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Abstract

Recently Beltran-Royo et.al presented a Semi-Lagrangean relaxation for the classical p-median location problem. The results obtained using the Semi-Lagrangean relaxation approach were quite impressive. In this paper we use a reformulation of the p-median problem in order to start from a formulation more suitable for Semi-Lagrangean relaxation and analyse the new approach on examples from the OR library.

Keywords: p-median Location, Lagrangean Relaxation, Mathematical Programing

1. Introduction

The p-median problem is a well studied integer programming problem. Over the years the problem has been approached by many authors using various mathematical programming methods. In a recent study of Beltran-Royo et al. use a Semi-Lagrangean approach to the p-median problem. Another recent study has benn conducted by Garcia et al., in which a reformulation is used which they name the radius formulation. In this paper we will make use of a reformulation of the p-median problem more suitable for Semi-Lagrangean relaxation. Apart from that we also show that the optimal Semi-Lagrangean dual variable have a very interesting economic interpretation.

The paper is organized as follows. In section 2 The standard formulation of the p-median problem is given followed by the reformulation that is to be used in the rest of the article. Section 3 gives a short description of the Semi-Lagrangean relaxation in general and its properties and describe the Semi-Lagrangean relaxation subproblem for the reformulated p-median model. In section 4 we illustrate the procedure on an example taken from the literature. Section 5 presents the computational results obtained on larger problem instances. Finally in section 6 we give conclusions that can be made from our investigation. We also comment on the similarities with our approach and the approach used by Garcia et. al.

2. The formulation of the p-median problem and a reformulation suitable for Semi-Lagrangean relaxation

The standard formulation of the p-median problem is as follows

$$Min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (1)$$

Subject to

$$\sum_{i=1}^{m} x_{ij} = 1 \; \forall \; j \in J \qquad (2)$$

$$x_{ij} \le y_i \qquad \forall \; j \in J, \; \forall \; i \in I \qquad (3)$$

$$\sum_{i=1}^{m} y_i = p \qquad (4)$$

$$x_{ij} \, , y_i \in \{0,1\} \qquad (5)$$

Where

P is the number of facilities to be opened

c_{ij}= cost of assigning customer j to facility i

y_i=1 if facility i is opened, 0 otherwise

x_{ij}=1 if customer js demand is satisfied from facility i, 0 otherwise

Equation (2) guarantees that all customers' demands are satisfied. Constraint (3) is the requirement that demand can only be satisfied from a facility that is open, constraint (4) states that exactly p facilities shall be opened and (5) are the integral requirements.

In the sequel we will use the following reformulation of the p-median location problem

$$Min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (1)$$

Subject to

$$\sum_{i=1}^{m} x_{ij} \le 1 \qquad (5)$$

$$x_{ij} \le y_i \qquad \forall \; j \in J, \; \forall \; i \in I \qquad (3)$$

$$\sum_{i=1}^{m} y_i = p \qquad (4)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = |J| \qquad (9)$$

$$x_{ij} \, , y_i \in \{0,1\} \qquad (4)$$

The formulations are clearly equivalent and if the p-median problem were to be solved by any regular method, the reformulation makes no sense. However as we shall show, if the solution approach is based on Semi-Lagrangean relaxation this formulation yields a more efficient relaxation procedure.

Avella has an alternative formulation, which is as follows

$Min \sum_{ij \in A} c_{ij}\, x_{ij}$ (12)

Subject to

$\sum_{i \in V\setminus\{i\}} x_{ij} + y_j = 1 \;\; \forall\, j \in V$ (13)

$x_{ij} \leq y_i \quad\quad \forall\, ij \in A$ (14)

$\sum_{j \in V} y_j$=p (15)

$y_j \in \{0,1\}$ j$\in V$ (16)

$x_{ij} \geq 0 \;\; ij \in A$ (17)

