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## Discussion paper

# Beyond the local mean-variance analysis in continuous time. The problem of non-normality

BY

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# Beyond the local mean-variance analysis in continuous time: The problem of non-normality.

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## Abstract

The paper investigates the effects of deviations from normality on the estimates of risk premiums and the real equilibrium, short-term interest rate in the conventional rational expectations equilibrium model of Lucas (1978). We consider a time-continuous approach, where both the aggregate consumption process as well as cumulative dividends from risky assets are assumed to be jump-diffusion processes. This approach allows for random jumps in the fundamental underlying processes at random time points. Preferences are time separable and additive. We derive testable expressions for these quantities, and confront these with 20. century sample estimates. Since there are non-linear components in the formulas for the risk premiums and the interest rate, we can readily explore what effect deviation from normality has on these quantities. Our results test the boundaries of the conventional model.

*KEYWORDS: Mean-variance analysis, Consumption based CAPM, Equilibrium real interest rate, The equity premium puzzle, jump-diffusions, Bivariate Normal Inverse Gaussian distribution.*

## Introduction

The paper develops an expression for the difference between average equity and debt returns, and an expression for average real debt return in equilibrium, using a dynamic model in continuous time containing jumps. These

expressions contain nonlinearities, which depart from the continuous type, as well as discrete time analysis. Thus the whole joint probability distribution of the growth rate of aggregate consumption and the return rate of a risky asset is required to calculate these quantities, not only the two first moments.

In this framework the standard approach would be equivalent to an assumption about joint normality between consumption and return rates. This also give different results from a mere mean-variance analysis, due to the nonlinearities. As is well known, speculative prices deviate from normality. Models based on the Pareto distribution were advocated by Mandelbrot (1963), who referred to much earlier works of P. Levy. A clear disadvantage with the models used by Mandelbrot is that there is no closed form for the probability density, and second order moments do not exist. Also, at the time there was not developed statistical tools for this kind of distributions. In order to take into account some stylized facts about asset prices, such as heavy tails and skewness, we consider a joint Normal Inverse Gaussian (NIG)-distribution for the simultaneous jump sizes in question. The combination of this type of distributional assumption with jump dynamics adds further insights to economic modeling under uncertainty. Advantages with this distribution are several: A joint probability density exists, variances and covariances exist, a moment generating function is available and statistical tools exist. The basic analytic framework builds on Aase (1993a-b).

A pure diffusion model is, for example, driven by the Brownian motion, a Gaussian stochastic process, and can be characterized by its first two "local" moment (rates). By Ito's lemma, all other processes, endogenous or exogenous, are of this type as well. The cause of the classical mean-variance dependence in the one period setting stems from either assumptions on the preferences, e.g., quadratic utility, or assumptions about return distributions, e.g., joint normality.

Recalling the related discussion between Borch, Feldstein and Tobin in 1969, Borch, for example, simply pointed out that the probability distribution of a random variable generally depends on more than just its two first moments. Similar remarks were made by Feldstein, and both authors illustrated possible shortcomings from restricting attention to only the two first moments in individual decision making under uncertainty.

Here we recall that it is a consequence of Carleman's Theorem (see e.g., Anderson (1958)) that, even in the case where a probability distribution has moments of all orders, knowledge of these moments is, in general, not enough to determine the entire probability distribution itself.

As an application of our approach we attempt to fit equity premiums and average debt returns derived from the model, to consumption and equity data of the 20. century used in the Mehra and Prescott (1985)-study.

Related work, using entirely different methods, include Hansen and Jagannathan (1991), Rietz (1988), Salyer (1998) and Barro (2006).

We work with a time additive and separable set of utility functions, the conventional assumption in this regard, and a key point is to confront the data with the resulting model using the type of framework explained above. Our results indicate how far it is possible to "stretch" the conventional model.

There is, of course, a large literature discussing different preferences, such as habit formation, in the present setting, recent references being Allais (2004) or Chen and Ludvigson (2004). See also the papers by Haug (2001), Constantinides (1990), Detemple and Zapatero (1991), Sundaresan (1989), and Kocherlacota (1990). Also recursive utility of the Epstein-Zin (1989-91) type (discrete time) and Duffie and Epstein (1991) (continuous time) is analyzed in the literature by many. Such models are calibrated to the same data set as we use by e.g., Aase (2015). We leave it to an epilogue to comment on this.

The paper is organized as follows: In Section 1 we present a short version of the economic model, where we explain the expressions for the equilibrium risk premiums and equilibrium interest rate using discontinuous dynamics in continuous time. In Section 2 we introduce the NIG-distribution, and calibrate our resulting model to the US-data of the 20. century. Section 3 concludes, and Section 4 is an epilogue.

## 1 The economic model

In this section we present the rudiments of a consumption based equilibrium model. Following e.g., Aase (2002), the consumption space  $L$  is the set of adapted processes  $c$  satisfying the integrability constraint  $E\left(\int_0^T c_t^2 dt\right) < \infty$  for some fixed time horizon  $T$ . In this economy there are  $m$  agents, each being characterized by a nonzero consumption endowment process  $e^i$  in the set  $L_+$  of non-negative processes in  $L$ , and by a strictly increasing utility function  $U_i(\cdot) : L_+ \rightarrow R$ .

Consider first an economy in which any consumption process  $c$  in  $L$  can be purchased at time zero at the price  $\Pi(c)$ , where  $\Pi(\cdot)$  is a strictly increasing, linear price functional. An allocation  $(c^1, c^2, \dots, c^m)$  is feasible if  $\sum_{i=1}^m c^i \leq \sum_{i=1}^m e^i$ , and an Arrow-Debreu equilibrium is a collection  $\{\Pi, (c^1, \dots, c^m)\}$  consisting of a price functional and a feasible allocation  $(c^1, c^2, \dots, c^m)$  such that, for each  $i$   $c^i$  solves

$$\sup_{c \in L_+} U_i(c) \quad \text{subject to} \quad \Pi(c) \leq \Pi(e^i). \quad (1)$$

We assume the utility functions to be time-additive with a representation  $U_i(c) = E \left[ \int_0^T u_i(c_t, t) dt \right], i = 1, 2, \dots, m$ . By the Riesz Representation Theorem the pricing functional is  $\Pi(c) = E \int_0^T \pi(t)c(t)dt$ , where the Riesz representation  $\pi$  is interpreted as the state price deflator, or pricing kernel. This quantity corresponds to the Arrow-Debreu state prices in units of probability.

