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Investment Flexibility in the Oil Industry

Optimizing Investment Decisions Applying Real Option Theory

Mats Strømmen Abelsen and Stig Ove Nystad

Supervisor: Michail Chronopoulos

Master thesis, Financial Economic

NORWEGIAN SCHOOL OF ECONOMICS

Preface

This thesis is written during the fall semester 2014 to conclude our Master of Science in the Economics and Business administration program at the Norwegian School of Economics, with major in Financial Economics.

The presented study would not have been possible to conduct without the guidance, corrections, feedback and suggestions provided by our advisor, ass. Professor Michail Chronopoulos. We have had several interesting discussions regarding the application of real option theory in the petroleum industry, and were guided how to apply the most appropriate approach for solving our case study. We are especially grateful for Chronopoulos' lectures on advanced topics in financial mathematics, and for helping us with Matlab® programming. Chronopoulos' expertise in real option theory was essential to ensure the quality of this study.

Mats Strømmen Abelsen and Stig Ove Nystad

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Abstract

The ambition of the study is to apply relevant real option theory to a specific investment decision in the oil industry. Investment flexibility is significant in this industry, but several companies rely on the standard net present value approach when valuing an investment possibility. The motivation of the study is to suggest a better way of optimizing investment decisions for appliance in the industry, where accounting for embedded options is the main focus.

Present study compares two mutually exclusive projects for an operating company in the oil industry. This company can choose between these two projects under price- and oil discovery uncertainty. The first embedded option considered is an option to expand an operating project. The second embedded option considered is an option to switch to another project.

The problem is solved by creating a comprehensive model through financial mathematics and programming in Matlab®. The model provides closed form solutions to the specific case study. The case study provides valuable insight in how the availability of the option to defer investment and the embedded options can alter an investment decision.

Analyzing the option to defer investment shows that the company should not invest immediately when the project provides a positive net present value, like a breakeven analysis would suggest. Instead, the company should wait until the oil price is at a higher threshold. The analysis proves that embedded options provide sufficient value to alter a company's investment decision. In addition, the uncertainty regarding an oil discovery process is proved to be a significant factor in the investment decision.

Through this study, companies in the industry are encouraged to account for the suggested embedded options in its investment decision. In addition, it is recommended to focus less on the standard net present value approach, and focus more on the flexibility the projects offer.

Table of Contents

List of Figures and tables

1. Introduction

This study performs a valuation of a sequential investment decision of two mutually exclusive projects under price- and oil discovery uncertainty. The work will be applied to a specific setting in the oil industry, where the goal is to optimize the operating company's investment decision in a monopoly setting.

Although the case is applied to a specific setting, the framework is applicable for other scenarios within the oil industry, and across other industries. Therefore, this study will help to provide insight in applying real option investment theory to relevant scenarios, and explain why accounting for investment flexibility is important when making investment decisions.

A net present value (NPV) approach is often applied by oil companies when making investment decisions. NPV does not account for investment flexibility in the decision-making process. Thus, this provides the possibility of mispricing an investment project, and in worst case a poor investment decision. It is shown how applying real option theory can improve the quality of a company's investment decision.

In the valuation process, it is assumed that the underlying variable, the oil price, follows a geometric Brownian motion (GBM) process. To value the options, a Decision Tree Analysis (DTA), Dynamic Programming (DP), will be applied.

The base case for the study is an operating oil company which can choose between two projects; project A or project B. These projects contain different characteristics. Project B is assumed to be a larger field than the field in project A, but investing in project B also requires higher investment costs. Both projects are prospect fields, i.e. already proven, and hence available for investment without conducting an oil discovery process. The case study analyses the investment decision as described above, and investigates how the option to defer investment affects the investment decision. It is assumed that project A has embedded options attached, and the goal is to analyze how these options affect the company's investment decision. The first embedded option is an option to expand project A into a nearby satellite field at a small cost. Even though the main fields are discovered, it is assumed that an oil discovery process for the expansion has not yet been conducted. However, an analysis of the situation where the expansion is certain, and the situation where it is uncertain, will be conducted for the sake of comparison. The final embedded option for project A, is to switch to project B after extracting the expanded project successfully.

The results found in this study confirm that flexibility may affect a company's investment decision. Compared to a traditional NPV analysis, it is shown that the option to defer investments have an impact on project values, and thereby investment decision. In addition, investment in project A becomes more attractive compared to deferring investment, and compared to investment in project B, when including embedded options. A sensitivity analysis shows that increased volatility makes deferring investment more attractive opposed to investing. The uncertainty regarding the expanded satellite field is a value-decreasing factor for investment in project A. Decreasing uncertainty proves that project A grows more attractive accordingly.

2. Structure

This chapter provides an overview over all chapters in the thesis, and a short summary of the main contents.

In Chapter 3, an overview of relevant previous research related to the topic is presented. It starts with presenting the origins of option theory and real option theory, before specifying the previous research which relates to this study.

As an introduction to the petroleum industry, the basics of oil production are presented in Chapter 4. The chapter starts with presenting an analysis of a typical oil production company's value chain. Thereafter, an analysis of risk factors associated with oil production is conducted, highlighted by the oil price volatility.

Chapter 5 describes the relevant theory applied in this study. It starts with the basics of financial- and real options, before addressing complex theories on stochastic processes, and suggests which method that can be applied to this model.

In Chapter 6, the applicability of the study is presented. The chapter starts with an industry research. This research shows the application of the suggested model in the petroleum industry, and reveals if there is a demand for these types of frameworks. Additionally, a review of the application of real options and which kind that is most relevant in the industry is conducted.

Chapter 7 provides a simple setup and description of the underlying problem this case study addresses.

Chapter 8 provides an overview of the notational framework and assumptions which are specific for the case study.

Chapter 9 describes the mathematical model and its application. The first section defines the mathematical framework. Then, it is assumed that the company can defer investment in both projects, but that project A does not have any embedded options. The next section builds on the first, but also accounts for the option to expand. Both situations with oil discovery certainty and uncertainty are reviewed. Finally, the scenario where all the described options are available is analyzed, with both oil discovery certainty and uncertainty.

The application of the mathematical model with numerical results is presented in Chapter 10. The parameter inputs are presented before a simple NPV approach is conducted. Then, the same steps as in Chapter 9 are followed by adding option elements for each section.

Chapter 11 provides a comparison and discussion of the numerical results from Chapter 10. It compares each scenario, and tests the model with a sensitivity analysis.

The model relies on several assumptions, and therefore has limitations. These are described in Chapter 12.

Finally, Chapter 13 concludes the study. The most relevant findings are presented, and the application of the framework for similar scenarios in the industry is discussed.

3. Previous Research

In academic research, there are several topics regarding investment optimizations. Option theory was derived from the two pioneers Black and Scholes (1973). Based on their findings, Myers (1977) took the approach a step further and developed a model for valuing options for real investment decisions. An important study by Brennan and Schwartz (1985) suggested a new model of how to value mining and other natural resource projects using self-financing portfolios. This study, combined with the origin option theory, has been used as source for further real option science. Same year, McDonald and Siegel (1986) used real options in analyzing the value of deferring an irreversible investment.

Majd and Pindyck (1987) analyzed how to make optimal investment, under a sequential investment process. Ekern (1988) developed a framework on how to evaluate projects in the petroleum industry, where a satellite field has several options for development and operations. Bjerksund and Ekern (1990) concluded in their research that the option to defer an investment in offshore fields is the most valuable real option. Some years later Dixit and Pindyck (1994) supported the findings due to the fact that investments in offshore fields require large irreversible investments. In addition, they developed an analytical framework for sequential investments, where one of the assumptions was that the output price follows a geometric Brownian motion process. Dixit (1993) analyzed and concluded that increasing returns and uncertainty makes it optimal to wait for the largest project when irreversible mutually exclusive projects has uncertain output price. Décamps, Mariotti and Villeneuve (2006) extended this research to include parameter restrictions where the optimal investment region is divided into two parts.

According to Laine (1997), options to abandon and defer are most valuable for marginal oil fields. Abandonment options are especially valuable in marginal oil fields since they are estimated to last in twelve months only. Schwartz and Moon (2000) extended Laine’s study by including research and development (R&D) projects. These projects accounted for both uncertain costs and uncertain value of the completed project. Miltersen and Schwartz (2007) developed a real option framework valuing oil projects under uncertain maturity and competition. In this research the analysis was linked to monopolistic and duopolistic models, and included both abandonment- and switch options. A recent study by Chronopoulos and Siddiqui (2014) used a sequential investment framework to determine the optimal timing for replacement of an emerging technology. The study assumed uncertainty in both the output price and the arrival of new versions.

The work of Chronopoulos and Siddiqui (2014) is especially important for this study. Parts of the reasoning and processes used in current study are an application of their work. The study is also inspired by Miltersen and Schwartz (2007). Unlike Chronopoulos and Siddiqui (2014), the study focuses more on the investment decision under several different scenarios, and is applied to the oil industry. The study focuses on other real options than Miltersen and Schwartz (2007), but in a similar setting.

4. Oil Production

This chapter analyses the companies in the oil industry more closely, focusing on oil production. To start, an overview of a typical value chain for these companies is presented along with the costs associated with it. Thereby, an overview of the companies' different risks is presented, with focus on the oil price volatility.

4.1 Value Chain

In the petroleum sector, the value chain has three main categories; upstream, midstream and downstream, (EY, 2013) as shown in Figure 4-1. Most important for this study is the upstream section, and the focus is on development and production. Development includes deciding whether developing or not, which is dependent of the estimated production from the project.

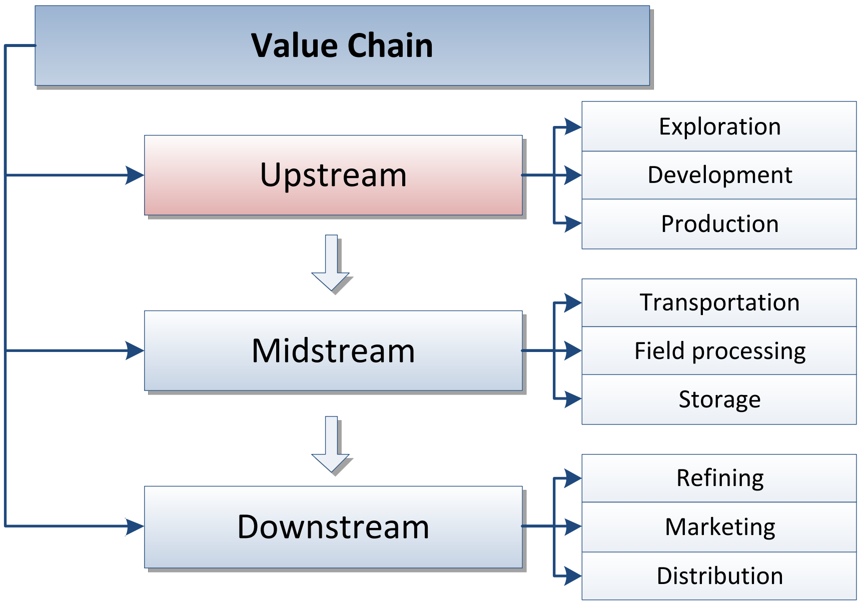

Figure 4-1: Value chain (EY, 2013)

A normal approach is that an international oil company has to bid for a license. When the company and its partners have won a bargain round for a license, a process for oil discoveries below the seabed is initiated. Oil discovery is hereby referred to as an exploration process. The exploration process is outsourced to companies offering seismic and electromagnetic services. If the exploration shows potential oil reservoirs, test drilling using mobile drilling units will be conducted for confirming the existence of the reservoirs.

If test drilling proves successful, further analyses of the reservoir size will be conducted by continued drilling verify the quantity and quality of the oil field. This is important for analyzing whether the project is profitable or not. If the projects prove profitable, the company takes on the project and the drilling of production wells. After finalizing drilling and completion of the wells, production starts. Regardless of the production well, oil service companies maintains and repairs the production equipment. The produced oil is transported onshore via pipelines or tankers. Produced oil is then sold for the spot price in the market, disregarding any hedging strategies (KPMG, 2013).

All these processes in the value chain produces cash flows, which makes it possible to estimate project values. The different incomes and expenditures that can occur are presented in Figure 4-2.

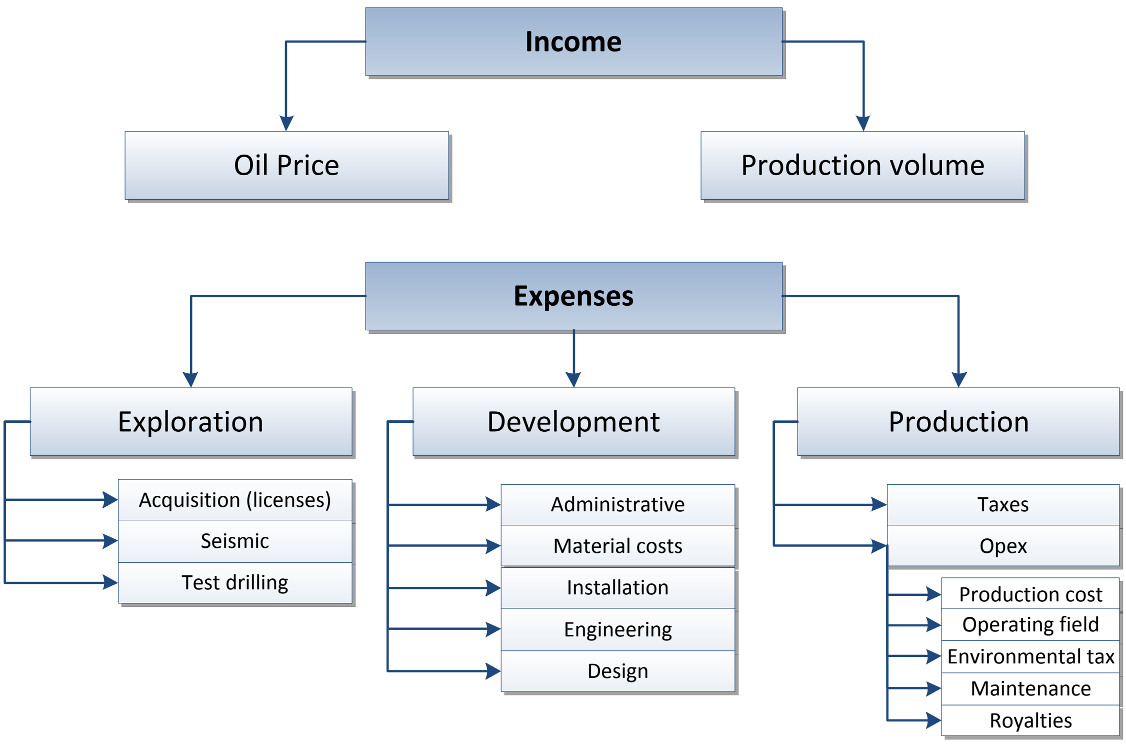

Figure 4-2: Income and expenses (Statoil, 2014)

4.2 Risks

Risks and uncertainties are keys to understand the value of flexibility in the petroleum industry. Approaching the subject from the industry's point of view, and at project level, firm specific risk is not considered to be mitigated through diversification, and should hence be acknowledged. This section provides an overview of the different uncertainties regarded important, both at project- and firm level. Figure 4-3 shows several different risks an oil company can be exposed to.

This study will limit project risks to the oil price volatility, but acknowledges that the company has more concerns when investing.

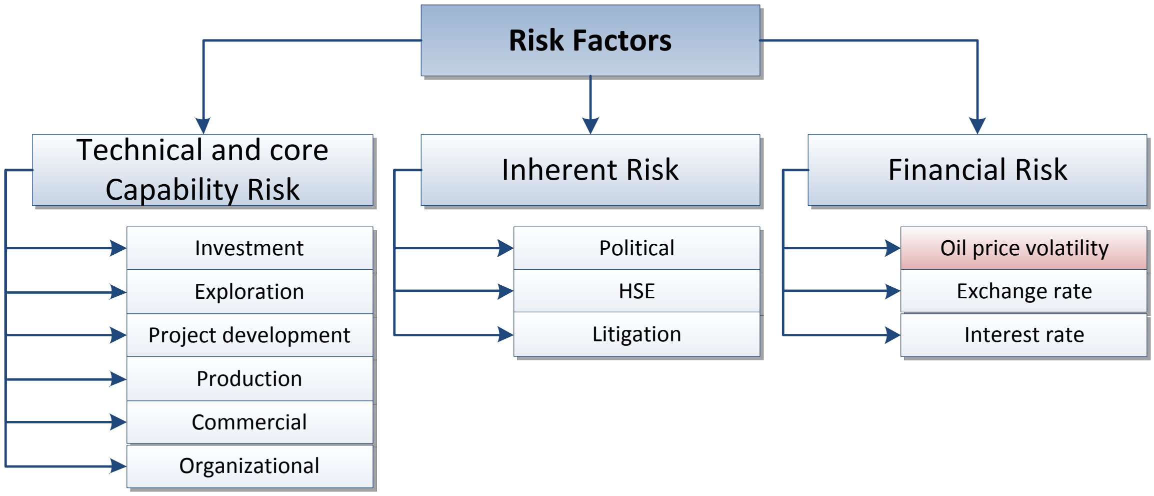

Figure 4-3: Risk factors (PTTEP, 2005).

4.2.1 Oil Price Volatility

The oil price is a critical risk that oil companies are facing. The oil price is the income per unit oil companies will get by selling their products. It is normally priced per barrel of crude oil, given in US dollars. As there are large costs attached to development and production, oil price is a risk which affects the company's performance.

In many cases, the oil price decides whether an investment is profitable or not. A normal standard for oil companies is to perform a breakeven analysis based on the oil price, and stop all investment if it is below a specific level. This breakeven price is normally different across different production fields. By example, Statoil's Johan Castberg project had a breakeven price of 85 USD per barrel as of October 6th 2013 (World Oil, 2013). This breakeven analysis mainly depends on the development costs, as development is the greatest cost in the value chain, and is assumed to be irreversible. One can expect development costs to rise unless new technology reduces the costs accordingly. The last decade has shown a smaller development in technology compared to development costs (Statistisk sentralbyrå, 2013). With increasing costs in the sector, one can expect the oil price to be increasingly critical for oil companies.

The oil price is subjected to factors beyond the company's control. Examples of price drivers are the relationship between demand and supply of the market, the Organization of the Petroleum Exporting Countries' (OPEC) production policy, geopolitical factors, oil reserves in individual countries, and the global climate (PTTEP, 2005). The petroleum industry is therefore at a tough state, especially due to concerns about the global climate and concerns about the scarcity of the crude oil. Figure 4-4 shows the monthly oil price fluctuations and development from 1987 to 2014.

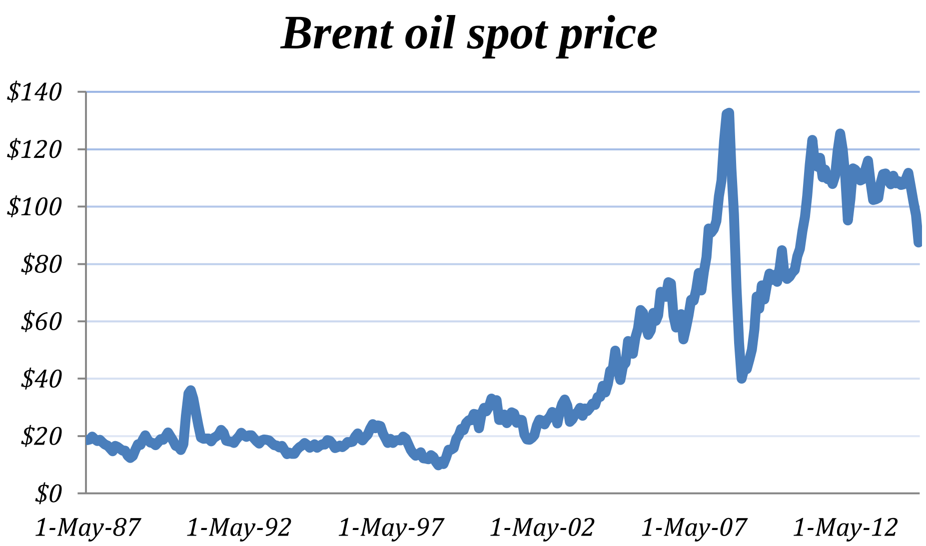

Figure 4-4: Historical oil price fluctuation 1987-2014 (US. Energy Information Administration, 2014).

One could argue that there has been an increasing demand for crude oil in this period, and that the price is sensitive to shocks in the economy. The volatility of the price fluctuations can also be observed from the graph. Some examples of annual volatility are presented in Table 4-1, where the volatility has been estimated based on historical monthly prices.

Time interval	Annualized volatility
1987 - 2014	30,45 %
2000 - 2014	29,56 %
2005 - 2014	27,98 %
2009 - 2014	21,03 %

Table 4-1: Examples annualized oil volatility

As of fall 2014, the oil price is at such a level that it is causing problems for several oil companies. Especially, as of December 2014, a recent drop below 70 USD per barrel is critical for many company's income, and makes them delay investment. According to oil-price.net, the situation might be caused by an oversupply in the current market (Oil-price.net, 2014).

5. Theory

This chapter presents the theoretical framework for addressing this study's research problem. First, background information regarding option basics will be presented. Thereafter, a review of Dynamic Programming and stochastic processes will be conducted.

5.1 Financial Options

To understand real options one should know the basics of financial options. As mentioned in Chapter 3, Black and Scholes pioneered option theory in 1973. Black and Scholes (1973) developed the seminal pricing model, called Black-Scholes formula. This is a method for pricing financial instruments of an underlying asset. This innovative tool has grown to become the fundament in modern option pricing.

Option trading is a contract between two parties. One part is the buyer, whereas the counterpart is the option writer. An option contract gives the holder a right, but not an obligation, to buy or sell an asset at a given price in the future. This price is defined as the strike price. An option to buy an asset is a call option, and the option to sell an asset is a put option. Call options are exercised if the spot price is above the strike price, also referred to as in the money. If the spot price is below the strike price, call options are out of the money whilst put options are in the money (Berk and DeMarzo, 2014). The payoff for call- and put options are illustrated in the following formula, and in Figure 5-1.

$$Payoff\ call\ option = max[spot\ price - strike\ price, 0]$$

$$Payoff\ put\ option = max[strike\ price - spot\ price, 0]$$

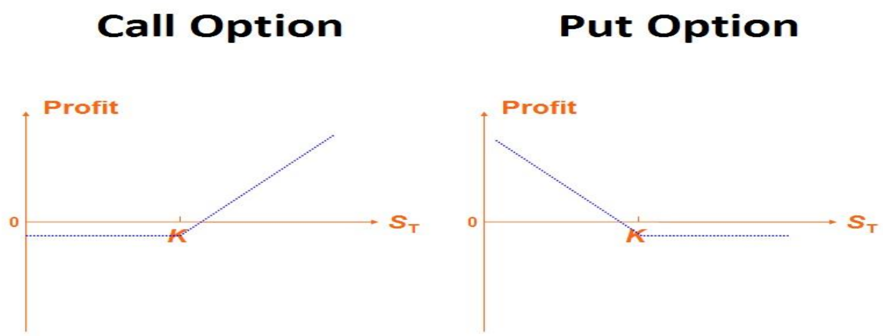

Figure 5-1: Call and put option payoff (Surly Trader, 2009).

