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Discussion paper

The Analysis of Leadership Formation in Networks Based on Shapley Value

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THE ANALYSIS OF LEADERSHIP FORMATION IN NETWORKS BASED ON SHAPLEY VALUE

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Abstract
In the given research we analyse how an agent can move towards leadership in a socio-economic network. For the node's (i.e., agent's) importance measure we use the Shapley value (SV) concept from the area of cooperative games. We consider SV as the node's centrality that corresponds to the significance of the agent within the socio-economic network. Using the polynomial algorithm developed by Aadithya, Ravindran, Michalak, & Jennings (2010) to compute SVs we analyze the way of creating new linkages to increase an agent's significance (i.e., importance) in networks.

Keywords: socio-economic networks, Shapley value, leadership

1. INTRODUCTION

The analysis of agent's (i.e., node's) importance in the domain of networks is one of the core ideas in socio-economic network analysis. Different evaluation methods exist. Degree (Freeman, 1979), betweenness (Anthonisse, 1971; Freeman, 1977), and closeness (Beauchamp, 1965; Sabidussi, 1966) are the most widely known metrics that assess the structural centralities of nodes. The algorithmic measures of node's authority are well represented in Kleinberg (1999) and Page, Brin, Motwani, & Winograd (1999), where the notion of authority is given based on the analysis of link structures. An interesting approach to characterize the role of nodes within the networks is given by Scripps & Esfahanian (2007), where the community-based metric in the symbiosis with the degree-based measure is introduced in the context of nodes' roles classification.

Another methodology for analyzing node's leadership and importance in networks is based on a game theoretic approach. Specifically, we employ the concept of the Shapley value (Shapley, 1952) from the area of cooperative games. We employ the Shapley value concept developed by Aadithya et al. (2010) in order to analyze how the nodes' leadership positions in networks can be strengthened by establishing new links. First, we consider basic network topologies since they are the base of any large-scale graph, and analyze the way to improve the influential power of nodes based on their Shapley value.

In section 2 we provide the description of the Shapley value and its interpretation in terms of networks. Specifically, we describe the approach made by Aadithya et al. (2010) giving an illustrative example.

Based on SV approach we represent the leadership formation algorithm that calculates the tuple of links to be established with initially the least influential node in order to make it as strong as possible. The formalized algorithm is represented in section 3.

Next, we apply our algorithm to four basic network topologies: "point-to-point", "star", "ring", and mesh, which are the base of any large-scale network structure. The results are presented in section 4.

In section 5 we apply the leadership formation method to two real-world networks.

2. SHAPLEY VALUE AS AN AGENT'S IMPORTANCE MEASURE

The Shapley value (Shapley, 1952) is a game theoretic approach that provides the solution for computing of players' gains in cooperative games where the players' contributions are non-equal. The formal definition of the Shapley value (SV) is well-described in Littlechild & Owen (1973) and Gul (1989). Specifically, the SV equation for the player *i* in the coalition game with *n* players is the following:

$$SV_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(v(S \cup \{i\}) - v(S)\right), \tag{1}$$

where:
N is a set of n players;
S is a coalition of players;
v is the characteristic function: $2^N \to \mathbb{R}$; $v(\emptyset) = 0$.
The Shapley value is characterized by different properties such as efficiency, symmetry, linearity etc. (Hart, 1989).
The special interest for employing the SV approach in the area of socio-economic network analysis is based on its use to measure the importance of nodes. In other words, SV is interpreted as the level of the nodes' importance within a network (Suri & Narahari, 2008; Gomez, González-Arangüena, Manuel, Owen, del Pozo, & Tejada, 2003).
We employ the computational approach of the Shapley value as a centrality measure in networks that was developed by Aadithya et al. (2010). They introduced the idea of Shapley value "in the domain of networks, where it is used to measure the importance of individual nodes, which is known as game theoretic network centrality" (Aadithya et al., 2010).
Consider graph *G(V,E)* and $v_i \in V$. All nodes (i.e., neighbors), which are reachable from v_i at most one hop within *G(V,E)* are denoted by $N_G(v_i)$. The degree of node v_i is defined by $deg_G(v_i)$. The SV interpretation for node v_i in *G(V,E)* according to Aadithya et al. (2010) is the following:

$$SV(v_i) = \sum_{v_j \in \{v_i\} \cup N_G(v_i)} \frac{1}{1+deg_G(v_j)}, \tag{2}$$

Based on equation (2) Aadithya et al. (2010) introduced an algorithm to calculate SVs for all nodes in the network:

SV-COMPUTING:
Input: Unweighted graph *G(V,E)*
Output: SVs of all nodes in *V(G)*
for each $v \in V(G)$ **do**
    ShapleyValue $[v]$= $\frac{1}{1+deg_G(v)}$;
    For each $u \in N_G(v)$ **do**
        ShapleyValue $[v]$ += $\frac{1}{1+deg_G(u)}$;
    end
end
return ShapleyValue;

The advantage of the given algorithm is the polynomial running time *O(V+E)* (Cormen, Leiserson, Rivest, & Stein, 2003) to compute SVs for all nodes in *G(V,E)* based on equation (2). To illustrate how the algorithm works we consider the trivial example (see Figure 1).