We assume enough smoothness on the utility functions, which are assumed strictly increasing and concave, so that the (1) is a nice optimization problem; the objective function is concave and the constraint is convex. For such problems the Kuhn-Tucker theorem says that, granted a suitable constraint qualification, any optimal solution  $c^i$  will be supported by a Lagrange multiplier  $\gamma_i$ . That is, there exists  $\gamma_i > 0$  such that the Lagrangian

$$\mathcal{L}_i(c; \gamma_i) := E \left( \int_0^T (u_i(c_t, t) - \gamma_i(\pi_t(c_t - e_t^i))) dt \right)$$

is maximal in  $c$  at  $c_t = c_t^i$  for all  $t \in [0, T]$  a.e. The nature of this maximum can be explored by equating the directional derivative of  $\mathcal{L}_i$  to zero in all feasible directions, i.e., for all  $c \in L$ , since the Lagrangian is maximized without constraints. Thus we get

$$\nabla \mathcal{L}_i(c^i; c) = \lim_{x \downarrow 0^+} \frac{\mathcal{L}_i(c^i + xc; \gamma_i) - \mathcal{L}_i(c^i; \gamma_i)}{x} = 0$$

for all  $c \in L$ . This condition translates to

$$E \left( \int_0^T (u'_i(c^i, t) - \gamma_i \pi_t) c_t dt \right) = 0$$

for all  $c \in L$ , which gives the first order conditions for each  $i = 1, 2, \dots, m$

$$u'_i(c^i, t) = \gamma_i \pi_t \quad \text{almost surely for almost all } t \in [0, T]. \quad (2)$$

Consider the real function  $u_\lambda$  defined by

$$u_\lambda(y, t) = \sup_{x \in R^m} \sum_{i=1}^m \lambda_i u_i(x_i, t) \quad \text{subject to} \quad \sum_{i=1}^m x_i \leq y, \quad (3)$$

for non-negative constants (agent-weights)  $\lambda_i$ . Suppose there exists an Arrow-Debreu equilibrium. Then the equilibrium allocations are Pareto optimal if and only if problem (3) has a solution, in which case the market, or *the representative agent*, has the “additive” utility function of the form

$$U_\lambda(c) = E \left[ \int_0^T u_\lambda(c_t, t) dt \right].$$

By the First Welfare Theorem the equilibrium allocation (2) is Pareto optimal. Thus the sup-convolution problem (3) has a solution  $u_\lambda(x, t)$ , where  $\lambda = (\lambda_1, \dots, \lambda_m)$ , each agent weight  $\lambda_i = \frac{1}{\gamma_i}$ , and the first order conditions can be written

$$\lambda_i u'_i(c^i, t) = u'_\lambda(e_t, t), \quad i = 1, 2, \dots, m, \quad (4)$$

where  $e_t = \sum_{i=1}^m e_t^i = \sum_{i=1}^m c^i = c(t)$  is the aggregate endowment process. Thus  $\pi_t = u'_\lambda(e_t, t)$  for all  $t \in [0, T]$ .

Since consumption processes can not be purchased the way we have suggested, or since Arrow-certificates do not in general exist, we introduce a securities market:

In order for the agents to have a possibility of obtaining Pareto optimal allocations of consumption, we assume there to be productive units, and ownership in these is determined each period in a competitive stock market. Shares are traded after payments of real dividends, at competitively determined prices  $X(t) = (X^0(t), X^1(t), \dots, X^N(t))$  at each time  $t$ , and the markets are open for trade at any time  $t \leq T$ . In our framework there are given an accumulated dividend process  $D = (D^0, D^1, \dots, D^N)$  of  $(N + 1)$  securities and a price process  $X$  such that the gains process, or adjusted price process,  $G = X + D$  is an Itô-diffusion process in  $R^{N+1}$ . We assume that  $D^0$  is a risk-less asset, having real price equal to  $X^0(t) = \beta_0 \exp\{\int_0^t r_u du\}$ , where  $r$  is a bounded short rate process.

Let  $\pi(t)$  be the spot-price process of the consumption good. A trading strategy  $\theta \in H^2(G)$  is said to finance a consumption process  $c$  if

$$\theta_t X_t = \int_0^t \theta_s dG_s - \int_0^t \pi_s c_s ds, \quad t \in [0, T], \quad (5)$$

and  $\theta_T X_T = 0$  (no remaining obligations at time  $T$ ), where  $H^2(G)$  is the set of predictable processes satisfying

$$E \left\{ \int_0^T \theta^2(t) d[G, G](t) \right\} < \infty, \quad (6)$$

$[G, G]$  being the quadratic variation process of  $G$ . The restriction of strategies to the set  $H^2(G)$  is to avoid arbitrage possibilities in continuous time, like e.g., the St. Petersburg game. One may loosely think of the restriction as limiting the number of trades in the finite time interval  $[0, T]$ .<sup>1</sup>

Given a security-price process  $X$  and a consumption-price process  $\pi$ , agent  $i$  solves the problem

$$\sup_{(c, \theta)} U_i(c), \quad \text{where } (c, \theta) \in L_+ \times H^2(G) \text{ and } \theta \text{ finances } (c - e^i). \quad (7)$$

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<sup>1</sup>Alternatively one could impose a borrowing constraint.

