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Part 5. Finance and econometrics

STACKELBERG EQUILIBRIA IN A CONTINUOUS-TIME VERTICAL CONTRACTING MODEL WITH UNCERTAIN DEMAND AND DELAYED INFORMATION

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Abstract

We consider explicit formulae for equilibrium prices in a continuous-time vertical contracting model. A manufacturer sells goods to a retailer, and the objective of both parties is to maximize expected profits. Demand is an Itô–Lévy process, and to increase realism, information is delayed. We provide complete existence and uniqueness proofs for a series of special cases, including geometric Brownian motion and the Ornstein–Uhlenbeck process, both with time-variable coefficients. Moreover, explicit solution formulae are given, so these results are operational. An interesting finding is that information that is more precise may be a considerable disadvantage for the retailer.

Keywords: Vertical contracting; stochastic differential game; delayed information; Itô–Lévy process

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1. Introduction

In a news-vendor problem a retailer orders goods from a manufacturer. Demand is a random variable, and the retailer aims to find an order quantity that maximizes expected profit. In the single-period problem only one such order is made; the multi-period problem is concerned with a sequence of orders. In this paper we consider the news-vendor problem in continuous time, where the discrete order quantity is replaced by an ordering rate, i.e. the number of items ordered per unit of time. The single-period problem dates back to Edgeworth (1888). The basic problem is very simple but appears to have a neverending number of variations. There is now a very large literature on such problems; for further reading, we refer the reader to the survey papers by Cachón (2003) and Qin *et al.* (2011).

In our paper, a retailer and a manufacturer write contracts for a specific delivery rate following a decision process in which the manufacturer is the leader who initially decides the wholesale price. Based on that wholesale price, the retailer decides on the delivery rate. We assume a Stackelberg framework, and, hence, ignore cases where the retailer can negotiate the wholesale price. The contract is written at time $t - \delta$, and goods are received at time t. (It is essential to assume that information is delayed, for if there is no delay, the demand rate is known, and the retailer's order rate is made equal to the demand rate.) Stackelberg games of this type have been studied in Øksendal *et al.* (2013) from which we use Theorem 3.2.2 to provide explicit formulae below for commonly used stochastic processes, namely, geometric Brownian motion (extended to a geometric Lévy process) and the Ornstein–Uhlenbeck (OU) process.

Stackelberg games for single-period news-vendor problems have been studied extensively by Lariviere and Porteus (2001), providing quite general conditions under which unique equilibria can be found. Multi-period news-vendor problems with delayed information have been discussed in several papers. Bensoussan *et al.* (2009) used a discrete-time approach and generalized several information delay models. Computational issues were not explored in their paper, and they only considered decision problems for inventory managers, disregarding any game theoretical issues. Calzolari *et al.* (2011) discussed the filtering of stochastic systems with fixed delay, indicating that problems with delay lead to nontrivial numerical difficulties even when the driving process is Brownian motion. Kaplan (1970) is a classical paper in which stochastic lead times in a multi-period problem are considered. Several authors have contributed to the discussion of stochastic lead times; we mention Song and Zipkin (1996).

The geometric Lévy process is fundamental in many models in physics, biology, and finance, because it is a natural extension to the case with random coefficients of an exponential growth model, as follows. If the relative growth rate in an exponential model is assumed random and represented by the sum of continuous noise (generated by Brownian motion) and jump noise (generated by a pure jump Lévy process), we arrive at a geometric Lévy process. Such processes represent natural generalizations to jumps of the classical geometric Brownian motion, which were introduced in Samuelson (1965) and later applied in the famous Black–Scholes market model by F. Black, M. Scholes and R. Merton. Regarding financial motivations and justifications for using extensions of the geometric Brownian motion to jump models based on Lévy processes, we refer the reader to Barndorff-Nielsen (1998), Eberlein (2009), and the references therein.

The OU process is a widely used model for any stochastic phenomenon exhibiting mean reversion. It is the unique nontrivial stochastic process that is stationary, Markovian, and Gaussian (see, e.g. Maller *et al.* (2009)). It is used in financial engineering as a model for the term structure of interest rates (see, e.g. Vasicek (1977)), and via other variants or generalisations as a model of financial time series with applications to option pricing, portfolio optimization, and risk theory; see, e.g. Nicolato and Vernardos (2003), Barndorff-Nielsen and Shephard (2001), Maller *et al.* (2009), and the references therein. The OU process can be thought of as a continuous-time interpolation of an autoregressive process of order 1 (AR(1) process), i.e. the series obtained by sampling OU processes at equally spaced times are autoregressive of the same order.

The paper is organized as follows. In Section 2 we formulate and discuss a general continuous-time news-vendor problem. In Section 3 we consider the case where the demand rate is given by geometric Brownian motion and provide explicit solutions for the unique equilibria that occur in that case. The result in the constant coefficient case is quite startling as it leads to an equilibrium where the manufacturer offers a constant price w and the retailer orders a fixed fraction of the observed demand rate. In Section 4 we discuss non-Markov jump diffusions and demonstrate that knowledge of the state of the system at time t is not sufficient to infer the optimal order quantity. In Section 5 we provide explicit formulae for the unique equilibria that occur when demand is given by an OU process with time variable coefficients. We also compute numerical values to compare the dynamic approach with a static approach where both parties (wrongly) believe that the demand rate has a static distribution. An interesting finding is that information that is more precise can be a considerable disadvantage to the retailer. Finally, in Section 6 we offer some concluding remarks. To make the paper easier to read, most details of the proofs are deferred to Appendix A.