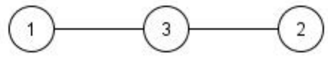

Figure 1. Example network for SVs computation

We calculate SVs for three nodes following the algorithm:

I. *For node 1:*

1) $\mathrm{SV}'(v_1) = \frac{1}{1+deg_G(v_1)} = \frac{1}{1+1} = \frac{1}{2}$

2) $u \in N_G(v_1): u \in \{\text{"}node\ 3\text{"}\}$:

$\mathrm{SV}(v_1) = \mathrm{SV}'(v_1) + \frac{1}{1+deg_G(\text{"}node\ 3\text{"})} = \frac{1}{2} + \frac{1}{1+2} = \frac{5}{6}$

Thus, $\mathrm{SV}(v_1) = \frac{5}{6}$

II. *For node 2:*

1) $\mathrm{SV}'(v_2) = \frac{1}{1+deg_G(v_2)} = \frac{1}{1+1} = \frac{1}{2}$

2) $u \in N_G(v_2): u \in \{\text{"}node\ 3\text{"}\}$:

$\mathrm{SV}(v_2) = \mathrm{SV}'(v_2) + \frac{1}{1+deg_G(\text{"}node\ 3\text{"})} = \frac{1}{2} + \frac{1}{1+2} = \frac{5}{6}$

Thus, $\mathrm{SV}(v_2) = \frac{5}{6}$

III. *For node 3:*

1) $\mathrm{SV}'(v_3) = \frac{1}{1+deg_G(v_3)} = \frac{1}{1+2} = \frac{1}{3}$

2) $u \in N_G(v_3): u \in \{\text{node 1}, \text{"}node\ 2\text{"}\}$:

$\mathrm{SV}(v_3) = \mathrm{SV}'^{(v_3)} + \frac{1}{1+deg_G(\text{node 1})} + \frac{1}{1+deg_G(\text{node 2})} = \frac{1}{2} + \frac{1}{1+1} + \frac{1}{1+1} = \frac{4}{3}$

Thus, $\mathrm{SV}(v_3) = \frac{4}{3}$

Obviously, node 3 has the highest SV, and nodes 1 and 2 have equal SVs. The given results satisfy the efficiency requirement (Hart, 1989).

3. LEADERSHIP FORMATION ALGORITHM

Based on the SV-COMPUTING presented by Aadithya et al. (2010) we developed the algorithm calculating the tuple of links that is to be established with initially the least influential node 'x' (i.e., node with the minimum SV in G(V,E)) in order to make it as strong as possible. In other words, we are testing the links between node 'x', that is chosen initially for the improvement in terms of SV, and the other nodes 'i' in a network.

Establishing the new link between node 'x' and node 'i' we consider two cases:

CASE 1:

We approve the newly established link between node 'x' and some other node 'i' if and only if SV-value increased for at least one of them and did not decrease for another one.

CASE 2:

We approve the newly established link if and only if SV-values increased for both nodes: 'x' and 'i'.

The following procedures implement the leadership formation of the least influential node in network G(V,E):

CASE 1:
LEADERSHIP-FORMATION:

```
SV-COMPUTING(G);
NODES-SORTING(G);
x = SL[n];
k = 1;
FOR i = k to n in SL:
    IF [edge (x, SL[i]) does not exist in G] AND [x ≠ SL[i]]:
    THEN: previous_SV(i) = SV(i);
          previous_SV(x) = SV(x);
          Establish test-edge (x, SL[i]);
          SV-COMPUTING (G);
          IF [previous_SV(i) < SV(i) AND previous_SV(x) ≤ SV(x)]
                                   OR
             [previous_SV(i) ≤ SV(i) AND previous_SV(x) < SV(x)]:
          THEN: approve edge (x, SL[i]);
                NODES-SORTING(G);
                IF x ≠ SL[i]:
                THEN: k = 1;
                ELSE: return SV(x);
                      Stop calculations;
          ELSE: Erase edge (x, SL[i]);
                Roll back to SV- and SL-results that exclude edge (x, SL[i]);
                k = k + 1;
    ELSE: k = k + 1;
return   SV(x)
```

CASE 2:
LEADERSHIP-FORMATION:

```
SV-COMPUTING (G);
NODES-SORTING(G);
x = SL[n];
k = 1;
FOR i = k to n in SL:
    IF [edge (x, SL[i]) does not exist in G] AND [x ≠ SL[i]]:
    THEN: previous_SV(i) = SV(i);
          previous_SV(x) = SV(x);
          Establish test-edge (x, SL[i]);
          SV-COMPUTING (G);
          IF [previous_SV(i) < SV(i) AND previous_SV(x) < SV(x)]:
          THEN: approve edge (x, SL[i]);
                NODES-SORTING(G);
                IF x ≠ SL[i]:
                THEN: k = 1;
                ELSE: return SV(x);
                      Stop calculations;
          ELSE: Erase edge (x, SL[i]);
                Roll back to SV- and SL-results that exclude edge (x, SL[i]);
                k = k + 1;
    ELSE: k = k + 1;
return   SV(x)
```

NOTATION:

NODES-SORTING(G):
For graph G(V,E) we have SVs for all n=|G.V| nodes based on the SV-COMPUTING(G) results. Applying one of the sorting algorithms, such as Quick Sort, Heap Sort or Merge Sort (Cormen et al., 2003), we sort all nodes in descending order based on the corresponding SVs. NODES-SORTING(G) returns the sorted list of nodes (SL), where SL[1] is the node with the max SV-value, and SL[n] is the node with the min SV-value.

