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Behavioral equilibrium and evolutionary dynamics in asset markets ☆

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ABSTRACT

This paper analyzes a dynamic stochastic equilibrium model of an asset market based on behavioral and evolutionary principles. The core of the model is a non-traditional game-theoretic framework combining elements of stochastic dynamic games and evolutionary game theory. Its key characteristic feature is that it relies only on objectively observable market data and does not use hidden individual agents' characteristics (such as their utilities and beliefs). A central goal of the study is to identify an investment strategy that allows an investor to survive in the market selection process, i.e., to keep with probability one a strictly positive, bounded away from zero share of market wealth over an infinite time horizon, irrespective of the strategies used by the other players. The main results show that under very general assumptions, such a strategy exists, is asymptotically unique and easily computable.

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1. Introduction

We develop a new dynamic stochastic equilibrium model of an asset market combining evolutionary and behavioral approaches. The classical financial DSGE theory going back to Kydland and Prescott (1982) and Radner (1972, 1982) (see Magill and Quinzii, 1996) relies upon the hypothesis of full rationality of market players, who are assumed to maximize their utilities or preferences subject to budget constraints, i.e., solve well-defined and precisely stated constrained optimization problems. The model we consider relaxes these assumptions and permits traders/investors to have a whole variety of patterns of behavior determined by their individual psychology, not necessarily describable in terms of utility maximization. Strategies may involve, for example, mimicking, satisficing, rules of thumb based on experience, etc. Strategies might be interactive—depending on the behavior of the others. Objectives might be of an evolutionary nature: survival (especially in crisis environments), domination in a market segment, fastest capital growth, etc. They might be relative—taking into account the performance of the others.

Models considered in this field – they are referred to as "EBF" (Evolutionary Behavioral Finance) models – combine elements of the theory of stochastic dynamic games and evolutionary game theory. The former offers the general notion of a strategy and the latter suggests the game solution concept: a *survival strategy*. In EBF frameworks, the process of market dynamics is described as a sequence of consecutive short-run equilibria determining equilibrium asset prices over each time period. The notion of a short-run price equilibrium is defined directly via the set of strategies of the market players specifying the patterns of their investment behavior (*behavioral equilibrium*).

The main focus of EBF is on investment strategies that survive in the market selection process, i.e., guarantee with probability one a positive, bounded away from zero share of market wealth over an infinite time horizon. Typical results show that such strategies exist, are asymptotically unique and easily computable. The computations do not require, in contrast with the classical DSGE, the knowledge of hidden agents' characteristics such as individual utilities and beliefs.

Fundamental contributions to the evolutionary modeling of financial markets were made in Anderson et al. (1988), Arthur et al. (1997), Blume and Easley (1992), Bottazzi et al. (2018, 2005), Bottazzi and Dindo (2013a,b), Brock et al. (2005), Coury and Sciubba (2012), Farmer (2002), Farmer and Lo (1999), Lo (2004, 2005, 2012, 2017), Lo et al. (2018), Sciubba (2005, 2006), and Zhang et al. (2014).

Financial DSGE models integrating evolutionary and behavioral approaches were proposed in Amir et al. (2011) and Amir et al. (2013). A survey describing the state of the art in EBF by 2016 and outlining a program for further research was given

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in Evstigneev et al. (2016). An elementary textbook treatment of the subject can be found in Evstigneev et al. (2015), Ch. 20. For a most recent review of the development of studies related to this area see Holtfort (2019). General perspectives of a synthesis of behavioral and mainstream economics based on the evolutionary approach are discussed in a recent paper by Aumann (2019).

EBF models invoke ideas related to behavioral economics and finance (Tversky and Kahneman, 1991; Shiller, 2003; Bachmann et al., 2018), evolutionary game theory (Weibull, 1995; Samuelson, 1997; Gintis, 2009; Kojima, 2006) and games of survival (Milnor and Shapley, 1957; Shubik and Thompson, 1959)[1]. Another important source for EBF is capital growth theory, or the theory of growth-optimal investments: Kelly (1956), Breiman (1961) and Algoet and Cover (1988), and others. For a textbook presentation of capital growth theory see Evstigneev et al. (2015), Ch. 17. The EBF models we deal with may be regarded as capital growth models with endogenous (formed in dynamic equilibrium), rather than exogenous (as in the classical theory) asset prices.

The present paper draws on the previous work of Amir et al. (2011), where a prototype of the model studied here was developed and some versions of the results we get in this paper were obtained. However, that study was conducted under very restrictive assumptions (equality of growth rates of the total volumes of all the assets and equality of investment rates of the market participants). Relaxing these assumptions required overcoming a number of conceptual and technical difficulties. Even the form of the main result on the existence of a survival strategy in the present, more general, setting differs substantially from that in Amir et al. (2011). Now this strategy is defined as a solution to a certain stochastic equation, in contrast with the previous, more specialized, model where it could be represented in an explicit form as the sum of a convergent series. For the proof of the existence and uniqueness of this solution we needed to develop new mathematical tools related to the ergodic theory of random dynamical systems: non-stationary stochastic Perron–Frobenius theorems (for stationary versions of these results see, e.g., Babaei et al. (2018)).

The structure of the paper is as follows. Section 2 describes the model. Section 3 states the main results. Section 4 discusses the EBF modeling approach, its characteristic features and applications. Section 5 contains some auxiliary propositions needed for the analysis of the model. Section 6 proves the main results. Appendix A includes routine proofs of a number of lemmas formulated in Section 6. Appendix B derives a non-stationary stochastic version of the Perron–Frobenius theorem used in this paper.

2. The model

We consider a market where $K \geq 2$ *assets* are traded. The market is influenced by random factors modeled in terms of an exogenous stochastic process $s_1, s_2, \ldots$, where s_t is a random element of a measurable space S_t ("state of the world" at date t). The market opens at date 0 and the assets are traded at all moments of time $t = 0, 1, 2, \ldots$. At each date $t = 1, 2, \ldots$ assets $k = 1, 2, \ldots, K$ pay dividends $D_{t,k}(s^t) \geq 0$ depending on the history $s^t := (s_1, \ldots, s_t)$ of states of the world up to date t. The functions $D_{t,k}(s^t)$ (as well as all other functions of s^t we will consider) are assumed to be measurable with respect to the product σ-algebra in the space $S_1 \times \cdots \times S_t$ and satisfy

$$\sum_{k=1}^{K} D_{t,k}(s^t) > 0 \text{ for all } t \geq 1 \text{ and } s^t. \tag{2.1}$$

This condition means that at each date in each random situation at least one asset yields a strictly positive dividend. The total volume (the number of units) of asset k available in the market at date $t \geq 0$ is $V_{t,k}(s^t) > 0$, where $V_{t,k}(s^t)$ is a measurable function of s^t. For $t = 0$, the number $V_{t,k} = V_{0,k} > 0$ is constant.

We denote by $p_t \in \mathbb{R}^K_+$ the vector of market prices of the assets. For each $k = 1, \ldots, K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, \ldots, p_{t,K})$ stands for the price of one unit of asset k at date $t \geq 0$. There are $N \geq 2$ *investors* (*traders*) acting in the market. A *portfolio* of investor i at date $t \geq 0$ is specified by a vector $x^i_t = (x^i_{t,1}, \ldots, x^i_{t,K}) \in \mathbb{R}^K_+$, where $x^i_{t,k}$ is the amount (the number of units) of asset k in the portfolio x^i_t. The scalar product $\langle p_t, x^i_t \rangle = \sum_{k=1}^K p_{t,k} x^i_{t,k}$ expresses the value of the investor i's portfolio x^i_t at date t in terms of the prices $p_{t,k}$. The *state of the market* at each date t is characterized by the set of vectors $(p_t, x^1_t, \ldots, x^N_t)$, where p_t is the vector of asset prices and $x^1_t, \ldots, x^N_t$ are the traders' portfolios.

At date $t = 0$ the investors have *initial endowments* $w^i_0 > 0$ ($i = 1, 2, \ldots, N$), that form their budgets at date 0. Investor i's budget at date $t \geq 1$ is

$$w^i_t(s^t) = \langle D_t(s^t) + p_t(s^t), x^i_{t-1}(s^{t-1}) \rangle,$$

where $D_t(s^t) = (D_{t,1}(s^t), \ldots, D_{t,K}(s^t))$. It consists of two components: the dividends $\langle D_t, x^i_{t-1} \rangle$ paid by the portfolio x^i_{t-1} and the market value $\langle p_t, x^i_{t-1} \rangle$ of x^i_{t-1} expressed in terms of the prices $p_t = (p_{t,1}, \ldots, p_{t,K})$ at date t.

For each $t \geq 0$, every trader $i = 1, 2, \ldots, N$ selects a vector of *investment proportions* $\lambda^i_t = (\lambda^i_{t,1}, \ldots, \lambda^i_{t,K})$ according to which i plans to distribute the available budget between assets. Vectors λ^i_t belong to the unit simplex

$$\Delta^K := \{(a_1, \ldots, a_K) \geq 0 : a_1 + \cdots + a_K = 1\}.$$

In terms of the game we are going to describe, the vectors λ^i_t represent the players' (investors') *actions* or *control variables*. The investment proportions at each date $t \geq 0$ are selected by the N traders simultaneously and independently, so that we deal here with a simultaneous-move N-person dynamic game. For $t \geq 1$, players' actions might depend, generally, on the history $s^t = (s_1, \ldots, s_t)$ of the realized states of the world and the *history of the game* $(p^{t-1}, x^{t-1}, \lambda^{t-1})$, where $p^{t-1} = (p_0, \ldots, p_{t-1})$ is the sequence of asset price vectors up to time $t-1$, and

$$x^{t-1} := (x_0, x_1, \ldots, x_{t-1}), \ x_l = (x^1_l, \ldots, x^N_l),$$

$$\lambda^{t-1} = (\lambda_0, \lambda_1, \ldots, \lambda_{t-1}), \ \lambda_l = (\lambda^1_l, \ldots, \lambda^N_l),$$

are the sets of vectors describing the portfolios and the investment proportions of all the players at all the dates up to $t-1$. The history of the game contains information about the *market history* – the sequence $(p_0, x_0), \ldots, (p_{t-1}, x_{t-1})$ of the states of the market – and about the actions λ^i_l of all the players (investors) $i = 1, \ldots, N$ at all the dates $l = 0, \ldots, t-1$. A vector $\Lambda^i_0 \in \Delta^K$ and a sequence of measurable functions with values in Δ^K

$$\Lambda^i_t(s^t, p^{t-1}, x^{t-1}, \ \lambda^{t-1}), \ t = 1, 2, \ldots$$

form an *investment (trading) strategy* Λ^i of trader i, specifying a *portfolio rule* according to which trader i selects investment proportions at each date $t \geq 0$. This is a general game-theoretic definition of a strategy, assuming full information about the history of the game, including the players' previous actions, and the knowledge of all the past and present states of the world.

Among general portfolio rules, we will distinguish those for which Λ^i_t depends only on s^t, rather than on the whole market history $(p^{t-1}, x^{t-1}, \lambda^{t-1})$. We will call such portfolio rules *basic*. They play an important role in the present work: the survival strategy we are going to construct will belong to this class.

[1] For a comprehensive discussion of game-theoretic aspects of EBF in a different but closely related model see Amir et al. (2013), Sections 1 and 6.

The essence of the main result (Theorem 2) lies in the fact that it indicates a relatively simple basic strategy, requiring a very limited volume of information and guaranteeing survival in competition with *any* other strategies which might use all theoretically possible information.

For each $k = 1, \ldots, K$, a sequence of functions $\alpha_{0,k}, \alpha_{1,k}(s^1), \alpha_{2,k}(s^2), \ldots$ is given characterizing transaction costs for buying asset k in the market under consideration. It is assumed that $0 < \alpha_{t,k} \le 1$. If an investor i allocates wealth $w^i_{t,k}$ to asset k at time t, then the value of the kth position of the i's portfolio will be $p_{t,k}x^i_{t,k} = \alpha_{t,k}w^i_{t,k}$. The amount $(1-\alpha_{t,k})w^i_{t,k}$ will cover transaction costs.