2. Continuous-time news-vendor problems

In this section we formulate a continuous-time news-vendor problem and use results in Øksendal *et al.* (2013) to describe explicitly a set of equations that we need to solve to find

Stackelberg equilibria. We assume that the demand rate for a good is given by an Itô–Lévy process starting from $D_0 = d_0 \in \mathbb{R}$ and otherwise of the form

$$\mathrm{d}D_t = \mu(t, D_t, \omega)\,\mathrm{d}t + \sigma(t, D_t, \omega)\,\mathrm{d}B_t + \int_{\mathbb{R}} \gamma(t, D_{t^-}, \xi, \omega)\,\widetilde{N}(\mathrm{d}t, \mathrm{d}\xi), \qquad t \in (0, T]. \tag{2.1}$$

Here B_t denotes a Brownian motion and $\widetilde{N}(\mathrm{d}t, \mathrm{d}\xi)$ is a compensated Poisson term. The coefficients μ, σ, and γ are assumed to satisfy standard conditions ensuring that (2.1) has a unique solution (see Øksendal and Sulem (2007)).

At time $t - \delta$ a retailer and a manufacturer negotiate a contract for items to be delivered at time t, where $\delta > 0$ is the delay time. The idea is that production takes time, and that the contract must be settled in advance. The manufacturer (leader) offers a wholesale price w_t per unit. On the basis of this wholesale price, the retailer (follower) chooses a delivery rate q_t. We assume that the retail price R is fixed. When the contract is written (at $t - \delta$), the demand at time t is unknown, so the contract must be based on information available at time $t - \delta$.

To formalize this setup, let $\mathcal{F}_t$ denote the σ-algebra generated by B_s and $\widetilde{N}(s, \mathrm{d}z)$, $0 \le s \le t$. Intuitively, $\mathcal{F}_t$ contains all the information up to time t. For the 'delayed' information, consider the σ-algebras $\mathcal{E}_t := \mathcal{F}_{t-\delta}$, $t \in [\delta, T]$. Both the retailer and the manufacturer are to base their actions on this delayed information. Technically, this means that q_t and w_t should be $\mathcal{E}_t$-measurable for every $t \ge 0$, i.e. q and w should be $\mathcal{E}$-predictable processes.

Assume that items can be salvaged at a unit price $S \ge 0$, and that items cannot be stored, i.e. they must be sold instantly or salvaged. Assuming that any sale occurs in the time period $\delta \le t \le T$, the retailer has an expected profit

$$J_2(w, q) = \mathbb{E}\biggl[\int_{\delta}^{T} [(R_t - S)\min\{D_t, q_t\} - (w_t - S)q_t]\,\mathrm{d}t\biggr]. \tag{2.2}$$

When the manufacturer has a constant production cost per unit M, his expected profit is

$$J_1(w, q) = \mathbb{E}\biggl[\int_{\delta}^{T} (w_t - M)q_t\,\mathrm{d}t\biggr]. \tag{2.3}$$

The profit functions (2.2) and (2.3) set up a stochastic Stackelberg game of the type studied in Øksendal *et al.* (2013).

2.1. Finding Stackelberg equilibria in the news-vendor model

It is well known that under conditions similar to our assumptions above, the discrete multi-period news-vendor model can be solved by an optimization pointwise in t. In a single-period news-vendor model with demand D, the retailer's order q_t must satisfy the equation

$$\mathbb{P}\{D \ge q\} = \frac{w - S}{R - S}. \tag{2.4}$$

If the demand process is Markov, it is reasonable to conjecture that the retailer at time $t - \delta$ should order a quantity corresponding to the distribution of D_t conditional on $\mathcal{E}_t$. Even when the process is non-Markovian, Theorem 3.2.2 of Øksendal *et al.* (2013), cited below, shows that, under reasonable conditions, the retailer's order that optimizes his expected profit is still the same conditional expectation.

Theorem 2.1. (Øksendal *et al.* (2013).) *Suppose that the pair $(\hat{w}, \hat{q})$ is a Stackelberg equilibrium for the news-vendor problem defined by (2.2) and (2.3). Assume that D_t as given by*

(2.1) has a continuous distribution. For any given w_t with $S < M \leq w_t \leq R$, let $q_t = \phi(w_t)$ denote the unique solution of

$$\mathbb{E}[(R-S)\,\mathbf{1}_{[0,D_t]}(q_t) - w_t + S \mid \mathcal{E}_t] = 0. \tag{2.5}$$

If the function

$$w_t \mapsto \mathbb{E}[(w_t - M)\phi(w_t) \mid \mathcal{E}_t] \tag{2.6}$$

has a unique maximum at $w_t = \hat{w}_t$ then $\hat{q}_t = \phi(\hat{w}_t)$.

Here $\mathbf{1}_{[0,D_t]}(q)$ denotes the indicator function for the interval $[0, D_t]$, i.e. a function that has the value 1 if $0 \leq q \leq D_t$ and 0 otherwise. To see why (2.5) always has a unique solution, note that w_t is $\mathcal{E}_t$-measurable and, hence, (2.5) is equivalent to

$$\mathbb{E}[\mathbf{1}_{[0,D_t]}(q_t) \mid \mathcal{E}_t] = \frac{w_t - S}{R - S}. \tag{2.7}$$

Existence and uniqueness of q_t then follow from monotonicity of the conditional expectation. Equation (2.7) is in fact the correct generalization of (2.4) to the continuous-time case. To avoid degenerate cases, we need to know that D_t has a continuous distribution. In the next sections we consider special cases where mostly we can write down explicit solutions to (2.5) and prove that (2.6) has a unique maximum.