3. Semi-Lagrangean Relaxation

The Semi-Lagrangean approach builds upon the well-known Lagrangean relaxation, but with the difference that when having equality constraint, the constraint is divided in two inequalities, namely a "greater than or equal to" inequality and a "less than or equal to" inequality. The former is relaxed and added to the objective function, while the latter is left as an inequality constraint in the sub-problem(Beltran et al. 2006). Mathematically, if we have a minimisation problem of the following type $z^* = \min_x \left\{ c^T x \mid Ax = b; x \in S := X \cap \mathbb{N}^n \right\}$ then, the Semi-Lagrangean function is written as $z_{SLR} = \max_\lambda \mathcal{L}_{SLR}(\lambda) = \max_\lambda \left\{ b^T \lambda + \min_x \left\{ (c - A^T \lambda)^T x \mid Ax \leq b; x \in S \right\} \right\}$.

It is proved by Beltran-Royo et al. that the Semi-Lagrangean relaxation closes the duality gap. The easiest way to see this is that the relaxation is the result of the intersection between the following two polytopes

$$Conv\{min_x(c^T x | Ax \leq b, x \in S)\} \cap Conv\{Ax \geq b\}$$

Which obviously is equal to $Conv\big\{\{minc^T x | Ax = b, x \in S\}\big\}$

The following theorem from Beltran-Royo et.al. gives the properties of the Semi-Lagrangean dual

Theorem: (Beltran-Royo et.al)

The following statements holds

1. The Semi-Lagrangean dual l(u) is concave and b-Ax(u) is a subgradient at u
2. L(u) is monotone and L(u´)≥L(u) if u´≥u with strict inequality if u´>u and u not belonging to U^*
3. $U^* = U^* \cup R_+^m$ thus U is unbounded
4. $If\ x(u) is\ such\ that\ Ax(u)b\ then\ u \in U^*$ and x(u) solves the original problem

5. $Conversely\ if\ u \in int(U^{*})$ then any minimizer x(u) is optimal in the original problem
6. $The\ Semi - Lagrangean\ relaxation\ closes\ the\ duality\ gap$

However there are some difficulties involved in calculating the optimal Semi-Lagrangean multipliers especially in the multi-dimensional case. The main problem is that the optimal Semi-Lagrangean prices are non-unique (in the multi-dimensional case). Moreover for large enough multipliers u x(u) will be a solution to the original problem and the relaxed problem is basically identical to the original problem. Also we are not looking for a maximum of the concave function L(u) rather we are looking for the "minimal" multiplier values for which L(u) reaches its maximal value.

Applying a Semi-Lagrangean relaxation to the reformulated UFL problem, relaxing the single equality constraint yields the following dual problem

Max L(u) subject to u≥0

Where L(u) is defined by the following optimization problem

$$Min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} - u\left(\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} - |J|\right) \quad (8)$$

Subject to

$$\sum_{i=1}^{m} x_{ij} \leq 1 \quad (5)$$

$$x_{ij} \leq y_i \qquad \forall\, j \in J,\ \forall\, i \in I \quad (3)$$

$$\sum_{i=1}^{m} y_i = p \quad (4)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \leq |J| \quad (9)$$

$$x_{ij}, y_i \in \{0,1\} \quad (4)$$

Note we have only one Semi-Lagrangean multiplier to search for as compared with the multi-dimensional search needed in Beltran-Royos et.al.s Semi-Lagrangean relaxation of the standard p-median model. This means that any one-dimensional search procedure can be used. Also the optimal Semi-Lagrangean price has a meaningful economic interpretation. The optimal Semi-Lagrangean price u^{*} is the price that has to be payed to be able to deliver to all customers. It should also be noted that it is easy to get an initial estimate on u since we know that all customer demand has to be satisfied. Hence a minimal u is obtained by $max_j min_i(c_{ij})$.

Also note that in the subproblems only alternatives ij with negative coefficients are of interest. Hence the subproblems will in most cases, have fewer 0/1 variables than the number of 0/1 variables in the original problem.

4. Illustrative Examples

As a first illustration we use the example from the book by Daskin.

