Essays on Optionality and Risk

Norwegian School of Economics

NHH

Giovanni Bruno

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To those who made my life richer, despite all my efforts...

'Nessuno ci impone di sapere, Adso. Si deve, ecco tutto, anche a costo di capire male'.

Umberto Eco, The Name of the Rose.

'Even as a youngster, though, I could not bring my self to believe that if knowledge presented danger, the solution was ignorance. To me, it always seemed that the solution had to be wisdom. You did not refuse to look at danger, rather you learned how to handle it safely'.

Isaac Asimov, The Caves of Steel.

Contents

Introduction

Optionality may be described as the existence of the right to do something without the obligation. It introduces asymmetries that have the capacity to distort the risk-return relationship of financial assets as well as the incentives of economic agents managing risk.

Therefore, the study of optionality has been of paramount importance for both the asset pricing and the corporate finance literature for many years.

In the following papers I study the relationship between optionality and risk from different perspectives.

In the first paper, we apply a two factor model to study the idiosyncratic-volatility puzzle, using the assumption that the levered equity of a firm is a call option on the underlying assets because of limited liability. We show, analytically and empirically, that the optionality introduced by leverage creates a negative cross-sectional relation between idiosyncratic volatility (at equity and asset level) and equitys expected returns. Furthermore, As known form previous literature, when systematic risk is time-varying, the parameters of the CAPM regression are biased. We show numerically and empirically that time variation of asset idiosyncratic volatility (ivol) translates into time variation of equity systematic risk and using a conditional CAPM model we attribute part of the risk adjusted performance of high minus low ivol portfolios to this variation in conditional CAPM equity betas.

In the second chapter I exploit a parsimonious model to describe how call options respond differently to changes in idiosyncratic versus systematic risk. The nonlinear payoff of call options creates a channel for the idiosyncratic risk of the underlying to affect not only the options idiosyncratic but also its systematic risk and as a consequence also its price and expected return. Using a simple two factor model we describe the price-risk-return relationship for call options, separating between the effects of systematic and

2

idiosyncratic risk. We show, analytically and numerically, that variations in the risk of the underlying may affect the call option quantities (price, expected return, risk and optimal exercise) through two main channels, the price-channel and the volatility-channel, and that these two channels have different, and sometimes opposite, effects.

Finally in the third chapter I study, how the relationship between exposure to a counterparty in a derivative contract, that can be seen as the payoff of an option, and counterparty risk is affected by different types of netting agreements, that offer the possibility to mitigate counterparty risk by netting the positions across different contracts. With bilateral netting, financial institutions can net the value of their positions with the same counterparty across several asset classes, but not across different counterparties. Instead, by accessing to central clearing counterparties (CCPs), financial institutions can benefit from multilateral netting that allows also netting across different counterparties. I show analytically how correlation across contracts exposures affects the incentive (evaluated as benefits from netting) of the financial institutions to resort to CCPs. Analyzing the incentive to use CCPs should help to identify a proper design of these institutions that are fundamental in the current financial regulation of several countries.

Expected Equity Returns *Should* Correlate with Idiosyncratic Risk*

Giovanni Bruno   Jørgen Haug

Abstract

Because levered equity is an option on the firm, variations in asset idiosyncratic risk (ivol) induces a negative relationship between equity ivol and expected returns. We show that the effect is caused by the nonlinear payoff of equity and the law of one price, and is present in all but risk-neutral economies. We test the cross-sectional predictions of our theory by contrasting the ivol-return relationship at the equity and asset levels. The ivol-return relationship is stronger for equity than for assets, and stronger for more levered firms—consistent with the theory. We test also the time-series implications of the theory. Time variation in asset ivol causes time variation in the option value of equity that translates into time varying risk factor loadings. Unconditional alpha subsequently becomes biased when asset ivol correlates with the market price of risk. We show empirically that a conditional CAPM that accounts for time variation in equity nonlinearity helps explain earlier findings that high-minus-low ivol-portfolios generate negative unconditional alpha.

JEL codes: G11, G12, G13, G32

Keywords: asset volatility, conditional CAPM, coskewness, covariance, equity elasticity, expected returns, idiosyncratic risk puzzle, leverage effect

1   Introduction

There is a growing literature that tries to explain the finding of Ang et al. (2006) that expected equity returns seem to be negatively related to equity idiosyncratic risk (ivol)— the "idiosyncratic volatility puzzle," recently summarized by Hou and Loh (2016). We argue that the observed negative relation is consistent with rational pricing of levered equity under the law of one price.

Figure 1: Expected equity returns and volatility
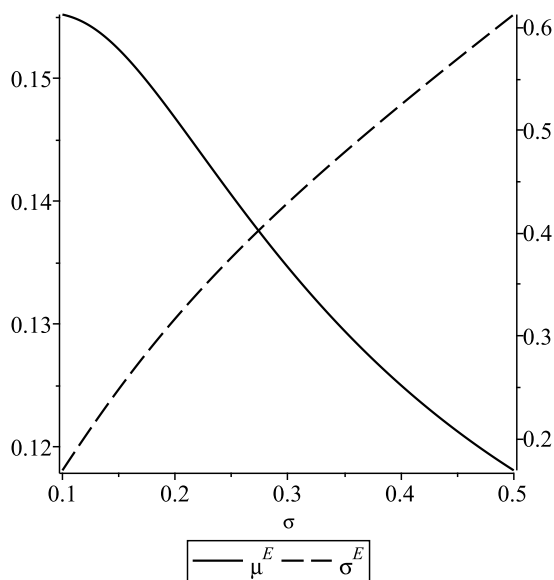
The figure graphs expected equity returns μ_E against the left axis and volatility σ_E against the right axis, as functions of asset volatility σ_A for a fixed level of systematic asset risk: Variations in asset and equity volatility away from their base values represent variations in idiosyncratic risk. The quantities are derived from a Black-Scholes-Merton model of the firm. Current value of assets is kept fixed at 10 with expected rate of return 10%, face value of debt is 5 with 3 years to expiry. The riskless rate is 2%.

The expected return on equity is determined by the covariance of its return with the stochastic discount factor (SDF). An increase in asset ivol increases the default risk of the firm—the likelihood of ending up in states in which the equity return has *zero covariance* with the SDF. Conditional on no default the covariance of equity returns with the SDF lines up with that of the underlying asset. The magnitude of the unconditional covariance

5

consequently falls as asset ivol increases, causing expected equity returns to fall. While the intuition behind the relationship between expected equity returns and asset ivol is simple, the response of equity ivol to asset ivol is more complex. To establish that equity expected returns and ivol have opposite responses to asset ivol we therefore consider a simple Black-Scholes-Merton (BSM) model of the firm (Black and Scholes, 1973; Merton, 1974), and prove the result through simple comparative statics. The formal analysis is summarized in Figure 1, which shows how expected equity returns and ivol respond to asset ivol for a firm with 50% leverage in terms of its face value of debt.

A key observation for our analysis is that the law of one price causes excess equity returns to be proportional to excess asset returns, $R^E = \epsilon R^A$. The ratio of the two returns equals the elasticity of equity value with respect to asset value (Black and Scholes, 1973). Elasticity has a negative and nonlinear response to volatility, and thus to ivol, and the strength of the response is increasing in leverage. The return restriction clearly extends to ratios of equity to asset total, systematic, and idiosyncratic risk. We use the BSM model to establish that while elasticity is decreasing in asset ivol, equity ivol—the product of elasticity and asset ivol—is increasing in asset ivol, as in Figure 1.

We investigate the implications of our theory in a sample of publicly traded firms for which we can use the model-free approach of Choi and Richardson (2016) to estimate firms' asset returns. By comparing the cross-sectional equity ivol-return relationship to that of assets we are able to control for all firm characteristics beyond the mechanism of interest.[1]

We utilize the law-of-one-price-restriction on equity and asset volatility to get a model-free estimate of elasticity as the ratio of equity to asset volatility. A sort on elasticity shows that the equity-asset return spread is significantly higher for the high than for the low elasticity portfolio. When we sort firms on asset ivol we find a positive relationship

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[1]Under the law of one price, economic effects that pertain to equity but not to debt must necessarily affect asset values.

with equity ivol and a negative relationship with equity expected returns. While the ivol-return relationship is thus negative at the equity level we find it to be weaker and *positive* at the asset level. This pattern is fully consistent with the implications of rational pricing of levered equity. Alternative explanations for the ivol-return relationship, like preferences for lottery-like payoffs, or market frictions that correlate with ivol, cannot easily explain the reversal between equity and assets. As elasticity is a function also of leverage, the preceding findings could be due to cross-sectional variations in leverage. To control for the effect of leverage we conduct a conditional double sort on leverage and asset ivol. We find that equity returns are still decreasing in the level of asset ivol, as predicted.

The idiosyncratic volatility puzzle pertains not only to the cross-section but also to the time-series of stock returns. Ang et al. (2006) document that a portfolio long in high-ivol stocks and short in low-ivol stocks generates negative unconditional alpha. As elasticity also scales equity beta a sufficient condition for unconditional alpha in our framework is that elasticity is time-varying and correlated with the market price of risk (Jagannathan and Wang, 1996). We address the time-series implications in two steps. We first establish that unconditional alpha is biased in a simple reduced-form extension of the BSM model, that allows for stochastic asset volatility. When asset beta is allowed to vary, it is sufficient that elasticity is a decreasing function of asset ivol in the cross section to generate biased alpha. Differences in levels of asset ivol causes differences in elasticity, which causes differences in the scaling of variations in asset systematic risk. We use simulations to verify that variations in asset ivol can generate sufficient variation in expected equity returns to match the observed unconditional alpha.

As a second step we investigate empirically to what extent asset ivol is an efficient instrument in estimating the conditional CAPM. In line with the predictions of the theory we find that the equity betas of portfolios with more equity nonlinearities respond more

strongly to time variation in asset ivol than those with less nonlinearity. The estimated conditional betas help reduce unconditional CAPM-alpha of the long-short ivol-portfolio by almost 37%.

The explanation for the observed equity ivol-return pattern most closely related to ours is due to Schneider et al. (2017), who also attribute the pattern to levered equity. They observe that leverage causes skewness in equity returns, and assume that coskewness is priced. By simulating BSM-style firms, using a suitably parametrized stochastic volatility model for firms' assets, they are able to reproduce the empirical pattern. We demonstrate through simulations of a mean-variance economy that the negative equity ivol-return relationship is present even when skewness is not priced. To provide additional support for the view that the pattern mainly is a mechanical result due to variations in elasticity we extend the higher moments decomposition of expected returns of Kraus and Litzenberger (1976) and Harvey and Siddique (2000). They essentially consider the SDF a nonlinear function of some risk factor, and approximate it through a low order series expansion. We observe that also equity returns may be nonlinear in the basic risk factor, and approximate also this quantity via a low order series expansion. This exercise allows us to explicitly study how the covariance and coskewness risk factor loadings respond to variations in ivol. A numerical investigation demonstrates that covariance contributes about *two orders* of magnitude more to expected equity returns than coskewness. The observed pattern is thus more likely due to the effect of ivol on elasticity, than the effect of ivol on priced skewness.

Our analysis has two key insights in common with Choi and Richardson (2016), and some of our results are better understood in light of their findings. We use their approach to estimate returns on assets. We also rely on the same basic insight that equity beta equals asset beta scaled by elasticity. Their analysis is concerned with the relationship between leverage, total and priced risk, and return. They argue that time variation in

8

leverage causes time variation in equity beta. We look at how unpriced risk at the asset level affects priced and unpriced risk at the equity level.

The remainder of the paper starts out in Section 2 with a formal analysis of the relationship between asset ivol and the equity ivol-return trade-off, in a setting of rational pricing of levered equity. Formal proofs are placed in Appendix A. Our main data set and the construction of firms' asset returns is presented in Section 3. We investigate our cross-sectional prediction, that ivol should be negatively related to returns at the equity level but not at the asset level, in Section 4. In Section 5 we discuss to what extent time varying risk factor loadings due to variations in elasticity can explain the negative unconditional alpha of high minus low ivol portfolios. Section 6 concludes.

2   Theory

To better understand the equity ivol-return relationship it is useful to express the equity price via its replicating, self-financing portfolio. To this end denote market prices of assets by A and assume the value of equity, E, is a function of the assets and time. It then generally holds that

$$dr^E = \frac{dE}{E} = \epsilon\, dr^A + (1 - \epsilon) r\, dt,$$

where ϵ is the elasticity of equity with regard to assets, r^A is the return on assets, and r is the riskless rate. Taking the conditional expectation and variance yields

$$\mu_E = \epsilon\mu_A + (1 - \epsilon)r, \tag{1}$$

$$\sigma_E = \epsilon\sigma_A. \tag{2}$$

9

The replicating portfolio view offers an alternative insight into the *mechanical* response of expected equity returns (1) to asset ivol. Changes in ivol causes changes in the elasticity, which simply corresponds to a change in the leverage of the replicating portfolio. Restrictions (1) and (2) imply the following immediate, model-free, testable predictions.

Proposition 1. *Assume that $\mu_A > r$. For a given level of asset idiosyncratic volatility, the expected return and idiosyncratic volatility of equity are increasing in elasticity.*

Proposition 1 is immediate because it considers changes only in the elasticity in (1) and (2)—which occur naturally in both the cross section and time series due to variations in leverage, volatility, interest rates, and time. To explain the negative relation between expected returns and ivol it becomes necessary to impose more structure. The observed relation can be explained only if an increase in asset ivol is not offset by the decrease in elasticity in (2). To this end consider the economy of Black and Scholes (1973) and Merton (1974), in which assets $\frac{dA}{A} = \mu_A\,dt + \sigma_A\,dW$ are financed by debt with face value K that expires at a fixed time in the future, with constant μ_A and σ_A.[2] The next result establishes that the expected equity rate of return is *negatively* related to equity ivol, even when there is no association at the asset level.

Theorem 1. *Assume that $\mu_A > r$ and that the firm is solvent in the sense that $A \geq Ke^{-rT}$. For a given level of leverage A/K, $\frac{\partial \mu_E}{\partial \sigma_A} < 0$ and $\frac{\partial \sigma_E}{\partial \sigma_A} > 0$ for all $\sigma_A > 0$.*

While the assumption that the firm is solvent is realistic, it is also nontrivial. It is easy to prove by example that the volatility of a call option is generally *not* increasing in the volatility of the underlying asset. The assumption of leverage being fixed amounts to

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[2]Equity of real firms clearly correspond to more intricate American derivatives, with potentially complex dynamics for the strike price, as the value of equity and the cost and structure of debt are endogenous quantities in general equilibrium. Extensions to dynamic capital structure models (as pioneered by Leland, 1994) are unlikely to affect the results much, however, at the relatively short time horizons relevant to the present study.

assuming that variations in asset volatility are caused by shifts in asset ivol only, which are mean preserving spreads of future asset values as in Merton (1973).[3]

Figure 1 illustrates Theorem 1 for a firm with face value of debt equal to 50% of its assets, when the debt matures in three years. Equity returns decrease uniformly while equity ivol increases uniformly as asset ivol increases. To better evaluate the likely strength of the relationship in a large cross section of firms Figure 2 recasts Figure 1 at more extreme levels of leverage—with 20% face value in Figure 2a and 80% face value in Figure 2b. In terms of market values A/E the leverage ratios are 1.23 and 3.55, and implied physical default risks are approximately 0% and 9% respectively. The market leverage ratios are consistent with our sample firms (as reported in Table 1 below), while the default probabilities are in line with the estimates of Huang and Huang (2012). Thus, while the graphs are produced by a highly stylized model of the firm, the chosen parameter values and implied quantities are in line with those of actual firms.

Figures 1 and 2 confirm that the relationship is robust to variations in leverage. They also demonstrate that empirical analyses should control for the varying strength across leverage, as the relationship becomes weaker as leverage decreases and equity becomes more asset-like. Figure 2a indicates the expected limiting relationship as the firm becomes fully equity financed: the graph for μ_E becomes horizontal at low levels of σ^A and converges to μ_A at high levels of σ_A, while σ_E approaches σ_A at high levels of σ_A. Notice also that the equity volatility graph lies above the 45° line, consistent with elasticity being bounded below by unity (Bergman et al., 1996). The distance to the 45° line decreases as asset ivol increases, which reflects that elasticity is a decreasing function of asset volatility.

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[3]Rasmusen (2007) considers the effects of more general changes in risk on option values.

Figure 2: Equity return and volatility for low and high leverage
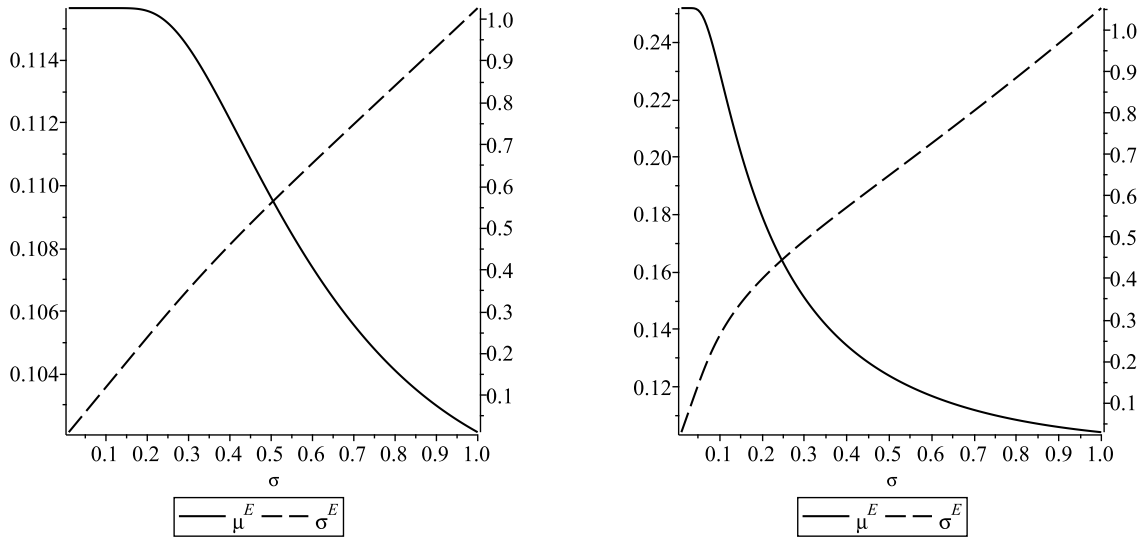
(a) Low leverage   **(b)** High leverage

Expected equity returns, μ_E, given by the decreasing graph, and equity volatility, σ_E, given by the increasing graph, are graphed against the volatility of the underlying assets, σ_A, for a fixed level of systematic asset risk. Variations in asset and equity volatility away from their base values thus represent variations in idiosyncratic risk. Both figures represent firms with current asset values of 10 with expected rates of return of 10%, debt that expires in three years, and a riskless rate of 2%. Figures 2a and 2b represent face values of debt of 2 and 8 respectively. In terms of market values A/E the corresponding leverage ratios are approximately 1.23 and 3.55, and implied physical default risks are about 0% and 9%.

Priced skewness

Schneider et al. (2017) argue that the relationship between equity returns and ivol is caused by how skewness in equity returns varies with equity ivol. An important step in their argument is to assume that skewness is priced—which is naturally the case in any but the simplest, linear asset pricing models. Priced skewness plays no role in the arguments leading up to Theorem 1 however. Our result relies only on the law of one price and the portfolio view of equity, as a linear combination of the underlying assets and a riskless asset. The portfolio weights respond to variations in ivol, essentially to reflect changes in default risk as elaborated on in the Introduction. To formally demonstrate that variations in elasticity in isolation is sufficient to generate the negative equity ivol-return relationship

we prove by example that the relationships established in Theorem 1 are not materially different in an economy where skewness is not priced. By considering returns over a non-trivial discrete time period this exercise has the added benefit of demonstrating that the relationship is invariant to the time scale.

Consider the SDF $m_{0,t} = e^{-rt} + b(X_{0,t} - E\{X_{0,t}\})$, which prices only the covariance of returns with the fundamental risk factor X, at a fixed time horizon t. If X is the return on the market portfolio and $b < 0$ then the traditional CAPM obtains. The value of equity, E_0, must satisfy the Euler equation defined by the SDF. To make as few changes to the economy as possible we maintain the structure of the equity payoff as $E_T = \max(A_T - K, 0)$ and let the asset payoff be lognormally distributed. Given E_0, discrete equity returns are simulated simply as $R_{0,T}^E = E_T/E_0$. We simulate one million scenarios for the system

$$X_T = e^{\left(\mu_X - \frac{\sigma_X^2}{2}\right)T + \sigma_X x_T},$$

$$A_T = e^{\left(\mu_A - \frac{\sigma_A^2}{2}\right)T + v_s x_T + v_i y_T},$$

where $\sigma_A = \sqrt{v_s^2 + v_i^2}$, and x and y are independent $N(0,T)$ random variables. Expected returns and volatilities are estimated as sample moments of the resultant return scenarios, using the parameter values $r = 0.02$, $\mu_X = 0.054$, $\sigma_X = v_s = 0.1275$, $b = -4.5$, and $T = 1$. The parameter values are chosen to ensure that the probability of $m \leq 0$ is negligible. The value of μ_X ensures that the Euler equation for X holds given σ_X, b, and r. Because the systematic risk factor loading of the asset equals that of X we get that $\mu_A = \mu_X$—the important aspect being that it is invariant to the ivol risk factor loading, v_i. The parameter space for (v_i, K) is set to match that of Figure 2, which gives rise to Figure 3. There are no qualitative differences in the ivol-return relationships across Figures 2 and 3. The mean-variance economy produces the expected limiting properties, but only for more

extreme values of asset ivol than those shown in Figure 3.

Figure 3: Equity return and volatility in a mean-variance economy
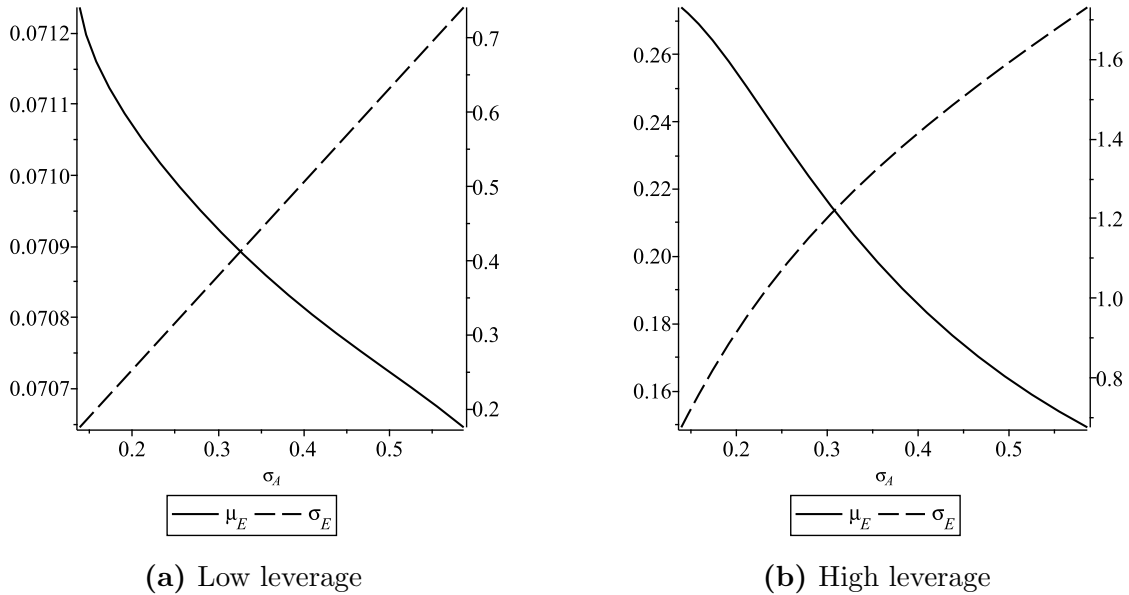
(a) Low leverage   **(b)** High leverage

Expected equity returns, μ_E, given by the decreasing graph, and equity volatility, σ_E, given by the increasing graph, are graphed against the volatility of the underlying assets, σ_A. Both figures represent firms with current asset values of 10 with expected rates of return of 5.4%, debt that expires in one year, and a riskless rate of 2%. Figures 2a and 2b represent face values of debt of 2 and 8 respectively. Variations in asset volatility and the response of equity volatility represent variations in idiosyncratic risk.

While the simulated mean-variance economy demonstrates that variations in elasticity is sufficient to generate the negative ivol-return relationship it is difficult to judge the relative contributions of priced skewness and elasticity: The affine specification of the SDF may easily misrepresent the strength of the relationship. To address this issue we next decompose expected equity returns into covariance and coskewness risk premia, relying on only very weak restrictions on the economy. The decomposition utilizes a simple second order approximation of the stochastic discount factor—common in the literature on priced skewness (going back to Kraus and Litzenberger, 1976)—in combination with a similar approximation of the value of equity.[4]

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[4]Our decomposition is most closely related to the conditional skewness model of Harvey and Siddique (2000). They assume equity returns exhibit stochastic volatility to generate skewness. In our setup skewed

Proposition 2. *Assume a stochastic discount factor* $m_{t+1} = e^{-r_t - \frac{1}{2}\lambda_t^2 - \lambda_t x_{t+1}}$, $R_{t+1}^A = \mu_t + \sigma_{At} z_{t+1}$, $z_{t+1} = \rho_t x_{t+1} + \sqrt{1-\rho_t^2} y_{t+1}$, *where* x *and* y *are independent, and* x *is symmetrically distributed around zero, for* r_t, $\mu_A(t)$, $\sigma_A(t)$ *in investors' date t information set, and* $\lambda = \frac{\mu_A - r}{\sigma_A}$.[5] *If the firm's equity value is a smooth function of the underlying asset value then*

$$\mu_E(t) - r_t \approx e^{-\frac{1}{2}\lambda_t^2} \left[\lambda_t \epsilon_t Cov_t\big(x_{t+1}, R_{t+1}^A\big) + \lambda_t^2 \frac{\gamma_t A_t}{\Delta_t} \epsilon_t Cov_t\big(x_{t+1}^2, (R_{t+1}^A)^2\big) \right]$$

$$= e^{-\frac{1}{2}\lambda_t^2} \left[f_t^v E_t \left\{ x_{t+1}^2 \right\} + f_t^s E_t \left\{ x_{t+1}^4 \right\} \right],$$

$$f_t^v = \rho_t \mu_{At}^e \epsilon_t,$$

$$f_t^s = -\frac{1}{4}(\rho_t \mu_{At}^e)^2 \frac{\gamma_t A_t}{\Delta_t} \epsilon_t,$$

where $\mu_A^e = \mu_A - r$, Δ, γ, *and* ϵ *are the delta, gamma, and elasticity of equity with respect to the value of assets.*

While asset returns do not exhibit skewness in Proposition 2, equity returns do so as a result of leverage. A quick inspection of the covariance and coskewness components, f^v and f^s, suggests that coskewness risk is at least an order of magnitude less important than covariance risk: $-\frac{f^s}{f^v} = \frac{1}{4}\mu_A^e \rho \frac{\gamma A}{\Delta}$. The magnitudes of asset risk premia, μ_A^e, and correlation, ρ, are both well below unity for stocks. Coskewness moreover depends on the elasticity of delta, $\frac{\gamma A}{\Delta}$. This delta-elasticity is strictly positive and typically smaller than four, even for firms with significant leverage.

To quantify the relative importance of the two components Figure 4 shows how f^v and f^s vary with ivol when the equity sensitivities Δ, γ, and ϵ are computed using the

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returns are caused by equity as a call on the firm's assets, which causes stochastic volatility. To our knowledge, earlier studies do not utilize the call option view of equity when decomposing expected returns into covariance and coskewness risk.

[5]The SDF can be viewed as the Radon-Nikodym derivative of the equivalent martingale measure with respect to the physical measure for equity. Hence the expression for λ.

Figure 4: Impact of covariance and coskewness risk on expected equity returns
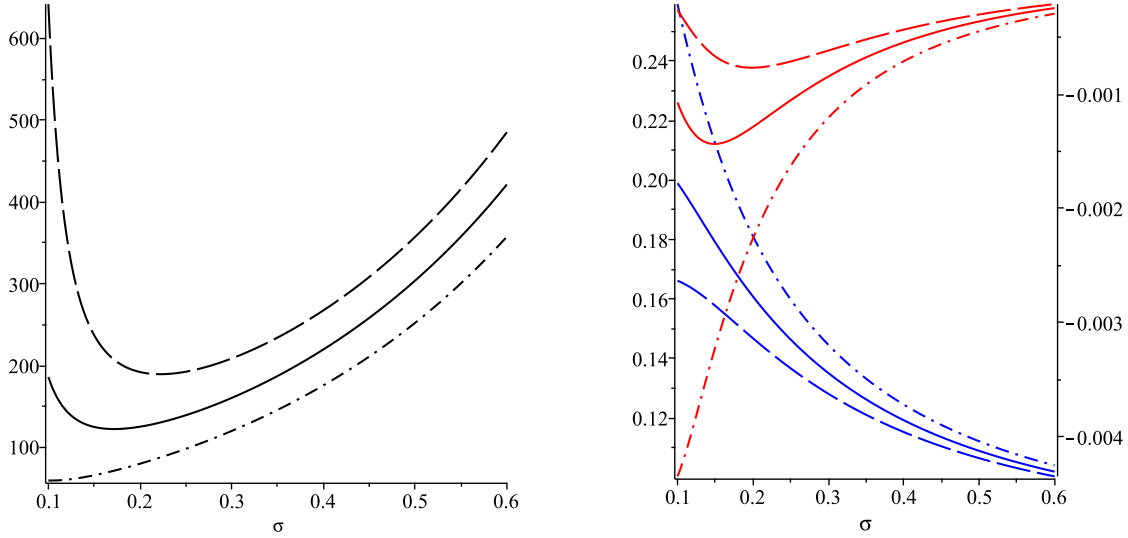
(a) Relative magnitude of coskewness to co-variance risk components, $-f^s/f^v$

(b) Covariance and coskewness risk components, $f^v > 0$ (blue) and $f^s < 0$ (red)

The figures graph the covariance and coskewness components f^v and f^s in Proposition 2, utilizing the BSM model for the equity sensitivities Δ, γ, and ϵ. Common parameter values are $A = 10$, $\mu_A = 0.10$, $r = 0.02$, $\rho = 0.5$, and three years to expiration of the firms' debt. The dashed, solid, and dash-dotted graphs correspond to $K/A = 0.2, 0.5,$ and 0.8 respectively.