An investor who is long in a call option, i.e. the buyer, has the opportunity to exercise the option. Thus, the counterpart who is short in the call option, i.e. the seller, is obligated to fulfill the contract if the option is exercised (Berk and DeMarzo, 2014). The option writer holds the risk of potentially having to finance the holders' gains. To compensate for this risk, the writer is awarded an option premium in advance, and therefore option value can never be negative (Berk and DeMarzo, 2014).

Within both call- and put options there are two main types; European and American. These can have limited maturity, or can be perpetual. A European option gives the owner an opportunity to exercise the option at a maturity date. An American option, on the other hand, gives the owner an opportunity to exercise the option up until the maturity date (Berk and DeMarzo, 2014). In general, American options are more difficult to handle, as European options only need to address the issue whether to exercise or not. An American option also has to address when to exercise. Due to the fact that American options can be exercised anytime, it cannot have a value below its European counterpart (Berk and DeMarzo, 2014).

5.2 Real Options

In financial theory, valuations traditionally use a standard discounted cash flow (DCF) method, and ultimately a net present valuation. This method is presented in the following equation (Berk and DeMarzo, 2014).

$$NPV(r, N) = \sum_{t=0}^{N} \frac{CF_t}{(1+r)^t} - I_0$$

Such methods does not account for flexibility in the investment process. Applying this approach therefore opens for the possibility of mispricing. The idea behind real option valuation is to apply framework from financial option theory to value projects with respect to the flexibility it offers (Koller, Goedhart and Wessels, 2010). A standard NPV-approach is usually a now-or-never approach, which is rejected if a negative net present value is obtained. Real option valuation offers flexibility, which considers that a negative NPV might be positive at another time due to changes in one or several variables (Koller et al., 2010). In Figure 5-2, drivers that contribute to additional value through flexibility, is presented.

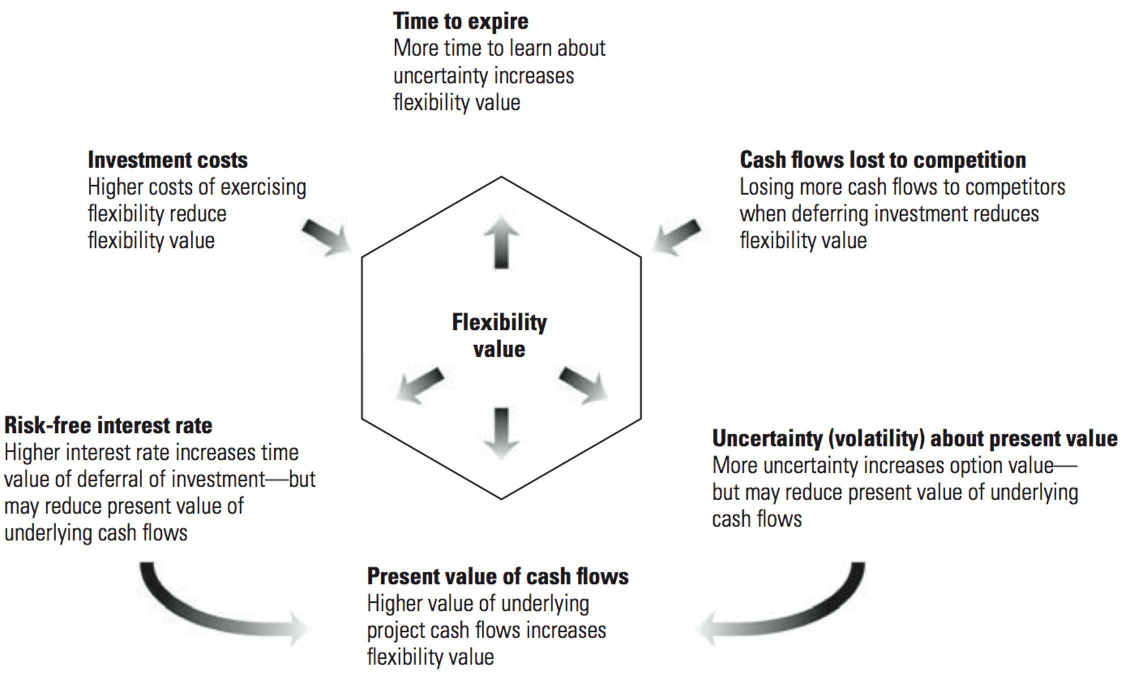

Figure 5-2: Drivers affecting value due to flexibility (Koller et al., 2010, p. 685)

Real options are often used for valuation of investment projects, and can appear as both European and American options. Including the value of flexibility is usually considered by using a Real-Option Valuation Method (ROV) or a Decision Tree Analysis (DTA) (Koller et al., 2010). The ROV-approach is similar to the valuation of financial options, and uses a model which replicates a portfolio similar to holding an option. The DTA approach considers different states in a decision tree, and discounts it with a subjective cost of capital, referred to as a discount rate. As will be explained in the next section, the DTA approach is the most viable approach for the cases studied in present report.

When accounting for flexibility, there are several possibilities which a manager or a company can consider. The following list presents the most normal kinds of options an investor should consider:

- Option to defer *Call*
- Option to abandon *Put*
- Option to switch *Call/Put*
- Option to expand *Call*
- Option to follow on *Call/Put*

5.3 Dynamic Programing

Dynamic Programming originates from Richard Bellman’s work from the 1950s, which describes a mathematical theory of optimal sequential decisions under uncertainty from the DTA process. In other words, DP breaks a given problem into different parts and combines the different solutions into an overall valuation, or a value function. DP moves systematically, and builds the best solutions as it goes. A common association with dynamic programming is decision trees, where the options are to reject or accept an action. One can work backwards towards the initial situation to find the most optimal solution. The DP approach makes calculations easier and less demanding than for example estimating large matrices when building the valuation model (Dixit and Pindyck, 1994).

Opposed to the method of Contingent Claims (CC), a constant risk-adjusted discount rate is used whereas CC uses a risk-neutral valuation (Dixit and Pindyck, 1994). DP solves the value using a constant discount rate that reflects the opportunity cost of capital. DP might be unclear on how the discount rate should be derived, and if it should be constant over time

(Dixit and Pindyck, 1994). On the other hand, an advantage of DP is that it is not dependent of having a variety of risky assets, which is required for perfectly replicate an uncertain investment. If the investment project does not have a twin asset, it is not possible to replicate. Hence, the investment project is illiquid, and not diversifiable.

Dynamic programing is ideal to optimize decisions under uncertainty like the casework described in this study. As oil reservoirs are illiquid assets, and not possible to replicate since all reservoirs are unique, CC is not the best suitable framework. Therefore, DP with a subjective discount rate is the preferred method in this case.

5.4 Stochastic Process

A stochastic process is a process where a variable follows a somewhat random pattern. According to Dixit and Pindyck (1994), at least one part has to be random for a process to be stochastic. A stochastic process is governed by probabilistic laws, which dictate its development over time. It is the opposite of a deterministic system as the process is a tool to describe infinite possibilities of the underlying variable's evolvement. Stochastic processes can have its form in either discrete time or continuous time.

The stochastic process in this study will be the explanation for the oil price development.

5.4.1 Standard Brownian Motion

A Brownian motion process, also known as a Wiener process, is the most common stochastic process. According to Dixit and Pindyck (1994), it has three important properties.

1. It follows a Markov process
2. It has independent increments
3. It is normally distributed

In a Markov process, the probability distribution for all future values depends on its current value only. It is unaffected by past values, i.e. it is without memory, and is not affected by any other information. The independent increments state that the probability distribution for changes is independent of any other time interval. The normal distribution of the process has a variance, which increases linearly with time (Dixit and Pindyck, 1994).

5.4.2 Standard Brownian Motion with Drift

One of the generalizations of the Wiener process is the standard Brownian motion with drift, which often is referred to as random walk. It is a stochastic differential equation (SDE), and the process is defined as in Equation (5.4.1) (Dixit and Pindyck, 1994).

$$dV = \mu dt + \sigma dz \tag{5.4.1}$$

This formula states that the value process is derived from one deterministic part and one stochastic part. For the stochastic part, σ defines the volatility of the underlying variable, and dz is the increment of a Wiener process. For the deterministic part, μ is the drift of the process. The drift can be characterized as a process following a trend, or a growth rate (Dixit and Pindyck, 2014). Figure 5-3 shows sampled Brownian motion with drift. The left hand side shows three samples, and the right hand side shows an optimal forecast with 66% confidence interval.

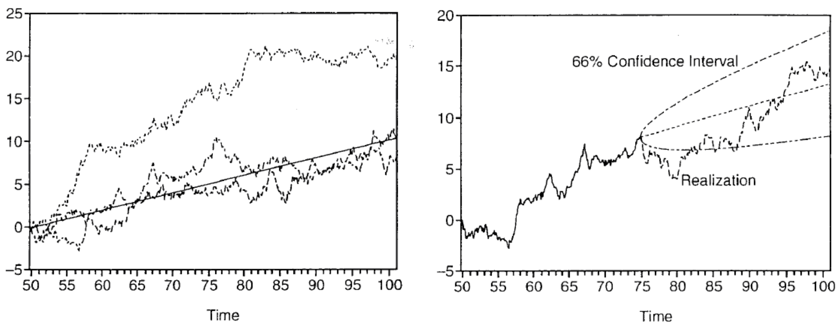

Figure 5-3: Drift (Dixit and Pindyck, 1994, p. 66-67)

5.4.3 Geometric Brownian Motion and Itôs Lemma

Unlike standard Brownian motion, geometric Brownian motion describes the evolution of a log-normal distributed variable. It is here a special case of Equation (5.4.1). Future values are log-normal distributed with a volatility that grows linearly with time. The process of a geometric Brownian motion SDE, can be written as in Equation (5.4.2) (Dixit and Pindyck, 1994).

$$dV = V\mu dt + V\sigma dz \tag{5.4.2}$$

Equation (5.4.2) assumes that today’s value of the project is known. V is the underlying variable and can be observed at any time, V_t. The continuous time stochastic process, V, is an Itô process. For more detail of the Itô process, see Dixit and Pindyck (1994). To determine the stochastic process of the oil price through a GBM and differentiate and integrate functions of Itô processes, one can apply Itô’s Lemma, where Itô’s Lemma can be understood as a Taylor series expansion (Dixit and Pindyck, 1994).

GBM is frequently used to model security prices, interest rates, output prices and other variables. Using this approach in modeling values of investment projects will therefore be suitable for this study. The model used assumes that the value process follows a GBM with drift, and that it can be expanded using Itôs Lemma.

5.4.4 Geometric Brownian Motion versus the Ornstein-Uhlenbeck Process

Another method commonly used for modeling the stochastic process of the oil price, is the Ornstein-Uhlenbeck process. This is often referred to as a Mean Reverting process (Dixit and Pindyck, 1994).

Geometric Brownian motion tend to wander far from the original starting point, which is realistic for some variables, but not for other (Dixit and Pindyck, 1994). Even if oil prices fluctuate randomly on short term, one could argue that the price over time is being drawn back towards the marginal cost of producing oil - a mean-reverting process. However, this study will not discuss which of these processes are the most realistic.

6. Applicability of Study

In this chapter, a study of the applicability of the casework is presented. It is shown how real options specifically can be applied to investment decisions in the petroleum industry. Firstly, a brief introduction of how the industry can choose to apply the different valuation approaches will be presented. In addition, an analysis of how real option theory creates value for the industry, and how it is applicable, will be discussed. Secondly, an analysis of whether oil companies actually expand existing projects or not, is presented. Thirdly, a review of the applicability of an option to switch to another project is conducted.

6.1 Valuation in the Industry

Amongst companies in the petroleum industry, there are differences regarding which practice is used in valuation of projects. From conversations with Rocksource ASA, a company operating on the Norwegian continental shelf, an anonymous employee implied that that use of real option theory as framework in decision strategies is not common practice. Instead, it practices NPV (Anonymous, 2014). Trusting this statement, they do not analyze which potential value options to expand productions or to switch to another project has.

6.1.1 Real Options in the Petroleum Industry

Real option theory is very applicable to investment decisions in the petroleum industry. It is an industry with high irreversible investment costs, and is very sensitive to the volatile oil price. Therefore, the value of flexibility is of great importance when making optimal investment decisions. This subsection will look closer at the application of the options listed in the Section, 5.2, and why it is important to consider this for the petroleum industry.

Option to Defer Investment

The option to defer an investment is equivalent to a financial call option on a stock (Koller et al., 2010). The strike price corresponds to the investment cost for developing an oil field. In other words, it is an option to invest at a later stage. Regarding an investment as an option, and not a now-or-never approach is a flexibility which should increase the value of an investment opportunity.

If the oil price when investing is low, a NPV approach would probably provide a negative value, and hence the considered project will be rejected at the time. This could lead to mispricing of the project. By waiting for better market conditions, the company can invest in the project at a later stage. This flexibility should be included in the valuation of the project to avoid mispricing.

This scenario is especially relevant today, December 2014, as the oil price is at a low level compared to the last five years. Rejected projects with regards to standard DCF, with a negative NPV today, might prove to be valuable in the future.

Option to Abandon a Project

The option to abandon a project is equivalent to a financial put option on a stock, with a strike price corresponding to the liquidation value of a project. This is a flexibility which is valuable when an investment project performs poorly (Koller et al., 2010). A standard NPV approach assumes that the processes of the projects are on-going, and therefore exaggerates a negative value which would provide a mispricing of a project.

If the oil price is at a low level, and the project contributes negatively for a company, abandoning the project may be an alternative. Abandonment can take place if the value of a project falls below its liquidation value (Koller et al., 2010). An abandonment of a producing oil field corresponds to shutting down production and selling the project to another market participant. This is because abandonment by removing drilling- and production facilities are very costly.

Option to Switch

The option to switch is equivalent to both a financial call- and put option, and the cost of switching corresponds to its exercise price. A switch option has two interpretations. The first interpretation is the option to set a project passive or active, where activating the project is a call option, and setting it passive is a put option. The second interpretation is an option to switch to another and more lucrative project after extracting the first one (Koller et al., 2010).

Miltersen and Schwartz (2007) describes the case where an investor can set a project passive or active based on the oil price fluctuations, or abandon the project if the price is very low. This switching gives the operating company and the partners flexibility to extract the resources when the market price is beneficial. Later in this study, the option to switch to another project will be discussed.

Option to Expand a Project

The option to expand a project is equivalent to a call option on a stock, and the corresponding exercise price is the investment cost of the expansion. The option provides a company the flexibility to make follow-on investments on a project.

In petroleum investment, expansion is very relevant, as expanding a developed field is less costly than installing new surface production facilities. When the original project is installed, and the resources extracted, the company could expand it to possible nearby satellite fields. As the production equipment already is installed, the only investment costs will be drilling new wells, and a possible upgrade of the production equipment.

Option to Follow-on Investments/Embedded Options

Technically, embedded options are options on options. It gives the holder or the issuer an option to perform a specified action in the future. Embedded options are phased investments where the management can invest at a later stage to exercise a new option.

Embedded options in the petroleum industry can by example be the expansion option and switch option mentioned earlier. If having an option to invest in a drilling project, a follow-on option can be an option to expand it later, or switch to another project.

6.2 Expanding Mature Fields by Including Nearby Satelite Fields

On the Norwegian continental shelf, the Norwegian Government offers licenses without bargain rounds to oil companies (The Norwegian Government, 2014). After receiving these licenses, oil companies can start exploring and extract oil in the designated areas. Each year Awards in Predefined Areas (APA) is offered. APA allows companies to conduct further exploration in well-established exploration areas on the Norwegian shelf. The Norwegian Government offers APA, and companies with its partners get a share of a license close to a mature field, fields that already are producing, and have done so for a significant time (The Norwegian Government, 2014). Companies must however do further data gathering, e.g. by seismic of the assigned satellite field to try to locate where it is optimal to do test drilling, and provide estimates of the field size. Such satellite fields are more likely to contain less production volume. Extracting oil in these smaller satellite fields using the installations of the main field, might be profitable since it reduces capital expenditures (CapEx) significantly. Quote by the Norwegian Government: “Small discoveries cannot justify stand-alone developments, but may have good profitability when they can exploit existing and planned process equipment and transportation systems, or be coordinated with other planned developments” (The Norwegian Government, 2014).

The Norwegian Petroleum Department (NPD) states licenses which are offered have a “drill or drop” condition, meaning that the license owner has up to three years to decide whether they shall drill an exploration well or abandon the license (The Norwegian Petroleum Department, 2014). This implies that this specific license has a three-year European call option to do exploration for the nearby fields. Assuming that every company is offered a license share each year, it is realistic to assume that an option to utilize these licenses can be viewed as perpetual American options.

6.3 Switching from One Project to Another

Oil reservoirs have finite lifetime. In order to make sure of an oil company’s long-term existence, continuous exploration for oil reservoirs must be done. Once a reservoir is extracted, the oil company can either choose to invest, or choose to downsize total productions. Should the company choose to reinvest, it can expand to nearby fields, abandon wells and re-drill new targets, or new wells can be drilled. Being offered a license for an alternative area makes it possible for an oil company to switch to the new reservoir when the ongoing field is emptied.

7. Defining the Problem

The ambition of this study is to analyse the effect real options have on an operating company's investment decision. To effectively analyse this effect, a specific case study will be applied. The case study will compare the optimal investment strategy when deferring an investment to a standard NPV analysis. Thereafter, the study will discuss how the embedded options alter this decision.

This study proposes a framework based on the work of Miltersen and Schwartz (2007), and Chronopoulos and Siddiqui (2014). A company optimizes its investment strategy in a monopoly setting where the company is the only operator with the option to invest in the projects. As all other investment problems, the ambition of this study is to suggest a better way of making investment decisions. Although the use of the ordinary NPV approach is common in this regard, this study will provide an overview based on real option theory. This chapter will provide a simple overview of the case study.

7.1 The Casework

The base case is that a company can choose between two projects, project A and B. These projects have some different characteristics. The projects are both mutually exclusive. Hence, a company can maximum take on one of these projects when making the investment decision.

7.2 Characteristics of the Projects

Project A and B has known reservoir sizes, which a company always knows are there. Neither of the projects has been invested in yet, and it is assumed that a company has an American perpetual option to invest in both projects. Hereby, this company will only invest in the projects at its optimal stopping point, at time t. The stopping time corresponds to the point where a company exercises the option to invest in the project, hereby called the optimal investment threshold.

The oil price is the random variable which provides value to the option of deferring an investment.

In the oil industry, it is reasonable to assume that developed oil fields do not have perpetual cash flows. Sometime during the extraction, the production volume will eventually vanish. Most likely, the extraction will decrease as the projects mature. When the project is maturing, the production volume decreases accordingly. Accounting for this, a suggestion from Dixit and Pindyck (1994) is applied. It is assumed that the values of the projects have infinite lifetimes with a value-decreasing factor, γ.

7.2.1 Embedded Options

Unlike project B, project A has embedded options attached. The only option project B has is the option to defer investment, which is also a feature for project A. The main characteristic of project B is that it has a larger oil reservoir, and can be extracted at a greater investment cost.

The first embedded option of project A is that a company has an option to expand production. This expansion is called project E. In practice, this means that after the production has started, the company has an option to do further exploration in the area nearby the main field. By using the developed field in place, the company can extract oil from nearby satellite fields at a far less CapEx than starting a new drilling process elsewhere.

The second embedded option of project A is a switch option. The idea for this option is based on the switch option suggested by Miltersen and Schwartz (2007). This study will extend this method. The switch option in this model is the possibility to switch from project A to project B after deciding to invest in both project A and project E. The company has to carry a new investment cost to activate project B.

8. Assumptions and Notation

In this study, an analysis of a price-taking decision-maker's (an investing company), decision is analyzed in m = 1, 2 or 3 different scenarios. A scenario defines how many options that are available for the investing company. The scenarios are: 1) Valuation of- and comparison between project A and project B, including the option to defer investment. 2) Valuation of project project A with option to defer and expand , and comparison with project B. 3) Valuation of project A with option to defer, expand and switch to project B, and comparison with project B.

At time $\tau_{n,k}^{(m)}$, the company is considering scenario *m*, operating project *n*, and has the option to invest in project *k*. Notice that when *n* = 0, the company is not operating any projects, and when *k* = 0, there are no further options available. At time $\tau_{n,k}^{(m)}$, the optimal investment threshold for project *k*, when operating project *n*, is ${P^*}_{n,k}^{(m)}$. By example, consider time $\tau_{0,A}^{(1)}$. The company considers scenario 1, operates no projects, and the subscript, *A*, denotes the option to invest in project A. At this time, it is optimal to invest in project A at its corresponding optimal investment treshold, ${P^*}_{0,A}^{(1)}$. Consider time $\tau_{A,E}^{(2)}$, where the company has the options from scenario 2. The company is currently operating project A, and has the option to expand to project E. This takes place at its corresponding optimal investment treshold, ${P^*}_{A,E}^{(2)}$.

At time $\tau_{n_v k}^{(m)}$, the company can choose to invest in either project *n* or project *k*, where *v* is defined as "or". Hence, $\tau_{n_v k}^{(m)}$ denotes the company's final investment decision in scenario *m*. By example, at time $\tau_{A_v B}^{(1)}$, the company can invest in either project A or project B, while having only the option to defer investment as defined in scenaro 1.

An option value function is denoted as $F_{n,k}^{(m)}(\cdot)$. It is the maximized expected NPV from investing in project *k* in scenario *m*, given that the company operates project *n*. $V_{n,k}^{(m)}(\cdot)$ denotes the expected project value function from operating the active project, *n*, in scenario *m*. If $k \neq 0$, the expected project value includes an embedded option to invest in project *k*. If the availability of an embedded option depends on the success of an exploration, the expected value of the project is denoted as $\overline{V}_{n,k}^{(m)}(\cdot)$.

The model is based on comprehensive financial mathematics. It is therefore not viable to include all calculations in the main part of the study. Therefore, in Chapter 9, it is often referred to the appendix. By example, referring to Equation (A-1.1) refers to Equation (1.1) in the appendix.