Here the Demand times distance matrix for the 12 node problem is as follows

	A	B	C	D	E	F	G	H	I	J	K	L
A	0	225	555	825	360	900	270	495	720	600	870	1005
B	150	0	220	400	380	520	330	480	420	550	610	610
C	444	264	0	216	192	360	492	336	240	696	468	468
D	990	720	324	0	612	216	1062	828	432	1116	774	612
E	120	190	80	170	0	180	125	60	120	235	185	215
F	1440	1248	720	288	864	0	1368	1008	288	1200	744	528
G	198	363	451	649	275	627	0	165	495	242	440	671
H	528	768	448	736	192	672	240	0	480	592	400	736
I	624	546	260	312	312	156	585	390	0	494	247	247
J	880	1210	1276	1364	1034	1100	484	814	836	0	418	880
K	1102	1159	741	817	703	589	760	475	361	361	0	399
L	1340	1220	780	680	860	440	1220	920	380	800	420	0

In order to select a reasonable value for the Semi-Lagrangean multiplier we proceed as follows. Given a value u for the Semi-Lagrangean multiplier, we can formulate a set covering problem with 10 variables and 10 constraints. The coefficients in the constraint matrix of this set covering problem are 1, if the cost in the cost matrix is less than u and 0 otherwise. We would like to select the first value for the Semi-Lagrangean multiplier such that the resulting relaxed p-median problems, having only variables for which the cost coefficients in the Semi-Lagrangean subproblems are negative includes a feasible solution to the original p-median problem. Solving the set covering problem with the objective function in which the number of median nodes is to be minimized leads to that the minimum possible value for the Semi-Lagrangean multiplier is equal to u=361

This means that the modified demand times distance matrix is

	A	B	C	D	E	F	G	H	I	J	K	L
A	-361	-136	X	X	-1	X	-46	X	X	X	X	X
B	-211	-361	-141	X	X	X	-31	X	X	X	X	X
C	X	-97	-361	-145	-169	-1	X	-25	-121	X	X	X
D	X	X	-37	-361	X	-145	X	X	X	X	X	X
E	-241	-171	-281	-191	-361	-181	-236	-301	-241	-126	-176	-146
F	X	X	X	-73	X	-361	X	X	-73	X	X	X
G	-163	X	X	X	-86	X	-361	-196	X	-119	X	X
H	X	X	X	X	-169	X	-121	-361	X	X	X	X
I	X	X	-101	-49	-49	-205	X	X	-361	X	-114	-114
J	X	X	X	X	X	X	X	X	X	-361	X	X
K	X	X	X	X	X	X	X	X	0	0	-361	X
L	X	X	X	X	X	X	X	X	X	X	X	-361

All variables with coefficients larger than 361 are not present in an optimal solution to the Semi-Lagrangean subproblems and can be deleted.

The optimal solution to the Semi-Lagrangean subproblem is to select nodes A,F,H,J and L and allocate the remaining nodes in the following way nodes C,E and G is allocated to H, node B is allocated to A, nodes D and I is allocated to F and finally node K is allocated to J. The objective function value for the Semi-Lagrangean relaxation is -2888 and hence the lower bound is 4332-2888=1444. Since this is the value of the current solution which is feasible, optimality has been proved. In this example the optimal Semi-Lagrangean multiplier value is equal to the value of the most costly assignment in the solution. This is however not always the case as the next illustrative example will show.

The data for the second illustrative example is as follows.

id	1	2	3	4	5	6	7	8	9	10
x	46.5	47.3	44.8	46.0	49.8	51.3	48.3	44.2	41.5	44.9
y	51.2	51.5	51.0	49.9	50.5	52.0	53.7	52.8	50.1	48.0
weight	249	155	233	209	215	245	224	219	249	257

The exact coordinates and weights of the basic units.