BSM formula. Figure 4a shows the graphs of $-\frac{f^s}{f^v}$ as functions of ivol, for three different levels of leverage, $K/A = 0.2, 0.5, 0.8$. Variations in the ratio is fully determined by the elasticity of equity delta, $\frac{\gamma A}{\Delta}$. Delta is typically of an order of magnitude larger than gamma, which in isolation causes the loading on covariance to be an order of magnitude larger than the loading on coskewness. As the ratio of the loadings moreover is proportional to the asset risk premium the loading on covariance ends up being more than two orders of magnitude larger than that of coskewness, even for the most levered firm, as confirmed by the Figure. These magnitudes materialize under a correlation with the 'basic risk factor' of $\rho = 0.5$, which must be considered an upper bound for all but perhaps very well diversified portfolios. Lower values of ρ has the knock-on effect of also lowering the asset expected return μ_A. Our parameter choice thus represent a best-case scenario for coskewness to

contribute to expected equity returns.

Figure 4b graphs the individual components, f^v and f^s. The graphs of covariance risk embodies Theorem 1, in that the covariance component falls uniformly with asset ivol, consistent with probability mass being shifted into default states for equity—states in which equity has zero covariance with the stochastic discount factor. The component is increasing in leverage, as expected. The graphs of the coskewness component verifies, not surprisingly, that coskewness is indeed priced—but at an economically insignificant level relative to covariance risk. The coskewness component increases in magnitude as moneyness decreases, as the nonlinear payoff of equity translates into a more heavily skewed equity return distribution. Coskewness is however not monotonic in ivol on the empirically relevant domain for test assets considered in the ivol literature (the portfolio of firms with low leverage exhibit an average σ_A of about 30% annualized, as reported in Table 1). In sum, Figure 4 demonstrates that the negative association between expected returns and ivol is due for the most part to the negative response of covariance risk to ivol (the mechanical "elasticity channel"), and to a much lesser degree by the response of coskewness risk (a "preference channel").

It is worth noting that the preceding analysis assumes returns on assets do not exhibit skewness, and thus biases the magnitude of skewness downwards in light of the results in Table 1 and the earlier findings of Choi and Richardson (2016). This does not invalidate our argument, however, as it is the relative responses of the risk factor loadings f^v and f^s that are relevant. It is unlikely that the conclusions drawn from Figure 4 will change if one allows for skewed asset returns—i.e., that the magnitude and response of equity delta, gamma, and elasticity to ivol is sensitive to asset skewness.

3 Data

To investigate to what extent the implications of the preceding theory bears out in the data we need observations of firms' asset returns. We follow Choi (2013) and Choi and Richardson (2016), and combine data from (i) CRSP for stock prices, (ii) Thomson Reuters Datastream for bond prices, (iii) the Fixed Income Security Database (FISD) from Mergent for information on corporate bonds, and (iv) Compustat for balance sheet data. Due to data availability our sample covers the period December 1997 to December 2016. We include only firms with at least $100 million in market assets (similarly to Choi and Richardson, 2016). There are on average 396 firms in our sample each month. Returns in the remainder of the paper are in excess of the riskless rate R. The riskless rate is from Kenneth French's database. For the remainder of the paper we moreover denote idiosyncratic volatility (ivol) of assets and equity by σ_A and σ_E. Unless otherwise noted asset (equity) portfolio statistics are computed using asset (equity) weights.

To construct the asset return time series we utilize that asset returns can be decomposed in the usual manner as

$$R_{t+1}^A = \frac{E_t}{E_t + D_t} R_{t+1}^E + \frac{D_t}{E_t + D_t} R_{t+1}^D,$$

where E and D are the market value of the firm's equity and debt, and R^E and R^D are the corresponding returns. For each firm we obtain daily data on equity values, E, from CRSP, by multiplying the share price with the amount of outstanding common shares, as well as returns, R^E. We approximate the market value of debt, D, by the market value of the firm's bonds.[6] We use FISD to obtain the identification number (ISIN) of the bond issued by each firm. The ISIN is next used to obtain the quoted clean price, P, their coupons,

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[6]Choi (2013) and Choi and Richardson (2016) use also Dealscan's data on firms' loans. We find only negligible differences in key statistics of the debt returns time series.

C, and their accrued interest, AI. Bond returns are then computed as

$$R_{t+1}^B = \frac{P_{t+1} + AI_{t+1} + C_{t+1} - (P_t + AI_t)}{P_t + AI_t}.$$

A firm's debt return, R^D, is next computed as a value weighted average of the return of each bond issue alive at that time. An attractive consequence of the weighting scheme is that a firm's less liquid bond issues—typically those of smaller magnitudes—have less impact on the debt returns. Finally, each day the market value of debt is obtained by multiplying recursively the debt returns with the latest observed book value of debt available from Compustat.[7]

Table 1 presents key statistics for our sample conditional on quintile leverage buckets. The buckets are constructed at the beginning of each month by sorting on contemporaneous financial leverage, A/E. Each entry represents the time series average for a quantile portfolio. Columns $\overline{A/E}$, $\bar{\beta}_E$, and $\bar{\beta}_A$ demonstrate that leverage seems unrelated to equity beta, and negatively related to asset beta, consistent with Choi (2013). Columns $\bar{\sigma}_E$ and $\bar{\sigma}_A$ show that leverage seems unrelated to equity ivol, but is monotonically decreasing in asset ivol, consistent with Choi and Richardson (2016). Overall, Table 1 shows that our sample captures significant cross-sectional variation in firms' leverage, risk, and return, and that it exhibits the same properties as that of Choi and Richardson.

4 Cross-sectional Implications

We begin this Section by developing hypotheses to capture the testable implications of Proposition 1 and Theorem 1, all of which are cross-sectional. We develop and test related time series implications in Section 5. Proposition 1 implies that expected equity returns

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[7]As in Choi (2013), we use book value of long term debt and debt in current liabilities.

should be increasing in elasticity, whereas expected asset returns should be unaffected by changes in elasticity. Cross-sectional variation in the equity returns should consequently be *stronger* than the variation in asset returns, as in *Hypothesis 1a:*

$$\mu_E^{H_\epsilon} - \mu_E^{L_\epsilon} > \mu_A^{H_\epsilon} - \mu_A^{L_\epsilon}, \tag{H--1a}$$

where H_ϵ and L_ϵ denotes the highest and lowest quintile portfolios sorted on elasticity. Similarly, equity ivol should be increasing in elasticity, as in *Hypothesis 1b:*

$$\sigma_E^{H_\epsilon} > \sigma_E^{L_\epsilon}. \tag{H--1b}$$

Theorem 1 implies that average equity returns should be cross-sectionally negatively correlated with equity ivol, but unrelated to asset ivol. Correlation is caused by the response of elasticity at the equity level due to leverage. While Theorem 1 assumes unit elasticity at the asset level, the relevant empirical test is that the correlation is negative and *stronger* for equity than for assets.[8] To this end we sort stocks into quintiles based on asset ivol, and compute the difference between the highest and lowest ivol quantiles. The difference should be higher for equity than for assets, as in *Hypothesis 2a:*

$$\mu_E^{H_\sigma} - \mu_E^{L_\sigma} < \mu_A^{H_\sigma} - \mu_A^{L_\sigma}. \tag{H--2a}$$

Theorem 1 also implies a positive relation between equity and asset ivol. Average equity ivol should thus increase with average asset ivol, leading to *Hypothesis 2b:*

$$\sigma_E^{H_\sigma} > \sigma_E^{L_\sigma}. \tag{H--2b}$$

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[8]The cash flow at the asset level may also be nonlinear in the basic risk factors, due for instance to real options (Brennan and Schwartz, 1985; McDonald and Siegel, 1985, 1986) or nonlinear tax schemes.

Furthermore, in order to relate cross-sectional variation in average equity returns to variations in elasticity we should find that the average elasticity of the high asset-ivol portfolios is lower than the elasticity of low asset-ivol portfolios, as in *Hypothesis 2c:*

$$\epsilon^{H_\sigma} < \epsilon^{L_\sigma}. \tag{H--2c}$$

Hypothesis (H–2c) provides an important implication for attributing the observed equity ivol-return pattern to the response of elasticity to asset ivol, which we can test by virtue of access to both asset and equity returns data. We investigate Hypotheses (H–2a)–(H–2c) via single sorts, as later reported on in Table 3.

Another implication of Theorem 1 is that the strength of the negative relation between equity returns and ivol is increasing in the sensitivity of elasticity. Because the slope of elasticity is steeper for options that are less in-the-money—the more levered firms—we investigate *Hypothesis 3:*

$$(\mu_E^{H_\sigma} - \mu_E^{L_\sigma})|\text{high leverage} < (\mu_E^{H_\sigma} - \mu_E^{H_\sigma})|\text{low leverage}. \tag{H--3}$$

We investigate (H–3) via a conditional double sort, as later reported on in Table 4.

For (H–1a) and (H–1b) we use a $1/0/1^9$ strategy (as in Ang et al., 2006) to sort equity and asset returns into quintile portfolios based on their previous month elasticity, and then compute the next month's excess returns for the portfolios. It follows from (2) that individual stock elasticities can be estimated as

$$\epsilon^i = \frac{\sigma_E^i}{\sigma_A^i}, \tag{3}$$

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[9]Let t be the date at which the strategy is initiated. A strategy N/W/M, is a strategy that uses information in N months (from date $t - W - N$ to $t - W$) to create the sorting, it has a waiting period of W months (from date $t - W$ to t) and it holds the portfolios for M months (from date t to $t + M$).

where σ_E^i and σ_A^i are the previous month's volatilities of equity and assets of firm i.[10]

Table 2 reports the results for the portfolios sorted on elasticity. Rows $\bar{\mu}_E$ and $\bar{\mu}_A$ report the average next period returns of the portfolios. The high minus low portfolio for equity yields a positive excess return of 0.648% per month, although the t-stat is below 2 (1.848). The high minus low asset portfolio offers instead a weakly negative excess return of 0.109%, which is statistically insignificant. As elasticity is increasing in leverage, the negative association between asset returns and elasticity can be understood by the finding of Choi and Richardson (2016) that firms' leverage is decreasing in asset volatility, coupled with the positive association between asset returns and volatility in Table 1. The lack of significance at the equity level is then not surprising in light of the equity-asset return restriction (1): Equity returns increase with elasticity, but as elasticity increases asset returns decrease—the latter effect partially offsetting the former. The difference return reported in row $\bar{\mu}_E - \bar{\mu}_A$ allows us to gauge the pure effect of elasticity on equity returns, controlling for the influence of other characteristics that may correlated with elasticity. The return spread is substantial (0.758% per month) and statistically significant (t-stat is 2.809), with sign in accordance with (H–1a). This spread uses different weights for the equity and asset legs. To verify that the effect remains for a more traditional long-short portfolio we use equity weights for both legs in row $\bar{\mu}_E - \bar{\mu}_A^e$. The spread remains substantial (0.543% per month) and statistically significant (2.432). Row $\bar{\sigma}_E$ confirms that equity ivol is significantly increasing in elasticity, in line with (H–1b). Finally, $\bar{\sigma}_A$ is decreasing in elasticity, which is again consistent with Choi and Richardson (2016) and elasticity increasing with leverage.

To address the remaining hypotheses we again follow Ang et al. (2006) and use a 1/0/1

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[10]We use previous month standard deviations of daily returns of equity and assets to compute monthly total volatilities. It is worth noting that Choi and Richardson (2016) estimate ϵ via the BSM formula, rather than utilizing the no-arbitrage restriction (2) to arrive at (3). Using the latter approach avoids a source of model error.

strategy to sort equity and asset returns into quintile portfolios based on their previous month's asset ivol. Ivol is estimated from previous-month daily returns, and we hold the quintile portfolios for one month. Asset and equity ivol are proxied by the standard deviation of the residuals from the three-factor model of Fama and French (1993).[11]

Before considering the remaining hypotheses we investigate to what extent our sample exhibits the negative equity ivol-return relationship. Panel A of Table 3 shows how equity and asset returns, and elasticity vary with equity ivol. The association between equity returns and ivol is weakly positive, unlike what Ang et al. (2006) report. This is not surprising, however, in light of recent findings by Linnainmaa and Roberts (2018) that show that also other "anomalies" are weaker in the time period of our sample. Notice, however, that sorting on equity ivol is equivalent to sorting on asset ivol *and* elasticity. There is thus a confounding effect due to elasticity, which indeed we find to be monotonically increasing in equity ivol. Theorem 1 is about the joint effect of asset ivol on equity ivol and expected returns, however, and we consider sorts on asset ivol in the following.

Panel B of Table 3 reports results for (H–2a)–(H–2c). Before considering the hypotheses observe that we find that the relation between average asset ivol $\bar{\sigma}_A$ and equity returns $\bar{\mu}^E$ is negative, consistent with Ang et al. (2006), but not significant. Again we attribute the lack of significance to using a smaller sample than theirs, for a different time period. On the other hand, we find that the relation between asset ivol, $\bar{\sigma}_A$, and average excess asset returns $\bar{\mu}_A$ is positive—consistent with some degree of nonlinearities at the asset level. Row $\bar{\mu}_E^a - \bar{\mu}_A^a$ reports the test for (H–2a) when using asset weights in forming the portfolio returns. As for the test reported in the previous Table, the double difference allows us to control for other characteristics that may be correlated with elasticity. Indeed, as the equity and asset high-minus-low portfolios are composed of the same firms the only source

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[11]Using the CAPM or the five factor model of Fama and French (2015) does not materially affect our results.

of difference in returns is elasticity. As predicted the average double difference is negative at -0.345% per month (-4.1% annually) and statistically significant, with a t-statistic of -2.568. To remove any possible contribution from differences in weights we test the double difference (H–2a) using equity weights for both equity and assets in the next row. Identical weights admits an interpretation of the double difference as the equity-asset spread on long-short asset-ivol portfolios. The resulting average double difference is monotonically decreasing in asset ivol, $\bar{\sigma}_A$. The difference between the quintiles with highest and lowest pre-formation $\bar{\sigma}_A$ is negative at -0.271% (-3.3% annually) and statistically significant with a t-stat of -3.783.

Table 3 also reports the weighted pre-formation ivol of equity, $\bar{\sigma}_E$, and the average elasticity, $\bar{\epsilon}$, computed as described in (3). Consistent with (H–2b) and (H–2c) we find that $\bar{\sigma}_E$ is uniformly increasing in $\bar{\sigma}_A$, while the elasticity is uniformly decreasing in $\bar{\sigma}_A$. The fact that elasticity is decreasing lends empirical support to the main mechanism at work in Theorem 1—that the negative relation between μ_E and σ_A is caused by the response of elasticity to asset ivol.

To test (H–3) we use a conditional double sort on leverage and asset ivol, $\bar{\sigma}_A$. We first sort firms into two leverage buckets. We then sort each leverage bucket into five quintiles according to the firms' asset ivol. Table 4 reports the outcome of the double sort. For each leverage bucket we report the average spread in average equity return between high and low asset ivol portfolios, as in (H–3); $\bar{\mu}_E^{H-L} = \bar{\mu}_E^H - \bar{\mu}_E^L$. Column three reports the double difference, between the high and the low leverage bucket. Consistent with the prediction of Theorem 1 we find that the difference is negative at -0.013% monthly (-15.6% annually) and statistically significant with a t-stat of -2.181. We also report elasticity spreads between high and low asset ivol, within each of the leverage buckets. The difference in elasticity *spreads* is negative and significant, consistent with ivol having

an important effect on elasticity when controlling for leverage.

5   Time-series Implications

The previous Section addresses how leverage creates a channel for ivol to cause cross-sectional variation in the covariance between equity returns and the SDF. Since the source of the negative equity ivol-return relationship is the mechanical response of equity elasticity to asset ivol one should expect to find a similar relationship also in the time series of firms' returns. We now turn to investigate to what extent the elasticity of levered equity can help explain that high ivol minus low ivol portfolios generate negative alpha, as first observed by Ang et al. (2006). For variations in elasticity to explain part of the observed alpha it is sufficient that elasticity is correlated with the SDF—the "market price of risk"—as equity beta can be expressed as the product of asset beta and elasticity. Unconditional estimates of alpha are then biased (Jagannathan and Wang, 1996). We proceed by first investigating the role of ivol and elasticity in a simple reduced form extension of the BSM model from Section 2, and use simulations to quantify the contribution of variations in elasticity. The simulation is based on an empirical parametrization of elasticity, which accounts for both time varying asset beta and ivol. As a second step we estimate the empirical contribution of time variation in asset beta and ivol in reducing unconditional alpha in our sample.

5.1   A Model with Stochastic Asset Volatility

Consider an SDF which is affine in the return on the market portfolio. If anything, this stacks the cards against being able to rationally explain alpha, as it rules out any contribution from priced skewness—however small. The realized excess return on the market

portfolio is given by

$$R_{t+1}^M = \lambda_t^M \sigma_M + \sigma_M x_{t+1}^M,$$

where λ_t^M is the market price of risk, σ_M is the volatility of R^M, and x_{t+1}^M is a mean zero random variable. The conditional market risk premium is $E_t\left\{R_{t+1}^M\right\} = \lambda_t^M \sigma_M$. We assume that the market price of risk λ_t^M has dynamics

$$\lambda_t^M = \lambda_M + \sigma_\lambda x_t^\lambda,$$

where λ_M is the average market price of risk, σ_λ is its volatility, and x_t^λ is mean zero and independent of x_{t+1}^M. The excess return of the underlying assets is

$$R_{t+1}^A = v_s R_{t+1}^M + v_t^i y_{t+1}^A$$

where y_{t+1}^A is mean zero and independent of x_{t+1}^M. Due to the affine specification of the SDF we have that $v_s = \beta_A \sigma_M$ where β_A is the traditional CAPM-beta. We keep the systematic risk factor loading v_s constant for now, while we allow the idiosyncratic risk factor loading v_t^i to be stochastic. It follows that the conditional risk premium of the asset is $E_t\left\{R_{t+1}^A\right\} = \lambda_t^M v_s$. We specify the dynamics of v_t^i as

$$v_t^i = v_i + \sigma_i y_t^i,$$

where v_i is the average exposure, σ_i the volatility and y_t^i is mean zero and correlated with x_t^λ with coefficient $\rho_{\lambda,i}$. From restriction (1) we know that equity returns are given by

$$R_{t+1}^E = \epsilon_t R_{t+1}^A.$$

To facilitate a formal analysis of the role of stochastic asset ivol we first consider the following simple reduced form elasticity

$$\epsilon_t = \epsilon_0(v_i) + \epsilon_2(v_t^i - v_i).$$

With $\epsilon_0'(v_i) < 0$ and $\epsilon_2 < 0$ this specification captures that elasticity is a decreasing function of asset ivol, in line with the BSM model. We use a more complex functional form that better captures the empirically observed characteristics of elasticity when we later simulate the model.

Bias in unconditional alpha

It follows from the above setup that the conditional and unconditional expected excess equity returns are $E_t\left\{R_{t+1}^E\right\} = \epsilon_t v_s \lambda_t^M$, and

$$E\left\{R_{t+1}^E\right\} = v_s \text{Cov}\left(\epsilon_t, \lambda_t^M\right) + v_s \epsilon_0(v_i)\lambda_M. \tag{4}$$

It is worth noting that while the cross-sectional analysis pertains to variations in the second term in (4), the time series effects materialize through variations in the first term: Unconditional expected equity returns are increasing in the covariance between elasticity and the market price of risk. We can express the first part of (4) as

$$v_s \text{Cov}\left(\epsilon_t, \lambda_t^M\right) = v_s \epsilon_2 v_i \sigma_\lambda \rho_{\lambda,i}. \tag{5}$$

Equation (4) shows that allowing for time varying asset ivol, the difference in unconditional expected equity returns between high and low ivol portfolios will be amplified by their difference in the covariance term in (5). Particularly important is the difference in

the sensitivity of elasticity to v_t^i. We show later, using simulations that match empirically observed quantities, that the magnitude of the sensitivity of elasticity to risk is significantly different for high and low ivol portfolios. We first demonstrate that our setup is consistent with the core assumptions of Jagannathan and Wang (1996).

The conditional equity CAPM beta is

$$\beta_t^E = \epsilon_t \beta_A = \epsilon_t \frac{v_s}{\sigma_M},$$

and the conditional equity excess return is

$$\mathrm{E}_t \left\{ R_{t+1}^E \right\} = \beta_t^E \mathrm{E}_t \left\{ R_{t+1}^M \right\}.$$

Define $\mathrm{E}_t \left\{ R_{t+1}^M \right\} = \mu_t^M$ and $\mathrm{E} \left\{ R_{t+1}^M \right\} = \bar{\mu}_M$, and similarly for R^E and β^E. The unconditional expected return is

$$\bar{\mu}_E = \mathrm{Cov} \left(\beta_t^E, \mu_t^M \right) + \bar{\beta}_E \bar{\mu}_M. \tag{6}$$

Using the CAPM regression to describe $\mathrm{E} \left\{ R_{t+1}^E \right\}$ we have

$$\bar{\mu}_E = \alpha_E^U + \beta_E^U \bar{\mu}_M. \tag{7}$$

Combining (6) and (7) with the previous definitions of β_t^E and μ_t^M we arrive at the following expression for the unconditional equity alpha

$$\alpha_E^U = v_s \mathrm{Cov} \left(\epsilon_t, \lambda_t^M \right) + \bar{\mu}_M \left(\bar{\beta}_E - \beta_E^U \right).$$

Unconditional equity alpha is consequently increasing in the covariance between elasticity and the market price of risk. The covariance is increasing in ϵ_2, as evident from equation

(5). We show below that the estimated ϵ_2 is significantly different between high and low ivol portfolios, which contributes to partially explain the spread in CAPM-alpha between high and low ivol portfolios.

Simulation: Time varying elasticity versus time varying asset beta

While the preceding analysis establishes that leverage opens a channel for asset ivol to affect unconditional alpha, through its effect on elasticity, it remains to determine the economic impact. A simple simulation exercise allows us to quantify the relative contribution of variations in elasticity and asset beta in generating unconditional alpha of long-short asset-ivol portfolios. To this end, systematic risk of assets is allowed to develop according to

$$v_t^s = v_s + \sigma_s x_t^s,$$

where x_t^s is correlated with x_t^λ, with correlation coefficient $\rho_{\lambda,s}$. We generalize the functional form for elasticity

$$\epsilon_t = \epsilon_0(\bar{\sigma}_A) \exp\left[\epsilon_1(v_t^s - v_s) + \epsilon_2(v_t^i - v_i) + \sigma_\epsilon y_t^\epsilon\right],$$

where $\bar{\sigma}_A$ is the average total asset volatility. This specification captures that elasticity varies with the level of priced risk, v^s, in addition to idiosyncratic risk. We also allow for a number of unspecified quantities that may affect elasticity through their effect on leverage. In addition to systematic and idiosyncratic risk, leverage may vary with the excess return and volatility of the market portfolio, the shape of the yield curve, default risk, etc. The effects of the latter variables are captured by the error term y^ϵ.

We estimate the parameter values for the simulation, for our high and low asset-ivol portfolios. Table 5 reports estimates of the unconditional CAPM regression for equity,

in Panel A, and regressions of portfolio elasticities on estimates of v_t^s and v_t^i, in Panel B. We use the estimates of elasticity and asset ivol at portfolio level that we report in Table 3. To determine v_t^s we multiply estimated portfolio asset betas with the monthly market volatility, estimated using daily data on a monthly basis. In the regression for elasticities we control for additional factors that may help capture $\rho_{\lambda,\epsilon}$, the residual correlation between y_t^ϵ and x_t^λ.[12] Observe from Panel A that we find that unconditional CAPM-alpha for the high ivol portfolio is negative while it is positive for the low ivol portfolio, consistent with Ang et al. (2006). It is apparent from the elasticity regression in Panel B that the high and low ivol portfolios have different exposure to the systematic and idiosyncratic risks factors. While the elasticity of the low ivol portfolio has both economically and statistically significant responses to v_t^s and v_t^i, the response of the high ivol portfolio is negligible. These findings are consistent not only with the theoretical prediction that elasticity is decreasing in volatility, but also with the empirical finding of Choi and Richardson (2016) that leverage is decreasing in asset volatility.

To assess the contribution of asset ivol in generating unconditional alpha, compared to time varying asset systematic risk, we try to replicate the alpha of the low ivol portfolio. To reduce the influence from extreme values of leverage we restrict ϵ to the interval $[1, 10]$. We draw all random variates from standard normal distributions, and run two sets of simulations, each consisting of 100 paths that cover 10,000 months. The two sets of simulations correspond to $\rho_{\lambda,\epsilon} = 0.5, 1.0$. In both sets we impose $\rho_{\lambda,s} = \rho_{\lambda,i} = -0.85$. Remaining parameters match empirically observed quantities: The parameters chosen for λ_t^M ensure that the average Sharpe ratio is 0.11 monthly, consistent with that observed for a broad market index during our sample period. We set the volatility of the market price

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[12]The factors used are the market volatility, market excess returns, the default spread, the difference of the yield of the AAA-corporate bonds and BAA-corporate bonds, as well as the term-spread—the difference between the ten and one year yields on treasury bonds, both obtained from the FRED database. The return on the market portfolio is from Kenneth French's database.

of risk equal to 0.1 which combined with $\sigma_M = 0.045$ implies a variation in the market risk premium equal to 0.0045 in line with previous studies (e.g. Lewellen and Nagel, 2006). The parameters governing systematic risk, idiosyncratic risk, and elasticity are set to be consistent with the ones observed for the low ivol portfolio.

Table 6 reports the results of the simulations. Table entries are averages across paths. As evident from column three in Panel A the model reproduces between 30% and 40% of the observed unconditional CAPM-alpha for the low ivol portfolio. From Tables 5 and 6 we can surmise that the unconditional alpha of low ivol portfolios is partially explained by the covariance of elasticity with the market price of risk. Elasticity of low ivol firms has higher sensitivities to time variation in risk, both systematic and idiosyncratic, which are correlated with λ_t^M, as reported in Panel B. Although comovement between the systematic risk loading and market price of risk affects also the underlying assets, the variation in systematic risk is not sufficient to generate significant alpha at asset level. The effect is amplified at the equity level through elasticity. Finally, residuals of elasticity may also correlate with the market price of risk, which increases further the fraction of unconditional alpha being explained.

5.2 Empirical Analysis

In this section we investigate to what extent elasticity affects time variation in risk exposure that is correlated with the market price of risk, and thus helps explain unconditional CAPM-alpha. We follow the two-stage approach of Boguth et al. (2011) to estimate the conditional CAPM, which ameliorates potential under- and over-conditioning biases in the conditional-beta estimates.

The first stage consists of regressions of estimated contemporaneous equity betas of

each test asset on a k-dimensional instrument, Z_{t-1},

$$\hat{\beta}_t^E = \delta_0 + \delta_1 \cdot Z_{t-1} + u_t. \tag{8}$$

The dependent variables $\hat{\beta}_t^E$ are obtained as value weighted averages of contemporaneous equity betas of each firm, estimated monthly using one month of daily observations. First-stage beta estimates $\underline{\beta}_t^E = \hat{\beta}_t^E - u_t$ are then used in the second stage, where we regress portfolio returns on the return of the market portfolio,

$$R_t^E = \alpha^C + (\phi_0 + \phi_1 \underline{\beta}_t^E) R_t^M + e_t. \tag{9}$$

The intersect α^C is the alpha of the conditional CAPM. The parameters ϕ_0 and ϕ_1 allow a rescaling of conditional beta, to help increase its predictive power. To stay as close as possible to the theory and to better judge the improvement in alpha from our instrumental approach we restrict $\phi_0 = 0$ and $\phi_1 = 1$. These restrictions correspond to assuming that the conditional beta estimates are unbiased and efficient (Boguth et al., 2011).

We use four sets of instruments in (8) that may capture important variation in elasticity and asset systematic risk. Because expected returns are affected by the volatility of the SDF we include in all sets the previous-month market volatility $\sigma_{M,t-1}$, obtained as the standard deviation of the daily excess returns on the market portfolio, multiplied by $\sqrt{22}$ to effectuate a monthly basis. In the first set of instruments we add firm characteristics that should affect asset beta and elasticity. Although theory predicts that elasticity should be a powerful predictor of equity beta we do not include it explicitly, as it does not allow us to distinguish between the effects of leverage and volatility on equity beta. Using elasticity directly also precludes identification of the interaction of systematic and idiosyncratic risk in scaling equity beta, as identified and discussed in Section 5.1: The theory predicts that

differences in the *level* of asset ivol across test assets cause differences in the sensitivity of elasticity to variations in systematic risk. To capture variations in elasticity we instead include asset ivol, computed as v_t^i multiplied by $\sqrt{22}$, and leverage $LEV_t = A_t/E_t$. To ease comparisons across risk factors we add asset systematic volatility v_t^s, given by the product between previous-month value-weighted asset beta and σ_t^M. Systematic volatility v_t^s should capture variations in both asset beta and elasticity. To better judge the role of asset ivol the second set of instruments omits v_t^i from the first set. Set three of instruments adds macroeconomic variables to σ_t^M, similar to those used in other papers (like Fama and French, 1989). We add the previous-month market return R_{t-1}^M, the default spread DEF_{t-1} computed as the difference in yields between BAA- and AAA-rated corporate bonds (obtained from the website of the FRED), and the term spread TS_{t-1} computed as the difference in yields between ten years and one year treasury bills (from the FRED website). Finally, the fourth set adds leverage to the first set.

Table 7 reports the results from the predictive regressions using instrument sets 1–3. The regression on Set 1 demonstrates that v_t^i is far more significant for the low ivol portfolio, which is the portfolio with higher elasticity. The signs of the coefficients are also different. For the low ivol portfolio the coefficient is -9.480 while it is positive and insignificant for the high ivol portfolio. Dropping v_t^i, in Set 2, reduces R^2 from 25.5% to 13.9% for the low ivol portfolio. Consistent with the theory that ivol is an important instrument for portfolios where elasticity is larger, there is only a minor reduction in R^2 for the high ivol portfolio, from 23.8% to 22.9%. We moreover find that the coefficient on v_t^s is positive for both portfolios, but lower for the low ivol portfolio.

When we use only macroeconomic variables, in Set 3, there is a sharp decline in R^2 for both portfolios. Market volatility, σ_t^M, appears with opposite signs across the two portfolios, and is significant for low ivol but insignificant for high ivol. While we have not

directly analyzed the role of how σ_t^M affects elasticity, it seems reasonable that it should have a negative sign and play a more prominent role in portfolios with higher elasticity. It is a source of uncertainty about the future value of equity, and is in this regard similar to volatility.