'x' is a node with the smallest SV. It is chosen for the leadership formation from the initial SL after the first SV-calculation.

SL[i] is the i-th node in the list of nodes sorted by the SV-values.

ALGORITHM's OVERVIEW:

Initially, we sort all nodes by their SV values. Consider the following: X is the least influential node and Y – initially the most influential node in the network. The process of the new link creation is the following: if X and Y are not connected then we link them; if X and Y are already connected then we link X with the next most influential node that becomes Y. Next, we measure their SVs. If SVs for both nodes of the newly established connection (i.e., link) has increased (case 2) or increased for at least one of them and did not decrease for another one (case 1) then we confirm the link establishment between X and current Y and update SV-values and SL-list. Otherwise, we erase the given link and connect X with the next node in the previously sorted list. We go through the sorted list of nodes until X gets the maximum possible SV (i.e., it becomes the most influential as possible). On each iteration (i.e., every time we link X with the next node Y in the sorted list) we consider only the SV-gains for X and for the current Y. We do not consider the SV-modifications of the Y-nodes that have already been included in the coalition of the previous iterations.

4. LEADERSHIP FORMATION IN DIFFERENT NETWORK TOPOLOGIES

Any large-scale network consists of the trivial topologies with different characteristics (Haddadi, Rio, Iannaccone, Moore, & Mortier, 2008):

- "point-to-point", or "line";
- "star";
- "ring";
- Mesh, i.e., topologies that are based on the previous three types.

According to the LEADERSHIP-FORMATION algorithm the main goal is to improve the node with the minimum SV continuously until its SV will become the maximum possible in the network. In other words, we analyze how the "weakest" node (i.e., with minimum SV) in the network can become the most influential as possible (i.e., getting the maximum possible SV). In order to approach this goal we have to establish new links in a way that will improve an SV of the weakest node based on the LEADERSHIP-FORMATION algorithm.

In this paper we show how the given mechanism works for the basic network topologies (listed above) due to the fact that any large-scale network consists of different combinations of these topologies. The results are represented in sections 4.1-4.4. In section 5 we illustrate our approach based on two real-world examples.

4.1 "Point-to-point" topology

"Point-to-point" is the simplest type of network topology. It is represented in Figure 2. Initially, we calculate SVs for all nodes following LEADERSHIP-FORMATION algorithm. The SVs sorted results are represented in Table 1.

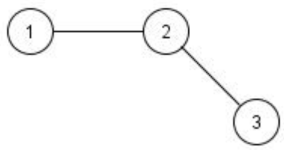

Figure 2. "Point-to-point" network topology in the initial state

Table 1. Initial SVs in the "point-to-point" topology

node	Shapley value
2	1.33
1	0.83
3	0.83

Nodes 1 and 3 are the weakest. We choose node 1 to improve its SV. Since the link with the most influential node 2 already exists we create the link with node 3 (Figure 3). We recalculate SVs for all nodes in the modified network. The results are represented in Table 2.

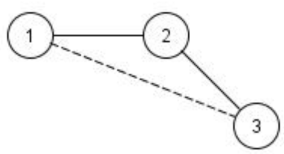

Figure 3. Modified "Point-to-point" network topology

Table 2. Updated SVs in the modified topology

node	Shapley value
1	1.00
2	1.00
3	1.00

Establishing the link between node 1 and node 3 we increase SVs for both of them by 0.17. Since the increment is positive we approve "node 1" – "node 3". As the result, we get the fully connected graph and no other links can be created. Node 1 has got the largest possible SV value and its influence in the given topology became as powerful as for nodes 2 and 3.

4.2 "Star" topology

This type of network topology is characterized by the existence of a hub that is the most central node in the network. All other nodes in the network are connected to the hub by "point-to-point" connections (see Figure 4). We calculate the initial SVs for the nodes. The sorted results are represented in Table 3.

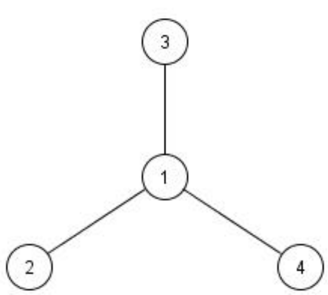

Figure 4. "Star" network topology in the initial state

Table 3. Initial SVs in the "star" topology

node	Shapley value
1	1.75
2	0.75
3	0.75
4	0.75

Nodes 2, 3 and 4 are equally weak. We choose node 2 to improve its leadership within the network. It is already linked with the most influential node 1. Hence, we connect it to the next node in the sorted list, which is node 3. The modified network structure is represented in Figure 5. Recalculating SVs for the network (see Figure 5) we get the results represented in Table 4.