The success of an exploration process is assumed to follow a Poisson process. Parameter λ denotes the intensity of the Poisson process and is independent of the continuous time, *t*. Hence, with probability λdt, the exploration will prove successful, and with probability $1 - \lambda dt$ it will prove unsuccessful (Dixit and Pindyck, 1994).

Initiating production of project *n* requires an investment cost, I_n. Keeping the project alive requires ongoing costs, which for simplicity is assumed to be included in the investment cost. Additionally, the output from the projects are called the production volume, which is denoted D_n. Finally, the underlying oil price at time *t*, P_t, is independent of the Poisson process and follows a geometric Brownian motion as described in Subsection 5.4.3, and shown in Equation (8.1).

$$dP = P\mu dt + P\sigma dz \tag{8.1}$$

9. Model

This chapter creates a model for valuation of the investment problem described in Chapter 7. It provides a framework for optimal investment decision-making.

It is assumed that an investing company has the option to either invest in project A or project B. To analyze which impact the embedded options have on the initial investment decision, a study of three separate scenarios is conducted.

1. Valuation of- and comparison between project A and project B, including the option to defer investment.
2. Valuation of project A with option to expand, and comparison with project B.
3. Valuation of project A with option to expand and to switch to project B, and comparison with project B.

Scenario 1

In the first scenario, the option to defer investment in both projects is analysed. The company can wait for better oil prices before deciding to invest. The analysed value functions are therefore the value functions of both projects, and the option value to invest in them. The scenario concludes with a framework for optimal decision-making when choosing between the projects. The option to choose between project A or B is denoted as $F_{A_v B}^{(1)}(P)$. The value functions for project A and B is denoted as $V_{A,0}^{(1)}(P)$ and $V_{B,0}^{(1)}(P)$, while their corresponding option values are denoted as $F_{0,A}^{(1)}(P)$ and $F_{0,B}^{(1)}(P)$. Valuing the value functions in scenario 1 implies that no embedded options are included, but only the option to defer investment.

Scenario 2

In the second scenario, the first scenario is expanded by adding the option to expand project A. This scenario is divided into two subsections. The first subsection assumes that the expansion is known, while the second subsection will take the uncertainty regarding an exploration process into account. Both sections conclude with a framework for optimal decision-making when choosing between project A and B. The investment decision where the existence of the expansion is known is denoted as $F_{A_v B}^{(2)}(P)$, while the situation where it is unknown is denoted as $\overline{F}_{A_v B}^{(2)}(P)$. Project A's value function must be redefined as it contains the value of having the option to expand to project E. The value of project A is denoted by $V_{A,E}^{(2)}(P)$ when the expansion is certain, and $\overline{V}_{A,E}^{(2)}(P)$ when the expansion is uncertain. The value function of the expansion is denoted as $V_{E,0}^{(2)}(P)$ and it's option value is $F_{A,E}^{(2)}(P)$. Valuing the value functions in scenario 2, implies that the option to expand is accounted for.

Scenario 3

In the third scenario, scenario two is expanded by accounting for the option to switch to project B. This scenario is into two subsections; where the expansion is certain, and where there has not been conducted an exploration process yet. The investment decision where the existence of the expansion is known is denoted as $F_{A_v B}^{(3)}(P)$, and the situation where the existence of the expansion is unknown is denoted as $\overline{F}_{A_v B}^{(3)}(P)$. Thereby, the value function of project A is denoted as $V_{A,E}^{(3)}(P)$ when the expansion is certain, and $\overline{V}_{A,E}^{(3)}(P)$ when there is uncertainty. Project E includes the option to switch to project B, thus the value function must be redefined. Therefore, the project value of the expansion is denoted as $V_{E,B}^{(3)}(P)$ and the option value is denoted as $F_{A,E}^{(3)}(P)$. In addition, the project value of project B when switching to it will be added, and is denoted $V_{B,0}^{(3)}(P)$ with the corresponding option value denoted as $F_{E,B}^{(3)}(P)$. Valuing the value functions in scenario 3, implies that all options are accounted for.

Estimating today's value using real option theory depends on future values. In order to determine the value functions in each scenario of the subsections, a backward process will be conducted, starting with the last project value in the scenario.

9.1 Valuation of- and Comparison between Project A and Project B, Including the Option to Defer Investment

This section considers the simplest scenario where project A does not have any embedded options, and compares its attractiveness opposed to project B. The company will has the option to defer investment in both projects. Technically, the modelling of these two project values is the same, and only the input values will differ. The calculations will therefore not distinguish between these two values until the final part of the section where an investment decision is made.

The section also provides an insight in the calculations, assumptions and formulas used in the model. These calculations are reusable for later stages in the study. Therefore, the calculations in the two coming subsections are of general character before it models specifically later in this chapter.

9.1.1 Project Value

The expected project values are here denoted as $V_{n,k}^{(m)}(P)$, where the project value is the expected net present value of investing in the projects.

$V_{n,k}^{(m)}(P)$ at time t can be expressed as the sum of the operating revenues, D_nP, over the interval t, $t+dt$, and the continuation value beyond $t+dt$, adjusted for investment costs, I_n. Notice that investment costs are fixed over the time period.

There is also a probability, γdt, that the project dies during the next short time interval. The γ represents this probability in Equation (9.1.1), and theoretically functions as a depreciation of the total project value (Dixit and Pindyck 1994). In this case, it is added to make sure that the resources has decreasing infinite profit flows. This factor is added because oil projects in reality have finite lifetimes. Not including a value decreasing parameter would hence overestimate the project values. The value of the projects with an option to defer investments is shown in Equation (9.1.1).

$$V_{n,k}^{(m)}(P) \; = D_nPdt + \varepsilon\left[V(P+dP)e^{-(\gamma+\rho)dt}\right] \qquad (9.1.1)$$

Equation (9.1.1) is built from the suggestions by Dixit and Pindyck (1994). The formula illustrates that $V_{n,k}^{(m)}(P)$ is dependent of a deterministic and stochastic part. In the stochastic part, dP is an increment of the Wiener process. dP can be expanded using Itô's Lemma. By reorganizing, $V_{n,k}^{(m)}(P)$ evolves into a non-homogenous ordinary differential equation (ODE). As proven in (A-1.1) – (A-1.4), it takes the form of Equation (9.1.2).

$$\frac{1}{2}\sigma^2 P^2 V_{n,k}^{(m)''}(P) + \mu P V_{n,k}^{(m)'}(P) - (\gamma + \rho) V_{n,k}^{(m)}(P) + D_n P \ = 0 \qquad (9.1.2)$$

In the value estimation, a homogenous ODE must be obtained. From substitutions, transformations, and reorganizations, Equation (9.1.2) converts into a homogenous ODE with constant coefficients, as proven in (A-1.4) – (A-1.9).

$$\frac{1}{2}\sigma^2 z'' + \left(\mu + \frac{1}{2}\sigma^2\right) z' + (\mu - \gamma - \rho) z = -D_n \qquad (9.1.3)$$

After further reorganization, Equation (9.1.3) can be written as a more intuitive interpretation, Equation (9.1.4). Evidence is shown in (A-1.9) – (A-1.12). Notice that the equation describes the present value of a perpetual stream of cash flows, but with a value decreasing factor, γ. It is important to be aware of that $0 < \rho \ > \mu$ must hold. If not, the project values will become either zero or negative.

$$V_{n,k}^{(m)}(P) = \int_0^\infty \frac{\gamma e^{-\gamma t} D_n P \left(1 - e^{-(\rho - \mu) t}\right)}{\rho - \mu} dt - I_n = \frac{D_n P}{\gamma + \rho - \mu} - I_n \qquad (9.1.4)$$

9.1.2 Option Value

This subsection provides the elements of the option to invest in the projects at time *t*, where $F_{n,k}^{(m)}(P)$ denotes the option value function. The fact that the option is perpetual is an element which derives from the assumption that the company operates in a monopoly world, and would never face competition about the oil reservoirs. The American element of the option gives the company flexibility to exercise the option at whichever time it finds optimal.

There are only two possible outcomes at time *t*:

1. Option is in or at the money: The company exercises the option
2. Option is out of the money: The company defer investing

If the option is in the money, the company gets the project value by investing, which was solved for in the previous subsection. The definition of what happens when the option is out of the money follows.

The option value follows the same processes as the project value and by expanding the stochastic process with Itô's lemma, the following expression stands.

$$\frac{1}{2}\sigma^2 P^2 F_{n,k}^{(m)''}(P) + \mu P F_{n,k}^{(m)'}(P) - \rho F_{n,k}^{(m)}(P) = 0 \tag{9.1.5}$$

The option element does obviously not have a profit flow, so it cannot have the same solution as the project value. According to Dixit and Pindyck (1994), this equation has the following boundary conditions:

$$\begin{aligned}
&(1): \quad F_{n,k}^{(m)}(0) = 0 \\
&(2): \quad F_{n,k}^{(m)}\left(P^{*}{}_{n}^{(m)}\right) = V_{n,k}^{(m)}(P^{*}{}_{n,k}^{(m)}) - I_n \\
&(3): \quad F'{}_{n,k}^{(m)}\left(P^{*}{}_{n,k}^{(m)}\right) = V'{}_{n,k}^{(m)}(P^{*}{}_{n,k}^{(m)})
\end{aligned} \tag{9.1.6}$$

$P^{*}{}_{n,k}^{(m)}$ is the optimal investment threshold for exercising the option. Boundary condition 1 states that the option to invest will be of no value when $P = 0$. Boundary condition 2 is the value-matching condition, and condition 3 is the smooth-pasting condition. The value-matching condition states that the value of the option must equal the net value obtained by exercising it, and the smooth-pasting condition states that the option value and the net value of the investment should meet tangentially at the optimal investment threshold (Dixit and Pindyck, 1994).

The equation for this option is a homogenous linear equation of second order, and the solution to this expression is a linear combination of any two linearly independent solutions, like the following expression suggested by Dixit and Pindyck (1994):

$$F_{n,k}^{(m)}(P) = A_{n,k}^{(m)} P^{\beta_1} + B_{n,k}^{(m)} P^{\beta_2} \tag{9.1.7}$$

$A_{n,k}^{(m)}$ and $B_{n,k}^{(m)}$ are here constants, but unknown at this point.

As shown in (A-1.16) – (A-1.20), Equation (9.1.5) can be written as the quadratic Equation (9.1.9).

$$\frac{1}{2}\sigma^2\beta'' + \left(\mu - \frac{1}{2}\sigma^2\right)\beta' - \rho\beta = 0 \tag{9.1.10}$$

This solves into the roots, β_1 and β_2:

$$\beta_1 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}}{\sigma^2} \tag{9.1.11}$$

$$\beta_2 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) - \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}}{\sigma^2} \tag{9.1.12}$$

As explained in the previous subsection, $0 < \rho > \mu$ is necessary for this model to make economic sense. This condition provides information stating that $\beta_1 > 1$, and $\beta_2 < 0$. When considering boundary condition 1, the constant with the negative root, $B_{n,k}^{(m)}$, should be given the value of zero to ensure that $F_{n,k}^{(m)}(P)$ goes to zero when P goes to zero (Dixit and Pindyck, 1994). Therefore, $B_{n,k}^{(m)}$ is set to zero, and the solution for the option to invest in the projects stands as Equation (9.1.13):

$$F_{n,k}^{(m)}(P) = \begin{cases} A_{n,k}^{(m)} P^{\beta_1} & , P < P^{*}{}_{n,k}^{(m)} \\ \dfrac{D_n P}{\gamma + \rho - \mu} - I_n & , P^{*}{}_{n,k}^{(m)} \leq P \end{cases} \tag{9.1.13}$$

$P^{*}{}_{n,k}^{(m)}$ is the optimal investment threshold where the option should be exercised. The first branch reflects the situation where the option is out of the money, $P < P^{*}{}_{n,k}^{(m)}$, and the company still has the option to invest at a later stage. The second branch shows the situation where the option is in the money, $P^{*}{}_{n,k}^{(m)} \leq P$. The company can extract the project value of the projects by investing I_n.

The solution for $F_{n,k}^{(m)}(P)$ has two unknowns, $A_{n,k}^{(m)}$and $P^{*}{}_{n,k}^{(m)}$. $A_{n,k}^{(m)} P^{\beta_1}$ is only valid when the option is out of the money, or in other words; when it is still optimal to hold the option and not to invest. It is clear that the incentive to invest rises with an increase in P. Therefore, there should be a point, $P^{*}{}_{n,k}^{(m)}$, where it is optimal to invest.

The value-matching and smooth-pasting conditions mentioned in the previous subsection will help determining these values.

Substituting $F_{n,k}^{(m)}(P) = A_{n,k}^{(m)} P^{\beta_1}$ into boundary condition (2) and (3) makes it possible to rewrite the value-matching and smooth-pasting conditions like the following equations:

(2) $$A_{n,k}^{(m)} {P^*}_{n,k}^{(m)\beta_1} = \frac{D_n {P^*}_{n,k}^{(m)}}{\gamma + \rho - \mu} - I_n \tag{9.1.14}$$

(3) $$\beta_1 A_{n,k}^{(m)} {P^*}_{n,k}^{(m)\beta_1} = \frac{D_n {P^*}_{n,k}^{(m)}}{\gamma + \rho - \mu} \tag{9.1.15}$$

Solving and rearranging, the two unknowns, $A_{n,k}^{(m)}$ and ${P^*}_{n,k}^{(m)}$, are presented in Equation (9.1.16) and (9.1.17)

$$ {P^*}_{n,k}^{(m)} = \frac{\beta_1 I_n (\gamma + \rho - \mu)}{D_n (\beta_1 - 1)} \tag{9.1.16}$$

$$A_{n,k}^{(m)} = \frac{D_n {P^*}_{n,k}^{(m)\,1-\beta_1}}{\beta_1 (\gamma + \rho - \mu)} \tag{9.1.17}$$

All proofs can be found in (A-1.23) – (A-1.30).

This concludes the general approach of the problem. The next subsection will apply the methods to the described scenario.

9.1.3 Appliying Model to Project A and B: Scenario 1, $\mathbf{F_{A_vB}^{(1)}(P)}$

After setting the foundation of this framework, it is time to be more specific. This subsection will first briefly present both investments separately, and secondly compare them to each other in a scenario where the company must choose between them. The values, which will be calculated is presented in Figure 9-1.

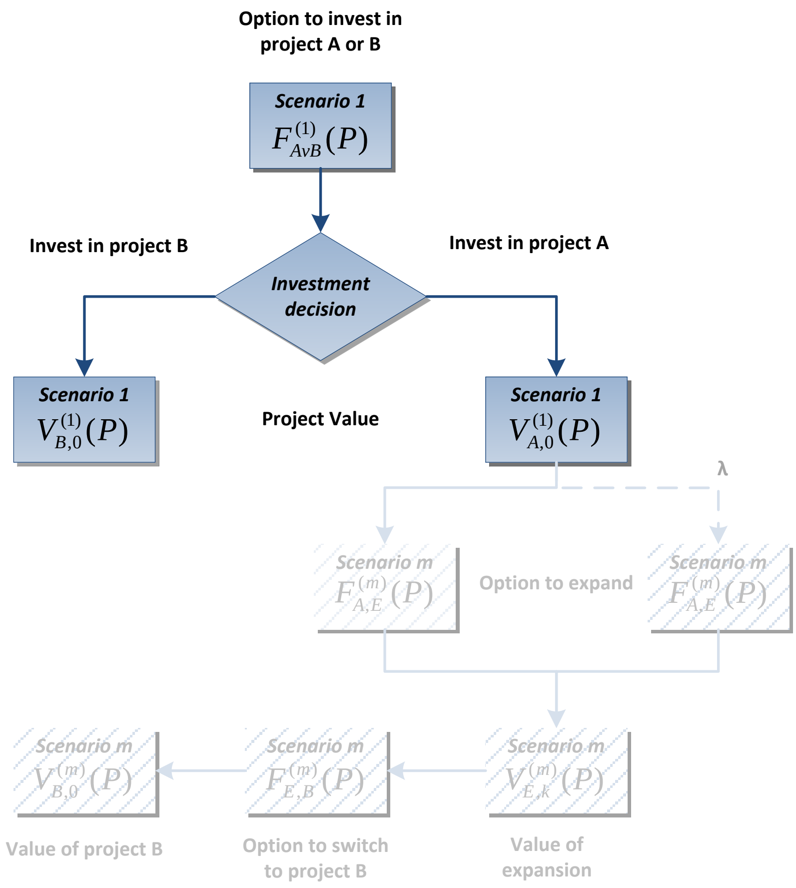

Figure 9-1: Value functions scenario 1

Using the framework from Subsection 9.1.1, the project values of project A and B, if the company invests, takes the following form.

$$V_{A,0}^{(1)}(P) = \frac{D_A P}{\gamma + \rho - \mu} - I_A \tag{9.1.18}$$

$$V_{B,0}^{(1)}(P) = \frac{D_B P}{\gamma + \rho - \mu} - I_B \tag{9.1.19}$$

When not considering the choice between the projects, and reusing the general approach from Subsection 9.1.2, the option to invest in project A and B must take the following forms.

$$F_{0,A}^{(1)}(P) = \begin{cases} A_{0,A}^{(1)} P^{\beta_1} & , P < P^{*(1)}_{0,A} \\ V_{A,0}^{(1)}(P) & , P^{*(1)}_{0,A} \leq P \end{cases} \tag{9.1.20}$$

$$F_{0,B}^{(1)}(P) = \begin{cases} A_{0,B}^{(1)} P^{\beta_1} & , P < P^{*(1)}_{0,B} \\ V_{B,0}^{(1)}(P) & , P^{*(1)}_{0,B} \leq P \end{cases} \tag{9.1.21}$$

In these equations $D_B > D_A$ and $I_B > I_A$ must hold due to the characteristics of the projects. It becomes clear investment in A and B will happen at different optimal thresholds. Since project B has greater fixed investment costs, this subsection assumes that project B's investment threshold must be higher than project A's.

This subsection regarded the value functions without including an investment decision. The next section will analyze the situation where the company can choose between project A and project B.

Investment Decision: Scenario 1, $F_{A_v B}^{(1)}(P)$

Consider a case where the price is very low, and the company has an option to invest in both projects. It will not invest in either of these projects until the price is high enough. The option is defined in the matter of an endogenous constant multiplied with the respective price, and the price is elevated with a positive root to define the possibility of the price increasing. In this scenario, option value $F_{0,A}^{(1)}(P)$ can help solving the endogenous constant and the optimal investment threshold via the value-matching and smooth-pasting conditions. The solutions for these are the following.

$$P^{*(1)}_{0,A} = \frac{\beta_1 I_A (\gamma + \rho - \mu)}{D_A(\beta_1 - 1)} \tag{9.1.22}$$

$$A_{0,A}^{(1)} = \frac{D_A {P^{*(1)}_{A}}^{1-\beta_1}}{\beta_1(\gamma + \rho - \mu)} \tag{9.1.23}$$

The option to invest is denoted as $A_{0,A}^{(1)}P^{\beta_1}$, while $P^*{}_{0,A}^{(1)}$ denotes the first investment threshold where it is optimal to invest in project A. The company will obviously not invest when the price is 0, so the option is defined in the interval $0 \leq P < P^*{}_{0,A}^{(1)}$.

The solution for $F_{0,B}^{(1)}(P)$ only describes an option value where positive price fluctuations are relevant, as the constant with the positive beta, $A_{0,B}^{(1)}$, is the only one considered. It seems unlikely that it is optimal to invest in one of the projects at every price between the optimal investment threshold of project A and the optimal investment threshold of project B. Thereby, the company has to expect two waiting regions in this case, where the optimal decision is to wait. The first waiting region is the price interval before investing in project A, and the second waiting region is the interval between project A and B where it is optimal to wait.

The second waiting region is in fact an option to invest in both projects. The option to invest in project A is at the money at a low threshold, and the option to invest in project B is at the money at a high threshold. The waiting region should therefore be expressed in a matter of the price both decreasing and increasing. If the scenario of the waiting region described above holds, the solution for $F_{0,B}^{(1)}(P)$ cannot be correct when choosing between project A and B, as it only considers the positive price fluctuations. The solution must include both the positive and negative solution of the quadratic function. The waiting region is therefore defined as $C_{0,A_vB}^{(1)}P^{\beta_1} + E_{0,A_vB}^{(1)}P^{\beta_2}$. If the price decreases sufficiently in the second waiting region, the company invests in project A. However, if the price increases sufficiently in the second waiting region, the optimal decision is to invest in project B.

Project B's value were previously defined as $V_{B,0}^{(1)}(P)$, with an optimal investment threshold, $P^*{}_{0,B}^{(1)}$. When the company faces its investment decision, the optimal threshold for investing directly in project B, $P^*{}_{\overline{0,B}}^{(1)}$, needs to be defined. The bar in the subscript implies that the threshold is different from when analyzing the project isolated. This is a result of having a second waiting region.

Consider a case where the price is very high. Since project B has higher production rate than project A, and the higher investment cost is fixed, project B should intuitively be preferred to project A at higher prices. The company will in other words invest directly in project B when $P^{*(1)}_{\overline{0,B}} \leq P$, where $P^{*(1)}_{\overline{0,B}} \neq P^{*(1)}_{0,B}$.

Consider that investing in project A is optimal in a certain interval, $P^{*(1)}_{0,A} \leq P < P^{*(1)}_{\overline{0,A}}$, where the bar is added to show that the thresholds are not equal. $P^{*(1)}_{\overline{0,A}} < P < P^{*(1)}_{\overline{0,B}}$ defines the region where it is optimal to defer investment, i.e. $C^{(1)}_{0,A_vB}P^{\beta_1} + E^{(1)}_{0,A_vB}P^{\beta_2}$. If the price drops to or below $P^{*(1,)}_{\overline{0,A}}$, the optimal investment decision is to invest in project A. However, if the price increase to or above $P^{*(1)}_{\overline{0,B}}$, investing in project B is the optimal decision.

Considering all possibilities of the investment decision, the solution takes the following form.

$$F^{(1)}_{A_vB}(P) = \begin{cases} A^{(1)}_{0,A}P^{\beta_1} & , P < P^{*(1)}_{0,A} \\ V^{(1)}_{A,0}(P) & , P^{*(1)}_{0,A} \leq P \leq P^{*(1)}_{\overline{0,A}} \\ C^{(1)}_{0,A_vB}P^{\beta_1} + E^{(1)}_{0,A_vB}P^{\beta_2} & , P^{*(1)}_{\overline{0,A}} \leq P \leq P^{*(1)}_{\overline{0,B}} \\ V^{(1)}_{B,0}(P) & , P^{*(1)}_{\overline{0,B}} \leq P \end{cases} \tag{9.1.24}$$

There are four unknowns in this equation, two optimal investment thresholds, $P^{*(1)}_{\overline{0,A}}$ and $P^{*(1)}_{\overline{0,B}}$, and two endogenous constants, $C^{(1)}_{0,A_vB}$ and $E^{(1)}_{0,A_vB}$. Solving it proves complex mathematically, and is therefore solved numerically. The answers depend on the inputs used, so the thresholds are presented in Chapter 10. The equations that needs to be solved is shown in (A-1.32) and (A-1.33).