The Equity and Asset columns of Table 8 report results for unconditional CAPM regressions, for the high-minus-low ivol portfolio. Intercepts are negative for both equity and assets at -0.687% and -0.350% respectively. Eliminating the effect of leverage thus reduces alpha by almost 50%, in line with the findings of Choi and Richardson (2016), who sort portfolios on equity and asset betas, book-to-market value, and size.

Columns (1), (2), and (3) of Table 8 report estimates of the conditional model (9) for the three sets of instruments. Including v_t^i increases the ability of the conditional model to explain unconditional equity alpha. The reduction in alpha is strongest for instrument Set 1, at -36.8%. Removing v_t^i, in Set 2, leads to a smaller reduction in alpha, at -32.7%. Note also that the R^2 of the regression decreases along with the t-stat of the slopes. Part of the importance of v_t^s as an instrument is due to its effect on elasticity. When we remove also v_t^s in Set 3, and use only macroeconomic instruments typically used for estimation of conditional models, the reduction in alpha is -15.6%. This reduction is materially weaker than that obtained by the alternative sets of instruments. The R^2 decreases as well, along with the t-stats of the slope coefficients.

Leverage versus volatility in elasticity variation

Table 8 demonstrates that time variation in equity beta explains a significant part of the spread in conditional alpha between high and low asset ivol portfolios. The Table also demonstrates that asset ivol is a significant instrument, especially so for the low ivol portfolio. The theory ascribe these findings to the effect of time variation in asset ivol

on elasticity. Elasticity is however also a function of leverage, which is stochastic. To investigate the relative contribution of these two sources of variation in elasticity, and thus to evaluate the strength of the link between asset ivol and elasticity, we regress our measure of elasticity (3) for the high and low ivol portfolios on the first and fourth sets of instruments. Both sets includes systematic and idiosyncratic volatility, while the second set adds leverage.

Panel A of Table 9 reports the results for the high ivol portfolio. As reported in Table 3 the average elasticity of this portfolio is 1.147, which is the lowest among all the quintile portfolios. Given the low elasticity of the portfolio, one should expect relatively low sensitivity to volatility. As predicted we indeed find that the coefficients on both risk factors are small in magnitude and the risk factors do not explain much of the variation in elasticity. R^2 increases drastically, from 13.2% to 67.6%, when we employ the alternative instruments that add leverage to the set of the covariates. Time variation in leverage is thus sufficient to explain a large part of the time variation in elasticity. Notice also from the Table that the standard deviation of elasticity $\sigma(\epsilon_t)$ is quite low at 0.074 monthly. Such low time variation implies that elasticity should not be able to contribute significantly to explain unconditional alpha for the high ivol portfolio, in line with the predictions of the theory.

The results for the low ivol portfolio in Panel B are orthogonal to that of the high ivol portfolio. The average elasticity, as reported in Table 3, is 1.731 and is thus the highest among the quintile portfolios. Adding leverage to the covariates of the elasticity regression does not contribute to explain variations in elasticity, however. Indeed, the R^2 is almost the same in the two regressions, at 25.7% without leverage and 27% with leverage. The coefficients on the risk factors are both large in magnitude and statistically strongly significant irrespective of the presence of leverage. We attribute this evidence to the cross

sectional sensitivity to volatility due to leverage. As documented by Choi and Richardson (2016) low asset ivol firms tend to have persistently high leverage, and as a consequence even small variations in asset risks generate strong responses in elasticity. Thus, while leverage does not exhibit sufficient variation in the time series to materially move elasticity, leverage is crucial in determining elasticity's response to variations in volatility.

The last four rows of Table 7 report the predictive regressions for equity betas using the fourth set of instruments. Adding leverage does not significantly change the results of the predictive regressions for either portfolio. Indeed, the R^2 improves only marginally and the coefficient estimates for the risk factors are largely unaffected. Adding leverage to the set of instruments thus does not improve the predictive power. For the high ivol portfolio this is as expected since variations in elasticity is not sufficiently strong to materially affect time-variation in equity beta. Equity beta is for this portfolio rather driven by variations in systematic volatility/asset beta. The finding that leverage does not improve the predictive power for the low ivol portfolio suggests that elasticity is the important source of variation in predicting equity beta, with some contribution from time varying asset beta (which also affects elasticity).

Table 10 reports results for the conditional CAPM for the high and low ivol portfolios with predicted equity betas obtained using the first and fourth sets of instruments, as reported in Table 7. We denote the associated predicted equity betas by $\underline{\beta}_E$ and $\underline{\beta}_E^{LEV}$ respectively. We constrain ϕ_0 and ϕ_1 in (9) as before. The estimation exercise confirms that adding leverage to the set of instruments does not materially improve the ability of the conditional model to explain the unconditional CAPM alphas of the ivol portfolios. For the high asset-ivol portfolio the percentage of explained alpha ($\alpha_E/\alpha_E^U - 1$) changes from -77.6% to -82.3%. For the low asset ivol portfolio $\alpha_E/\alpha_E^U - 1$ changes from -18.0% to -18.5%. Overall, the percentage of unconditional spread in CAPM alpha changes from

−36.8% to −38.6%. The performance of the model in terms of R^2 does not improve, and actually decreases for the high ivol portfolio. The results suggests that variations in elasticity is predominantly due to variations in volatility rather than variations in leverage. This finding is consistent with the behavior of elasticity in the BSM model and our extension of it to stochastic volatility.

6   Conclusion

The law of one price forces equity excess returns to be proportional to asset excess returns, with the constant of proportionality equal to the elasticity of equity with respect to assets. The elasticity is a function of several firm characteristics, including asset volatility. Because equity elasticity is uniformly decreasing in volatility, expected equity returns are decreasing in asset ivol—both in the cross section and in the time series. Equity ivol is on the other hand increasing in asset ivol. An observed negative equity ivol-return relationship is consequently fully consistent with rationally priced corporate liabilities that reflect variations in asset ivol. We formally prove the presence of this mechanism, and verify that its empirical implications bear out in a sample of returns on publicly traded firms' equity and assets.

While our analysis is based on a view of equity as a call option on assets, the basic premise of the analysis is that the cash-flow that supports the price of study is *nonlinear* in some basic, priced risk factors. The implications of the analysis thus go beyond equity: One should expect to find correlation between ivol and expected returns for any derivative with non-trivial nonlinearities, be it real or financial. Ivol is thus truly a "characteristic" in asset pricing models that do not explicitly take nonlinearities into account, and is neither an "anomaly factor" nor a "priced risk factor." Indeed, the *absence* of ivol-return correlation may indicate deviations from the law of one price.

A   Proofs

Proof of Proposition 1. The first part of the result is immediate from (1). To show that the total risk restriction (2) also applies to idiosyncratic risk, decompose $\sigma_A^2 = v_s^2 + v_i^2$, where v_s and v_i are systematic and idiosyncratic risk factor loadings. The idiosyncratic risk factor loading of equity is $\epsilon v_i = \epsilon \sqrt{\sigma_A^2 - v_s^2} \propto \epsilon$. $\qquad\square$

Proof of Theorem 1. The value of equity is $E = A\mathrm{N}(d) - Ke^{-rT}\mathrm{N}(d')$, with the usual definition of d and $d' = d - \sigma_A\sqrt{T}$. Solvency of the firm implies that $d \geq 0$.

Using that $\mu_A > r$ it is sufficient for the first claim to show that $\frac{\partial\epsilon}{\partial\sigma_A} < 0$. We have that

$$
\frac{\partial\epsilon}{\partial\sigma_A} = \frac{\partial}{\partial\sigma_A}\frac{AE_A}{E} = \frac{A}{E}n(d)\sqrt{T}\left(1 - \frac{d}{\sigma_A\sqrt{T}} - \epsilon\right),
$$

where E_A is the partial derivative of equity with respect to assets. Proposition 2 of Bergman et al. (1996) establishes that generally $\epsilon \geq 1$, which in our special case takes the form $\epsilon = \frac{A\mathrm{N}(d)}{A\mathrm{N}(d) - Ke^{-rT}\mathrm{N}(d')} > 1$.

Define $R(d) = \frac{\mathrm{N}(d)}{n(d)} > 0$, and $f(d) = R(d) - d$. To show the second claim we need to show that

$$
\begin{aligned}
\frac{\partial\sigma_E}{\partial\sigma_A} &= \frac{\partial\epsilon}{\partial\sigma_A}\sigma_A + \epsilon = \frac{A}{E^2}\left\{n(d)\frac{\partial d}{\partial\sigma_A}E - \mathrm{N}(d)\frac{\partial E}{\partial\sigma_A}\right\}\sigma_A + \frac{A\mathrm{N}(d)}{E} \\
&= \frac{\epsilon}{R(d)}\left\{R(d) - d - \frac{\mathrm{N}(d - \sigma_A\sqrt{T})\sigma_A\sqrt{T}}{e^{\sigma_A\sqrt{T}d - \frac{1}{2}\sigma_A^2 T}\mathrm{N}(d) - \mathrm{N}(d - \sigma_A\sqrt{T})}\right\} \\
&\triangleq \frac{\epsilon}{R(d)}\left\{f(d) - g(d)\right\}
\end{aligned}
$$

is strictly positive for all $d \geq 0$ (and thus for all $\sigma_A > 0$).

It is easy to verify that $f(0) - g(0) > 0$. Consider therefore the change in f and g as d

increases beyond 0. First,

$$f'(d) = \sqrt{\frac{\pi}{2}} e^{\frac{1}{2}d^2} N(d) d$$

confirms that f is uniformly strictly increasing on $\mathbb{R}_{++}$. Second, let $\gamma(d) \triangleq e^{\sigma_A \sqrt{T} d - \frac{1}{2}\sigma_A^2 T}$ and observe that $n(d') = \gamma(d)n(d)$ iff $\gamma(d) = n(d')/n(d)$. Hence, with $h = \sigma_A \sqrt{T}$

$$g(d) = h \frac{N(d-h)}{\gamma(d)N(d) - N(d-h)} = \frac{R(d-h)}{\frac{1}{h}[R(d) - R(d-h)]} > 0,$$

which equals the ratio of R to the average slope of R over $[d-h, d]$. To establish that g is decreasing it is thus sufficient to demonstrate that R increases at a lower rate than R'. We have that

$$R'(x) = \frac{n^2(x) + xN(x)n(x)}{n^2(x)} = 1 + xR(x),$$

$$R''(x) = R(x) + xR'(x),$$

which establishes that $R'(x) < R''(x)$ on $[1, \infty)$. For later use, observe also that $R'(x), R''(x) > 0$ on $\mathbb{R}_+$.

It remains to demonstrate that $\forall\, x \in (0,1)$

$$R''(x) = R(x) + xR'(x) > R'(x) \iff R(x) > (1-x)R'(x),$$

or equivalently, by Taylor's Theorem, that

$$R(0) + R'(0)x + \frac{1}{2}R''(0)x^2 + o(x^3) > (1-x)R'(x).$$

The remainder term $o(x^3) > 0$ because $o(x^3) = \frac{1}{6}R'''(z)z^3$ for some $z \in [0, x]$ and $R'''(z) = 2R'(z) + zR''(z) > 0 \ \forall\, z \in [0, x]$. Using that $R'(0) = 1$ and $R''(0) = R(0)$ it is therefore

sufficient to establish the inequality

$$R(0) + x + \frac{1}{2}R(0)x^2 > (1-x)R'(x) = 1 - x + x(1-x)R(x),$$

or equivalently

$$[R(0) - 1] + 2x + \frac{1}{2}R(0)x^2 > x(1-x)R(x). \tag{10}$$

Because $R(0) = \sqrt{\frac{\pi}{2}} > 1$ inequality (10) holds if

$$2 + \frac{1}{2}R(0)x > (1-x)R(x). \tag{11}$$

Observe now that (11) holds at 0: $2 > R(0)$. Because the left hand side of (11) is increasing in x it is sufficient to establish that the right hand side is decreasing in x. For this to be true we need

$$\frac{\partial}{\partial x}(1-x)R(x) = (1-x)R'(x) - R(x) < 0$$

iff

$$[R(x) - 1](1-x) + x^2 R(x) > 0,$$

which is true for all $x \in (0,1)$. $\qquad \square$

Proof of Proposition 2. Using second order Taylor expansions of $m_{t+1} = m(x_{t+1})$ around $\mathrm{E}\{x_{t+1}\}$, and of $E_{t+1} = C(A_{t+1})$ around A_t, where C is a sufficiently smooth function,

yields

$$\mu_E(t) - r_t \approx -\frac{\mathrm{Cov}_t\big(m_t' x_{t+1} + \frac{1}{2}m_t'' x_{t+1}^2, \Delta_t(A_{t+1} - A_t) + \frac{1}{2}\gamma_t(A_{t+1} - A_t)^2\big)}{\mathrm{E}_t\{m_{t+1}\}\, C(A_t)},$$

$$= -\frac{m_t'}{\mathrm{E}_t\{m_{t+1}\}}\epsilon_t \mathrm{Cov}_t\big(x_{t+1}, R_{t+1}^A\big) - \frac{1}{2}\frac{m_t'}{\mathrm{E}_t\{m_{t+1}\}}\frac{\gamma_t A_t}{\Delta_t}\epsilon_t \mathrm{Cov}_t\big(x_{t+1}, (R_{t+1}^A)^2\big)$$

$$- \frac{1}{2}\frac{m_t''}{\mathrm{E}_t\{m_{t+1}\}}\epsilon_t \mathrm{Cov}_t\big(x_{t+1}^2, R_{t+1}^A\big) - \frac{1}{4}\frac{m_t''}{\mathrm{E}_t\{m_{t+1}\}}\frac{\gamma_t A_t}{\Delta_t}\epsilon_t \mathrm{Cov}_t\big(x_{t+1}^2, (R_{t+1}^A)^2\big),$$

where $m_t' = -\lambda_t m_t|_{x_t=0}$ and $m_t'' = \lambda_t^2 m_t|_{x_t=0}$. Using the structure of R^A and the symmetry of x the second and third covariance terms are proportional to $\mathrm{E}_t\{x_{t+1}^3\} = \mathrm{E}_t\{(-x_{t+1})^3\} = -\mathrm{E}_t\{(x_{t+1})^3\}$ iff $\mathrm{E}_t\{x_{t+1}^3\} = 0$. $\qquad\square$

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Table 1: Summary statistics

The sample consists of monthly observations for the period 1997:12–2016:12. We sort firms in leverage quintiles at the beginning of each month and value weigh returns within each quantile. Each entry represents the time series average for a quantile portfolio. Subscripts E and A in column headings designate equity and assets. Column two reports the value weighted leverage of each quintile. Columns three and four report average returns, columns five and six report pre-sorting CAPM beta, columns seven and eight report pre-sorting ivol relative to the three factor model of Fama and French, columns nine and ten report volatility, and columns 11 and 12 report skewness.

Leverage quintile	Leverage, $\overline{A/E}$	$\bar{\mu}_E$	$\bar{\mu}_A$	$\bar{\beta}_E$	$\bar{\beta}_A$	$\bar{\sigma}_E$	$\bar{\sigma}_A$	$\overline{TVOL}_E$	$\overline{TVOL}_A$	$\overline{Skew}_E$	$\overline{Skew}_A$
L	1.064	0.634%	0.611%	0.968	0.903	6.656%	6.189%	9.144%	8.486%	-0.345	-0.353
2	1.199	0.453%	0.428%	0.898	0.739	6.498%	5.324%	8.792%	7.145%	-0.448	-0.456
3	1.395	0.491%	0.438%	0.939	0.671	6.762%	4.789%	9.143%	6.417%	-0.545	-0.654
4	1.809	0.456%	0.332%	0.958	0.524	7.125%	3.962%	9.582%	5.229%	-0.208	-0.543
H	4.006	1.042%	0.379%	1.070	0.413	9.300%	3.857%	11.970%	4.845%	0.046	-0.806

Table 2: Portfolios sorted on elasticity

Each month we sort firms' equity and asset returns into five quintiles based on their previous month elasticity, computed for each firm using (3) and daily equity and asset returns. We next determine beginning-of-the-month equity and asset market weights to compute the following portfolio statistics. The first row reports average equity elasticity (3), using equity weights. Row σ_E^i reports average equity ivol for each bucket, and the difference between high and low elasticity portfolios. Rows $\bar{\sigma}_E$ and $\bar{\sigma}_A$ report average equity and asset ivol for each bucket. Rows $\bar{\mu}^E$ and $\bar{\mu}^A$ report average monthly excess returns for equity and assets, and test the difference of high minus low elasticity portfolios. Row $\bar{\mu}_E - \bar{\mu}_A$ reports the double difference in average equity and asset returns across high minus low $\bar{\epsilon}$ portfolios. This exercise is repeated in the next row, using equity weights in average asset portfolio returns, $\bar{\mu}_A^e$.

	L	2	3	4	H	$H - L$
$\bar{\epsilon}$	1.052	1.173	1.354	1.709	3.129	
$\bar{\mu}_E$	0.566%	0.463%	0.483%	0.418%	1.21%	0.648%
						(1.848)
$\bar{\mu}_A$	0.557%	0.411%	0.419%	0.382%	0.448%	-0.109%
						(-0.425)
$\bar{\mu}_E - \bar{\mu}_A$						0.758%
						(2.809)
$\bar{\mu}_E - \bar{\mu}_A^e$	0.013%	0.021%	0.058%	0.082%	0.556%	0.543%
						(2.432)
$\bar{\sigma}_E$	6.769%	6.386%	6.660%	7.050%	9.018%	2.248%
						(10.912)
$\bar{\sigma}_A$	6.445%	5.423%	4.893%	4.139%	3.336%	-3.081%
						(-23.150)

Table 3: Portfolios sorted on ivol

Each month we sort firms' equity and asset returns into five quintiles based on their previous month ivol, computed for each firm using the Fama-French three-factor model and daily asset returns. We next determine beginning-of-the-month equity and asset market weights to compute the following portfolio statistics. Panel A sorts firms on their equity ivol σ_E, while Panel B sorts on asset ivol, σ_A. Rows $\bar{\sigma}_E$ and $\bar{\sigma}_A$ report the average equity and asset ivol of each bucket. Rows $\bar{\mu}^E$ and $\bar{\mu}^A$ report average monthly excess returns for equity and asset portfolios, and test the difference of high minus low ivol portfolios. Row $\bar{\mu}_E - \bar{\mu}_A$ reports the double difference in average equity and asset returns across high minus low $\bar{\sigma}_A$ portfolios. This exercise is repeated in the next row, using equity weights in average asset portfolio returns, $\bar{\mu}_A^e$. The last row reports average equity elasticity (3), using equity weights.

	L	2	3	4	H	$H-L$
			Panel A: Sorting on $\bar{\sigma}_E$			
$\bar{\sigma}_E$	3.925%	5.771%	7.656%	10.350%	17.604%	13.678%
$\bar{\mu}_E$	0.442%	0.634%	0.714%	0.250%	0.808%	0.366%
						(0.654)
$\bar{\mu}_A$	0.395%	0.547%	0.583%	0.256%	0.560%	0.164%
						(0.404)
$\bar{\epsilon}$	1.258	1.297	1.330	1.405	1.643	0.384
						(3.619)
			Panel B: Sorting on $\bar{\sigma}_A$			
$\bar{\sigma}_A$	2.478%	3.768%	5.106%	6.986%	12.211%	
$\bar{\sigma}_E$	3.962%	4.892%	6.227%	8.326%	13.959%	9.996%
						(45.946)
$\bar{\mu}_E$	0.771%	0.342%	0.576%	0.257%	0.543%	−0.227%
						(−0.448)
$\bar{\mu}_A$	0.474%	0.322%	0.538%	0.303%	0.591%	0.117%
						(0.261)
$\bar{\mu}_E - \bar{\mu}_A$						−0.345%
						(−2.586)
$\bar{\mu}_E - \bar{\mu}_A^e$	0.215%	0.023%	0.022%	−0.007%	−0.049%	−0.271%
						(−3.783)
$\bar{\epsilon}$	1.731	1.297	1.215	1.19	1.147	−0.583
						(−11.350)

Table 4: Double sorting on leverage and asset ivol

The table reports the result of a conditional double sort. At the beginning of each month we first sort firms according to their market leverage A/E. Within each leverage bucket we then sort firms according to their asset ivol, σ_A, computed from daily observations over the previous month. For each quintile we compute the equity weighted excess returns. Row $\bar{\mu}_E^{H-L}$ reports the difference in average excess returns between the highest and lowest asset ivol quintile, conditional on either the high leverage bucket (column H) or the low leverage bucket (column L). We also compute the equity weighted elasticities for each $\bar{\sigma}_A$ quintile. The second row reports the values of the difference in elasticities between the highest and the lowest asset ivol quintile. Column $H-L$ reports the difference between the high and low leverage columns.

	Leverage		
	H	L	$H-L$
$\bar{\mu}_E^{H-L}$	-0.744%	0.558%	-1.302%
	(-1.288)	(1.084)	(-2.181)
$\bar{\epsilon}_H - \bar{\epsilon}_L$	-0.976	-0.117	-0.860
	(-6.622)	(-6.896)	(-5.949)

Table 5: Variations in equity alpha and elasticities across asset ivol buckets

Panel A reports unconditional CAPM regressions for the high and low asset ivol portfolios, using monthly excess returns. Panel B reports regressions of average portfolio elasticities on risk factor loadings v_t^s, v_t^i, and a set of controls: the market return, the volatility of the market, the term and default spreads, for which we do not report coefficient estimates. Risk factor loadings v_t^s are computed by multiplying the estimated portfolio beta, obtained from monthly CAPM regressions on daily data, with the volatility of the market; also estimated on a monthly base using daily data. Risk factor loadings v_t^i and ϵ_t are obtained from monthly estimates, reported in Table 3. t-statistics are reported in parentheses.

Panel A: Unconditional CAPM regression			
	$\hat{\alpha}_E^U$	$\hat{\beta}_E^U$	
$R_{t+1}^{E,H}$	-0.216%	1.537	
	(-0.665)	(21.615)	
$R_{t+1}^{E,L}$	0.470%	0.606	
	(2.159)	(12.801)	
Panel B: Elasticity regression			
	$\hat{\epsilon}_0$	$\hat{\epsilon}_1$	$\hat{\epsilon}_2$
$\log(\epsilon_t^H)$	0.042	-0.327	-0.170
	(2.931)	(-1.531)	(-1.015)
$\log(\epsilon_t^L)$	0.035	-13.355	-10.988
	(0.686)	(-7.963)	(-4.449)

Table 6: Alpha bias with time varying elasticity and asset beta

The table reports the results from two sets of simulations, each based on 100 paths and 10,000 months. Panel A reports average monthly unconditional alpha and beta of CAPM regressions for equity and asset returns. The last column reports the ratio of the simulated to the empirically estimated unconditional alphas. Panel B reports the average coefficients from regressions of simulated elasticities on simulated risk factor loadings v_t^s and v_t^i. Parameters used are $\lambda_M = 0.11$, $\sigma_M = 0.045$, $\sigma_\lambda = 0.1$, $v_s = 0.025$, $\sigma_s = 0.0129$, $v_i = 0.025$, $\sigma_i = 0.0125$, $\epsilon(\bar{\sigma}_A) = 1.4$, $\epsilon_1 = \epsilon_2 = -12.5$, $\eta_\epsilon = 0.3$, $\rho_{\lambda,s} = \rho_{\lambda,i} = -0.85$.

	Panel A: Unconditional CAPM simulation			
	$\rho_{\lambda,\epsilon}$	$\hat{\alpha}^{SIM}$	$\hat{\beta}^{SIM}$	$\frac{\hat{\alpha}^{SIM}}{\hat{\alpha}_E^U}$
R_{t+1}^E	1.0	0.194%	0.951	0.411
	0.5	0.137%	0.905	0.292
R_{t+1}^A	1.0	0.000%	0.547	0.000
	0.5	0.000%	0.547	0.000
	Panel B: Elasticity simulation			
		$\hat{\epsilon}_0$	$\hat{\epsilon}_1$	$\hat{\epsilon}_2$
$\log(\epsilon_t)$	1.0	0.441	-17.179	-17.557
	0.5	0.416	-13.543	-13.763

Table 7: Predicted equity betas

The table reports the results of predictive regressions (8) across four different sets of instruments. We regress the high (H) and low (L) ivol portfolios' contemporaneous value-weighted equity betas on three sets of instruments. The first set of instruments consists of market volatility σ_{t-1}^M, portfolio systematic risk v_{t-1}^s, and portfolio idiosyncratic risk v_{t-1}^i. The second set of instruments omits v_{t-1}^i. The third set consists of only the macro-variables σ_{t-1}^M, R_{t-1}^M, the default spread DEF_{t-1}, and term spread TS_{t-1}. Set 4 adds leverage LEV_{t-1} to Set 1. We obtain LEV_{t-1} as the equity value weighted leverage A_{t-1}/E_{t-1} of the firms in the portfolio. For each regression we report the R^2 in the last column, and t-stats in parentheses.

Set	Portfolio	Intercept	v_{t-1}^s	v_{t-1}^i	σ_{t-1}^M	DEF_{t-1}	TS_{t-1}	R_{t-1}^M	LEV_{t-1}	R^2
	H	1.197	8.625	1.641	-10.992					0.238
		(13.913)	(6.671)	(1.641)	(-7.185)					
(1)	L	0.878	5.961	-9.480	-1.120					0.255
		(25.090)	(4.907)	(-5.907)	(-1.806)					
	H	1.315	9.550		-10.544					0.229
		(27.681)	(8.179)		(-6.979)					
(2)	L	0.716	7.280		-3.118					0.139
		(30.631)	(5.684)		(-5.585)					
	H	1.424			2.065	-21.256	5.095	0.176		0.034
		(19.556)			(1.588)	(-2.492)	(2.019)	(0.778)		
(3)	L	0.697			-2.044	8.311	2.624	-0.555		0.088
		(21.814)			(-3.583)	(2.221)	(2.369)	(-2.017)		
	H	0.764	8.958	1.606	-11.892				0.386	0.252
		(3.278)	(6.917)	(1.616)	(-7.502)				(1.995)	
(4)	L	0.830	6.298	-9.371	-1.295				0.027	0.258
		(13.029)	(4.953)	(-5.820)	(-1.991)				(0.900)	

50

Table 8: The performance of unconditional and conditional models

This table reports estimates of the unconditional and conditional CAPM, for high-minus-low asset ivol portfolios for both equity and asset returns. The second and third columns report estimates for the unconditional CAPM. For the conditional model we consider only equity returns. We estimate the intercept with the constraint that the slope equals 1 for the return of the market multiplied by the conditional beta of the high ivol portfolio, and -1 for the low ivol portfolio. Predicted betas are obtained using the three sets of instruments considered in Table 7. For each regression we report coefficients and t-stat (in parenthesis). Row $\alpha/\alpha_E^U - 1$ reports the reduction in unconditional α attained by the conditional model. The last row reports R^2.

	Test assets:	Unconditional Equity	Asset	Set:	Conditional, equity (1)	(2)	(3)
α		-0.687%	-0.350%		-0.434%	-0.461%	-0.579%
		(-1.610)	(-1.010)		(-1.036)	(-1.088)	(-1.362)
R^M		0.928	0.946				
		(9.977)	(12.490)				
$\underline{\beta}_E^H R^M$					1	1	1
					(3.973)	(3.266)	(1.835)
$\underline{\beta}_E^L R^M$					-1	-1	-1
					(-1.997)	(-1.649)	(-0.943)
$\alpha_A/\alpha_E - 1$			-48.9%				
$\alpha_E/\alpha_E^U - 1$					-36.8%	-32.7%	-15.6%
R^2		0.305	0.408		0.335	0.316	0.307

Table 9: Leverage versus volatility in elasticity

The table reports time-series regressions of equity value weighted elasticity ϵ_t on the instrumental variables in Set 1 and 4 in Table 7. Panel A and B report results for the high and low ivol portfolios respectively. The first set consists of v_t^s, v_t^i, and σ_t^M. The second set adds portfolio leverage, LEV_t. We report coefficients and t-stats (in parenthesis) of the estimates. We also report the R^2 and monthly standard deviations of each portfolio elasticity, $\sigma(\epsilon_t)$.

Set	Intercept	v_t^s	v_t^i	σ_t^M	LEV_t	R^2	$\sigma(\epsilon_t)$
			Panel A: High asset ivol				
(1)	1.127	−0.453	−0.237	1.520		13.2%	
	(65.691)	(−1.759)	(−1.191)	(4.984)			0.074
(4)	0.611	−0.055	−0.279	0.447	0.460	67.6%	
	(21.323)	(−0.351)	(−2.288)	(2.298)	(19.335)		
			Panel B: Low asset ivol				
(1)	2.154	−34.907	−35.248	23.881		25.7%	
	(14.060)	(−6.567)	(−5.019)	(8.792)			0.785
(4)	1.703	−31.741	−34.225	22.241	0.262	27.0%	
	(6.148)	(−5.742)	(−4.890)	(7.865)	(1.946)		

Table 10: Leverage versus volatility in the conditional CAPM

The table reports results for the conditional model (9) for equity returns on the high and low asset-ivol portfolios. We denote predicted equity betas obtained using the first set of instruments described in Table 7 by $\underline{\beta}_E$, and those obtained using the fourth set by $\underline{\beta}_E^{LEV}$. The fourth set of instruments adds leverage to the first. We constrain the slope of the regressions to unity. For each regression we report estimated coefficients and t-stats (in parenthesis), conditional alphas, improvement relative to unconditional CAPM alpha $\alpha_E/\alpha_E^U - 1$, and R^2.

	α_E	$\underline{\beta}_E$	$\underline{\beta}_E^{LEV}$	$\alpha_E/\alpha_E^U - 1$	R^2
H	-0.048%	1		-77.6%	68.0%
	(-0.150)	(19.849)			
L	0.385%	1		-18.0%	45.7%
	(1.830)	(15.278)			
$H-L$	-0.434%			-36.8%	33.5%
	(-1.036)				
H	-0.038%		1	-82.3%	67.7%
	(-0.110)		(19.823)		
L	0.383%		1	-18.5%	45.7%
	(1.820)		(15.277)		
$H-L$	-0.421%			-38.6%	32.7%
	(-0.999)				

Options and Risk[*]

Giovanni Bruno   Jørgen Haug

Abstract

We propose a parsimonious general equilibrium extension of the Black-Scholes economy that helps clarify how options' prices, expected returns, risk exposure, and optimal exercise policies respond to variations in the risk exposure of the underlying asset. The model allows one to separate the effects from changes in idiosyncratic versus systematic risk. Among the new insights we establish are that *i*) call option prices typically respond negatively to increases in systematic risk, *ii*) call options' expected returns are monotonically decreasing in idiosyncratic risk, and *iii*) the optimal exercise date of an American call can be pushed backwards in time in response to an increase in systematic risk—decreasing the value of waiting to invest. The effects of a change in risk on options is generally ambiguous because it affects their prices through two key channels—the volatility channel and the price channel—and a change in systematic risk causes a repricing of the underlying asset that may dominate the volatility channel. The comparative statics are robust to the presence of stochastic volatility, and thus yield internally consistent implications not only for the cross-section of options but also for the time-series of a particular option.