Note that (2.5) is an equation defined in terms of the conditional expectation. Conditional statements of this type are in general difficult to compute: the challenge is then to state the result in terms of unconditional expectations.

3. Explicit formulae for geometric Brownian motion

In this section we offer explicit formulae for the equilibria that occur when the demand rate is given by a geometric Brownian motion. We first consider the case with constant coefficients, and then extend the results to the case with time-dependent, deterministic coefficients. We also discuss a non-Markovian case where demand is given by a geometric Lévy process.

3.1. Geometric Brownian motion with constant coefficients

In this section we assume that D_t is a geometric Brownian motion with constant coefficients, i.e. that

$$\mathrm{d}D_t = aD_t\,\mathrm{d}t + \sigma D_t\,\mathrm{d}B_t, \tag{3.1}$$

where a and σ are constants. Equation (3.1) has the explicit solution $D_t = D_0 \exp((a - \frac{1}{2}\sigma^2)t + \sigma B_t)$, and it is easy to see that

$$D_t = D_{t-\delta}\,\exp\big((a - \tfrac{1}{2}\sigma^2)\delta + \sigma(B_t - B_{t-\delta})\big). \tag{3.2}$$

The explicit form of (3.2) makes it possible to write down a closed-form solution to (2.5) as in the proposition below (see Appendix A for the proof).

Proposition 3.1. *Let $\Phi\colon [M, R] \mapsto \mathbb{R}$ denote the function*

$$\Phi(w) = \exp\bigg[\bigg(a - \frac{1}{2}\sigma^2\bigg)\delta + \sqrt{\delta\sigma^2}\,G^{-1}\bigg(1 - \frac{w - S}{R - S}\bigg)\bigg], \tag{3.3}$$

where G^{-1} denotes the inverse of the standard normal distribution, and define the function $\Psi\colon [M, R] \mapsto \mathbb{R}$ by $\Psi(w) = (w - M)\Phi(w)$. The function Ψ has a unique maximum with *a strictly positive function value. At time $t-\delta$ the retailer should observe $y = D_{t-\delta}$, and a unique Stackelberg equilibrium occurs for*

$$w_t^* = \arg\max\{\Psi\}, \qquad q_t^* = y\,\Phi(\arg\max\{\Psi\}).$$

The equilibria resulting from this situation are quite surprising. We see that the wholesale price is in fact constant. Consequently, the manufacturer need not observe demand at time $t-\delta$ to settle the price. In fact, she can write a contract with set wholesale price for the whole sales period. The retailer needs to observe demand, but his strategy is very simple: observe demand D and order a fixed fraction of D.

As is clear from the proof, these properties originate from the multiplicative scaling of geometric Brownian motion, i.e. if the initial condition is scaled by a multiplicative factor, any sample path is scaled by the same factor. Critical fractiles are scaled accordingly, and as a consequence the optimal wholesale price will not change. It is the same type of effect driving the classical Merton (1969) portfolio problem in finance: if the risky asset is a constant coefficient geometric Brownian motion, the optimal policy is to keep a fixed fraction in the risky asset.

3.2. Geometric Brownian motion with variable coefficients

Assume now that D_t is a geometric Brownian motion with variable deterministic coefficients, i.e. that

$$\mathrm{d}D_t = a(t)D_t\,\mathrm{d}t + \sigma(t)D_t\,\mathrm{d}B_t,$$

where $a(t)$ and $\sigma(t)$ are given deterministic functions. Then the following holds (see Appendix A for the proof).

Proposition 3.2. *For $t \in [\delta, T]$, let $\Phi_t\colon [M, R] \mapsto \mathbb{R}$ denote the function*

$$\Phi_t(w) = \exp\left[\hat{a}(t) + \hat{\sigma}(t)\,G^{-1}\left(1 - \frac{w-S}{R-S}\right)\right],$$

where

$$\hat{a}(t) = \int_{t-\delta}^{t}\left[a(s) - \frac{1}{2}\sigma^2(s)\right]\mathrm{d}s, \qquad \hat{\sigma}(s) = \sqrt{\int_{t-\delta}^{t}\sigma^2(s)\,\mathrm{d}s},$$

and define the function $\Psi_t\colon [M, R] \mapsto \mathbb{R}$ by $\Psi_t(w) = (w-M)\Phi_t(w)$. The function Ψ_t has a unique maximum with a strictly positive function value. At time $t-\delta$ the retailer should observe $y = D_{t-\delta}$, and a unique Stackelberg equilibrium occurs when

$$w_t^* = \arg\max\{\Psi_t\}, \qquad q_t^* = y\,\Phi_t(\arg\max\{\Psi_t\}).$$

Comparison with the case of constant coefficients shows that the wholesale price w is no longer constant. Nevertheless, we see that the equilibria are defined in terms of two deterministic functions $\arg\max\{\Psi_t\}$ and $\Phi_t(\arg\max\{\Psi_t\})$. As in the constant coefficient case, the manufacturer need not observe demand, but can settle wholesale prices upfront for the whole sales period.