This gives us the following matrix of weighted distances

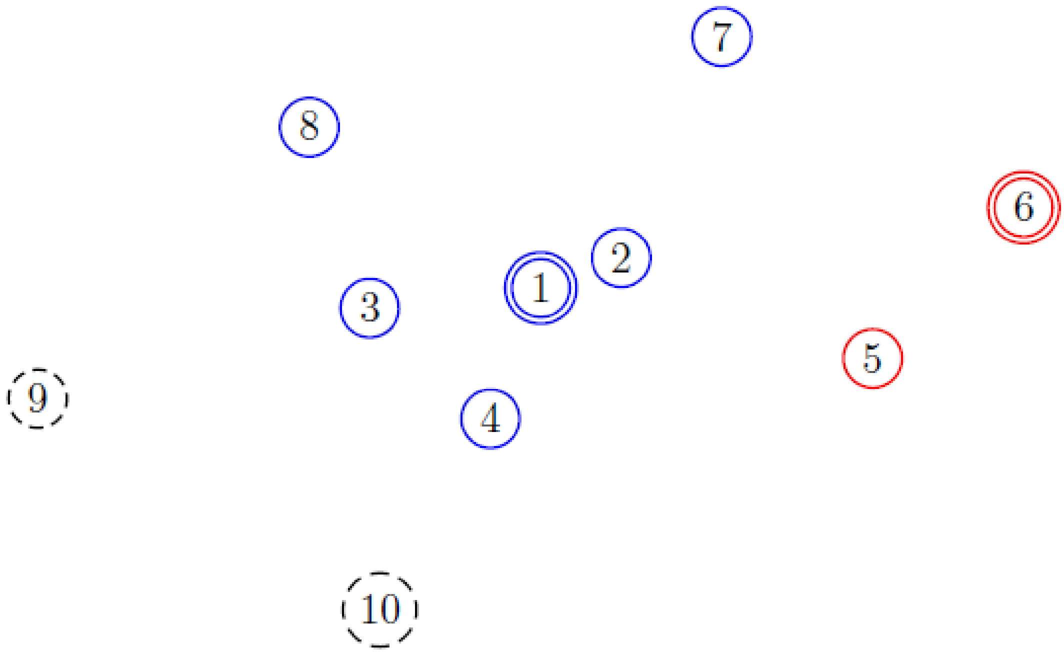
	1	2	3	4	5	6	7	8	9	10
1	0	212.7	426.2	346.8	840.0	1211.69	767.1	697.6	1274.8	890.8
2	132.4	0	395.2	319.5	417.4	624.825	374.6	521.0	924.8	657.8
3	398.8	594.0	0	379.3	1170.8	1532.32	1030.0	442.1	797.0	699.4
4	291.1	430.9	340.2	0	804.0	1191.48	928.3	713.4	941.4	458.8
5	725.3	578.9	1080.4	827.1	0	456.084	759.8	1301.6	1786.6	1182.7
6	1192.2	987.6	1611.2	1396.7	519.7	0	844.8	1750.5	2445.7	1849.1
7	690.1	541.3	990.2	995.0	791.6	772.394	0	940.3	1723.5	1486.7
8	613.6	736.2	415.5	747.5	1325.8	1564.74	919.3	0	836.2	1062.3
9	1274.8	1485.7	851.7	1121.6	2069.1	2485.64	1915.8	950.8	0	995.1
10	919.5	1090.7	771.4	564.2	1413.7	1939.63	1705.7	1246.7	1027.0	0

The matrix of weighted distances.

