^r)$ is when $\phi_{k+1} = C_k$, where C_k is a constant. A scenario in which all the memory elements are constants, will create identical repeated games at different periods.

The same structure is employed to decouple the nested optimization problems of the manufacturer. Notice that as in the single-period case in (3), each r_k^* is obtained as a function of manufacturing price at k, i.e. $r_k^* = r_k^*(w_k)$.

$$\max_{\mathbf{w}_n} \overline{\Pi}^m(\mathbf{w}_n) = \max_{\mathbf{w}_n} \sum_{k=1}^{n} \alpha_k \Phi_k(\mathbf{r}_k^*) \overline{\pi}_k^m(w_k) \tag{41}$$

$$J_k^m(\mathbf{w}_k) = \sum_{i=k}^{n} \alpha_i \Phi_i(\mathbf{r}_i^*) \overline{\pi}_i^m(\mathbf{w}_i) \tag{42}$$

$$\max_{w_k} J_k^m(\mathbf{w}_k) = \alpha_k \Phi_k(\mathbf{r}_{k-1}^*) \overbrace{\left[\overline{\pi}_k^m(w_k) + \phi_{k+1}\big(r_k^*(w_k)\big)\mathcal{F}_k^m\right]}^{\mathfrak{J}_k^m(w_k)} \tag{43}$$

Where $\mathcal{F}_k^m$ in (43) is the scaled expected value of future (time interval within $\{k+1, \cdots, n\}$) discounted profit. When the backward induction process reaches the kth period, $\mathcal{F}_k^m$ has already been calculated. This makes $\mathfrak{J}_k^m$ a function of only w_k.

$$\begin{gathered} \mathcal{F}_k^m = \frac{\alpha_{k+1}}{\alpha_k}\, \overline{\pi}_{k+1}^m(w_{k+1}) + \cdots + \frac{\alpha_n}{\alpha_k}\, \overline{\pi}_n^m(w_n) \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*) \\ \mathcal{F}_n^m = 0 \end{gathered} \tag{44}$$

Finally, we can decouple the nested $n-$variable optimization problem into n single variable optimization problems.

$$\max_{w_k} \overline{\Pi}^m(\mathbf{w}_n) \equiv \max_{w_k} J_k^m(\mathbf{w}_k) \equiv \max_{w_k} \mathfrak{J}_k^m(w_k) \tag{45}$$

Anaglogous to the retailer's case, the manufacturer's part of the multi-period bilevel optimization is outlined in the follwing recusrsive procedure.

$$\begin{gathered} \mathcal{F}_n^m = 0 \quad \text{no future earning after } n \\ \max_{w_k} \mathfrak{J}_k^m(w_k) = \max_{w_k} \left[\overline{\pi}_k^r(w_k) + \phi_{k+1}\big(r_k(w_k)\big)\mathcal{F}_k^m\right] \quad k = n, \cdots, 1 \rightarrow \text{ yields } w_k^* \\ \mathcal{F}_{k-1}^m = \frac{\alpha_k}{\alpha_{k-1}}\, \mathfrak{J}_k^m(w_k^*) \quad k = n, \cdots, 2 \end{gathered} \tag{46}$$

Finding the numerical values of w_k^*s allows us follow the procedure outline in (3) in reverse order and calculate the numerical values of $r_k^*(w_k^*)$s which in turn yield q_k^*s. It is now evident that the results of (45) and (39), (w_n^*, r_n^*, q_n^*) are the optimal decision variables of the holistic objective function and not those of individual myopic ones.

We state the final results of this section in the following two theorems.

Theorem 3.1.

Let n be the number of periods and assume that the uncertain demand at period k is given by

$$D_k(\boldsymbol{r}_k) = \Phi_k(\boldsymbol{r}_{k-1})\Big(\mu_k(r_k) + \sigma_k(r_k)\epsilon_k\Big) \tag{47}$$

where

$$\Phi_1(\cdot) = \phi_1(\cdot) = 1, \quad \Phi_k(\boldsymbol{r}_{k-1}) = \prod_{i=1}^{k} \phi_i(r_{i-1})$$

and where ϵ_ks are continously distributed with $\mathrm{E}[\epsilon_k] = 0$ *and* $\mathrm{Var}[\epsilon_k] = 1$ *for all k. with $f_{\epsilon_k} > 0$ a.e. on their supports. If for each k the single-period Stackelberg problem below has an equilibrium at r_k^* and w_k^**

$$\begin{aligned} \mathfrak{J}_k^r &= \overline{\pi}_k^r + \phi_{k+1}(r_k)\mathcal{F}_k^r \\ \mathfrak{J}_k^m &= \overline{\pi}_k^m + \phi_{k+1}(r_k)\mathcal{F}_k^m \end{aligned} \tag{48}$$

where $\mathcal{F}_k^r$ and $\mathcal{F}_k^m$ are found recursively from:

$$\begin{aligned} &\mathcal{F}_n^r = 0, \quad \mathcal{F}_k^m = 0 \\ &\mathcal{F}_{k-1}^r = \frac{\alpha_{k-1}}{\alpha_k}\,\mathfrak{J}_k^r(r_k^*), \quad \mathcal{F}_{k-1}^m = \frac{\alpha_{k-1}}{\alpha_k}\,\mathfrak{J}_k^m(w_k^*), \quad k = n, \cdots, 2 \end{aligned} \tag{49}$$

and

$$\begin{aligned} \overline{\pi}_k^r &= (r_k - w_k - c_{r_k})\mu_k(r_k) + (r_k - s_k - b_k)\,\sigma_k(r_k)\int_{\underline{\epsilon}_k}^{z_k} t f_\epsilon(t)dt \\ \overline{\pi}_k^m &= \mu_k\left(r_k^*(w_k)\right)\left(w_k - c_{m_k}\right) + \sigma_k\left(r_k^*(w_k)\right)\Big[(z_k^*(w_k)\Big(w_k - c_{m_k} - \frac{r_k^* - w_k - c_{r_k}}{r_k^* - s_k - b_k}\Big) \\ &\quad + b_k\int_{\underline{\epsilon}_k}^{z_k^*} t f_\epsilon(t)dt\Big] \end{aligned} \tag{50}$$

then the bilevel (Stackelberg) optimization problem

$$\begin{aligned} \overline{\Pi}^r &= \sum_{k=1}^{n} \alpha_k \mathrm{E}[\pi_k^r] = \sum_{k=1}^{n} \alpha_k \Phi_k(\boldsymbol{r}_{k-1})\overline{\pi}_k^r \\ \overline{\Pi}^m &= \sum_{k=1}^{n} \alpha_k \mathrm{E}[\pi_k^m] = \sum_{k=1}^{n} \alpha_k \Phi_k(\boldsymbol{r}_{k-1}^*)\overline{\pi}_k^m \end{aligned} \tag{51}$$

has an equlibrium at $\boldsymbol{r}_n^* = [r_1^*, \cdots, r_n^*]$ and $\boldsymbol{w}_n^* = [w_1^*, \cdots, w_n^*]$.
The optimal order quantity at k is then calculated as below.

$$q_k^* = \Phi_k(\boldsymbol{r}_{k-1}^*)\left[\mu_k(r_k^*) + \sigma_k(r_k^*)\,F_{\epsilon_k}^{-1}\Big(\frac{r_k^* - w_k^* - c_{r_k}}{r_k^*}\Big)\right] \tag{52}$$

Next, we prove that the results of Theorem 3.1 are subgame perfect.

Proposition 3.2.

The equilirbium obtained in Theorem.3.1 is subgame perfect. That is, subsets of the equilibrium results covering the time interval between an arbitrary period j and n, i.e. $[r_j^*, \cdots, r_n^*]$ and

$[w_j^*, \cdots, w_n^*]$ *and, a fortiori, their resulting* $[q_j^*, \cdots, q_n^*]$ *will also constitue an equilibrium for the corresponding subgame of the original problem, covering that time-inetrval:*

$$\begin{aligned} J_j^r &= \alpha_j \Phi_j(\boldsymbol{r}_{j-1}) \overline{\pi}_j^r(r_j) + \cdots + \alpha_n \Phi_n(\boldsymbol{r}_{n-1}) \overline{\pi}_n^r(r_n) \\ J_j^m &= \alpha_j \Phi_j(\boldsymbol{r}_{j-1}^*) \overline{\pi}_j^r(w_j) + \cdots + \alpha_n \Phi_n(\boldsymbol{r}_{n-1}^*) \overline{\pi}_n^r(w_n) \end{aligned} \tag{53}$$

Proof. (By induction)

We have to prove that if $\{r_j^*, \cdots, r_n^*\}$ and $\{w_j^*, \cdots, w_n^*\}$ are subsets of the equilibrium results for $[\overline{\Pi}^r, \overline{\Pi}^m, 1:n]$, then they also constitue an equilibrium for $[J_j^r, J_j^m, j:n]$.