4. Geometric Lévy processes

In this section we compute explicit Stackelberg equilibria in cases where the demand is given by a non-Markovian process. Consider first the case where demand is given by

$$\mathrm{d}D_t = (\alpha_1 + \alpha_2 B_t)D_t\,\mathrm{d}t + \sigma D_t\,\mathrm{d}B_t, \tag{4.1}$$

where α_1, α_2, and σ are constants. Solving (4.1) leads to

$$D_t = D_{t-\delta} \exp\left[-\frac{1}{2}\sigma^2\delta + \sigma(B_t - B_{t-\delta}) + \int_{t-\delta}^{t} (\alpha_1 + \alpha_2 B_s)\,\mathrm{d}s\right].$$

An additional difficulty arises here because the term $\int_{t-\delta}^{t} \alpha_2 B_s\,\mathrm{d}s$ is not independent of $\mathcal{E}_t$, reflecting the non-Markovian nature of the process. To compute the conditional expectation, we need to rewrite the expression. Integration by parts gives

$$D_t = D_{t-\delta}\,\mathrm{e}^{\delta(\alpha_1+\alpha_2 B_{t-\delta})} \exp\left[-\frac{1}{2}\sigma^2\delta + \int_{t-\delta}^{t} (\alpha_2(t-s)+\sigma)\,\mathrm{d}B_s\right].$$

This separates the expression into a product where the first factor is $\mathcal{E}_t$-measurable, while the second factor is log-normal and independent of $\mathcal{E}_t$. Using the same separation technique as before, it is then straightforward to find an explicit solution to (2.5), and existence and uniqueness of the corresponding Stackelberg problem follow as in the proof of Proposition 3.1. This technique is in fact applicable to quite general processes. A geometric Lévy process is a solution of a stochastic differential equation of the form

$$\mathrm{d}D(t) = D(t-)\left(a(t,\omega)\,\mathrm{d}t + \sigma(t,\omega)\,\mathrm{d}B_t + \int_{\mathbb{R}} \gamma(t,z,\omega)\widetilde{N}(\mathrm{d}t,\mathrm{d}z)\right). \tag{4.2}$$

If we assume that $D(0) = D_0 > 0$ and $\gamma(t,z) > -1$, the solution satisfies $D_t \geq 0$ for all t. The explicit solution of (4.2) is

$$\begin{aligned} D_t = D_0 \exp\bigg[\int_0^t \bigg(a(s,\omega) - \frac{1}{2}\sigma^2(s,\omega) + \int_{\mathbb{R}_0} [\log[1+\gamma(s,z,\omega)] - \gamma(s,z,\omega)]\,\nu(\mathrm{d}z)\bigg)\,\mathrm{d}s \\ + \int_0^t \sigma(s,\omega)\,\mathrm{d}B_s + \int_0^t\int_{\mathbb{R}_0} \log[1+\gamma(s,z,\omega)]\,\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\bigg]. \end{aligned} \tag{4.3}$$

Now make the additional assumption that

$$a(s,\omega) = \alpha_1(s) + \alpha_2(s)B_s(\omega), \qquad \sigma(s,\omega) = \sigma(s), \qquad \gamma(s,z,\omega) = \gamma(s,z), \tag{4.4}$$

where σ and γ are given deterministic functions, while the growth rate $a(s,\omega)$ depends on ω as well as t, and α_1 and α_2 are given deterministic functions. For each fixed t, consider the random variable X_t defined by

$$\begin{aligned} X_t = \exp\bigg[\int_{t-\delta}^t \bigg(\int_s^t \alpha_2(u)\,\mathrm{d}u + \sigma(s)\bigg)\,\mathrm{d}B_s \\ + \int_{t-\delta}^t \bigg(-\frac{1}{2}\sigma^2(s) + \int_{\mathbb{R}_0} [\log[1+\gamma(s,z)] - \gamma(s,z)]\,\nu(\mathrm{d}z)\bigg)\,\mathrm{d}s \\ + \int_{t-\delta}^t \int_{\mathbb{R}_0} \log[1+\gamma(s,z)]\,\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\bigg]. \end{aligned} \tag{4.5}$$

We can then state the following proposition (see Appendix A for the proof).

Proposition 4.1. *Assume that demand D_t is a geometric Lévy process given by (4.2), where the coefficients satisfy (4.4). Let F_t denote the cumulative distribution of X_t given by (4.5),* *and, for each fixed t, let F_t^{-1} denote the inverse function of F_t. For each $t \in [\delta, T]$, define the functions*

$$\Phi_t(w) = F_t^{-1}\left(1 - \frac{w - S}{R - S}\right), \qquad \Psi_t(w) = (w - M)\Phi_t(w). \tag{4.6}$$

At time $t - \delta$ the retailer should observe both the demand rate $y = D_{t-\delta}$ and $z = B_{t-\delta}$, and a Stackelberg equilibrium occurs when

$$w_t^* = \arg\max\{\Psi_t\}, \qquad q_t^* = y \, \exp\left[\int_{t-\delta}^{t} (1 + z)\alpha_1(s) \mathrm{d}s\right] \Phi_t(\arg\max\{\Psi_t\}). \tag{4.7}$$

Note that the value of z can be found from the growth rate $\alpha_1(t-\delta) + \alpha_2(t-\delta)B_{t-\delta}$. If α_1 and α_2 are constants then the factor $\exp[\int_{t-\delta}^{t}(1+z)\alpha_1(s)\,\mathrm{d}s]$ is the correction to be expected if the growth rate is to remain constant at the level observed at time $t - \delta$. Note that the structure of the solution is quite similar to the case covered in Proposition 3.2. The manufacturer has a pricing strategy defined in terms of a deterministic function. The retailer should observe the demand rate, adjust it by the observed growth rate, and multiply the adjusted number by a deterministic fraction.