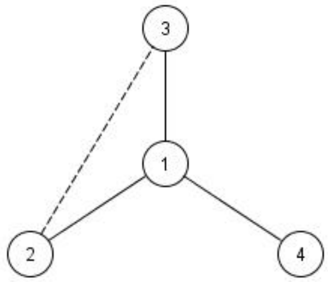

Figure 5. First modification of the "Star" network topology

Table 4. SVs after the first modification

node	Shapley value
1	1.42
2	0.92
3	0.92
4	0.75

Based on Table 4 it is obvious that nodes 2 and 3 have improved their leadership positions: SVs increased by 0.17 for both of them. It means that we approve the link "node 2" – "node 3".
Since SV of node 2 is less than SV of node 1, we have to establish the new link with the node from the updated SL-list. Links "node 2 – node 1" and "node 2 – node 3" already exist. Sequentially, we establish the link with node 4. The resulted structure is represented in Figure 6. We recalculate SVs for the newly modified structure. The results are represented in Table 5.

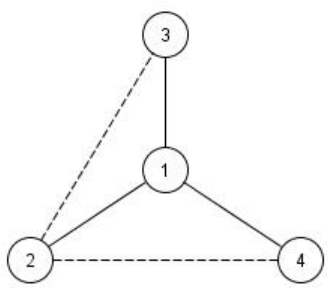

Figure 6. Second modification of the "Star" network topology

Table 5. SVs after the second modification

node	Shapley value
1	1.17
2	1.17
3	0.83
4	0.83

Link "node 2 – node 4" improves the SVs for both nodes: ΔSV("node 2") = +0.25 and ΔSV("node 4") = +0.08. Therefore, we approve the link "node 2 – node 4". On this step we get the maximum SV for node 2, and its leadership position is the highest as well as for the hub (i.e., node 1).

4.3 "Ring" topology

"Ring" topology is characterized by the sequential "point-to-point" connections of odd or even number of nodes forming the cycle. First, we consider the structure with an even number of nodes, which is represented in Figure 7. Initially, we calculate SVs for the graph. Sorted results are represented in Table 6.

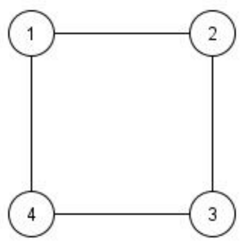

Figure 7. The initial state of the "Ring" network topology with even number of nodes

Table 6. Initial SVs in the "Ring" topology with even number of nodes

node	Shapley value
1	1.00
2	1.00
3	1.00
4	1.00

According to Table 6, all nodes in the "Ring" topology have the equal level of leadership within the network. We select node 2 to improve its position by creating the link with node 4. The modified topology is represented in Figure 8. We recalculate SVs for the graph represented in Figure 8 (see Table 7).

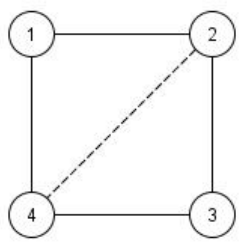

Figure 8. Modified "Ring" network topology with even number of nodes

Table 7. Updated SVs in the modified topology

node	Shapley value
2	1.17
4	1.17
1	0.83
3	0.83

Creating the link between nodes 2 and 4 we increase SVs for both of them by 0.17. Since the increment is positive we approve the link "node 2" – "node 4".
Obviously, node 2 achieved the highest influence within the network as well as node 4.

Next, we consider the "ring" structure with an odd number of nodes (see Figure 9). We calculate the initial SVs. Results are represented in Table 8.

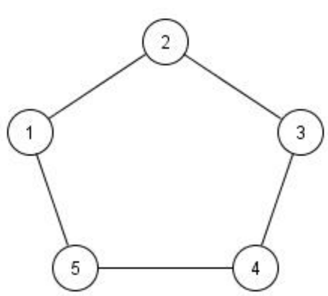

Figure 9. The initial state of the "Ring" network topology with odd number of nodes

Table 8. Initial SVs in the "Ring" topology with odd number of nodes

node	Shapley value
1	1.00
2	1.00
3	1.00
4	1.00
5	1.00

According to Table 8 SVs of all nodes are equal. We choose node 1 in the list to improve its leadership position. We create the link "node 1 – node 3" (see Figure 10) and recalculate the SVs based on the updated structure (see Table 9).

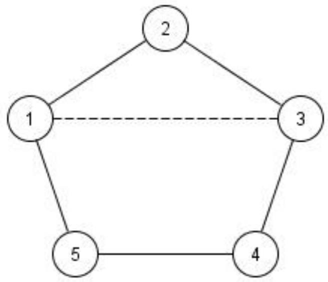

Figure 10. First modification of the "Ring" network topology with odd number of nodes

Table 9. SVs after the first modification

node	Shapley value
1	1.17
3	1.17
4	0.92
5	0.92
2	0.83

We approve link "node 1 – node 3", because the SVs for both nodes have been improved: ΔSV("node 1") = +0.17 and ΔSV("node 3") = +0.17

Next, we test the link "node 1 – node 4". The updated structure is represented in Figure 11 and the resulting SVs are represented in Table 10.

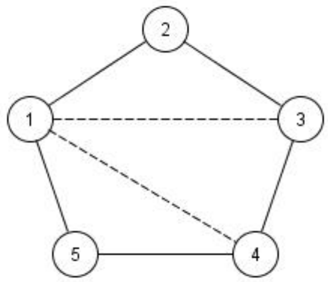

Figure 11. Second modification of the "Ring" network topology with odd number of nodes

Table 10. SVs after the second modification

node	Shapley value
1	1.37
3	1.03
4	1.03
2	0.78
5	0.78

According to Table 10, we have a positive SV improvement for nodes 1 and 4. We get the maximum SV for the node 1 (i.e., SV=1.37), and its leadership position is the highest in the network.