Beginning from the final period, we analyse the two agents' equilibrium problem. In the expressions for both J_k^r and $\overline{\Pi}^r$ the variable r_n appears in $\overline{\pi}_n^r(r_n)$ only. The same logic is applicable to the manufacturer's solution procedure.

$$\max_{r_n} J_k^r \equiv \max_{r_n} \overline{\pi}_n^r \equiv \max_{r_n} \Pi^r$$

$$\max_{w_n} J_k^m \equiv \max_{w_n} \overline{\pi}_n^m \equiv \max_{w_n} \Pi^m$$

Thus, at n the conclusion is obvious. The rest of the proof for an arbitrary k, $j < k < n$ has been argued in detail within the discussion resulting in (39) and (45). □

In Section 5, we will use the subgame perfection of the open-loop equilibrium in the analysis of the closed-loop equilibrium in a price-postponement scenario.

4 Post-observation Equilibrium: Postponing the Order Quantity

In this section, we analyze the closed-loop equilibrium in an order-postponement scenario. Similar to the open-loop analysis, we begin by studying the single-period case and later generalize the approach for the memory-based multi-period problem.

At each period analyze the order-postponement scenario in two steps, happening before and after the realization of the demand uncertainty. At the beginning of the period, both decision-makers are aware that the order-quantity will be sent to the retailer after demand uncertainty has been resolved. That is, they both know that $q = D(r, \epsilon)$. Thus they both consider the following equation in their further calculations.

$$\begin{aligned} \min(D, q) &= D \quad \text{sales} \\ \mathcal{S} = \mathrm{E}[\min(D, q)] &= \mathrm{E}[D] = \mu \quad \text{expected sales} \end{aligned} \tag{54}$$

In the first step at each period, both the retailer and the manufacturer substitute (54) in their respective optimization expression as outlined in Section 3.1. The rest of the precedure is exactly the same as the one in the open-loop equilibrium solution process.

$$\begin{aligned} \pi^r &= (r - c_r - w)q \quad \pi^m = (w - c_m)q \\ \overline{\pi}^r &= (r - c_r - w)\mu(r) \quad \max_r \overline{\pi}^r \to r^*(w) \\ \overline{\pi}^m &= (w - c_m)\mu\big(r^*(w)\big) \quad \max_w \overline{\pi}^m \to w^*, r^* \end{aligned} \tag{55}$$

The manufacturer, then, sets w^* and the retailer sets r^* as her own retail price. Note that at the end of this open-loop solution, the optimal order quanity will be $q^* = \mu(r^*)$.

However, instead of ordering q^* items, the retailer postpones ordering until after she has observed demand uncertainty $\hat{\epsilon}$. After observing $\hat{\epsilon}$, at the second step the retailer orders $\hat{q}$ items to the manufacturer.

$$\hat{q} = \mu(r^*) + \sigma(r^*)\,\hat{\epsilon} = D(r^*, \hat{\epsilon}) \tag{56}$$

The real profit for the two players is then calculated as below.

$$\pi^r = (r^* - w^* - c_r)\,\hat{q} \tag{57}$$

$$\pi^m = (w^* - c_m)\,\hat{q} \tag{58}$$

Note that as the retailer's order quantity addresses the entire demand, there is no need to consider salvage price and buy back contract in the profit expressions.

4.1 Multi-period Equilibria

In this section we generalize the two step single-period optimization procedure for a multi-period scenario. Similar to the single-period scenario described in (55), at the first step in each period k, the manufacturer and the retailer set their corresponding prices obtained from the open-loop equilibria solutions. The retailer, however, postpones her order quantity until after demand randomness at that period has been resolved; i.e. $\hat{\epsilon}_k$ has been observed.

At the second step, when the retailer observes demand uncertainty $\hat{\epsilon}_k$, it is obvious that in order to optimize her local-in-time profit, i.e. $(r_k^* - w_k^* - c_{r_k})\,\hat{q}_k$, she must pick the highest possible value for $\hat{q}_k$ which will be $D_k = \Phi_k(\mathbf{r}_{k-1}^*)\Big(\mu_k(r_k^*) + \sigma_k(r_k^*)\hat{\epsilon}_k\Big)$. Notice that $r_k^* - w_k^* - c_{r_k} > 0$, and this net price is now fixed so the retailer cannot influence the future demand by her choice of price. Hence she optimizes local profit by maximizing local sale. Thus the optimal order quantity for the retailer is equal to real demand.

$$\hat{q}_k = D_k = \Phi_k(\mathbf{r}_{k-1}^*)\left[\mu_k(r_k^*, k) + \sigma_k(r_k^*, k)\,\hat{\epsilon}_k\right] \tag{59}$$

The manufacturer on the other hand, may either benefit from or be adversely impacted by the retailer's deviation from q_k^* depending on whether $\hat{q}_k > q_k^*$ or $\hat{q}_k < q_k^*$ respectively. Moreover, a deviation from q_k^* to $\hat{q}_k$ does not change the initial condition at $k+1$th period, i.e. the memory function $\Phi_{k+1}(\mathbf{r}_k^*)$. Thus in the beginning of the $k+1$th period the manufacturer faces exactly the same expected profit optimization problem as the corresponding one in the open-loop equilibrium problem. In other words, a change in order quantity at period k does not affect the *expected* profit of the manufacturer in the future periods because the retailer has not deviated from the r_k^* obtained by open-loop equilibrium solution. Besides the manufacturer has no strategic means to influence the occurrence of $\hat{\epsilon}_k$. Thus she will not deviate from previously calculated w_{k+1}^*. Hence the results $[w_k^*, r_k^*, \hat{q}_k]$ constitute the *ex-post* equilibrium state at k. At the end of the nth period, the real total discounted profit for the manufacturer and retailer will be as below.

$$\Pi^r = \sum_{k=1}^{n} \alpha_k (r_k^* - w_k^* - c_{r_k})\, \hat{q}_k \tag{60}$$

$$\Pi^m = \sum_{k=1}^{n} \alpha_k (w_k^* - c_{m_k})\, \hat{q}_k \tag{61}$$

4.2 Comparison between open loop and closed loop profits

We consider a scenario with two different retailers facing the same uncertain demand (ϵ_k). One of the two retailers does not postpone her declaration of order quantity; instead she adheres to the pre-observation optimal order quantity, q_k^*. The other retailer postpones her declaration of optimal order quantity $\hat{q}_k$ until after observation of $\hat{\epsilon}_k$. In this hypothetical scenario, they both face the same $\hat{\epsilon}_k$. We refer to the (real) profit obtained by the non-postponing retailer as $\pi_{OL_k}^r$ (open-loop profit) and to the postponing one's as $\pi_{CL_k}^r$ (closed-loop profit).

Below we show that for the retailers, the closed-loop profit is always greater than or equal to the open-loop profit. Hence, for the retailer, it is always beneficial to postpone her declaration of the order quantity until after she has observed demand uncertainty.

Theorem 4.1.

Between two retailers who will face the same uncertain demand D_k, the profit obtained by the one who postpones her order quantify $\hat{q}_k$ until she observes the demand uncertainty $\hat{\epsilon}_k$ is higher than or equal to that of the retailer who instead of postponing, adheres to the order quantity obtained from the open-loop equilibrium $q_k^* = \mathrm{E}[D_k] = \Phi_k(\boldsymbol{r}_{k-1}^*)\mu_k(r_k^*)$.

Proof.