Suppose that at date 0 each investor i has selected some investment proportions $\lambda^i_0 = (\lambda^i_{0,1}, \ldots, \lambda^i_{0,K}) \in \Delta^K$. Then the amount allocated to asset k by trader i is $\lambda^i_{0,k}w^i_0$, where $w^i_0 > 0$ is the i's initial endowment, so that the value of the holding of asset k in the i's portfolio is $\alpha_{0,k}\lambda^i_{0,k}w^i_0$. Thus the value of the total holding of asset k in all the investors' portfolios amounts to $\alpha_{0,k}\sum_{i=1}^N \lambda^i_{0,k}w^i_0$. It is assumed that the market is always in equilibrium (asset supply is equal to asset demand), which makes it possible to determine the equilibrium price $p_{0,k}$ of each asset k from the equations

$$p_{0,k}V_{0,k} = \alpha_{0,k}\sum_{i=1}^N \lambda^i_{0,k}w^i_0, \ \ k = 1, 2, \ldots, K. \tag{2.2}$$

On the left-hand side of (2.2) we have the total value $p_{0,k}V_{0,k}$ of all the assets of the type k in the market (recall that the total amount of asset k at date 0 is $V_{0,k}$). The investment proportions $\lambda^i_0 = (\lambda^i_{0,1}, \ldots, \lambda^i_{0,K})$ chosen by the traders at date 0 determine their portfolios $x^i_0 = (x^i_{0,1}, \ldots, x^i_{0,K})$ at date 0 by the formula

$$x^i_{0,k} = \frac{\alpha_{0,k}\lambda^i_{0,k}w^i_0}{p_{0,k}}, \ \ k = 1, 2, \ldots, K, \ \ i = 1, \ldots, N. \tag{2.3}$$

Assume now that all the investors have chosen their investment proportion vectors $\lambda^i_t = (\lambda^i_{t,1}, \ldots, \lambda^i_{t,K})$ at date $t \ge 1$. Then the equilibrium of asset supply and demand determines the market clearing prices

$$p_{t,k}V_{t,k} = \alpha_{t,k}\sum_{i=1}^N \lambda^i_{t,k}\langle D_t + p_t, x^i_{t-1}\rangle, \ \ k = 1, \ldots, K. \tag{2.4}$$

The investment budgets $\langle D_t + p_t, x^i_{t-1}\rangle$ of the traders $i = 1, 2, \ldots, N$ are distributed between assets in the proportions $\lambda^i_{t,k}$, so that the kth position of the trader i's portfolio $x^i_t = (x^i_{t,1}, \ldots, x^i_{t,K})$ is

$$x^i_{t,k} = \frac{\alpha_{t,k}\lambda^i_{t,k}\langle D_t + p_t, x^i_{t-1}\rangle}{p_{t,k}}, \ \ k = 1, \ldots, K, \ \ i = 1, \ldots, N. \tag{2.5}$$

Note that the price vector p_t is determined implicitly as the solution to the system of Eqs. (2.4).

Define

$$\gamma_{t,k}(s^t) = V_{t,k}(s^t)/V_{t-1,k}(s^{t-1}).$$

The number $\gamma_{t,k}$ characterizes the speed of growth of the total volume $V_{t,k}$ of asset k. It can be shown (see Proposition 1 in Section 5) that a non-negative vector $p_t(s^t)$ satisfying Eqs. (2.4) exists and is unique (for any s^t and any feasible x^i_{t-1} and λ^i_t) as long as the following condition holds

$$\alpha_{t,k}(s^t) < \gamma_{t,k}(s^t) \text{ for all } t \ge 1 \text{ and all } s^t. \tag{2.6}$$

This condition is implied by the basic assumptions under which the results of this paper are obtained (see Section 4). Note that if there are no transaction costs, i.e. $\alpha_{t,k} = 1$, then (2.6) means that the total volumes of all the assets grow in time at a strictly positive rate. In another extreme case, when $\gamma_{t,k} = 1$, i.e. $V_{t,k}$ is constant in t, condition (2.6) requires that $\alpha_{t,k} < 1$, i.e. the transaction cost rate is non-zero. This property – termed in Mathematical Finance "efficient market friction" (see, e.g., Kabanov and Safarian (2009), p. 117) – plays an important role in various models with transaction costs, excluding phenomena like the Saint Petersburg paradox. In our context it is indispensable since in those cases when this assumption does not hold, a short-run equilibrium might fail to exist.

Given a strategy profile $(\Lambda^1, \ldots, \Lambda^N)$ of investors and their initial endowments $w^1_0, \ldots, w^N_0$, we can generate a path of the market game by setting

$$\lambda^i_0 = \Lambda^i_0, \ \ i = 1, \ldots, N, \tag{2.7}$$

$$\lambda^i_t = \Lambda^i_t(s^t, p^{t-1}, x^{t-1}, \ \lambda^{t-1}), \ \ t = 1, 2, \ldots, \ \ i = 1, \ldots, N, \tag{2.8}$$

and by defining p_t and x^i_t recursively according to Eqs. (2.2)–(2.5). The random dynamical system described defines step by step the vectors of investment proportions $\lambda^i_t(s^t)$, the equilibrium prices $p_t(s^t)$ and the investors' portfolios $x^i_t(s^t)$ as measurable vector functions of s^t for each moment of time $t \ge 0$. Thus we obtain a random path of the game

$$(p_t(s^t); x^1_t(s^t), \ldots, x^N_t(s^t); \lambda^1_t(s^t), \ldots, \lambda^N_t(s^t)), \ \ t \ge 0, \tag{2.9}$$

as a vector stochastic process in $\mathbb{R}^K_+ \times \mathbb{R}^{KN}_+ \times \mathbb{R}^{KN}_+$.

The above description of asset market dynamics requires clarification. Eqs. (2.3) and (2.5) make sense only if $p_{t,k} > 0$, or equivalently, if the aggregate demand for each asset (under the equilibrium prices) is strictly positive. Those strategy profiles which guarantee that the recursive procedure described above leads at each step to strictly positive equilibrium prices will be called *admissible*. In what follows, we will deal only with such strategy profiles. The hypothesis of admissibility guarantees that the random dynamical system under consideration is well-defined. Under this hypothesis, we obtain by induction that on the equilibrium path all the portfolios $x^i_t = (x^i_{t,1}, \ldots, x^i_{t,K})$ are non-zero and the wealth

$$w^i_t := \langle D_t + p_t, x^i_{t-1}\rangle \tag{2.10}$$

of each investor is strictly positive. Further, by summing up Eqs. (2.5) over $i = 1, \ldots, N$, we find that

$$\sum_{i=1}^N x^i_{t,k} = \frac{\alpha_{t,k}\sum_{i=1}^N \lambda^i_{t,k}\langle D_t + p_t, x^i_{t-1}\rangle}{p_{t,k}} = \frac{p_{t,k}V_{t,k}}{p_{t,k}} = V_{t,k} \tag{2.11}$$

(the market clears) for every asset k and each date $t \ge 1$. The analogous relations for $t = 0$ can be obtained by summing up Eqs. (2.3). Thus for every equilibrium state of the market $(p_t, x^1_t, \ldots, x^N_t)$, we have $p_t > 0$, $x^i_t \ne 0$ and (2.11).

We give a simple sufficient condition for a strategy profile to be admissible. This condition will hold for all the strategy profiles we shall deal with in the present paper, and in this sense it does not restrict generality. Suppose that some trader, say trader 1, uses a portfolio rule that always prescribes to invest into all the assets in strictly positive proportions $\lambda^1_{t,k}$. Then a strategy profile containing this portfolio rule is admissible. Indeed, for $t = 0$, we get from (2.2) that $p_{0,k} \ge \alpha_{0,k}V^{-1}_{0,k}\lambda^1_{0,k}w^1_0 > 0$ and from (2.3) that $x^1_0 = (x^1_{0,1}, \ldots, x^1_{0,K}) > 0$ (coordinatewise). Assuming that $x^1_{t-1} > 0$ and arguing by induction, we obtain

$$\langle D_t + p_t, x^1_{t-1}\rangle \ge \langle D_t, x^1_{t-1}\rangle > 0$$

in view of (2.1), which in turn yields $p_t > 0$ and $x^1_t > 0$ by virtue of (2.4) and (2.5), as long as $\lambda^1_{t,k} > 0$.

3. The main results

Let $(\Lambda^1, \ldots, \Lambda^N)$ be an admissible strategy profile of the investors. Consider the path (2.9) of the random dynamical system generated by this strategy profile and the given initial endowments w_0^i. We are primarily interested in the long-run behavior of the *relative wealth* or the *market shares* $r_t^i := w_t^i/W_t$ of the traders, where w_t^i is the investor i's wealth at date $t \geq 0$ and $W_t := \sum_{i=1}^N w_t^i$ is the *total market wealth*. We shall say that a portfolio rule Λ, or an investor i using it, *survives* with probability one if $\inf_{t\geq 0} r_t^i > 0$ almost surely (a.s.). This means that for almost all realizations of the process of states of the world $s_1, s_2, \ldots$, the market share of investor i using Λ is bounded away from zero by a strictly positive random variable.

Definition. Let us say that a portfolio rule Λ is a *survival strategy* if any investor using it survives with probability one irrespective of what portfolio rules are used by the other investors.

We will construct a strategy Λ^* which, as we shall prove, will be a survival strategy. Put

$$\rho_{t,k} := \frac{\alpha_{t,k}}{\gamma_{t,k}} = \frac{\alpha_{t,k}V_{t-1,k}}{V_{t,k}},\ \ t \geq 1,\ \ k = 1, \ldots, K.$$

Define the *relative dividends* of the assets $k = 1, \ldots, K$ by

$$R_{t,k} = R_{t,k}(s^t) := \frac{D_{t,k}(s^t)V_{t-1,k}(s^{t-1})}{\sum_{m=1}^K D_{t,m}(s^t)V_{t-1,m}(s^{t-1})},\quad k = 1, \ldots, K,\ \ t \geq 1, \tag{3.1}$$

and put $R_t(s^t) := (R_{t,1}(s^t), \ldots, R_{t,K}(s^t))$. The strategy $\Lambda^* = (\lambda_t^*(s^t))_{t\geq 0}$, where $\lambda_t^* = (\lambda_{t,1}^*, \ldots, \lambda_{t,K}^*)$, is defined as the basic strategy satisfying the equation

$$E_t[\rho_{t+1,k}\lambda_{t+1,k}^* + (1 - \sum_{m=1}^K \rho_{t+1,m}\lambda_{t+1,m}^*)R_{t+1,k}] = \lambda_{t,k}^* \text{ (a.s.)},\ \ k = 1, \ldots, K. \tag{3.2}$$

Here $E_t(\cdot) = E(\cdot|s^t)$ stands for the conditional expectation given s^t. We will provide conditions under which the strategy Λ^* exists and is unique up to stochastic equivalence, i.e. if $\Lambda = (\lambda_t(s^t))_{t\geq 0}$ is another solution to (3.2), then $\lambda_t^* = \lambda_t$ (a.s.) for all t.

Throughout the paper we will assume that the following conditions hold:

(A.1) There exist constants $\upsilon > 0$ and $l \geq 0$ such that for each t and k, we have

$$\max_{1\leq m\leq l} R_{t+m,k} \geq \upsilon. \tag{3.3}$$

(A.2) There exist strictly positive constants κ and α such that for all k, t

$$\alpha \leq \rho_{t,k} \leq 1 - \kappa. \tag{3.4}$$

Theorem 1. *Under assumptions (A.1) and (A.2), a solution $(\lambda_t^*)_{t\geq 0}$ to Eq. (3.2) exists and is unique up to stochastic equivalence. There exists a constant $\delta > 0$ such that $\lambda_{t,k}^* \geq \delta$.*

For a proof of Theorem 1 see Appendix B, Theorem B.2.

Let us discuss the meaning of Eq. (3.2). Suppose for the moment that the growth rates of all the assets are the same, so that

$$\rho_{t,1} = \rho_{t,2} = \cdots = \rho_{t,K} = \rho_t. \tag{3.5}$$

In this case, Eq. (3.2) takes on the following form

$$E_t[\rho_{t+1}\lambda_{t+1,k}^* + (1 - \rho_{t+1})R_{t+1,k}] = \lambda_{t,k}^* \text{ (a.s.)}, \tag{3.6}$$

and it admits an explicit solution. The kth coordinate $\lambda_{t,k}^*$ of the vector λ_t^* can be represented as the conditional expectation of the sum of the series

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} \rho_t^l R_{t+l,k}\ , \tag{3.7}$$

where

$$\rho_t^l := \begin{cases} 1 - \rho_{t+l}\,, & \text{if } l = 1, \\ \rho_{t+1}\rho_{t+2}\ldots\rho_{t+l-1}(1 - \rho_{t+l}), & \text{if } l > 1. \end{cases} \tag{3.8}$$

Note that in view of (3.4), the series of random variables

$$\sum_{l=1}^{\infty} \rho_t^l = (1 - \rho_{t+1}) + \rho_{t+1}(1 - \rho_{t+2}) + \rho_{t+1}\rho_{t+2}(1 - \rho_{t+3}) + \cdots$$

converges uniformly, and its sum is equal to one. Therefore the series of random vectors $\sum_{l=1}^{\infty} \rho_t^l R_{t+l,k}$ in (3.7) converges uniformly to a random vector belonging the unit simplex Δ^K, so that the right-hand side of (3.7) is well-defined. The proof of Eq. (3.7) will be given in Proposition 5.

Assume that $\rho_t = \rho$ is constant. Then formula (3.7) can be written as

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty}[(1 - \rho)\rho^{l-1}R_{t+l,k}]. \tag{3.9}$$

Further, if the random elements s_t are independent and identically distributed (i.i.d.) and the relative dividends $R_{t,k}(s^t) = R_k(s_t)$ depend only on the current state s_t and do not explicitly depend on t, then $E_tR_k(s_{t+l}) = ER_k(s_t)$ $(l \geq 1)$, and so

$$\lambda_{t,k}^* = ER_k(s_t), \tag{3.10}$$

which means that the strategy Λ^* is formed by the sequence of vectors $(ER_1(s_t), \ldots, ER_K(s_t))$ (constant and independent of t and s^t). Note that in this special case, the formula (3.10) for Λ^* does not involve the factor ρ.

Formulas (3.7), (3.9) and (3.10) reflect two general principles in Financial Economics:

(a) The strategy Λ^* prescribes the allocation of wealth among assets in the proportions of their fundamental values—the expectations of the future relative (discounted, weighted) dividends.