A *security-spot market equilibrium* is a collection  $\{X; \pi; (c^i, \theta^i), 1 \leq i \leq m\}$ , such that, given the security-price process  $X$  and the consumption-price process  $\pi$ , for each agent  $i$ ,  $(c^i, \theta^i)$  solves (7) and markets clear;  $\sum_{i=1}^m \theta_0^i = 0$ ,  $\sum_{i=1}^m \theta_n^i = \theta_n^M$ ,  $n = 1, 2, \dots, N$ , and  $\sum_{i=1}^m (c^i - e^i) = 0$ . Here  $\theta_n^M(t)$ ,  $n = 1, 2, \dots, N$  is the value-weighted market portfolio at each time  $t \in [0, T]$ .

The idea is that we can implement a security-spot market equilibrium in an Arrow-Debreu equilibrium, if the latter exists (see e.g., Radner (1972), Duffie and Huang (1985), and Duffie (1986)). Usually this requires a complete market structure, which we shall not require in general, but instead rely on the assumption that the initial endowments  $e^i$  of the agents are in shares of the firms, and assume HARA-type felicity functions  $u_i$ , in which case it is known that the security-spot market equilibrium allocation is Pareto optimal. This is sometimes termed an "essentially" complete market.

In the following we assume that there exists a representative agent equilibrium, and our results are exact in a single agent economy. With our choice of felicity index of the CRRA-type, alternatively it is sufficient that all the agents have the same relative risk aversions.

We close the system with the assumption of *rational expectations*: The market clearing prices  $(X, \pi)$  implied by consumer behavior is assumed to be the same as the price function  $(X, \pi)$  on which consumer decisions are based. The main analytic issue is then the determination of equilibrium price behavior.

## 1.1 Discontinuous dynamics

In this section we introduce discontinuous dynamics for the exogenously given processes  $c$  of aggregate consumption and  $D$  of the cumulative dividends of the risky assets. We assume that the aggregate consumption  $c$  and the dividend process  $D$  of a risky asset are given by the dynamic growth equations

$$\frac{dc(t)}{c(t-)} = \mu_c(t)dt + \sigma_c(t)dB(t) + \int_{\mathcal{Z}} \gamma_c(t, z) \tilde{N}(dz, dt), \quad (8)$$

and

$$dD(t) = \mu_D(t)dt + \sigma_D(t)dB(t) + \int_{\mathcal{Z}} \gamma_D(t, z) \tilde{N}(dz, dt), \quad (9)$$

Here  $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$  is an  $l$ -dimensional compensated Poisson random measure of the underlying  $l$ -dimensional Levy process, and  $B(t)$  is an independent  $d$  dimensional, standard Brownian motion.  $N(dz, dt)$  is a random Poisson measure, or simply a counting process. We shall limit ourselves to the case when  $l = 2$ , where the two independent, underlying

Levy-processes are assumed to have finite variances.  $\nu(dz)$  is the Levy measure.

The reason we choose our primitive processes to be in  $L^2$  is that state prices, belonging to the dual space, are then also in  $L^2$ . Thus we avoid unnecessary technical complications, in particular with regard to the representation of the underlying jump processes.

The terms  $\sigma_D$ ,  $\sigma_c$ , and  $\gamma_D$ ,  $\gamma_c$  may all be vectors/matrices of appropriate dimensions, depending upon circumstances.

For this type of processes we may perform several kinds of relevant analyses, including: Optimal stopping, stochastic control, the stochastic maximum principle, impulse control, singular control, chaos expansion and Malliavin calculus. There is an extended Ito's lemma, a Girsanov-type theorem, and statistical inference is available, etc.

## 1.2 A General Pricing Formula

Let  $(S, D)$  represent any given primitive security with real price process  $S$  and accumulated dividends process  $D$ . In Aase (2002) it is demonstrated that a security-spot market equilibrium is characterized as follows: The real market value  $S$  at each time  $t$  satisfies

$$S(t) = \frac{1}{u'(c_t, t)} E \left\{ \int_t^T \left( u'(c_s, s) dD(s) + d[D, u'](s) \right) \mid \mathcal{F}_t \right\}. \quad (10)$$

Here  $u'$  is the marginal utility of the representative agent,  $c_t := \sum_{i=1}^m c_t^i$  is the aggregate consumption process in the market, and  $[D, u']$  is the realized square covariance process between accumulated dividends and the marginal utility process. With jumps the quadratic covariation of two processes  $X$  and  $Y$  is given by

$$\begin{aligned} [X, Y](t) = & \int_0^t (\sigma_X(s)\sigma_Y(s) + \int_{\mathcal{Z}} \gamma_X(s, \zeta)\gamma_Y(s, \zeta)\nu(d\zeta)) ds \\ & + \int_0^t \int_{\mathcal{Z}} \gamma_X(s, \zeta)\gamma_Y(s, \zeta)\tilde{N}(ds, d\zeta). \end{aligned}$$

The additional realized square covariance term may be surprising to some, and at first glance does not seem to follow from Lucas' (1978)-model. However, by paying close attention to the information constraint when dividends are paid and prices adjusted, it is shown in Aase (2005) that this formula follows directly from the discrete time Lucas framework, and is shown to be of particular importance when jumps are present. Among other things



it is pointed out that this term does not even vanish when  $D$  and  $c$  are deterministic.

This term can also be of importance in the continuous model when the dividend process follows an Itô-diffusion with a nonvanishing diffusion term  $\sigma_D(t)$ . In Aase (2002) the pricing relation (10) was taken as the main starting point in deriving both the equilibrium interest rate and the equilibrium risk premium.

### 1.3 The equilibrium risk premium and the short-term interest rate

In the conventional model the felicity index  $u$  has the separable form  $u(c, t) = \frac{1}{1-\gamma} c^{1-\gamma} e^{-\rho t}$ . The parameter  $\gamma$  is the representative agent's relative risk aversion and  $\rho$  is the utility discount rate, or the impatience rate, and  $T$  is the time horizon. These parameters are assumed to satisfy  $\gamma > 0$ ,  $\rho \geq 0$ , and  $T < \infty$ .