JEL codes: G12, G13, G31

Keywords: Option pricing, general equilibrium, idiosyncratic risk, systematic risk.

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[*]Both authors are at the Department of Finance, NHH Norwegian School of Economics, and can be contacted by email at giovanni.bruno@nhh.no or jorgen.haug@nhh.no.

1   Introduction

It is well known that call and put option prices are increasing in volatility (Merton, 1973, Theorem 8). A basic premise of this analysis is that the value of the underlying asset (hereafter simply "asset") does not respond to variations in volatility. We know from the basic present value restriction that this premise amounts to implicitly assuming that variations in volatility of the asset are altogether due to variations in *idiosyncratic* risk (ivol)—or a knife-edge case where variations in the required rate of return is exactly offset by variations in the asset's expected future value. This leaves open the question of how option *prices* respond to non-trivial variations in systematic risk, both in the cross-section of options and in the time-series of a particular option. Earlier studies moreover leave open the question of how the option *risk-return* relationship responds to variations in the risk exposure of the asset—be it idiosyncratic or systematic.

We address the preceding questions by introducing a minimal general equilibrium extension to the economy of Black and Scholes (1973). Rather than exogenously specifying the price of the asset, we specify the asset's cash flow together with a state price density (SPD, a.k.a. stochastic discount factor in discrete time). This allows us to jointly price the asset and its derivatives. While our main focus is European call options, we also discuss important deviating properties of European puts, and the optimal exercise policy of American calls. To stay as close as possible to the analysis of Black and Scholes (1973) and Merton (1973) we let uncertainty in the SPD be generated by a Brownian motion. A novel and crucial aspect of our specification is that the cash flow process is exposed to an additional Brownian motion, which is independent of that of the SPD. This two-factor structure of the cash flow allows us to independently consider the effects of variations in exposure to systematic and idiosyncratic risk.

The thrust of our analysis derives from a simple generalization of the Black-Scholes-Merton (BSM) formula (Lemma 1). In the BSM economy an increase in risk increases call

and put option prices via a widening of the distribution of the asset, which increases the probability of in-the-money (ITM) states—the *volatility channel*. Our economy allows additionally for a repricing of the asset. An increase in systematic risk consequently shifts the distribution to the left, which in isolation reduces call values—the *price channel*. Variations in systematic risk clearly also causes variations in volatility (total risk). A shift in systematic risk consequently activates both channels while a shift in ivol involves only the volatility channel. We establish sufficient conditions for which the price channel dominates the volatility channel for ITM calls (Theorem 2).

The option risk-return relationship responds to variations in the risk exposure of the asset in ways that are surprising relative to conventional wisdom. We establish in a complementary paper that increases in ivol of the asset causes expected ITM call returns to decrease and call ivol to increase (Bruno and Haug, 2018, whose model is a reduced form version of the present setup). In the present paper we demonstrate that this result does not extend to out-of-the-money (OTM) calls, for which also the call ivol falls with that of the asset (Figures 1 and 2). While call systematic risk typically is increasing in the systematic risk of the asset, call ivol is decreasing (Figure 8).

Our main results derive from a model with constant parameters, and are therefore internally consistent in the cross section. We establish that the qualitative properties hold also for a particular option by studying the more general case with stochastic risk exposure—stochastic volatility. Variations in the initial conditions of the risk factor loadings provide results that are consistent with the comparative statics with constant parameters.

Finally, we study the effect of variations in risk exposure on the optimal exercise policy of American calls on the dividend paying asset. We demonstrate that increases in systematic risk or economy-wide risk shrinks the distance between the asset price and the exercise boundary. Contrary to conventional wisdom (McDonald and Siegel, 1986)

an increase in volatility may thus cause firms to invest *earlier* rather than later, and thereby *reduces* the value of waiting to invest (Figure 10).

The observation that Merton's result—that call prices increase with volatility—is not general is not new. Indeed, Merton (1974) observes that variations in risk may coincide with variations in correlation with the "market portfolio" (i.e., the SPD), without pursuing it further. Jagannathan (1984) makes the related observation that variations in risk may shift payoffs into states that are more or less valuable to investors, even when the shifts are mean preserving spreads. He considers a simple one-period, four-state example with two distinct calls that do not pay off in the same states of nature. The less risky option is dearer even when the assets have identical prices. The difference in the options' prices are due to an assumption that the state prices are higher in states where the less risky option pays off. As Jagannathan's intention is to stay as close to Merton's analysis as possible he considers only cases where the price of the asset does not respond to variations in risk. The formal analysis therefore does not consider general equilibrium effects on call option prices (despite Jagannathan making several apt remarks in that direction). Our analysis adds to his in that we explicitly distinguish between systematic and idiosyncratic risk, in that we provide sufficient conditions for a call to be a decreasing function of systematic risk, and in addressing the time series effects for a particular option.

Rasmusen (2007) considers more general definitions of "riskier distribution" than mean preserving spreads and derives sufficient conditions under which call prices increase with risk. His results assume risk neutral agents, and that the expected value of the asset is unaffected by variations in risk. Rasmusen thus shares with Jagannathan the implicit assumption that variations in risk do not affect the value of the asset.

Kim (1992) considers how call option prices respond to risk in general equilibrium. He assumes in the thrust of his analysis that the asset and the SPD depend on *one*

common risk factor. An increase in the volatility of the risk factor thus affects not only the volatility of the asset but also the volatility of the SPD. His main conclusion is that call option prices generally respond ambiguously to increases in risk.

Unlike Jagannathan and Rasmusen, but like Kim, we allow risk to affect call option prices also through its repricing effect on the asset—the price channel. Unlike Kim, however, our setup allows us to decompose risk into systematic and idiosyncratic risk. The decomposition allows us to arrive at sharper results than the aforementioned studies, and to clarify the economic role of systematic versus idiosyncratic risk in option pricing. By being clear about the role of systematic versus idiosyncratic risk we arrive at clearer implications for applications in asset pricing and corporate finance that rely on the response of call prices to changes in risk. Finally, none of the aforementioned papers discuss the response of option expected returns, systematic, idiosyncratic, and total risk (volatility).

Bailey and Stulz (1989) consider the effect of risk on call prices through the riskless rate channel, by considering the valuation of index options with stochastic short rates that are negatively correlated with index volatility.[1] They provide a numerical example of how an increase in volatility of the index causes a fall in the price of the call option. When risk goes up, the riskless rate goes down, thus increasing the present value of the strike price. They demonstrate that the net effect on call option value can be negative. Because the valuation effects are minor even for carefully chosen parameter values, we choose to shut down this channel in our study, so as not to confound simple discounting effects with the more fundamental repricing problem. As in Jagannathan (1984) and Rasmusen (2007), Bailey and Stulz' analysis does not take the price channel into account.

The major difference in the analyses can be understood by considering what types of

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[1]One can view Kim's analysis as a generalization of Bailey and Stulz (1989). Variations in risk in Kim's model also works through the riskless rate channel, as variations in asset risk coincide with variations in SPD risk in his framework. The volatility of the SPD feeds into the riskless rate via precautionary savings motives.

questions they can answer. All of the aforementioned studies can answer which of two options will be dearer, when they are written on two different assets, with the same price but different risk exposure. Only by taking the more robust general equilibrium view of the problem can we answer the additional question of how the value of a particular call option responds to a change in risk of its asset—as a change in risk of an asset necessarily requires a reassessment of its value. Kim's analysis allows for the latter channel, but does not allow the riskiness of the asset to change while keeping the riskiness of the economy unchanged. Thus, his framework does not allow for comparative statics without also changing the entire economy. As he works with a constant coefficients setup, the comparative statics are internally consistent only in a cross section of economies.

We proceed by presenting our economy in Section 2, in which we derive the general equilibrium expression for a European call when the SPD and cash flow generating processes are lognormal. In Section 3 we use comparative statics to study how the option price and risk-return respond to variations in risk. Section 4 extends the analysis to a setting with stochastic volatility. This extension allows us to revisit the comparative statics results of the previous section in an internally consistent framework for a particular option. In Section 5 we discuss how optimal exercise of an American call responds to variations in risk. Section 6 concludes.

2 The Economy

We represent the sources of risk in the economy by a pair of uncorrelated standard Brownian motions, W^1 and W^2. The Brownian motions are defined relative to a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ that satisfies the usual conditions, where $\mathbb{F}$ is the collection of information sets generated by the Brownian motions, $\mathcal{F}_t = \sigma(\{W_u^1, W_u^2\}_{u=0}^t)$, $t \in \mathbb{R}_+$. We use the short-hand notation $\mathrm{E}_t\{\cdot\} = \mathrm{E}\{\cdot | \mathcal{F}_t\}$ throughout, and assume frictionless markets and the absence of arbitrage opportunities.

There is a risky asset, and a riskless asset that offers a rate of return r. There are also derivatives whose payoffs depend on the price of the risky asset (hereafter, simply "the asset"). While our main focus is European call options, we also consider European put options and American call options when they offer relevant distinct insights.[2] To allow repricing of the asset to affect option prices we specify the dynamics of the cash flow accruing to the owners of the asset, and let its price be determined endogenously with those of its derivatives. Prices are determined by the absence of arbitrage opportunities that implies the existence of a state price density π. We assume that the joint dynamics of the SPD and the cash flow generating process are governed by the stochastic differential equations (SDEs)

$$d\pi_t = a_t \pi_t \, dt + b_t \pi_t \, dW_t^1, \tag{1}$$

$$dX_t = g_t X_t \, dt + v_{1,t} X_t \, dW_t^1 + v_{2,t} X_t \, dW_t^2. \tag{2}$$

The parameter $b < 0$ determines the riskiness of the economy, g is the growth rate of the cash flow, v_1 and v_2 determine the exposure of the cash flow to systematic and idiosyncratic risk, and the riskless rate is $r = -\frac{a\pi}{\pi} = -a$. This framework allows for separate choices of the overall riskiness of the economy, through b, and the riskiness of the asset, through the *risk factor loadings* $v = (v_1, v_2)$.[3] The two-factor structure allows us to separately study the effects of systematic risk, through v_1, and idiosyncratic risk, through v_2. Because the systematic risk of the asset is determined not only by v_1 but also by b we refer to v_1 as the 'systematic risk factor loading.' The risk factor loadings can take values in $\mathbb{R}$, provided asset volatility is non-degenerate in that $\sigma = \sqrt{v_1^2 + v_2^2} > 0$. The sign of v_1 determines the sign of the asset's "beta", relative to the systematic risk

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[2]The framework we suggest makes it simple to extend the analysis to most types of derivatives, although the comparative statics we derive do not necessarily extend to a particular claim.

[3]The analysis of Kim (1992) is nested in the above framework, by letting $v_1 = k_1 b$ and $v_2 = k_2 b$ for suitable constants k_1 and k_2.

factor W^1. The sign of v_2 has no bearing on its economic interpretation in our limited setting. In a more elaborate economy the sign of v_2 across firms would indicate for instance how the firms are related in the supply chain: good news for a firm may well be bad news for one of its suppliers.

The equilibrium prices of the asset and its European call are now immediate, as both must satisfy Euler equations determined by the SPD. To arrive at concrete results we consider the case of constant coefficients, until we generalize to stochastic volatility in Section 4. It moreover simplifies the analysis to assume the asset pays out the cash flow rate X_t during $[0, \infty)$.[4]

Lemma 1. *Assume constant parameters a, b, g, v_1, v_2, and that $g < \mu \triangleq r - bv_1$.[5] The date 0 price of the asset is*

$$S_0 = \frac{X_0}{\mu - g}, \tag{3}$$

and μ is its required rate of return. The price of a European call with strike price K and expiration date $\tau > 0$ can be expressed as

$$C_0 = \frac{X_0}{\delta} e^{-\delta\tau} N(d) - K e^{-r\tau} N(d') \tag{4}$$

where

$$d = \frac{\ln\left(\frac{X_0}{K}\right) - \ln(\delta) + \left(r - \delta + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \tag{5}$$

$\delta = \mu - g$, $d' = d - \sigma\sqrt{\tau}$, and $\sigma = \sqrt{v_1^2 + v_2^2}$.

Proof. Consider first an asset with a payout rate of X during $[0, T] \subset \mathbb{R}_+$. Because state

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[4]It is easy to check that all of the results hold also for an asset that pays out X_T at a future date T, no sooner than the expiration date of the option. The case where the asset pays out a rate on $[0, T]$, $T < \infty$, is analytically difficult, as a finite sum (integral) of lognormals is not lognormal. While our results surely hold for a rich class of stochastic processes, it is useful to consider the geometric Brownian motion because of its analytic tractability, and not the least because of its close relation to the analyses of Black and Scholes (1973) and Merton (1973).

[5]We use $\triangleq$ to highlight equality by definition.

price deflated gains are P-martingales the price of the asset is given by

$$S_0 = \mathrm{E}_0 \left\{ \int_0^T \frac{\pi_t}{\pi_0} X_t \, dt \right\} = \int_0^T \mathrm{E}_0 \left\{ \frac{\pi_t}{\pi_0} X_t \right\} dt = \frac{X_0}{g - \mu} \left[\mathrm{e}^{(g-\mu)T} - 1 \right],$$

where the second equality follows from Fubini's Theorem. The third equality is immediate from the independence of W^1 and W^2. The Williams-Gordon-Shapiro growth model (3) obtains by letting $T \to \infty$. The state price density restriction on equilibrium returns imply that $\mu = r - bv_1$ is the required rate of return of S (Duffie and Zame, 1989). To price the call option observe that the call's payoff is in the span of the riskless and underlying assets. The continuous cash flow payout corresponds to a constant dividend yield of $\delta = \mu - g$. Because the riskless rate is constant and the underlying asset is a geometric Brownian motion the formula of Black and Scholes (1973) and Merton (1973) thus obtains. Because W^1 and W^2 are independent, the volatility of the asset is $\sigma = \sqrt{v_1^2 + v_2^2}$. $\qquad\square$

Not surprisingly, (3) verifies that the price of the asset is independent of idiosyncratic risk,[6] v_2, and decreasing in systematic risk, v_1, as $b < 0$. Thus only the economy-wide risk, b, and the asset's systematic risk factor loading, v_1, can affect the call price through the price channel. The appearance of the formula of Black and Scholes (1973) and Merton (1973) in (4) is not at all surprising or new, in light of the consumption-based argument of Rubinstein (1976). What the Lemma does, however, is to make precise the joint determination of the two prices in equilibrium within the two-factor specification (1)–(2).

To better understand the risk-return relationship between a derivative and its underlying we use Itô's lemma to express the option's risk factor loadings and volatility

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[6]This is of course generally not true, as a change in ivol risk may affect the value of $\mathrm{E}_0 \{ X_t \}$, as discussed by Rasmusen (2007). In the parametric case we consider, changes in risk are in the Rothschild and Stiglitz (1970) sense of mean preserving spreads (at the cash flow growth rate level), as in Merton (1973) (at the asset rate of return level).

as

$$v_1^c = \epsilon v_1, \tag{6}$$

$$v_2^c = \epsilon v_2, \tag{7}$$

$$\sigma^c = \epsilon\sqrt{v_1^2 + v_2^2} = \epsilon\sigma, \tag{8}$$

where ϵ is the elasticity of the derivative value with respect to the value of the asset (Black and Scholes, 1973)—for our European call $S_0 \mathrm{e}^{-\delta\tau}\mathrm{N}(d)/C_0$. The state price restriction on expected rates of return (as in Duffie and Zame, 1989) implies that the expected return on the derivative equals

$$\mu^c = r - \frac{\sigma_\pi}{\pi} \cdot \frac{\sigma_c}{c} = r - bv_1\epsilon, \tag{9}$$

where σ_π and σ_c are the dispersion terms of the SDEs for the state price density and option price respectively.

3 The Effects of Changes in Risk

We proceed to analyze the effects of changes in risk via simple comparative statics that apply formally to the cross section of assets and their derivatives. While this type of analysis is not internally consistent in the time series of a given asset, it yields valuable insights also in this case, that we verify in an internally consistent but less transparent setup in Section 4.

We first consider the effects of a change in idiosyncratic risk, and then the effects of changes in systematic and economy-wide risk. Several insights are based on proof by example rather than formally stated proofs. Table 1 summarizes the base-case parameter values that we use for the former.

Table 1: Base-case Parameter Values

The asset is parametrized by its initial value S_0, its growth rate g, volatility σ, systematic risk factor loading v_1, and idiosyncratic risk factor loading v_2, where $\sigma = \sqrt{v_1^2 + v_2^2}$. The call is parametrized by its strike price K and time to expiration τ. The economy-wide parameters are the riskless rate r and the volatility of the stochastic discount factor $|b|$. The parameters imply that the asset has an expected rate of return of $\mu = 0.10$. The initial condition for the cash flow (2) is $X_0 = S_0(\mu - g)$.

Asset parameters					Call parameters		Economy-wide parameters	
S_0	g	σ	v_1	v_2	K	τ	r	b
10	0.05	0.40	0.16	0.367	10	0.25	0.02	-0.50

3.1 Variations in asset idiosyncratic risk

A change in idiosyncratic risk (ivol) does not affect the price of the asset. Because the price channel does not come into play the effect on the call price is that of Merton (1973):

$$\frac{\partial C_0}{\partial v_2} = S_0 e^{-\delta\tau} \mathrm{n}(d) \frac{v_2}{\sigma} \sqrt{\tau}. \tag{10}$$

While the derivative is negative when the risk factor loading is negative, the call price is of course uniformly increasing in volatility, $|v_2|$, as expected.

While the call price response to asset ivol is as expected, the response of the risk-return trade-off is less obvious. Figure 1 demonstrates how the call's risk-return tradeoff (6)–(9) responds to variations in asset ivol, v_2, for at- and out-of-the-money calls (ATM, OTM). In-the-money (ITM) calls have the same response as ATM calls, and are not reported. The Figure demonstrates that the expected rate of return, and thus also the systematic risk of the call are uniformly *decreasing* in the ivol of the asset, independent of moneyness. The intuition is simple: An increase in asset ivol causes an increase in the probability of ending up OTM. The covariance between the call payoff and the SPD is zero conditional on OTM states. Increasing asset ivol thus reduces the magnitude of the unconditional covariance with the SPD, and consequently the amount of call systematic risk. Figure 1a confirms that call ivol, v_2^c, is in contrast uniformly *increasing* in v_2 for ATM and ITM calls (as formally proven in Bruno and Haug, 2018). Figure 1b

demonstrates that OTM call ivol has a more complex non-uniform response, increasing at low levels of v_2 while decreasing at higher levels.

Figure 1: The effect of ivol on the call risk-reward trade-off

(a) At-the-money

(b) Out-of-the-money
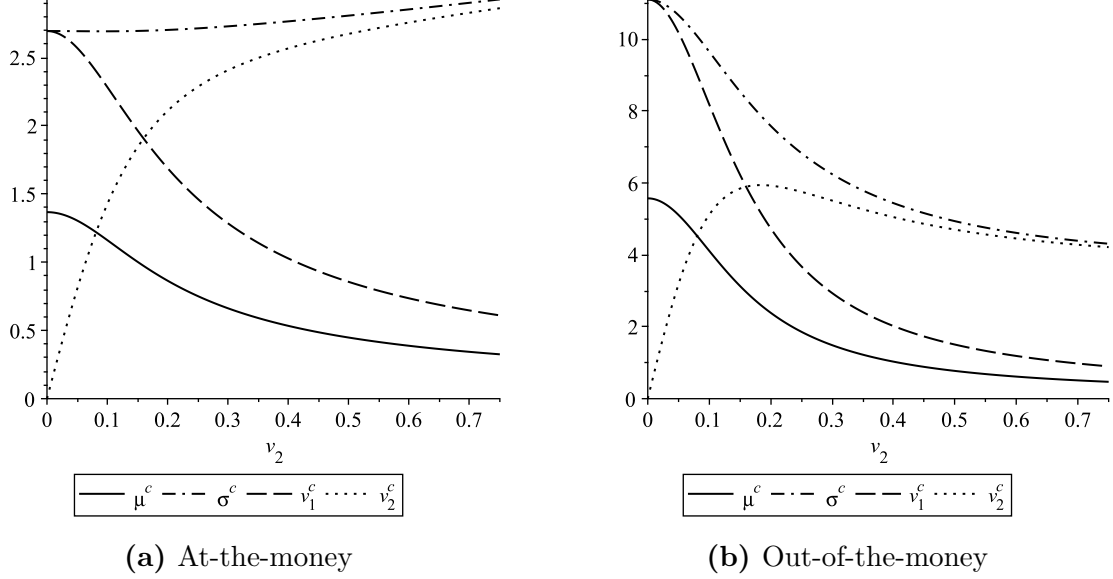
The figures are generated using $b = -0.5$, $g = 0.05$, $X = 0.5$, $r = 0.02$, $\tau = 0.25$, $\sigma = 0.4$. The OTM option has strike $K = 1.5S$.

Theorem 1 formalizes some of the discussed features. To reduce the parameter space we base the result on the concept of forward moneyness (as in e.g. Hull and White, 1987).

Theorem 1. *Let $F_{0,\tau}$ be the forward price of the asset and $m = F_{0,\tau}/K$.*

1. *If $\mu > r$ then $\forall\ m > 0\ \frac{\partial \mu^c}{\partial v_2} < 0$.*

2. *$\forall\ m \geq 1$, $\frac{\partial v_2^c}{\partial v_2} > 0$ and $\frac{\partial \sigma^c}{\partial v_2} > 0$.*

3. *$\forall\ m > 0$, $\frac{\partial v_2^c}{\partial v_2} > 0$ on $[0, \underline{v}_2) \cup (\overline{v}_2, \infty)$.*

Proof. Consider first the claim that $\frac{\partial \mu^c}{\partial v_2} < 0$. From (9) and $\mu > r$ it is sufficient to show that $\frac{\partial \epsilon}{\partial v_2} < 0$. The derivative

$$\frac{\partial \epsilon}{\partial v_2} = \frac{Se^{-\delta\tau}}{c^2}\left[\mathrm{n}(d)\frac{\partial d}{\partial v_2}c - \mathrm{N}(d)\frac{\partial c}{\partial v_2}\right] = \frac{Se^{-\delta\tau}}{c}\mathrm{n}(d)\frac{v_2}{\sigma}\sqrt{\tau}\left(1 - \frac{d}{\sigma\sqrt{\tau}} - \epsilon\right)$$

65

is strictly negative iff $1 - \frac{d}{\sigma\sqrt{\tau}} - \epsilon < 0$, which can be rearranged as

$$\frac{Se^{-\delta\tau}}{Ke^{-r\tau}} > \frac{N(d')}{N(d)}\frac{d'}{d}.$$

Now use that $Se^{-\delta\tau}n(d) = Ke^{-r\tau}n(d')$ to get the restriction $\frac{N(d)d}{n(d)} > \frac{N(d')d'}{n(d')}$. Because $R(d) \triangleq \frac{N(d)}{n(d)}$ is increasing in d the inequality holds for every $d > d' = d - \sqrt{(v_1^2 + v_2^2)\tau}$, and thus $\forall\, v_2 \in \mathbb{R}$.[7]

The second claim that $\partial\sigma^c/\partial v_2 > 0$ when $m \geq 1$ follows from Theorem 1 in Bruno and Haug (2018), by a suitable change of variables. Their Theorem also establishes that systematic risk, v_1^c, is decreasing in v_2. Using that $(v_2^c)^2 = (\sigma^c)^2 - (v_1^c)^2$ thus establishes that $\partial v_2^c/\partial v_2 > 0$.

For the third claim, observe first that $\sigma > 0$ ensures that $\epsilon > 1$ at $v_2 = 0$. From (7) we thus have that $v_2^c|_{v_2=0} = 0$ and $v_2^c|_{v_2>0} > 0$. Continuity of $v_2^c(v_2)$ ensures the existence of $\underline{v}_2 > 0$.

With the change of variables $h = \sigma\sqrt{\tau} = \sqrt{h_1^2 + h_2^2} > 0$ we have that

$$d = \frac{\ln(m)}{h} + \frac{1}{2}h \implies \lim_{h_2 \to \infty} d = \infty,$$
$$d' = \frac{\ln(m)}{h} - \frac{1}{2}h \implies \lim_{h_2 \to \infty} d' = -\infty.$$

Next, define $f(h_2) = R(d) - \left(\frac{h_2}{h}\right)^2 d$ and $g(h_2) = \left(\frac{h_2}{h}\right)^2 h\frac{R(d')}{R(d)-R(d')}$. The effect on call

---

[7]The function $R(d)$ is related to the Mills ratio (a.k.a. inverse hazard function) m by $R(d) = m(-d)$ (see for instance Greene, 1993).

ivol is:

$$
\begin{aligned}
\frac{\partial v_2^c}{\partial v_2} &= \epsilon + v_2 \frac{Se^{-\delta\tau}}{c} \mathrm{n}(d) \frac{v_2}{\sigma} \sqrt{\tau} \left(1 - \frac{d}{\sigma\sqrt{\tau}} - \epsilon \right) \\
&= \epsilon \left\{ 1 - \frac{\mathrm{n}(d)}{\mathrm{N}(d)} \frac{v_2^2}{\sigma} \sqrt{\tau} \left[\frac{Se^{-\delta\tau}\mathrm{N}(d)}{c} - 1 + \frac{d}{\sigma\sqrt{\tau}} \right] \right\} \\
&= \epsilon \left\{ 1 - \frac{\mathrm{n}(d)}{\mathrm{N}(d)} \frac{v_2^2}{\sigma} \sqrt{\tau} \left[\frac{e^{-r\tau}K\mathrm{N}(d')}{c} + \frac{d}{\sigma\sqrt{\tau}} \right] \right\} \\
&= \frac{\epsilon}{R(d)} \left\{ R(d) - \left(\frac{h_2}{h} \right)^2 d - \frac{h_2^2}{h} \frac{\mathrm{N}(d')}{\frac{\mathrm{n}(d')}{\mathrm{n}(d)}\mathrm{N}(d) - \mathrm{N}(d')} \right\} \\
&= \frac{\epsilon}{R(d)} \left\{ R(d) - \left(\frac{h_2}{h} \right)^2 d - \left(\frac{h_2}{h} \right)^2 \frac{R(d')}{\frac{1}{h}[R(d) - R(d')]} \right\} \\
&\triangleq \frac{\epsilon}{R(d)} \left\{ f(h_2) - g(h_2) \right\}.
\end{aligned}
$$

Because $\lim_{d\to-\infty} R(d) = 0$ it follows that $\lim_{h_2\to\infty} g(h_2) = 0$ and consequently

$$
\lim_{h_2\to\infty} \frac{\partial v_2^c}{\partial v_2} = \lim_{h_2\to\infty} \frac{\epsilon}{R(d)} [f(h_2) - g(h_2)] = \lim_{h_2\to\infty} \epsilon \lim_{h_2\to\infty} \frac{R(d) - d}{R(d)} = 1,
$$

which establishes $\bar{v}_2 > \underline{v}_2$. $\qquad\square$

Corollary 1. *For m as in Theorem 1, $\frac{\partial v_1^c}{\partial v_2} < 0$ for all $m > 0$.*

Bruno and Haug (2018) prove that ATM and ITM call ivol is uniformly increasing in asset ivol. Figure 1 proves by example that said result does not extend to OTM calls, for common maturities. Figure 1b shows that there exists an open interval for which call ivol is *decreasing* in asset ivol. While we are not able to analytically characterize the open interval, becuase $\partial v_2^c / \partial v_2$ is highly nonlinear in v_2, Figure 2 illustrates it for short- and long-lived calls. Regions D represent combinations of ivol and moneyness for which call ivol has a negative response to asset ivol. As the equations that characterize the boundaries becomes highly sensitive to rounding error when the probability of ending up ITM becomes sufficiently low (determined by the ratio of moneyness to uncertainty

$\sigma\sqrt{\tau}$), we consider moneyness S/K above 0.45.