Let p=2. Proceeding in the same way as before to find the minimum possible Semi-Lagrangean multiplier, i.e., by checking if the generated set covering problem has a cover with the value of at least p gives us the initial multiplier value u=851.711

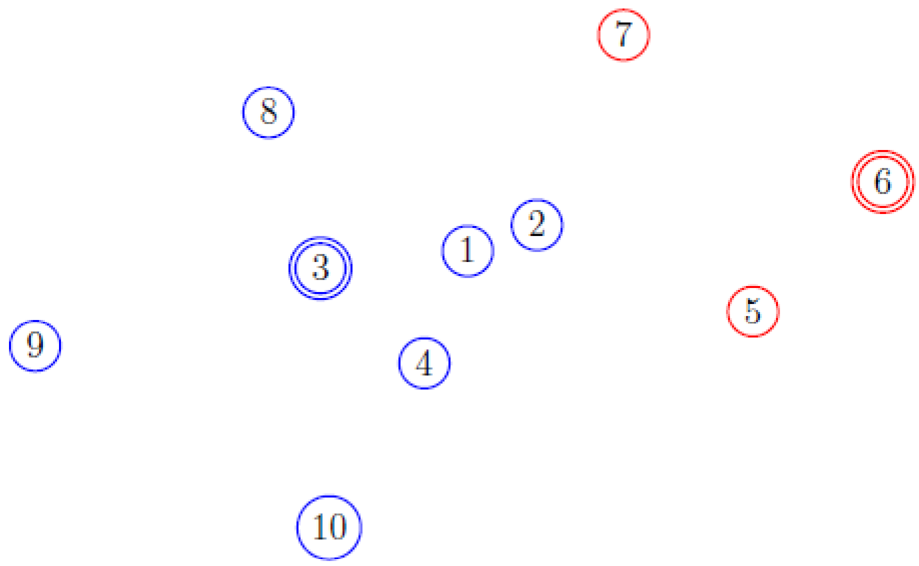
Solving the Semi-Lagrangean subproblem for this u gives us the optimal objective function value of -4231.5958 and a lower bound of 4285.5..

In this solution there are two units left unassigned, 9 and 10, and the medians selected are 1 and 6. The solution is illustrated graphically in this figure



(b) Optimal solution of the Lagrangian subproblem for p-median problem with $\lambda = 851.711 + \epsilon$.

In order to generate a feasible solution and an upper bound we proceed as follows. We solve the strengthened relaxation in which the constraints for node 9 and 10 are equalities instead of inequalities. Note that this relaxation will also provide us with a lower bound to the optimal p-median problem. The objective function value for the strengthened relaxation is -4088.351 and the lower bound is thus 8571.1-4088.351=4428.76 which is the value of the current feasible solution and hence optimality has been proved. The solution is illustrated graphically in the figure with node 3 and 6 as medians.



(a) Optimal solution for the (uncapacitated) p-median problem

If we instead continue with the original Semi-Lagrangean relaxation we need to increase the multiplier value further in order to prove optimality and generate the optimal 2-median solution. For a Semi-Lagrangean multiplier value of 940.267 optimality is proved. As can be seen from this example normally the Semi-Lagrangean subproblem will give us a lower bound for the optimal solution value and leave some of the locations not assigned to any of the chosen medians.

The question is of course how to select the initial Semi-Lagrangean multiplier value and if the optimal Semi –lagrangean multiplier has a usable economic interpretation.

4. An Economic Interpretation of the procedure

Looking at the Semi-Lagrangean subproblem

$$Min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} - u(\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} - |J|) \quad (8)$$

Subject to

$$\sum_{i=1}^{m} x_{ij} \leq 1 \quad (5)$$

$$x_{ij} \leq y_i \quad \forall j \in J, \ \forall i \in I \quad (3)$$

$$\sum_{i=1}^{m} y_i = p \quad (4)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \leq |J| \quad (9)$$

$$x_{ij}, y_i \in \{0,1\} \quad (4)$$

We can observe that he Semi-Lagrangean sub problem can be thought of as a procedure to search for a market price in a market in which customers located in different places should be served from p locations. In order to serve the various markets the goods has to be transported from one of the open sources to the customer. Each customer has a demand for one unit, hence the total demand is $|J|$.

The optimal Semi-Lagrangean price u* corresponds to the market price for which all customer demands are satisfied. At a lower price than u* one or more of the customers will be left unserved. An interesting question is how the market price, optimal Semi-Lagrangean multiplier varies with the number of medians p?