5. The OU process

In this section we discuss equilibrium prices for the OU process. We extend the results from Proposition 4.1.1 of Øksendal *et al.* (2013) to the case with time-variable coefficients, and also report the results of a numerical experiment where we compare the performances of static versus dynamic pricing strategies.

5.1. Explicit formulae for the OU process

In this section we offer explicit formulae for the equilibria that occur when the demand rate is given by an OU process, extending our earlier work from the constant coefficient case to the case

$$\mathrm{d}D_t = a(t)(\mu(t) - D_t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t,$$

where $a(t)$, $\mu(t)$, and $\sigma(t)$ are given deterministic functions. The increased flexibility is important in applications since it allows for scenarios where the mean reversion level μ can have a time-variable trend. The basic result can be summarized as follows (see Appendix A for the proof).

Proposition 5.1. *For each $t \in [\delta, T]$ and $y \in \mathbb{R}$, let $\Phi_{t,y} \colon [M, R] \mapsto \mathbb{R}$ denote the function*

$$\Phi_{t,y}(w) = y\,\mathrm{e}^{-A_{t-\delta,t}} + \hat{\mu}(t) + \hat{\sigma}(t) G^{-1}\left(1 - \frac{w - S}{R - S}\right), \tag{5.1}$$

where, for $s < t$, $A_{st} = \int_s^t a(u)\,\mathrm{d}u$ and

$$\hat{\mu}(t) = \int_{t-\delta}^{t} a(s)\mu(s)\mathrm{e}^{-A_{st}}\,\mathrm{d}s, \qquad \hat{\sigma}(t) = \sqrt{\int_{t-\delta}^{t} \sigma^2(s)\mathrm{e}^{-2A_{st}}\,\mathrm{d}s}, \tag{5.2}$$

and define the function $\Psi_{t,y} \colon [M, R] \mapsto \mathbb{R}$ by $\Psi_{t,y}(w) = (w - M)\Phi_{t,y}(w)$. If $\Phi_{t,y}(M) > 0$, the function $\Psi_{t,y}$ has a unique maximum where it is strictly positive. At time $t - \delta$ the parties should observe $y = D_{t-\delta}$, and a unique Stackelberg equilibrium occurs at

$$w_t^* = \begin{cases} \arg\max\{\Psi_{t,y}\} & \text{if } \Phi_{t,y}(M) > 0, \\ M & \text{otherwise,} \end{cases} \qquad q_t^* = \begin{cases} \Phi_{t,y}(\arg\max\{\Psi_{t,y}\}) & \text{if } \Phi_{t,y}(M) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remarks. (a) The condition $\Phi_{t,y}(M) > 0$ has an obvious interpretation. The manufacturer cannot offer a wholesale price w lower than the production cost M. If $\Phi_{t,y}(M) \leq 0$, it means that the retailer is unable to make a positive expected profit even at the lowest wholesale price the manufacturer can offer. When that occurs, the retailer's best strategy is to order $q = 0$ units. When the retailer orders $q = 0$ units, the choice of w is arbitrary. However, the choice $w = M$ is the only strategy that is increasing and continuous in y.

(b) Note that the structure of the equilibria is quite different from the case with geometric Brownian motion. Contrary to that case, the manufacturer now needs to observe the market to compute wholesale prices.

5.2. Numerical examples for the OU process

In this section we compare the performance of the dynamic approach with a scenario where the retailer believes that demand has a constant distribution D. A constant coefficient OU process

$$D_t = D_0\,\mathrm{e}^{-at} + \mu(1 - \mathrm{e}^{-at}) + \sigma\mathrm{e}^{-at}\int_0^t \mathrm{e}^{as}\,\mathrm{d}B_s$$

is ergodic in the sense that observations along any sample path approach the normal distribution $N(\mu, \sigma^2/(2a))$. Assuming that the retailer believes the demand rate has a static distribution D and that he has observed that demand rate for long enough prior to ordering, he will conclude that D is $N(\mu, \sigma^2/(2a))$. If the manufacturer knows that the retailer will order according to a static $N(\mu, \sigma^2/(2a))$ distribution, a fixed value for w can be computed so as to optimize the expected profit.

To examine dynamic and static approaches, we sampled paths of the OU process using the parameters $\mu = 100$, $\sigma = 12$, $a = 0.05$, and $D_0 = 100$. We then computed the accumulated profits

$$\int_\delta^T [(R - S)\min\{D_t, q_t\} - (w_t - S)q_t]\,\mathrm{d}t, \qquad \int_\delta^T (w_t - M)q_t\,\mathrm{d}t$$

for three different values of δ using the values $R = 10$, $M = 2$, $S = 1$, and $T = 100 + \delta$, in conjunction with the following four different strategies:

(i) static approach as defined above;

(ii) dynamic approach as defined by Proposition 5.1;

(iii) static cooperative approach using $w_t = M$;

(iv) dynamic cooperative approach using $w_t = M$.

Adding the expected profits in (2.2) and (2.3), it is easily seen that, when $w_t = M$ (in which case the manufacturer has zero profit), the optimal policy for the retailer also maximizes the expected profit for the supply chain. The order quantity in the dynamic cooperative case is then found from (2.7) with $w_t = M$, leading to $q_t^* = \Phi_{t,y}(M)$, where $\Phi_{t,y}$ is given by (5.1). The static cooperative case is equivalent to a single-period news-vendor problem, leading to a constant order rate.