4.4 Mixed topology

Mixed topology is represented by different topological combinations described in sections 4.1 – 4.3. For the analysis we choose a symmetric mixed topology that includes "point-to-point", "star" and "ring" based sub-graphs. The given network is represented in Figure 12. We calculate SVs and represent the sorted results in Table 11.

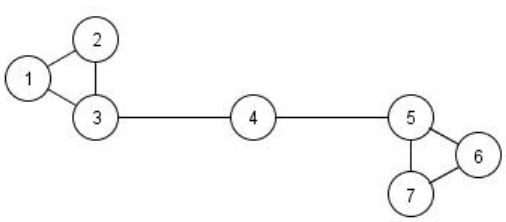

Figure 12. Mixed network topology in the initial state

Table 11. Initial SVs in the mixed topology

node	Shapley value
3	1.25
5	1.25
1	0.92
2	0.92
6	0.92
7	0.92
4	0.83

According to Table 11 node 4 is the least influential (SV=0.83). Since node 4 is already linked with the leading nodes 3 and 5, we connect it with node 1 that is the next most influential node in the sorted list. The resulting network is represented in Figure 13. The recalculated and sorted SVs are represented in Table 12.

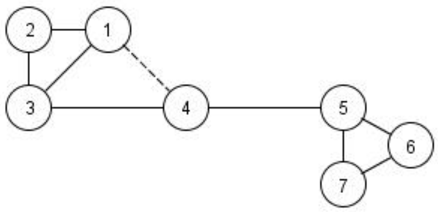

Figure 13. First modification of the mixed network topology

Table 12. SVs after the first modification

node	Shapley value
5	1.17
1	1.08
3	1.08
4	1.00
6	0.92
7	0.92
2	0.83

Link "node 4 – node 1" improves the leadership positions for both nodes: ΔSV("node 4") = +0.17 and ΔSV("node 1") = +0.16. Therefore, we approve link "node 4 – node 1".
Node 4 improved its SV, but nodes 1, 3 and 5 are still more powerful. Therefore, we create the link with the next node from the sorted list. Links "node 4 – node 1", "node 4 – node 3" and "node 4 – node 5" already exist. Therefore, we connect to the next node from the sorted list, which is node 6. The modified graph is represented in Figure 14. Recalculating SVs for the modified graph we get the results represented in Table 13.

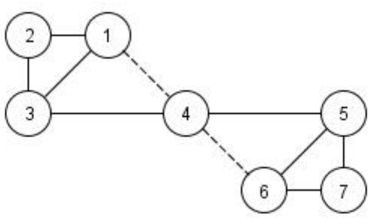

Figure 14. Second modification of the mixed network topology

Table 13. SVs after the second modification

node	Shapley value
4	1.20
1	1.03
3	1.03
5	1.03
6	1.03
2	0.83
7	0.83

Based on the results in Table 13 we get a positive SV growth for both nodes: ΔSV("node 4") = +0.20 and ΔSV("node 6") = +0.11. Therefore, we approve link "node 4" – "node 6".
Based on two modifications of the mixed topology we conclude that node 4 became the most influential by creating two connections: with node 1 and node 6, respectively.

It is important to notice that, running the LEADERSHIP-FORMATION algorithm for different topologies, we have the results that satisfy Case 1 and Case 2 previously described in Section 3. However, as shown in the next Section, the computational process can be different depending on which case is employed.

5. TESTING ON THE REAL-LIFE NETWORKS

5.1. The Florentine network of marriages

We illustrate our approach based on the network of marriages between the most powerful families in 15th century Florence (Padgett & Ansell, 1993; Jackson, 2010). The network's structure is represented in Figure 15 and the initial SVs are given in Table 14. We represent how the family with the weakest leadership position can become the most powerful in the network following case 1 and case 2 of the LEADERSHIP-FORMATION algorithm.