Figure 2: OTM call ivol decreasing in asset ivol

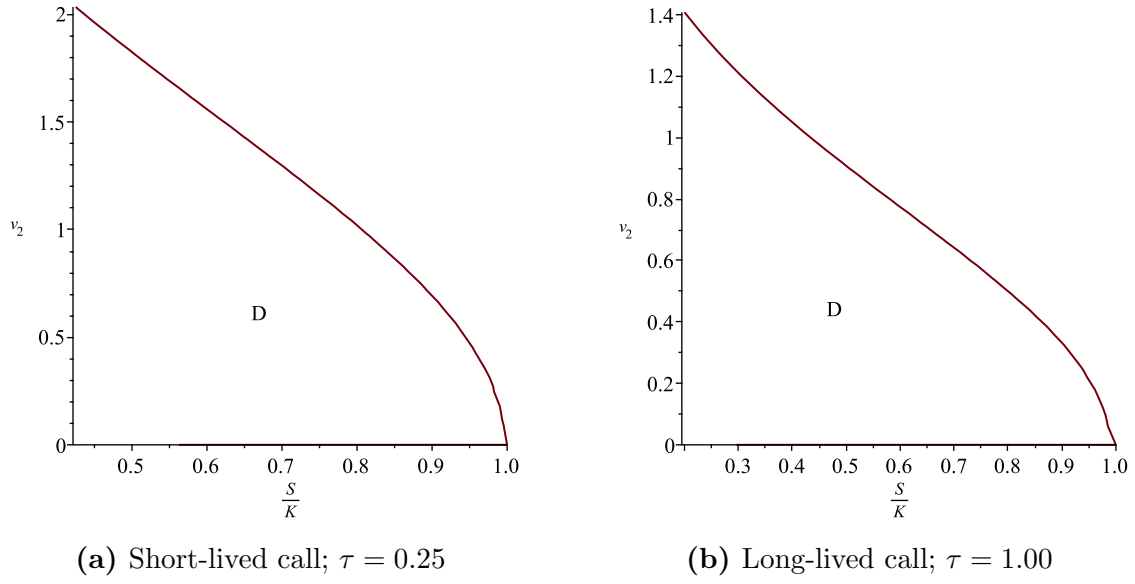**(a)** Short-lived call; $\tau = 0.25$ **(b)** Long-lived call; $\tau = 1.00$
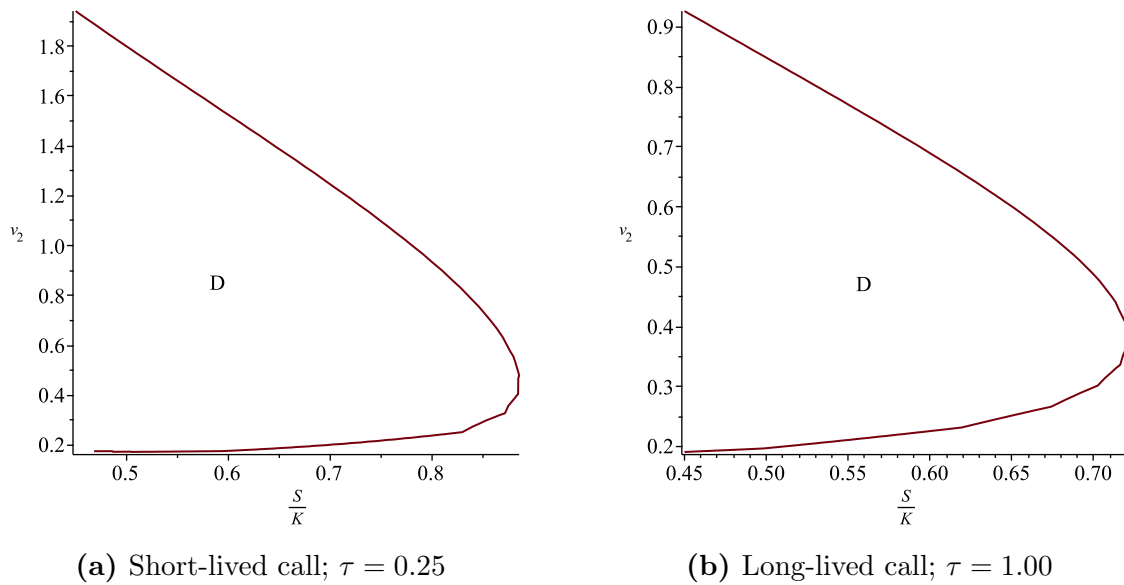
Regions D are combinations of asset ivol v_2 and moneyness S/K for which call ivol $v_2^c = \epsilon v_2$ is decreasing in v_2. Remaining parameters are as in Table 1.

While Figure 2 reports the sign of the response of call ivol to asset ivol, Figure 3 reports the sign of the response of call volatility to asset ivol. Region D in Figure 3a represent combinations of moneyness and asset ivol for which call volatility has a negative response to asset ivol. The call has three months to expiration. While the response is positive for very high levels of asset ivol, the graph suggests that short-lived call options' response is negative in the economically relevant part of the parameter space. Figure 3b repeats the exercise for a longer lived option—with one year to expiration in this case. A longer time to expiration makes the call more asset like. Consequently the region in which call volatility decreases in asset ivol shrinks.

3.2 Variations in asset systematic risk

We first consider simple comparative statics for variations in the systematic risk factor loading v_1, for which the ensuing repricing of the asset causes variations in moneyness.

Figure 3: OTM call volatility decreasing in asset ivol

**(a)** Short-lived call; $\tau = 0.25$ **(b)** Long-lived call; $\tau = 1.00$

Regions D are combinations of asset ivol v_2 and moneyness S/K for which call volatility $\sigma^c = \epsilon\sqrt{v_1^2 + v_2^2}$ is decreasing in v_2. Remaining parameters are as in Table 1.

The observed effects are thus due to the joint effect of variations in systematic risk and moneyness. We subsequently consider comparative statics that control for moneyness, while still allowing for general equilibrium responses to v_1, and thus ensure the price channel is active.

Figure 4 shows how call total, idiosyncratic, and systematic risk and expected return varies with the systematic risk of the asset. The graphs are generated for a call that is ATM for $v_1 = 0.16$. While all quantities have a positive response to increases in v_1 at realistic levels, ivol has a pronounced drop at relatively lower levels of v_1. Large values of v_1 corresponds to low levels of moneyness and thus causes a response in ivol similar to that observed in Figure 1b: Ivol of OTM calls has a negative response to increases in the total risk of the asset beyond a threshold risk level. Figure 4b shows that the other quantities have uniformly positive responses to v_1 when the maturity of the call is extended to five years.

Figure 4 suggests that the expected return and risk factor loadings are increasing

Figure 4: The effect of systematic risk on call option expected returns and risk

(a) Short-lived call; $\tau = 0.25$ **(b)** Long-lived call; $\tau = 5.00$
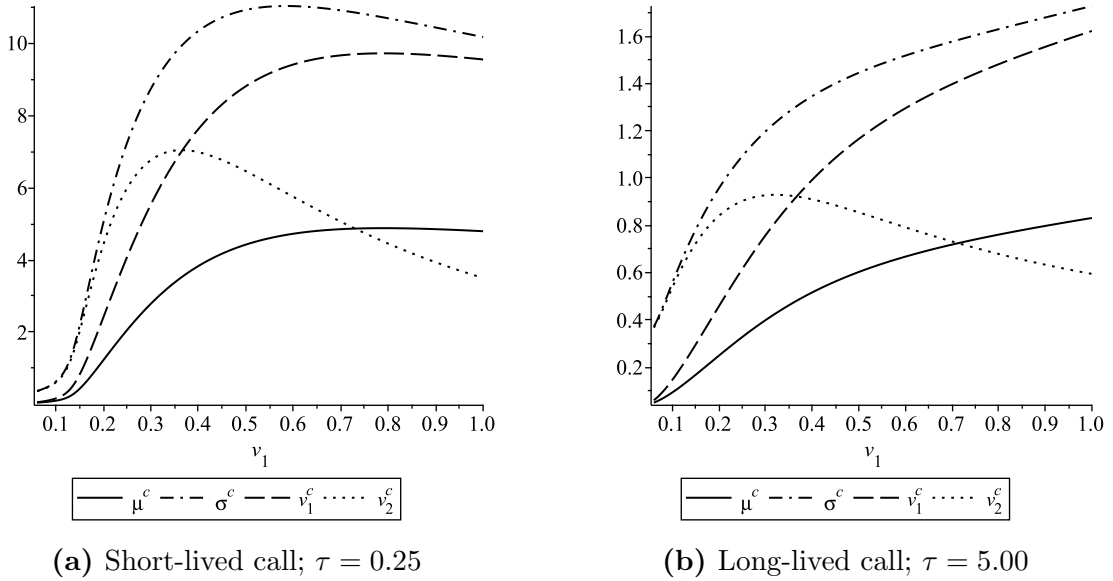
The Figures reports graphs for a European call's expected return, μ^c, volatility σ^c, and systematic and idiosyncratic risk factor loadings v_1^c and v_2^c, against the systematic risk factor loading of the asset v_1. The spot price of the asset S_0, and thus also moneyness, varies with v_1. The calls are ATM at $v_1 = 0.16$. Remaining parameter values are as in Table 1.

on some domain $[0, \overline{v}_1)$, and that $\overline{v}_1$ is increasing in maturity, as calls become more asset-like as the time to expiration increases. The Figure moreover suggests the general property that the domains on which μ^c and v_1^c are increasing contain the corresponding domains for σ^c and v_2^c. Moreover, the domains for μ^c and v_1^c coincide as μ^c is a strictly positive transformation of v_1. We formally verify that this is a general property in the present economy in the next result.

Proposition 1. *Let* $\underline{v}_1 = \frac{g-r}{-b}$, *and consider the connected sets* $A = \left\{ v_1 > \underline{v}_1 : \frac{\partial v_2^c}{\partial v_1} > 0 \right\}$, $B = \left\{ v_1 > \underline{v}_1 : \frac{\partial \sigma^c}{\partial v_1} > 0 \right\}$, *and* $C = \left\{ v_1 > \underline{v}_1 : \frac{\partial q}{\partial v_1} > 0, q = \mu^c, v_1^c \right\}$ *that contains the smallest possible* v_1. *We have that* $A \subset B \subset C$, *and* $\inf A = \inf B = \inf C = \underline{v}_1$.

It is worth noting that the role of the lower limit $\underline{v}_1$ in Proposition 1 is asset specific: The asset price (3) clearly relies crucially on assumptions about the cash flow process. The qualifier "contains the smallest possible v_1" ensures that we do not consider extremely high values of v_1, as all quantities except v_2^c are increasing for sufficiently high,

but economically irrelevant v_1.

Proof. ϵ has an inverted U-shape as a function of v_1: For low levels of v_1 the call becomes deep ITM. Moneyness decreases as v_1 increases, which increases ϵ, but as v_1 increases further ϵ decreases as the likelihood of ending up ITM increases due to the magnitude of volatility. Recall that $v_2^c = \epsilon v_2$ while $v_1^c = \epsilon v_1$ and $\sigma^c = \epsilon \sqrt{v_1^2 + v_2^2}$. On the set $D = \{v_1 : \frac{\partial \epsilon}{\partial v_1} < 0\}$ we have that v_2^c is strictly decreasing in v_1. By continuity v_1^c and σ^c are strictly increasing on a non-empty subset of D.

It remains to show that $\inf A = \inf B = \inf C = \underline{v}_1$. We have that $\lim_{v_1 \to \underline{v}_1} S_0 = \infty$ and thus that $\lim_{v_1 \to \underline{v}_1} \mathrm{N}(d) = \lim_{v_1 \to \underline{v}_1} \mathrm{N}(d') = 1$. Consequently $\lim_{v_1 \to \underline{v}_1} \epsilon = 1$, the lower bound of ϵ (Bergman et al., 1996). Because $\epsilon(\underline{v}_1) \neq \epsilon(\underline{v}_1 + x)$ for $x > 0$ the desired property follows. Continuity of ϵ implies the sets are connected. $\square$

Table 2 confirms that the non-monotonic responses to v_1 are caused by large changes in v_1 causing shifts in moneyness that fundamentally changes the properties of the call—making it more or less asset-like. ITM and OTM calls are defined relative to $v_1 = 0.1$, so that increases in v_1 beyond 0.1 moves the call OTM. The first column reports the values we use for economy-wide risk, b, and maturity, τ. The range of values of v_1 for which both μ^c and σ^c are increasing in v_1 is larger than the range for which v_2^c is increasing, across all parametrizations, consistent with Proposition 1. Indeed, μ^c and σ^c have negative responses to v_1 only for values of v_1 that causes the call to be deep OTM. The table demonstrates that the non-monotonic responses for short-lived calls are due to ϵ having an inverted U-shaped response to v_1.

While the value of b is not important for σ^c, it is important for μ^c. Indeed, for high values of $-b$, we observe that the expected returns increase for almost all levels of v_1 and the spread in expected returns is higher. On the other hand the level of τ is important for both μ^c and σ^c. Indeed, the effect of changes in v_1 on σ^c and μ^c is more prominent for shorter maturities—calls that are less asset-like.

Table 2 confirms that the increase in expected return and volatility due to an increase in v_1 is enhanced by ϵ's response to v_1, except for deep OTM options. For the deep OTM calls the elasticity exhibits a negative response. The negative response of ϵ is however compensated for in μ^c and σ^c because $\mu^c = \epsilon\mu = \epsilon(r - bv_1)$ and $\sigma^c = \epsilon\sigma = \epsilon\sqrt{v_1^2 + v_2^2}$. Thus, even when ϵ falls, μ^c and σ^c may still still exhibit a positive response to increases in v_1. The idiosyncratic risk factor loading $v_2^c = \epsilon v_2$, on the other hand, has an unambiguous negative response to v_1 whenever $\partial\epsilon/\partial v_1 < 0$.

Consider next the effect of variations in systematic risk on the call price. A call price behaves increasingly like the asset price as moneyness or time to expiration increases. Trivially, only the price channel is at work for the asset. For deep ITM calls, or calls with long time to expiration, the price channel thus dominates the volatility channel. OTM calls derive their value largely from the possibility that they end up ITM. This possibility has a non-negligible probability precisely because of the volatility of the asset. The volatility channel consequently becomes more important as moneyness decreases. The next result formalizes this simple intuition.

Theorem 2. *If* $-b > (\mu - g)\sqrt{\tau}$ *then* $\frac{\partial C_0}{\partial v_1} < 0 \ \forall \ F_{0,\tau} > K$.

Proof. We have that

$$\frac{\partial c_0}{\partial v_1} = \frac{b}{\mu - g}S_0 e^{-\delta\tau}N(d) + b\tau S_0 e^{-\delta\tau}N(d) + S_0 e^{-\delta\tau}n(d)\frac{v_1}{\sigma}\sqrt{\tau},$$

which is strictly negative iff

$$-\frac{b}{\delta}N(d) - b\tau N(d) > n(d)\frac{v_1}{\sigma}\sqrt{\tau},$$

iff

$$-\frac{b}{\delta\sqrt{\tau}}\frac{N(d)\sigma}{v_1} - b\tau\frac{N(d)\sigma}{n(d)v_1} > 1.$$

Consider the first term on the left hand side of this inequality. For $K < F_{0,\tau}$ we have that $\frac{N(d)}{n(d)} > 1$. Because $\frac{\sigma}{v_1} \geq 1$ the term is strictly larger than unity. The second term is strictly positive. $\square$

Theorem 2 identifies ITM forward as a sufficient condition in terms of the moneyness of the call. The restriction $-b > (\mu - g)\sqrt{\tau}$ implies that a negative price response is more common the higher the economy-wide risk, b, the smaller the asset systematic risk is relative to the economy-wide risk, and the lower the dividend payout rate $\delta = \mu - g$.

Corollary 2. *Let p_0 be the price of a European put. It is always the case that $\frac{\partial p_0}{\partial v_1} > 0$.*

Proof. $p_0 = C_0 + K e^{-r\tau} - S_0$ and $\Delta = \frac{\partial C_0}{\partial \sigma} \in (0, 1)$. $\square$

A shift from idiosyncratic to systematic risk that does not affect overall riskiness has a negative impact through the price channel, while no impact through the volatility channel. The next result formalizes this rather obvious result.

Proposition 2. *For a given level of volatility σ,*

$$\frac{\partial C_0}{\partial v_1}\bigg|_\sigma = S_0 e^{-\delta\tau} N(d) \frac{b}{\mu - g} < 0.$$

Proof. The result is immediate from $v_2^2 = \sigma^2 - v_1^2$, $S_0 e^{-\delta\tau} n(d) = K e^{-r\tau} n(d')$, and that $\frac{\partial d}{\partial b} = \frac{\partial d'}{\partial b}$. $\square$

A reasonable concern is that the above results apply only in very special circumstances, or that the effects are economically small. Figure 5 graphs call prices as functions of systematic risk, for three levels of moneyness, using the base-case parameter values of Table 1. The Figure shows that all option prices uniformly decline in value as systematic risk increases from zero, for all degrees of moneyness, and for reasonable

levels of systematic risk.[8] Figure 5a shows the relationship for the economically relevant domain of systematic risk, for which monotonicity holds. Figure 5b includes also unrealistically high levels of systematic risk, and demonstrates that the relationship is not uniformly monotone on $\mathbb{R}_+$. For sufficiently high values of v_1 the sufficient condition in Theorem 2 does not hold, and indeed even the call labled ITM exhibit a positive response. The reason is simply that ITM is defined relative to $v_1 = 0.16$, and values significantly higher than this causes a sufficiently strong repricing of the asset to shift the call out of the money. Figure 5b thus highlights the necessity to control for moneyness to better understand the price impact from smaller magnitude shocks to systematic risk.

Figure 5: Call option prices and systematic risk

**(a)** Economically relevant domain　　　　　　**(b)** Extended domain
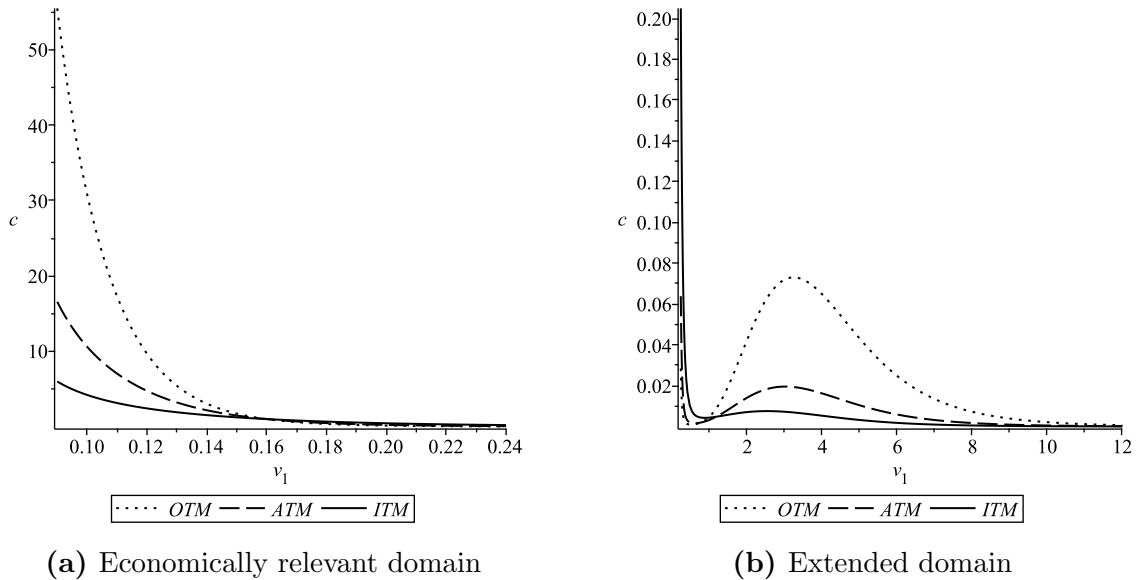
The graphs report option values relative to the base case option value, $\frac{C(v_1)}{C_0}$, for different levels of the systematic risk factor loading v_1. The current price of the asset is $S_0 = 10$ at the base-case value $v_1 = 0.16$, and the graphs represent three degrees of base-case moneyness; $K = 1.5S$ (OTM), $K = S$ (ATM), and $K = 0.5S$ (ITM).

While we do not report the formal results in the paper, Figure 6 shows the same relationship as in Figure 5 for an asset that offers a one-time payout of X_T (with $X_T \sim$

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[8]For an asset that pays out X_T in the near future, for instance at $T = 1$, ATM and OTM options start to increase in value in response to increases in v_1, at 'medium' levels of systematic risk. For these types of short-lived assets there is thus a U-shaped relationship even at economically meaningful levels of systematic risk. Monotone decline in value sets in at about $T = 2$ for the base-case parameters.

log N) at dates $T = 1$ or $T = 10$, in Figures 6a and 6b respectively. The call has time to expiration $\tau = 0.25$ in either case. The different responses to increases in systematic risk in the Figures is caused by the relative importance of the price and volatility channels in the two assets. Systematic risk affects long-lived assets more than short-lived assets when risk is assumed to increase with time—as in our setup. In Figure 6a systematic risk has a limited impact on the price of the asset, causing the price channel to be dominated by the volatility channel. In the more realistic case of Figure 6b the repricing becomes sufficiently strong for the price channel to dominate the volatility channel. The relationship converges to that of Figure 5a as the life span of the asset is extended.

Figure 6: Call option prices and systematic risk for short-lived assets

(a) $T = 1$ **(b)** $T = 10$
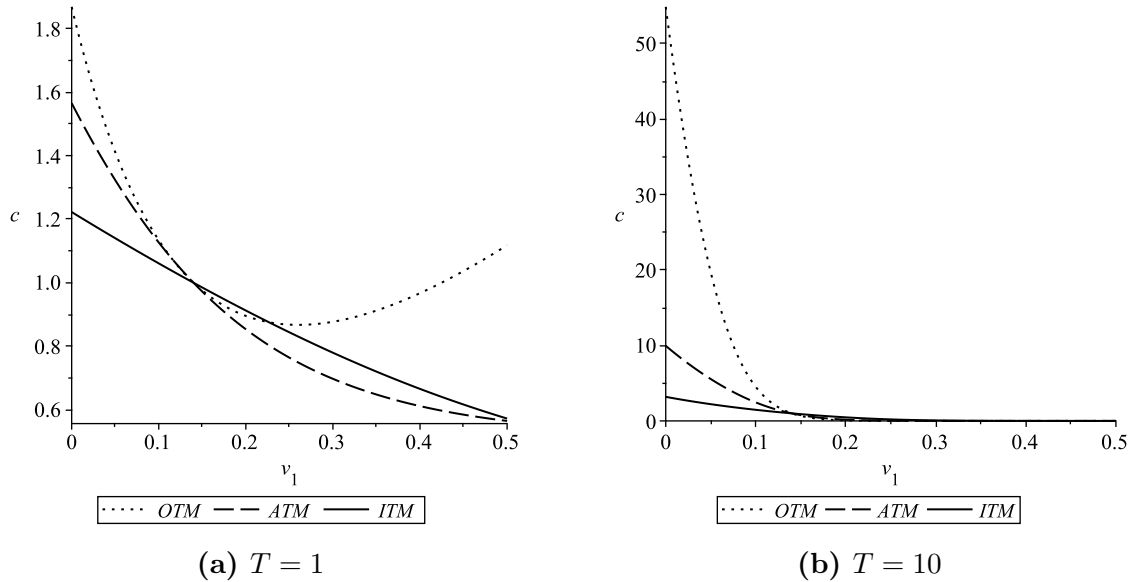
The graphs report call option values c for different levels of the systematic risk factor loading v_1, when the asset pays out X_T at date $T = 1$ or $T = 10$. The current price of the underlying is set equal to $S_0 = X_0 \exp\{-(\mu - g)T\} = 10$ for the base-case value $v_1 = 0.16$. The graphs represent three degrees of moneyness; $K = 15$ (OTM), $K = 10$ (ATM), and $K = 5$ (OTM). Time to expiration of the call is $\tau = 0.25$.

It is evident from Lemma 1 that the economy-wide risk parameter b—the 'volatility' of the state price deflator—interacts in a crucial way with the systematic risk factor loading v_1. It is thus pertinent to ask if the impression conveyed by Figure 5, that the price channel generally dominates the volatility channel, relies crucially on the level

of economy-wide risk. Table 3 reports ratios of call option prices $C(X_0, K, b, v_1)$ for different levels of economy-wide risk, systematic risk factor loadings, and moneyness. For each level of economy-wide risk, moneyness is defined relative to the scenario with a systematic risk factor loading $v_1 = 0.16$. In the low risk case, with $b = -0.30$, all calls change more in value as v_1 changes, than in the cases of $b = -0.50$ and $b = -0.70$. The reason is apparent from (3), which demonstrates that the effect of v_1 is large when $\mu = r - bv_1$ is close to the cash flow growth rate g. While the table verifies that the effect of the systematic risk factor loading is weaker the higher the economy-wide risk, it also demonstrates that the price channel does indeed dominate the volatility channel even for what seems to be realistic levels of b.[9] The effects are economically significant not only for the relatively large four point shocks to the systematic risk factor loadings, centered around 0.16, but also for the smaller one point changes.

In real settings changes in risk are typically more complex than what we account for in the preceding analysis. It is therefore relevant to ask what *joint* changes to idiosyncratic and systematic risk increases or decreases option value—a simple relationship would serve as a useful empirical proxy. We can answer this question by characterizing the joint changes that leave option value unchanged. Let $C(\theta)$ represent (4) evaluated at some parameter θ. The relationship of interest is given by the equation $C(v_1, v_2) = C_0 \in \mathbb{R}_{++}$. The problem is thus equivalent to that of computing implied volatility. Because the equation is highly transcendental we cannot express the exact relationship analytically.

Figure 7 shows level curves $C(v_1, v_2) = C_0$ of options prices in (v_1, v_2)-space for three levels of moneyness. Theorem 2 implies that combinations (v_1, v_2) above (below) the level curves represent changes in risk that increase (decrease) option value. For deep ITM options the volatility channel is largely ineffective, which causes ivol to have a negligible

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[9]The case of $b = -0.30$ corresponds to a standard consumption-based model with a relative risk aversion of 10, and *per capita* consumption volatility of 0.03: Let $\pi_t = e^{-\beta t}e_t^{-\gamma}$ (as in Rubinstein, 1976; Lucas, 1978; Breeden, 1979), where *per capita* consumption $de_t = \mu_e e_t \, dt + \sigma_e e_t \, dW_t^1$. We then have that $a = -\beta - \gamma\mu_e + \frac{1}{2}\gamma(1 + \gamma)\sigma_e^2$ and $b = -\gamma\sigma_e$.

effects on the call price. For deep OTM options the volatility channel is dominant, which causes systematic risk to have a much weaker effects on the call price, and consequently a flatter level curve. The effects are consistent with the earlier intuition of how moneyness determines how "asset-like" the call is.

Figure 7: Level curves in (v_1, v_2) space
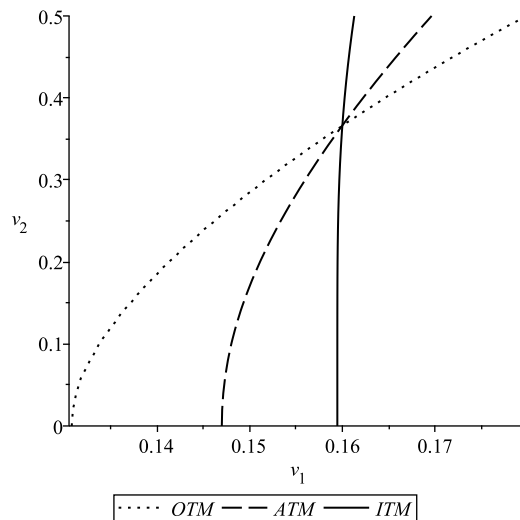
The graphs report combinations of v_1 and v_2 that ensure $C(v_1, v_2; K) = C_0$ for a suitable 'observed' price C_0. The current price of the asset is $S_0 = 10$ and the graphs represent three degrees of moneyness; $K = 1.5S$ (OTM), $K = S$ (ATM), and $K = 0.5S$ (ITM).

The pure risk response: Controlling for moneyness

It is evident from the preceding analysis that qualitative properties of comparative statics depend crucially on moneyness. An important insight from Lemma 1 is that an increase in systematic risk causes a repricing of the asset that may offset the benefits from the associated increase in volatility. But the repricing of the asset also affects moneyness. To better understand the pure risk-response we next formally control for moneyness in the comparative statics. The previous comparative statics fit situations with very large variations in systematic risk, while the present comparative statics correspond to situations with only marginal variations.

Figure 8 shows graphs for the call expected rate of return μ^c, volatility σ^c, and

systematic and idiosyncratic risk factor loadings v_1^c and v_2^c as functions of the systematic risk factor loading of the asset v_1. The graphs are generated for short and long maturity OTM calls, at $\tau = 0.25, 1.0$, with $S/K = 0.8$. It is clear from (7) that v_2^c is simply a scaled version of the call elasticity, and is decreasing in v_1 in both cases. The call's expected return and systematic risk are uniformly increasing in that of the asset, consistent with the situation when moneyness is allowed to vary with v_1, in Figure 4. The response of call ivol differs from the earlier situation though, in that it is now uniformly decreasing in v_1. Call volatility behaves differently across Figures 8a and 8b. While it is decreasing at reasonable levels of v_1 for the short-lived call, it becomes uniformly increasing for the call with one year to maturity. The qualitative features of the long-lived call in Figure 8b hold when maturity is extended, and is also representative for ITM calls irrespective of maturity.

Figure 8: The effect of systematic risk on call expected returns and risk, with fixed moneyness

(a) Short-lived OTM call; $\tau = 0.25$ (b) Long-lived OTM call; $\tau = 1.00$
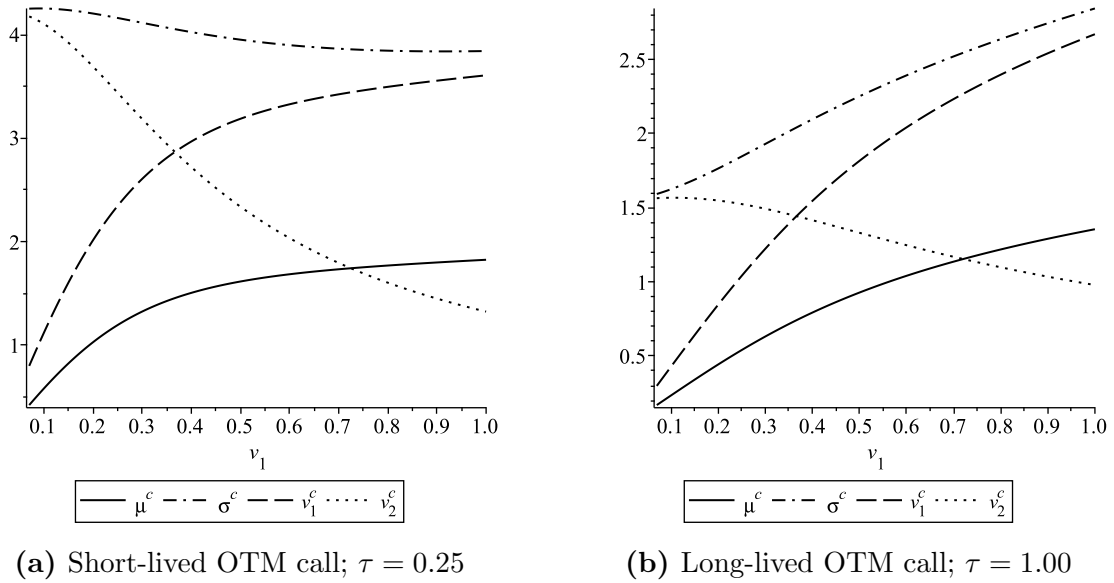
The Figures report graphs for a European call's expected return, μ^c, volatility σ^c, and systematic and idiosyncratic risk factor loadings v_1^c and v_2^c, against the systematic risk factor loading of the asset v_1. Moneyness S/K is held fixed at 0.8. Remaining parameter values are as in Table 1.

Consider next the general equilibrium response of the call price when controlling for

moneyness. We compute the partial derivative of the call price with respect to v_1 using that $m = \frac{F_{0,\tau}}{K} = \frac{X_0}{r - bv_1 - g} e^{(bv_1 + g)\tau}$, but hold forward moneyness m fixed when evaluating the partial derivative. The first step ensures that the partial derivative captures the price channel impact. The second step ensures that the reported effects are pure risk responses, and not confounded by variations in moneyness.