We assume that sales take place in the time intervals $[\delta, 100 + \delta]$. The length of the sales period is then 100 regardless of the value of δ. This makes it easier to compare performances using different values of δ. The results averaged over 1000 sample paths are reported in Table 1. It can be seen from this table that the dynamic approach favours the manufacturer, with it being more favourable the shorter the delay. At $\delta = 30$, the effect of the dynamic approach is close to being wiped out. The same conclusions apply to the supply chain, i.e. a dynamic approach offers

TABLE 1: Average profits (1000 realizations) under various strategies.

Strategy	Manufacturer	Retailer	Supply chain
	Delay $\delta = 1$		
Static approach	42 830	12 729	55 559
Dynamic approach	61 356	4073	65 429
Static cooperation	—	—	73 251
Dynamic cooperation	—	—	77 766
	Delay $\delta = 7$		
Static approach	42 830	12 457	55 286
Dynamic approach	48 592	9438	58 030
Static cooperation	—	—	73 029
Dynamic cooperation	—	—	74 838
	Delay $\delta = 30$		
Static approach	42 830	12 074	54 903
Dynamic approach	43 225	11 882	55 106
Static cooperation	—	—	72 648
Dynamic cooperation	—	—	72 794

improved profits and the improvement is larger when the delay is shorter. Note, however, that the retailer has a distinct disadvantage under the dynamic approach, and that this disadvantage is larger the shorter the delay.

In a cooperative setting, a dynamic approach can reward both the retailer and the manufacturer. Profits can be shared, which leads to an improved position for both parties. In a noncooperative equilibrium, more precise information can be to the disadvantage of the retailer. This is due to the Stackelberg structure of the game. With more precise information, the leader has more control and can take a larger share of the profits. In the limit $\delta \to 0$, the leader is in full control. The retailer will then order the observed demand rate regardless of the price. The manufacturer offers a price marginally close to R taking all profit in the limit. See also Taylor and Xiao (2010) for an interesting discussion of the single-period case.

6. Concluding remarks

We have provided explicit formulae for equilibrium prices in a continuous-time news-vendor model. Complete existence and uniqueness results have been stated for widely used processes like geometric Brownian motion and the OU process, both with time-variable coefficients. We have also outlined how to obtain explicit expressions when demand is given by a geometric Lévy process with time-variable, deterministic coefficients, including cases with random coefficients. To our knowledge, path properties of this kind have not previously been discussed in the literature.

Of particular interest is the structure of the equilibria for a geometric Brownian motion with constant coefficients. In this case the manufacturer offers a fixed wholesale price, while the retailer orders a fixed fraction of the observed demand. This result is clearly a parallel to Merton's classical result on optimal investment in a risky and a secure asset, where the optimal policy is to keep a fixed fraction in the risky asset.

From an applied point of view, we believe that the numerical results in Section 5.2 are of general interest. We demonstrate that the retailer suffers a distinct disadvantage from having more information, and that this disadvantage is bigger the more precise the information. Such issues may have important political implications, in particular in electricity markets, and we believe that our model offers new insights into the mechanisms governing equilibria in such markets.

Appendix A

In this appendix we provide proofs for all unproved statements given in Sections 3, 4, and 5. We start with a nontrivial estimate for the standard normal distribution G in the function

$$h_m(z) := z(1 - G(z)) - G'(z) - mz, \qquad z \in \mathbb{R}, \tag{A.1}$$

where $0 \le m \le 1$.

Lemma A.1. *The function $h_m(z)$ is negative for all finite $z \in \mathbb{R}$.*

This property, given in Lemma 4.1.2 of Øksendal *et al.* (2013), is crucial to our proof of the uniqueness of the maxima in our results. It can be proved by showing that $0 > h_0(z) > h_0(0)$ for $z > 0$ because, for such z, $h_1(-z) = h_0(z) > h_m(z)$, and $h_1(z) < h_m(z)$ for $z < 0$. For $z > 0$, $\sqrt{2\pi}[1 - G(z)] = \int_z^\infty \mathrm{e}^{-x^2/2}\,\mathrm{d}x < \int_z^\infty (x/z)\mathrm{e}^{-x^2/2}\,\mathrm{d}x = \sqrt{2\pi}\,G'(z)/z$, whence $h_0(z) < 0$ as required.

Proof of Proposition 3.1. We readily see from (3.2) that $q_t \le D_t$ is equivalent to the inequality

$$\ln\biggl(\frac{q_t}{D_{t-\delta}}\biggr) - \biggl(a - \frac{1}{2}\sigma^2\biggr)\delta \le \sigma(B_t - B_{t-\delta}).$$

The left-hand side here is $\mathcal{E}_t$-measurable, while the right-hand side is normally distributed and independent of $\mathcal{E}_t$. It is then straightforward to prove that

$$\mathbb{E}[\mathbf{1}_{[0,D_t]}(q_t) \mid \mathcal{E}_t] = 1 - G\biggl(\frac{\ln(q_t/D_{t-\delta}) - (a - \sigma^2/2)\delta}{\sqrt{\sigma^2\delta}}\biggr).$$