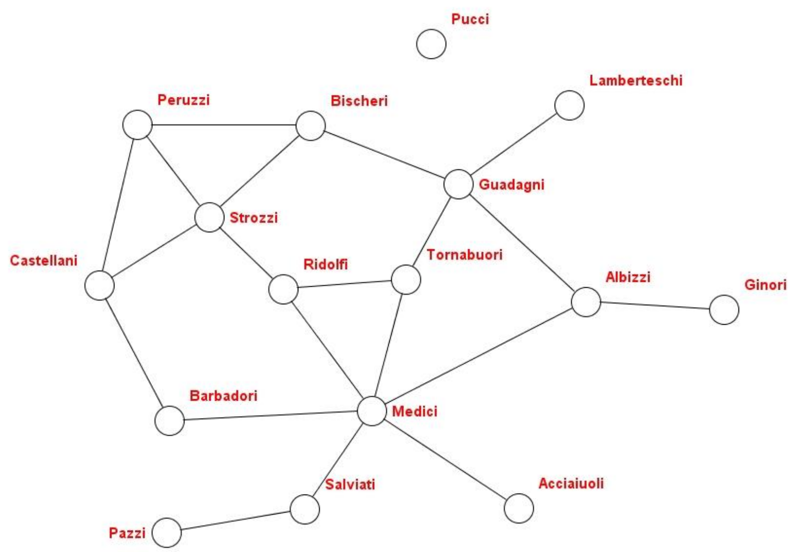

Figure 15. Florentine network of marriages

Table 14. The initial SVs for the Florentine network of marriages

NODE	SV	NODE	SV
Acciaiuoli	0.643	Medici	2.060
Albizzi	1.093	Pazzi	0.833
Barbadori	0.726	Peruzzi	0.950
Bischeri	0.900	Pucci	1.000
Castellani	1.033	Ridolfi	0.843
Ginori	0.750	Salviati	0.976
Guadagni	1.450	Strozzi	1.200
Lamberteschi	0.700	Tornabuori	0.843

CASE 1: LEADERSHIP-FORMATION

According to Table 14, node "Acciaiuoli" has the weakest position in the network (SV=0.643). Following the algorithm we establish the link with the most powerful node from the SV-based sorted list. Node "Medici" is the most powerful in the network (SV("Medici"=2.06)), but the link "Acciaiuoli" – "Medici" already exists. Therefore, we check the link with the second most powerful node, which is "Guadagni". The link "Acciaiuoli" – "Guadagni" is approved, because the node "Guadagni" improved its SV from 1.45 to 1.75, and SV("Acciaiuoli") has not changed.

Following the algorithm, we test the corresponding links. The results for all established links, sorted according to the approval order, are represented in Table 15.

Table 15.CASE 1: SVs based on the established links in the Florentine network of marriages

LINK		**Shapley Value**		
Start node	**End node**	**Start node**	**End node**	
			before	**after**
Acciaiuoli	Guadagni	0.643	1.45	1.75
Acciaiuoli	Strozzi	0.726	1.2	1.417
Acciaiuoli	Albizzi	0.876	1.060	1.210
Acciaiuoli	Castellani	1.043	1.000	1.117
Acciaiuoli	Salviati	1.269	0.976	1.036
Acciaiuoli	Peruzzi	1.451	0.867	0.942
Acciaiuoli	Ridolfi	1.637	0.810	0.871

CASE 2: LEADERSHIP-FORMATION

Since links with node "Acciaiuoli" can be approved if and only if both nodes, that form the link, improve their SVs then links "Acciaiuoli" – "Guadagni" and Acciaiuoli" – "Strozzi" are not approved on the first two corresponding iterations. This is due to the fact that SVs for "Acciaiuoli" have not improved. The first approved link is with the third most powerful node "Albizi". The link "Acciaiuoli" – "Albizi" is approved, because node "Acciaiuoli" improved its SV from 0.643 to 0.676 and node "Albizi" – from 1.093 to 1.376. The results for all further established links are represented in Table 16.

Table 16.CASE 2: SVs based on the established links in the Florentine network of marriages

LINK		**Shapley Value**		
Start node	**End node**	**Start node**	**End node**	
			before	**after**
Acciaiuoli	Albizzi	0.676	1.093	1.376
Acciaiuoli	Guadagni	0.760	1.400	1.617
Acciaiuoli	Strozzi	0.876	1.200	1.367
Acciaiuoli	Castellani	1.043	1.000	1.117
Acciaiuoli	Salviati	1.269	0.976	1.036
Acciaiuoli	Peruzzi	1.451	0.867	0.942
Acciaiuoli	Ridolfi	1.637	0.810	0.871

According to Tables 15-16, node "Acciaiuoli" has achieved the leader's position with SV=1.637. The updated network's structure is represented in Figure 16.