We have to show that $\pi_{CL_k}^r \geq \pi_{OL_k}^r$.

$$\pi^r_{CL_k} = r^*_k D_k - w^*_k \hat{q}_k \overset{(D_k=\hat{q}_k)}{=} (r^*_k - w^*_k) D_k$$

$$\pi^r_{OL_k} = r^*_k \min\left(\overbrace{\Phi_k(\mathbf{r}^*_{k-1})\left[\mu_k(r^*_k) + \sigma_k(r^*_k)\,\hat{\epsilon}_k\right]}^{D_k},\, q^*_k\right) - w_k q^*_k = r^* \min(D_k, q^*_k) - w^*_k q^*_k$$

Comparing with a no-postponement scenario, we have to analyze two possible situations.

1 - When the retailer under-orders: $q^*_k < D_k$, $\min(D_k, q^*_k) = q^*_k$

Then $\pi^r_{OL_k} = (r^*_k - w^*_k) q^*_k < (r^*_k - w^*_k) D_k = \pi^r_{CL_k}$

2- When the retailer over-orders: $q^*_k > D_k$, $\min(D_k, q^*_k) = D^*_k$

$\pi^r_{OL_k} = r^*_k D_k - w^*_k q^*_k < r^*_k D_k - w^*_k D_k = \pi^r_{CL_k}$

Note that in the proof above, we compared $\hat{q}_k = D_k$ with a general $q^*_k \neq \hat{q}_k$. We did not use the fact that $q^*_k = \mathrm{E}[D_k]$ which stems from the a priori knowledge of the decision-makers about an order-postponement taking place in the second step. The proof thus shows that for a given r^*_k, an order quantity equal to the resulting uncertain demand will outperform any other arbitrary order quantity, including the one prescribed by the open-loop solution.

Equality ($\pi^r_{CL_k} = \pi^r_{OL_k}$) happens when the mean of demand is equal to the real demand, $q^*_k = \mathrm{E}[D_k] = D_k$. □

Remark 4.2.

Notice that while the order quantities and retail prices prescribed by the open-loop equilibrium guarantee a non-negative *expected* profit for the retailer, the *real* retail profit can become negative in extreme over-ordering cases, i.e when $q^*_k \gg D_k$. In contrast, order quantity postponement guarantees an always-positive profit for the retailer.

Corollary 4.3.

In the hypothetical scenario described in theorem 4.1, the holistic profit for the postponing retailer is higher than or equal to that of the non-postponing retailer.

Proof.

$$\Pi^r_{CL} = \sum_{k=1}^{n} \alpha_k \pi^r_{CL_k} \geq \sum_{k=1}^{n} \alpha_k \pi^r_{OL_k} = \Pi^r_{OL}$$

□

4.2.1 Postponement and Channel Profit

Corollary 4.4.

In a hypothetical scenario with two price-setting decentralized channels, the aggregate channel profit *for a channel with an order-postponing retailer is higher than or equal to that of the channel with a non-postponing retailer.*

Proof.
We denote the channel profit for the postponing channel at period k by $\pi^c_{P_k}$ and for the non-postponing channel by $\pi^c_{NP_k}$.

$$\pi^c_{P_k} = (r^*_k - c_{m_k})D_k$$

$$\pi^c_{NP_k} = r^*_k \min(D_k, q^*_k) - c_{m_k} q^*_k$$

$$\begin{aligned}\pi^c_{P_k} - \pi^c_{NP_k} &= (r^*_k - c_{m_k})D_k - r^*_k \min(D_k, q^*_k) + c_{m_k} q^*_k \\ &\geq (r^*_k - c_{m_k})\big(D_k - \min(D_k, q^*_k)\big) \geq 0\end{aligned}$$

□

Remark 4.5. Corollary 4.4 shows that in an order-postponement scenario, despite the fact that the manufacturer may lose potential profits due to postponement, the channel always benefits from postponement.

Corollary 4.6.
In a hypothetical scenario with two price-setting centralized channels, the channel that postpones supplying the market until after demand uncertainty has been resolved will benefit higher than or equal to a non-postponing channel.

Proof.
It suffices to show that the profit expression for centralized channels is identical to that of a retailer.

$$\pi^c_k = \pi^r_k + \pi^m_k = r \min(D_k, q_k) - c_{m_k} q_k$$

We observe that a centralized channel is equivalent to a retailer who has to pay only a given manufacturing cost c_{m_k} at each period. Thus the result of Theorem 4.1 is applicable to centralized channels. □

5 Price Postponement

In this section, we analyse another closed-loop variant of the problem, in which the retailer postpones the announcement of retail price until after the demand uncertainty has been resolved. We use essentially the same notations for the model variables and parameters as those in Section 3. We use $\hat{r}_k$, and $\hat{q}_k$ to denote the optimal retail price and order quantity, respectively.

Here, again the two players start from the open-loop equilibrium solutions and obtain $\mathbf{r}_n^*$, $\mathbf{w}_n^*$, and $\mathbf{q}_n^*$. At the beginning of the first period the manufacturer sets w_1^* and the retailer orders $\hat{q}_1 = q_1^*$. But the retailer postpones the announcement of the retail price $\hat{r}_1$ until after she observes $\hat{\epsilon}_1$. In sections 5.2 and 5.3.1 we solve the equilibrium problems for each player to obtain the optimal post-observation decision variables at an arbitrary period k.

Furthermore, since in the price-postponement scenario the entire demand is not necessarily addressed by the retailer, for the sake of generality we must also consider a (possibly time-dependent) salvage price for the retailer, and a buy back contract between the two agents.

While any non-zero buy back price in a single-period setting is desirable for the retailer, as it reduces her expected loss due to uncertainty, the manufacturer may try to hedge the amount of possible loss she may incur. In our game setting, the manufacturer who is the leader imposes the following constraint on the buy back price to ensure a non-negative profit at each period:

$$0 \leq b_k \leq w_k - c_{m_k}. \tag{62}$$

5.1 Observing the Feedback: Closing the Loop

In this scenario, at the beginning of the kth period the manufacturer sets the w_k^* and the retailer orders $\hat{q}_1 = q_k^*$ items. However, the retailer postpones her declaration of the retail price until after she has observed the demand uncertainty $\hat{\epsilon}_k$.

It should be noted that while in the ex-ante analysis of the no-postponement equilibria states, we used the dynamic programming method known as *backward induction*, here in the ex-post analysis of price-postponement scenario we use a forward induction approach. Thereby, we incorporate the newly-revealed information in the form of feedback signals into the decision-making process. This is due to the fact that we now change future demand by our postponement.

5.2 Post-observation Bilevel Optimization

In our analysis of the retail price-postponement scenario, we divide the decision-making process into two steps. First, at the beginning of each period k, both the retailer and the manufacturer solve the expected profit optimization (equilibrium) problem in a Stackelberg framework within the time interval $\{k, \cdots, n\}$. The manufacturer then declares the equilibrium wholesale price and the retailer submits her order quantity to the manufacturer. However, the retailer does not declare her retail price to the market. Instead, she postpones doing so until after she observes demand uncertainty.

In the second step, having observed $\hat{\epsilon}_k$, the retailer incorporates this new information and solves the equilibrium problem anew while considering the manufacturer's response for the next periods. That is, after observing $\hat{\epsilon}_k$ the retailer tries to find optimal retail prices within $\{k, \cdots, n\}$ while being subject to the optimality of the wholesale prices within $\{k+1, \cdots, n\}$. The equilibrium solution will provide the retailer with her post-observation optimal retail price vector $[\hat{r}_k, \cdots, \hat{r}_n]$. Then she declares the first element of her newly found optimal price vector, $\hat{r}_k$, to the market.