Hence, it follows from (2.7) that

$$q_t = D_{t-\delta}\exp\biggl[\biggl(a - \frac{1}{2}\sigma^2\biggr)\delta + \sqrt{\delta\sigma^2}\,G^{-1}\biggl(1 - \frac{w - S}{R - S}\biggr)\biggr] =: D_{t-\delta}\,\Phi(w),$$

where Φ as here is also given by (3.3). With this order quantity q_t, the expected profit for the manufacturer is

$$\mathbb{E}[D_{t-\delta}(w_t - M)\Phi(w_t)] =: \mathbb{E}[D_{t-\delta}\Psi(w_t)],$$

where Ψ as here is also given below (3.3). In general, w_t can be a random variable. If $w^* = \arg\max\{\Psi\}$ then, by definition and the nonnegativity of Ψ,

$$\mathbb{E}[D_{t-\delta}\Psi(w_t)] \le \mathbb{E}[D_{t-\delta}]\,\Psi(w^*),$$

with equality if $w_t = w^*$. Therefore, w^* is optimal.

It remains to prove that $\arg\max\{\Psi\}$ is unique. Putting $b = \sqrt{\delta\sigma^2} > 0$, it follows that Ψ is proportional to a function of the form

$$w \mapsto (w - M)\exp\biggl[b\,G^{-1}\biggl(1 - \frac{w - S}{R - S}\biggr)\biggr]. \tag{A.2}$$

The mapping $w \mapsto z$ defined by $w = R - (R - S)G(z)$ is one–one and monotone, with $-\infty < z < G^{-1}(1 - m)$ and $m := (M - S)/(R - S)$ for which $0 < m < 1$. Substitution in the right-hand side of (A.2) shows that Ψ is proportional to

$$(R - S)(1 - m - G(z))\mathrm{e}^{bz} =: (R - S) f_m(z).$$

Here $R > S$, and the nonnegative function f_m is twice differentiable on its range $Z := (-\infty, G^{-1}(1 - m))$. Since $f_m(z) \to 0$ at both ends of Z, and $f'_m(G^{-1}(1 - m)-) < 0$, f_m has a strictly positive maximum on Z, at z_0 say, with $f'_m(z_0) = 0$. Note that

$$f'_m(z) = -G'(z)\mathrm{e}^{bz} + (1 - m - G(z))b\mathrm{e}^{bz} < (b - z)(1 - m - G(z))\mathrm{e}^{bz}, \tag{A.3}$$

the inequality coming from Lemma A.1 below, while using the fact that $G''(z) = -z\,G'(z)$ simplifies the expression for f''_m to

$$f''_m(z) = (z - b)G'(z)\mathrm{e}^{bz}. \tag{A.4}$$

Now let z_1 satisfy $f'_m(z_1) = 0$. Then, by (A.3) and the positivity of $1 - m - G(z)$ on Z, we must have $b > z_1$, and then, by (A.4), z_1 must be a local maximum. If $f'_m(z'_1) = 0$ for some $z'_1 \neq z_1$ then between these two zeros of f'_m, both local maxima, there must be a local minimum, at z_2 say, and differentiability then implies that $f'_m(z_2) = 0$, which in turn implies that z_2 is a local maximum so we have a contradiction. Thus, $z_1 = z_0$ is the unique local (and global) maximum of f'_m on Z.

It follows from Theorem 2.1 that w^* is the only candidate for a Stackelberg equilibrium. To see that this candidate is indeed a Stackelberg equilibrium, we argue as follows. Since w^* equals $\arg\max\{\Psi\}$, any w_t other than $\arg\max\{\Psi_{D_{t-\delta}}\}$ must lead to strictly lower expected profit at time t. As demand does not depend on w_t, no lower expected profit at any time can be compensated by any higher expected profit later on. Hence, if the statement $w_t = \arg\max\{\Psi_{D_{t-\delta}}\}$ almost surely $\lambda \times \mathbb{P}$ (λ denotes the Lebesgue measure on $[0, T]$) is false, any such strategy must lead to strictly lower expected profit. The same argument applies for the retailer, and, hence, a unique Stackelberg equilibrium always exists in this case.

Proof of Proposition 3.2. In the case of variable coefficients,

$$D_t = D_{t-\delta} \exp\biggl[\int_{t-\delta}^{t}\biggl[\mu(s) - \frac{1}{2}\sigma^2(s)\biggr]\mathrm{d}s + \int_{t-\delta}^{t}\sigma(s)\,\mathrm{d}B_s\biggr]. \tag{A.5}$$

Put

$$\hat{\mu}(t) = \int_{t-\delta}^{t}\bigl[\mu(s) - \tfrac{1}{2}\sigma^2(s)\bigr]\mathrm{d}s, \qquad \hat{\sigma}^2(s) = \int_{t-\delta}^{t}\sigma^2(s)\,\mathrm{d}s.$$

Because the exponent in (A.5) is normally distributed and independent of $\mathcal{E}_t$,

$$\mathbb{E}[\mathbf{1}_{[0,D_t]}(q_t) \mid \mathcal{E}_t] = 1 - G\biggl(\frac{\ln(q_t/D_{t-\delta}) - \hat{\mu}(t)}{\hat{\sigma}(t)}\biggr).$$

It follows from (2.7) that

$$q_t = D_{t-\delta} \exp\biggl[\hat{\mu}(t) + \hat{\sigma}(t)\,G^{-1}\biggl(1 - \frac{w - S}{R - S}\biggr)\biggr].$$

With this order quantity, the expected profit for the manufacturer is

$$\mathbb{E}\left[D_{t-\delta}\,(w_t - M)\exp\left[\hat{\mu}(t) + \hat{\sigma}(t)\,G^{-1}\left(1 - \frac{w-S}{R-S}\right)\right]\right].$$

The calculations in the proof of Proposition 3.1 can now be repeated line by line for each fixed t, proving the general case in Proposition 3.2.