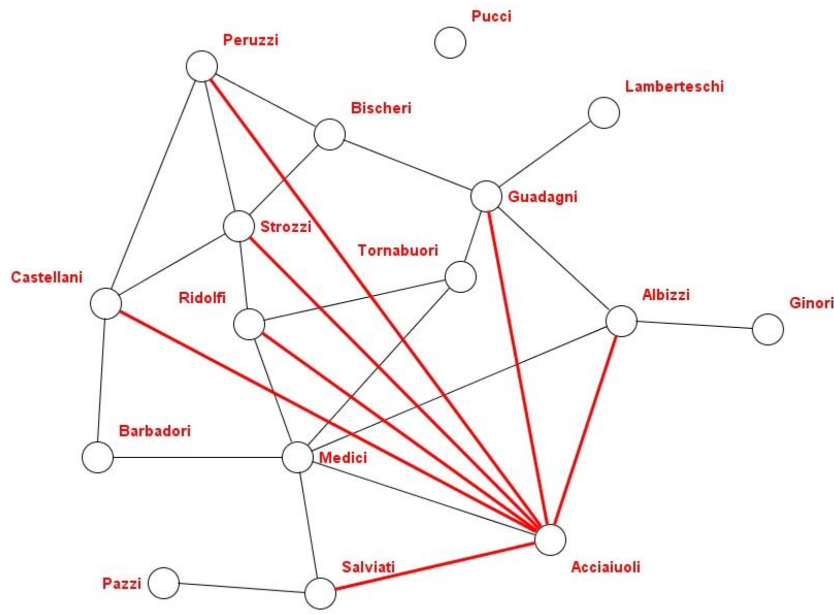

Figure 16. The updated Florentine network of marriages

5.2. Coauthorship network of the Norwegian School of Economics

We illustrate the given Shapley-based leadership formation approach based on the coauthorship network of the Norwegian School of Economics (Belik & Jornsten, 2014). Its structure is represented in Appendix I. The given network is based on the scientific collaboration between the faculty members at the Norwegian School of Economics (NHH) and their international academic publication records according to the ISI Web of Science for the period 1950 – Spring, 2014.
Network's nodes correspond to the NHH faculty members, and the edges correspond to the existing joint publications. More detailed information regarding the NHH coauthorship network is represented in Belik & Jornsten (2014). The initial SVs calculated for the network are represented in Table 17.

Table 17. The initial SVs for the NHH coauthorship network

NODE	SV	NODE	SV	NODE	SV
node 1	0.912	node 37	0.7	node 84	0.667
node 3	0.778	node 39	1.067	node 85	0.783
node 4	0.778	node 40	0.625	node 86	0.976
node 9	1.408	node 45	1.017	node 94	0.726
node 10	0.917	node 50	1.033	node 97	0.7
node 14	2.079	node 52	0.667	node 98	0.843
node 16	0.875	node 53	0.992	node 102	0.833
node 18	0.75	node 58	0.708	node 108	1.617
node 21	0.667	node 60	0.875	node 109	1.843
node 22	0.746	node 61	0.833	node 111	0.843
node 24	0.875	node 65	1.492	node 112	0.676
node 25	1.496	node 67	0.708	node 120	0.75
node 26	1.733	node 68	1.587	node 122	1.583
node 27	0.875	node 69	0.983	node 129	0.75
node 29	0.917	node 70	1.992	node 130	0.667
node 30	1.246	node 73	1.35	node 134	0.833
node 31	**0.611**	node 76	0.667	node 137	1.417
node 33	0.726	node 78	1.176	node 138	1.033
node 34	0.817	node 81	1.117	node 142	0.667

CASE 1: LEADERSHIP-FORMATION

According to Table 17 node 31 has the lowest SV=0.611. We choose this node to improve its leadership position within the network following the algorithm.

First, we check the link with node 14 that has the highest SV=2.079. However, link "node 31 – node 14" already exists in the initial network. Next, we check the second most powerful node 70 (SV=1.992). We recalculate SVs for the network establishing link "node 31 – node 70". As a result, we do not approve this link because SV("node 31") decreased from 0.611 to 0.556. The same situation occurs when we create the links with the third and the fourth most powerful nodes in the network. Specifically, we do not approve links "node 31 – node 109" and "node 31 – node 26" because SV("node 31") is decreased from 0.611 to 0.569 and from 0.611 to 0.587, respectively.

Connecting node 31 to the fifth most powerful node 108 we get a positive result. We approve link "node 31 – node 108", because node 108 improved its SV from 1.617 to 1.917, and SV("node 108") has not decreased.

Next, we follow the LEADERSHIP-FORMATION algorithm reporting only the cases when the links are approved. This is in order to avoid comments regarding the existing links and the links that do not give the SV improvement. The detailed results are represented in Table 18.

Table 18. Case 1: SVs based on the established links in the NHH coauthorship network

LINK		Shapley Value		
			End node	
Start node	End node	Start node	before	after
31	108	0.611	1.617	1.917
31	70	0.639	1.992	2.228
31	109	0.714	1.843	2.025
31	26	0.823	1.733	1.876
31	68	0.942	1.587	1.706
31	122	1.125	1.583	1.658
31	25	1.222	1.496	1.593
31	65	1.354	1.478	1.554
31	137	1.544	1.417	1.458
31	9	1.68	1.371	1.43
31	73	1.816	1.35	1.403
31	30	1.936	1.208	1.262
31	78	2.098	1.158	1.192

CASE 2: LEADERSHIP-FORMATION

Following case 2 algorithm, node 31 has the smallest SV on the initial computational step. We do not approve the links with nodes 14, 70, 109 and 26. In contrast to case 2 we do not approve link 31-108, because both nodes should improve SVs, but SV(“node 108”) did not change. The first approved link is “node 31” – “node 122”, because ΔSV(“node 31”) = +0.033 and ΔSV(“node 122”) = +0.283. The results for the approved links are represented in Table 19.

Table 19. Case 1: SVs based on the established links in the NHH coauthorship network

LINK		Shapley Value		
			End node	
Start node	End node	Start node	before	after
31	122	0.644	1.583	1.867
31	70	0.672	1.992	2.228
31	109	0.747	1.843	2.025
31	26	0.857	1.733	1.876
31	108	1.000	1.617	1.726
31	68	1.125	1.587	1.688
31	25	1.593	1.496	1.593
31	65	1.354	1.478	1.554
31	137	1.544	1.417	1.458
31	9	1.680	1.371	1.430
31	73	1.816	1.350	1.403
31	30	1.936	1.208	1.262
31	78	2.098	1.158	1.192

According to Table 19, node 31 has achieved the leader's position with SV=2.098. The updated network's structure is represented in Appendix II.