This investment decision concludes the modelling of scenario 1. The next section analyses how the option to expand alters the company’s investment decision from Equation (9.1.24).

9.2 Valuation of Project A with Option to Expand, and Comparison with Project B

This section considers the scenario where project A has the option to expand to project E. At the point of expansion, project A is already drilled and extracted.

In this section, two situations are presented. Both situations assume that the company has already invested in project A. The first situation assumes that an exploration has been conducted successfully, while the other assumes that the exploration process has not yet been conducted. In both situations, it is concluded how the option to expand affects optimal investment choice between project A or B, starting with the first situation.

9.2.1 Project A versus Project B when the Expansion is Certain: Scenario 2, $\mathbf{F}_{\boldsymbol{A_v}\mathbf{B}}^{(\mathbf{2})}(\mathbf{P})$

To provide a better overview, this subsection assumes that the company already knows that the oil reservoir of the expansion is there. Hence, there is no need for an exploration process, and the company can choose to expand the project whenever it is optimal. The scenario is presented in Figure 9-2.

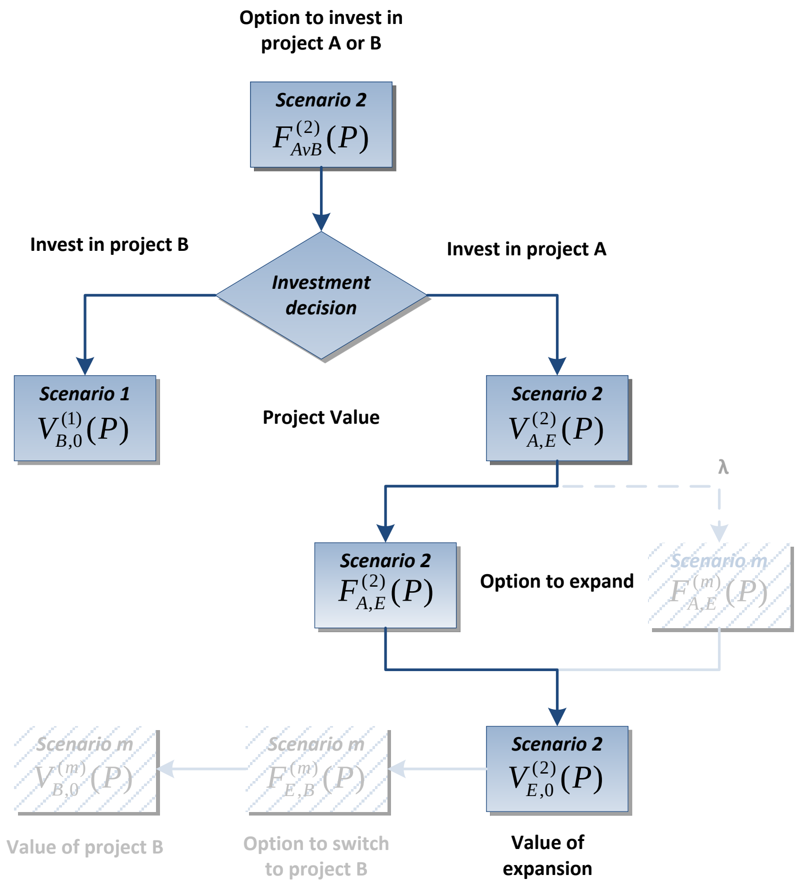

Figure 9-2: Project A vs. project B when expansion is certain in scenario 2

Project Value of Expansion: Scenario 2, $V^{(2)}_{E,0}(P)$

This subsection assumes that the company has already utilized the option to expand project A. Following the same GBM process as earlier, the value function of the expansion is defined in the same matter as in Section 9.1.

However, the project value has some different characteristics. I_E is the investment cost specific for the expansion. Subtracting I_A implies that investing in A must take place before expanding. D_E refers to the production volume of both project A, and the expanded field, E.

$$V^{(2)}_{E,0}(P) \quad = \frac{D_E P}{\gamma + \rho - \mu} - (I_A + I_E) \qquad (9.2.1)$$

Option Value of Expansion: Scenario 2, $F^{(2)}_{A,E}(P)$

As well as in the option to invest in project A, the option to expand is also considered as a perpetual American option, The company can either decide to expand or wait for better market conditions. If the option is in the money, the company can extract the value of the expansion, $V_{E,0}^{(2)}(P)$, and if it is out of the money, it still has the option to invest at a later stage. However, what differs from previous calculations is that when the option is out of the money, the company still gets the cash flows from the ongoing production in project A, which is already invested in.

Applying the same arguments and boundary conditions as in Equation (9.1.6), the option value to expand takes the following form.

$$F_{A,E}^{(2)}(P) = \begin{cases} V_{A,0}^{(1)}(P) + A_{A,E}^{(2)} P^{\beta_1} & , P < P^{*(2)}_{A,E} \\ V_{E,0}^{(2)}(P) & , P^{*(2)}_{A,E} \leq P \end{cases} \tag{9.2.2}$$

The first branch in Equation (9.2.2) is the interval where the option is out of the money, and the second branch is where the option is in the money.

In Equation (9.2.2), two new unknowns needs to be solved; the endogenous constant $A_{A,E}^{(2)}$, and the optimal investment threshold, $P^{*(2)}_{A,E}$. Applying the value-matching and smooth-pasting conditions, the two branches in $F_{A,E}^{(2)}(P)$ can be solved with respect to the two unknowns.

As shown in (A-2.1) – (A-2.3), solving for the two unknowns provides the following answers.

$$A_{A,E}^{(2)} = \frac{{P^{*(2)}_{A,E}}^{(1-\beta_1)}(D_E - D_A)}{\beta_1(\gamma + \rho - \mu)} \tag{9.2.3}$$

$$P^{*(2)}_{A,E} = \frac{\beta_1 I_E(\gamma + \rho - \mu)}{(\beta_1 - 1)(D_E - D_A)} \tag{9.2.4}$$

Investment Decision: Scenario 2, $F_{A_v B}^{(2)}(P)$

This subsection goes back to the initial investment decision where the company has the option to invest in project A or project B. The main difference in this subsection compared to Section 9.1 is that project A should be more attractive as it has the option to expand the production. The investment in project A exclusively will be analyzed before investigating how the embedded options will change optimal investment decision in project A or B.

When the company invests in project A, the option to expand project A must be included to provide sufficient information for optimal decision-making. Compared to Section 9.1, the value changes if the company decides to invest in project A, as it has the option to expand. The company will receive the project value of A, and the option to expand it, denoted as $V_{A,E}^{(2)}(P)$, which is the same as $F_{A,E}^{(2)}(P)$ when $P < P^{*(2)}_{A,E}$. The solution for the project value of project A included the option to expand is presented in Equation (9.2.5).

$$V_{A,E}^{(2)}(P) = \frac{D_A P}{\gamma + \rho - \mu} - I_A + A_{A,E}^{(2)} P^{\beta_1} \tag{9.2.5}$$

Equation (9.2.5) states that the company gets the value of project A plus the option to expand if it decides to invest in project A. In this matter, the option to invest in project A is the following.

$$F_{0,A}^{(2)}(P) = \begin{cases} A_{0,A}^{(2)} P^{\beta_1} & , P < P^{*(2)}_{0,A} \\ V_{A,E}^{(2)}(P) & , P^{*(2)}_{0,A} \leq P \end{cases} \tag{9.2.6}$$

The first branch of the equation states that if the option is out of the money, the company still has the option to invest in project A, because of the perpetual element of the option. If the option is in the money, the company can extract the value of project A, and have the option to expand.

Here, the endogenous constant, $A_{0,A}^{(2)}$ and the optimal investment threshold, $P_{0,A}^{*(2)}$ are unknowns, which is solved via the value-matching and smooth-pasting conditions as the following equations.

$$A_{0,A}^{(2)} = \left(\frac{1}{P_{0,A}^{*(2)}}\right)^{\beta_1} \left[\frac{D_A P_{0,A}^{*(2)}}{\gamma + \rho - \mu} + A_{A,E}^{(2)} P_{0,A}^{*(2)} - I_A\right] \tag{9.2.7}$$

$$P_{0,A}^{*(2)} = \frac{\beta_1 I_A (\gamma + \rho - \mu)}{D_A (\beta_1 - 1)} \tag{9.2.8}$$

Hereby, one can consider the optimal investment decision of choosing either project A or B. The option to invest in project B remains unchanged compared to Subsection 9.1.3. So following the same reasoning and assumptions as earlier, the company's optimal investment decision takes the following form.

$$F_{A_vB}^{(2)}(P) = \begin{cases} A_{0,A}^{(2)} P^{\beta_1} & , P < P^{*}{}_{0,A}^{(2)} \\ V_{A,E}^{(2)}(P) & , P^{*}{}_{0,A}^{(2)} \leq P \leq P^{*}\frac{(2)}{0,A} \\ C_{0,A_vB}^{(2)} P^{\beta_1} + E_{0,A_vB}^{(2)} P^{\beta_2} & , P^{*}\frac{(2)}{0,A} \leq P \leq P^{*}\frac{(2)}{0,B} \\ V_{B,0}^{(1)}(P) & , P^{*}\frac{(2)}{0,B} \leq P \end{cases} \qquad (9.2.9)$$

Equation (9.2.9) consists of four unknowns, the two endogenous constants, $C_{0,A_vB}^{(2)}$ and $E_{0,A_vB}^{(2)}$, and two investment thresholds, $P^{*}\frac{(2)}{0,A}$ and $P^{*}\frac{(2)}{0,B}$, which has to be solved numerically. The equations solving these unknowns are presented in (A-2.5) – (A-2.6).

$F_{A_vB}^{(2)}(P)$ represent the initial investment decision an investing company has when there is no uncertainty regarding the existence of project E. In the next subsection, a corresponding analysis will be given in a scenario where uncertainty of the explorations success is added.

9.2.2 Project A versus Project B when the Expansion is Uncertain: Scenario 2, $\mathbf{F_{A_vB}^{(2)}(P)}$

This subsection approaches project E from the point of view when there is uncertainty regarding the option to expand project A. When the company decides to invest in project A, it does still not know if there are satellite fields nearby, but that there is a possibility to explore for it. The exploration process will reveal a reservoir with some probability, λ. There are two possible outcomes.

1. The exploration reveals a new satellite field nearby, λ.
2. The exploration does not reveal any new satellite fields, (1-λ)

From Subsection 9.2.1 the project value, $V_{E,0}^{(2)}(P)$, and the option value, $F_{A,E}^{(2)}(P)$, is already presented in the case with certainty, and will remain the same also in this subsection. However, the uncertainty factor will affect the attractiveness of investing in project A as there is uncertainty about the existence of project E. The case with uncertainty is presented in Figure 9-3, and will be accounted for in the next subsection.

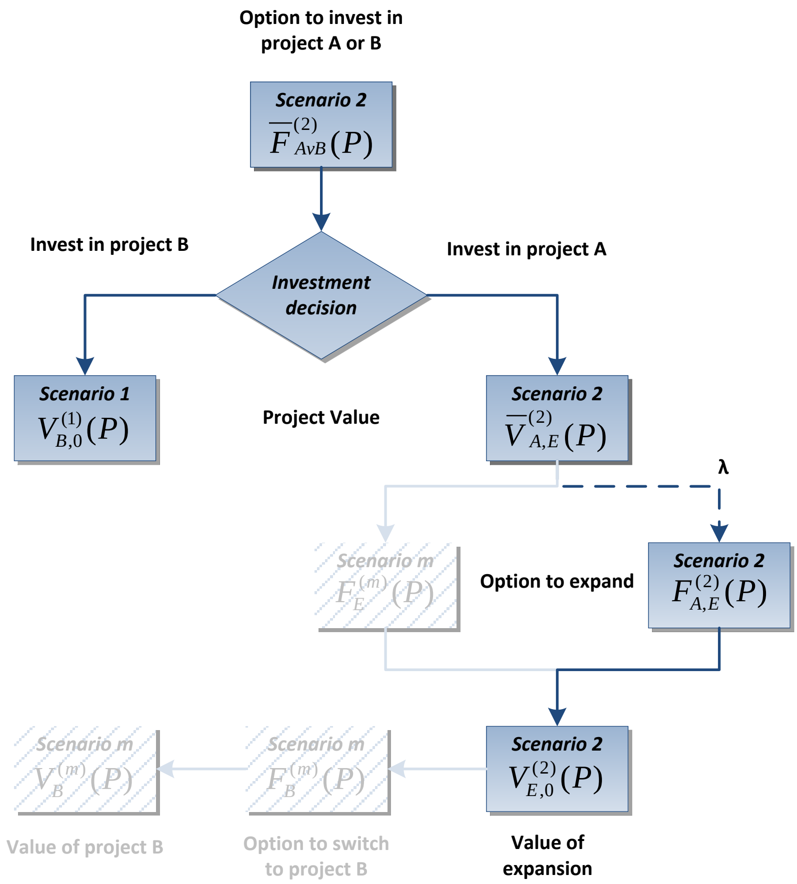

Figure 9-3: Project A vs. project B when expansion is uncertain in state 2

Modelling the Uncertainty Effect on Project A: Scenario 2, $\overline{V}^{(2)}_{A,E}(P)$

This subsection models what effect the uncertainty factor has on the value function of project A. It is assumed that if the exploration process is not successful, the company still has the option to continue exploring. This assumption is a simplification which might seem unrealistic at first, as there is a relatively small area the exploration process can happen in. Hence, it seems unlikely that the first exploration process will miss the point where the reservoir is. This assumption is justified by technology development. It is possible that today's technology is not sufficient to reveal reservoirs deeper in the ground, but that a future development of the technology will.

According to Chronopoulos and Siddiqui (2014), the project value of project A with uncertainty can be written as in Equation (9.2.10), where $\overline{V}^{(2)}_{A,E}(P)$ denotes the value function.

$$\overline{V}_{A,E}^{(2)}(P) = (D_A P dt - \rho I_A dt) + (1-\rho dt)\lambda dt \varepsilon\left[F_{A,E}^{(2)}(P+dP)\right] + (1-\rho dt)(1-\lambda dt)\varepsilon\left[\overline{V}_{A,E}^{(2)}(P+dP)\right] \tag{9.2.10}$$

The first part of Equation (9.2.10) represents the ongoing process of project A. The second part represents the event of λ, where the exploration process is successful, and the company gets the option to invest in the expanded project. The third part states that if the exploration is unsuccessful, the company can still continue exploring. If $\lambda = 0$, the company has the option to continue exploring and operate project A.

As shown in (A-2.7) – (A-2.9), solving this equation, and expanding the stochastic part with Itôs Lemma, gives the following solution.

$$\frac{1}{2}\sigma^2 P^2 \overline{V}_{A,E}^{(2)''}(P) + \mu P \overline{V}_{A,E}^{(2)'}(P) - (\rho+\lambda)\overline{V}_{A,E}^{(2)}(P) + D_A P - \rho I_A + \lambda F_{A,E}^{(2)}(P) = 0 \tag{9.2.11}$$

Equation (9.2.11), has two solutions, as $F_{A,E}^{(2)}(P)$ is defined both in- and out of money in the previous subsection. The evidence can be found in (A-2.10) – (A-2.11), and the result is presented in Equation (9.2.12).

$$\overline{V}_{A,E}^{(2)}(P) = \begin{cases} V_{A,0}^{(1)}(P) + A_{A,E}^{(2)} P^{\beta_1} + \underline{A}_{A,E}^{(2)} P^{\delta_1} & , P < P*_{A,E}^{(2)} \\ \dfrac{P[\lambda D_E + (\gamma+\rho-\mu)D_A]}{(\rho+\lambda-\mu)(\gamma+\rho-\mu)} - \dfrac{\lambda I_E}{\rho+\lambda} - I_A + \underline{B}_{A,E}^{(2)} P^{\delta_2} & , P*_{A,E}^{(2)} \le P \end{cases} \tag{9.2.12}$$

It is important to notice that the optimal investment threshold is independent of the exploration process. In other words, an increase in λ does have impact on the likelihood of the exploration process being successful, but not when to exercise the option to invest. Thus, $P*_{A,E}^{(2)}$ is defined at the same threshold as in Equation (9.2.4) .

$V_{A,0}^{(1)}$, $A_{A,E}^{(2)}$ and $P*_{A,E}^{(2)}$ are known from Subsection 9.2.1, and the first part in the second branch is the solution of $V_{A,E}^{(2)}(P)$ adjusted for uncertainty, λ. In other words, there are two unknowns in this equation, $\underline{A}_{A,E}^{(2)}$ and $\underline{B}_{A,E}^{(2)}$. The bar is used to separate these values from $A_{A,E}^{(2)}$ and $B_{A,E}^{(2)}$.

The idea regarding the two new endogenous constants is provided by Chronopoulos and Siddiqui (2014). The intuition behind the introduction of these is an expected lag period from a successfully conducted exploration process, and until the investment can take place.

In the first branch of Equation (9.2.12), the constant, $\underline{A}_{A,E}^{(2)}$, is an adjustment for positive fluctuations in the oil price since the option is not available at the time an exploration has been concluded successfully. Respectively, $\underline{B}_{A,E}^{(2)}$ represents an adjustment for negative fluctuations in the oil price. Since the option in this branch is in the money, there is a possibility that the oil price drops to a point where it is out of the money before the investment is implemented. These unknowns can be solved for via the value-matching and smooth-pasting conditions. Applying the solutions suggested by Chronopoulos and Siddiqui (2014), the specific answers in this case study is presented in Equation (9.2.13) and (9.2.14).

$$\underline{A}_{A,E}^{(2)} = \frac{{P^*}_E^{(2)^{-\delta_1}}}{(\delta_2-\delta_1)}\left[\frac{\lambda(\delta_2-1)(D_E-D_A){P^*}_E^{(2)}}{(\rho+\lambda-\mu)(\gamma+\rho-\mu)} - \frac{\delta_2\lambda I_E}{\rho+\lambda} - (\delta_2-\beta_1)A_{A,E}^{(2)}{P^*}_{A,E}^{(2)^{\beta_1}}\right] \tag{9.2.13}$$

$$\underline{B}_{A,E}^{(2)} = \frac{{P^*}_E^{(2)^{-\delta_2}}}{(\delta_1-\delta_2)}\left[\frac{\lambda(1-\delta_1)(D_E-D_A){P^*}_E^{(2)}}{(\rho+\lambda-\mu)(\gamma+\rho-\mu)} + \frac{\delta_1\lambda I_E}{\rho+\lambda} + (\delta_1-\beta_1)A_{A,E}^{(2)}{P^*}_{A,E}^{(2)^{\beta_1}}\right] \tag{9.2.14}$$

To find the two values of δ, Equation (9.2.11) can by following the same arguments as (A-1.16) – (A-1.20) be viewed as the quadratic function presented in Equation (9.2.15).

$$\frac{1}{2}\sigma^2\delta'' + \left(\mu-\frac{1}{2}\sigma^2\right)\delta' + (\rho-\lambda)\delta = 0 \tag{9.2.15}$$

This solves into Equation (9.2.16) and (9.2.17) where $0 < \rho > \mu$, and $\delta_1 > 1$, $\delta_2 < 0$.

$$\delta_1 = \frac{\left(\frac{1}{2}\sigma^2-\mu\right)+\sqrt{\left(\mu-\frac{1}{2}\sigma^2\right)^2+2\sigma^2(\rho+\lambda)}}{\sigma^2} \tag{9.2.16}$$

$$\delta_2 = \frac{\left(\frac{1}{2}\sigma^2-\mu\right)-\sqrt{\left(\mu-\frac{1}{2}\sigma^2\right)^2+2\sigma^2(\rho+\lambda)}}{\sigma^2} \tag{9.2.17}$$

Intuitively, λ=0 should make $\delta_1=\beta_1$ and $\delta_2=\beta_2$, which proves correct in the quadratic function.

Investment Decision: Scenario 2, $\overline{F}^{(2)}_{A_v B}(P)$

Finally, this subsection provides the decision framework for the company when the exploration is not conducted. Following the same reasoning as earlier, analyzing the optimal investment decision in project A first, the option value of project A takes the following form.

$$\overline{F}^{(2)}_{0,A}(P) = \begin{cases} \overline{A}^{(2)}_{0,A} P^{\beta_1} & , P < \overline{P}^{*(2)}_{0,A} \\ \overline{V}^{(2)}_{A,E}(P) & , \overline{P}^{*(2)}_{0,A} \leq P \end{cases} \tag{9.2.18}$$

Unlike when the expansion is known, the second branch defines when the option to invest in project A is in the money, the company will have the out of money value of $\overline{V}^{(2)}_{A,E}(P)$. This is because the company gets the value from investing in project A, and an embedded option to expand after exploration has been conducted successfully. Thereafter, if the company explores, it gets the expected value of project E. The solution of $\overline{F}^{(2)}_{0,A}(P)$ therefore leaves two unknowns, the endogenous constant, $\overline{A}^{(2)}_{0,A}$, and the optimal threshold level, $\overline{P}^{*(2)}_{0,A}$. These values are determined numerically via the following value-matching and smooth-pasting conditions.

$$\overline{A}^{(2)}_{0,A} {\overline{P}^{*(2)}_{0,A}}^{\beta_1} = V^{(1)}_{A,0} \overline{P}^{*(2)}_{0,A} + A^{(2)}_{A,E} {\overline{P}^{*(2)}_{0,A}}^{\beta_1} + \underline{A}^{(2)}_{A,E} {\overline{P}^{*(2)}_{0,A}}^{\delta_1} \tag{9.2.19}$$

$$\beta_1 \overline{A}^{(2)}_{0,A} {\overline{P}^{*(2)}_{0,A}}^{\beta_1} = \frac{D_A \overline{P}^{*(2)}_{0,A}}{\gamma + \rho - \mu} + \beta_1 A^{(2)}_{A,E} {\overline{P}^{*(2)}_{0,A}}^{\beta_1} + \delta_1 \underline{A}^{(2)}_{A,E} {\overline{P}^{*(2)}_{0,A}}^{\delta_1} \tag{9.2.20}$$

When choosing between project A or B, the solution remains similar to Subsection 9.2.1. However, the value-decreasing effect of λ gives different optimal thresholds, $\overline{P}^{*(2)}_{\overline{0,A}}$ and $\overline{P}^{*(2)}_{\overline{0,B}}$, and endogenous constants, $\overline{C}^{(2)}_{0,A_v B}$ and $\overline{E}^{(2)}_{0,A_v B}$, which must to be solved. The investment decision takes the following form.

$$\overline{F}^{(2)}_{A_v B}(P) = \begin{cases} \overline{A}^{(2)}_{0,A} P^{\beta_1} & , P < \overline{P}^{*(2)}_{0,A} \\ \overline{V}^{(2)}_{A,E}(P) & , \overline{P}^{*(2)}_{0,A} \leq P \leq \overline{P}^{*(2)}_{\overline{0,A}} \\ \overline{C}^{(2)}_{0,A_v B} P^{\beta_1} + \overline{E}^{(2)}_{0,A_v B} P^{\beta_2} & , \overline{P}^{*(2)}_{\overline{0,A}} \leq P \leq \overline{P}^{*(2)}_{\overline{0,B}} \\ V^{(1)}_{B,0}(P) & , \overline{P}^{*(2)}_{\overline{0,B}} \leq P \end{cases} \tag{9.2.21}$$

For the remaining four unknowns, the same method with numerical iterations is applied. The equations solving these unknowns are presented in (A-2.13) – (A-2.14).