We begin the analysis of the equilibrium problem from the first period and using forward induction reasoning delineate a general optimization procedure for all periods. At the first step in the first period, both the retailer and the manufacturer solve the equilibrium problem aimed at maximizing their own respective expected holistic profit while subject to the optimality of the other player's solution. Thus they obtain the results of the open-loop equilibrium, i.e. $\{\mathbf{r}_k^*, \mathbf{q}_k^*, \mathbf{w}_k^*\}$. Therefore at $k = 1$ the manufacturer proceeds with declaring w_1^* and the retailer orders q_1^* items. However, instead of declaring r_1^* to the market, the retailer waits for the uncertainty of demand, ϵ_1 to be resolved. In the second step and after observing $\hat{\epsilon}_1$, the retailer (and the manufacturer) solve the following equilibrium problem to obtain the optimal retail prices.

$$
\begin{aligned}
&\max_{\mathbf{r}_n} \Pi^r \\
&\max_{w_2, \cdots, w_n} J_2^m \\
\Pi^r &= \pi_1^r(r_1, w_1^*, q_1^*) + \cdots + \alpha_k \Phi_k(\mathbf{r}_{k-1})\, \overline{\pi}_k^r(r_k, w_k, q_k) \\
&\quad + \cdots + \alpha_n \Phi_n(\mathbf{r}_{n-1})\, \overline{\pi}_n^r(r_n, w_n, q_n) = \pi_1^r + J_2^r \\
J_2^m &= \alpha_2 \Phi_2\big(\hat{r}_1(w_1)\big)\, \overline{\pi}_2^m(w_2) + \cdots + \alpha_n \Phi_n\big(\hat{r}_1(w_1), \cdots, \hat{r}_{n-1}(w_{n-1})\big)\, \overline{\pi}_n^m(w_n) \\
\text{where } \pi_1^r &= (r_1 - s_1 - b_1) \min \overbrace{\Big(\mu_1(r_1) + \sigma_1(r_1)\hat{\epsilon}_1\Big)}^{D_1(r_1)} + (s_1 + b_1 - c_{r_1} - w_1^*) q_1^*
\end{aligned}
\tag{63}
$$

Note that the only difference between the retailer's problem expression in (63) and the one in (32) is in the first term, where the expected value of the profit in the first period $\overline{\pi}_1^r$ is replaced by the real profit π_1^r. Thus the retailer, having observed $\hat{\epsilon}_1$, tries to find the vector of optimal retailer prices $\hat{\mathbf{r}}_n$ to optimize the sum of her real profit at the first period π_1^r and the expected (discounted) profits in the future J_2^r.

To solve (63) we use the backward induction reasoning again. Starting with the retailer's problem in the last period n we observe that in order to obtain $\hat{r}_n$ from (63) the retailer has to solve (31) once again. This means that $\hat{r}_n(w_n)$ equals $r_n^*(w_n)$ which was obtained in the pre-observation optimization. In general, going backward in time from period n to 2, the retailer will face the exact same optimization problems as the ones in the pre-observation analysis, i.e. $\hat{r}_k(w_k) = r_k^*(w_k)$, $k \in \{2, \cdots, n\}$. However, when the backward induction reaches the first period, it will face the only term in the objective function which is different from the corresponding one in (32), i.e. π_1^r. Thus, in general the optimal $\hat{r}_1$ is different from r_1^*.

$$\max_{r_1} \Pi^r = \pi_1^r(r_1) + \phi_2(r_1) \times \underbrace{\left[\frac{\alpha_2}{\alpha_1}\overline{\pi}_2^r(r_2^*) + \cdots + \frac{\alpha_n}{\alpha_k}\overline{\pi}_n^r(r_n^*)\prod_{i=3}^{n}\phi_i(r_{i-1}^*)\right]}_{\mathcal{F}_1^r \text{ future expected profit, given (obtained from pre-observation analysis)}} \tag{64}$$

Therefore the vector of optimal decision variables for the retailer after observing $\hat{\epsilon}_1$ is $[\hat{r}_1, r_2^*, \cdots, r_n^*]$ where $\hat{r}_1$ is obtained from (64) and r_k^*s ($k = 2 \cdots n$) are equal to the ones obtained from the pre-observation optimization problems.

Now we proceed to the manufacturer's part of the equilibrium (63), considering the effect of the new retail pricing scheme on future (time interval $\{2, \cdots, n\}$) demand.

$$\begin{gathered}\max_{w_2,\cdots,w_n} J_2^m = \max_{w_2,\cdots,w_n}\left[\alpha_2\Phi_2(\hat{r}_1)\,\overline{\pi}_2^m(w_2) + \cdots + \alpha_n\Phi_n(\hat{r}_1, r_2^*, \cdots, r_{n-1}^*)\,\overline{\pi}_n^m(w_n)\right]\\ \text{where } \hat{r}_1 = \hat{r}_1(w_1^*), r_k^* = r_k^*(w_k) \quad 1 < k\end{gathered} \tag{65}$$

where each $\overline{\pi}_k^m$ is calculated from (26) and (28).

Analogously, observing that the term w_n appears only in the profit expression for the final period $\overline{\pi}_n^m$, we start the backward induction process from the nth period.

$$\max_{w_n} J_2^m \equiv \max_{w_n} \overline{\pi}_n^m \tag{66}$$

But this problem has already been solved in the open-loop analysis and it will yield the same optimal decision variable as before, i.e. w_n^*. Going backward in time, in general, at each period $j \in \{2, \cdots, n\}$ the manufacturer faces the optimization problem (67). Note that for this arbitrary period j we have $\max_{w_j} J_2^m \equiv \max_{w_j} J_j^m$. This is due to the result of the Proposition 3.2 about the subgame perfection of the equilibrium aimed at maximization of the expected profits on time interval between 2 and n.

$$\begin{gathered}\max_{w_j} J_2^m \equiv \max_{w_j} J_j^m = \max_{w_j} \alpha_k\Phi_j(\tilde{\mathbf{r}}_{j-1})\overbrace{\left[\overline{\pi}_k^m(w_j) + \phi_{j+1}\big(r_k^*(w_j)\big)\mathcal{F}_j^m\right]}^{\mathfrak{I}_k^m(w_j)}\\ \text{where } \tilde{\mathbf{r}}_{j-1} = \{\hat{r}_1(w_1^*), r_2^*(w_2), \cdots, r_{j-1}^*(w_{j-1})\}, \quad \Phi_j(\tilde{\mathbf{r}}_{j-1}) = \phi_2(\hat{r}_1)\prod_{i=3}^{j}\phi_i(r_{i-1}^*)\end{gathered} \tag{67}$$

$$\begin{gathered}\mathcal{F}_j^m = \frac{\alpha_{j+1}}{\alpha_j}\,\overline{\pi}_{j+1}^m(w_{j+1}) + \cdots + \frac{\alpha_n}{\alpha_j}\,\overline{\pi}_n^m(w_n)\prod_{i=k+2}^{n}\phi_i(r_{i-1}^*)\\ \mathcal{F}_n^m = 0\end{gathered} \tag{68}$$

Again we have $\max_{w_j} J_j^m \equiv \max_{w_j} \mathfrak{J}_j^m$. Plus, when solving $\max_{w_j} \mathfrak{J}_j^m$ we observe that the choice of $\hat{r}_1$ does not affect $\mathcal{F}_j^m$. Therefore the results of $\max_{w_j} \mathfrak{J}_j^m$ will be exactly as equal to ones obtained by the open-loop solutions. Thus comparing 67 with 43 and 44) we can conclude that after the observation of $\hat{\epsilon}_1$ and declaration of $\hat{r}_1$ in the first period, the manufacturer's optimal price vector for the rest of the periods (from 2 to n) does not change.

However, $\phi_2(\hat{r}_1)$ in (68) will scale $\mathfrak{J}_j^m$ differently from $\phi_2(r_1^*)$ in the corresponding open-loop equilibrium. Hence while the same w_j^*s will come out of the two equilibrium problems, the expected values of the total profits will be different due to different memory elements.

After analysing the two-step solution for the players in the period 1, we try to find a general solution procedure at a period k. The players arrive at period k with the memory function containing the already declared $\hat{\mathbf{r}}_{k-1}$. In the first step they have to solve the following bilevel optimization (Stackelberg equilibrium) problem.

$$\begin{aligned} &\max_{r_k,\cdots,r_n} J_k^r \\ &\max_{w_k,\cdots,w_n} J_k^m \end{aligned} \tag{69}$$

From Proposition 3.2 we know that the equilibrium aimed at maximization of the expected profits is subgame perfect. Hence, in the first step, each decision maker obtains a subset of her original open-loop equilibrium results; i.e. $[r_k^*, \cdots, r_n^*]$ and $[w_k^*, \cdots, w_n^*]$. Thus, at the first step in period k, the manufacturer declares w^* and the retailer orders $\hat{q}_k = \Phi_k(\hat{\mathbf{r}}_{k-1}) \left[\mu_k(r_k^*) + \sigma_k(r_k^*)\, F_{\epsilon_k}^{-1}\left(\frac{r_k^* - w_k^* - c_{r_k}}{r_k^*}\right)\right]$.