When jumps are included the risk premium ( $\mu_R - r$ ) of any risky security labeled  $R$  (for "risky") is given by

$$\mu_R(t) - r_t = \gamma \sigma_{Rc}(t) - \int_{\mathcal{Z}} ((1 + \gamma_c(t, \zeta))^{-\gamma} - 1) \gamma_R(t, \zeta) \nu(d\zeta). \quad (11)$$

Here  $r_t$  is the equilibrium real interest rate at time  $t$ , and the term  $\sigma_{Rc}(t) = \sum_{i=1}^d \sigma_{R,i}(t) \sigma_{c,i}(t)$  is the covariance rate between returns of the risky asset and the growth rate of aggregate consumption at time  $t$ , a measurable and adaptive process satisfying standard conditions. The dimension of the Brownian motion is  $d > 1$ . Underlying the jump dynamics we have  $\{N_j\}$ ,  $j = 1, 2, \dots, l$  independent Poisson random measures with Levy measures  $\nu_j$  coming from  $l$  independent (1-dimensional) Levy processes. The possible time inhomogeneity in the jump processes is expressed through the terms denoted  $\gamma_{R,j}(t, \zeta_j)$  for the risky asset under consideration, and  $\gamma_{c,j}(t, \zeta_j)$  for the aggregate consumption process, both measuring the jump sizes. Here also jump frequencies at time  $t$  are embedded. The "mark space" is  $\mathcal{Z} = \mathbb{R}^l$  in this paper, where  $\mathbb{R} = (-\infty, \infty)$ . Thus the last term in (11) is short-hand notation for the following

$$\sum_{j=1}^l \int_{\mathbb{R}} ((1 + \gamma_{c,j}(t, \zeta_j))^{-\gamma} - 1) \gamma_{R,j}(t, \zeta_j) \nu(d\zeta_j).$$

This is a continuous-time version of the consumption-based CAPM, allowing for jumps at random time points. Similarly the expression for the risk-free,

real interest rate is

$$r_t = \rho + \gamma \mu_c(t) - \frac{1}{2} \gamma (\gamma + 1) \sigma'_c(t) \sigma_c(t) - \left( \gamma \int_{\mathcal{Z}} \gamma_c(t, \zeta) \nu(d\zeta) + \int_{\mathcal{Z}} ((1 + \gamma_c(t, \zeta))^{-\gamma} - 1) \nu(d\zeta) \right). \quad (12)$$

In the risk premium (11) the last term stems from the jump dynamics of the risky asset and aggregate consumption, while in (12) the last two terms have this origin. These results follow from Aase (1993a,b).

The process  $\mu_c(t)$  is the annual growth rate of aggregate consumption and  $(\sigma'_c(t) \sigma_c(t))$  is the annual variance rate of the consumption growth rate, both at time  $t$ , again dictated by the Ito-isometry. Both these quantities are measurable and adaptive stochastic processes, satisfying usual conditions. The return processes as well as the consumption growth rate process in this paper are also assumed to be ergodic processes, implying that statistical estimation makes sense.

Notice that in the model is the instantaneous correlation coefficient between returns and the consumption growth rate given by

$$\kappa_{Rc}(t) = \frac{\sigma_{Rc}(t)}{\|\sigma_R(t)\| \cdot \|\sigma_c(t)\|} = \frac{\sum_{i=1}^d \sigma_{R,i}(t) \sigma_{c,i}(t)}{\sqrt{\sum_{i=1}^d \sigma_{R,i}(t)^2} \sqrt{\sum_{i=1}^d \sigma_{c,i}(t)^2}},$$

and similarly for other correlations given in this model. Here  $-1 \leq \kappa_{Rc}(t) \leq 1$  for all  $t$ . With this convention we can equally well write  $\sigma'_R(t) \sigma_c(t)$  for  $\sigma_{Rc}(t)$ , and the former does *not* imply that the instantaneous correlation coefficient between returns and the consumption growth rate is equal to one. Prime means transpose.

Similarly the term  $\sum_{j=1}^l \int_{\mathbb{R}} \gamma_{R,j}(t, \zeta_j) \gamma_{c,j}(t, \zeta_j) \nu(d\zeta_j)$  is the covariance rate at time  $t$  between returns of the risky asset and the growth rate of aggregate consumption stemming from the discontinuous dynamics. We use the shorthand notation  $\int_{\mathcal{Z}} \gamma_R(t, \zeta) \gamma_c(t, \zeta) \nu(d\zeta)$  for this term as well.

Using a Taylor series expansion, the risk premium is approximately

$$\mu_R(t) - r_t = \gamma \left( \sigma_{Rc}(t) + \int_{\mathcal{Z}} \gamma_R(t, \zeta) \gamma_c(t, \zeta) \nu(d\zeta) \right) - \frac{1}{2} \gamma (\gamma + 1) \int_{\mathcal{Z}} \gamma_R(t, \zeta) \gamma_c^2(t, \zeta) \nu(d\zeta) + \dots \quad (13)$$

and an approximation for the interest rate is

$$r_t = \rho + \gamma\mu_c(t) - \frac{1}{2}\gamma(1 + \gamma)\left(\sigma'_c(t)\sigma_c(t) + \int_{\mathcal{Z}} \gamma_c^2(t, \zeta)\nu(d\zeta)\right) + \frac{1}{6}\gamma(\gamma + 1)(\gamma + 2) \int_{\mathcal{Z}} \gamma_c^3(t, \zeta)\nu(d\zeta) - \dots \quad (14)$$

Here the term  $\int_{\mathcal{Z}} \gamma_c^2(t, \zeta)\nu(d\zeta)$  is the variance rate of the consumption growth rate at time  $t$ , stemming from the discontinuous dynamics, so that the total consumption variance rate is  $\sigma'_c(t)\sigma_c(t) + \int_{\mathcal{Z}} \gamma_c^2(t, \zeta)\nu(d\zeta)$  at time  $t$ . Similarly the total covariance rate between returns of the risky asset and the consumption growth rate is  $\sigma_{Rc}(t) + \int_{\mathcal{Z}} \gamma_R(t, \zeta)\gamma_c(t, \zeta)\nu(d\zeta)$ .