(b) The portfolio rule Λ^* defined in terms of the relative dividends provides an investment recommendation in line with the CAPM principles, emphasizing the role of the market portfolio (see, e.g., Evstigneev et al., 2015, Chapter 7).

In this connection it should be emphasized that instead of the traditional weighing assets according to their prices, the weights in the definition of Λ^* are based on fundamentals, so that Λ^* is an example of *fundamental indexing* (Arnott et al., 2008).

As we have already noted, EBF can be viewed as an extension of the classical capital growth theory (Kelly, 1956; Breiman, 1961; Algoet and Cover, 1988, and others) to the case of endogenous asset prices and returns. In the classical setting, a central role is played by the famous Kelly portfolio rule (Kelly, 1956) guaranteeing the fastest asymptotic growth rate of wealth in the long run. The Kelly rule is obtained by the maximization of the expected logarithm of the portfolio return. It can be shown (see the next section) that in the present model survival is equivalent to the fasted relative growth of wealth in the long run. Therefore Λ^* may be viewed as a counterpart of the Kelly portfolio rule in the present model. However, in the game-theoretic model at hand, where the performance of a strategy depends not only on the strategy itself but on the whole strategy profile, Λ^* cannot be obtained as a solution to a single-agent optimization problem with a logarithmic or any other objective functional.

It should be noted that in the case of different $\rho_{t,k}$, when condition (3.5) does not hold, we cannot provide an explicit formula, like (3.7), for the strategy Λ^*. However, we can suggest an algorithm for computing Λ^* converging at an exponential rate. This algorithm is actually contained in the proof of the existence and uniqueness of a solution to Eq. (3.2), see Appendix B, formulas (B.9) and (B.10).

The main results of the paper are formulated in Theorems 2 and 3.

Theorem 2. *The portfolio rule Λ^* is a survival strategy.*

As we have already noted, the portfolio rule Λ^* belongs to the class of basic portfolio rules: the investment proportions $\lambda_t^*(s^t)$ depend only on the history s^t of the process of states of the world and do not depend on the market history.

Note that the class of basic strategies is *sufficient* in the following sense. Any sequence of vectors $r_t = (r_t^1, \ldots, r_t^N)$ ($r_t = r_t(s^t)$) of market shares generated by some strategy profile $(\Lambda^1, \ldots, \Lambda^N)$ can be generated by a strategy profile $(\lambda_t^1(s^t), \ldots, \lambda_t^N(s^t))$ consisting of basic portfolio rules. The corresponding vector functions $\lambda_t^i(s^t)$ can be defined recursively by (2.7) and (2.8), using (2.2)–(2.5). Thus it is sufficient to prove Theorem 2 only for basic portfolio rules; this will imply that the portfolio rule (3.7) survives in competition with any, not necessarily basic, strategies.

The following result shows that the survival portfolio rule Λ^* is unique in the class of all basic strategies.

Theorem 3. *If there exists another basic survival strategy $\Lambda = (\lambda_t)$, then:*

$$\sum_{t=0}^{\infty} \|\lambda_t^* - \lambda_t\|^2 < \infty \text{ (a.s.)}.$$

It is not known whether this result remains valid for the class of general, not necessarily basic, strategies. This question remains open; it indicates an interesting direction for further research. Some examples pertaining to a different, but closely related, model might suggest a conjecture that the answer to this question is negative (see Amir et al., 2013, Section 5).

Proofs of Theorems 2 and 3 are given in the remainder of the paper.

4. Discussion

In this section we discuss the EBF approach, the model under consideration and the results obtained.

1. Marshallian temporary equilibrium. In the general methodological perspective, the modeling framework at hand relies upon the Marshallian (Marshall, 1949) principle of temporary equilibrium. The dynamics of the asset market in this framework are similar to the dynamics of the commodity market as outlined in the classical treatise by Marshall (1949) (Book V, Chapter II "Temporary Equilibrium of Demand and Supply"). The ideas of Marshall were developed in the framework of mathematical economics by Samuelson (1947). As it was noticed by Samuelson and discussed in detail by Schlicht (1985), in order to study the process of market dynamics by using the Marshallian "moving equilibrium method," one needs to distinguish between at least two sets of economic variables changing with different speeds. Then the set of variables changing slower (in our case, the set $x_t = (x_t^1, \ldots, x_t^N)$ of investors' portfolios) can be temporarily fixed, while the other (in our case, the asset prices p_t) can be assumed to rapidly reach the unique state of partial equilibrium. Samuelson (1947), pp. 321–323, writes about this approach:

> I, myself, find it convenient to visualize equilibrium processes of quite different speed, some very slow compared to others. Within each long run there is a shorter run, and within each shorter run there is a still shorter run, and so forth in an infinite regression. For analytic purposes it is often convenient to treat slow processes as data and concentrate upon the processes of interest. For example, in a short run study of the level of investment, income, and employment, it is often convenient to assume that the stock of capital is perfectly or sensibly fixed.

As it follows from the above citation, Samuelson thinks about a hierarchy of various equilibrium processes with different speeds. In our model, it is sufficient to deal with only two levels of such a hierarchy. We leave the price adjustment process leading to the solution of the partial equilibrium problem (2.4) beyond the scope of the model. It can be shown, however, that this equilibrium will be reached at an exponential rate in the course of a naturally defined *tâtonnement* procedure. This can be demonstrated by using the contraction property of the operator (5.1) involved in the equilibrium pricing equation (2.4). Our framework makes it possible to admit a whole spectrum of mechanisms leading to an equilibrium in the short run. In reality, various auction-type mechanisms are used for the purpose of equilibrating bids and offers, resulting in market clearing. An analysis of several types of such mechanisms and their implications for the structure of trading in financial markets has been performed by Bottazzi et al. (2005).

A rigorous mathematical treatment of the above multiscale approach, involving "rapid" and "slow" variables, is provided within continuous-time settings in the theory of *singular perturbations*, see e.g. Smith (1985) and Kevorkian and Cole (1996). In connection with economic modeling, questions of this kind are considered in detail in the monograph by Schlicht (1985). The equations on pp. 29–30 in Schlicht (1985) are direct continuous-time (deterministic) counterparts of our Eqs. (2.4) and (2.5).

The term "temporary equilibrium" was apparently coined for the first time by Marshall. However, in the last decades this term has been associated basically with a different, non-Marshallian notion, going back to Lindahl (1939) and Hicks (1946). This notion was developed in formal settings by Grandmont, Hildenbrand and others, see Grandmont (1988, 1977) and Grandmont and Hildenbrand (1974). The characteristic feature of the Lindahl–Hicks temporary equilibrium is the idea of *forecasts* or *beliefs* about the future states of the world, which the market participants possess and which are formalized in terms of stochastic kernels (transition functions) conditioning the distributions of future states of the world upon the agents' private information. A comprehensive discussion of this direction of research is provided by Magill and Quinzii (2003). In this work, we pursue a completely different approach. Our model might indirectly take into account agents' forecasts or beliefs, but they can be only implicitly reflected in the agents' investment strategies. We do not need to model in formal terms how the market players form, update and use these beliefs in their investment decisions.

For further comments on the comparison of the financial DSGE models based on the traditional Walrasian paradigm and those relying upon the EBF approach, see Amir et al. (2020), Section 7.

2. In order to survive you have to win! One might think that the focus on survival substantially restricts the scope of the analysis, since "one should care about survival only if things go wrong". It turns out, however, that the class of survival strategies in most of the EBF models coincides with the class of unbeatable strategies performing in the long run not worse in terms of wealth accumulation than any other strategies competing in the market. To demonstrate this let us reformulate the notion of a survival strategy in terms of the wealth processes w_t^i of the market players $i = 1, 2, \ldots, N$. Survival of a portfolio rule Λ^1 used by player 1 means that $w_t^1 \geq c \sum_{i=1}^N w_t^i$, where c is a strictly positive random variable. The last inequality holds if and only if

$$w_t^i \leq C w_t^1, \; i = 1, \ldots, N, \tag{4.1}$$

where C is some random variable. Property (4.1) expresses the fact that the wealth of any player i using any strategy Λ^i cannot grow asymptotically faster than the wealth of player 1 who uses the strategy Λ^1. If this is the case, the portfolio rule Λ^1 is called *unbeatable*. Thus survival strategies are those and only those that are unbeatable: in order to survive, you have to win!

For a general definition and discussion of the notion of an unbeatable strategy as a game solution concept see Amir et al. (2013), Section 6.

3. Evolutionary portfolio theory. One of the sources of motivation for EBF has always been related to quantitative applications of the results to portfolio selection problems. The data of EBF models needed for quantitative financial analysis are essentially the same as those needed for the applications of the theory of derivative securities pricing (e.g. the Black–Scholes formula) in Mathematical Finance/Financial Engineering. They do not need the knowledge, or the algorithms for revealing, hidden agents' characteristics such as their utilities and beliefs. The model and the results are described in operational terms and require only statistical estimates of objectively observable asset data.

A crucial role in the applications of EBF to portfolio selection is played by the discovery of investment factors that deliver returns in excess of the market. For example, Basu (1977) found the so-called value factor, according to which investing into equities with a high book-to-market ratio delivers higher returns than the market. Banz (1981) found that the same is true if one invests into equities with small market capitalization. Carhart (1997) found the momentum factor according to which investing in equities that have recently gone up delivers excess returns. Moreover even though by now hundreds of investment factors have been proposed, Harvey et al. (2016) have shown that only a few factors are needed to understand the dynamics of equity returns. The current state of these discoveries is summarized in the Fama and French (2015) five-factor model. According to these empirical results, the return of every portfolio selection strategy can be decomposed into its allocation to a few investment factors. Thus, it is natural to model the dynamics of equity markets by modeling the dynamic interaction of those investment factors. And this is what EBF is perfectly suited for. In the EBF framework, an investment factor defines a strategy determining the corresponding investment proportions. Note that investment factors are not based on individuals' utility functions and subjective probabilities! EBF can then be used to compute what impact the increase in relative wealth corresponding to one factor has on any other factor. In particular, the impact of a factor on itself gives a model-based measure of the capacity of the factor. This is very practical information since investors should avoid being stuck in crowded strategies. Also, when a certain investment factor gets fashionable this has cross impacts on other factors that one can compute based on the EBF model. For example, in recent years investing according to ESG (Environmental, Social, and Corporate Governance) criteria has become fashionable, and the EBF approach shows that this has a strong negative impact on the momentum factor. Finally, based on this approach one can compute the dynamics of the relative wealth, so that one can use the EBF model to determine which investment factors survive in the long run. A first paper systematically developing these ideas and opening up a new realm of fruitful applications of EBF to portfolio selection problems has recently been published in the Journal of Portfolio Management (Hens et al., 2020).

5. Auxiliary propositions

In this section we prove several auxiliary propositions needed for the analysis of the model at hand. The first proposition establishes the existence and uniqueness of an equilibrium price vector at each date $t \geq 0$.

Proposition 1. *Let assumption (2.6) hold. Let $x_{t-1} = (x_{t-1}^1, \ldots, x_{t-1}^N)$ be a set of vectors $x_{t-1}^i \in \mathbb{R}_+^K$ satisfying (2.11). Then for any s^t there exists a unique solution $p_t \in \mathbb{R}_+^K$ to Eqs. (2.4). This solution is measurable with respect to all the parameters involved in (2.4).*

Proof of Proposition 1. Fix some t and s^t and consider the operator transforming a vector $p = (p_1, \ldots, p_K) \in \mathbb{R}_+^K$ into the vector $q = (q_1, \ldots, q_K) \in \mathbb{R}_+^K$ with coordinates

$$q_k = \alpha_{t,k} V_{t,k}^{-1} \sum_{i=1}^N \lambda_{t,k}^i \langle D_t + p, x_{t-1}^i \rangle. \tag{5.1}$$

This operator is contracting in the norm $\|p\|_V := \sum_k |p_k| V_{t-1,k}$. Indeed, by virtue of (2.6) we have

$$\beta := \max_{k=1,\ldots,K} \{\alpha_{t,k} V_{t-1,k} V_{t,k}^{-1}\} < 1,$$

and so

$$\|q - q'\|_V = \sum_{k=1}^K |q_k - q_k'| V_{t-1,k} \leq$$
$$\sum_{k=1}^K \alpha_{t,k} V_{t-1,k} V_{t,k}^{-1} \sum_{i=1}^N \lambda_{t,k}^i |\langle p - p', x_{t-1}^i \rangle|$$
$$\leq \beta \sum_{i=1}^N \sum_{k=1}^K \lambda_{t,k}^i |\langle p - p', x_{t-1}^i \rangle| =$$
$$\beta \sum_{i=1}^N |\langle p - p', x_{t-1}^i \rangle| \leq \beta \sum_{i=1}^N \sum_{m=1}^K |p_m - p_m'| x_{t-1,m}^i =$$
$$\beta \sum_{m=1}^K \sum_{i=1}^N |p_m - p_m'| x_{t-1,m}^i = \beta \sum_{m=1}^K |p_m - p_m'| V_{t-1,m}$$
$$= \beta \|p - p'\|_V \;,$$

where the last but one equality follows from (2.11). By using the contraction principle, we obtain the existence, uniqueness and measurability of the solution to (2.4). □

In the next proposition, we derive a system of equations governing the dynamics of the market shares of the investors given their admissible strategy profile $(\Lambda^1, \ldots, \Lambda^N)$. Consider the path (2.9) of the random dynamical system generated by $(\Lambda^1, \ldots, \Lambda^N)$ and the sequence of vectors $r_t = (r_t^1, \ldots, r_t^N)$, where r_t^i is the investor i's market share at date t.