This subsection concludes the scenario where the operating company only has one embedded option. The next section will add a new embedded option, the option to switch to project B.

9.3 Valuation of Project A Including all Embedded Options, and Comparison with Project B

This section considers the scenario where project A has both the option to expand, and the option to switch to project B after investing in project A and project E. Intuitively, having the option to switch to project B after extracting the expanded project, makes the initial investment in project A more attractive compared to a direct investment in project B.

As in Section 9.2, this section analyses both the situation where the expansion is certain, and where an exploration process must be conducted, starting with the certain situation. Following the framework presented in Section 9.1, only the final expressions are presented.

9.3.1 Project A Versus Project B when the Expansion is Certain: Scenario 3, $\mathbf{F}^{(3)}_{\mathbf{A_vB}}(\mathbf{P})$

It is assumed that the value of the expansion is known and does not require an exploration process. This scenario, including all options is described in Figure 9-4.

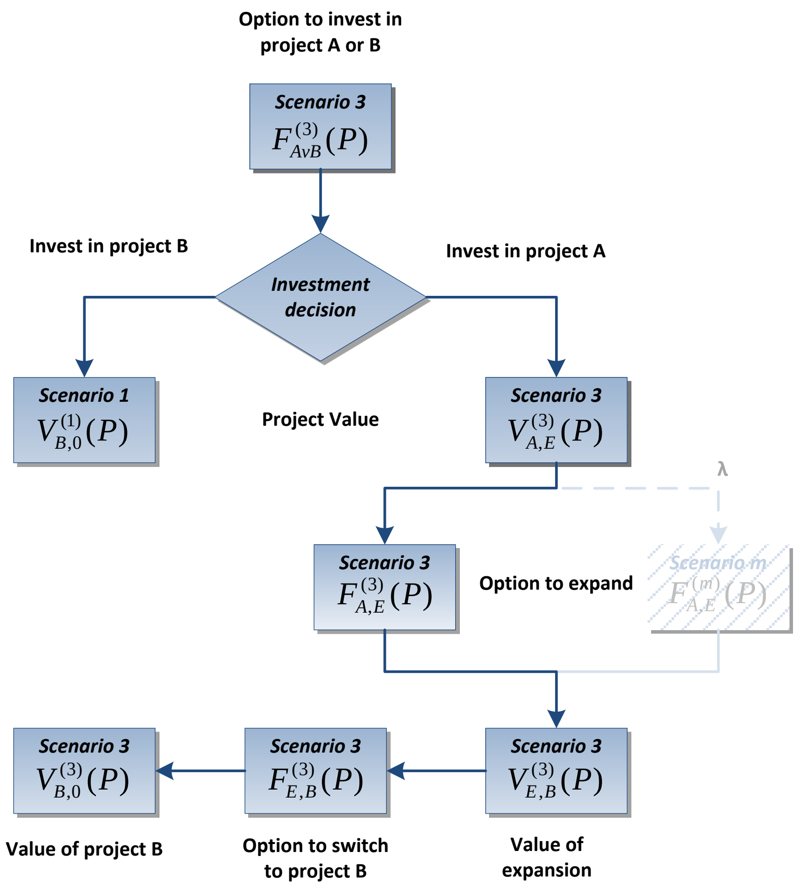

Figure 9-4: Project A vs. project B when expansion is certain in scenario 3

Value of Project B: Scenario 3, $V^{(3)}_{B,0}(P)$

This subsection defines project value of B in scenario 3. It is the value function if the company decides to switch to project B after extracting both project A and the expansion. This calculation assumes that the company already has utilized the option to switch to project B and gets the project value from switching. The value function is defined in the same matter as in Subsection 9.1.1, so this part states the expression from the process.

$$V^{(3)}_{B,0}(P) = \frac{D_B P}{\gamma + \rho - \mu} - (I_A + I_E + I_B) \tag{9.3.1}$$

Option Value of Switching to Project B: Scenario 3, $F_{E,B}^{(3)}(P)$

Having the option to switch to project B is considered as a perpetual American option. Applying the same arguments as in Subsection 9.1.2, the option value takes the following form.

$$F_{E,B}^{(3)}(P) = \begin{cases} \dfrac{D_E P}{\gamma + \rho - \mu} - (I_A + I_E) + A_{E,B}^{(3)} P^{\beta_1} & , P < P_{E,B}^{*(3)} \\ V_{B,0}^{(3)}(P) & , P_{E,B}^{*(3)} \leq P \end{cases} \tag{9.3.2}$$

The first branch implies the scenario where the option is out of the money. The company gets perpetual cash flows from the expanded project, and the option to switch to project B later. When the option is in the money, as in the second branch, the company switches to project B and extract its project value, $V_{B,0}^{(3)}(P)$.

As investment in project B only applies after investing in project A and project E, it is exercised at a different optimal threshold, $P_{E,B}^{*(3)}$, and the option value will have a new endogenous constant, $A_{E,B}^{(3)}$. Applying the value-matching and smooth-pasting conditions, these are solved as the following equations.

$$A_{E,B}^{(3)} = \frac{{P^*}_{E,B}^{(3)\,(1-\beta_1)}(D_B - D_E)}{\beta_1(\gamma + \rho - \mu)} \tag{9.3.3}$$

$${P^*}_{E,B}^{(3)} = \frac{\beta_1 I_B (\gamma + \rho - \mu)}{(\beta_1 - 1)(D_B - D_E)} \tag{9.3.4}$$

Project Value of Expansion: Scenario 3, $V_{E,B}^{(3)}(P)$

The project value of the expansion follows the same arguments as in Subsection 9.2.1. However, the value increases because company also has the option to switch to project B, $A_{E,B}^{(3)} P^{\beta_1}$. The project value function is shown in the following equation.

$$V_{E,B}^{(3)}(P) = \frac{D_E P}{\gamma + \rho - \mu} - (I_A + I_E) + A_{E,B}^{(3)} P^{\beta_1} \tag{9.3.5}$$

Equation (9.3.5) states that if the company invests in the expansion, it gets the project value of the expansion plus the option to switch to project B.

Option Value of Expansion: Scenario 3, $F_{A,E}^{(3)}(P)$

The option value of expanding production is seemingly similar to the value of the option to expand in Subsection 9.2.1. However, one must take into account the option to switch to project B. The option value is presented in Equation (9.3.6)

$$F_{A,E}^{(3)}(P) = \begin{cases} \dfrac{D_A P}{\gamma + \rho - \mu} - I_A + A_{A,E}^{(3)} P^{\beta_1} & , P < P^{*(3)}_{A,E} \\ V_{E,B}^{(3)}(P) & , P^{*(3)}_{A,E} \leq P \end{cases} \tag{9.3.6}$$

The first branch of Equation (9.3.6) states that if the option is out of the money, the company still operates project A, and has the option to expand. The second branch states that if the option to invest is in the money, the company invests and extracts the value of the expansion. Different from scenario 2, the project value, $V_{E,B}^{(3)}(P)$, and the option value, $A_{A,E}^{(3)} P^{\beta_1}$, of the expansion includes the option to switch to project B.

As a result of the change in the project value of the expansion, it is clear that the optimal threshold and endogenous constant will differ from Subsection 9.2.1. Using value-matching and smooth-pasting conditions, these are solved as in Equation (9.3.7) and (9.3.8).

$$A_{A,E}^{(3)} = \left(\frac{1}{P^{*(3)}_{A,E}}\right)^{\beta_1} \left[\frac{P^{*(3)}_{A,E}(D_E - D_A)}{\gamma + \rho - \mu} + A_{E,B}^{(3)} {P^{*(3)}_{A,E}}^{\beta_1} - I_E\right] \tag{9.3.7}$$

$$P^{*(3)}_{A,E} = \frac{\beta_1 I_E (\gamma + \rho - \mu)}{(\beta_1 - 1)(D_E - D_A)} \tag{9.3.8}$$

Investment Decision: Scenario 3, $F_{A_v B}^{(3)}(P)$

In this subsection, the company faces the initial investment decision of choosing project A or project B. The initial investment decision of investing in project A or B differs from Subsection 9.2.1. This is because project A has the option to both expand and switch to project B. Following the same arguments as in Subsection 9.1.1, and accounting for the option to expand, project A’s value is defined as the following.

$$V_{A,E}^{(3)}(P) = \frac{D_A P}{\gamma + \rho - \mu} - I_A + A_{A,E}^{(3)} P^{\beta_1} \tag{9.3.9}$$

Its corresponding option value has the following value function.

$$F_{0,A}^{(3)}(P)=\begin{cases}A_{0,A}^{(3)}P^{\beta_1} & ,P<P^{*(3)}_{0,A}\\ V_{A,E}^{(3)}(P) & ,P^{*(3)}_{0,A}\leq P\end{cases} \tag{9.3.10}$$

The option value implies that when the option is in the money, the company has the value of project A and the option expand to project E, which again has the option to switch to project B. Investing in project A is done at or above the optimal investment threshold, $P^{*(3)}_{0,A}$, and the endogenous constant is given as $A_{0,A}^{(3)}$. These unknowns are solved via the value-matching and smooth-pasting conditions.

$$A_{0,A}^{(3)}=\left(\frac{1}{P^{*(3)}_{0,A}}\right)^{\beta_1}\left[\frac{P^{*(3)}_{0,A}D_A}{\gamma+\rho-\mu}+A_{A,E}^{(3)}P^{*(3)\beta_1}_{0,A}-I_A\right] \tag{9.3.11}$$

$$P_{0,A}^{*(3)}=\frac{\beta_1 I_A(\gamma+\rho-\mu)}{D_A(\beta_1-1)} \tag{9.3.12}$$

Finally, given that the exploration is known, the company's optimal investment decision takes the following form.

$$F_{A_vB}^{(3)}(P)=\begin{cases}A_{0,A}^{(3)}P^{\beta_1} & ,P<P^{*(3)}_{0,A}\\ V_{A,E}^{(3)}(P) & ,P^{*(3)}_{0,A}\leq P\leq P^{*\frac{(3)}{0,A}}\\ C_{0,A_vB}^{(3)}P^{\beta_1}+E_{0,A_vB}^{(3)}P^{\beta_2} & ,P^{*\frac{(3)}{0,A}}\leq P\leq P^{*\frac{(3)}{0,B}}\\ V_{B,0}^{(1)}(P) & ,P^{*\frac{(3)}{0,B}}\leq P\end{cases} \tag{9.3.13}$$

Equation (9.3.13) consists of four unknowns, the two endogenous constants, $C_{0,A_vB}^{(3)}$ and $E_{0,A_vB}^{(3)}$, and two optimal investment thresholds, $P^{*\frac{(3)}{0,A}}$ and $P^{*\frac{(3)}{0,B}}$, which are solved numerically. The equations solving these unknowns are presented in (A-3.2) – (A-3.3).

$F_{A_vB}^{(3)}(P)$ represent the initial investment decision an investing company has when there is no uncertainty regarding the exploration process. The next subsection describes the situation where there is uncertainty regarding the success of the exploration process.

9.3.2 Project A Versus Project B when the Expansion is Uncertain: Scenario 3, $\overline{\mathbf{F}}^{(3)}_{\mathbf{A_vB}}(\mathbf{P})$

This subsection accounts for uncertainty about the expansion. From Subsection 9.3.1, the value of project B after switching from project E, and the option value of switching to project B, was calculated. These values are unaffected by the exploration process. However, as in Subsection 9.2.2, a new value for project A must be conducted as the uncertainty affects the option to expand, and thereby the option to switch to project B. The modeling is presented in Figure 9-5.

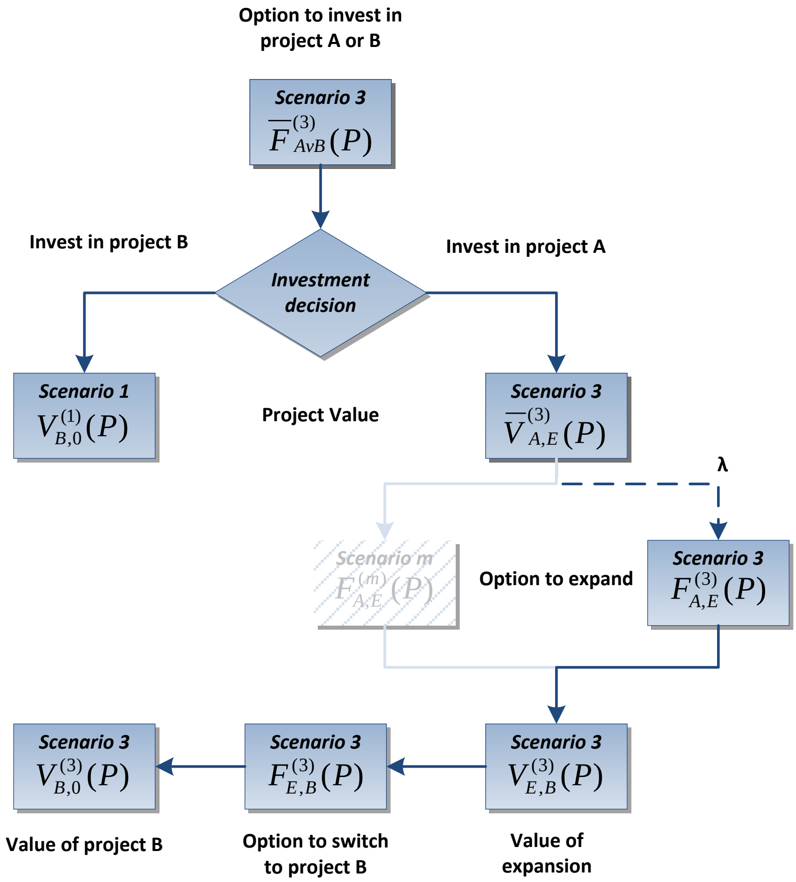

Figure 9-5: Project A vs. project B when expansion is uncertain in scenario 3

Modelling the Uncertainty Effect on Project A: Scenario 3, $\overline{V}_{A,E}^{(3)}(P)$

This subsection follows the same process as in Subsection 9.2.2, but the solution has to reflect the option to switch to project B. Considering this, the value of project A when there is uncertainty about the expansion takes the following form.

$$\overline{V}_{A,E}^{(3)}(P) = \begin{cases} V_{A,0}^{(1)}(P) + A_{A,E}^{(3)}P^{\beta_1} + \underline{A}_{A,E}^{(3)}P^{\delta_1} & , P < P^{*(3)}_{A,E} \\ \dfrac{P[\lambda D_E + (\gamma + \rho - \mu)D_A]}{(\rho + \lambda - \mu)(\gamma + \rho - \mu)} - \dfrac{\lambda I_E}{\rho + \lambda} - I_A & \\ \qquad + A_{E,B}^{(3)}P^{\beta_1} + \underline{B}_{A,E}^{(3)}P^{\delta_2} & , P^{*(3)}_{A,E} \leq P \end{cases} \tag{9.3.14}$$

The first branch is similar to Equation (9.2.12), but the option to expand, $A_{A,E}^{(3)}P^{\beta_1}$, also reflects a value to switch to project B. In the second branch, there is added a new endogenous constant, $A_{E,B}^{(3)}P^{\beta_1}$, compared to Equation (9.2.12). It reflects the option to switch from project E to project B. $A_{E,B}^{(3)}$ was estimated in Equation (9.3.3). Applying the solutions suggested by Chronopoulos and Siddiqui (2014), the solutions for the two unknowns, $\underline{A}_{A,E}^{(3)}$ and $\underline{B}_{A,E}^{(3)}$, are in this case study presented in Equation (9.3.15) and (9.3.16) when applying the value-matching and smooth-pasting conditions. Notice that the optimal investment threshold, $P^{*(3)}_{A,E}$, is the same as in Equation (9.3.8).

$$\underline{A}_{A,E}^{(3)} = \frac{{P^{*(3)}_{A,E}}^{-\delta_1}}{(\delta_2 - \delta_1)} \left[\frac{\lambda(\delta_2 - 1)(D_E - D_A)P^{*(3)}_{A,E}}{(\rho + \lambda - \mu)(\gamma + \rho - \mu)} - \frac{\delta_2 \lambda I_E}{\rho + \lambda} - (\delta_2 - \beta_1)A_{A,E}^{(3)}{P^{*(3)}_{A,E}}^{\beta_1} \right. $$
$$\left. + (\delta_2 - \beta_1)A_{E,B}^{(3)}{P^{*(3)}_{A,E}}^{\beta_1} \right] \tag{9.3.15}$$

$$\underline{B}_{A,E}^{(3)} = \frac{{P^{*(3)}_{A,E}}^{-\delta_2}}{(\delta_1 - \delta_2)} \left[\frac{\lambda(1 - \delta_1)(D_E - D_A)P^{*(3)}_{A,E}}{(\rho + \lambda - \mu)(\gamma + \rho - \mu)} - \frac{\delta_2 \lambda I_E}{\rho + \lambda} + (\delta_1 - \beta_1)A_{A,E}^{(3)}{P^{*(3)}_{A,E}}^{\beta_1} \right. $$
$$\left. - (\delta_1 - \beta_1)A_{E,B}^{(3)}{P^{*(3)}_{A,E}}^{\beta_1} \right] \tag{9.3.16}$$

Investment Decision: Scenario 3, $\overline{F}^{(3)}_{A_vB}(P)$

The option to invest in project A, under uncertainty regarding the expansion, takes the following form.

$$\overline{F}^{(3)}_{0,A}(P) = \begin{cases} \overline{A}^{(3)}_{0,A} P^{\beta_1} & , \overline{P}^{*(3)}_{0,A} < P \\ \overline{V}^{(3)}_{A,E}(P) & , P \leq \overline{P}^{*(3)}_{0,A} \end{cases} \tag{9.3.17}$$

The option to invest leaves two unknowns, the optimal investment threshold, $\overline{P}^{*(3)}_{0,A}$, and the endogenous constant, $\overline{A}^{(3)}_{0,A}$. These values are determined numerically via the following value-matching and smooth-pasting conditions.

$$\overline{A}^{(3)}_{0,A} \overline{P}^{*(3)^{\beta_1}}_{0,A} = V^{(1)}_{A,0} \overline{P}^{*(3)}_{0,A} + A^{(3)}_{A,E} \overline{P}^{*(3)^{\beta_1}}_{A} + \underline{A}^{(3)}_{A,E} \overline{P}^{*(3)^{\delta_1}}_{0,A} \tag{9.3.18}$$

$$\beta_1 \overline{A}^{(3)}_{0,A} \overline{P}^{*(3)^{\beta_1}}_{0,A} = \frac{D_A \overline{P}^{*(3)}_{0,A}}{\gamma + \rho - \mu} + \beta_1 A^{(3)}_{A,E} \overline{P}^{*(3)^{\beta_1}}_{0,A} + \delta_1 \underline{A}^{(3)}_{A,E} \overline{P}^{*(3)^{\delta_1}}_{0,A} \tag{9.3.19}$$

At the end, when choosing between project A or B, the company faces the following option for optimal investment.

$$\overline{F}^{(3)}_{A_vB}(P) = \begin{cases} \overline{A}^{(3)}_{0,A} P^{\beta_1} & , P < \overline{P}^{*(3)}_{0,A} \\ \overline{V}^{(3)}_{A,E}(P) & , \overline{P}^{*(3)}_{0,A}, \leq P \leq \overline{P}^{*(3)}_{\overline{0,A}} \\ \overline{C}^{(3)}_{0,A_vB} P^{\beta_1} + \overline{E}^{(3)}_{0,A_vB} P^{\beta_2} & , \overline{P}^{*(3)}_{\overline{0,A}} \leq P \leq \overline{P}^{*(3)}_{\overline{0,B}} \\ V^{(1)}_{B,0}(P) & , \overline{P}^{*(3)}_{\overline{0,B}} \leq P \end{cases} \tag{9.3.20}$$

As in the previous section, the four unknowns, $\overline{P}^{*(3)}_{\overline{0,A}}$, $\overline{P}^{*(3)}_{\overline{0,B}}$, $\overline{C}^{(3)}_{0,A_vB}$ and $\overline{E}^{(3)}_{0,A_vB}$ will be solved numerically. The equations solving these unknowns are presented in (A-3.10) – (A-3.11).

10. Numerical Results

In this chapter, numerical values are applied to the model developed in Chapter 9. It provides a sufficient amount of information for the company to make an optimal decision at time t, but with some limitations regarding the assumptions for the model as mentioned in Chapter 7.

After presenting the chosen input values, a base case scenario where the company will choose based on a standard NPV analysis is presented. Thereafter, a presentation of the same steps and procedure as in Chapter 9, starting with the simplest scenario where project A does not have embedded options is given. Thereafter, the option to expand and the option to switch to project B are added. The application of the model is done in Matlab® where all the necessary formulas from the previous chapter are applied to get numerical results.

10.1 Parameter Values

Without insight in specific projects in the oil industry, it is not possible to derive perfectly reusable numbers for the industry. However, the model is reusable for similar problem sets with more realistic inputs. Therefore, given the break-even oil price in the industry, as discussed in Section 4.2.1, the inputs based on this, but are adjusted to make the analysis as realistic as possible. The parameter values chosen are presented Table 10-1. These values will remain constant throughout the chapter if not anything else is clearly specified.