In Table 1 we present the key summary statistics of the data in Mehra and Prescott (1985), of the real annual return data related to the S&P-500, denoted by  $M$ , as well as for the annualized consumption data, denoted  $c$ , and the government bills, denoted  $b$  <sup>2</sup>.

	Expectat.	Standard dev.	Covariances
Consumption growth	1.83%	3.57%	$\text{cov}(M, c) = .002226$
Return S&P-500	6.98%	16.54%	$\text{cov}(M, b) = .001401$
Government bills	0.80%	5.67%	$\text{cov}(c, b) = -.000158$
Equity premium	6.18%	16.67%	

Table 1: Key US-data for the time period 1889-1978. Discrete-time compounding.

Here we have, for example, estimated the covariance between aggregate consumption and the stock index directly from the data set to be .00223. This gives the estimate .3770 for the correlation coefficient <sup>3</sup>.

Since our development is in continuous time, we have carried out standard adjustments for continuous-time compounding, from discrete-time compounding. The results of these operations are presented in Table 2. This gives, e.g., the estimate  $\hat{\kappa}_{Mc} = .4033$  for the instantaneous correlation coefficient  $\kappa(t)$ . The overall changes are in principle small, and do not influence our comparisons to any significant degree, but are still important.

Interpreting the risky asset  $R$  as the value weighted market portfolio  $M$  corresponding to the S&P-500 index, equations (13) and (14) are two equations in two unknowns that can provide estimates of the two preference

<sup>2</sup>There are of course newer data by now, but these retain the same basic features. If our model can explain the data in Table 1, it can explain any of the newer sets as well.

<sup>3</sup>The full data set was provided by Professor Rajnish Mehra.

	Expectation	Standard dev.	Covariances
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{Mc} = .002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	

Table 2: Key US-data for the time period 1889-1978. Continuous-time compounding.

parameters by the "method of moments". Ignoring the higher order terms in each of these equations, the result is  $\gamma = 26.3$  and  $\rho = -.015$ , i.e., a relative risk aversion of about 26 and an impatience rate of minus 1.5%.

In order to illustrate what a risk aversion of 26 really means, consider a random variable  $X$  with probability distribution given in Table 3: The equation

$$E\{u(100 + X)\} := u(100 + e_u)$$

defines its certainty equivalent  $e_u$  at initial fortune 100 for the utility function  $u$ . If  $u$  is of power type  $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ , the certainty equivalent  $e_u$  is illustrated in Table 3 for some values of  $\gamma$ .

X	0	100
Probability	0.5	0.5
$\gamma = 0$	$e_u = 50.00$	
$\gamma = 1$	$e_u = 41.42$	
$\gamma = 2$	$e_u = 33.33$	
$\gamma = 3$	$e_u = 26.49$	
$\gamma = 4$	$e_u = 21.89$	
$\gamma = 5$	$e_u = 17.75$	
$\gamma = 17$	$e_u = 4.42$	
$\gamma = 20$	$e_u = 3.71$	
$\gamma = 22$	$e_u = 3.55$	
$\gamma = 26$	$e_u = 2.81$	

Table 3: Certainty equivalents of  $X$  for CRRA-utility.

As can be seen, a relative risk aversion of 26 corresponds to a rather low certainty equivalent.

One aim of this paper is to investigate if the non-linearities contained in the expressions in (11) and (12) might change these numbers somewhat in the "right" direction. This is the topic of the next sections.

## 2 Joint Normal Inverse Gaussian jumps

### 2.1 Introduction

In order to truly take into account some of the stylized facts about speculative prices, we propose to use a joint Normal Inverse Gaussian (NIG)-distribution for the simultaneous jump sizes of the stock index and aggregate consumption. In order to focus on the essential features of this distribution, we leave out the continuous diffusion part. If the data are yearly as in our case, the jump part may describe the whole dynamics by simply setting the frequency  $\lambda$  equal to one. This corresponds to a discrete time model on the average, but with the analytical tools of the continuous time marked point process framework. A major advantage with this approach is that there is no need to separate the jumps from the continuous paths in the data, a task which may prove challenging in practice.

The Normal Inverse Gaussian distribution was brought to the attention of workers in empirical finance by Barndorff-Nielsen (1997). It fits fat-tailed and skewed data very well and is analytically tractable. The manner in which we utilize this distribution in the following does not appear elsewhere.

### 2.2 The equity premium and the short rate with pure jumps

In order to explain our approach, we start with an expression for the risk premium

$$\mu_R - r = -\lambda \int_{-1}^{\infty} \int_{-1}^{\infty} ((1 + z_c)^{-\gamma} - 1) z_R dF(z_c, z_R) \quad (15)$$

where  $z_c$  refers to the aggregate consumption variable,  $z_R$  signify the return on the stock index, and where  $F(z_c, z_R)$  is the joint probability distribution function of these two variables, assumed to have a density function  $f$ .