Proposition 2. *The following equations hold:*

$$w_{t+1}^i = \sum_{k=1}^K (\rho_{t+1,k} \langle \lambda_{t+1,k}, w_{t+1} \rangle + D_{t+1,k} V_{t,k}) \frac{\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle}, \tag{5.2}$$

$i = 1, \ldots, N, \; t \geq 0.$

Proof of Proposition 2. From (2.4) and (2.5) we get

$$p_{t,k} = \alpha_{t,k}V_{t,k}^{-1}\sum_{i=1}^{N}\lambda_{t,k}^{i}\langle p_t + D_t, x_{t-1}^{i}\rangle$$

$$= \alpha_{t,k}V_{t,k}^{-1}\sum_{i=1}^{N}\lambda_{t,k}^{i}w_t^{i} = \alpha_{t,k}V_{t,k}^{-1}\langle\lambda_{t,k}, w_t\rangle,$$

$$x_{t,k}^{i} = \frac{\alpha_{t,k}\lambda_{t,k}^{i}w_t^{i}}{p_{t,k}} = \frac{\alpha_{t,k}\lambda_{t,k}^{i}w_t^{i}}{\alpha_{t,k}V_{t,k}^{-1}\langle\lambda_{t,k}, w_t\rangle} = \frac{V_{t,k}\lambda_{t,k}^{i}w_t^{i}}{\langle\lambda_{t,k}, w_t\rangle},$$

where $t \geq 1$, $w_t := (w_t^1, \ldots, w_t^N)$ and $\lambda_{t,k} := (\lambda_{t,k}^1, \ldots, \lambda_{t,k}^N)$. Consequently, we have

$$w_{t+1}^{i} = \sum_{k=1}^{K}(p_{t+1,k} + D_{t+1,k})x_{t,k}^{i}$$

$$= \sum_{k=1}^{K}\left(\frac{\alpha_{t+1,k}\langle\lambda_{t+1,k}, w_{t+1}\rangle}{V_{t+1,k}} + D_{t+1,k}\right)\frac{V_{t,k}\lambda_{t,k}^{i}w_t^{i}}{\langle\lambda_{t,k}, w_t\rangle} =$$

$$\sum_{k=1}^{K}\left(\frac{\alpha_{t+1,k}\langle\lambda_{t+1,k}, w_{t+1}\rangle V_{t,k}}{V_{t+1,k}} + D_{t+1,k}V_{t,k}\right)\frac{\lambda_{t,k}^{i}w_t^{i}}{\langle\lambda_{t,k}, w_t\rangle} =$$

$$\sum_{k=1}^{K}(\langle\lambda_{t+1,k}, w_{t+1}\rangle\rho_{t+1,k} + D_{t+1,k}V_{t,k})\frac{\lambda_{t,k}^{i}w_t^{i}}{\langle\lambda_{t,k}, w_t\rangle},$$

where, we recall, $\rho_{t+1,k} = \alpha_{t+1,k}V_{t,k}/V_{t+1,k}$. □

Consider the model with two traders ($N = 2$) using strategies $\Lambda^i = (\lambda_{t,k}^i(s^t))$, $i = 1, 2$, and denote by x_t the ratio of their market shares:

$$x_t = \frac{r_t^1}{r_t^2} = \frac{w_t^1}{w_t^2}.$$

Recall that the *relative dividends* $R_{t,k}(s^t)$ of the assets $k = 1, \ldots, K$ are defined by (3.1), and $R_t(s^t)$ denotes the vector $(R_{t,1}(s^t), \ldots, R_{t,K}(s^t))$. Further, let us define for $i = 1, 2$,

$$U_{t+1}^{i} := 1 - \sum_{k=1}^{K}\rho_{t+1,k}\lambda_{t+1,k}^{i} = \sum_{k=1}^{K}(1 - \rho_{t+1,k})\lambda_{t+1,k}^{i}\,. \tag{5.3}$$

Proposition 3. *The sequence x_t is generated by the following random dynamical system*

$$x_{t+1} = x_t\frac{\sum_{k=1}^{K}[\rho_{t+1,k}\lambda_{t+1,k}^{2} + R_{t+1,k}U_{t+1}^{2}]\dfrac{\lambda_{t,k}^{1}}{\lambda_{t,k}^{1}x_t + \lambda_{t,k}^{2}}}{\sum_{k=1}^{K}[\rho_{t+1,k}\lambda_{t+1,k}^{1} + R_{t+1,k}U_{t+1}^{1}]\dfrac{\lambda_{t,k}^{2}}{\lambda_{t,k}^{1}x_t + \lambda_{t,k}^{2}}}$$

$$(t = 0, 1, \ldots). \tag{5.4}$$

Proof of Proposition 3. Let $i \in \{1, 2\}$ and $j \in \{1, 2\}$, $j \neq i$. By virtue of Proposition 2, we have (5.2). Then

$$W_{t+1} = w_{t+1}^{1} + w_{t+1}^{2} = \sum_{k=1}^{K}\rho_{t+1,k}\left\langle\lambda_{t+1,k}, w_{t+1}\right\rangle + \sum_{k=1}^{K}D_{t+1,k}V_{t,k}\,,$$

and so

$$\sum_{k=1}^{K}D_{t+1,k}V_{t,k} = W_{t+1} - \sum_{k=1}^{K}\rho_{t+1,k}\left\langle\lambda_{t+1,k}, w_{t+1}\right\rangle = \left\langle w_{t+1}, U_{t+1}\right\rangle. \tag{5.5}$$

Indeed,

$$\left\langle w_{t+1}, U_{t+1}\right\rangle = \sum_{l=1}^{2}w_{t+1}^{l}(1 - \sum_{k=1}^{K}\rho_{t+1,k}\lambda_{t+1,k}^{l})$$

$$= W_{t+1} - \sum_{k=1}^{K}\rho_{t+1,k}\sum_{l=1}^{2}w_{t+1}^{l}\lambda_{t+1,k}^{l}$$

$$= W_{t+1} - \sum_{k=1}^{K}\rho_{t+1,k}\langle w_{t+1}, \lambda_{t+1,k}\rangle.$$

By using the definition of the relative dividends $R_{t+1,k}$, we can rewrite formula (5.2) as follows:

$$w_{t+1}^{i} = \sum_{k=1}^{K}\left[\rho_{t+1,k}\left\langle\lambda_{t+1,k}, w_{t+1}\right\rangle + \left\langle w_{t+1}, U_{t+1}\right\rangle R_{t+1,k}\right]\frac{\lambda_{t,k}^{i}w_t^{i}}{\left\langle\lambda_{t,k}, w_t\right\rangle}.$$

Further, consider the expression in the brackets above:

$$\rho_{t+1,k}\left\langle\lambda_{t+1,k}, w_{t+1}\right\rangle + \left\langle w_{t+1}, U_{t+1}\right\rangle R_{t+1,k}$$
$$= w_{t+1}^{i}\left(\rho_{t+1,k}\lambda_{t+1,k}^{i} + R_{t+1,k}U_{t+1}^{i}\right) +$$
$$w_{t+1}^{j}\left(\rho_{t+1,k}\lambda_{t+1,k}^{j} + R_{t+1,k}U_{t+1}^{j}\right). \tag{5.6}$$

This, combined with (5.2), yields

$$w_{t+1}^{i} = w_{t+1}^{i}\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i} + R_{t+1,k}U_{t+1}^{i}\right]\frac{\lambda_{t,k}^{i}w_t^{i}}{\left\langle\lambda_{t,k}, w_t\right\rangle} +$$
$$w_{t+1}^{j}\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{j} + R_{t+1,k}U_{t+1}^{j}\right]\frac{\lambda_{t,k}^{i}w_t^{i}}{\left\langle\lambda_{t,k}, w_t\right\rangle},$$

and so

$$w_{t+1}^{i}\left(1 - \sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i} + R_{t+1,k}U_{t+1}^{i}\right]\frac{\lambda_{t,k}^{i}w_t^{i}}{\left\langle\lambda_{t,k}, w_t\right\rangle}\right) =$$
$$w_{t+1}^{j}\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{j} + R_{t+1,k}U_{t+1}^{j}\right]\frac{\lambda_{t,k}^{i}w_t^{i}}{\left\langle\lambda_{t,k}, w_t\right\rangle}. \tag{5.7}$$

Finally, note that

$$\frac{\lambda_{t,k}^{i}w_t^{i}}{\left\langle\lambda_{t,k}, w_t\right\rangle} = 1 - \frac{\lambda_{t,k}^{j}w_t^{j}}{\left\langle\lambda_{t,k}, w_t\right\rangle},$$

and consequently, the expression in the parentheses in Eq. (5.7) can be written as:

$$1 - \sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i} + R_{t+1,k}U_{t+1}^{i}\right]\left(1 - \frac{\lambda_{t,k}^{j}w_t^{j}}{\left\langle\lambda_{t,k}, w_t\right\rangle}\right) =$$
$$1 - \sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i} + R_{t+1,k}U_{t+1}^{i}\right]$$
$$+ \sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i} + R_{t+1,k}U_{t+1}^{i}\right]\frac{\lambda_{t,k}^{j}w_t^{j}}{\left\langle\lambda_{t,k}, w_t\right\rangle} =$$
$$\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i} + R_{t+1,k}U_{t+1}^{i}\right]\frac{\lambda_{t,k}^{j}w_t^{j}}{\left\langle\lambda_{t,k}, w_t\right\rangle},$$

where the last equality follows from (5.3). Thus we obtain from (5.7)

$$\frac{w_{t+1}^{i}}{w_{t+1}^{j}} = \frac{w_t^{i}}{w_t^{j}}\frac{\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{j} + R_{t+1,k}U_{t+1}^{j}\right]\dfrac{\lambda_{t,k}^{i}}{\left\langle\lambda_{t,k}, w_t\right\rangle}}{\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i} + R_{t+1,k}U_{t+1}^{i}\right]\dfrac{\lambda_{t,k}^{j}}{\left\langle\lambda_{t,k}, w_t\right\rangle}},$$

which completes the proof. □

The next proposition shows that it is sufficient to consider the case when $N = 2$, i.e., the general model can be reduced to the case of two investors.

Proposition 4. *In the model with two investors $i = 1, 2$ using the strategies Λ and $\tilde{\Lambda}$, respectively, the wealth w_t^1 of the first player coincides with the wealth w_t^1 of the first player in the original model, and the wealth $\tilde{w}_t^2$ of the second "aggregate" investor coincides with the total wealth $w_t^2 + \cdots + w_t^N$ of the group of $N-1$ investors $i = 2, \ldots, N$ in the original model.*

Proof of Proposition 4. Define

$$\tilde{w}_t^2 = w_t^2 + \cdots + w_t^N, \tag{5.8}$$

$$\tilde{\lambda}_{t,k}^2 = \frac{\lambda_{t,k}^2 w_t^2 + \cdots + \lambda_{t,k}^N w_t^N}{\tilde{w}_t^2} \quad (k = 1, 2, \ldots, K). \tag{5.9}$$

We have

$$\tilde{\lambda}_{t,k}^2 \geq 0, \ \sum_{k=1}^K \tilde{\lambda}_{t,k}^2 = \frac{w_t^2 + \cdots + w_t^N}{w_t^2 + \cdots + w_t^N} = 1,$$

which means that the vector $\tilde{\lambda}_t^2 := (\tilde{\lambda}_{t,1}^2, \ldots, \tilde{\lambda}_{t,K}^2)$ belongs to the unit simplex Δ^K. Let us regard $\tilde{\lambda}_t^2 = (\tilde{\lambda}_{t,1}^2, \ldots, \tilde{\lambda}_{t,K}^2)$ as the vector of investment proportions of an "aggregate investor", whose wealth is $\tilde{w}_t^2 = w_t^2 + \cdots + w_t^N$. The sequence of vectors $\tilde{\lambda}_t^2 = \tilde{\lambda}_t^2(s^t)$ defines a portfolio rule, which will be denoted by $\tilde{\Lambda}$. Note that

$$\tilde{\lambda}_{t,k}^2 \tilde{w}_t^2 = \lambda_{t,k}^2 w_t^2 + \cdots + \lambda_{t,k}^N w_t^N, \tag{5.10}$$

and so

$$\begin{aligned} \langle \lambda_{t,k}, w_t \rangle &= \lambda_{t,k}^1 w_t^1 + \tilde{\lambda}_{t,k}^2 \tilde{w}_t^2, \\ \langle \lambda_{t+1,k}, w_{t+1} \rangle &= \lambda_{t+1,k}^1 w_{t+1}^1 + \tilde{\lambda}_{t+1,k}^2 \tilde{w}_{t+1}^2. \end{aligned} \tag{5.11}$$

Recall that the dynamics of wealth of N investors is governed by the system of equations

$$w_{t+1}^i = \sum_{k=1}^K (\rho_{t+1,k} \langle \lambda_{t+1,k}, w_{t+1} \rangle + D_{t+1,k} V_{t,k}) \frac{\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle},$$
$$i = 1, 2, \ldots, N$$