At the second step, after the retailer observes $\hat{\epsilon}_k$ the following bilevel equation has to be solved.

$$\max_{r_k,\cdots,r_n} \alpha_k \Phi_k(\hat{\mathbf{r}}_{k-1}) \pi_k^r(r_k) + J_{k+1}^r \qquad \text{over } k, \cdots, n \tag{70}$$

$$\max_{w_{k+1},\cdots,w_n} J_{k+1}^m = \max_{w_{k+1},\cdots,w_n} \sum_{i=k+1}^{n} \alpha_i \Phi_i(\tilde{\mathbf{r}}_{i-1}) \overline{\pi}_i^r(w_i) \qquad \text{over } k+1, \cdots, n$$

$$\text{where } \tilde{\mathbf{r}}_{i-1} = [\hat{\mathbf{r}}_{k-1}, r_k, \cdots, r_{i-1}] \tag{71}$$

Similarly, starting the backward induction from the final period, it is evident that from period n to $k+1$ the retailer will face the exact same optimization problems as the ones in the pre-observation analysis. The only term in the entire objective function which is different from its corresponding term in (32) is π_k^r (the real profit at k which has replaced its own expected value, $\overline{\pi}_k^r$). Therefore the retailer's optimization problem boils down to the following.

$$J_k^r = \alpha_k \underbrace{\Phi_k(\hat{\mathbf{r}}_{k-1})}_{\text{price history}} \overbrace{\left(\pi_k^r(r_k) + \phi_{k+1}(r_k) \underbrace{\left[\frac{\alpha_{k+1}}{\alpha_k} \overline{\pi}_{k+1}^r(r_{k+1}) + \cdots + \frac{\alpha_n}{\alpha_k} \overline{\pi}_n^r(r_n) \prod_{i=k+2}^{n} \phi_i(r_{i-1}) \right]}_{\mathcal{F}_k^r = \text{expected (future) values, given (obtained from pre-observation problem)}} \right)}^{\mathfrak{J}_k^r(r_k)}$$

$$\max_{r_k} J_k^r \equiv \max_{r_k} \mathfrak{J}_k^r \tag{72}$$

Note that by the time the backward induction process reaches the kth period $\mathcal{F}_k^r$ in (72), i.e. the future expected profit, is already calculated and is treated as a constant. Solving the single-variable optimization problem in (72) yields $\hat{r}_k$ while the rest of the optimal retail prices remain equal to those obtained in the pre-observation (open-loop) optimization problem. Thus at the second step in the kth period, the retailer obtains her optimal decision variables $[\hat{r}_k(w_k^*), r_{k+1}^*(w_{k+1}), \cdots, r_n^*(w_n)]$ as functions of corresponding manufacrting prices.

In order to find the numerical values of $\hat{r}_k(w_k)$ and the rest of the optimal retail prices, the retailer has to solve the manufacturer's problem of an optimal response for the next periods.

$$\max_{w_{k+1},\cdots,w_n} J_{k+1}^m$$

$$\begin{aligned} J_{k+1}^m &= \alpha_{k+1}\Phi_{k+1}(\hat{\mathbf{r}}_k)\overline{\pi}_{k+1}^m + \cdots + \alpha_n \Phi_k(\hat{\mathbf{r}}_k) \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*)\overline{\pi}_n^m \\ &= \alpha_{k+1}\Phi_{k+1}(\hat{\mathbf{r}}_k)\left[\overline{\pi}_{k+1}^m + \cdots + \frac{\alpha_n}{\alpha_{k+1}} \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*)\overline{\pi}_n^m\right] \end{aligned} \tag{73}$$

The numerical results for optimal wholesale prices are obtained using the recursive solution procedure delineated below.

$$\begin{aligned} J_j^m &= \alpha_j \Phi_j(\tilde{\mathbf{r}}_{j-1})\left[\overline{\pi}_j^m(w_j) + \phi_{j+1}(r_j^*)\mathcal{F}_j^m\right] \quad k+1 \le j \le n \\ \tilde{\mathbf{r}}_{j-1} &= (\hat{\mathbf{r}}_k, r_{k+1}^* \cdots, r_{j-1}^*) \Rightarrow \Phi_j(\tilde{\mathbf{r}}_{j-1}) = \prod_{i=1}^{k+1} \phi_i(\hat{r}_{i-1}) \prod_{i=k+2}^{j} \phi_i(r_{i-1}^*) \end{aligned} \tag{74}$$

$$\begin{aligned} \mathcal{F}_n^m &= 0 \\ \mathcal{F}_j^m &= \frac{\alpha_{j+1}}{\alpha_j} \overline{\pi}_{j+1}^m(w_{j+1}) + \cdots + \frac{\alpha_n}{\alpha_j} \overline{\pi}_n^m(w_n) \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*) \end{aligned} \tag{75}$$

Again, comparing (73) with (43) and (44), and using the result of Proposition 3.2, it is straightforward to see that after declaration of $\hat{r}_k$ at period k, and when solving manufacturer's optimization problem for the time interval $\{k+1, \cdots, n\}$ the backward induction process will yield the same

$\{w^*_{k+1}, \cdots, w^*_n\}$ as those obtained in the open-loop equilibrium problem. However, the manufacturer's expected profit will be different from the results of the open-loop solutions. This is due to the scaling factor $\Phi_k(\hat{\mathbf{r}}_{k-1})$ which in general will be different from $\Phi_k(\mathbf{r}^*_{k-1})$ in (43). The results of this section are expressed in the following theorem.

Theorem 5.1.

In a retail price postponement scenario where the retailer and the manufacturer face the uncertain demand described in Theorem 3.1, the retailer at each period k postpones the declaration of her price until after observing demand uncertainty ϵ_k.

Assuming that there exists an equilibrium state $[\boldsymbol{r}^*_k, \boldsymbol{w}^*_k]$ for the open-loop problem described in Theorem 3.1, if the following objective function has a global maximum, $\hat{r}_k$,

$$\begin{aligned}
&\mathfrak{J}^r_k(r_k) = \pi^r_k(r_k) + \phi_{k+1}(r_k)\mathcal{F}^r_k \\
&\text{where } \mathcal{F}^r_k = \frac{\alpha_{k+1}}{\alpha_k}\,\overline{\pi}^r_{k+1}(r^*_{k+1}) + \cdots + \frac{\alpha_n}{\alpha_k}\,\overline{\pi}^r_n(r^*_n)\prod_{i=k+2}^{n}\phi_i(r^*_{i-1}) \quad \text{given value}
\end{aligned}$$

then the closed-loop problem of price postponement has an equilibrium with the following optimal decision variables.

$$\begin{aligned}
\hat{\boldsymbol{r}}_n &= [\hat{r}_1, \cdots, \hat{r}_n] \\
\boldsymbol{w}^*_n &= [w^*_1, \cdots, w^*_n] \\
\hat{\boldsymbol{q}}_n &= [\hat{q}_1, \cdots, \hat{q}_n] \\
\text{where } \hat{q}_k &= \Phi_k(\hat{\boldsymbol{r}}_{k-1})\left[\mu_k(r^*_k) + \sigma_k(r^*_k)\,F^{-1}_{\epsilon_k}\left(\frac{r^*_k - w^*_k - c_{r_k}}{r^*_k}\right)\right] \\
&= \frac{\Phi_k(\hat{\boldsymbol{r}}_{k-1})}{\Phi_k(\boldsymbol{r}^*_{k-1})}q^*_k
\end{aligned}$$

5.3 Comparison between open loop and closed loop profits

At the end of period n, the set of post-observation optimal retail prices, $[\hat{\mathbf{r}}_n]$ is the result of the optimization problem $\max_{\mathbf{r}_n} \Pi^r$ considering the real values of π^r_ks. Whereas the set of pre-observation optimal retail prices, $[\mathbf{r}^*_n]$ is the result of optimization $\max_{\mathbf{r}_n} \overline{\Pi}^r$ considering the expected values of the profits at each period $\overline{\pi}^r_k$s. Thus it is trivial that in a hypothetical n-period scenario where two retailers face the same ϵ_k at each period k, the one that postpones the declaration of her prices ($\hat{r}_k$s) until after observation of each ϵ_k gains higher profit compared to the retailer who adheres to sub-optimal r^*_ks. In other words, in a price-postponement scenario, because $\hat{r}_k$s are the results of the real profit optimizations, any other set of decision variables (including the set of r^*_ks) will be sub-optimal. Therefore we have $\Pi^r_{CL} \geq \Pi^r_{OL}$ where Π^r is the total discounted real profit gained through n periods.