Proof of Proposition 4.1. From (4.3) and (4.4), it follows that

$$\begin{aligned} D_t = D_{t-\delta}\exp\bigg[\int_{t-\delta}^{t}\bigg(&\alpha_1(s) + \alpha_2(s)B_s(\omega) - \frac{1}{2}\sigma^2(s) \\ &+ \int_{\mathbb{R}_0}[\log[1+\gamma(s,z)] - \gamma(s,z)]\,\nu(\mathrm{d}z)\bigg)\,\mathrm{d}s \\ &+ \int_{t-\delta}^{t}\sigma(s)\,\mathrm{d}B_s + \int_{t-\delta}^{t}\int_{\mathbb{R}_0}\log[1+\gamma(s,z)]\,\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\bigg]. \end{aligned}$$

In this expression the term $\exp[\int_{t-\delta}^{t}[\alpha_1(s) + \alpha_2(s)B_s(\omega)]\,\mathrm{d}s]$ is usually *not* independent of $\mathcal{E}_t$. Changing the order of integration we see that

$$\begin{aligned} &\exp\left[\int_{t-\delta}^{t}[\alpha_1(s) + \alpha_2(s)B_s(\omega)]\,\mathrm{d}s\right] \\ &\quad = \exp\left[\int_{t-\delta}^{t}[\alpha_1(s) + \alpha_2(s)B_{t-\delta}]\,\mathrm{d}s + \int_{t-\delta}^{t}\int_{s}^{t}\alpha_2(u)\,\mathrm{d}u\,\mathrm{d}B_s\right], \end{aligned}$$

from which it follows that

$$D_t = D_{t-\delta}\,\exp\left[\int_{t-\delta}^{t}[\alpha_1(s) + \alpha_2(s)B_{t-\delta}]\,\mathrm{d}s\right]X_t,$$

where X_t is given by (4.5). Here the first two terms are $\mathcal{E}_t$-measurable, while the term X_t is independent of $\mathcal{E}_t$. It is then straightforward to see that (4.6) and (4.7) follow from (2.7).

Proof of Proposition 5.1. The statement $q_t \le D_t$ is equivalent to the inequality

$$q_t - \left(D_{t-\delta}\mathrm{e}^{-A_{t-\delta,t}} + \int_{t-\delta}^{t} a(s)\mu(s)\mathrm{e}^{-A_{st}}\mathrm{d}s\right) \le \int_{t-\delta}^{t}\sigma(s)\mathrm{e}^{-A_{st}}\,\mathrm{d}B_s.$$

Using the Itô isometry, we see that the right-hand side has expected value 0 and variance $[\hat{\sigma}(t)]^2 := \int_{t-\delta}^{t}\sigma^2(s)\mathrm{e}^{-2A_{st}}\,\mathrm{d}s$ (cf. (5.2)). It is then straightforward to see that

$$\mathbb{E}[\mathbf{1}_{[0,D_t]}(q_t) \mid \mathcal{E}_t] = 1 - G\left(\frac{q_t}{\hat{\sigma}(t)} - \hat{y}\right),$$

where

$$\hat{y} = \frac{D_{t-\delta}\mathrm{e}^{-A_{t-\delta,t}} + \int_{t-\delta}^{t} a(s)\mu(s)\mathrm{e}^{-A_{st}}\,\mathrm{d}s}{\hat{\sigma}(t)},$$

and (5.1) follows trivially from (2.7).

It remains to prove that the function $\Psi_{t,y}$ has a unique maximum if $\Phi_{t,y}(M) > 0$. Observe that $\Psi_{t,y}$ is proportional to

$$(w - M)\left[\hat{y} + G^{-1}\left(1 - \frac{w - S}{R - S}\right)\right].$$

Make the same substitutions as below (A.2), namely, $w = R - (R - S)G(z)$ and $m = (M - S)/(R - S)$ so $m \in (0, 1)$. With these changes we see that $\Psi_{t,y}$ is proportional to

$$(1 - m - G(z))(\hat{y} + z) =: f_m(z).$$

The condition $\Phi_{t,y}(M) > 0$ is equivalent to $\hat{y} + G^{-1}(1 - m) > 0$, and the condition $w \geq M$ is equivalent to $z \leq G^{-1}(1 - m)$. For fixed m and $\hat{y} \in \mathbb{R}$, consider the function $f_m(z)$ on the interval $Z := \{z\colon -\hat{y} \leq z \leq G^{-1}(1 - m)\}$. If $\hat{y} + G^{-1}(1 - m) > 0$, Z is nondegenerate and nonempty, and

$$f'_m(z) = -G'(z)(\hat{y} + z) + (1 - m - G(z)).$$

Since $f'_m(-\hat{y}) > 0$ and $f_m(-\hat{y}) = f_m(G^{-1}(1 - m)) = 0$, the function f_m must have at least one strictly positive maximum on Z.

We can now mimic the argument around (A.3) and (A.4) in the proof of Proposition 3.1 to prove that the maximum of f_m is unique. It follows from Theorem 2.1 that this is the only candidate for a Stackelberg equilibrium. Similarly, the argument used at the end of the proof of Proposition 3.1 shows that a unique Stackelberg equilibrium exists in this case also.

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