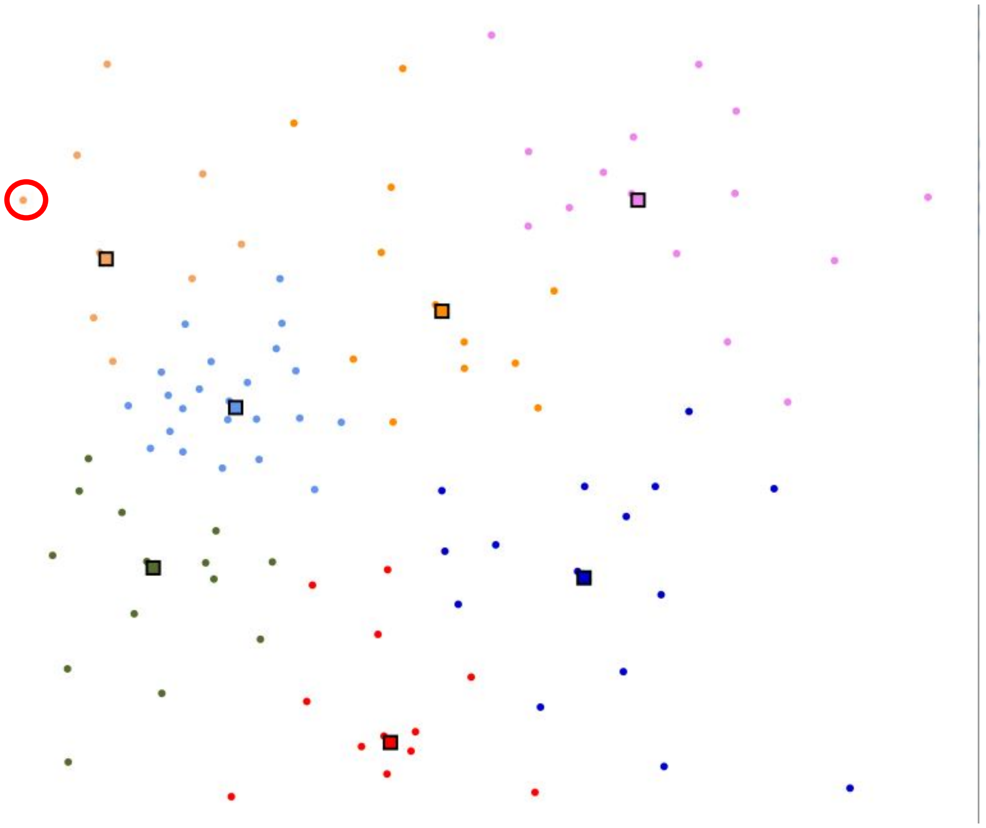
In order to find that out we conducted some numerical experiments on larger p-median problems.

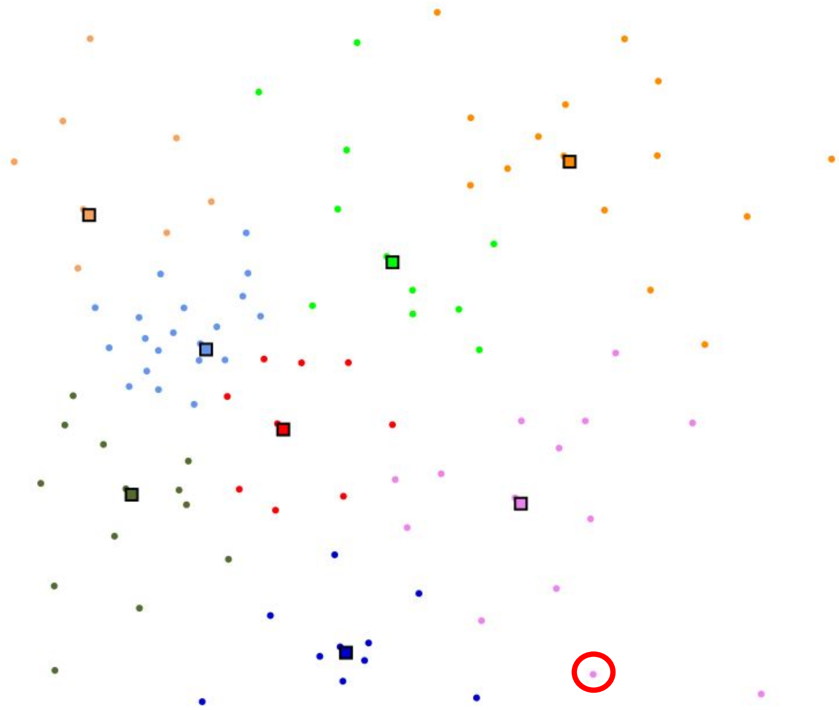
We also show how the market price varies as a function of the value of p. For this purpose, two problems were considered one with 100 and one with 300 basic points.