Figure 9 traces the partial derivative $\left.\frac{\partial C_0}{\partial v_1}\right|_m$ for three levels of moneyness. Figure 9a demonstrates that the call price has a uniformly negative response to increases in systematic risk on the economically relevant domain, and that the strength of the response is monotonically decreasing. Small increases in systematic risk of the asset will thus always depress the call price. While it may be hard to visualize this for the OTM call, its graph indeed stays negative, but close to zero. Figure 9b extends the graphs to a larger domain to illustrate the general mathematical properties of the call price response. In particular the graph for the OTM call warrants extra attention, as it gives the impression of being positive for relatively low values of v_1. The root of the OTM graph has a value of about 30% though—an unrealistically high level of *systematic* volatility for any asset.[10]

To formally study the call price response, it is useful to control for moneyness by letting the strike price adjust to keep moneyness constant. This again allows us to look at the general equilibrium effects of a small change in systematic risk for a given level of moneyness. A literal, internally consistent interpretation of this situation is a cross section of calls on different underlying assets, identical levels of moneyness κ, and identical parameter values except for v_1. To this end define

$$c_0^* = S_0 \left[e^{-\delta\tau} \mathrm{N}(d) - \kappa e^{-r\tau} \mathrm{N}(d') \right],$$

---

[10]The S&P 500 experienced volatility of above 30% in the run-up to the LTCM crisis. This index is however far removed from the true 'market portfolio,' and necessarily includes a non-trivial level of idiosyncratic risk, v_2. Moreover, the OTM call in Figure 9 represents an extreme degree of moneyness not on offer in actual markets. For a call 0.8 OTM forward, rather than 0.5 as in the graph, the root is above 60%.

Figure 9: Call option price sensitivity to systematic risk, with fixed moneyness

(a) Economically relevant domain **(b)** Extended domain
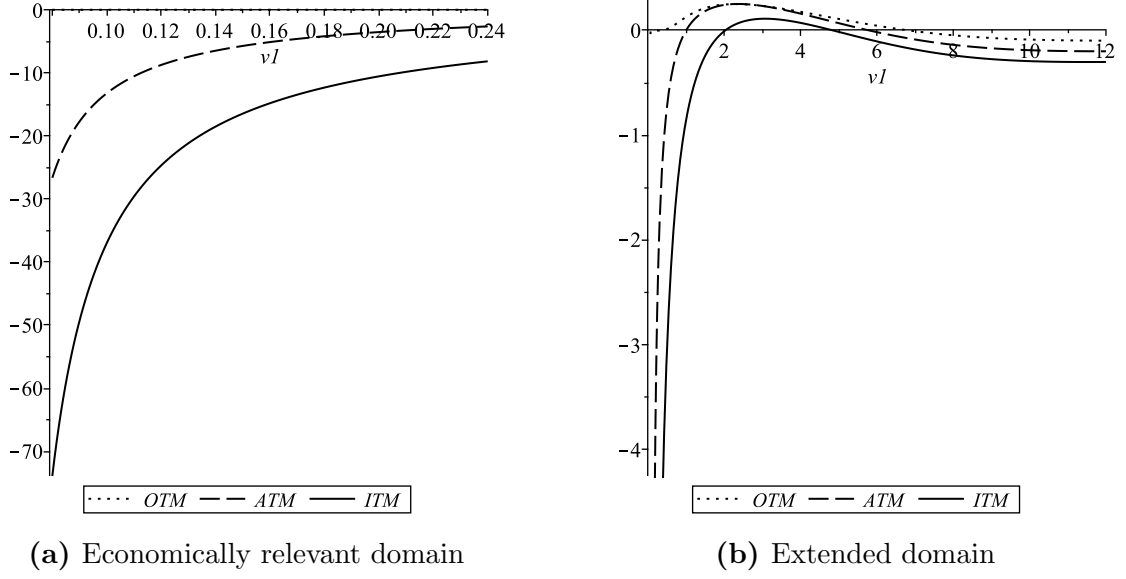
The graphs represent systematic Vega for fixed forward moneyness, $\left.\frac{\partial C_0}{\partial v_1}\right|_m$. The graphs are generated using the base-case parameter values in Table 1, for three degrees of moneyness; $m = 0.5$ (OTM), $m = 1.0$ (ATM), and $m = 1.5$ (ITM).

where κ is a constant, and $S_0 = X_0/\delta$ and $\delta = r - bv_1 - g$ as in Lemma 1. An increase in v_1 causes a positive contribution from the volatility channel as it widens the asset price distribution, which increases the likelihood of ending up ITM. The increase in v_1 has an offsetting effect through the price channel as it increases the P-discount rates of the asset and the call payoff—reflected in a reduction in S_0. The strength of the price channel depends on how asset-like the call is, formalized in the following result.

Proposition 3. *If $S_0 \geq K$ and $\tau \geq \frac{1}{b^2}$ then c_0^* is decreasing in v_1.*

Proof. The partial derivative with respect to v_1 is

$$\frac{\partial c_0^*}{\partial v_1} = \frac{b}{\mu - g} S_0 \left[e^{-\delta\tau} N(d) - \kappa e^{-r\tau} N(d') \right] + b\tau S_0 e^{-\delta\tau} N(d)$$
$$+ S_0 e^{-\delta\tau} n(d) \frac{\partial d}{\partial v_1} - S_0 \kappa e^{-r\tau} n(d') \left(\frac{\partial d}{\partial v_1} - \frac{v_1}{\sigma} \sqrt{\tau} \right).$$

Observe first that $S_0 e^{-\delta\tau} n(d) = S_0 \kappa e^{-r\tau} n(d')$, which implies that $\frac{\partial c_0^*}{\partial v_1}|_{v_1=0} < 0$. Consider

hereafter $v_1 \neq 0$. Since we in addition can write $S_0 \left[\mathrm{e}^{-\delta\tau}\mathrm{N}(d) - \kappa\mathrm{e}^{-r\tau}\mathrm{N}(d') \right] = c_0^*$ we can rewrite the previous expression as

$$\frac{b}{\mu - g}c_0^* + b\tau S_0 \mathrm{e}^{-\delta\tau}\mathrm{N}(d) + S_0 \mathrm{e}^{-\delta\tau}\mathrm{n}(d)\frac{v_1}{\sigma}\sqrt{\tau},$$

which is negative iff

$$-b\left[\frac{\mathrm{e}^{-\delta\tau}\mathrm{N}(d) - \kappa\mathrm{e}^{-r\tau}\mathrm{N}(d')}{\mu - g}\right] - b\tau\mathrm{e}^{-\delta\tau}\mathrm{N}(d) > \mathrm{e}^{-\delta\tau}\mathrm{n}(d)\frac{v_1}{\sigma}\sqrt{\tau}$$

iff

$$-b\left[\frac{\mathrm{e}^{-\delta\tau}\mathrm{N}(d) - \kappa\mathrm{e}^{-r\tau}\mathrm{N}(d')}{\mu - g}\right]\frac{1}{\mathrm{e}^{-\delta\tau}\mathrm{n}(d)\frac{v_1}{\sigma}\sqrt{\tau}} - b\sqrt{\tau}\frac{\mathrm{N}(d)\sigma}{\mathrm{n}(d)v_1} > 1.$$

Observe first that the first term is strictly positive. Observe next that $\sigma/v_1 \geq 1$ (assuming without loss of generality that $v_1 \geq 0$). If $\kappa < 1$ then $\frac{\mathrm{N}(d)}{\mathrm{n}(d)} > 1$. Hence, for $-b\sqrt{\tau} \geq 1$ the result holds. $\square$

Proposition 3 shows call prices are decreasing in systematic risk given adequate time to expiration and moneyness. The parameter restrictions represent sufficient conditions for the call to be adequately asset-like for the price channel to dominate the volatility channel. The Proposition implies a similar result for c_0, complementing Theorem 2.

Corollary 3. *If c_0^* is decreasing in v_1 then c_0 is also decreasing in v_1*

Proof. Consider the case in which the starting strike price of c_0^* is equal to the strike price of c_0. A positive change in v_1 determines a negative variation in S_0. The strike price in c_0^* is an increasing function of S_0, $K^* = \kappa S_0$, while in c_0 the strike price is fixed at K. Observe that c_0 and c_0^* are both decreasing functions of their strike price, and that K^* is decreasing in v_1. Consequently, if c_0^* is decreasing in v_1, the negative response of c_0 will be at least as strong as that of c_0^*. $\square$

3.3 Variations in economy-wide risk

Consider first the effect of variations in economy-wide risk, $-b$, on the call's expected return and risk. Table 4 quantifies the relationships for two levels of moneyness that we define relative to $b = -0.5$, with ITM at $S/K = 1.25$, OTM at $S/K = 0.80$, and maturities of 0.25 and 1.00. There is a uniform positive response to an increase in economy-wide risk, across all cases. Notice that an increase in $|b|$ has a knock-on effect on μ^c, which increases not only because the asset expected return increases, $\mu = r - bv_1$, but also because of an increase in elasticity and $\mu^c = \epsilon\mu$. The positive response in call total risk and idiosyncratic risk factor loadings is due to the increase in elasticity. The option behaves more like the asset as maturity increases, evident in the weaker response of elasticity for the longer maturity $\tau = 1.00$.

The intuition for the effect of variations in economy-wide risk on price levels is simple and well known. We verify the intuition in the present economy for completeness. An increase in the magnitude of b causes a reduction in the state price density across all states, which causes all prices to fall. For a given volatility of the asset there is no offsetting increase in value from the volatility channel, and the value of the call consequently decreases.[11]

Proposition 4. *For given risk factor loadings (v_1, v_2),*

$$\frac{\partial C_0}{\partial - b}\bigg|_{v_1, v_2} = -S_0 e^{-\delta\tau} N(d) v_1 \left(\frac{1}{\mu - g} + \tau \right) < 0.$$

Proof. The result is immediate from $S_0 e^{-\delta\tau} n(d) = K e^{-r\tau} n(d')$, and that $\frac{\partial d}{\partial b} = \frac{\partial d'}{\partial b}$. □

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[11] Kim (1992) considers a situation of joint changes in v_1 and b. We discuss how his setup is a special case of the present economy, in Appendix A.

4 Stochastic Volatility

The basis for the earlier analysis—Lemma 1—assumes constant risk factor loadings. The preceding comparative statics are consequently internally consistent only in the cross section. We next formally verify that the earlier comparative statics implications apply also in the time series by extending (2) to incorporate stochastic volatility in the cash flow process, while maintaining the assumption of constant parameters of the state price density π in (1). To this end we assume the following dynamics for the squared risk factor loadings:

$$dv_{1,t}^2 = \kappa_1(\bar{v}_1^2 - v_{1,t}^2)\,dt + \eta_1 v_{1,t}\left(\rho_1\,dW_t^1 + \sqrt{1 - \rho_1^2}\,dW_t^{v,1}\right), \tag{11}$$

$$dv_{2,t}^2 = \kappa_2(\bar{v}_2^2 - v_{2,t}^2)\,dt + \eta_2 v_{2,t}\left(\rho_2\,dW_t^2 + \sqrt{1 - \rho_2^2}\,dW_t^{v,2}\right), \tag{12}$$

where $W_t^1, W_t^2, W_t^{v,1}$ and $W_t^{v,2}$ are independent Brownian motions. The dynamics assumed for the risk factor loadings are identical to those of Heston (1993), where κ_j governs the speed of adjustment, $\bar{v}_j$ is the long-run mean, η_j is the volatility of the squared risk factor, and the ρ_j's are the correlation parameters. The earlier comparative statics correspond to variations in the initial conditions of (11) and (12). Compared to the analysis in Section 3 the stochastic volatility framework captures rational expectations about future reversals of shocks to risk exposure. This more general framework thus allows us to assess to what extent the earlier comparative statics overstate the pricing impact from shocks to the risk exposure of the asset.

Monte Carlo simulation

Because our objective is not to derive an option pricing formula, and the current setup is even more complex than that of for instance Heston (1993), we use Monte Carlo simulation to determine call prices. We first utilize the physical probability measure P

to obtain the date 0 equilibrium price of the asset. We next work under the equivalent martingale measure Q to obtain the call price. The first step, under P, allows us to incorporate the price channel into the analysis.

To determine the equilibrium price of the asset we simulate (1), (2), (11), and (12) under P. We use $M = 10000$ paths and $T_S = DY$ dates, with $D = 250$ and $Y = 5$. Each period, of equal length Δt, corresponds to a day. We use the Milstein discretization scheme for the risk factor loadings, which is known to produce high quality approximations for mean reverting processes (Glasserman, 2004), and a simple first order Euler scheme for the cash-flow and state price density processes.

The discounted cash-flows along path j is given by $h(j,t) = \frac{\pi_t(j)}{\pi_0} X_t(j)$. The date 0 price of the asset is consequently given by

$$S_0 = \frac{1}{M} \sum_{j=1}^{M} \sum_{t=1}^{T_S} h(j,t). \tag{13}$$

Using S_0 as the initial condition for the SDE of S we proceed in the usual manner to simulate S_τ under Q, and discount corresponding call payoffs at the riskless rate. As in Heston (1993) the relevant Q dynamics are given by

$$dS_t = (r - \delta_t)S_t \, dt + v_{1,t}S_t \, dW_{1,t}^Q + v_{2,t}S_t \, dW_{2,t}^Q,$$

$$dv_{1,t}^2 = \kappa_1^Q(\bar{v}_1{}^{Q2} - v_{1,t}^2) \, dt + \eta_1 v_{1,t} \left(\rho_1 \, dW_t^{Q,1} + \sqrt{1 - \rho_1^2} \, dW_t^{Q,v,1} \right),$$

$$dv_{2,t}^2 = \kappa_2^Q(\bar{v}_2{}^{Q2} - v_{2,t}^2) \, dt + \eta_2 v_{2,t} \left(\rho_2 \, dW_t^{Q,2} + \sqrt{1 - \rho_2^2} \, dW_t^{Q,v,2} \right),$$

where $\delta_t = \mu_t - g$ and $\mu_t = r - b\,v_{1,t}$. As in Heston (1993) the risk-adjusted parameters for the risk factor loadings are given by $\kappa_j^Q = \kappa_j + \lambda_j$ and $\bar{v}_j{}^{Q2} = \kappa_j \bar{v}_j{}^2 / \kappa_j^Q$, where λ_j is the market prices of risks for risk factor $j = 1, 2$. To determine if the preceding comparative statics are invariant to stochastic volatility it is sufficient to consider the

simplest case, where $\lambda_j = 0$ and $\rho_j = 0$, which causes the dynamics of the risk factor loadings to be identical under P and Q.

To obtain call returns we simulate the asset price under P, again using S_0 from (13) as the initial condition. The dynamics of the ex-dividend price is

$$dS_t = gS_t\, dt + v_{1,t}S_t\, dW_t^1 + v_{2,t}S_t\, dW_t^2, \tag{14}$$

which we discretize with a first order Euler scheme.

We simulate M paths for S_t for $\tau_c < T_S$ dates, with $\tau_c \Delta t = \tau$—the maturity date of the option. The call return is simply $R_{0,\tau}^c = C_\tau / C_0 - 1$, where C_0 is approximated as described above, and $C_\tau = (S_\tau - K)^+$ with S_τ simulated under P. The M independent return observations $R_{0,\tau}^c$ allow us to easily compute expected option returns and return variances.

Comparative statics

We next investigate how the pricing of a particular option responds to shocks to the risk exposure of the asset. Such shocks correspond to variations in the initial conditions of the risk factor loadings (11) and (12), $v_{0,1}$ and $v_{0,2}$.

Table 5 reports results from the Monte Carlo simulation. For each combination $(v_{0,1}\ v_{0,2})$ the table reports moneyness, call price, excess expected return, volatility, and idiosyncratic risk.[12] We report the latter values for three cases: A benchmark case with constant risk factor loadings (corresponding to the setup in Section 3), the case

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[12]We compute the call option's discrete returns from 0 to τ over each path j as $R_\tau^{c,j} = \frac{C_\tau^j}{c_0}$, the excess expected return as $\mu_\tau^c = \frac{1}{M}\sum_{j=1}^M R_\tau^{c,j} - e^{r\tau}$, the standard deviations of returns as $\sigma_\tau^c = \sqrt{\frac{1}{M-1}\sum_{j=1}^M (R_\tau^{c,j} - \mu_\tau^c)^2}$. For the ivol of call returns we use the restriction on total volatility of the option $\sigma_\tau^c = \sqrt{\beta_{R_\tau^c,\pi_\tau}^2 \sigma(\pi_\tau)^2 + (v_{2,\tau}^c)^2}$, where $\beta_{R_\tau^c,\pi_\tau}$ is the ratio between the covariance of the call's returns with the SPD and the variance of the SPD ($\sigma(\pi_\tau)^2$). We estimate $\beta_{R_\tau^c,\pi_\tau}$ as $\frac{\mu_\tau^c}{\sigma(\pi_\tau)^2}$. We also tried to estimate beta as the slope of the regression of the call's returns on the SPD at time τ, and we did not find significant differences between the two methods.

with stochastic risk factor loadings and slow adjustment and the case with stochastic risk factor loadings and fast adjustment. The overall impression is that the presence of stochastic volatility does not materially affect the comparative statics in the simpler model in Section 3.

In the upper half of Table 5 systematic risk varies between 10% and 30% while ivol is fixed at 20%. An increase in systematic risk, $v_{0,1}$, depreciates the call price through its repricing effect on the asset. On the other hand, expected call returns and volatility are increasing in $v_{0,1}$ as expected in light of Figure 4. The responses are independent of the speed of adjustment of volatility, κ_j. Even a temporary shock to systematic risk may thus materially shift call prices in the *opposite* direction of what is usually assumed in the option pricing literature. The presence of stochastic risk factor loadings does not prevent the price channel to dominate the volatility channel, as is the case also in the benchmark case with fixed risk exposure.

In the lower part of Table 5 ivol varies between 10% and 30% while systematic risk is fixed at 15%. In line with Merton's (1973) model free result (when applied to ivol), positive shocks to ivol inflates call prices. The table again reproduces the perhaps surprising negative response of call expected returns and volatility.[13] We also observe that the call's ivol has a positive response, which is consistent with what we observe in Figures 1a and 1b, where v_2^c is always increasing in v_2 when the interval of variation for v_2 does not contain unrealistically high values. As in the static volatility framework, the volatility channel dominates the price channel. While ivol may in general trigger the price channel—expected future cash flows generally depend on the level of total/idiosyncratic risk—the cash flow process (2) is sufficiently linear in idiosyncratic risk for this not to be the case. The irrelevance of ivol for the value of the asset in the present economy is evident in the moneyness column, which equals unity across all values of $v_{0,2}$.

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[13]The negative response of volatility (total risk) of the call occurs because the calls under consideration are ATM spot/*OTM forward*. This is consistent with the response in the fixed risk exposure model, as demonstrated in Section 3.1. For calls that are ITM forward the call volatility is increasing in $v_{0,2}$.

We also observe that the effects are more pronounced for low speed of adjustment κ_j, as the effect of changes in the initial conditions are more persistent for low κ_j. For high speed of adjustment the impact of the initial condition is absorbed earlier, and prices of *European* options with the same levels of long-run risk will converge as the time to expiration, τ, is extended. For our particular analysis, the change in speed of adjustment has a similar effect to changing the maturity of the option. Indeed, there is more time to absorb the effect of a shock to initial risks exposure with longer times to expiration. For $\kappa_j = 0.1$ there is for instance a -73% (91%) variation in c_0 and a 36% (-14%) difference in μ_τ^c when $v_{0,1}(v_{0,2})$ varies from its lowest to its highest value. When $\kappa_j = 1.5$ there is a -49% (74%) variation in c_0 and a 26% (-12%) difference in μ_τ^c as $v_{0,1}(v_{0,2})$ changes from the lowest to the highest value.

5 On The Value of Waiting to Invest

A pertinent question is how changes in systematic risk affect the "value of waiting to invest" (McDonald and Siegel, 1986). We address this question through simple comparative statics of American call options when the asset has fixed risk exposures. Because the asset pays out a continuous cash flow stream there is a non-trivial parameter space for which early exercise is optimal. The question can thus be rephrased as asking if the optimal exercise boundary is materially affected by changes in systematic risk versus changes in idiosyncratic risk.

We use a simple implicit finite difference scheme to illustrate the relationship between the American (Bermudan) call option price, its optimal exercise boundary, and the systematic risk parameters.[14] The scheme is based on (2) and (3) from Lemma 1 as the model for the asset, which yields the following version of Black and Scholes' (1973)

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[14]There are well known weaknesses of the implicit finite difference scheme. These deficiencies do not affect our conclusions in any way, however, and we choose simplicity and transparency over numerical accuracy and convergence properties.

partial differential equation (PDE)

$$rc = c_t + c_S(rS - X) + \frac{1}{2}c_{SS}\sigma^2 S^2,$$

where subscripts denote partial derivatives of $C_t = c(t, S)$. The PDE is solved subject to the usual condition that $c(t, S) = (S - K)^+$ on the free boundary and at the expiration date $t = \tau$ (McKean, 1965).

Figure 10 reports optimal exercise boundaries, S^*, for the base case parameters of Table 1, as well as for levels of the systematic risk factor loading 20% below and above the base case. Figure 10a may give the impression that there is nothing new in how systematic risk affects the boundary: An increase in systematic risk from 0.128 through 0.160 to 0.192—corresponding to volatilities σ of about 0.388, 0.400, and 0.414—causes the boundary to shift downwards, which in isolation pushes the optimal exercise of the option forwards in time. The value of waiting thus seems to increase with risk. Notice, however, that this picture represents only the effect of the volatility channel. To take also the price channel into account, Figure 10b plots the exercise boundaries relative to the current price of the asset, $\frac{S^*}{S_0}$. The ordering of the boundaries are inverted relative to Figure 10a, because the price of the asset shifts *closer* to the optimal exercise boundary as the systematic risk factor loading increases. In other words, the price channel—which moves the price closer to the exercise boundary, *ceteris paribus*—dominates the volatility channel—which moves the exercise boundary away from the current price of the asset, *ceteris paribus*. Indeed, immediate exercise becomes optimal in the riskiest scenario. An increase in risk, which is due to an increase only in systematic risk, thus causes a *decrease* in the value of waiting.[15] Figure 11 confirms that the economy-wide risk parameter b

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[15]To the extent that government policy affects systematic risk, it is thus not true that politicians that increase uncertainty by postponing important decisions as a rule cause firms to postpone investment. Delaying economy-wide decisions may in fact bring investments forward—and destroy value in the process.

trivially plays the same role as v_1, as it affects options prices only through the price channel.

Figure 10: Optimal exercise boundaries and systematic risk factor loadings

(a) Exercise boundaries S^* **(b)** Exercise boundaries $\frac{S^*}{S_0}$
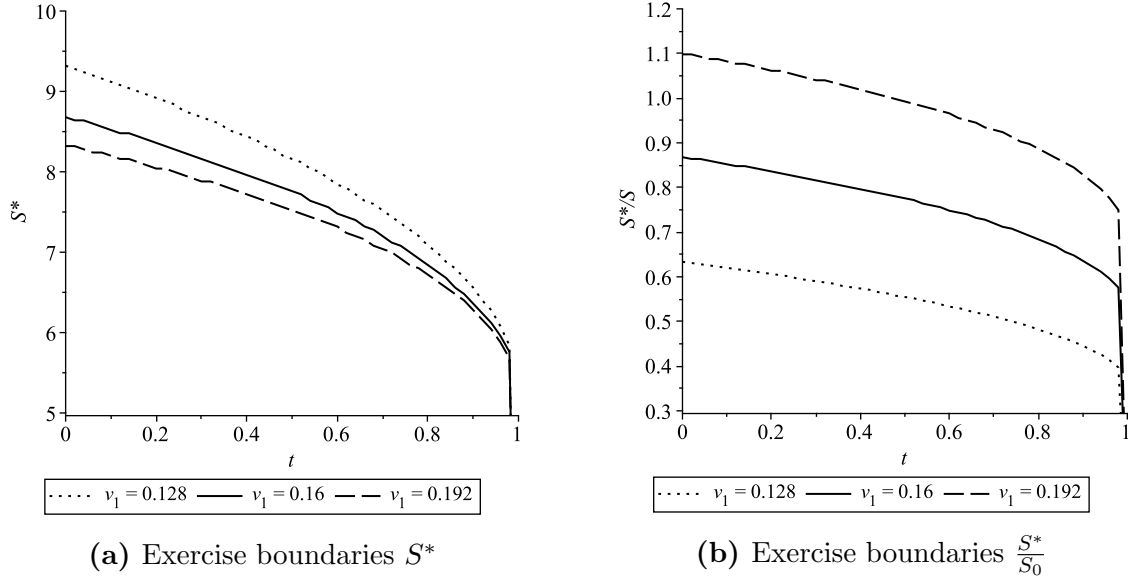
The plots correspond to systematic risk factors loadings of $v_1 = 0.128, 0.16$, and 0.192, with total risk of 0.4 when $v_1 = 0.16$. All plots are computed using 50 periods and 500 price levels, equally spaced from 0 to 20. The strike price is $K = 5$, and all options stay deeply in-the-money with initial prices of 14.71, 10.00, and 7.58 respectively.

Without providing any restrictions on the parameter space, we conjecture that the value of waiting is decreasing in systematic risk for long-lived assets. The intuition behind this should be clear from the analysis of European options, for which the price channel dominates the volatility channel for all but the very short-lived assets (and for economically irrelevant parameter values).

6   Conclusion

We study the response of call options' prices and risk-return trade-off to variations in the risk exposure of the underlying asset, which has distinct exposures to idiosyncratic and systematic risk factors. The analysis is carried out in a simple general equilibrium extension of the Black-Scholes economy, that allows the underlying asset to be priced

Figure 11: Optimal exercise boundaries and economy-wide risk

(a) Exercise boundaries S^* **(b)** Exercise boundaries $\frac{S^*}{S_0}$
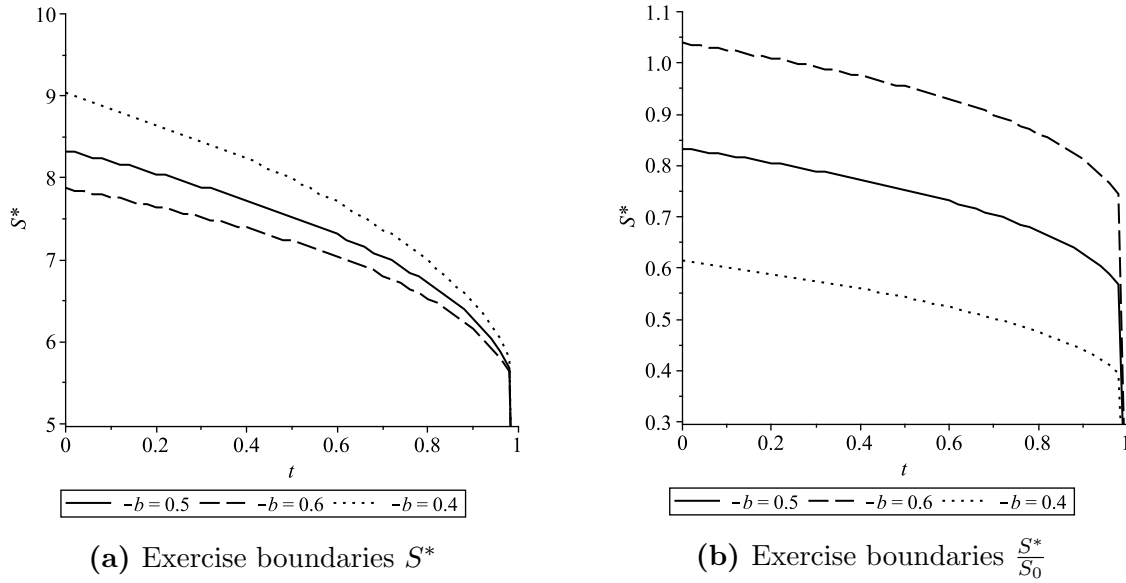
The plots correspond to economy-wide risks of $-b = 0.4, 0.5$, and 0.6. All plots are computed using 50 periods and 500 price levels, equally spaced from 0 to 20. The strike price is $K = 5$, and all options stay deeply in-the-money with initial prices of 14.71, 10.00, and 7.58 respectively.

jointly with its derivatives. Variations in ivol of the underlying affects derivatives only through the traditional *volatility channel*, while variations in systematic risk additionally affect derivates through the *price channel*: A change in systematic risk necessarily requires a repricing of the asset.

We establish several novel insights through a combination of formal (constructive) proofs and proofs by example: Call prices may decline when volatility increases due to an increase in systematic risk of the underlying; call expected returns are generally decreasing in idiosyncratic risk of the underlying, but increasing in systematic risk; call idiosyncratic risk is decreasing in the underlying's systematic risk, but not necessarily increasing in idiosyncratic risk when deep out-of-the-money; the value of waiting to invest is typically decreasing in the systematic risk of the underlying.

The main thrust of the analysis is carried out as comparative statics for a model with constant parameters, and is internally consistent in the cross section of options. To address to what extent the main insights hold also in the time series of a particular

option we extend the analysis to allow for stochastic risk factor loadings—stochastic volatility. Changes in the initial conditions of the risk factor loadings generate responses similar to those outlined in the constant parameter case.

A Relation to Kim (1992)

The analysis of Kim (1992) can be cast within the current framework by allowing $b = f(v_1)$, for instance $b = B + Iv_1$ with $B < 0$ and $I \in \mathbb{R}$. The interpretation of this factor structure is that a change in systematic risk of a firm is caused by an industry-wide change in risk, or for very small $|I|$ that the asset's cash flow uncertainty determines part of the uncertainty of the state price deflator.[16] In this economy a change in v_1 causes a change also in the volatility of the state price deflator, whose magnitude depends on the 'industry exposure' I.

Proposition 5. *If $b = B + Iv_1$ then*

$$\frac{\partial C_0}{\partial v_1} = S_0 e^{-\delta\tau} \left[N(d) \frac{B + 2Iv_1}{\mu - g} + \frac{v_1}{\sigma} \sqrt{\tau} n(d) \right] = S_0 e^{-\delta\tau} \left[N(d) \frac{b + I\rho\sigma}{\mu - g} + \rho\sqrt{\tau} n(d) \right].$$

If $I = 0$ then Proposition 5 is equivalent to Theorem 2. The role of I is to strengthen the role of the price channel when $I < 0$ and weaken it when $I > 0$. These effects are intuitive as a negative value of I causes an increase in the risk factor loading v_1 to coincide with an increase in economy-wide risk.

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[16]This is the case in general equilibrium if the asset is in positive supply and its cash flow is not in the span of the other assets.