In the following it will be an advantage to consider the model in exponential, rather than in the stochastic exponential form. We therefore make the substitution  $1 + z_i = e^{x_i}$ ,  $i = c, R$  which leads to the following expression

$$\mu_R - r = -\lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{-\gamma x_c} - 1) (e^{x_R} - 1) dG(x_c, x_R) \quad (16)$$

where  $G$  has density function  $g$  given by

$$g(x_c, x_R) = f(z_c(x_c), z_R(x_R))J(x_c, x_R)$$

and where the  $J$  is the Jacobian

$$J(x_c, c_R) = \text{mod} \begin{vmatrix} e^{x_c} & 0 \\ 0 & e^{x_R} \end{vmatrix} = e^{x_c+x_R}$$

Letting  $M(\mathbf{u}) = M(u_c, u_R)$  be the moment generating function

$$M(u_c, u_R) = E\{e^{u_c X_c + u_R X_R}\}$$

where the random vector  $\mathbf{X} = (X_c, X_R)'$  represents the joint jump sizes in the consumption growths and the returns of the stock index. The risk premium can be written

$$\begin{aligned} \mu_R - r = & -\lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\gamma x_c + x_R} dG(x_c, x_R) \\ & + \lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\gamma x_c} dG(x_c, x_R) + \lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x_R} dG(x_c, x_R) - \lambda \end{aligned} \quad (17)$$

By employing the moment generating function  $M$ , this can be expressed as

$$\mu_R - r = -\lambda(M(-\gamma, 1) - M(-\gamma, 0) - M(0, 1) + 1). \quad (18)$$

Our assumption taken later on will be that the distribution  $G$  is bivariate Normal Inverse Gaussian (NIG), for which we have a convenient closed form expression for the corresponding moment generating function  $M$ . Note that this development assumes that the moment generating function exists, and therefore excludes the more extreme heavy tailed distributions of stable Paretian type.

Next we consider the equilibrium short-term interest rate  $r$ . The starting point in the case of pure jumps is the following

$$r = \rho + \gamma\mu_c - \lambda\left(\gamma \int_{-1}^{\infty} z_c dF_c(z_c) + \int_{-1}^{\infty} ((1+z_c)^{-\gamma} - 1) dF_c(z_c)\right) \quad (19)$$

where  $F_c$  is the marginal distribution function of the consumption jumps. Using the substitution  $1+z_c = e^{x_c}$  this equation becomes

$$r = \rho + \gamma\mu_c - \lambda\left(\gamma \int_{-\infty}^{\infty} (e^{x_c} - 1) dG_c(x_c) + \int_{-\infty}^{\infty} (e^{-x_c\gamma} - 1) dG_c(x_c)\right) \quad (20)$$

where  $G_c$  is the marginal distribution for the consumption jumps in the exponential version of the model. In terms of the moment generating function  $M$  the expression for  $r$  becomes

$$r = \rho + \gamma\mu_c - \lambda\left(\gamma(M(1, 0) - 1) + M(-\gamma, 0) - 1\right). \quad (21)$$

The variance per time unit of the consumption process is given by

$$\lambda \int_{-1}^{\infty} z_c^2 dF_c(z_c) = \lambda \int_{-\infty}^{\infty} (e^{x_c} - 1)^2 dG_c(x_c) = \lambda(M(2, 0) - 2M(1, 0) + 1), \quad (22)$$

and the corresponding variance of any risky asset is

$$\lambda \int_{-1}^{\infty} z_R^2 dF_R(z_R) = \lambda \int_{-\infty}^{\infty} (e^{x_R} - 1)^2 dG_R(x_R) = \lambda(M(0, 2) - 2M(0, 1) + 1). \quad (23)$$

In order to get a yardstick from which to compare results using the NIG-distribution, consider first the joint normal distribution and the associated moment generating function given by

$$M^N(s, t) = \exp \left\{ \frac{1}{2} \sigma_1^2 s^2 + \mu_1 s + \frac{1}{2} \sigma_2^2 t^2 + \mu_2 t + st \sigma_1 \sigma_2 \rho_n \right\}$$

where  $\rho_n$  is the correlation coefficient.

In our calibrations, because of the log-transformation (16), it will be necessary to consider a log transformation and use log returns. The relevant summary statistics are given in Table 4. Notice that this table is not a mere transformation of Table 1, but developed from the the original data set used in the Mehra and Prescott (1985)-study, by taking logarithms of the relevant yearly quantities, and basing the statistical analysis on these transformed data points<sup>4</sup>.

	Expectat.	Standard dev.	Covariances
Consumption growth	1.75%	3.55%	$\text{cov}(M, c) = .002268$
Return S&P-500	5.53%	15.84%	$\text{cov}(M, b) = .001477$
Government bills	0.64%	5.74%	$\text{cov}(c, b) = -.000149$
Equity premium	4.89%	15.95%	

Table 4: Key US-data for the time period 1889-1978 in terms of log returns of discrete-time compounding.

Calibrating to the data summarized in Table 1 and Table 4, using the above equations (18) and (21) we obtain the following  $(\gamma, \rho) = (24.27, -0.044)$ . The results we get below must then really be compared to these, and not to the pair  $(\gamma, \rho) = (26.3, -0.015)$  of the last section. This way we will see what

<sup>4</sup>We have obtained the original data set from Professor R. Mehra. For example, a log return is not obtained simply adjusted as  $\mu - (1/2)\sigma^2$  from Table 1, which would be (almost) true if returns and growth rates of consumption were normally distributed. We observe deviations from normality in the data, although not substantial ones.

the deviations from normality really gives us. The pair (24.27, -0.044) shows us what the jump model contributes to in isolation, in a normal universe. It indicates an improvement for risk aversion parameter, while the subjective discount rate is still negative.