It is important to notice that applying the Leadership-Formation algorithm we get the same SVs for both cases (i.e., Case 1 and Case 2) in the corresponding networks tested in this section. The only difference is the order of the links' approvals.

5. CONCLUSION

In this paper we analysed how to improve nodes' influential power in networks based on a game theoretic approach. We employed the Shapley value concept adapted by Aadithya et al. (2010) to measure a nodes' leadership in the domain of networks. We represented the leadership formation algorithm based on two different approaches of the link's approval. Four basic network topologies were analyzed: "point-to-point", "ring", "star", and mesh. We showed how the leadership position of the initially "weakest" node improves over the iterative structural modifications of different topological types. We also gave the illustrative examples of how the Shapley-based leadership formation algorithm works in two real networks. First, we applied our method to the classical example of the Florentine network of marriages. Based on our computations we represented the set of links that could potentially help the weakest family become the leader in the network. The second example is a modern network of coauthorship relations at NHH. Here we showed which relations (i.e., coauthorship) should be established by the weakest faculty member to become the most powerful in the network.
The given analysis is important in terms of large-scale networks where it is critically important to use calculation techniques with polynomial running time. The analysis of how the leadership formation proceeds in the basic network topologies is essential for understanding how influential power can be estimated and improved in different types of large-scale networks.

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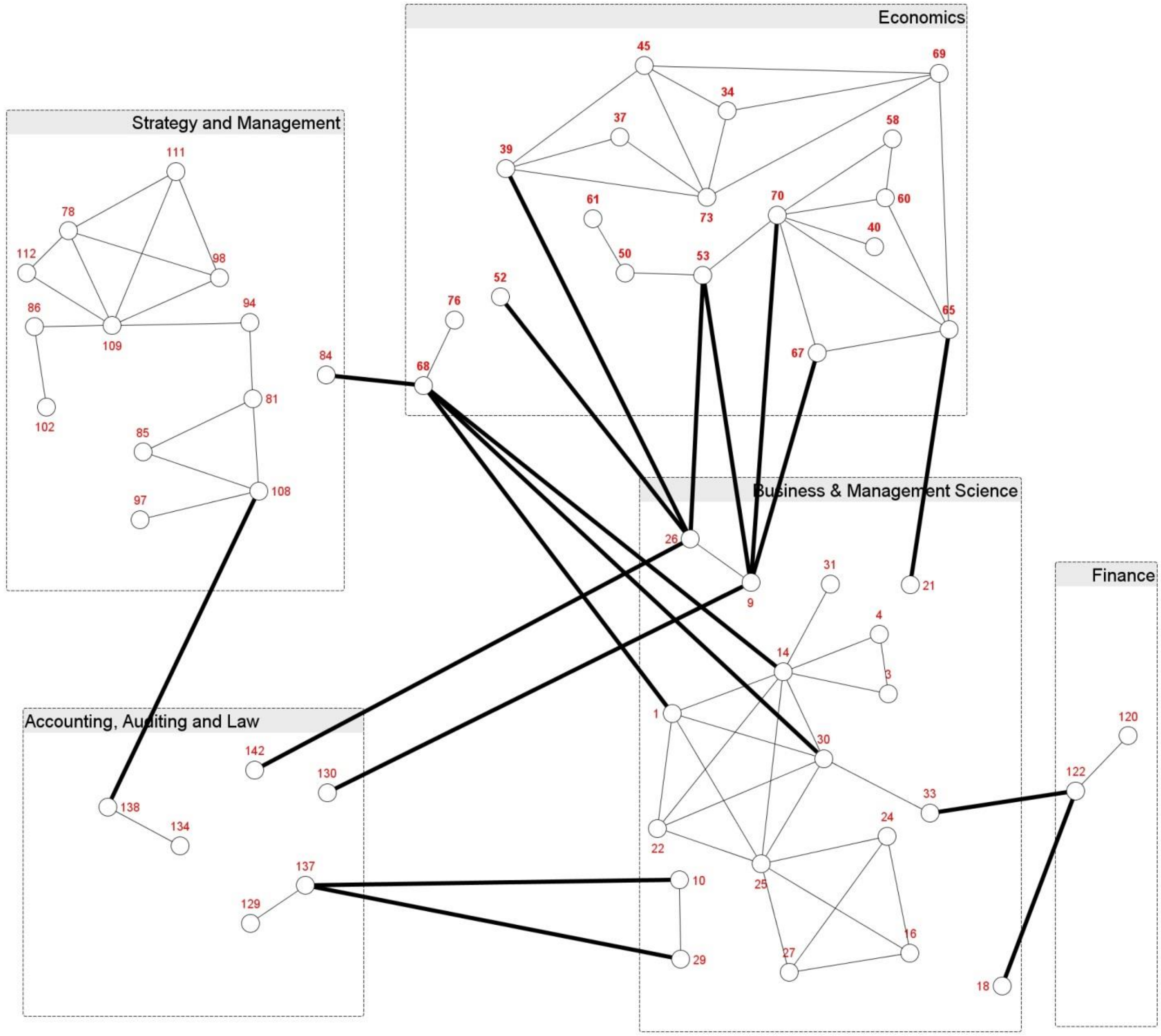
Appendix I

The initial NHH coauthorship network



Appendix II

The updated NHH coauthorship network