Parameter Values	Notations	Input Values	Quantity Factor
Subjective discount rate	ρ	10	*% per year*
Instantaneous drift of the value process	μ	1	*% per year*
Instantaneous volatility of the value process	σ	20	*% per year*
Production volume project A	D_A	3	*Million barrels per year*
Production volume project B	D_B	4	*Million barrels per year*
Production volume expansion	D_e	0.5	*Million barrels per year*
Combined production volume of project A and expansion	D_E	3.5	*Million barrels per year*
Total cost of project A	I_A	1100	*Million USD*
Total cost of project B	I_B	1900	*Million USD*
Total cost of expansion	I_E	400	*Million USD*
Value decreasing factor	γ	3	*% per year*
Poisson intensity (probability of exploration succuess)	λ	1	*% per year*
Possible range of oil price	x	0 - 150	*USD*

Table 10-1: Parameter Values

Notice that $I_B > I_A > I_E$, $D_B > D_A > D_e$ and $\rho > \mu > 0$. These are key assumptions for this model to make sense. The chosen price range of *0 – 150 USD*, and the volatility of 20% is based on historical prices of oil per barrel as shown in Subsection 4.2.1.

10.2 Investment Decision Applying NPV

For the sake of discussion, simplicity and a better overview, it is for now assumed that the operating company makes decisions based on a standard DCF valuation, and ultimately a NPV. Hereby, the company will not consider deferring investment, but invest as long as the project produces a positive NPV. The preferred project is the project with the highest NPV at the time a valuation has been made. Figure 10-1 shows how the company invests based on the presented price range.

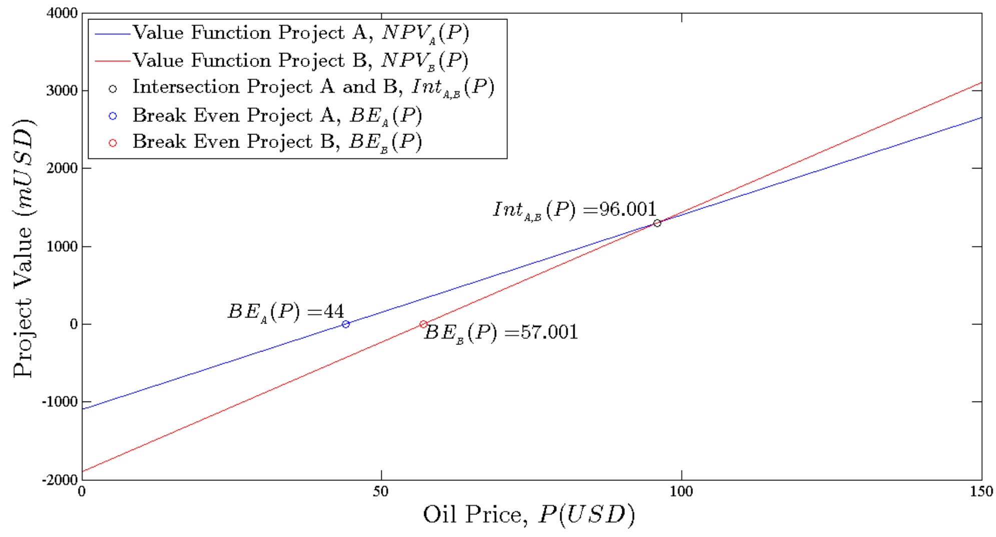

Figure 10-1: Investing based on NPV (generated by Matlab®)

The figure describes the total NPV, represented by the solid lines, of investing in project A and B at different price thresholds. There are shed light on three spots; $BE_A(P)$, $BE_B(P)$ and $Int_{A,B}$. $BE_A(P)$ describes project A's value function where the NPV of investing is zero. This price threshold corresponds to the project's breakeven price when the company makes decisions based on NPV. $BE_B(P)$ corresponds to project B's breakeven price. Finally, $Int_{A,B}$ is the intersection price between the value functions of project A and B, where they are equally profitable. Project A is preferred to project B when $P < Int_{A,B}$ and project B is preferred to project A when $Int_{A,B} < P$. Project A is the preferred project at lower prices because of a smaller investment cost, while project B is more profitable at higher prices because of a higher production volume. Thus, project B has a higher breakeven price before considering an investment.

This analysis does not consider the flexibility to wait, expand or switch, and could therefore lead to mispricing of the projects, and possibly a wrong investment decision. Adding flexibility to the analysis will be accounted for in the following sections.

10.3 Project A and B with Option to Defer Investments

Applying the model from Section 9.1, one can take a closer look at how the option to defer investment affects the decision-making compared to the NPV analysis. This section analyses the scenario where the company has the option to defer the investment, but no embedded options.

10.3.1 Optimized Investment in Project A and B

Consider a scenario where the company can invest in project A and B; the company is not choosing between the projects. Applying real option theory, the company only invests at or above the optimal investment threshold of the respective projects. The scenario is presented in Figure 10-2.

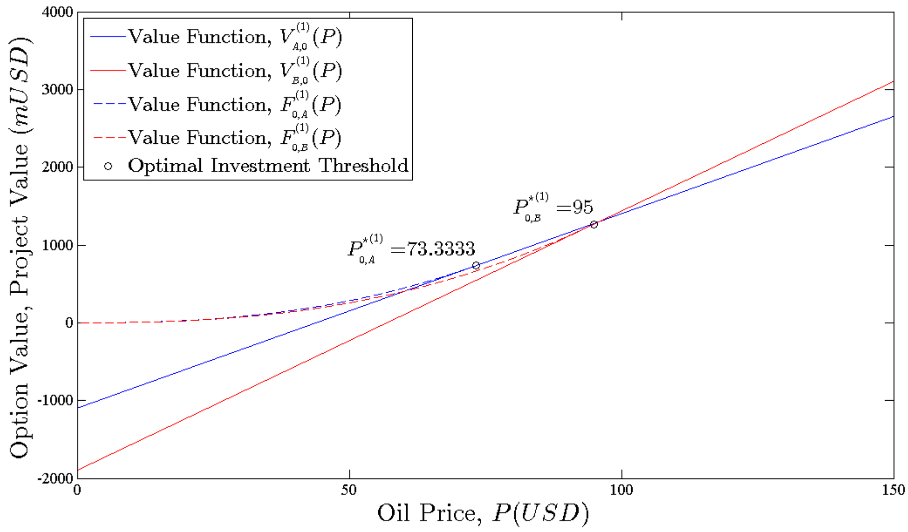

Figure 10-2: Optimized investment timing in project A and B (generated by Matlab®)

The solid lines are the total project value of investing in project A and B. The two dashed lines represent the option value functions to invest. Considering the smooth-pasting condition, it is discussed that the optimal investment threshold of the options is the price where the project value and the option value meets tangentially, here $P_{0,A}^{(1)}$ and $P_{0,B}^{(1)}$. The optimal investment thresholds in project A are when $73.33 \leq P$, and correspondingly when $95 \leq P$ for project B. As expected, due to the characteristics of the projects, it is confirmed that $P_{0,A}^{(1)} < P_{0,B}^{(1)}$. Interestingly, both investment thresholds are above their breakeven prices from the previous section.

10.3.2 Investment Decision

Consider the scenario where the company optimizes its investment opportunity by choosing between project A or B, corresponding to the option value function derived from Subsection 9.1.3. The solution of the value function $F_{A_vB}^{(1)}(P)$ is illustrated in Figure 10-3.

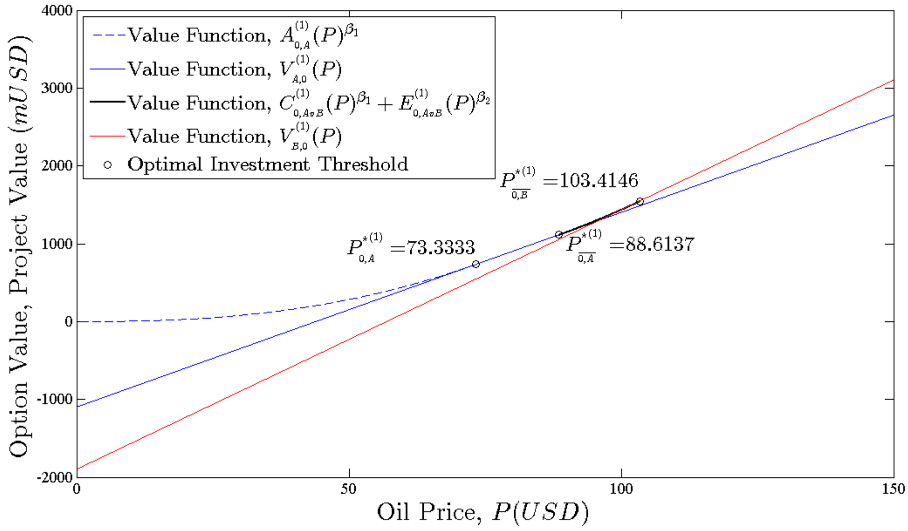

Figure 10-3: Optimized investment timing in project A or B in scenario 1 (generated by Matlab®)

The company should expect two waiting regions in this case, i.e. the price regions where the optimal decisions is to wait for another price rather than investing.

The first waiting region is defined where $P < P_{0,A}^{(1)}$, illustrated with the dashed line. When $P < 73.33$, the optimal decision for the company is to defer investment until $73.33 \leq P$, and then invest immediately in project A when $73.33 \leq P \leq 88.61$. The second waiting region, illustrated by the bold and black line, is defined when it is optimal to wait and see how the price will fluctuate rather than invest directly in project A. The company should defer investment and consider the price fluctuations in the price region $88.61 < P < 103.41$. If the price drops to or below 88.61, the company should invest in project A. If the price increases to 103.41, it is optimal to invest in project B.

The company should in other words not consider investing at all before the price is 73.33. This threshold is significant higher than the breakeven price of project A at 44 from the NPV analysis. Another point is that project B should not be preferred to project A at lower prices than 103.41, compared to the NPV intersection point at 96.01, as shown in Figure 10.1.

This section proves that having the option to defer investment does affect a company's investment decision. The next sections consider the scenario where project A has embedded options.

10.4 Project A with Option to Expand Versus Project B

This section corresponds to the model developed in Section 9.2, where the company has one embedded option; the option to expand project A. Thereby, project A has two options attached, the option to defer investment and the option to expand the project. Both situations from Section 9.2 are analysed. The situation with certainty about the expansion is regarded first, and thereby compared to the situation with uncertainty from an exploration process.

10.4.1 Choosing between Project A or B when the Expansion is Certain

Following the same reasoning as in Subsection 9.2.1, the decision between project A or B when the company knows that the expansion is there is analysed. This corresponds to $\lambda = 1$. Having the option to expand project A should make investing more attractive compared to investing in project B, and deferring investment. This subsection will analyze to which extent it does. The solution of the option value function, $F_{A_v B}^{(2)}(P)$ is illustrated in Figure 10-4.

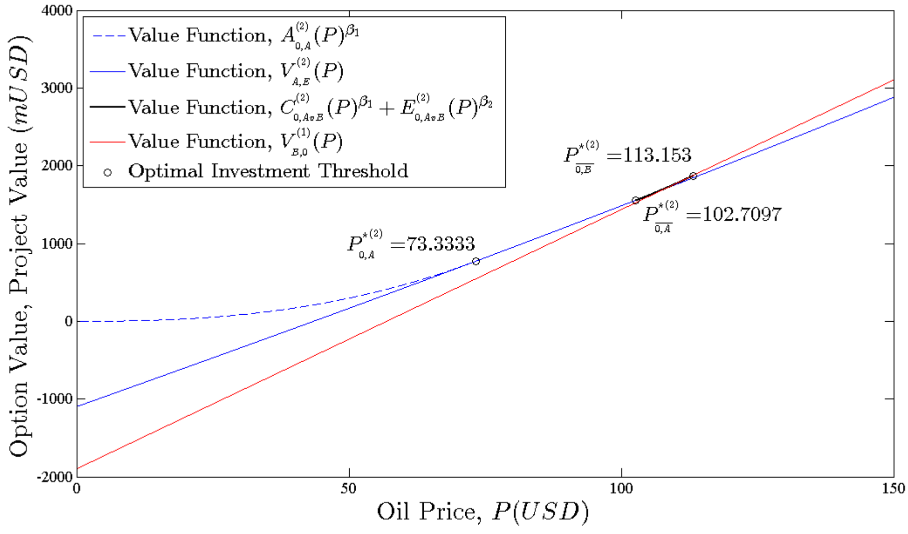

Figure 10-4: Optimized investment in project A or B when expansion is certain in scenario 2 (generated by Matlab®)

The option to expand project A does make the investment opportunity more attractive. The price range where it is optimal to invest in project A has increased to $73.33 \leq P \leq 102.71$. The reason is that the project value function, $V^{(2)}_{A,E}(P)$, and option value, $A^{(2)}_{0,A}(P)$, includes the option to expand at a later stage. The graph lines for these functions are therefore steeper compared to the respective value functions in Figure 10-3. This makes investing in project A more attractive relative to project B, which is now optimal at a higher price threshold, $113.15 \leq P$. The second waiting region is also in a lesser interval compared to the previous section, as investment is more attractive than deferring investment.

Notice that the first optimal investment threshold for project A is not affected. This is because both the option to invest in project A, and the project value of A are increasing with the option value to expand the project. As they both increase with the same amount, they will still meet tangentially at the same point as previously. The project value and option value at this point is however higher because of the new option.

10.4.2 Choosing between Project A or B when the Expansion is Uncertain

This subsection relates to the model calculated in Subsection 9.2.2, where the expansion is uncertain. The company does not know if the expansion exists, but can perform an exploration process to reveal its existence. Figure 10-5 presents the solution for $\overline{F}^{(2)}_{A_v B}(P)$, assuming $\lambda = 0{,}01$.

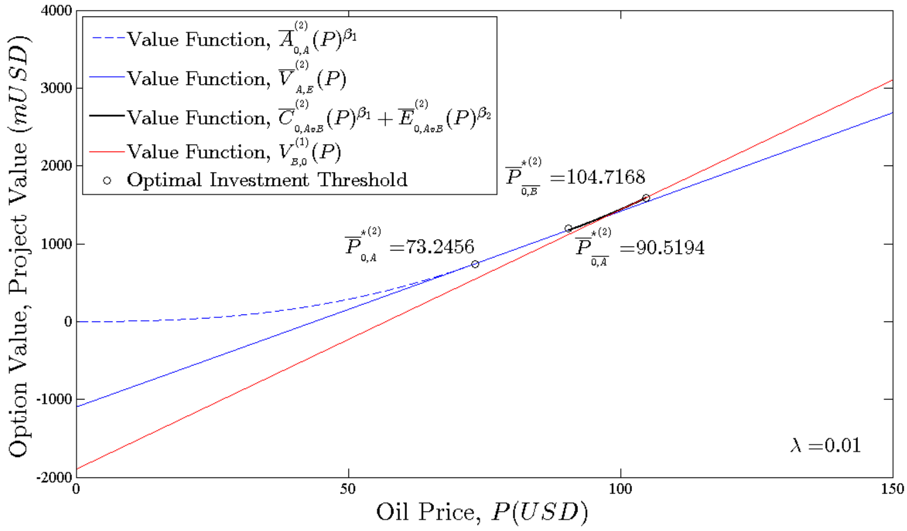

Figure 10-5: Optimized investment timing in project A or B when uncertain expansion in scenario 2 (generated by Matlab®)

The company should invest in project A when $73.25 \leq P \leq 90.52$, and only invest in project B when $104.72 \leq P$. Comparing the result with Figure 10-4, considering uncertainty regarding the expansion makes investing in project A less attractive. Simultaneously, the incentive to choose project B increases. Having the possibility to explore for expansion is in fact attractive, as the price range for investing in project A increases compared to not having it.

One interesting point is that $\overline{P}^{(2)}_{0,A} < P^{(2)}_{0,A}$. This is because of the introduction of the new endogenous constants, $\underline{A}^{(2)}_{A,E}$ and $\underline{B}^{(2)}_{A,E}$. As these are adjustments for an expected lag period from a successfully conducted exploration process, they affect the project- and option values. $\underline{A}^{(2)}_{A,E}$ affects the out of the money option value function, and $\underline{B}^{(2)}_{A,E}$ affects the project value.

As $\underline{A}_{A,E}^{(2)} \neq \underline{B}_{A,E}^{(2)}$, the option value function and project value function will no longer meet tangentially at the same optimal investment threshold as in Figure 10-4.

10.5 Project A Including all Options Versus Project B

This final section analyses the scenario where the value of project A includes all options discussed in present study. If the company decides to invest in project A, the project includes the option to expand it, and switching to project B at a later stage. This section divides between two situations; where the expansion is certain, and where there is uncertainty regarding an exploration process.

10.5.1 Choosing between Project A and B, when the Expansion is Certain

This subsection follows the model derived from Subsection 9.3.1. As there is certainty regarding the expansion, the company also knows that it can switch to project B after extracting the expanded project. The solution to the investment decision, $F_{A_v B}^{(3)}(P)$, is presented in Figure 10-6.

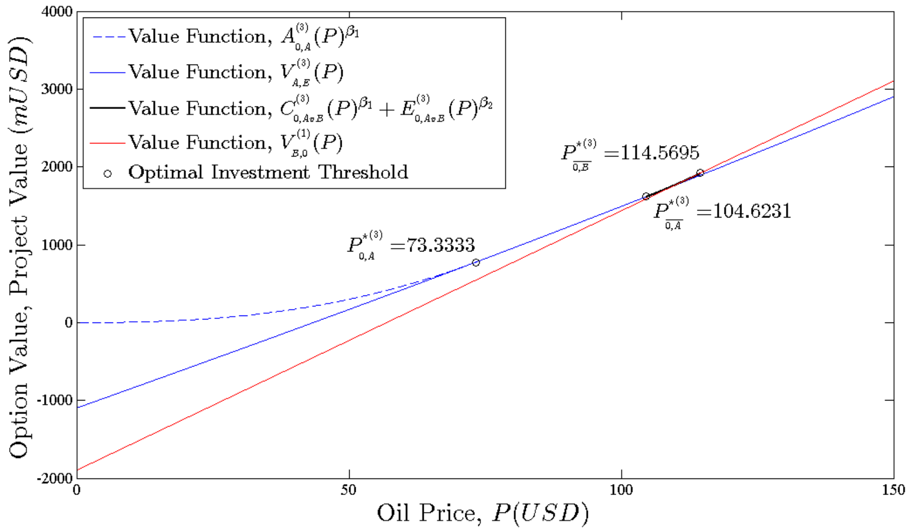

Figure 10-6: Optimized investment in project A or B when certain expansion in scenario 3 (generated by Matlab®)

Investing in project A is more attractive when considering the possibility to switch to project B. However, if the price is high enough, here specifically $114.57 \leq P$, it is still optimal to invest directly in project B. From Figure 10-2, it is proved that if the company does not have the choice between project A or B, the optimal investment threshold in project B is $95 \leq P$, which corresponds to the optimal price threshold to switch to project B. When $114.57 \leq P$, the company will not invest in project A, including all embedded options, but rather invest directly in project B.

10.5.2 Choosing between Project A and B, when the Expansion is Uncertain

This subsection applies the model calculated in Subsection 9.3.2 where there is uncertainty about the expansion. Adding an uncertainty parameter thereby affects the value of the switch option indirectly. Figure 10-7 provides the solution for the investment decision, $\overline{F}_{A_v B}^{(3)}(P)$.

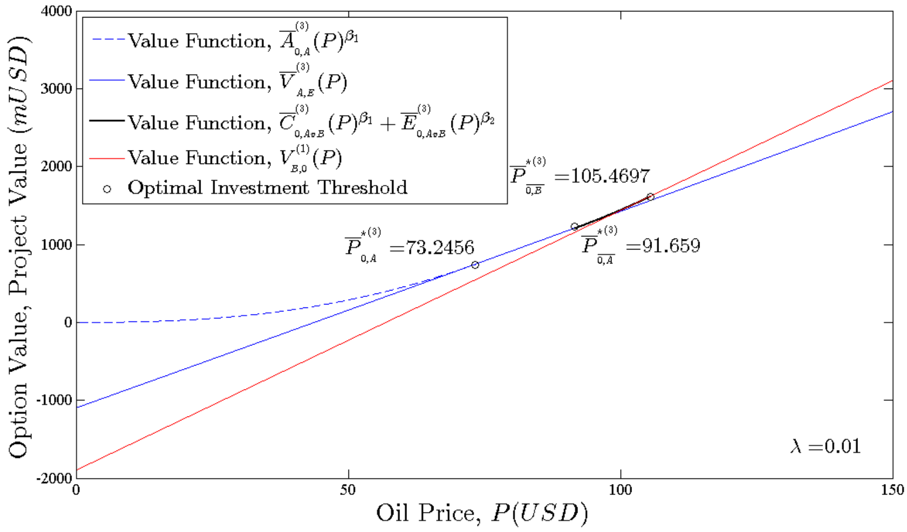

Figure 10-7: Optimized investment in project A or B when uncertain expansion in scenario 3 (generated by Matlab®)

Considering uncertainty in the calculations makes investing in project A less attractive compared to the solution in Figure 10-6. However, compared to not having this option, it is more likely that the company should choose project A rather than project B. The company will invest in project A when $73.25 \leq P \leq 105.47$ under exploration uncertainty, and thus the interval decreases compared to the situation with no exploration uncertainty.

This chapter has provided the numerical results for the case study. The next chapter will proceed with a discussion of the obtained results, and see how the models react to alterations of key parameters.

11. Comparison and Discussion of the Numerical Results

Chapter 10 provided interesting results which may provide investing companies valuable inputs for its investment decisions. In this chapter, the numerical results are discussed more thoroughly and compared across the different scenarios. Finally, a sensitivity analysis shows how the solutions change if the volatility parameter is altered.

11.1 Embedded Option's Effect on Investment Decision when Expansion is Certain

The three different scenarios with full certainty regarding the expansion is analysed first. For each scenario a new option is added, and thereby the results can be compared. A comparison based on the results from Chapter 10 is presented in Table 11-1.

Scenario	Optimal Investment Project A	Waiting Region	Optimal Investment Project B
NPV approach	$44 \leq P < 95$	$0 < P < 44$	$96.01 \leq P$
Scenario 1	$73.33 \leq P \leq 88.61$	$88.61 < P < 103.41$	$103.41 \leq P$
Scenario 2	$73.33 \leq P \leq 102.71$	$102.71 < P < 113.15$	$113.15 \leq P$
Scenario 3	$73.33 \leq P \leq 104.62$	$104.62 < P < 114.57$	$114.57 \leq P$

Table 11-1: Comparison results NPV and scenario 1-3 without uncertainty

Comparing the three scenarios studied to the standard NPV approach, there are great differences in the investing regions. The reason is that the NPV approach does not open for any waiting regions as long as the NPV is positive. The only waiting region in the NPV approach is from $P = 0$ until the project NPV is zero, here at $P = 44$. If the NPV approach considers the investment to be a now-or-never situation, project A will be rejected at $44 > P$. Opening for flexibility in the investment decision results in longer waiting regions, and the optimal price regions are therefore shorter.