(see (5.2)). By summing up these equations over $i = 2, 3, \ldots, N$ and using (5.8), (5.10) and (5.11), we get

$$w_{t+1}^1 = \sum_{k=1}^K [\rho_{t+1,k} (\lambda_{t+1,k}^1 w_{t+1}^i + \tilde{\lambda}_{t+1,k}^2 \tilde{w}_{t+1}^2) + D_{t+1,k} V_{t,k}] \times \frac{\lambda_{t,k}^1 w_t^1}{\lambda_{t,k}^1 w_t^1 + \tilde{\lambda}_{t,k}^2 \tilde{w}_t^2},$$

$$\tilde{w}_{t+1}^2 = \sum_{k=1}^K [\rho_{t+1,k} (\lambda_{t+1,k}^1 w_{t+1}^1 + \tilde{\lambda}_{t+1,k}^2 \tilde{w}_{t+1}^2) + D_{t+1,k} V_{t,k}] \times \frac{\tilde{\lambda}_{t,k}^2 \tilde{w}_t^2}{\lambda_{t,k}^1 w_t^1 + \tilde{\lambda}_{t,k}^2 \tilde{w}_t^2},$$

which completes the proof. □

Proposition 5. *Under assumption (3.5), the portfolio rule $\Lambda^* = (\lambda_{t,k}^*)$ can be computed by formula (3.7).*

Proof of Proposition 5. Suppose (3.7) holds. Let us verify (3.6). We have

$$E_t(\rho_{t+1} \lambda_{t+1,k}^*) = E_t(\rho_{t+1} E_{t+1} \sum_{l=1}^\infty \rho_{t+1}^l R_{t+1+l,k}) =$$
$$E_t(E_{t+1} \sum_{l=1}^\infty \rho_{t+1} \rho_{t+1}^l R_{t+1+l,k}) = E_t(\sum_{l=1}^\infty \rho_{t+1} \rho_{t+1}^l R_{t+1+l,k}),$$

and so

$$E_t[\rho_{t+1} \lambda_{t+1,k}^* + (1 - \rho_{t+1}) R_{t+1,k}]$$
$$= E_t[\sum_{l=1}^\infty \rho_{t+1} \rho_{t+1}^l R_{t+1+l,k} + (1 - \rho_{t+1}) R_{t+1,k}] =$$
$$E_t(\sum_{l=1}^\infty \rho_t^{l+1} R_{t+l+1,k} + \rho_t^1 R_{t+1,k}) = E_t \sum_{l=1}^\infty \rho_t^l R_{t+l,k} = \lambda_{t,k}^*$$

because $1 - \rho_{t+1} = \rho_t^1$ and

$$\rho_t^{l+1} = \rho_{t+1} \rho_{t+2 \ldots} \rho_{t+l} (1 - \rho_{t+l+1}) = \rho_{t+1} \rho_{t+1}^l$$

for $l \geq 1$. □

6. Proofs of the main results

In this section, proofs of Theorems 2 and 3 are given. The plan of the proofs is as follows. Proposition 4 shows that we can consider, without loss of generality, the case of two investors. This reduces the dimension of the original random dynamical system from a general N to $N = 2$. Proposition 3 describes a one-dimensional system which governs the evolution of the ratio $x_t = r_t^1 / r_t^2$ of the market shares of the two investors, and thus reduces the dimension of the problem to 1. Our goal is to show that the random sequence (x_t) defined recursively by (5.4) is bounded away from zero almost surely. To this end it turns out to be convenient to take a "step back" and to increase the dimension to K (the number of assets). Assuming that the first trader uses the investment proportions $\lambda_{t,k}^1 = \lambda_{t,k}^*(s^t)$ prescribed by the portfolio rule Λ^* and the second trader employs investment proportions $\lambda_{t,k}^2 = \lambda_{t,k}(s^t)$ specified by some other portfolio rule Λ, we introduce the following change of variables

$$y_t^k = \lambda_{t,k} / x_t, \ k = 1, \ldots, K, \tag{6.1}$$

and define $y_t := (y_t^1, \ldots, y_t^K)$. We examine the dynamics of the random vectors $y_t = y_t(s^t)$ implied by the system (5.4). The norm $|y_t| := \sum_k |y_t^k|$ of the vector $y_t \geq 0$ is equal to $\sum_k (\lambda_{t,k} / x_t) = 1/x_t$, and what we need is to show that $1/|y_t|$ is bounded away from zero (a.s.). To prove this, we construct a stochastic Lyapunov function—a function of y_t which forms a non-negative supermartingale (ζ_t) along a path (y_t) of the system at hand (see Lemma 3). By using the supermartingale convergence theorem, we prove that the stochastic process ζ_t converges (a.s.), which implies that it is bounded (a.s.). We complete the proof of Theorem 2 by showing that the boundedness of ζ_t implies that $x_t = 1/|y_t|$ is bounded away from zero. By using the above techniques, together with some additional considerations, we complete this section with a proof of Theorem 3.

We begin the realization of the plan outlined with two lemmas containing inequalities involving the variables y_t^k defined by (6.1). Define the non-negative random variables

$$Y_t := \ln(1 + |y_t|) = -\ln r_t^1, \tag{6.2}$$

$$Z_{t,k} := \ln(1 + \frac{y_t^k}{\lambda_{t,k}^*}) = \ln\left(1 + \frac{r_t^2 \lambda_{t,k}}{r_t^1 \lambda_{t,k}^*}\right), \tag{6.3}$$

$$\gamma_{k,m}^{t+1} = \frac{1 + y_{t+1}^m / \lambda_{t+1,m}^*}{1 + y_t^k / \lambda_{t,k}^*}. \tag{6.4}$$

In particular, we have $\gamma_{k,k}^{t+1} = (1 + y_{t+1}^k / \lambda_{t+1,k}^*) / (1 + y_t^k / \lambda_{t,k}^*)$.

Later in the proofs, the following two equalities will be employed:

$$\ln \gamma_{k,m}^{t+1} = Z_{t+1,m} - Z_{t,k} \tag{6.5}$$

and

$$\frac{\lambda_{t+1,m}\lambda^*_{t,k}|y_{t+1}| - \lambda^*_{t+1,m}y^k_t}{\lambda^*_{t,k}+y^k_t} = \lambda^*_{t+1,m}\left(\gamma^{t+1}_{k,m}-1\right). \tag{6.6}$$

The algebraic identity (6.6) can be checked quite easily, and we leave its proof to the reader. In particular, if $m = k$ then (6.6) takes on the following form:

$$\frac{\lambda_{t+1,k}\lambda^*_{t,k}|y_{t+1}| - \lambda^*_{t+1,k}y^k_t}{\lambda^*_{t,k}+y^k_t} = \lambda^*_{t+1,k}\left(\gamma^{t+1}_{k,k}-1\right).$$

Proofs of Theorems 2 and 3 are based on Lemmas 1–4 which we formulate below and prove in Appendix A.

Let us define a function

$$f(x) = \frac{(x-1)\ln x}{x+2}, \tag{6.7}$$

which will be helpful in estimating some logarithmic expressions.

Lemma 1. *The function $f(x)$ is non-negative, has a unique root $x = 1$ and satisfies*

$$x - 1 \geq \ln x + f(x),\ \ x \in (-\infty, +\infty). \tag{6.8}$$

Lemma 2. *The following inequality holds:*

$$\sum_{k=1}^{K}\lambda^*_{t+1,k}Z_{t+1,k} + \sum_{m=1}^{K}\sum_{k=1}^{K}R_{t+1,k}(1-\rho_{t+1,m})\lambda^*_{t+1,m}f\left(\gamma^{t+1}_{k,m}\right) \leq \sum_{k=1}^{K}\rho_{t+1,k}\lambda^*_{t+1,k}Z_{t,k} + U^*_{t+1}\sum_{k=1}^{K}R_{t+1,k}Z_{t,k}\,. \tag{6.9}$$

Put

$$\zeta_t := \sum_{k=1}^{K}\lambda^*_{t,k}Z_{t,k} + \sum_{m=1}^{K}\sum_{k=1}^{K}R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m}f\left(\gamma^{t}_{k,m}\right). \tag{6.10}$$

Lemma 3. *The sequence of random variables ζ_t ($t \geq 1$) is a non-negative supermartingale, and we have*

$$\zeta_t - E_t\zeta_{t+1} \geq \sum_{m=1}^{K}\sum_{k=1}^{K}R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m}f\left(\gamma^{t}_{k,m}\right) \geq 0 \text{ (a.s.)}. \tag{6.11}$$

Lemma 4. *Let ζ_t be a supermartingale such that $\inf_t E\zeta_t > -\infty$. Then the series of non-negative random variables $\sum_{t=0}^{\infty}(\zeta_t - E_t\zeta_{t+1})$ converges (a.s.).*

In what follows, in the proofs of Theorems 2 and 3 as well as Lemmas 1–4, we will sometimes omit "a.s." where it does not lead to ambiguity.

Proof of Theorem 2. By Lemma 4, the sequence ζ_t defined in (6.10) is a non-negative supermartingale. Therefore it converges (a.s.), and hence it is bounded above (a.s.) by some random constant C:

$$C \geq \zeta_t = \sum_{k=1}^{K}\lambda^*_{t,k}Z_{t,k} + \sum_{m=1}^{K}\sum_{k=1}^{K}R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m}f\left(\gamma^{t}_{k,m}\right) \geq \sum_{k=1}^{K}\lambda^*_{t,k}Z_{t,k} = \sum_{k=1}^{K}\lambda^*_{t,k}\ln\left(1+\frac{r^2_t\lambda_{t,k}}{r^1_t\lambda^*_{t,k}}\right).$$

Here, we used the non-negativity of the function f established in Lemma 1 and the non-negativity of $R_{t,k}$, $\lambda^*_{t,m}$ and assumption (A.2).

Recall that by virtue of Theorem 1, $\lambda^*_{t,k} \geq \delta$ for any t, k. Therefore $C/\delta \geq \ln\left(1 + r^2_t\lambda_{t,k}/r^1_t\lambda^*_{t,k}\right)$ for all t, k, and there exists some random variable H such that $H \geq 1 + r^2_t\lambda_{t,k}/r^1_t\lambda^*_{t,k}$ for all t, k. Furthermore, there exists some k such that $\lambda_{t,k} \geq 1/K$ (since $\sum_{k=1}^{K}\lambda_{t,k} = 1$). For this k the following inequality holds:

$$H \geq 1 + \frac{r^2_t\lambda_{t,k}}{r^1_t\lambda^*_{t,k}} \geq 1 + \frac{r^2_t}{r^1_t\lambda^*_{t,k}K} \geq 1 + \frac{r^2_t}{r^1_tK} = 1 + \frac{(1-r^1_t)}{r^1_tK},$$

which implies $r^1_t \geq (K(H-1)+1)^{-1} = \tau$. □

Proof of Theorem 3. The proof of this theorem consists in several steps. We outline these steps here and provide details of the arguments in Appendix A.

1st step. We first show that

$$\sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{k=1}^{K}R_{t,k}(\gamma^t_{k,m}-1)^2 < \infty. \tag{6.12}$$

2nd step. From (6.12) we deduce that

$$\sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{k=1}^{K}R_{t,k}\left(y^m_t/\lambda^*_{t,m} - y^k_{t-1}/\lambda^*_{t-1,k}\right)^2 < \infty. \tag{6.13}$$

3rd step. At this step, by using (6.13), we obtain:

$$\sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{k=1}^{K}\left(y^m_t/\lambda^*_{t,m} - y^k_t/\lambda^*_{t,k}\right)^2 < \infty.$$

This series can be estimated as:

$$\infty > \sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{k=1}^{K}\left(\frac{y^m_t}{\lambda^*_{t,m}} - \frac{y^k_t}{\lambda^*_{t,k}}\right)^2 = \sum_{t=0}^{\infty}\left(\frac{r^2_t}{r^1_t}\right)^2\sum_{m=1}^{K}\sum_{k=1}^{K}\left(\frac{\lambda_{t,m}}{\lambda^*_{t,m}} - \frac{\lambda_{t,k}}{\lambda^*_{t,k}}\right)^2 \geq \sum_{t=0}^{\infty}\left(\frac{r^2_t}{r^1_t}\right)^2\sum_{k=1}^{K}\left(\frac{\lambda_{t,m}}{\lambda^*_{t,m}} - \frac{\lambda_{t,k}}{\lambda^*_{t,k}}\right)^2 \tag{6.14}$$

for each m. This fact will be used at the next step.