## 2.3 Specifications of the NIG-distribution

We now go on to specify the bivariate NIG distribution. Although there is no canonical definition of a multivariate Normal Inverse Gaussian (NIG) distribution, the most common one is obtained by a mean-variance mixture of a multivariate Normal distribution with respect to the inverse Gaussian distribution (IG). This is convenient, as it leads to a relatively simple expression for its moment generating function, which may be taken as the definition of the distribution itself. Staying as close as possible to the notation in the literature this is given by

$$M(\mathbf{u}) = \exp(\mathbf{u}'\boldsymbol{\mu} + \delta(\sqrt{\alpha^2 - \boldsymbol{\beta}'\boldsymbol{\Delta}\boldsymbol{\beta}} - \sqrt{\alpha^2 - (\boldsymbol{\beta} + \mathbf{u})'\boldsymbol{\Delta}(\boldsymbol{\beta} + \mathbf{u})})) \quad (24)$$

This is referred to as the *NIG* ( $\boldsymbol{\mu}$ ,  $\delta$ ,  $\alpha$ ,  $\boldsymbol{\beta}$ ,  $\boldsymbol{\Delta}$ ) distributions where  $\alpha$  and  $\delta$  are non-negative scalars,  $\boldsymbol{\beta}$  and  $\boldsymbol{\mu}$  vectors and  $\boldsymbol{\Delta} = (\Delta_{ij})$  a positive definite matrix which, in order to have parameter identifiability, may be taken to have determinant 1, without loss of generality. Moreover, feasible values must be so that  $\gamma_0^2 = \alpha^2 - \boldsymbol{\beta}'\boldsymbol{\Delta}\boldsymbol{\beta} > 0$ , with subscript zero to avoid confusion with the previous  $\gamma$ . Essentially  $\delta$  and  $\alpha$  relate to variance and peak/tail behavior and  $\boldsymbol{\beta}$  to skewness and  $\boldsymbol{\Delta}$  to covariation, but it is a bit more complicated. A no-skewness NIG distribution obtains when  $\boldsymbol{\beta} = \mathbf{0}$  and then  $\gamma_0 = \alpha$ . We get distributions approaching Gaussianity as  $\alpha$  tends to infinity. The largest departure from Gaussianity obtains when  $\alpha$  is close to zero. However, both the skewness and covariation may affect the lower bound of feasible  $\alpha$ 's by the requirement  $\gamma_0 > 0$  above. Other parameterizations of the distribution exist with more direct parameter interpretation at the expense of more complicated formulas.

The expectation in our multivariate NIG-distribution is

$$E(\mathbf{X}) = \boldsymbol{\mu} + \frac{\delta}{\gamma_0} \cdot \boldsymbol{\Delta}\boldsymbol{\beta} \quad (25)$$

so that  $E(\mathbf{X}) = \boldsymbol{\mu}$  only in the no-skewness case. The covariance matrix is

$$\boldsymbol{\Sigma} = \frac{\delta}{\gamma_0^3} \cdot (\gamma_0^2 \boldsymbol{\Delta} + \boldsymbol{\Delta}\boldsymbol{\beta}\boldsymbol{\beta}'\boldsymbol{\Delta}) \quad (26)$$



which, in the no-skewness case, simplifies to

$$\Sigma = \frac{\delta}{\alpha} \cdot \Delta$$

The marginal distributions are univariate NIG, i.e.  $NIG(\mu_i, \delta_i, \alpha_i, \beta_i)$ . However, the marginal parameters are not just picks from the multivariate parameters, but have to be determined by formulas omitted here, see Lillestøl (2000). For instance, the marginal  $\alpha_i$ 's and  $\beta_i$ 's are affected jointly by  $\beta$  and  $\Delta$ .

We now fit the the bivariate NIG-model to the data pair (consumption growth, S&P 500 return) referred to in Table 1, and Table 2. The vectors of observations and parameters are indexed by  $(c, R) = (1, 2)$ , but note that  $\mu = (\mu_1, \mu_2)$  should not be confused with the earlier meaning of  $\mu_c$  and  $\mu_R$ .

Maximum likelihood estimation by the R-package *ghyp* of Breymann and Lüthi (2013) gave the following result:

$$\mu = (0.050, 0.068) \quad \beta = (-37.28, 2.47)$$

$$\Delta = \begin{pmatrix} 0.2213 & 0.4332 \\ 0.4332 & 5.3662 \end{pmatrix}$$

Moreover,  $\alpha = 35.45$  and  $\delta = 0.1431$ . The large  $\alpha$  indicates just moderate heavy tail/peakedness and the  $\beta$ -vector indicates some negative skewness. Marginally the negative skewness pertains to both variables, cf. the note made above on marginalization. These marginal features are of course apparent from histogram plots. The estimates of the corresponding expectation and covariance matrix are:

$$E\mathbf{X} = (0.0175, 0.0553) \quad \Sigma = \begin{pmatrix} 0.0012 & 0.0021 \\ 0.0021 & 0.0244 \end{pmatrix}$$

from which we obtain the estimates of the standard deviations (0.0352, 0.1561) and correlation 0.3748. This shows a close to exact fit with the estimates in Table 6 based on just matching the corresponding empirical quantities.

Let us illustrate the use of this by taking equations (18) and (21) and solving for  $(\gamma, \rho)$ , in case of  $\lambda = 1$ , for fixed equity premium  $\mu_R - r = 0.06$ , interest rate  $r = 0.008$  and expected consumption growth  $\mu_c = 0.018$ , i.e. the quantities in Table 5. We then get the estimates  $(\gamma, \rho) = (22.2, 0.0083)$ , i.e. a large  $\gamma$  and a  $\rho$  slightly below 1%. Note that with this data we get  $\rho > 0$  if and only if  $\gamma > 21.8$ . The sensitivity with respect to the other parameters may be studied as well. It is also of interest to see how sensitive the solution

$(\rho, \gamma)$  is to parameter changes. Some insight may be obtained by perturbing each parameter estimate, one at a time. Table 5 shows some examples which relate mainly to each of less Gaussianity, larger variation and more skewness, respectively, keeping the other attributes fixed.

Perturbation	$\gamma$	$\rho$
None	22.2	0.008
$\alpha - 5$	17.7	0.058
$\delta + 0.05$	18.3	0.053
$\beta_1 - 5$	19.9	0.029

Table 5: Sensitivity of solution to parameter perturbation.