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Table 2: European call option expected return and risk

The table reports expected returns, μ^c, volatility, σ^c, idiosyncratic risk factor loading v_2^c, and elasticity, ϵ, for call options that are ITM or OTM at systematic risk factor loading $v_1 = 0.1$. Column one reports economy-wide risk—SPD volatility, $-b$, and the call maturity, τ. The remaining parameter values are $r = 0.02$, $g = 0.02$, $v_2 = 0.2$, $X_0 = 0.5$, and $K = 9.0$ and $K = 11.5$ for ITM and OTM calls respectively. We use (3) to compute S_0.

b, τ	v_1	ITM μ^c	σ^c	v_2^c	ϵ	OTM μ^c	σ^c	v_2^c	ϵ
	0.10	0.55	1.76	1.57	7.89	1.19	3.80	3.40	17.01
	0.15	2.40	6.34	5.07	25.36	3.49	9.20	7.36	36.81
	0.20	4.05	9.55	6.75	33.78	5.18	12.22	8.64	43.20
	0.25	5.12	11.31	7.06	35.34	6.20	13.70	8.56	42.80
$b = -0.5$	0.30	5.75	12.20	6.77	33.86	6.76	14.34	7.95	39.77
$\tau = 0.25$	0.35	6.09	12.60	6.25	31.26	7.01	14.51	7.19	35.99
	0.40	6.25	12.70	5.68	28.41	7.09	14.43	6.45	32.26
	0.45	6.29	12.65	5.13	25.69	7.07	14.21	5.77	28.86
	0.50	6.27	12.50	4.64	23.22	6.98	13.93	5.17	25.87
	0.55	6.20	12.30	4.20	21.03	6.86	13.61	4.65	23.27
	0.10	0.37	1.18	1.06	5.30	0.53	1.71	1.53	7.66
	0.15	0.89	2.34	1.87	9.37	1.13	2.99	2.39	11.96
	0.20	1.32	3.12	2.20	11.03	1.58	3.73	2.64	13.21
	0.25	1.61	3.57	2.23	11.15	1.86	4.12	2.57	12.89
$b = -0.5$	0.30	1.79	3.81	2.11	10.58	2.03	4.31	2.39	11.96
$\tau = 1$	0.35	1.90	3.93	1.95	9.77	2.12	4.38	2.17	10.88
	0.40	1.96	3.99	1.78	8.92	2.16	4.39	1.96	9.83
	0.45	1.99	4.00	1.62	8.13	2.17	4.37	1.77	8.88
	0.50	2.00	3.99	1.48	7.42	2.17	4.33	1.60	8.04
	0.55	2.00	3.97	1.35	6.79	2.15	4.28	1.46	7.31
	0.10	0.99	1.85	1.66	8.30	2.12	3.96	3.54	17.73
	0.15	4.48	6.59	5.27	26.39	6.44	9.48	7.58	37.92
	0.20	7.68	9.88	6.98	34.94	9.76	12.55	8.87	44.39
	0.25	9.85	11.68	7.30	36.50	11.87	14.08	8.79	43.98
$b = -1$	0.30	11.18	12.60	6.99	34.96	13.08	14.74	8.17	40.88
$\tau = 0.25$	0.35	11.94	13.01	6.45	32.28	13.70	14.93	7.40	37.03
	0.40	12.33	13.13	5.87	29.37	13.95	14.86	6.64	33.23
	0.45	12.49	13.09	5.31	26.58	13.98	14.65	5.95	29.76
	0.50	12.50	12.94	4.80	24.04	13.88	14.37	5.34	26.70
	0.55	12.42	12.75	4.35	21.79	13.70	14.07	4.80	24.04
	0.10	0.69	1.30	1.16	5.83	1.00	1.86	1.66	8.33
	0.15	1.75	2.57	2.06	10.31	2.20	3.24	2.59	12.97
	0.20	2.66	3.42	2.42	12.10	3.15	4.05	2.86	14.32
	0.25	3.30	3.91	2.44	12.23	3.77	4.48	2.79	13.99
$b = -1$	0.30	3.71	4.18	2.32	11.61	4.16	4.69	2.60	13.01
$\tau = 1$	0.35	3.97	4.32	2.14	10.73	4.39	4.78	2.37	11.86
	0.40	4.12	4.39	1.96	9.82	4.51	4.80	2.14	10.74
	0.45	4.21	4.41	1.79	8.97	4.57	4.79	1.94	9.72
	0.50	4.26	4.41	1.63	8.19	4.59	4.75	1.76	8.83
	0.55	4.28	4.39	1.50	7.51	4.58	4.70	1.60	8.04

Table 3: European call option prices and systematic risk

The table reports ratios of European call option values $C(X_0, K, b, v_1)/C(X_0, K, b, 0.16)$ based on Lemma 1, for varying levels of systematic risk due to economy-wide risk b, in (1), and the systematic risk factor loading v_1, in (2). The last three columns report ratios for 20% out-, at-, and in-the-money options defined relative to the cases $v_1 = 0.16$.

b	v_1	$\mu - r$	S_0	OTM	ATM	ITM
	0.12	0.036	30.00	8.45	12.79	19.90
	0.15	0.045	12.00	1.64	1.85	2.07
-0.30	0.16	0.048	10.00	1.00	1.00	1.00
	0.17	0.051	8.57	0.61	0.55	0.49
	0.20	0.060	6.00	0.15	0.10	0.07
	0.12	0.060	16.67	3.54	4.74	6.36
	0.15	0.075	11.11	1.37	1.47	1.58
-0.50	0.16	0.080	10.00	1.00	1.00	1.00
	0.17	0.085	9.09	0.73	0.68	0.64
	0.20	0.100	7.14	0.29	0.23	0.18
	0.12	0.084	15.19	3.10	4.04	5.24
	0.15	0.105	10.93	1.33	1.42	1.50
-0.70	0.16	0.112	10.00	1.00	1.00	1.00
	0.17	0.119	9.21	0.75	0.71	0.67
	0.20	0.140	7.45	0.33	0.26	0.22

In column three $\mu - r = -bv_1$. The remaining fixed parameter values are $r = 0.02$, $g = 0.05$, $v_2 \approx 0.37$, and $\tau = 1$. The cash flow initial value X_0 is set using (3) to ensure that $S_0|_{v_1=0.16} = 10$ across the three levels of b. Strike prices are $K = 12, 10, 8$ for the OTM, ATM, and ITM columns.

Table 4: European call option expected returns-risk and economy-wide risk

The table reports call expected returns and risk based on (6)–(9), for varying levels of economy-wide risk b in (1). We report moneyness S/K, the call expected return μ^c, volatility σ^c, and systematic and idiosyncratic risk factor loadings v_1^c and v_2^c. We consider two base-case levels of moneyness, $S/K = 1.25$ (ITM) and $S/K = 0.8$ (OTM), defined relative to $b = -0.5$.

τ	b	ITM					OTM				
		S_0/K	μ^c	σ^c	v_1^c	v_2^c	S_0/K	μ^c	σ^c	v_1^c	v_2^c
	-0.30	1.93	0.12	0.53	0.33	0.41	1.24	0.26	1.28	0.80	1.00
	-0.35	1.70	0.15	0.62	0.39	0.49	1.09	0.46	2.02	1.26	1.58
	-0.40	1.52	0.21	0.76	0.47	0.59	0.97	0.77	3.01	1.88	2.35
	-0.45	1.37	0.29	0.97	0.61	0.76	0.87	1.18	4.13	2.58	3.23
0.25	-0.50	1.25	0.42	1.29	0.80	1.01	0.80	1.67	5.29	3.30	4.13
	-0.55	1.14	0.61	1.74	1.08	1.36	0.73	2.22	6.42	4.01	5.01
	-0.60	1.06	0.88	2.31	1.44	1.80	0.67	2.84	7.52	4.70	5.87
	-0.65	0.98	1.22	2.97	1.86	2.32	0.63	3.50	8.57	5.35	6.69
	-0.70	0.92	1.63	3.69	2.31	2.88	0.59	4.20	9.57	5.98	7.47
	-0.75	0.86	2.10	4.44	2.77	3.47	0.55	4.95	10.53	6.57	8.22
	-0.30	1.93	0.12	0.54	0.34	0.42	1.24	0.21	1.03	0.64	0.80
	-0.35	1.70	0.16	0.64	0.40	0.50	1.09	0.30	1.29	0.80	1.01
	-0.40	1.52	0.21	0.76	0.48	0.60	0.97	0.41	1.57	0.98	1.22
	-0.45	1.37	0.27	0.91	0.57	0.71	0.87	0.54	1.85	1.16	1.45
1.00	-0.50	1.25	0.35	1.07	0.67	0.84	0.80	0.68	2.13	1.33	1.67
	-0.55	1.14	0.45	1.25	0.78	0.98	0.73	0.84	2.41	1.50	1.88
	-0.60	1.06	0.56	1.44	0.90	1.12	0.67	1.02	2.68	1.67	2.09
	-0.65	0.98	0.68	1.63	1.02	1.27	0.63	1.21	2.94	1.83	2.29
	-0.70	0.92	0.82	1.83	1.14	1.43	0.59	1.41	3.19	1.99	2.49
	-0.75	0.86	0.97	2.02	1.26	1.58	0.55	1.62	3.43	2.14	2.67

The remaining fixed parameter values are as in Table 1.

Table 5: Effects from variations in stochastic risk exposure

The table reports the results from the Monte Carlo simulations described in Section 4, for three risk exposure cases. The base case $\kappa_j = 0, \eta_j = 0$ is the fixed risk exposure case of Section 3, while $\kappa_j = 0.1, \eta_j = 0.2$ and $\kappa_j = 1.5, \eta_j = 0.2$ correspond to slow and fast mean-reversion of the risk factor exposures. Columns 1 and 2 hold initial conditions for the systematic and idiosyncratic risk factor loadings (11) and (12) respectively. The top half of the table considers variations in $v_{0,1}$ for fixed $v_{0,2}$, while the lower half reverses which parameter is fixed. Columns S_0/K reports moneyness, C_0 reports call prices, μ_τ^c reports expected period $[0, \tau]$ returns, σ_τ^c the corresponding standard deviations of call returns, and $v_{2,\tau}^c$ the corresponding idiosyncratic risk. Equilibrium prices of the asset S_0 are estimated using $M = 10{,}000$ independent paths for $T_S = 5D$ days, with $D = 250$. The maturity of the option is $\tau = \tau_c/D$ in which $\tau_c = 66$—about three months. Remaining parameters used are $r = 0.02, g = 0.05$, $X_0 = 0.1, b = -0.5, \eta_1 = \eta_2 = 0.2, \rho_1 = \rho_2 = 0, \bar{v}_1 = 0.15$ and $\bar{v}_2 = 0.2$.

		$\kappa_j = 0, \eta_j = 0$					$\kappa_j = 0.1, \eta_j = 0.2$					$\kappa_j = 1.5, \eta_j = 0.2$				
$v_{0,1}$	$v_{0,2}$	S_0/K	c_0	μ_τ^c	σ_τ^c	$v_{2,\tau}^c$	S_0/K	c_0	μ_τ^c	σ_τ^c	$v_{2,\tau}^c$	S_0/K	c_0	μ_τ^c	σ_τ^c	$v_{2,\tau}^c$
0.10	0.20	1.00	5.43	0.15	1.69	1.58	1.00	5.15	0.15	1.71	1.60	1.00	5.19	0.16	1.73	1.61
0.12	0.20	0.97	4.12	0.20	2.04	1.88	0.98	4.43	0.19	1.91	1.77	0.99	4.77	0.19	1.87	1.72
0.14	0.20	0.95	3.17	0.25	2.44	2.24	0.97	3.80	0.22	2.15	1.96	0.98	4.39	0.21	2.01	1.83
0.16	0.20	0.92	2.46	0.30	2.88	2.64	0.95	3.28	0.26	2.41	2.18	0.97	4.05	0.24	2.17	1.95
0.18	0.20	0.90	1.93	0.35	3.38	3.10	0.94	2.84	0.30	2.69	2.42	0.96	3.76	0.27	2.33	2.08
0.20	0.20	0.88	1.56	0.40	3.91	3.61	0.92	2.47	0.34	3.00	2.68	0.95	3.50	0.30	2.50	2.22
0.22	0.20	0.86	1.27	0.44	4.48	4.15	0.91	2.17	0.38	3.33	2.97	0.94	3.28	0.32	2.67	2.35
0.24	0.20	0.84	1.06	0.48	5.07	4.72	0.89	1.92	0.42	3.66	3.28	0.93	3.08	0.35	2.84	2.50
0.26	0.20	0.82	0.90	0.53	5.69	5.31	0.87	1.71	0.45	4.01	3.60	0.92	2.91	0.38	3.02	2.64
0.28	0.20	0.80	0.78	0.57	6.30	5.91	0.86	1.54	0.48	4.37	3.94	0.91	2.76	0.40	3.20	2.79
0.30	0.20	0.78	0.69	0.62	6.95	6.54	0.84	1.40	0.51	4.72	4.28	0.90	2.63	0.42	3.37	2.95
0.15	0.10	1.00	3.70	0.31	1.85	1.38	1.00	3.67	0.30	1.87	1.47	1.00	4.02	0.28	1.86	1.51
0.15	0.12	1.00	3.98	0.29	1.84	1.43	1.00	3.89	0.28	1.86	1.51	1.00	4.23	0.26	1.85	1.53
0.15	0.14	1.00	4.28	0.27	1.83	1.48	1.00	4.16	0.26	1.85	1.54	1.00	4.47	0.25	1.85	1.55
0.15	0.16	1.00	4.60	0.25	1.82	1.52	1.00	4.46	0.24	1.84	1.57	1.00	4.73	0.24	1.84	1.58
0.15	0.18	1.00	4.94	0.24	1.81	1.55	1.00	4.77	0.23	1.83	1.60	1.00	5.01	0.22	1.83	1.60
0.15	0.20	1.00	5.29	0.22	1.80	1.58	1.00	5.11	0.21	1.82	1.62	1.00	5.31	0.21	1.82	1.62
0.15	0.22	1.00	5.66	0.21	1.80	1.60	1.00	5.47	0.20	1.82	1.64	1.00	5.62	0.20	1.82	1.64
0.15	0.24	1.00	6.04	0.19	1.79	1.62	1.00	5.84	0.18	1.81	1.66	1.00	5.95	0.19	1.81	1.65
0.15	0.26	1.00	6.42	0.18	1.79	1.64	1.00	6.22	0.17	1.81	1.68	1.00	6.29	0.18	1.81	1.67
0.15	0.28	1.00	6.81	0.17	1.79	1.66	1.00	6.62	0.16	1.81	1.69	1.00	6.63	0.17	1.81	1.68
0.15	0.30	1.00	7.20	0.16	1.79	1.67	1.00	7.02	0.15	1.81	1.70	1.00	6.99	0.16	1.81	1.69

CCP's Multilateral Netting with Payoffs Correlation across Members

Giovanni Bruno[*]

Abstract

Central clearing of derivatives negotiated in over-the-counter (OTC) markets, has been proposed as one of the main pillars of the financial regulation, developed after the crisis of the 2007-2008, in order to improve financial stability. I study the impact of the clearing of derivatives through a central counterparty on the expected exposure generated by several derivative asset classes negotiated on the OTC markets. As showed in previous literature, the effect in terms of exposure is determined by the trade-off between bilateral and multilateral netting opportunities. I find that correlation across members decreases multilateral netting efficiency and I provide a closed form solution to evaluate the relation between benefit of multilateral netting and correlation of payoffs across members. I also find that less diversified members have less incentive to clear and I show a rational way of maximizing netting efficiency when there are multiple central counterpaties.

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Keywords: CCP, Netting, Counterparty Risk, Derivatives, Systemic Risk.

[*]The Norwegian School of Economics (NHH). giovanni.bruno@nhh.no

1 Introduction

During the last two decades the derivatives markets have grown exponentially. As illustrated in D.Duffie (2011), they have the potential of improving the overall efficiency of the financial markets as well of creating important frictions. Financial derivatives are traded on regulated exchanges or in Over-The-Counter (OTC) markets. Exchanges offer higher liquidity and mitigation of credit risk, while OTC markets allow to trade less standardized contracts and are usually traded bilaterally between counterparties. In 1986, the notional outstanding on the OTC markets and on the exchanges were almost the same (about \$500 billion)[1]. By 1995, the notional outstanding in the OTC markets was 5 times higher than the exchanges and in 2005 this ratio was still the same (see J.Gregory (2010)).

The systemic importance for the stability of the financial markets of the OTC transactions has been a major concern for regulators in the last years, in particular for their embedded counterparty risk, that is the risk that one party to a derivative contract is not able to fulfill its obligations. The default of a major dealer bank[2] threatens the financial positions of its counterparties, whose creditworthiness is questioned by other counterparties, generating a destabilized financial system that might lead to a default cascade as described in D.Duffie (2011), K.R.French et al. (2010) and S.Battiston et al. (2012). In order to deal with the risk produced by OTC transactions, regulators in US and EU issued respectively the Dodd-Frank Act and the European Market Infrastructure Regulation (EMIR). One of the main objectives of this new regulatory framework is to decrease systemic risk by shifting the OTC transactions to a market where trades are executed throughout a Central Clearing Counterparty (CCP).

A CCP is a financial institution which stands between the two sides of a trade, in

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[1] Source: J.Gregory (2010)

[2] A commercial bank authorized to buy and sell government securities including federal reserve and municipal bonds.

particular when a contract is cleared, it is replaced with two other contracts (of opposite sides) where the CCP is the buyer for the original seller and the seller for the original buyer[3]. The most relevant characteristic of this trading mechanism is that the CCP guarantees the payoff of each part of the contract from the default of its counterparty. However, as C.Pirrong (2009) points out, counterparty risk is not completely eliminated, but it is shifted from each counterparty to the CCP

The quantification of the counterparty risk has three main components:

- the exposure, that is the max between the payoff of the contract and zero,

- the default probability of the counterparty,

- the wrong way risk, that is the correlation between the first two components.

The quantification of the counterparty risk is relevant for the pricing as well as for the computation of the capital requirements, to which the banks are subject due to the previous mentioned regulation. In this work I analyze the impact of the CCP on the exposure.

The exposure on a derivative contract can be decreased using Netting agreements[4]. As showed in D.Duffie and H.Zhu (2011) and R.Cont and T.Kokholm (2014), the impact of the introduction of the central clearing in terms of exposure depends on a trade-off between two types of netting opportunities: bilateral and multilateral netting. The former is obtained through the original trading mechanism in the OTC markets between each pairs of conterparties, which can net each other positions across several asset classes. The latter, instead, is obtained through the CCP, which can net the positions across entities on all the cleared asset classes (see Figure 1). D.Duffie and H.Zhu (2011) also

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[3]See C.Pirrong (2009).

[4]A Netting agreement is a contract between two counterparties that allows the aggregation of all the transactions between them. For instance, if two counterparties are linked by a CDS and a swap contract, they can agree to settle on the base of the net value of the two positions in case of default of one of the two counterparties, instead of considering the value of the contracts separately (see Bielecki et al. (2011)).

show the conditions under which clearing a certain asset class generates a positive effect in terms of exposure[5], quantified using the expected exposure (EE) as risk measure. The present work contributes to the literature showing how correlation of payoffs across members decreases the multilateral netting efficiency.

The main objective for the introduction of the central clearing on the OTC markets is to enhance financial stability. However, the contribution of the CCP to systemic risk is ambiguous. Indeed, the introduction of the CCP modifies the architecture of the financial network (as shown in Figure 1). If we consider just the cleared contracts, the CCP creates an highly dense interconnected network, also known as complete network, since, through the CCP, the losses due to a negative liquidity shock are shared across all the members, therefore, each member of the system is exposed to all the other members'losses. In F.Allen and D.Gale (2000) the authors show that such kind of network minimizes the risk of a financial contagion since the losses of a bank are mutualized across more creditors, minimizing the negative impact of the liquidity shock. In a more recent paper, D.Acemoglu et al. (2015) dispute the role of the interconnections. The authors show that as long as the magnitude of the liquidity shock is below a certain limit threshold, the complete network is less fragile, confirming what was found in F.Allen and D.Gale (2000). However, if the liquidity shock is above the limit threshold, the complete network is most dangerous since it allows for a fast propagation of the shock across all the members of the system and it maximizes the number of default in the network.

In the present work I do not consider default probabilities and systemic risk. The main contribution is to analyse how correlation of derivative payoffs across members contributes to modify the overall exposure of each member given that the optimal infrastructure of the network in terms of systemic risk might be of several kinds. Using the theoretical framework presented in D.Duffie and H.Zhu (2011), I show in section

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[5]The new regulatory framework asks to limit the exposure of the banks, in order to prevent a large loss as a result of a single institution. Therefore, the lower is the exposure the better it is.

4 that if the exposures of member i to different members are correlated, the benefit of multilateral netting is decreased in terms of expected exposure (EE) and expected shortfall (ES). The payoffs of the contracts with different members can be correlated because such members might be exposed to similar risks and therefore they need to use similar hedging strategies to manage them, as showed in an empirical work Benoit et al. (2015). For instance, member j and m may be both exposed to the credit risk of the same bond and enter both in a CDS contract with member i. Therefore, in this case the exposure of i to j and to m would be positively correlated. As showed in E.Farhi and J.Tirole (2012), correlated risk exposures might be explained by the incentive of the banks to maximize the likelihood of being bailed out.

The intuition is that a positive correlation between exposures across members decreases the positive effect of multilateral netting since the netting will be more difficult because the payoffs will move in the same direction. This is relevant for the risk management of the banks because a different exposure may affect the cost of collateral and have an impact on capital requirements.

In section 5 I investigate the different benefit from clearing for two counterparties entering in a new derivative contract whose payoff has different correlations with the other derivatives to which the the two counterparties are exposed. Under the assumption of heterogenous correlations, the members that are less diversified have less incentive to clear. The intuition is that the payoff of a new derivative contract that has different correlations with the payoffs of the other derivatives in the portfolios of the two counterparties, gives a different contribution to the overall exposure of the two counterparties and the one that will benefit the most will have an higher incentive to clear the new contract. This is relevant for the fee policy of the CCP, which may have an incentive to over charge the members that are doing the best job at diversifying because they have an higher incentive to clear.

Finally, in section 6, I show the most efficient way to clear when there are multiple CCPs and there is correlation across members, since having a single CCP might endanger the overall system because it would be completely dependent from a single *too big* and *too interconnected to fail* financial institution.
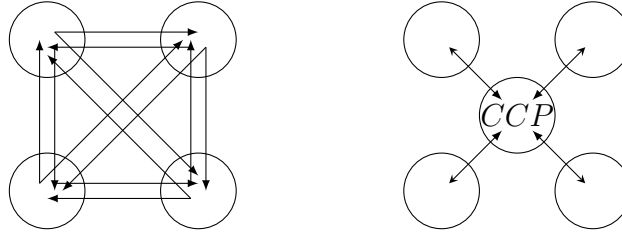

Figure 1: Interbanking member networks: on the left without CCP and with CCP on the right.

2 The Power of Netting

Netting is one of the typical clauses used in the derivative contracts to decrease counterparty risk. In order to understand the power of bilateral netting ,which is attainable in the OTC markets, let's consider a toy example.

Let the member[6] i be exposed to the member j on a CDS for 150\$ (which means that j owes 150\$ to i) and that member j is exposed to i on a swap contract for 100\$, using the netting agreement the net position will be an exposition from i to j for 50\$. Therefore, in case j would default, i would lose 50\$. Instead, if i was the one incurring in bankruptcy, j would still owe 50\$. Now let's introduce the CCP, if the two members decide to clear just the CDS contract, then the bilateral netting (through asset classes) will not be possible anymore, therefore there will be an exposition of i to the CCP of 150\$ and an exposition of j to i of 100\$. On the other hand, there is the power of multilateral netting. In order to understand the latter, let's consider a different situation, a market

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[6]The terms bank, member and sometimes entity will be used interchangeably.

104

with three members (i, j and m) where i is exposed to j for 100\$, j is exposed to m for 100\$ and m is exposed to i for 100\$ and that all the expositions are on CDS contracts; in this case the bilateral netting is completely ineffective, but if all the CDS contract are cleared throughout the same CCP then the net exposition of each member will be 0.

Netting is an important feature to take in consideration for the determination of both EE and ES. I use EE for this analysis since is one of the main measures used by the Basel regulation for the determination of the banking risk and in order to be consistent with the analysis performed in D.Duffie and H.Zhu (2011). Furthermore, I use ES in order to check the robustness of the results. We may define the EE as the amount expected to be lost if the counterparty defaults under the physical probability measure P (see J.Gregory (2010)), analytically

$$EE = E\left[\max\left(X, 0\right)\right], \tag{1}$$

where X is the payoff of the derivative contract and $E[.]$ denotes the expectation.

ES is the conditional expected exposure above the quantile q of the distribution under the physical probability measure P, analytically

$$ES = E\left[\max\left(X, 0\right) | X > q^{th} quantile\right]. \tag{2}$$

The exposure's computation captures the fact that, in the case of default of the counterparty, the bank will still owe the full value of the derivatives if the position has a positive value for the counterparty. Instead, if the position has a positive value and there is a default of the counterparty, the bank will not be able to recover the full amount.

Correlation between payoffs may significantly affect the power of netting. Let's consider for instance two random payoffs X_1 and X_2 jointly normally distributed with means μ_1 and μ_2, standard deviations σ_1 and σ_2 and correlation ρ. Allowing for netting

between the 2 positions gives the following EE

$$EE_{Net} = E\left[\max(X_1 + X_2, 0)\right] \tag{3}$$

$$= \sigma\phi(\mu/\sigma) + \mu\Phi(\mu/\sigma). \tag{4}$$

where Φ is the standard normal cumulative distribution, ϕ is its density and $\mu = \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$ are the mean and variance of the sum of $X_1 + X_2$ (See D.Brigo et al. (2013) for a comprehensive explanation). If netting is not allowed the EE is

$$EE_{NoNet} = E\left[\max(X_1, 0) + \max(X_2, 0)\right] \tag{5}$$

$$= \sigma_1\phi(\mu_1/\sigma_1) + \mu_1\Phi(\mu_1/\sigma_1) + \sigma_2\phi(\mu_2/\sigma_2) + \mu_2\Phi(\mu_2/\sigma_2). \tag{6}$$

Assuming for simplicity that $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2$, the difference between the two EE above is

$$EE_{NoNet} - EE_{Net} = \frac{\sigma_1}{\sqrt{\pi}}(\sqrt{2} - \sqrt{1 + \rho}). \tag{7}$$

The expression above shows the power of netting of decreasing EE and we can notice that it depends on the correlation parameter ρ, indeed if ρ increases also EE_{Net} increases until $\rho = 1$ where $EE_{Net} = EE_{NoNet}$ and therefore the power of netting is neutralized.

3 Independent Exposures

In this section I obtain the change in terms of EE and ES due to the introduction of the CCP, using the same setting presented in D.Duffie and H.Zhu (2011).

Firstly, I describe the setting. There is an OTC market with N dealer banks and K asset classes (such as credit, interest rates, commodities, equities) in which the members evaluate in 0 the expected value of a payoff due at the final date T. X_{ij}^k represents

the amount that the entity j owes to i at the final date T for the asset class k and it is uncertain. The uncertainty about this amount is generated by changes in the mark to market of the derivatives as well in their notionals. The *exposure* is given by the $\max(X_{ij}^k, 0)$. In this section, I assume that X_{ij}^k are independent across members of the market (i.e. for all i and j) and asset classes k. Furthermore, they are identically normally distributed with 0 expected value and standard deviation equals to σ[7].

Bilateral netting allows for netting across asset classes with the same counterparty but not across members. Therefore, before introducing a CCP the EE for each member i is

$$\phi_{N,K}^i = \sum_{j \neq i} E\left[\max\left(\sum_{k=1}^K X_{ij}^k, 0\right)\right], \tag{8}$$

where

$$z = \sum_{k=1}^K X_{ij}^k \sim \mathcal{N}(0, \sigma\sqrt{K}), \tag{9}$$

Therefore, from expression (8) we may obtain

$$\phi_{N,K}^i = \sum_{j \neq i} \int_0^{+\infty} z f(z) d(z) \tag{10}$$

$$= \sum_{j \neq i} \frac{1}{\sigma\sqrt{2\pi K}} \int_0^{+\infty} z e^{-\frac{z^2}{2\sigma^2 K}} d(z) \tag{11}$$

$$= (N-1)\sigma\sqrt{\frac{K}{2\pi}}. \tag{12}$$

Now let us introduce the CCP for clearing a single asset class, for instance l. With central clearing, positions are netted across members for the cleared asset class, therefore the

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[7]Some of these assumptions will be relaxed later.

expected exposure of entity i to the CCP is

$$\gamma_{N,l}^i = E\left[\max\left(\sum_{j\neq i} X_{ij}^l, 0\right)\right],\tag{13}$$

where

$$w = \sum_{j\neq i} X_{ij}^l \sim \mathcal{N}(0, \sigma\sqrt{N-1}).\tag{14}$$

From expression (13) we may obtain

$$\gamma_{N,l}^i = \int_0^{+\infty} wf(w)d(w)\tag{15}$$

$$= \frac{1}{\sigma\sqrt{2\pi(N-1)}} \int_0^{+\infty} we^{-\frac{w^2}{2\sigma^2(N-1)}}d(w)\tag{16}$$

$$= \sigma\sqrt{\frac{N-1}{2\pi}}.\tag{17}$$

The total EE in this case for the member i is given by the sum of $\gamma_{N,l}^i$ plus the expected exposure on the remaining $K-1$ asset classes that are not cleared. Therefore, there will be a reduction in the total exposure if

$$\Delta_{EE}^i = \phi_{N,K}^i - (\gamma_{N,l}^i + \phi_{N,K-1}^i) > 0.\tag{18}$$

Now, we can also compute the ES in the 2 cases indicated above.