4th step. Next we prove the following estimate for the sum involved in (6.14):

$$\sum_{k=1}^{K}\left(\lambda_{t,m}/\lambda^*_{t,m} - \lambda_{t,k}/\lambda^*_{t,k}\right)^2 \geq \left(\lambda_{t,m}/\lambda^*_{t,m} - 1\right)^2. \tag{6.15}$$

Finally, by using (6.14) and inequality (6.15), we conclude

$$\infty > \sum_{t=0}^{\infty}\left(\frac{r^2_t}{r^1_t}\right)^2\sum_{m=1}^{K}\sum_{k=1}^{K}\left(\frac{\lambda_{t,m}}{\lambda^*_{t,m}} - \frac{\lambda_{t,k}}{\lambda^*_{t,k}}\right)^2 \geq \sum_{t=0}^{\infty}\left(\frac{r^2_t}{r^1_t}\right)^2\sum_{m=1}^{K}\left(\frac{\lambda_{t,m}}{\lambda^*_{t,m}} - 1\right)^2 = \sum_{t=0}^{\infty}\left(\frac{r^2_t}{r^1_t}\right)^2\sum_{m=1}^{K}\left(\frac{\lambda_{t,m}-\lambda^*_{t,m}}{\lambda^*_{t,m}}\right)^2 \geq \sum_{t=0}^{\infty}\left(\frac{r^2_t}{r^1_t}\right)^2\sum_{m=1}^{K}\left(\lambda_{t,m}-\lambda^*_{t,m}\right)^2 = \sum_{t=0}^{\infty}\left(\frac{r^2_t}{r^1_t}\right)^2\left|\left|\lambda_t - \lambda^*_t\right|\right|^2 \geq \sum_{t=0}^{\infty}\phi^2\left|\left|\lambda_t - \lambda^*_t\right|\right|^2,$$

where $\phi > 0$ is a random variable such that $r^2_t/r^1_t \geq \phi$, which exists because $\Lambda = (\lambda_t)$ is a survival strategy. □

Appendix A

Proof of Lemma 1. We first observe that for $0 < x \le 1$ we have $2(x-1)(x+1)^{-1} \ge \ln x$. Hence

$$(x-1) \ge \frac{(x+1)\ln x}{2} \ge \ln x + \frac{(x-1)\ln x}{x+2} = \ln x + f(x). \tag{A.1}$$

On the other hand, for $x \ge 1$ we have $(x-1)(x+1)/(2x) \ge \ln x$. Therefore

$$x - 1 \ge \frac{2x}{x+1}\ln x \ge \ln x + \frac{(x-1)\ln x}{x+2} = \ln x + f(x). \tag{A.2}$$

By combining (A.1) and (A.2) we obtain (6.8). Clearly $f(x)$ is non-negative and if $x \ne 1$, then $f(x) = (x-1)\ln x/(x+2) \ne 0$, and so if $x \ne 1$, then $f(x) > 0$. □

Proof of Lemma 2. From formula (5.4) and (5.3) with $\lambda^1_{t,k} = \lambda^*_{t,k}$ and $\lambda^2_{t,k} = \lambda_{t,k}$, we obtain

$$x_{t+1} = x_t \frac{\sum_{k=1}^K [\rho_{t+1,k}\lambda_{t+1,k} + R_{t+1,k}U_{t+1}]\dfrac{\lambda^*_{t,k}}{\lambda^*_{t,k}x_t + \lambda_{t,k}}}{\sum_{k=1}^K [\rho_{t+1,k}\lambda^*_{t+1,k} + R_{t+1,k}U^*_{t+1}]\dfrac{\lambda_{t,k}}{\lambda^*_{t,k}x_t + \lambda_{t,k}}}.$$

Consequently,

$$\sum_{k=1}^K [\rho_{t+1,k}\lambda^*_{t+1,k} + R_{t+1,k}U^*_{t+1}]\frac{\lambda_{t,k}}{\lambda^*_{t,k}x_t + \lambda_{t,k}}$$
$$= \sum_{k=1}^K [\rho_{t+1,k}\frac{\lambda_{t+1,k}}{x_{t+1}} + \frac{R_{t+1,k}}{x_{t+1}}U_{t+1}]\frac{\lambda^*_{t,k}x_t}{\lambda^*_{t,k}x_t + \lambda_{t,k}}.$$

By using the notation $y^k_t = \lambda_{t,k}/x_t$ and the fact that $|y_t| = 1/x_t$, we rewrite the above formula as

$$\sum_{k=1}^K [\rho_{t+1,k}\lambda^*_{t+1,k} + R_{t+1,k}U^*_{t+1}]\frac{y^k_t}{\lambda^*_{t,k} + y^k_t}$$
$$= \sum_{k=1}^K [\rho_{t+1,k}y^k_{t+1} + R_{t+1,k}\,|y_{t+1}|\,U_{t+1}]\frac{\lambda^*_{t,k}}{\lambda^*_{t,k} + y^k_t},$$

which yields

$$\sum_{k=1}^K \rho_{t+1,k}\frac{\lambda^*_{t,k}y^k_{t+1} - \lambda^*_{t+1,k}y^k_t}{\lambda^*_{t,k} + y^k_t} + \sum_{k=1}^K R_{t+1,k}\frac{U_{t+1}\lambda^*_{t,k}|y_{t+1}| - U^*_{t+1}y^k_t}{\lambda^*_{t,k} + y^k_t} = 0. \tag{A.3}$$

Recalling the definition of U_{t+1} (5.3), we notice that

$$\frac{U_{t+1}\lambda^*_{t,k}|y_{t+1}| - U^*_{t+1}y^k_t}{\lambda^*_{t,k} + y^k_t} = \sum_{m=1}^K (1 - \rho_{t+1,m})\lambda^*_{t+1,m}\left(\gamma^{t+1}_{k,m} - 1\right),$$

where $\gamma^{t+1}_{k,m}$ comes from (6.4). Then using (6.4) and (6.6), we write (A.3) as

$$\sum_{k=1}^K \rho_{t+1,k}\lambda^*_{t+1,k}\left(\gamma^{t+1}_{k,k} - 1\right) + \sum_{m=1}^K\sum_{k=1}^K R_{t+1,k}(1-\rho_{t+1,m})\lambda^*_{t+1,m}\left(\gamma^{t+1}_{k,m} - 1\right) = 0. \tag{A.4}$$

The first sum in (A.4) can be estimated by using the well-known inequality $\gamma^{t+1}_{k,k} - 1 \ge \ln\gamma^{t+1}_{k,k}$. To estimate the second sum let us employ Lemma 1: $\gamma^{t+1}_{k,m} - 1 \ge \ln\gamma^{t+1}_{k,m} + f(\gamma^{t+1}_{k,m})$. Then we have

$$\sum_{k=1}^K \rho_{t+1,k}\lambda^*_{t+1,k}\ln\gamma^{t+1}_{k,k} + \sum_{m=1}^K\sum_{k=1}^K R_{t+1,k} \times (1-\rho_{t+1,m})\lambda^*_{t+1,m}(\ln\gamma^{t+1}_{k,m} + f(\gamma^{t+1}_{k,m})) \le 0.$$

Recall that $\ln\gamma^{t+1}_{k,m} = Z_{t+1,m} - Z_{t,k}$, and so

$$\sum_{k=1}^K \rho_{t+1,k}\lambda^*_{t+1,k}(Z_{t+1,k} - Z_{t,k}) + \sum_{m=1}^K\sum_{k=1}^K R_{t+1,k}(1-\rho_{t+1,m})\lambda^*_{t+1,m}(Z_{t+1,m} - Z_{t,k} + f(\gamma^{t+1}_{k,m})) \le 0.$$

This implies

$$\sum_{k=1}^K \lambda^*_{t+1,k}Z_{t+1,k} + \sum_{m=1}^K\sum_{k=1}^K R_{t+1,k}(1-\rho_{t+1,m})\lambda^*_{t+1,m}f\left(\gamma^{t+1}_{k,m}\right) \le \sum_{k=1}^K \rho_{t+1,k}\lambda^*_{t+1,k}Z_{t,k} + U^*_{t+1}\sum_{k=1}^K R_{t+1,k}Z_{t,k}.$$

This inequality obtained is nothing but the one in (6.9), which completes the proof. □

Proof of Lemma 3. By virtue of Lemma 1, the function f is non-negative, and so we have

$$\sum_{m=1}^K\sum_{k=1}^K R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m}f\left(\gamma^t_{k,m}\right) \ge 0. \tag{A.5}$$

This implies that $\zeta_t \ge 0$. By taking the conditional expectation $E_t(\cdot)$ of both sides of (6.9), we obtain the following chain of relations:

$$E_t\zeta_{t+1} = E_t\left[\sum_{k=1}^K \lambda^*_{t+1,k}Z_{t+1,k} + \sum_{m=1}^K\sum_{k=1}^K R_{t+1,k} \times (1-\rho_{t+1,m})\lambda^*_{t+1,m}f\left(\gamma^{t+1}_{k,m}\right)\right] \le$$
$$E_t\sum_{k=1}^K [\rho_{t+1,k}\lambda^*_{t+1,k}Z_{t,k} + U^*_{t+1}R_{t+1,k}Z_{t,k}] =$$
$$\sum_{k=1}^K Z_{t,k}E_t\left[\rho_{t+1,k}\lambda^*_{t+1,k} + U^*_{t+1}R_{t+1,k}\right] = \sum_{k=1}^K Z_{t,k}\lambda^*_{t,k} \tag{A.6}$$

where the last equality follows from the definition of $\lambda^*_{t,k}$. By using (A.5) and (A.6), we find

$$E_t\zeta_{t+1} \le E_t\zeta_{t+1} + \sum_{m=1}^K\sum_{k=1}^K R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m}f\left(\gamma^t_{k,m}\right)$$
$$\le \sum_{k=1}^K Z_{t,k}\lambda^*_{t,k} + \sum_{m=1}^K\sum_{k=1}^K R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m}f\left(\gamma^t_{k,m}\right) = \zeta_t. \tag{A.7}$$

To complete the proof that ζ_t is a supermartingale it is sufficient to prove that the random variable

$$\zeta_1 = \sum_{k=1}^K \lambda^*_{1,k}Z_{1,k} + \sum_{m=1}^K\sum_{k=1}^K R_{1,k}(1-\rho_{1,m})\lambda^*_{1,m}f\left(\gamma^1_{k,m}\right)$$

is bounded. To this end we notice that

$\gamma^1_{k,m} = (1+y^m_1/\lambda^*_{1,m})/(1+y^k_0/\lambda^*_{0,k}) \le 1+y^m_1/\lambda^*_{1,m} \le 1+r^2_1/r^1_1\delta$ (a.s.)

because $\lambda^*_{1,k} \ge \delta$ (see Theorem 1). Now it remains only to show that r^1_1 is bounded away from zero by a strictly positive constant.

From Eq. (5.4) we get

$$\frac{r^1_1}{r^2_1} = \frac{r^1_0}{r^2_0}\frac{\sum_{k=1}^K[\rho_{1,k}\lambda_{1,k}+R_{1,k}U_1]\dfrac{\lambda^*_{0,k}}{\lambda^*_{0,k}x_0+\lambda_{0,k}}}{\sum_{k=1}^K[\rho_{1,k}\lambda^*_{1,k}+R_{1,k}U^*_1]\dfrac{\lambda_{0,k}}{\lambda^*_{0,k}x_0+\lambda_{0,k}}} =: \frac{r^1_0}{r^2_0}\frac{A}{B}. \tag{A.8}$$

Since r^2_0 is a strictly positive constant, it is sufficient to show that the nominator of the above fraction, which we denote by A, is bounded away from zero and the denominator, denoted by B, is bounded above. We have

$$A \ge U_1\sum_{k=1}^K R_{1,k}\frac{\lambda^*_{0,k}}{\lambda^*_{0,k}x_0+\lambda_{0,k}} \ge \kappa\sum_{k=1}^K R_{1,k}\frac{\lambda^*_{0,k}}{\lambda^*_{0,k}x_0+\lambda_{0,k}}$$

$\ge \kappa\lambda^*_{0,k}/(K(\lambda^*_{0,k}x_0+\lambda_{0,k})) \ge \kappa\delta/(K(\delta x_0+1)) =: \bar{A}$.

The second inequality holds because $U_1 \ge \kappa$ by the definition of U_i (see (5.3)) and assumption (A.2). The third inequality is valid since there exists k such that $R_{1,k} \ge 1/K$ (because $\sum_{k=1}^K R_{1,k} = 1$) and the whole sum is not less than one summand. It remains only to observe that B is bounded above:

$$B \le \sum_{k=1}^K[\rho_{1,k}\lambda^*_{1,k}+R_{1,k}U^*_1] = 1.$$

Finally, $r^1_1 = r^2_1Ax_0/B = (1-r^1_1)Ax_0/B \ge (1-r^1_1)\bar{A}x_0$, which yields $r^1_1 \ge \bar{A}x_0/(\bar{A}x_0+1)$. Thus, r^1_1 is bounded away from zero, therefore $Z_{1,k}$ and $\gamma^1_{k,m}$ are bounded above, which implies the boundedness of $f(\gamma^1_{k,m})$ and hence the boundedness of ζ_1. The proof is complete. □

Proof of Lemma 4. The random variables $\eta_t := \zeta_t - E_t\zeta_{t+1}$ are non-negative by the definition of a supermartingale. Further, we have

$$\sum_{t=0}^{T-1}E\eta_t = \sum_{t=0}^{T-1}(E\zeta_t - E\zeta_{t+1}) = E\zeta_0 - E\zeta_T,$$

and so the sequence $\sum_{t=0}^{T-1}E\,\eta_t$ is bounded because $\sup_T(-E\zeta_T) = -\inf_T E\,\zeta_T < +\infty$. Therefore the series of the expectations $\sum_{t=0}^{\infty}E\ \eta_t$ of the non-negative random variables η_t converges, which implies $\sum_{t=0}^{\infty}\eta_t < \infty$ a.s. because $E\sum_{t=0}^{\infty}\eta_t = \sum_{t=0}^{\infty}E\ \eta_t$ (the last equality holds for any sequence $\eta_t \ge 0$). The proof is complete. □

The remainder of the Appendix provides details of the proof of Theorem 3 (steps 1 to 4).