The results for the problem with 100 basic points, (the data for this problem can be found in the appendix), is as follows

Areas	Median	optLambda	obj
100	4	84,6577	2401,74
100	5	75,9394	2149,38
100	6	75,9394	1966,18
100	7	57,0797	1794,45
100	8	57,2712	1684,85
100	9	55,9870	1580,19
100	10	52,9555	1489,29
100	11	46,5587	1401,78
100	12	43,2219	1310,33
100	13	40,8280	1248,12
100	14	38,8826	1172,82
100	15	37,2251	1111,16

As can be seen the market price is almost always decreasing as a function of p. However, in this example the market price increases when p is increased from 7 to 8. The increase is not large and the reason for the increase is the discrete nature of the p-median problem. It turns out to be more expensive to assign the last basic unit when p=8 than it is when p=7. A graphical illustration of the solutions for p=7 and p=8 is shown in the figures below.



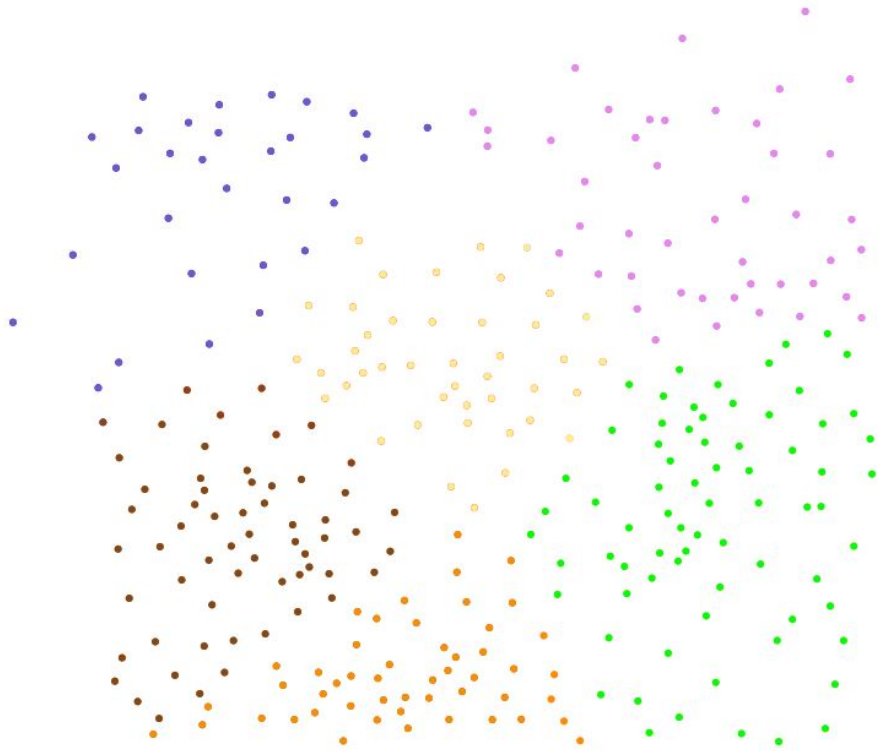


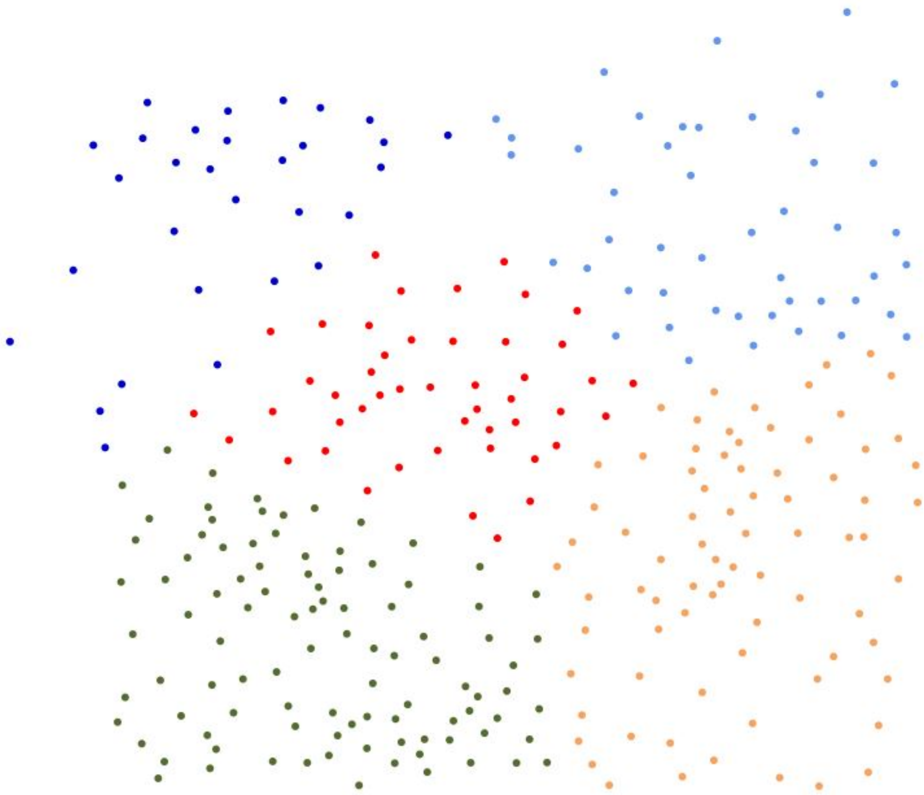
For the 300 basic unit problem the results are as follows

#Area	#Median	optLambda	obj
300	5	116,8030	3422,1
300	6	120,2950	3109,23
300	7	83,4527	2822,1
300	8	83,5651	2602,37
300	9	62,5821	2418,81
300	10	62,5821	2296,82
300	11	62,5821	2183,86
300	12	59,9683	2077,32
300	13	55,0258	1982,52
300	14	51,9112	1897,6

Here the market price increases when p is increased from 7 to 8 as before with a small increase. However, the increase in market price when p is increased from 5 to 6 is substantial.

In the figures below, we give a graphical illustration of what happens.





5. Conclusions

In this article, we have presented a Semi-Lagrangean relaxation method for the p-median problem. We use a reformulation more suitable for Semi-Lagrangean relaxation. Since the reformulated and relaxed problem has only one multiplier, no subgradient procedure is needed to update the multiplier value. Also the number of potential multiplier values is limited to the number of different costs in the cost matrix. In order to find the initial value for the Semi-Lagrangean multiplier a set covering problem is studied. It is shown that for a more restricted version of the Semi-Lagrangean relaxation, the optimal multiplier value is equal to the most costly assignment in the optimal solution. However, in the ordinary Semi-Lagrangean relaxation of the reformulated problem the optimal Semi-Lagrangean multiplier value is equal to the price that has to be payed in order for all basic units demand to be fulfilled.

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