In absence of clearing ES will be

$$\phi_{N,K}^{i,ES} = \sum_{j\neq i} E\left[\max\left(\sum_{k=1}^K X_{ij}^k, 0\right) \mid \sum_{k=1}^K X_{ij}^k \geq Z\right]\tag{19}$$

$$= \sum_{j\neq i} \int_{-\infty}^{+\infty} zf(z|z \geq Z)d(z).\tag{20}$$

Since

$$E(z|z \geq Z) = \frac{E[z\mathbf{1}_{z \geq Z}]}{P(z \geq Z)}, \tag{21}$$

we can rewrite (20) as

$$= \sum_{j \neq i} \frac{1}{\sigma\sqrt{2\pi K}} \int_{Z}^{+\infty} z e^{-\frac{z^2}{2\sigma^2 K}} d(z) \frac{1}{P(z \geq Z)} \tag{22}$$

$$= (N-1)\sigma\sqrt{\frac{K}{2\pi}} e^{-\frac{z^2}{2\sigma^2 K}} \frac{1}{P(z \geq Z)} \tag{23}$$

To simplify notation let us define

$$d_1 = \frac{Z}{\sigma\sqrt{K}}, \tag{24}$$

therefore (23) is

$$\phi_{N,K}^{i,ES} = (N-1)\sigma\sqrt{\frac{K}{2\pi}} e^{-\frac{d_1^2}{2}} \frac{1}{P(z \geq Z)} \tag{25}$$

In the second case the ES respect to the CCP $\gamma_{N,l}^{i,ES}$ is

$$E\left[\max\left(\sum_{j \neq i} X_{ij}^l, 0\right) | w \geq W\right] = \int_{W}^{+\infty} w f(w|w \geq W) d(w) \tag{26}$$

$$= \frac{1}{\sigma\sqrt{2\pi(N-1)}} \int_{W}^{+\infty} w e^{-\frac{w^2}{2\sigma^2(N-1)}} d(w) \frac{1}{P(w \geq W)} \tag{27}$$

$$= \sigma\sqrt{\frac{N-1}{2\pi}} e^{-\frac{W^2}{2\sigma^2(N-1)}} \frac{1}{P(w \geq W)} \tag{28}$$

$$= \sigma\sqrt{\frac{N-1}{2\pi}} e^{-\frac{d_2^2}{2}} \frac{1}{P(w \geq W)} \tag{29}$$

where

$$d_2 = \frac{W}{\sigma\sqrt{N-1}}. \tag{30}$$

If

$$\alpha = P(w \geq W) = P(z \geq Z) \tag{31}$$

Then

$$d_1 = d_2 = d. \tag{32}$$

The clearing of the asset class l will reduce the ES if

$$\Delta_{ES}^i = \phi_{N,K}^{i,ES} - (\gamma_N^{i,ES} + \phi_{N,K-1}^{i,ES}) > 0 \tag{33}$$

$$= (N-1)\sigma\sqrt{\frac{K}{2\pi}}\frac{e^{-\frac{d^2}{2}}}{\alpha} - \left(\sigma\sqrt{\frac{N-1}{2\pi}}\frac{e^{-\frac{d^2}{2}}}{\alpha} + (N-1)\sigma\sqrt{\frac{K-1}{2\pi}}\frac{e^{-\frac{d^2}{2}}}{\alpha}\right) \tag{34}$$

$$= \frac{e^{-\frac{d^2}{2}}}{\alpha}\Delta_{EE}^i \tag{35}$$

As showed in D.Duffie and H.Zhu (2011) the reduction in EE, due to the clearing of an asset class, is increasing in the number of members, since the possibilities of multilateral netting across members increase. However, the reduction is decreasing in the number of asset classes, because an high numbers of asset classes increases the bilateral netting efficiency. From a comparison of Δ_{EE}^i and Δ_{ES}^i, we can see from Figure 2, that with the expected shortfall the number of members of the market needed in order to obtain a positive effect from the clearing of asset class l is the same that the one in terms of EE. However, the ES is a more sensitive measure.

Figure 2: Benefit from central clearing and number of members with Independent exposures.
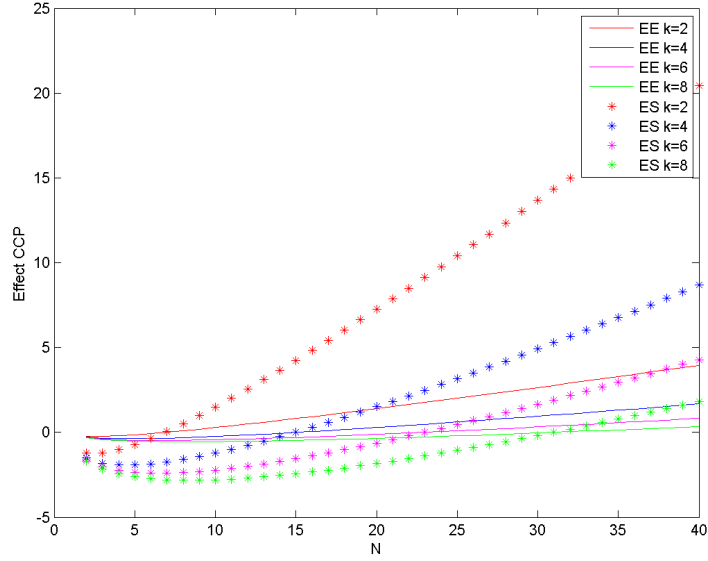
This figure represents the relationship between Δ_{EE}^i and Δ_{ES}^i (in which ES is at 95%) and the number of dealer banks for several numbers of asset classes K.

4   Correlation of exposures across members

In this section I relax the assumption of independence and analyse the effect of the correlation between payoffs across members on the EE and the ES of each member. In other words I study the case in which $E\left[X_{ij}^k X_{im}^k\right] \neq 0$. Furthermore, I assume that such correlation is constant. The latter assumption facilitates calculations and allows to focus on the impact that correlation has on multilateral netting efficiency. I maintain the assumptions of normal distribution, zero expected value and equal standard deviation for the payoffs, but I relax the assumption of independence between members, in particular

for the covariance we have the following

$$\text{Cov}\left(X_{ij}^k, X_{im}^k\right) = E\left[X_{ij}^k X_{im}^k\right] \tag{36}$$

$$= \rho_{j,m}^{i,k}\sigma^2 \tag{37}$$

$$= \rho\sigma^2, \tag{38}$$

assuming that

$$\rho_{j,m}^{i,k} = \rho \quad \forall \quad j, m, i, k. \tag{39}$$

Proposition 1. *The reduction in expected exposure and expected shortfall due to the introduction of a CCP is decreasing in the correlation across members.*

Proof. Let us consider now the case in which member i clears just the asset class l and compares the EE with the ES in the case in which there is no clearing. The EE in the case of no clearing is

$$\phi_{N,K}^i = \sum_{j\neq i} E\left[\max\left(\sum_k^K X_{ij}^k, 0\right)\right] \tag{40}$$

$$= (N-1)\sigma\sqrt{\frac{K}{2\pi}}. \tag{41}$$

The EE in the case of clearing of the asset class l is

$$\phi_{N,K-1}^i + \gamma_{N,l}^i, \tag{42}$$

in which

$$\gamma_{N,l}^i = E\left[\max\left(\sum_{j\neq i} X_{ij}^l, 0\right)\right] \tag{43}$$

$$= \sqrt{\frac{S}{2\pi}}. \tag{44}$$

Here S is the variance of $\sum_{j\neq i} X_{ij}^l$ and is given by

$$S = \mathrm{Var}\left(\sum_{j\neq i} X_{ij}^l\right) \tag{45}$$

$$= E\left[\left(\sum_{j\neq i} X_{ij}^l\right)^2\right] \tag{46}$$

$$= \sum_{j\neq i} E\left[(X_{ij}^l)^2\right] + 2\sum_{j\neq i}\sum_{m\neq i,j} E\left[X_{ij}^l X_{im}^l\right] \tag{47}$$

$$= (N-1)\sigma^2 + 2\sum_{j\neq i}\sum_{m\neq i,j} \mathrm{Cov}\left(X_{ij}^l, X_{im}^l\right). \tag{48}$$

Using the assumptions stated above we have

$$S = (N-1)\sigma^2 + 2\sum_{j\neq i}\sum_{m\neq i,j} \rho\sigma^2 \tag{49}$$

$$= \sigma^2(N-1) + 2\frac{(N-1)!}{2!(N-3)!}\rho\sigma^2 \tag{50}$$

$$= \sigma^2\left((N-1) + \frac{(N-1)!}{(N-3)!}\rho\right). \tag{51}$$

Therefore,

$$\gamma_{N,l}^i = \frac{\sigma}{\sqrt{2\pi}}\sqrt{(N-1) + \frac{(N-1)!}{(N-3)!}\rho} \tag{52}$$

is an increasing function of ρ. Therefore, the benefit of the multilateral netting is

$$\Delta^i_{EE} = \phi^i_{N,K} - (\phi^i_{N,K-1} + \gamma^i_{N,l}), \tag{53}$$

which reduces when ρ increases. $\qquad\square$

The relation between the benefit of multilateral netting and ρ it is shown in Figure 3.

Figure 3: Benefit from central clearing and correlation of payoffs across members.
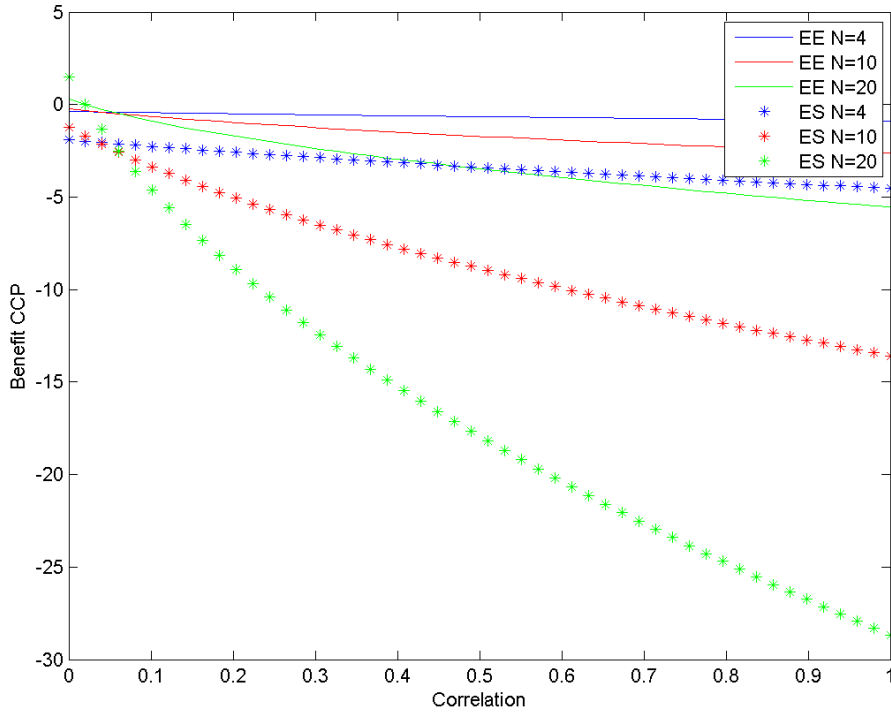
This figure represents the relationship between the variation in Expected Exposure (solid line) and Expected Shortfall (stars) due to the central clearing of asset class k and the average correlation ρ for different numbers of members N in the network.

Given the normality assumption for the payoffs, it can been shown that 35 holds also in this case, therefore for the ES at 95% the benefit from the introduction of the CCP on

the asset class k is

$$\Delta^i_{ES} = \frac{e^{-\frac{d^2}{2}}}{\alpha} \Delta^i_{EE}, \tag{54}$$

where $d = 1.644$ and $P = 0.05$. As we can see from Figure 3, also the benefit in terms of ES reduces but it is more sensitive. Indeed it can be shown that

$$\mid \frac{\partial \Delta_{ES}}{\partial \rho} \mid \geq \mid \frac{\partial \Delta_{EE}}{\partial \rho} \mid \tag{55}$$

5   Heterogenous correlations

In the previous section I assumed a constant correlation across members, however, it might be that some derivatives'payoffs with some members are more correlated than others. This might affect the incentive to clear a specific derivative for some members.

For instance, let's consider the toy example in Figure 4. The members A and B have a derivative on asset class l with payoff X^l_{AB}[8]. Both, A and B, have derivatives on the same asset class l with other two members of the system j_1 and j_2, but the payoffs of the derivative whose A is exposed ($X^l_{Aj_1}$ and $X^l_{Aj_2}$) have a correlation of ρ with the payoff of the new derivative X^l_{AB}, whereas $X^l_{Bj_1}$ and $X^l_{Bj_2}$ are uncorrelated with X^l_{AB}. As I showed in the previous section, correlation decreases the benefit of multilateral netting efficiency. Therefore, depending on the level of ρ, it may be more convenient for A not to clear the derivative with B[9]. On the other hand, B has an higher incentive to clear such a derivative in terms of EE. The result is that the member that is doing a better job in terms of diversification (in this case B) has an higher incentive to clear, at least in term of EE.

Proposition 2. *Given two counterparties entering in a derivative contract, the benefit*

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[8]From the perspective of B would be $X^l_{BA} = -X^l_{AB}$.
[9]Because it might be the case that the benefit from bilateral netting efficiency is higher.

Figure 4: Central clearing of payoffs with heterogeneous correlation.
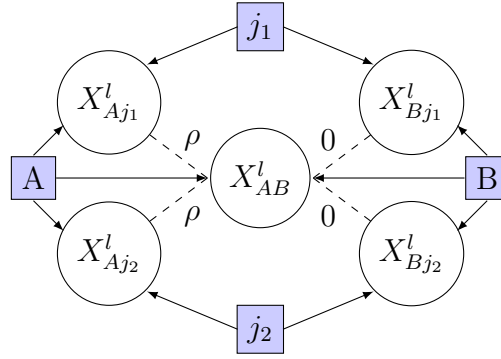
This figure represents a network of counterparties, A,B, j_1 and j_2, having exposure to derivatives in the asset class l, in which X_{kz}^s represents the exposure of the counterpary A to the counterparty z on the asset class s. Furthermore, ρ represents the correlation between payoffs linked by the dashed line.

from clearing the contract will be higher for the most diversified counterparty.

Proof. Consider the case in which the payoff that the member c owes to i is correlated with the payoffs that the other members owe to i and that the other payoffs are uncorrelated between each other. The covariances between payoffs are

$$E[X_{ic}^l X_{ij}^l] = \rho\sigma^2, \quad \forall \quad j \tag{56}$$

$$E[X_{im}^l X_{ij}^l] = 0, \quad \forall \quad m, j. \tag{57}$$

Where c represents the member who generates the correlation between the payoffs of i on the asset class l. In words, I assume that the correlation of a payoff of a derivative signed between i and a generic member j on the asset class l (X_{ij}^l) with the payoff signed between i and c on the same asset class (X_{ic}^l) is always equal to ρ, while all the other payoffs are uncorrelated between each other. Furthermore, I keep the assumptions of normally distributed payoffs and that all the payoffs have equal standard deviation.

The variance of the exposures of i with all the other members on the cleared asset

class l is

$$E\left[\left(\sum_{j\neq i}X_{ij}^l\right)^2\right] = E\left[\sum_{j\neq i}(X_{ij}^l)^2\right] + E\left[\sum_{j\neq i,c}\sum_{m\neq i,c}X_{ij}^lX_{im}^l\right] + E\left[\sum_{j\neq i,c}X_{ic}^lX_{ij}^l\right] \quad (58)$$

$$= (N-1)\sigma^2 + (N-2)\rho\sigma^2 \quad (59)$$

$$= \sigma^2((N-1) + (N-2)\rho). \quad (60)$$

Therefore if the member i clears all the positions on the asset class l the EE is the following

$$EE_a = E\left[\max\left(\sum_{j\neq i}X_{ij}^l, 0\right)\right] + \sum_{j\neq i}E\left[\max\left(\sum_{k\neq l}X_{ij}^k, 0\right)\right] \quad (61)$$

$$= \frac{\sigma}{\sqrt{2\pi}}\sqrt{(N-1) + (N-2)\rho} + (N-1)\sigma\sqrt{\frac{K-1}{2\pi}} \quad (62)$$

Whereas the total EE if the member i clears all the position but those with the member c is

$$EE_b = E\left[\max\left(\sum_{j\neq i,c}X_{ij}^l, 0\right)\right] + \sum_{j\neq i,c}E\left[\max\left(\sum_{k\neq l}X_{ij}^k, 0\right)\right] + E\left[\max\left(\sum_{k}^{K}X_{ic}^k, 0\right)\right] \quad (63)$$

$$= \sigma\sqrt{\frac{N-2}{2\pi}} + (N-2)\sigma\sqrt{\frac{K-1}{2\pi}} + \sigma\sqrt{\frac{K}{2\pi}}. \quad (64)$$

The difference between EE_a and EE_b represents the difference in total EE due to the decision to clear the position with the member c and it is increasing in ρ. $\qquad\square$

As we can see from Figure 5, when correlation increases also expected exposures increases. Therefore, the more a derivative contributes to the diversification of the member the higher is the incentive of the member to clear that contract in terms of EE. This is an aspect to take in consideration for the determination of the fees charged by the

117

Figure 5: Benefit from central clearing of positions with member c and its correlation with the exposures of other members.
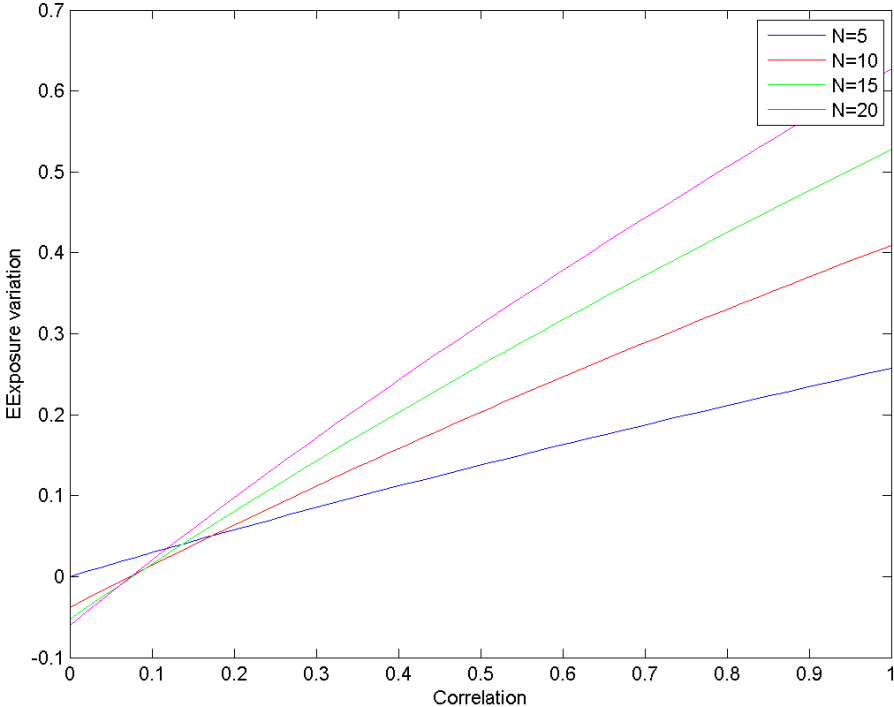
This figure represents the benefit from the decision of clearing the exposures with the member c. On the vertical axis there is the difference $EE_a - EE_b$ and on the horizontal axis the correlation of the exposures with c with the exposures with the other members.

CCP. Indeed, the CCP might charge more to those members that are $well-diversified$ and have an higher incentive to clear, while it might apply a lower fee for those members that are $under-diversified$ and such a situation might have a negative effect on the risk management of the members.

6   Multiple CCPs

It has been shown in D.Duffie and H.Zhu (2011) that it is always better to clear using one CCP instead of multiple CCPs in terms of multilateral netting efficiency. The intuition is that a single CCP can net across all the members of the system and this maximizes the multilateral netting efficiency. However, using just one CCP to clear all the OTC derivatives might risk to create a financial institution $too\ big$ and $too\ interconnected\ to\ fail$, putting too much systemic risk on a single institution. Furthermore, a member of the system might be willing to diversify across multiple CCPs in order not to get too much exposed to the default of a single CCP[10]. Let's consider the examples in Figures 6 and 7. With a single CCP, the generic member of the system i clears all his derivatives using a single CCP (CCP_a). If CCP_a defaults, the member i loses all his payoffs. On the contrary, with multiple CCPs, if CCP_a defaults, i will lose just a part of his payoffs. Therefore, it is interesting to study how a member should divide her derivatives' exposures across multiple CCPs in order to minimize the loss in terms of multilateral netting efficiency.

Firstly, I show that a single CCP maximizes multilateral netting efficiency, as already shown in D.Duffie and H.Zhu (2011). Consider that the member i clears the transactions on the asset class l with all his counterparties on the same CCP. The EE is

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[10]Although, a troubled CCP may enjoy a bailout from the government, since the its default would represent a major danger for financial stability and therefore banks may consider the default of a CCP as an highly remote event.
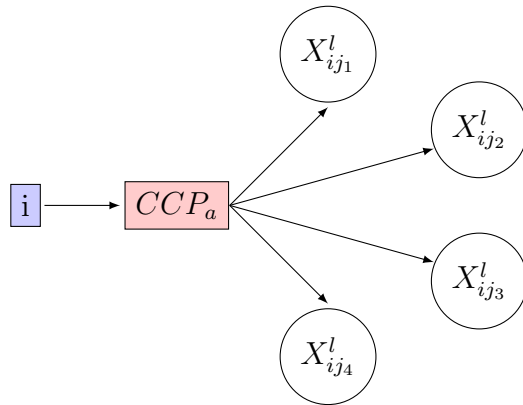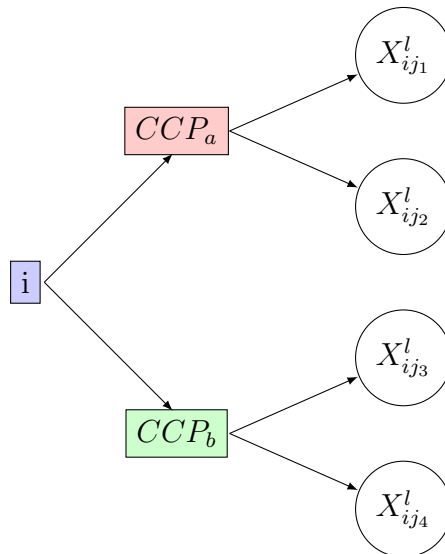
Figure 6: Clearing with a single CCP.

Figure 7: Clearing with 2 CCPs.

$$U_i = E\left[\max\left(\sum_{j\neq i} X_{ij}^l, 0\right)\right] + \phi_{N,K-l}. \tag{65}$$

$$\tag{66}$$

Instead, if i clears the transactions on the asset class l using two different CCPs dividing his counterparties in two groups α and β, the EE is

$$\hat{U}_i = E\left[\max\left(\sum_{j\neq i\in\alpha} X_{ij}^l, 0\right)\right] + E\left[\max\left(\sum_{m\neq i\in\beta} X_{im}^l, 0\right)\right] + \phi_{N,K-l}. \tag{67}$$

$$\tag{68}$$

Since $E[\max(.)]$ is a convex function, we have

$$\hat{U}_i \geq U_i \tag{69}$$

because of Jensen's inequality.

Let be $j = 1, 2 \in \alpha$, $m = 3, 4 \in \beta$ and ρ_a and ρ_b being respectively the correlation between $1, 2$ and $3, 4$. I use the assumptions of normality, zero expected value and equal standard deviation for the payoffs. Furthermore, I assume that the exposures between members of different groups are uncorrelated (for instance $Corr\left(X_{i1}^l, X_{i3}^l\right) = 0$). If i clears all the transactions under the same CCP the EE is

$$U_i = \sqrt{\frac{s}{2\pi}} + \phi_{N,K-l}, \tag{70}$$

where

$$s = E\left[\left(X_{i1}^l + X_{i2}^l + X_{i3}^l + X_{i4}^l\right)^2\right] \tag{71}$$

$$= \sigma^2(4 + 2\rho_a + 2\rho_b). \tag{72}$$

Consider the case A, in which the member i clears the positions on the asset class l for 1 and 3 in the first CCP and the positions with 2 and 4 on the second CCP. The EE is

$$\hat{U}_i^A = \sqrt{\frac{s_1}{2\pi}} + \sqrt{\frac{s_2}{2\pi}} + \phi_{N,K-l}, \tag{73}$$

in which

$$s_1 = s_2 = 2\sigma^2. \tag{74}$$

Instead, in the case B, in which the member i clears the positions on the asset class l for 1 and 2 in the first CCP and the positions with 3 and 4 on the second CCP, the EE is

$$\hat{U}_i^B = \sqrt{\frac{s_3}{2\pi}} + \sqrt{\frac{s_4}{2\pi}} + \phi_{N,K-l}, \tag{75}$$

where

$$s_3 = 2\sigma^2 + 2\rho_a\sigma^2, \tag{76}$$

and

$$s_4 = 2\sigma^2 + 2\rho_b\sigma^2, \tag{77}$$

In Figure 8 we can notice that the difference $\Delta U_A = \hat{U}_i^A - U_i$ is always positive and it is equal to zero only for $\rho_a = \rho_b = 0$. Therefore, as stated in D.Duffie and H.Zhu (2011) the EE with multiple CCPs is always higher than the EE with a single CCP. However, in the second case the difference $\Delta U_B = \hat{U}_i^B - U_i$ is always higher respect to the first kind of division, as it is shown in Figure 9. Indeed, in the second case the lower
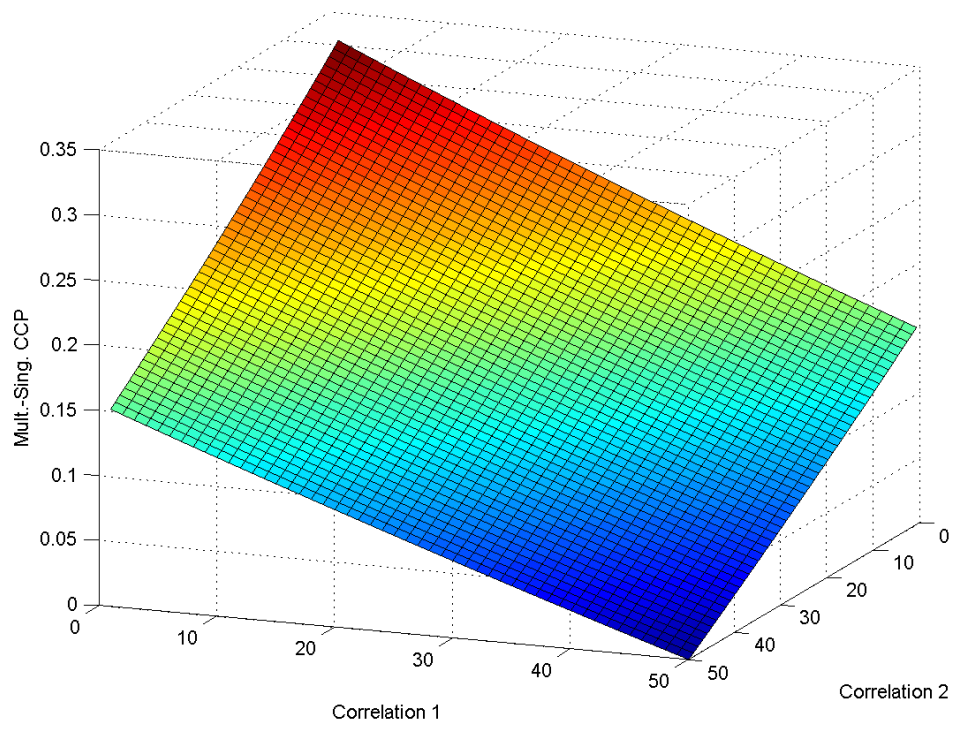
Figure 8: Relationship between ΔU_A and the correlations in the 2 groups of members.

performance of the multiple CCPs respect to the single CCP in terms EE is exacerbated from the correlation between the cleared members that generates a further loss in terms of multilateral netting efficiency. Therefore, In case of multiple CCPs, it is better to clear under different CCPs those counterparties that are less correlated between each other.
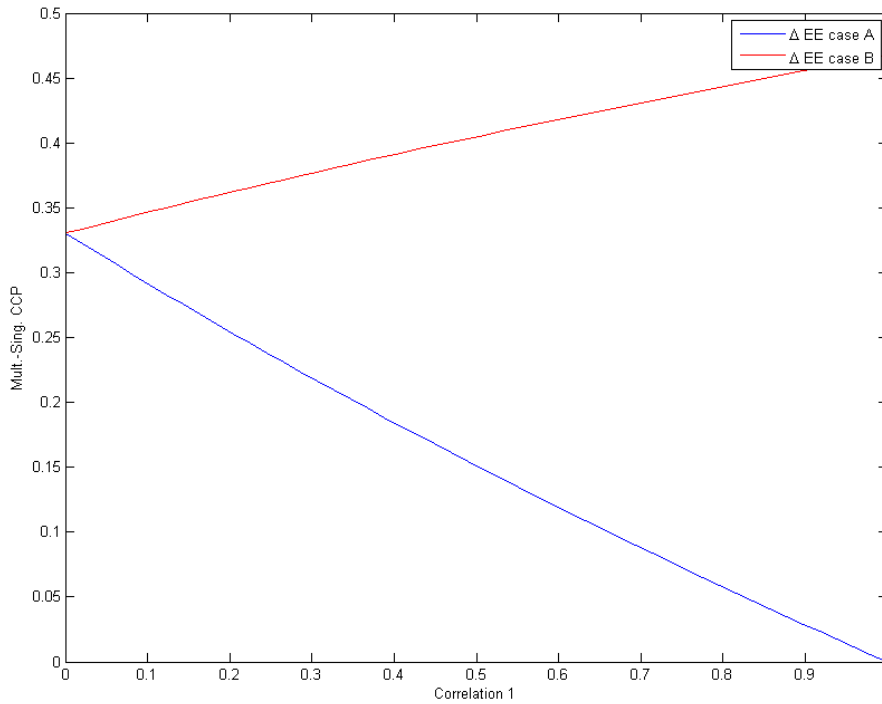

Figure 9: Relationship between ΔU_A, ΔU_B and the correlations in the 2 groups of members given that $\rho_a = \rho_b$.

7   Conclusion

I have shown that correlation of exposures across members affects the trade off between bilateral and multilateral netting. In particular, a positive correlation across members decreases the benefit from multilateral netting evaluated in terms of EE and ES, therefore it reduces the incentive to clear. I have also shown that members that are less diversified

124

have less incentive to clear. Future research might investigate if the latter point have implications on the fee policy of the CCPs and the risk management of the members.

Finally, I consider a market with multiple CCPs. Although, a single CCP is always better of multiple CCPs in terms of netting efficiency (as showed by D.Duffie and H.Zhu (2011), a network depending on a single CCP might actually increase the overall systemic risk. Therefore, I obtained an efficient method of using multiple CCPs, from the perspective of a clearing member, when there is correlation across members. Future research may investigate how competition across multiple CCPs may affect the incentive of the members to clear their derivative positions.

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