5.3.1 Closed-loop Optimization for the Manufacturer

In general, in the closed-loop optimization scenario, at each period k the retailer enjoys the strategic means to find an optimal $\hat{r}_k$ maximizing the sum of her current profit and expected future profits. Whereas the manufacturer always faces the structurally identical (though differently scaled) expected profit optimization.

At each period k after observing $\hat{\epsilon}_k$, the retailer deviates from the previously obtained equilibrium price r^*_k by declaring $\hat{r}_k$ instead. Due to the structure of the memory functions, this new pricing scheme will affect the future demand and thereby the future earnings for both the retailer and the manufacturer. The retailer's optimization problem as generalized in Theorem 5.1 is tailored such that an optimal $\hat{r}_k$ will maximize the sum of the current real profit and expected future profits. Thus after declaring each $\hat{r}_k$, it is the manufacturer's turn to modify her own optimal pricing scheme for the future considering the effects of the retail prices on future demand and expected earnings.

Comparing to the non-postponement solutions, the retailer always benefits from postponing her retail price. Whereas the manufacturer's may either benefit or lose potential profit compared to the non-postponement case, depending on the structure of demand mean and variance, and different realizations of the uncertain demand. In Section 6.3, we provide simulated examples with price-postponement having different effects on the manufacturer's profit. In that section, we provide a hypothetical scenario (example 3) wherein the entire channel is worse off due to price postponement.

6 Numerical Implementation of the Model

In this section, we illustrate the theoretical results and implement the solution algorithms discussed in Sections 3, 4, and 5. In the examples analyzed in this section, we use Cobb-Douglas demand functions.

In the examples we use a truncated and re-normalized normal distribution function for ϵs to ensure that the negative noise terms do not cause the entire demand to become negative.

The theoretical results for each sections, as expressed in Theorem 3.1, Section 4.1, and Theorem 5.1 consider all model parameters and functional forms, including μ_ks and σ_ks to be time-dependent. However, for illustration purposes, we consider only the case in which the same functions are used for all periods, although using varying functions does not make the problem any harder to solve.

It should be noted that the main purpose of this section is merely to familiarize the reader with the implantation of the solution algorithms and the type of results that they can potentially produce. The examples we present are particularly simple, and the mathematical features describing the market structure are merely speculative and not the results of empirical studies.

To take full advantage of the model, one should try to vary scaling factors and functional forms in a systematic way. This makes it is possible to model a wide range of economic contexts. A full discussion of the model and all the variations it can cover, is, however, beyond the scope of this paper.

6.1 The Open-loop Equilibria Solutions

Following the order by which the scenarios were presented, we begin by providing an example of the open-loop equilibria wherein optimization takes place based on the expected values of discounted profits within a number of periods.

• **Example 1 (boosting the demand through initial free distribution)**

For the first example, we consider the following scaled demand function.

$$d_k(r_k) = \frac{1000}{r_k^2} + 10\epsilon_k \tag{76}$$

Multiplicative memory functions scale the future demand such that an increase in the current retail price decreases the future demand. Thus the memory function at period $k+1$ which will scale the future demand D_{k+1} is monotonically decreasing with respect to the retail price at all previous periods.

$$\forall k \in \{1, \cdots n\} \quad \frac{\partial \Phi_{k+1}(r_1, \cdots, r_n)}{\partial r_k} = \frac{\partial \prod_{k=1}^{n} \phi_{k+1}(r_k)}{\partial r_k} < 0 \tag{77}$$

This means that the memory element at $k+1$ must be monotonically decreasing with respect to r_k.

$$\frac{\partial \phi_{k+1}(r_k)}{\partial r_k} < 0 \tag{78}$$

Here, for illustration purpose, we use the following functional structure for memory elements

$$\phi_{k+1}(r_k) = 1 + \gamma_k(\kappa_k - r_k) \tag{79}$$

where $\gamma_k \geq 0$, the memory strength factor at period k, is a given parameter. The given parameter $\kappa_k \geq 0$ can be interpreted as a price cap; i.e., any initial price above κ_k reduces demand, whereas demand is more likely to increase if $r_k < \kappa_k$. If the scaling factor is negative, maxima are turned

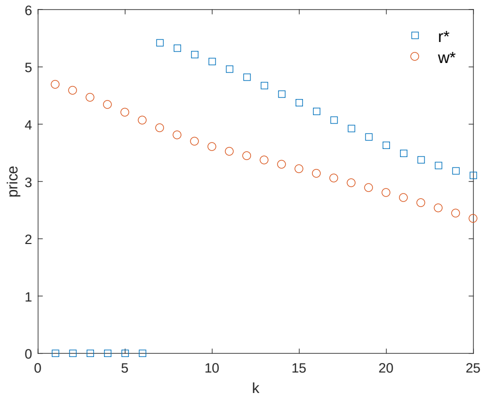

Figure 1: Equilibrium State Retail and Wholesale Prices

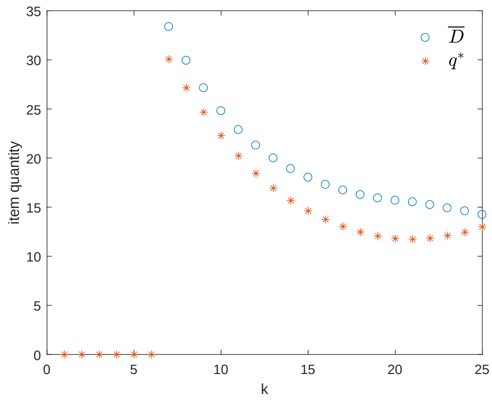

Figure 2: Equilibrium State Order Quantities and Demand Means

into minima. Hence, if $\phi_{k+1}(r_k) \leq 0$, the optimal order q_{k+1} is zero. To avoid this problem, we consider

$$\phi_{k+1}(r_k) = [1 + \gamma_k(\kappa_k - r_k)]^+. \tag{80}$$

For simplicity, we set $\gamma_k = 0.01$, $\kappa_k = 5$, $\alpha_k = 1$, $c_{m_k} = 2$, $c_{r_k} = 0$, $s_k = 1$, and $b_k = 0$ for all ks. The number of periods, n, is set to be 25.

The decision variables at the equilibrium state obtained from Theorem 3.1 are given in Figures 1 and 2. Figure 1 illustrates r^*_k and $w{*}_k$ and Figure 1 shows q^*_k $\overline{D}_k = \Phi_k \, \mu_k$ at each period. We observe that the holistic optimization algorithm prescribes the retailer to set $r^*_{1,2,3} = 0$ and $q^*_{1,2,3} \approx 0$. A strategy of this type makes good sense economically; it corresponds to a situation in which a small number of items ($q \approx 0$) are given away for free at earlier periods to create increased interest for the product in the next periods. We observe the resulting boost in demand mean $\overline{D}$ in Figure 1 to begin at period 4. The expected profits for the two suppliers are: $\overline{\Pi}^r = 1140.14$ and $\overline{\Pi}^m = 799.6$.

6.2 Order Postponement Scenarios

In this section we implement the two-step optimization algorithm delineated in Section 4.1. In the first step, the retailer and the manufacturer both aware of a forthcoming order-postponement have to solve a specific open-loop problem with assumptions described in (54) and (55).

In order to provide illustrative examples of different scenarios that may happen in the second step, we simulate different realizations of the stochastic variable ϵ_k. We create these ϵ_ks based on a given truncated normal distribution and normalized as discussed in Section 2.

- **Example 2.**

$$
\begin{aligned}
&d_k = \mu_k(r_k) + \sigma_k(r_k)\,\epsilon_k \quad \mu_k = \frac{1000}{r_k^3},\ \sigma_k(r_k) = \frac{10}{r_k} \\
&\alpha_k = 0.95^{k-1},\ \gamma_k = 0.01,\ \kappa_k = 5, c_{m_k} = 2, c_{r_k} = 0, s_k = 1, n = 25
\end{aligned} \tag{81}
$$

We use the same functional structure in (80) for the memory elements in this analysis as well. Note that because both the suppliers are aware that there will be an order-postponement, there is no buy-back feature embedded in their contract. Besides, because the retailer will always address the demand, no salvage price is needed in the model.