1st step. Since investor 1 uses the strategy Λ^*, by virtue of Lemma 3 the sequence ζ_t defined by (6.10) is a non-negative supermartingale. By using inequality (6.11) and Lemma 4 we obtain

$$\sum_{t=0}^{\infty}\sum_{m=1}^K\sum_{k=1}^K R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m}f(\gamma^t_{k,m}) < \infty \text{ (a.s.)}.$$

By assumption (A.2), we have $(1-\rho_{t,m}) \ge \varkappa > 0$, and since $\lambda^*_{t,m} \ge \delta$, the above inequality implies

$$\sum_{t=0}^{\infty}\sum_{m=1}^K\sum_{k=1}^K R_{t,k}f(\gamma^t_{k,m}) < \infty \text{ (a.s.)}. \tag{A.9}$$

Let us show that if (A.9) converges, then the following series converges as well:

$$\sum_{t=0}^{\infty}\sum_{m=1}^K\sum_{k=1}^K R_{t,k}(\gamma^t_{k,m}-1)^2 < \infty \text{ (a.s.)}. \tag{A.10}$$

To this end it is sufficient to verify that for some random variable $\theta > 0$, we have

$$G_t := \frac{\sum_{m=1}^K\sum_{k=1}^K R_{t,k}f(\gamma^t_{k,m})}{\sum_{m=1}^K\sum_{k=1}^K R_{t,k}(\gamma^t_{k,m}-1)^2} \ge \theta > 0.$$

To prove this we observe that

$$\min_{m,k}\frac{\ln\gamma^t_{k,m}}{(\gamma^t_{k,m}+2)(\gamma^t_{k,m}-1)} = \min_{m,k}\frac{(\gamma^t_{k,m}-1)\ln\gamma^t_{k,m}}{(\gamma^t_{k,m}-1)^2(\gamma^t_{k,m}+2)} \le G_t.$$

Note that the function $\ln x(x-1)^{-1}(x+2)^{-1}$ is non-increasing and hence it achieves its minimum on $(0, M]$ at M. Furthermore,

$$\begin{aligned}\gamma^t_{k,m} &= \frac{1+y^m_{t+1}/\lambda^*_{t+1,m}}{1+y^k_t/\lambda^*_{t,k}} \le 1+\frac{y^m_{t+1}}{\lambda^*_{t+1,m}}\\ &= 1+\frac{r^2_{t+1}\lambda_{t+1,m}}{r^1_{t+1}\lambda^*_{t+1,m}} \le 1+\frac{1-\tau}{\tau\delta} =: M,\end{aligned} \tag{A.11}$$

where the last inequality holds by virtue of Theorem 2 and because $\lambda^*_{t,m} \ge \delta$. Since $\gamma^t_{k,m} \le M$ for each t, k, m, we get

$$G_t \ge \min_{t,k,m}\frac{\ln\gamma^t_{k,m}}{(\gamma^t_{k,m}-1)(\gamma^t_{k,m}+2)} \ge \frac{\ln M}{(M-1)(M+2)}.$$

Thus we have proved that the series (A.10) converges.

2nd step. Using (6.6) we can see that the following inequality holds:

$$\begin{aligned}(\gamma^t_{k,m}-1)^2 &= \left(\frac{y^m_t/\lambda^*_{t,m}-y^k_{t-1}/\lambda^*_{t-1,k}}{1+y^k_{t-1}/\lambda^*_{t-1,k}}\right)^2\\ &\ge \frac{1}{M^2}\left(\frac{y^m_t}{\lambda^*_{t,m}}-\frac{y^k_{t-1}}{\lambda^*_{t-1,k}}\right)^2\end{aligned}$$

because $1+y^k_{t-1}/\lambda^*_{t-1,k} \le M$ in view of (A.11). Hence, the series

$$\sum_{t=0}^{\infty}\sum_{m=1}^K\sum_{k=1}^K R_{t,k}\left(y^m_t/\lambda^*_{t,m}-y^k_{t-1}/\lambda^*_{t-1,k}\right)^2 < \infty \tag{A.12}$$

converges.

3rd step. Let us denote $a^m_t = y^m_t/\lambda^*_{t,m}$ and $b^t_{m,k} = a^m_t - a^k_{t-1}$. In this new notation,

$$\sum_{t=0}^{\infty}\sum_{m=1}^K\sum_{k=1}^K R_{t,k}(b^t_{m,k})^2 < \infty$$

for any k, m. Now recall that $\sum_{k=1}^K R_{t,k} = 1$ and hence for all t there exists at least one k such that $R_{t,k} \ge 1/K$. Denote this k by k^*_t. Clearly, we have

$$\infty > \sum_{t=0}^{\infty}\sum_{k=1}^K R_{t,k}\sum_{m=1}^K(b^t_{m,k})^2 \ge \sum_{t=0}^{\infty}\frac{1}{K}\sum_{m=1}^K(b^t_{m,k^*_t})^2.$$

Fix m and m'. Then it is easy to see that

$$\sum_{t=0}^{\infty}(b^t_{m,k^*_t})^2 < \infty \text{ and } \sum_{t=0}^{\infty}(b^t_{m',k^*_t})^2 < \infty. \tag{A.13}$$

Observe that the following equalities hold

$$b^t_{m,k^*_t} - b^t_{m',k^*_t} = a^m_t - a^{k^*_t}_{t-1} - a^{m'}_t + a^{k^*_t}_{t-1} = a^m_t - a^{m'}_t,$$

and so

$$\sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{m'=1}^{K}\left(a_t^m - a_t^{m'}\right)^2 = \sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{m'=1}^{K}\left(b^t_{m,k_t^*} - b^t_{m',k_t^*}\right)^2.$$

Furthermore

$$\left(b^t_{m,k_t^*} - b^t_{m',k_t^*}\right)^2 = \left|(b^t_{m,k_t^*})^2 + (b^t_{m',k_t^*})^2 - 2b^t_{m,k_t^*}b^t_{m',k}\right| \le$$
$$(b^t_{m,k_t^*})^2 + (b^t_{m',k_t^*})^2 + 2\left|b^t_{m,k_t^*}b^t_{m',k_t^*}\right| \le 2\left((b^t_{m,k_t^*})^2 + (b^t_{m',k_t^*})^2\right)$$

since $2\left|b^t_{m,k_t^*}b^t_{m',k_t^*}\right| \le (b^t_{m,k_t^*})^2 + (b^t_{m',k_t^*})^2$. Therefore, since both series $\sum_{t=0}^{\infty}(b^t_{m,k_t^*})^2$ and $\sum_{t=0}^{\infty}(b^t_{m',k_t^*})^2$ converge (see (A.13)), then for all pairs m, m', we have

$$\sum_{t=0}^{\infty}\left(b^t_{m,k_t^*} - b^t_{m',k_t^*}\right)^2 < \infty,$$

and consequently,

$$\sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{m'=1}^{K}\left(b^t_{m,k_t^*} - b^t_{m',k_t^*}\right)^2$$
$$= \sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{k=1}^{K}\left(y_t^m/\lambda^*_{t,m} - y_t^k/\lambda^*_{t,k}\right)^2 < \infty. \tag{A.14}$$

4th step. Consider two cases: (i) $\lambda_{t,m}/\lambda^*_{t,m} \ge 1$ and (ii) $\lambda_{t,m}/\lambda^*_{t,m} \le 1$. In the first case, among the $K-1$ fractions $\lambda_{t,k}/\lambda^*_{t,k}$ $(k \ne m)$ we can find at least one with $\lambda_{t,m'}/\lambda^*_{t,m'} \le 1$. Otherwise, $\lambda_{t,m'}/\lambda^*_{t,m'} > 1$ for all $m' \ne m$, i.e., $\lambda_{t,m'} > \lambda^*_{t,m'}$ $(m' \ne m)$ and $\lambda_{t,m} \ge \lambda^*_{t,m}$. Then we get $1 = \sum_{k=1}^{K}\lambda_{t,k} > \sum_{k=1}^{K}\lambda^*_{t,k} = 1$, which is a contradiction. By the same argument, we can show in the second case that if $\lambda_{t,m}/\lambda^*_{t,m} \le 1$, then there exists m' satisfying $\lambda_{t,m'}/\lambda^*_{t,m'} \ge 1$.

Thus, we have proved that for each m there exists m' such that either $\lambda_{t,m}/\lambda^*_{t,m} \ge 1 \ge \lambda_{t,m'}/\lambda^*_{t,m'}$ or $\lambda_{t,m'}/\lambda^*_{t,m'} \ge 1 \ge \lambda_{t,m}/\lambda^*_{t,m}$, Consequently,

$$\left|\lambda_{t,m'}/\lambda^*_{t,m'} - \lambda_{t,m}/\lambda^*_{t,m}\right| \ge \left|\lambda_{t,m}/\lambda^*_{t,m} - 1\right|,$$

which implies

$$\sum_{k=1}^{K}\left(\lambda_{t,m}/\lambda^*_{t,m} - \lambda_{t,k}/\lambda^*_{t,k}\right)^2 \ge \left(\frac{\lambda_{t,m}}{\lambda^*_{t,m}} - \frac{\lambda_{t,m'}}{\lambda^*_{t,m'}}\right)^2 \ge \left(\frac{\lambda_{t,m}}{\lambda^*_{t,m}} - 1\right)^2.$$

Appendix B

The purpose of this Appendix is to prove Theorem B.2, which implies the existence and uniqueness of the Λ^* strategy playing a central role in this work (see the definition in Section 3). We will deduce Theorem B.2 from Theorem B.1, which represents a non-stationary version of the stochastic Perron–Frobenius theorem, see Babaei et al. (2018) and references therein. In turn, Theorem B.1 will be obtained as a consequence of a chain of auxiliary results formulated in Lemmas B.1–B.2 and Propositions B.1–B.3 below.

Denote by $\mathcal{M}^n$ $(n > 1)$ the set of $n\times n$ matrices $B \ge 0$ such that $Bx \ne 0$ for all $x \in Q := \{x : 0 \ne x \ge 0\}$. For $x = (x^1, \dots, x^n) \in R^n$, define $|x| = |x^1| + \dots + |x^n|$, $x^0 = x/|x|$, and, for $B \in \mathcal{M}^n$, put

$$\kappa(B) = \max_{x,y\in Q}|(Bx)^0 - (By)^0|.$$

Let $\phi(B)$ denote the ratio of the smallest and the greatest elements of the matrix B.

Lemma B.1. *Let $B_1, B_2, \dots, B_k \in \mathcal{M}^n$. If $B_i > 0$ and $n > 1$, then*

$$\kappa(B_k \dots B_1) \le \rho_1^{-1}\delta_1\dots\delta_{k-1}, \tag{B.1}$$

where

$$\rho_i = n^{-2}\phi(B_i)\phi(B_{i+1}),\ \ \delta_i = (1 - 2\rho_i).$$

For a proof of this result see Evstigneev (1974), Lemma 1.

Put $\Delta = \{x = (x_1, \dots, x_n) : x_j \ge 0, \sum x_j = 1\}$. Let $\mathcal{D}^n$ denote the set of matrices B in $\mathcal{M}^n$ representing linear transformations of R^n that map Δ into itself. For $\delta > 0$ we will denote by $\mathcal{D}^n_\delta$ the set of matrices $B \in \mathcal{D}^n$ whose elements are not less than δ.

Lemma B.2. *Let $B_1, B_2, \dots, B_k \in \mathcal{D}^n_\delta$. Then*

$$\kappa(B_k \dots B_1) \le M\rho^{k-1}, \tag{B.2}$$

where $M = n^2\delta^{-2}$ and $\rho = 1 - n^{-2}\delta^2$.

Proof. This is immediate from (B.1) because

$$1 \ge \phi(B_i) \ge \delta,\ \ \rho_i = n^{-2}\phi(B_i)\phi(B_{i+1}) \ge n^{-2}\delta^2,$$
$$\rho_1^{-1} \le n^2\delta^{-2} = M,$$

$$\delta_i = 1 - 2\rho_i \le 1 - n^{-2}\delta^2 = \rho.\quad \square$$

Let $B_1, B_2, \dots$ be a sequence of matrices in $\mathcal{D}^n$.