Comparing scenario 1 to scenario 2, the option to expand makes investing in project A more attractive. The end of the interval increases from 88.61 to 102.71, and implies that investing is optimal at a greater range of price thresholds. The waiting period is moved further up in the price region as a direct consequence of the increased attractiveness of project A. This applies for the optimal investment threshold of project B as well, because project B's attractiveness is reduced compared to project A.

As shown in scenario 3, the option to switch to project B does provide additional value to project A, illustrated by an increase in the optimal investment interval. However, the effect is not as great as the effect of the expansion. Compared to scenario 2, the end of the interval increases from 102.71 to 104.62. Correspondingly, the waiting region is shorter and moved further up in the price region. Investing in project B is here only optimal at or above 114.57.

Flexibility with certain expansion makes project A more attractive compared to project B. The next section provides the same analysis, but in addition considers the exploration uncertainty.

11.2 Embedded Option's Effect on Investment Decision when Expansion is Uncertain

This section analyses the investment decision when accounting for the uncertainty regarding the expansion. This applies for scenario 2 and scenario 3, and a comparison between the scenarios with uncertainty is presented in Table 11-2.

Scenario	Optimal Investment Project A	Waiting Region	Optimal Investment Project B
Scenario 2 w/uncertainty	$73.25 \leq P \leq 90.52$	$90.52 < P < 104.72$	$104.72 \leq P$
Scenario 3 w/uncertainty	$73.25 \leq P \leq 91.66$	$91.66 < P < 105.47$	$105.47 \leq P$

Table 11-2: Comparison scenario 1-2 with uncertainty

When comparing the two scenarios with uncertainty to scenario 1, the embedded options makes investing in project A more attractive also after adjusting for exploration uncertainty. More interestingly, project A in both scenario 1 and 2 is less attractive after adjusting for exploration uncertainty compared to the situation where the expansion is certain. This result was expected, but one can see that the top threshold of project A is reduced significantly, and may have great impact on investment decisions.

The model assumes that $\lambda = 0.01$, and this has impact on the reduced price thresholds discussed above. It is intuitively difficult to imagine what a realistic input parameter for λ should be, as it reflects the likelihood of an exploration process being successful. A realistic parameter value for the Poisson intensity could be critical for the company when optimizing investment decisions. If $\lambda = 0$, the company will only look at the project value of A without embedded options in its decision-making. If $\lambda = 1$, then $\overline{F}^{(m)}_{A_v B}(P) = F^{(m)}_{A_v B}(P)$ if correcting for the lag period, and investing in project A is hence more valuable. Therefore, Figure 11-1 presents what happens if λ increases for both these scenarios.

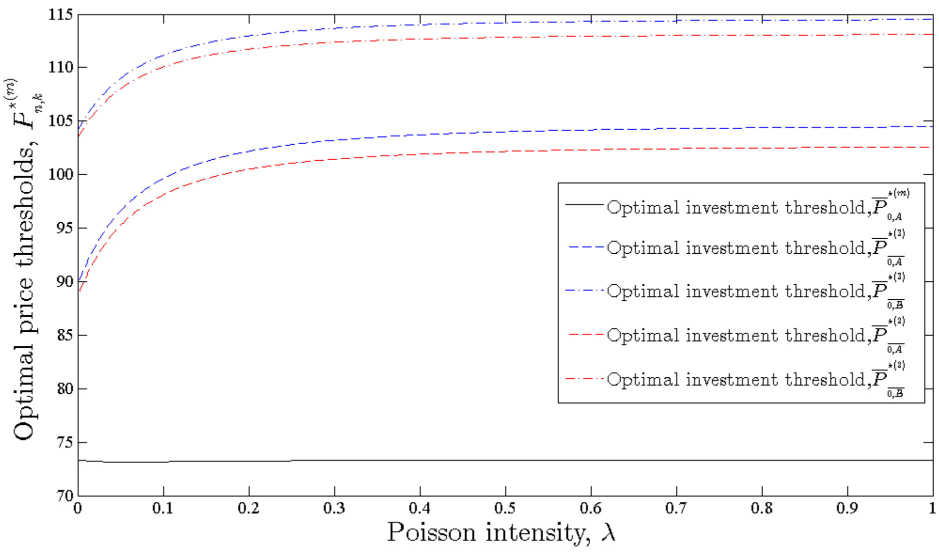

Figure 11-1: Sensitivity analysis Poisson intensity (generated by Matlab®)

Figure 11-1 describes the interval of λ from $0 \leq \lambda \leq 1$, and how the optimal investment thresholds react to the likelihood of an exploration process being successful. As previously argued, $\overline{P}^{*(2)}_{0,A}$ and $\overline{P}^{*(3)}_{0,A}$ is unaffected by the Poisson intensity if correcting for the lag period, and the figure confirms this as it remains nearly constant around 73.25 in the interval.

Intuitively, an increase in λ makes investment in project A more attractive, as there is a greater probability that the company can exercise both the option to expand and the option to switch to project B. The dashed lines in the figure confirm this. The price ranges for optimal investment in project A increases accordingly in both scenarios. The investment thresholds for project B in both scenarios increases respectively, as investing in project A is more attractive, while the second waiting period is moved parallel to this.

In addition to λ, the volatility affects a company's investment decision. The effect of this parameter is analyzed in the next section.

11.3 Volatility Sensitivity of the Model

The presented results assumed an oil price volatility of 20%.When considering option valuation, the volatility parameter is arguable the most important input. If there is no oil price fluctuation, there is no value in the flexibility of deferring an investment. As argued in Chapter 5, there is uncertainty in the estimation of this parameter based on the length of the estimation period. In addition, the calculations are based on monthly spot prices, when others might use daily price fluctuations. In addition, these are historical prices, so there is no guarantee that they will represent future price fluctuations.

This section provides an overview of what occurs when the volatility is altered in the situations with and without certainty of the expansion. The first example is presented in Figure 11-2 when expansion is certain.

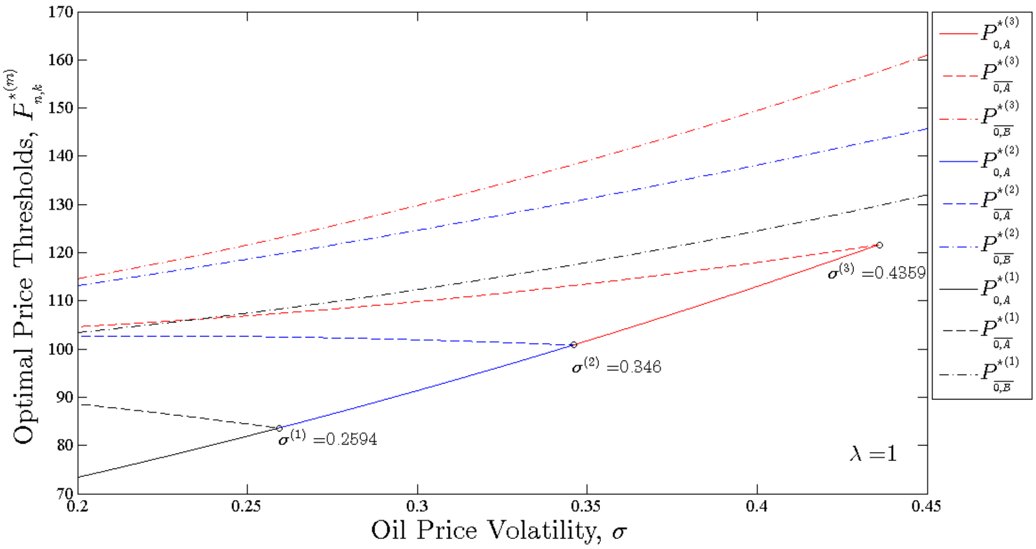

Figure 11-2: Volatility sensitivity when certain expansion (generated by Matlab®)

The figure describes the evolution of the three price thresholds in all three scenarios, with full certainty regarding the existence of the expansion, i.e. $\lambda = 1$. The graph samples every threshold in a volatility region from $20\% \leq \sigma \leq 45\%$. The grey lines are from scenario 1, the blue lines are from scenario 2, and the red lines are from scenario 3.

The base case with volatility at 20% is at the intersection point between the graphs and the y-axis. Intuitively, an increased volatility makes deferring investment more attractive as the option value increases. As shown in the figure, this also applies in this model. When volatility increases, the first optimal investment threshold for project A, $P^{*}{}^{(m)}_{0,A}$, and the optimal investment threshold for project B, $P^{*}\frac{(m)}{0,B}$, increases. This is because it requires a higher investment threshold for it to be optimal with an immediate investment at higher volatilities. Correspondingly, the second investment threshold of investment in project A, $P^{*}\frac{(m)}{0,A}$, decreases with increasing volatility. A decrease in this price threshold makes the second waiting region larger, and thereby confirms that waiting grows more attractive.

A point to note is where $P^{*}{}^{(m)}_{0,A}$ and $P^{*}\frac{(m)}{0,A}$ intersects at all scenarios. The intersection points are at $\sigma^{(1)} = 25.9\%$ in scenario 1, at $\sigma^{(2)} = 34.6\%$ in scenario 2, and at $\sigma^{(3)} = 43.6\ \%$ in scenario 3. As the interval between these two investment thresholds represent the region where the company should invest in project A, the intersection points are at the highest volatility possible for considering investment in project A. At every threshold where $\sigma^{(m)} < \sigma$, it will never be optimal to invest in project A. The company should therefore defer investment until $P^{*}\frac{(3)}{0,B} \leq P$, and invest directly in project B.

By observing the implied volatilities from the different scenarios, the effect embedded options have on the investment decision is shown. It is clear from Figure 11-2 that the required volatility for never considering investment in project A, increases when adding embedded options to the project. Adding both the option to expand, and the option to switch, requires an increase from $\sigma = 25.9\%$ to $\sigma = 43.6\%$ for project A to never be optimal. The optimal investment threshold for project B increases accordingly, as project A is more attractive when including the embedded options.

Figure 11-2 accounts only for the scenarios where there is certainty regarding the existence of the option to expand project A. Figure 11-3 shows the same case, but including uncertainty about the expansion, i.e. $\lambda = 0.01$.

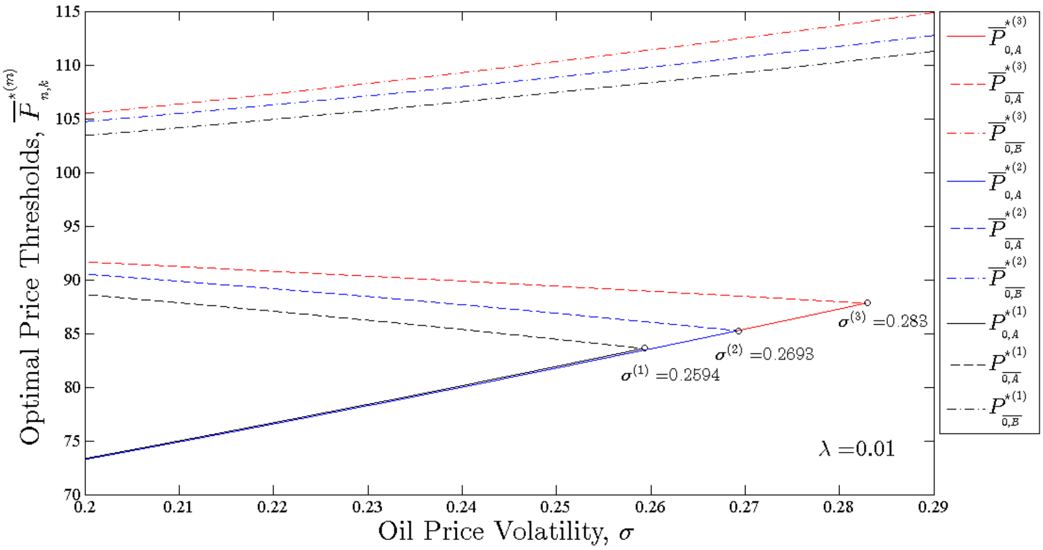

Figure 11-3: Volatility sensitivity when uncertain expansion (generated by Matlab®)

The same intuition applies for Figure 11-3, but in a lesser scale. As there is uncertainty regarding the availability of the embedded options, the level of the required implied volatility decreases. Adding both the option to expand and the option to switch requires an increase from $\sigma = 25.9\%$ to $\sigma = 28.3\%$ for project A to never be optimal, compared to $\sigma = 43.6\%$ under certainty. Based on this result, accounting for an uncertain parameter makes deferring investment more attractive.

12. Limitations and Simplifications

The proposed framework is developed to show how accounting for flexibility can affect a company's investment decision. This is done associated with several assumptions and simplifications. Therefore, the framework is not a final answer to these kinds of decision problems. The solution can be applied to several topics, but should include more elements to be completely viable for use in the oil industry. In this chapter, limitations which an investing company should consider in the investment process are listed first. Thereafter, other methods which can be applicable, and topics for future work based on this study is proposed.

12.1 Sector Simplifications

Present study proposes simplifications to the industry, which an operating company may consider including when making an investment decision.

It is assumed that the company operates in a monopoly setting. In reality, there is competition about licenses. To make the scenarios more realistic, the model should consider this.

As explained in Section 4.2, an operating company has to consider a variety of different risks. This study only considers the oil price volatility. It is arguably the most important variable, but accounting for other risks might affect investment decisions.

12.2 Limitations in the Model

This case study only considers three options; the option to defer investment, the option to expand, and the option to switch to another project. The embedded options are also only considered at one of the projects in this case. In reality, an investing company has more options than the options considered in this case study. One relevant example is the option to abandon a project. If the company seeks to avoid all possibilities of mispricing, all the options available in both projects must be accounted for.

The proposed model does not consider any types of convenience yield for holding the option rather than investing. The convenience yield in these situations should reflect the cost of deferring investment. Hence, the value of waiting can be overrated in present model.

Another assumption is that all costs are fixed in this model. Costs are factors which are uncertain, and are often a matter of available technology at the time. New technology can reduce investment costs in the future, and hence favouring deferring an investment. Costs can also be reduced by a decrease in the commodity prices, such as the steel price.

The model considers changes in the oil price up until the time of investment only. There is also only one investment decision in this regard. After investing, a perpetual model is applied. In reality, oil price fluctuations can provide the investing company possibilities to optimize investment decisions, after deciding to invest in the first place. One of these possibilities is the option to set a project passive or active at certain levels, as proposed by Miltersen and Schwartz (2007).

Applying GBM opposed to the Ornstein-Uhlenbeck process is a subject which is open for discussion. One can argue for both theories, and they can provide different answers.

12.3 Suggestions for Future Work

As there are limitations and simplifications in the proposed framework, this study opens for further research. It would be interesting if the framework were applied to other settings in the petroleum industry, or in other sectors as well. Therefore, this chapter concludes by listing some possible topics related to the study, which should be followed up in future work.

- Accounting for other options, especially abandonment option, and convenience yield
- Adding a third project to the problem
- Applying the Ornstein-Uhlenbeck process to the setting
- Accounting for cost uncertainty in the model
- Investment decision under competition

13. Conclusion

The ambition of present study was to apply relevant real option theory to a specific investment decision in the oil industry. The meaning was to suggest a better way of optimizing investment decisions when choosing between projects. Accounting for flexibility in the petroleum industry is of great importance, and several studies encourage operating companies to pay more attention to the subject. Even though there is significant investment flexibility in oil projects, several companies still rely on the standard NPV approach which may result in mispricing of projects.

The model suggested in this study emphasized the importance of including the value of flexibility when making an investment decision, and highlighted the importance of embedded options. The model described how an investing company effectively could consider these options in a decision-making process. Specifically, an option to expand a project and an option to switch to another project were the main focus.

The solutions to the case study clearly indicate that accounting for flexibility could alter a company's investment decision. Analyzing the option to defer investment showed that the company should not invest immediately when the project provides a positive net present value, like a breakeven analysis would suggest. Instead, the company should wait until the oil price is at a higher threshold. In addition, the analysis proved that embedded options provide sufficient value to alter a company's investment decision significantly. Another finding was that uncertainty regarding an exploration process is an important value-decreasing factor which must be acknowledged in the process.

The study raised questions which could be important for the oil industry, as well as other sectors. When considering large irreversible investment decisions, mispricing of the projects may lead to the wrong investment decision. While the study did not offer a conclusive answer to a total investment problem, it provided valuable insight in how to address them.

It would be helpful to do further research in this area, and the study suggested several important topics to the matter. Based on the findings, it is recommended that the industry analyses the effect of the embedded options proposed when investing. Accounting for flexibility is a critical factor which operating companies should consider in the process.

Appendix

1 Valuation of- and comparison between project A and project B, including the option to defer investment

1.1 Project value,

The oil price, *P*, follows a Geometric Brownian motion process (A-1.1)

$$dP = P\mu dt + P\sigma dz \tag{A-1.1}$$

The value of the outcome is dependent of a deterministic and a stochastic part, *dP*.

$$V_{n,k}^{(m)}(P) = D_n P dt + \varepsilon\left[V(P + dP)e^{-(\gamma+\rho)dt}\right] \tag{A-1.2}$$

Substituting (A-1.1) into (A-1.2) and expanding the stochastic process using Itôs Lemma, (A-1.2) into. An Itô's Lemma expansion can be perpetual by using more than two primes, but this is an unnecessary procedure as all parts with primes higher than two are so small that they do not add value. Thus, primes above two can be excluded.

$$\begin{aligned} V_{n,k}^{(m)}(P) = \frac{1}{2}\sigma^2 P^2 V_{n,k}^{(m)''}(P)dt + \mu V_{n,k}^{(m)} V'(P)dt - (\gamma + \rho)V_{n,k}^{(m)}(P)dt \\ + D_n P dt + V_{n,k}^{(m)}(P) \end{aligned} \tag{A-1.3}$$

Simplifying, dividing by *dt* and accounts for the below factors, the non-homogenous ordinary differential equation (ODE) below appears in Equation (A-1.4).

- $dz = \epsilon_t\sqrt{dt}$, where ϵ_t has zero mean and unit standard deviation. Thus dz = 0
- dt^2 converges towards zero faster than dt, thus $dt^2 = 0$
- $dtdx$ also converges towards zero faster than dt and thus $dtdx = 0$
- dz^2 converges towards dt and thus $dz^2 = dt$

$$\frac{1}{2}\sigma^2 P^2 V_{n,k}^{(m)''}(P) + \mu P V_{n,k}^{(m)'}(P) - (\gamma + \rho)V_{n,k}^{(m)}(P) + D_n P \;= 0 \tag{A-1.4}$$

To obtain a particular solution to the non-homogenous ODE (A-1.4), *t* is set to *log(p)*, and the below bullet points is substituted into (A-1.4), which solves into (A-1.7)

- $P\frac{dV_{n,k}^{(m)}}{dP} = \mathrm{P}\frac{dV_{n,k}^{(m)}}{dt}\frac{dt}{dP} = \mathrm{P}\frac{dV_{n,k}^{(m)}}{dt}\frac{1}{P} = \frac{dV_{n,k}^{(m)}}{dt}$
- $P^2\frac{d^2V_{n,k}^{(m)}}{dP^2} = P^2\frac{d}{dP}\left(\frac{1}{P}\frac{dV_{n,k}^{(m)}}{dt}\right) =$

$P^2\left(\frac{1}{P^2}\right)\frac{dV_{n,k}^{(m)}}{dt} + P^2\frac{1}{P}\frac{d}{dP}\frac{dV_{n,k}^{(m)}}{dt} - \frac{dV_{n,k}^{(m)}}{dt} + \mathrm{P}\frac{d}{dt}\left(\frac{dV_{n,k}^{(m)}}{dt}\right)\frac{dt}{dV_{n,k}^{(m)}} = -\frac{dV_{n,k}^{(m)}}{dt} + \frac{d^2V_{n,k}^{(m)}}{dt^2}$

$$\Rightarrow \frac{1}{2}\sigma^2\left(-\frac{dV_{n,k}^{(m)}}{dt} + \frac{d^2V_{n,k}^{(m)}}{dt^2}\right) + \mu P\left(\frac{dV_{n,k}^{(m)}}{dt}\right) - (\lambda+\rho)V_{n,k}^{(m)}(P) + D_nP = 0 \quad \text{(A-1.5)}$$

$$\Rightarrow \frac{1}{2}\sigma^2 V_{n,k}^{(m)''} - \frac{1}{2}\sigma^2 V_{n,k}^{(m)'} + \mu V_{n,k}^{(m)'} - (\lambda+\rho)V_{n,k}^{(m)}(P) + D_nP = 0 \quad \text{(A-1.6)}$$

$$\Rightarrow \frac{1}{2}\sigma^2 V_{n,k}^{(m)''} + \left(\mu - \frac{1}{2}\sigma^2\right)V_{n,k}^{(m)'} - (\gamma+\rho)V_{n,k}^{(m)} + D_ne^t = 0 \quad \text{(A-1.7)}$$

First, V is transformed into ze^t and after grouping the terms the following appears.

$$\begin{aligned}&\frac{1}{2}\sigma^2(z''e^t + z'e^t + z'e^t + ze^t)\\ &\quad + \left(\mu - \frac{1}{2}\sigma^2\right)(z'e^t + ze^t) - (\gamma+\rho)ze^t = -D_ne^t\end{aligned} \quad \text{(A-1.8)}$$

Further reorganizing solves into a homogenous ODE:

$$\frac{1}{2}\sigma^2 z'' + \left(\mu + \frac{1}{2}\sigma^2\right)z' + (\mu - \gamma - \rho)z = -D_n \quad \text{(A-1.9)}$$

In the next stage, z is set to c, where c is constant. A natural interpretation of $z = c =$ constant, is that $z'' = z' = 0$. Thus, remaining from Equation (A-1.9) is:

$$(\mu - \gamma - \rho)z = -D_n \Rightarrow z = \frac{D_n}{\gamma + \rho - \mu} \quad \text{(A-1.10)}$$

Now, transforming back $V = ze^t$ and a closed form solution is obtained:

$$z = \frac{D_n}{\gamma + \rho - \mu} \Rightarrow V = \frac{D_ne^t}{\gamma + \rho - \mu} = \frac{D_nP}{\gamma + \rho - \mu} \quad \text{(A-1.11)}$$

This can also be written as a more intuitive interpretation, Equation (A-1.12), as the value of a perpetual stream of cash flows with value decreasing factor, γ. Notice that $0 < \rho > \mu$.

$$V_{n,k}^{(m)}(P) = \int_0^{\infty} \frac{\gamma e^{-\gamma T} D_n P\left(1 - e^{-(\rho-\mu)T}\right)}{\gamma + \rho - \mu} d\tau - I_n = \frac{D_n P}{\gamma + \rho - \mu} - I_n \qquad \text{(A-1.12)}$$

1.2 Option value

The option to invest in project A or B in scenario 1 is given in (A-1.13).

$$F_{n,k}^{(m)}(P) = \begin{cases} \varepsilon\left[F_{n,k}^{(m)}(P + dP)e^{-\rho dt}\right] & , P < P^{*(m)}_{n,k} \\ V_{n,k}^{(m)}(P) & , P^{*(m)}_{n,k} \leq P \end{cases} \qquad \text{(A-1.13)}$$

Expanding the stochastic part in the first branch using Itôs Lemma, (A-1.13) evolves into the homogenous second-order differential Equation, (A-1.16)

$$F_{n,k}^{(m)}(P) = \frac{1}{2}\sigma^2 P^2 F_{n,k}^{(m)''}(P)dt + \mu P F_{n,k}^{(m)'}(P)dt - \rho F_{n,k}^{(m)}(P)dt + F_{n,k}^{(m)}(P) \qquad \text{(A-1.14)}$$

$$\Rightarrow \qquad \frac{1}{2}\sigma^2 P^2 F_{n,k}^{(m)''}(P)dt + \mu P F_{n,k}^{(m)'}(P)dt - \rho F_{n,k}^{(m)}(P)dt = 0 \qquad \text{(A-1.15)}$$

$$\Rightarrow \qquad \frac{1}{2}\sigma^2 P^2 F_{n,k}^{(m)''}(P) + \mu P F_{n,k}^{(m)'}(P) - \rho F_{n,k}^{(m)}(P) = 0 \qquad \text{(A-1.16)}$$

As (A-1.16) is an equation of second-order, the general solution (A-1.17) can be expressed as a linear combination of any two independent solutions,

$$F_{n,k}^{(m)}(P) = A_{n,k}^{(m)} P^{\beta_1} + B_{n,k}^{(m)} P^{\beta_2} \qquad \text{(A-1.17)}$$

where $A_{n,k}^{(m)}$, $B_{n,k}^{(m)}$ is to be determined, and β_1 > 1 and β_2 < 0 is known. $F_{n,k}^{(m)}(P)$ must satisfy the following three boundary conditions in (A-1.18):

$$\begin{aligned} &(1): \quad F_{n,k}^{(m)}(0) = 0 \\ &(2): \quad F_{n,k}^{(m)}(P^*) = V_{n,k}^{(m)}(P^*) - I_n \\ &(3): \quad F'^{(m)}_{n,k}(P^*) = V'^{(m)}_{n,k}(P^*) \end{aligned} \qquad \text{(A-1.18)}$$

When boundary condition (1) is adhered and due to $\beta_2 < 0$, $B_{n,k}^{(m)}(P) = 0$, $F_{n,k}^{(m)}(P)$ takes the following form.

$$F_{n,k}^{(m)}(P) = A_{n,k}^{(m)} P^{\beta_1} \qquad \text{(A-1.19)}$$

Finding the solutions for β_1 and β_2, the particular solution to (A-1.16), *t* is set to *log(p),* and the below bullet points is substituted into (A-1.16). Also, $F_{n,k}^{(m)}$ is substituted by β at the end. This solves into the quadratic function (A-1.20).