At the beginning of the first step, having solved the open-loop equilibrium problem, we obtain the following results.

$$\overline{\Pi}^r = 242.23, \overline{\Pi}^m = 202.46$$

Figure 3 illustrates the optimal retail and wholesale prices that the suppliers set at the first step, in the beginning of each period.

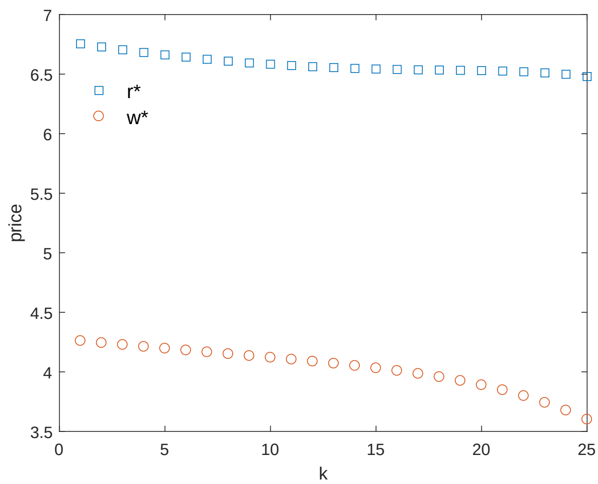

Figure 3: Equilibrium State Retail and Wholesale Price at k

At each period, the retailer postpones her order-quantity until after demand uncertainty at that period is resolved. Figure 4 shows a possible realization of uncertain demand stemmed from a simulated set of ϵ_ks. The retailer sets her $\hat{q}_k$s as equal to these realizations. The figure also shows the expected values of demand at each period. If the retailer had not decided to postpone her order-quantity, she would have adhered to these expected values as her q_k^*s. In other words, the non-postponing retailer described in Theorem 4.1 would order the following amount at each period.

$$q_k^* = \overline{D}_k = \mathrm{E}[D_k] = \Phi_k(\mathbf{r}_{k-1}^*)\,\mu_k(r_k^*).$$

The demand realization shown in Figure 4 results in the following real profits for the two hypothetical channels described in Theorem 4.1 and its Corollary 4.4. The subscripts P and NP denote postponement and no-postponement, and the superscripts r, m, c denote retailer, manufacturer, and channel, respectively.

$$\Pi^r_{NP} = 207.6 \qquad \Pi^m_{NP} = 202.46 \qquad \Pi^c_{NP} = 410.05$$
$$\Pi^r_{P} = 258.47 \qquad \Pi^m_{P} = 215.53 \qquad \Pi^c_{P} = 474.00$$

The results of this simulated scenario show that facing this set of ϵ_ks, both the retailer and the manufacturer, and a fortiori, the whole channel benefit from order postponement. We also observe that in a no-postponement scenario, the real profit obtained by the manufacturer equals her expected profit: $\Pi^m_{NP} = \overline{\Pi}^m$. This is due to the fact that in no-postponement scenario, the manufacturer receives the order quantity at the beginning of each period and thus does not share the risk of facing an uncertain demand.

Different realizations of demand uncertainty ϵ_k may indeed cause different real profits for the channel members. Iterating the simulation, in Figure 5, we illustrate the results of a different realization of the uncertain demand. Similarly, we consider two channels facing this realization of demand, one with a postponing retailer and one with a retailer who adheres to the open-loop solutions. In this realization, the real profits for these two hypothetical channels and their individual members are as follows.

$$\Pi^r_{NP} = 127.97 \qquad \Pi^m_{NP} = 202.46 \qquad \Pi^c_{NP} = 330.43$$
$$\Pi^r_{P} = 222.96 \qquad \Pi^m_{P} = 185.31 \qquad \Pi^c_{P} = 408.27$$

In this case, the manufacturer does not benefit from order postponement. Despite her loss, the whole channel still benefits from order postponement. This observation is consistent with the result of Corollary 4.4.

6.3 Price Postponement Scenarios

In this section we provide examples of price postponement scenarios and implement the two-step optimization algorithm discussed in Section 5. Because in a price postponement scenario, the retailer does not necessarily address the entire demand, for the sake of generality, we have to consider non-zero salvage prices and buy-back rates in the profit optimization expressions.

The buy-back rates b_ks can be considered as either given model parameters or be expressed as functions of the decision variables which are to be found. For example, the expression in (62) states that the manufacturer may offer a, possibly too conservative, buy-back rate of $b_k = w_k - c_{m_k}$ to the

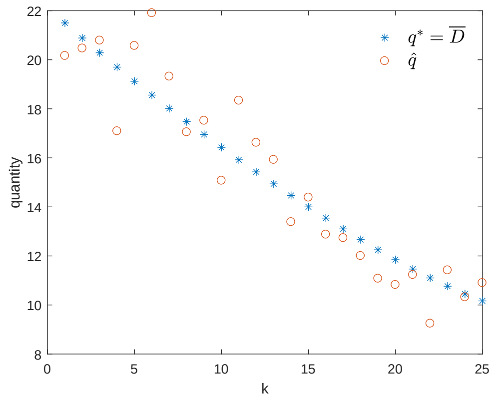

Figure 4: Stochastic Demand Realization and Expected Demand

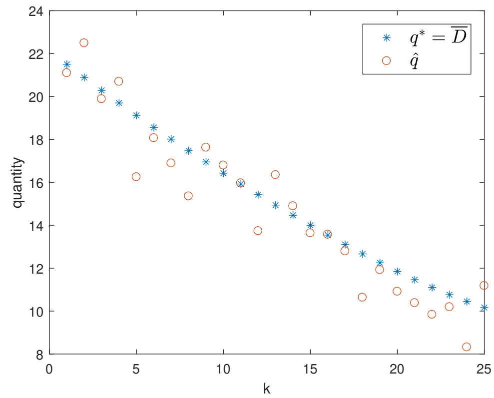

Figure 5: Stochastic Demand Realization and Expected Demand

retailer. While the retailer benefits from any non-zero buy-back rate, this specifically constrained buy-back rate guarantees a non-negative local-in-time profit for the manufacturer at every period.

It should be noted that the solution procedures described in Theorems 3.1 and 5.1 can take each form of the buy-back rate (wether a constant or a function of the decision variables).

In the examples, following the procedure outlined in Theorem 5.1, at the first step, we provide the open-loop solution upon which the suppliers base their initial decision variables, i.e. the wholesale price and the order quantity $w_k^*, \Phi_k(\hat{\mathbf{r}}_{k-1})q_k^*$. Next, we simulate a vector of ϵ_ks based on a given truncated normal distribution and normalized as stated in Section 2. The retailer then observes this demand uncertainty and finds her optimal retail price $\hat{r}_k$ accordingly.

• Example 3. Given buy back prices

We consider the following demand structure and parameters for the model. We also use the same functional structure in (80) for the memory elements.

$$\begin{gathered} d_k = \mu_k(r_k) + \sigma_k(r_k)\,\epsilon_k \quad \mu_k = \frac{1000}{r_k^2},\, \sigma_k(r_k) = \frac{10}{r_k} \\ \alpha_k = 0.95^{k-1},\, \gamma_k = 0.01,\, \kappa_k = 5 \\ c_{m_k} = 2, c_{r_k} = 0, b_k = 1.1 > s_k = 1, n = 25 \end{gathered} \tag{82}$$

The open-loop (first-step) solution results for this scenario are given in Figures 6 and 7. The corresponding expected values for the profits are as below.

$$\overline{\Pi}^r = 74.55, \overline{\Pi}^m = 77.07$$

Now that both the channel members have obtained the open loop solutions, at each period, the retailer updates her objective function after demand uncertainty is resolved. She then declares her