Proposition B.1. *There exists a sequence $(y_t^*)_{\ge 0}$ such that $y_t^* \in \Delta$ and*

$$y_t^* = B_{t+1}y_{t+1}^*,\ \ t \ge 0. \tag{B.3}$$

Proof. Put $\Delta^\infty = \Delta \times \Delta \times \cdots$ and $\mathcal{Y} = R^n \times R^n \times \cdots$. Let us introduce in $\mathcal{Y}$ the product topology: $(y_t^m)_{t\ge 0} \to (y_t)_{t\ge 0}$ if and only if $y_t^m \to y_t$ for all t. Then $\mathcal{Y}$ is a topological locally convex vector space and Δ^∞ is a compact convex set in $\mathcal{Y}$. Consider the mapping $\mathfrak{B} : \mathcal{Y} \to \mathcal{Y}$ transforming $(y_t)_{t\ge 0}$ into $(B_{t+1}y_{t+1})_{t\ge 0}$. This mapping is continuous and transforms Δ^∞ into itself. Consequently, by the Schauder–Tychonoff theorem (e.g. Zeidler, 1986) it has a fixed point $y^* = \mathfrak{B}y^*$, which proves the proposition. $\square$

For each $t \ge 1$ and $j \ge 0$ denote $B_t^{t+j} = B_t \dots B_{t+j}$. For any $y = (y_t) \in \Delta^\infty$ denote by $\mathfrak{B}_t^m(y)$ the tth term of the sequence $\mathfrak{B}^m(y) \in \Delta^\infty$, where $\mathfrak{B}^m(y)$ is the mth iterate of the mapping $\mathfrak{B}$. Clearly, if we put $y_t^m = \mathfrak{B}_t^m(y)$ $(t \ge 0)$, then

$$y_t^1 = B_{t+1}y_{t+1} = B_{t+1}^{t+1}y_{t+1},$$
$$y_t^2 = B_{t+1}y_{t+1}^1 = B_{t+1}B_{t+2}y_{t+2} = B_{t+1}^{t+2}y_{t+2}, \dots,$$

$$y_t^m = B_{t+1}B_{t+2}\dots B_{t+m}y_{t+m} = B_{t+1}^{t+m}y_{t+m},\ \ t \ge 0.$$

Proposition B.2. *Suppose there exist an integer $l \ge 0$ and a real number $\delta > 0$ such that for any $t \ge 1$ the matrix B_t^{t+l} belongs to $\mathcal{D}^n_\delta$. Then the solution $y^* = (y_t^*)_{t\ge 0}$ to Eq. (B.3) is unique, and for every $t \ge 0$, the sequence $y_t^m = \mathfrak{B}_t^m(y)$ converges to y_t^* uniformly in $y \in \Delta^\infty$.*

Proof. Uniqueness follows from convergence. To prove the uniform convergence of y_t^m we estimate the distance between y_t^m and y_t^* by using (B.2). Define

$$H_j = B^{t+jl+j}_{t+(j-1)l+j},\ j \ge 1.$$

For $m \ge l+1$ denote by $k = k(m)$ the greatest natural number such that $kl + k \le m$ and put

$$C_t^{t+m} = \begin{cases} B^{t+m}_{t+kl+k+1}, & kl+k < m \\ Id, & kl+k = m \end{cases}.$$

Then we have

$$B_{t+1}^{t+m} = B_{t+1}^{t+l+1}B_{t+l+2}^{t+2l+2}B_{t+2l+3}^{t+3l+3}\dots B^{t+kl+k}_{t+(k-1)l+k}B^{t+m}_{t+kl+k+1} = H_1\dots H_k C_t^{t+m}$$

and

$$y_t^* = B_{t+1}^{t+m} y_{t+m}^* = H_1 ... H_k C_t^{t+m} y_{t+m}^*,$$
$$y_t^m = B_{t+1}^{t+m} y_{t+m} = H_1 ... H_k C_t^{t+m} y_{t+m}.$$

Thus, in view of (B.2),

$$|y_t^* - y_t^m| \le M\rho^{k-1}$$

because $H_j \in D_\delta^n$. Therefore $y_t^m = \mathfrak{B}_t^m(y) \to y_t^*$ uniformly in y, since $k = k(m) \to \infty$ as $m \to \infty$. □

Suppose that the matrices $B_t = B_t(\omega) \in \mathcal{D}^n$ are random, i.e., $B_t(\omega)$ for each $t = 1, 2, \ldots$ is a measurable matrix function on the probability space $(\Omega, \mathcal{F}, P)$. Assume the following condition holds:

($\mathcal{B}$) For some $l \ge 0$ and $\delta > 0$, the matrix $B_t^{t+l}(\omega)$ belongs to $\mathcal{D}_\delta^n$ a.s. for all $t \ge 1$.

Proposition B.3. *Under assumption ($\mathcal{B}$), there exists a sequence $(y_t^*)_{t\ge 0}$ of measurable vector functions $y_t^*(\omega)$ with values in Δ such that*

$$B_{t+1} y_{t+1}^* = y_t^*, \ t \ge 0 \ (a.s.). \tag{B.4}$$

The solution $(y_t^*)_{t\ge 0}$ to Eq. (B.4) is unique, and we have $y_t^*(\omega) \ge \delta e$ (a.s.). There exists a set $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) = 1$ such that for every $t \ge 0$ and $\omega \in \Omega_1$ the sequence $y_t^m(\omega) = \mathfrak{B}_t^m(y)(\omega)$ converges to $y_t^*(\omega)$ uniformly in $y \in \Delta^\infty$.

The uniqueness is understood in terms of stochastic equivalence: if $(y_t^{**}(\omega))_{t\ge 0}$ is another such sequence, then $y_t^{**}(\omega) = y_t^*(\omega)$ (a.s.) for all t.

Proof of Proposition B.3. By assumption, there exists a set $\Omega_1 \in \mathcal{F}$ of full measure such that for $\omega \in \Omega_1$ the matrix $B_t^{t+l}(\omega) \in \mathcal{D}_\delta^n$ for all $t \ge 1$. Take any $\omega \in \Omega_1$ and apply Proposition B.2. We obtain that there exists a sequence of vector functions $y_t^*(\omega)$ with values in Δ satisfying (B.3) for $\omega \in \Omega_1$. Fix some $d \in \Delta$ and define $y_t^*(\omega)$ as d for $\omega \in \Omega \backslash \Omega_1$. Then (B.4) will hold almost surely. Observe that the functions $y_t^*(\omega)$ are measurable because according to Proposition B.2 (applied to $y := (d, d, \ldots) \in \Delta^\infty$), we have

$$y_t^*(\omega) = \lim_{m\to\infty} B_{t+1}^{t+m}(\omega) d \text{ for } \omega \in \Omega_1.$$

To prove uniqueness suppose there is another sequence $\hat{y} = (\hat{y}_t(\omega))_{t\ge 0}$ satisfying (B.4) almost surely. Then by virtue of Proposition B.2, $\mathfrak{B}_t^m(\hat{y})(\omega) \to y_t^*(\omega)$ for $\omega \in \Omega_1$. On the other hand, $\mathfrak{B}_t^m(\hat{y})(\omega) = \hat{y}_t(\omega)$ (a.s.), and so $y_t^*(\omega) = \hat{y}_t(\omega)$ (a.s.).

Finally, $y_t^*(\omega) \ge \delta e$ (a.s.) because $y_t^*(\omega) = B_{t+1}^{t+l+1}(\omega) y_{t+l+1}^*(\omega)$, where $y_{t+l+1}^*(\omega) \in \Delta$ and $B_{t+1}^{t+l+1}(\omega) \in \mathcal{D}_\delta^n$ (a.s.). □

Let $A_1(\omega), A_2(\omega), \ldots$ be a sequence of random matrices. Consider the following condition:

($\mathcal{A}$) For each $t \ge 1$, the matrix $A_t(\omega)$ depends $\mathcal{F}_t$-measurably on ω, and there exist $l \ge 0$ and $\delta > 0$, such that the matrix $A_t^{t+l}(\omega) := A_t(\omega) \ldots A_{t+l}(\omega)$ belongs to $\mathcal{D}_\delta^n$ a.s. for all $t \ge 1$.

Theorem B.1. *Under assumption ($\mathcal{A}$), there exists a sequence $(x_t^*(\omega))_{t\ge 0}$ of vector functions with values in Δ such that $x_t^*(\omega)$ is $\mathcal{F}_t$-measurable and*

$$E_t A_{t+1} x_{t+1}^* = x_t^* \ (a.s.), \ t \ge 0. \tag{B.5}$$

This sequence is unique up to stochastic equivalence, and we have

$$x_t^* \ge \delta e \ (a.s.). \tag{B.6}$$

Proof. Fix any (non-random) matrix $B^1 \in \mathcal{D}_\delta^n$ and define

$$B_1 = B^1, \ B_t := A_{t-1}, \ t \ge 2. \tag{B.7}$$

By applying Proposition B.3 to the sequence of matrices $(B_t)_{t\ge 1}$ defined by (B.7), we obtain that there exists a sequence $y_t^*(\omega)$, $t \ge 1$, of measurable vector functions with values in Δ such that

$$A_t y_{t+1}^* = y_t^*, \ t \ge 1. \tag{B.8}$$

Define

$$x_t^* = E_t y_{t+1}^*, \ t \ge 0.$$

From (B.8) we get

$$E_t A_{t+1} y_{t+2}^* = E_t y_{t+1}^*, \ t \ge 0.$$

Therefore

$$E_t A_{t+1} x_{t+1}^* = E_t A_{t+1} E_{t+1} y_{t+2}^* = E_t E_{t+1} A_{t+1} y_{t+2}^* = E_t y_{t+1}^* = x_t^*,$$

and so the sequence $(x_t^*)_{t\ge 0}$ satisfies (B.5).

Suppose there is another sequence $(\hat{x}_t)_{t\ge 0}$ satisfying $E_t A_{t+1} \hat{x}_{t+1} = \hat{x}_t$ (a.s.) for all $t \ge 0$. Then we have

$$\hat{x}_t = E_t A_{t+1} \hat{x}_{t+1} = E_t A_{t+1} E_{t+1} A_{t+2} \hat{x}_{t+2}$$
$$= E_t E_{t+1} A_{t+1} A_{t+2} \hat{x}_{t+2} = E_t A_{t+1} A_{t+2} \hat{x}_{t+2} \text{ (a.s.)}.$$

Continuing this process, we get

$$\hat{x}_t = E_t A_{t+1} \ldots A_{t+m} \hat{x}_{t+m} = E_t A_{t+1}^{t+m} \hat{x}_{t+m} \text{ (a.s.)}. \tag{B.9}$$

By using Proposition B.3, we obtain $A_{t+1}^{t+m} \hat{x}_{t+m} = B_{t+2}^{t+1+m} \hat{x}_{t+m} \to y_{t+1}^*$ (a.s.), consequently,

$$\hat{x}_t = E_t A_{t+1}^{t+m} \hat{x}_{t+m} \to E_t y_{t+1}^* = x_t^* \text{ (a.s.)}, \tag{B.10}$$

and so $\hat{x}_t = x_t^*$ (a.s.). □

We conclude this Appendix by formulating and proving Theorem B.2—the result on the existence and uniqueness of the Λ^* strategy in the model studied in the present paper. Let $(\rho_t)_{t\ge 1}$ be a sequence of $\mathcal{F}_t$-measurable random vectors $\rho_t = (\rho_{t,1}, \ldots, \rho_{t,n})$ such that $0 \le \rho_{t,i} \le 1$, and $(R_t)_{t\ge 1}$ a sequence of $\mathcal{F}_t$-measurable random vectors $R_t = (R_{t,1}, \ldots, R_{t,n})$ satisfying

$$R_t \ge 0, \ \sum_{i=1}^n R_{t,i} = 1.$$

Recall that Λ^* was defined as the solution to Eq. (3.2). To prove that this solution exists and is unique let us define for each $t \ge 0$ the linear operator A_{t+1}:

$$(A_{t+1}x)_i = \rho_{t+1,i} x_i + (\sum_{m=1}^n x_m - \sum_{m=1}^n \rho_{t+1,m} x_m) R_{t+1,i}$$
$$= \rho_{t+1,i} x_i + \sum_{m=1}^n (1 - \rho_{t+1,m}) x_m R_{t+1,i}.$$

This operator transforms Δ into itself, and for $x \in \Delta$ we have

$$(A_{t+1}x)_i = \rho_{t+1,i} x_{t+1,i} + (1 - \sum_{m=1}^n \rho_{t+1,m} x_{t+1,m}) R_{t+1,i}.$$

Consequently, Eq. (3.2) can be written in the form (B.5) (with obvious changes in notation).

Let us introduce the following condition.

($\mathcal{R}$) There exist constants $\gamma > 0$ and $l \ge 0$ such that for each t and i, we have

$$\max_{1\le m\le l} R_{t+m,i} \ge \gamma. \tag{B.11}$$

Theorem B.2. *Suppose that condition* $(\mathcal{R})$ *holds and there exists a constant* $\theta > 0$ *such that* $\min\{\rho_{t,i},\ 1-\rho_{t+1,i}\} \geq \theta$ *for all* t *and* i. *Then a solution* $(x_t^*)_{t\geq 0}$*to Eq.* (B.5) *exists, is unique up to stochastic equivalence, and satisfies* (B.6) *for some* $\delta > 0$.

Proof. Take any $x \in \Delta$ and define recursively $x_{t+1} = A_{t+1}x$, and $x_{t+m+1} = A_{t+m+1}x_{t+m}$. Then we have

$$(A_{t+1}x)_i = \rho_{t+1,i}x_i + \sum_{m=1}^{n}(1-\rho_{t+1,m})x_m R_{t+1,i} \geq \theta(x_i + R_{t+1,i}),$$

which yields

$$\begin{aligned} x_{t+l} &\geq \theta^l x_{t,i} + \theta^l R_{t+1,i} + \theta^{l-1}R_{t+2,i} + \cdots + \theta R_{t+l,i} \\ &\geq \theta^l \max_{1\leq m\leq l} R_{t+m,i} \geq \theta^l \gamma. \end{aligned}$$

Thus condition $(\mathcal{A})$ holds with $\delta := \theta^l\gamma$, and so Theorem B.2 follows from Theorem B.1. □

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