The crucial question is whether the perturbations above are within reasonable margins of estimation error. This may be explored by taking samples of size of the original data ( $n=90$ ) from the estimated distribution. Repeated sampling  $m=1000$  times gave the following insights: The distribution of the estimate of the parameter  $\alpha$  is very wide with the given tilted value at the low end, and with a high risk of overestimation, i.e., claim more Gaussianity than there is. For the estimate of the parameter  $\delta$  we have a distribution with quartile range about 0.08, with the given tilted value at the upper quartile. For the estimates of the parameters  $\beta_1$  and  $\beta_2$  we have a distribution with long tails to the left and to the right respectively, and with the given tilted value of  $\beta_1$  well within the quartile range. Thus the tilted values in Table 5 are not at all unrealistic as a result of estimation error. In practice more than one parameter may be in error at the same time and the solution further removed. However, there may be compensating correlations between estimates so that the changes do not add up. Here a slight positive correlation between the estimates of  $\alpha$  and  $\delta$  works this way.

These examples show subjective rates which are still positive and a slightly reduced relative risk aversion below 20. It is still high, but we see that the NIG-model allows improvement, taking heavy tails and skewness into account. For the continuous time model with no jumps the results of the calibration are  $(\gamma, \rho) = (26.3, -.015)$ . As a real comparison, showing the effects of the NIG-assumption, the results in Table 5 should be compared to the one from the last section using joint normality in the pure jump model: There we found  $(\gamma, \rho) = (24.3, -.044)$ .

We have studied the equity premium issue in a statistical context, where the relevant quantities are expressed by parameters in an underlying distribution or data generating process representing the "true world". Many analysts have studied the issue within the Gaussian model or just matching

moments with no model at all, at best implicitly justified in case of Gaussianity. A puzzling result may then simply be due to a "wrong" model. We have demonstrated above that by leaving the Gaussian world we may get more reasonable estimates than the conventional ones. Since the estimate of the relative risk aversion  $\gamma$  is still too large, we do not have a full explanation of the equity premium puzzle. However, we have indicated that it may be room for further improvement, although not substantial ones. Our analysis also indicates that samples of the size of the Mehra-Prescott data may be too small to arrive at substantial conclusions.

### 3 Conclusions

We have introduced jump dynamics in the "noise term" of the dynamic stochastic differential equations for the aggregate consumption process and the dividend processes of the risky assets. As a result, the equilibrium relations for the short rate and the risk premium could no longer be fully described by the two first moments only. We demonstrate that this gives some added flexibility in modeling; for example, it brings us outside the local mean-variance framework, permitting us to utilize other properties of a joint probability distribution than merely its two first moments.

Our discussion reveals that any model will have difficulties explaining both the classical equity premium puzzle and the corresponding risk-free rate puzzle of the last century, using the above framework of time additive and separable utility functions. Nevertheless, the analysis shows that with jump components included, this opens up several possibilities related to the interpretations of the classical model. The Taylor series approximations indicate that the jump model is just the right extension of the continuous model, related to the issues we study: the risk premiums and the equilibrium short term interest rate.

When considering US-data of the last century, we found it convenient to use a pure jump version of frequency one per year. The risk premiums as well as the short rate can then be expressed by moment generating functions, not only first and second order moments. This allows us to explore if deviations from normality is important in explaining these data. We use a joint probability distribution for the growth rate of the aggregate consumption process and the return rate of the S&P-500 index which can capture both skewness, kurtosis and heavy tails - the Normal Inverse Gaussian (NIG)-distribution, and compare to the results when the joint normality is assumed.

First, the non-linearity imposed by the marked point processes brings the risk aversion from about  $\hat{\gamma} = 26.3$  to 24.27 and the impatience rate from

$\hat{\rho} = -.015$  to  $-.044$  (the latter hardly an improvement).

Second, the deviations from normality brings the relative risk aversion from about  $\hat{\gamma} = 24$  to  $\hat{\gamma} = 22$ , and the impatience rate from  $\hat{\rho} = -.044$  to  $\hat{\rho} = .0083$ . In the best possible interpretation of our results, we obtain  $\hat{\gamma} = 17$  and  $\hat{\rho} = .058$  within the sampling errors of the data. Beyond this we can not move the parameter estimates much. This brings the results in the right direction, but not really enough to conclude that the conventional model with expected utility solves this puzzle.

## 4 Epilogue

The conventional asset pricing model, the consumption-based capital asset pricing model (CCAPM) of Lucas (1978) and Breeden (1979), assumes a representative agent with a utility function of consumption that is the expectation of a sum, or a time integral, of future discounted utility functions. The model has been criticized for several reasons. First, it does not perform well empirically, as we have seen in this paper. Our approach mitigates this statement. Second, the usual specification of utility can not separate the risk aversion from the elasticity of intertemporal substitution, while it would clearly be advantageous to disentangle these two conceptually different aspects of preference. Third, while this representation seems to function well in deterministic settings, and for *timeless* situations, it is not well founded for *temporal* problems (derived preferences do not in general satisfy the substitution axiom, e.g., Mossin (1969)).

Recursive utility has been introduced in the discrete time model by Epstein and Zin (1989-91) and in continuous time by Duffie and Epstein (1992a-b) which elaborates the foundational work by Kreps and Porteus (1978), Epstein and Zin (1989) and Chew and Epstein (1991) of recursive utility in dynamic models. Models based on recursive utility have become popular, and the numerous contributions can not be mentioned here. We limit ourselves to the paper by Aase (2015), who discuss recursive utility in a jump diffusion setting. In this paper it is shown that the puzzle can be solved even without resorting to the non-linearities caused by the jump parts, but taking these into consideration certainly does not make things worse.

Recursive utility in the Lucas (1978)-model can be said to give a solution the equity premium puzzle in discrete, or in continuous time models, with or without jumps. There may be others.

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