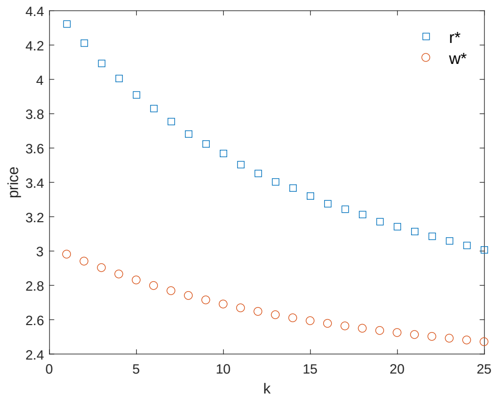

Figure 6: Open-loop Equilibrium Prices

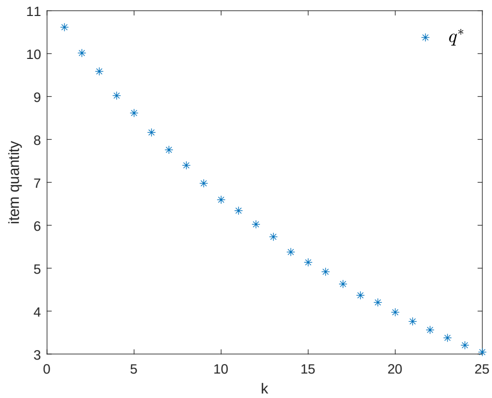

Figure 7: Open-loop Equilibrium Order Quantities

optimal retail price $\hat{r}_k$ to the market. The optimal prices for a simulated scenario are given in Figure 8. Analogous to the analysis in the previous section, we consider two hypothetical channels facing the same set of ϵ_ks at each period. In one channel (denoted by the subscript NP), the retailer does not postpone her declaration of the retail price, i.e. she always adheres to r_k^*. In the other channel (denoted by the subscript P), the retailer postpones her decision on retail price after demand uncertainty is resolved and then declares $\hat{r}_k$ instead.

$$\Pi^r_{NP} = 81.93 \qquad \Pi^m_{NP} = 126.60 \qquad \Pi^c_{NP} = 208.54$$
$$\Pi^r_{P} = 87.87 \qquad \Pi^m_{P} = 119.04 \qquad \Pi^c_{P} = 206.90$$

In this simulated example, while the retailer benefits from postponing her retail prices, the manufacturer loses potential profits, causing the whole channel worse off by price postponement.

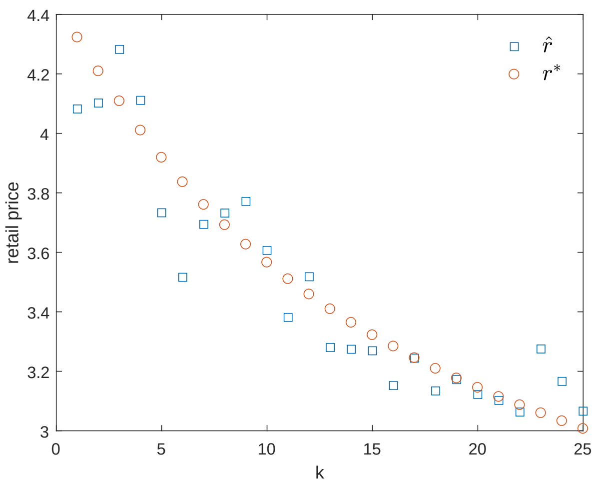

Figure 8: Open-loop and Closed-loop Optimal Retail Prices

- **Example 4. Buy back prices as functions of decision variables**

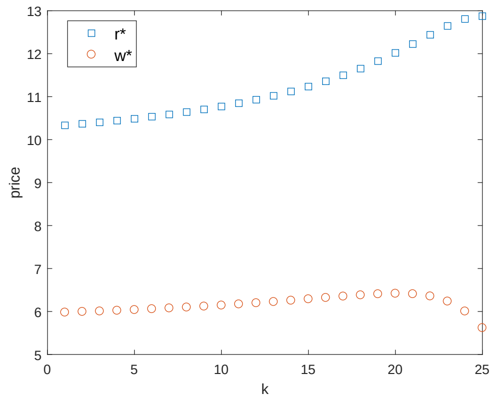

Figure 9: Open-loop Equilibrium Prices

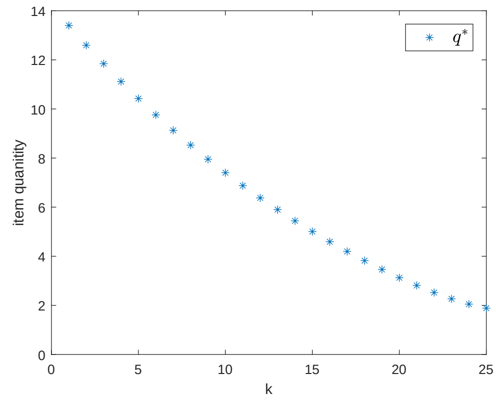

Figure 10: Open-loop Equilibrium Order Quantities

In this example we use the following functional structures and parameters for the uncertain demand. Instead of using a given buy back price at each period, we let the manufacturer to subject the buy back price she is offering, according to the discussion in the beginning of this section.

$$\begin{gathered} d_k = \mu_k(r_k) + \sigma_k(r_k)\,\epsilon_k \quad \mu_k = \frac{1000}{r_k^2},\ \sigma_k(r_k) = \frac{5\mu_k}{r_k} \\ \alpha_k = 0.95^{k-1},\ \gamma_k = 0.01,\ \kappa_k = 5, c_{m_k} = 2, c_{r_k} = 0, n = 25 \\ \text{buy-back rate subject to} \quad b_k = w_k - c_{m_k} \end{gathered} \tag{83}$$

Similarly, first, the two suppliers solve the open-loop equilibrium problems and obtain the following results. The decision variables r_k^*, w_k^*, q_k^* for all periods are given in Figures 9 and 10.

$$\overline{\Pi}^r = 308.98, \overline{\Pi}^m = 306.80$$

Now, at each period, the retailer postpones her decision on retail price until demand uncertainty is resolved and then declares $\hat{r}_k$. The optimal prices for a specific scenario based on a simulated realization of ϵ_ks is given in Figure 10. The corresponding profits for two hypothetical postponing and non-postponing channels facing the same realization of ϵ_ks are given below.

$$\Pi^r_{NP} = 355.59 \qquad \Pi^m_{NP} = 341.75 \qquad \Pi^c_{NP} = 697.34$$

$$\Pi^r_{P} = 387.42 \qquad \Pi^m_{P} = 420.51 \qquad \Pi^c_{P} = 807.93$$

In this scenario, both the retailer and the manufacturer benefit from price postponement.

7 Concluding remarks

In this paper, we have developed analytical tools to use postponement strategies in multi-period supply chain optimization problems. In the first part of the paper, we developed an explicit solution

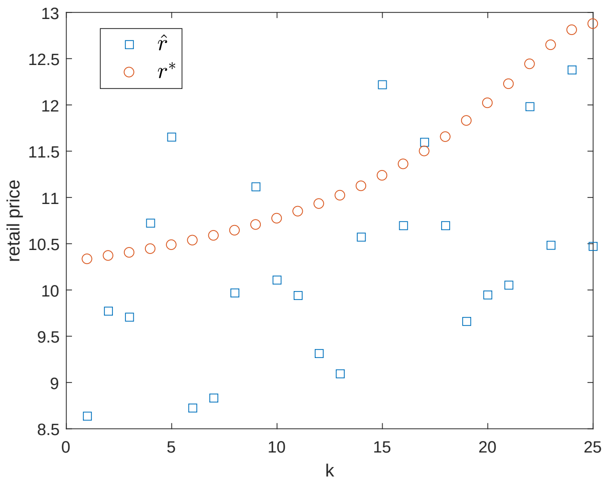

Figure 11: Open-loop and Closed-loop Optimal Retail Prices

algorithm for the problem of finding equilibria without employing postponement strategies. In this part, the agents are risk-neutral and try to maximize their expected profits within a given time period.

Next, we allowed for the gradual addition of extra information and enhanced the decision variables obtained in the previous part accordingly.

The importance of our analytical solution algorithms lies in that not only do they provide us with theoretical results in various scenarios, but also they yield concrete numerical solutions for a wide variety of multi-period bilevel optimization problems.

Due to flexility and generality of our model which allows different functional forms to represent the market features, and also due to its high level of non-autonomy with respect to the parameters and variables, it can be applicable to different economic and management contexts.

An interesting next step would be adding an extra dimension to the equilibrium problem at each period. This can be done, for example, by the inclusion of buy back price as a decision variable, where the b_ks are considered as unknowns and the equilibria problems are solved to obtain b_k^*s as well as the other optimal decision variables.

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NHH

NORGES HANDELSHØYSKOLE
Norwegian School of Economics

Helleveien 30
NO-5045 Bergen
Norway

T +47 55 95 90 00
E nhh.postmottak@nhh.no
W www.nhh.no