- $P\frac{dF_{n,k}^{(m)}}{dP} = \mathrm{P}\frac{dF_{n,k}^{(m)}}{dt}\frac{dt}{dP} = P\frac{dF_{n,k}^{(m)}}{dt}\frac{1}{P} = \frac{dF_{n,k}^{(m)}}{dt}$
- $P^2\frac{d^2}{dP^2} = P^2\frac{d}{dP}\left(\frac{1}{P}\frac{dF_{n,k}^{(m)}}{dt}\right) = P^2\left(\frac{1}{P^2}\right)\frac{dF_{n,k}^{(m)}}{dt} + P^2\frac{1}{P}\frac{d}{dP}\frac{dF_{n,k}^{(m)}}{dt} =$

  $-\frac{dV}{dt} + \mathrm{P}\frac{d}{dt}\left(\frac{dF_{n,k}^{(m)}}{dt}\right)\frac{dt}{dF_{n,k}^{(m)}} = -\frac{dF_{n,k}^{(m)}}{dt} + \frac{d^2F_{n,k}^{(m)}}{dt^2}$

$$\frac{1}{2}\sigma^2\beta'' + \left(\mu - \frac{1}{2}\sigma^2\right)\beta' - \rho\beta = 0 \qquad \text{(A-1.20)}$$

Solving this quadratic solution with respect to β, the two roots are as follow in Equation (A-1.21) and (A-1.22).

$$\beta_1 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}}{\sigma^2} \qquad \text{(A-1.21)}$$

$$\beta_2 = \frac{\left(\frac{1}{2}\sigma^2 - \mu\right) - \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}}{\sigma^2} \qquad \text{(A-1.22)}$$

Finally, this leaves the company with the following option to invest in either project A or B.

$$F(V) = \begin{cases} A_{n,k}^{(m)} V^{\beta_1} & , P < P^{*}{}_{n,k}^{(m)} \\ \dfrac{D_n P}{\gamma + \rho - \mu} - I_n & , P^{*}{}_{n,k}^{(m)} \leq P \end{cases} \qquad \text{(A-1.23)}$$

Using value-matching and smooth-pasting conditions (boundary condition (2) and (3)) in (A-1.18), the two unknowns, $A_{n,k}^{(m)}$ and $P^{*}{}_{n,k}^{(m)}$ in (A-1.23) can be solved as a set of equations with two unknowns, Equation (A-1.24) and (A-1.25).

(2)
$$A_{n,k}^{(m)} {P^*}_{n,k}^{(m)\beta_1} = \frac{D_n {P^*}_{n,k}^{(m)}}{\gamma + \rho - \mu} - I_n \tag{A-1.24}$$

(3)
$$\beta_1 A_{n,k}^{(m)} {P^*}_{n,k}^{(m)\beta_1} = \frac{D_n {P^*}_{n,k}^{(m)}}{\gamma + \rho - \mu} \tag{A-1.25}$$

In the following, a step-by step solution will be presented on how to solve the set of equations with the two unknowns. First, Equation (A-1.25) is reorganized to (A-1.26)

(3)
$$A_{n,k}^{(m)} {P^*}_{n,k}^{(m)\beta_1} = \frac{D_n {P^*}_{n,k}^{(m)}}{\beta_1(\gamma + \rho - \mu)} \tag{A-1.26}$$

Setting (A-1.24) equal to (A.1.26), $A_{n,k}^{(m)} {P^*}_{n,k}^{(m)\beta_1}$ cancels out and after reorganizing, the following expression is left.

$$-\frac{D_n {P^*}_{n,k}^{(m)}}{\gamma + \rho - \mu} + \frac{D_n {P^*}_{n,k}^{(m)}}{\beta_1(\gamma + \rho - \mu)} = -I_n \tag{A-1.27}$$

After factorizing the left hand side, and multiplying the whole expression by -1, the expression can be simplified as follow:

$$\frac{D_n {P^*}_{n,k}^{(m)}(\beta_1 - 1)}{\beta_1(\gamma + \rho - \mu)} = I_n \tag{A-1.28}$$

Solving the expression with respect to ${P^*}_{n,k}^{(m)}$ gives the final answer to the optimal threshold, ${P^*}_{n,k}^{(m)}$, in Equation (A-1.29)

$${P^*}_{n,k}^{(m)} = \frac{\beta_1 I_n(\gamma + \rho - \mu)}{D_n(\beta_1 - 1)} \tag{A-1.29}$$

As the optimal threshold is solved, the solution can be used to estimate $A_{n,k}^{(m)}$ by substituting ${P^*}_{n,k}^{(m)}$ into the smooth-pasting condition, (A-1.25). After reorganizing, $A_{n,k}^{(m)}$ is presented in (A-1.30)

$$A_{n,k}^{(m)} = \frac{D_n {P^*}_{n,k}^{(m)(1-\beta_1)}}{\beta_1(\gamma + \rho - \mu)} \tag{A-1.30}$$

Investment decision: Scenario 1, $F^{(1)}_{A_v B}(P)$

$$F^{(1)}_{A_v B}(P) = \begin{cases} A^{(1)}_{0,A} P^{\beta_1} & , P < P^{*(1)}_{0,A} \\ V^{(1)}_{A,0}(P) & , P^{*(1)}_{0,A} \le P \le P^{*}\frac{(1)}{0,A} \\ C^{(1)}_{0,A_v B} P^{\beta_1} + E^{(1)}_{0,A_v B} P^{\beta_2} & , P^{*}\frac{(1)}{0,A} \le P \le P^{*}\frac{(1)}{0,B} \\ V^{(1)}_{B,0}(P) & , P^{*}\frac{(1)}{0,B} \le P \end{cases} \tag{A-1.31}$$

In Equation (A-1.31), $A^{(1)}_{0,A} P^{\beta_1}$ and $P^{*(1)}_{0,A}$, are solved via value-matching and smooth-pasting between the first and second branch, while $P^{*}\frac{(1)}{0,A}$, $P^{*}\frac{(1)}{0,B}$, $C^{(1)}_{0,A_v B} P^{\beta_1}$ and $E^{(1)}_{0,A_v B} P^{\beta_2}$ are solved numerically via the second, third, and fourth branch. Value-matching and smooth-pasting between the second and third branch are presented in (A-1.32), and value-matching and smooth-pasting between the third and fourth branch are presented in (A-1.33).

$$\begin{gathered} \frac{D_A P^{*}\frac{(1)}{0,A}}{\gamma + \rho - \mu} - I_A = C^{(1)}_{0,A_v B} P^{*}\frac{(1)}{0,A}^{\beta_1} + E^{(1)}_{0,A_v B} P^{*}\frac{(1)}{0,A}^{\beta_2} \\ \frac{D_A P^{*}\frac{(1)}{0,A}}{\gamma + \rho - \mu} = \beta_1 C^{(1)}_{0,A_v B} P^{*}\frac{(1)}{0,A}^{\beta_1} + \beta_2 E^{(1)}_{0,A_v B} P^{*}\frac{(1)}{0,A}^{\beta_2} \end{gathered} \tag{A-1.32}$$

$$\begin{gathered} C^{(1)}_{0,A_v B} P^{*}\frac{(1)}{0,B}^{\beta_1} + E^{(1)}_{0,A_v B} P^{*}\frac{(1)}{0,B}^{\beta_2} = \frac{D_B P^{*}\frac{(1)}{0,B}}{\gamma + \rho - \mu} - I_B \\ \beta_1 C^{(1)}_{0,A_v B} P^{*}\frac{(1)}{0,B}^{\beta_1} + \beta_2 E^{(1)}_{0,A_v B} P^{*}\frac{(1)}{0,B}^{\beta_2} = \frac{D_B P^{*}\frac{(1)}{0,B}}{\gamma + \rho - \mu} \end{gathered} \tag{A-1.33}$$

2 Valuation of project A with option to expand, and comparison with project B

2.1 Project A versus Project B when the expansion is certain: Scenario 2, $\mathbf{F}^{(2)}_{\mathbf{A_vB}}(\mathbf{P})$

Option value of expansion: Scenario 2, $F^{(2)}_{A,E}(P)$

$$F^{(2)}_{A,E}(P) = \begin{cases} V^{(1)}_{A,0}(P) + \varepsilon\left[F^{(2)}_{A,E}(P + dP)e^{-\rho dt}\right] \\ V^{(1)}_{A,E}(P) \end{cases} = \begin{cases} V^{(1)}_{A,0}(P) + A^{(2)}_{A,E}P^{\beta_1} & , P < P^{*(2)}_{A,E} \\ V^{(2)}_{A,E}(P) & , P^{*(2)}_{A,E} \le P \end{cases} \quad \text{(A-2.1)}$$

By applying value-matching and smooth-pasting conditions between the two branches in (A-2.1), the two unknowns, $A^{(2)}_{A,E}$ and $P^{*(2)}_{A,E}$ can be solved following the same framework as Equation (A-1.23) – (A-1.30) and is presented in Equation (A-2.2) and (A-2.3).

$$A^{(2)}_{A,E} = \frac{{P^{*(2)}_{A,E}}^{(1-\beta_1)}(D_E - D_A)}{\beta_1(\gamma + \rho - \mu)} \quad \text{(A-2.2)}$$

$$P^{*(2)}_{A,E} = \frac{\beta_1 I_E(\gamma + \rho - \mu)}{(\beta_1 - 1)(D_E - D_A)} \quad \text{(A-2.3)}$$

Investment decision: Scenario 2, $F^{(2)}_{A_vB}(P)$

$$F^{(2)}_{A_vB}(P) = \begin{cases} A^{(2)}_{0,A}P^{\beta_1} & , P < P^{*(2)}_{0,A} \\ V^{(2)}_{A,E}(P) & , P^{*(2)}_{0,A} \le P \le P^{*\frac{(2)}{0,A}} \\ C^{(2)}_{0,A_vB}P^{\beta_1} + E^{(2)}_{0,A_vB}P^{\beta_2} & , P^{*\frac{(2)}{0,A}} \le P \le P^{*\frac{(2)}{0,B}} \\ V^{(1)}_{B,0}(P) & , P^{*\frac{(2)}{0,B}} \le P \end{cases} \quad \text{(A-2.4)}$$

In Equation (A-2.4), $A^{(2)}_{0,A}P^{\beta_1}$ and $P^{*(2)}_{0,A}$, are solved via value-matching and smooth-pasting between the first and second branch, while $P^{*\frac{(2)}{0,A}}$, $P^{*\frac{(2)}{0,B}}$, $C^{(2)}_{0,A_vB}P^{\beta_1}$ and $E^{(2)}_{0,A_vB}P^{\beta_2}$ are solved numerically via the second, third, and fourth branch. Value-matching and smooth-pasting between the second and third branch are presented in (A-2.5), and value-matching and smooth-pasting between the third and fourth branch are presented in (A-2.6).

$$\frac{D_A P^{*(2)}_{0,A}}{\gamma+\rho-\mu} - I_A + A^{(2)}_{A,E} P^{*(2)\,\beta_1}_{0,A} = C^{(2)}_{0,A_vB} P^{*(2)\,\beta_1}_{0,A} + E^{(2)}_{0,A_vB} P^{*(2)\,\beta_2}_{0,A}$$
$$\frac{D_A P^{*(2)}_{0,A}}{\gamma+\rho-\mu} + \beta_1 A^{(2)}_{A,E} P^{*(2)\,\beta_1}_{0,A} = \beta_1 C^{(2)}_{0,A_vB} P^{*(2)\,\beta_1}_{0,A} + \beta_2 E^{(2)}_{0,A_vB} P^{*(2)\,\beta_2}_{0,A} \quad \text{(A-2.5)}$$

$$C^{(2)}_{0,A_vB} P^{*(2)\,\beta_1}_{0,B} + E^{(2)}_{0,A_vB} P^{*(2)\,\beta_2}_{0,B} = \frac{D_B P^{*(2)}_{0,B}}{\gamma+\rho-\mu} - I_B$$
$$\beta_1 C^{(2)}_{0,A_vB} P^{*(2)\,\beta_1}_{0,B} + \beta_2 E^{(2)}_{0,A_vB} P^{*(2)\,\beta_2}_{0,B} = \frac{D_B P^{*(2)}_{0,B}}{\gamma+\rho-\mu} \quad \text{(A-2.6)}$$

2.2 Project A versus Project B when the expansion is uncertain: Scenario 2, $\overline{\mathbf{F}}^{(2)}_{\mathbf{A_vB}}(\mathbf{P})$

Modelling the uncertainty effect on project A: Scenario 2, $\overline{V}^{(2)}_{A,E}(P)$

$$\overline{V}^{(2)}_{A,E}(P) = (D_A P dt - \rho I_A dt) + (1-\rho dt)\lambda dt \varepsilon\left[F^{(2)}_{A,E}(P+dP)\right] + (1-\rho dt)(1-\lambda dt)\varepsilon\left[\overline{V}^{(2)}_{A,E}(P+dP)\right] \quad \text{(A-2.7)}$$

Expanding the stochastic part on the right hand side of (A-2.7) using Itôs Lemma, the equation evolves into (A-2.8)

$$\begin{aligned}\overline{V}^{(2)}_{A,E}(P) &= D_A P - \rho I_A \\ &+ (1-(\rho+\lambda)dt)\left[\overline{V}^{(2)}_{A,E}(P)dt + \frac{1}{2}\sigma^2 P^2 \overline{V}^{(2)''}_{A,E}(P)dt + \mu P \overline{V}^{(2)'}_{A,E}(P)dt\right] \\ &+ (1-\rho dt)\lambda dt\left[F^{(2)}_{A,E}(P)dt + \frac{1}{2}\sigma^2 P^2 F^{(2)''}_{A,E}(P)dt + \mu P F^{(2)'}_{A,E}(P)dt\right]\end{aligned} \quad \text{(A-2.8)}$$

Reorganizing and simplifying (A-2.8), the differential equation that describes $\overline{V}^{(2)}_{A,E}(P)$ is presented in (A-2.9)

$$\frac{1}{2}\sigma^2 P^2 \overline{V}^{(2)''}_{A,E}(P) + \mu P \overline{V}^{(2)'}_{A,E}(P) - (\rho+\lambda)\overline{V}^{(2)}_{A,E}(P) + D_A P - \rho I_A + \lambda F^{(2)}_{A,E}(P) = 0 \quad \text{(A-2.9)}$$

As $F^{(2)}_{A,E}(P)$ has two solutions, when $P < P^{*(2)}_{A,E}$ and $P^{*(2)}_{A,E} \le P$, the solution to $\overline{V}^{(2)}_{A,E}(P)$ can be written as:

$$\begin{cases} \frac{1}{2}\sigma^2 P^2 \overline{V}_{A,E}^{(2)''}(P) + \mu P \overline{V}_{A,E}^{(2)'}(P) - (\rho+\lambda)\overline{V}_{A,E}^{(2)}(P) \\ \qquad + \frac{D_A P}{\gamma+\rho-\mu} - I_A + \lambda A_{A,E}^{(2)} P^{\beta_1} = 0 & , P < P^{*(2)}_{A,E} \\ \frac{1}{2}\sigma^2 P^2 \overline{V}_{A,E}^{(2)''}(P) + \mu P \overline{V}_{A,E}^{(2)'}(P) - (\rho+\lambda)\overline{V}_{A,E}^{(2)}(P) \\ \qquad + D_A P - I_A + \frac{\lambda D_E}{\gamma+\rho-\mu} - \lambda I_E = 0 & , P^{*(2)}_{A,E} \leq P \end{cases} \tag{A-2.10}$$

Using the framework presented in (A-1.4) – (A-1.12), the solution for $\overline{V}_{A,E}^{(2)}(P)$ is given in (A-2.11). Notice, when following the same reasoning as in (A-1.17), the terms containing negative exponents, β_2 and δ_2 can be ruled out in the top part of (A-2.11) and the term containing the positive exponent, δ_1 in the bottom part of (A-2.11) can be ruled out.

$$\overline{V}_{A,E}^{(2)}(P) = \begin{cases} V_{A,0}^{(1)}(P) + A_{A,E}^{(2)} P^{\beta_1} + \underline{A}_{A,E}^{(2)} P^{\delta_1} & , P < P^{*(2)}_{A,E} \\ \frac{P[\lambda D_E + (\gamma+\rho-\mu)D_A]}{(\rho+\lambda-\mu)(\gamma+\rho-\mu)} \\ \qquad - \frac{\lambda I_E}{\rho+\lambda} - I_A + \underline{B}_{A,E}^{(2)} P^{\delta_2} & , P^{*(2)}_{A,E} \leq P \end{cases} \tag{A-2.11}$$

Investment decision: Scenario 2, $\overline{F}_{A_v B}^{(2)}(P)$

$$\overline{F}_{\mathrm{A_v B}}^{(2)}(P) = \begin{cases} \overline{A}_{0,A}^{(2)} P^{\beta_1} & , P < \overline{P}^{*(2)}_{0,A} \\ \overline{V}_{A,E}^{(2)}(P) & , \overline{P}^{*(2)}_{0,A} \leq P \leq \overline{P}^{*(2)}_{\underline{0,A}} \\ \overline{C}_{0,A_v B}^{(2)} P^{\beta_1} + \overline{E}_{0,A_v B}^{(2)} P^{\beta_2} & , \overline{P}^{*(2)}_{\underline{0,A}} \leq P \leq \overline{P}^{*(2)}_{\underline{0,B}} \\ V_{B,0}^{(1)}(P) & , \overline{P}^{*(2)}_{\underline{0,B}} \leq P \end{cases} \tag{A-2.12}$$

In Equation (A-2.12), $\overline{A}_{0,A}^{(2)} P^{\beta_1}$ and $\overline{P}^{*(2)}_{0,A}$, are solved via value-matching and smooth-pasting between the first and second branch, while $\overline{P}^{*(2)}_{\underline{0,A}}$, $\overline{P}^{*(2)}_{\underline{0,B}}$, $\overline{C}_{0,\mathrm{A_v B}}^{(2)} P^{\beta_1}$ and $\overline{E}_{0,\mathrm{A_v B}}^{(2)} P^{\beta_2}$ are solved numerically via the second, third, and fourth branch. Value-matching and smooth-pasting between the second and third branch are presented in (A-2.13), and value-matching and smooth-pasting between the third and fourth branch are presented in (A-2.14).

$$\frac{D_A \overline{P}^{*(2)}_{\overline{0,A}}}{\gamma+\rho-\mu} - I_A + A^{(2)}_{A,E} \overline{P}^{*(2)}_{\overline{0,A}}{}^{\beta_1} + \underline{A}^{(2)}_{A,E} \overline{P}^{*(2)}_{\overline{0,A}}{}^{\delta_1} = \overline{C}^{(2)}_{0,A_vB} \overline{P}^{*(2)}_{\overline{0,A}}{}^{\beta_1} + \overline{E}^{(2)}_{0,A_vB} \overline{P}^{*(2)}_{\overline{0,A}}{}^{\beta_2}$$

$$\frac{D_A \overline{P}^{*(2)}_{\overline{0,A}}}{\gamma+\rho-\mu} + \beta_1 A^{(2)}_{A,E} \overline{P}^{*(2)}_{\overline{0,A}}{}^{\beta_1} + \delta_1 \underline{A}^{(2)}_{A,E} \overline{P}^{*(2)}_{\overline{0,A}}{}^{\delta_1} = \beta_1 \overline{C}^{(2)}_{0,A_vB} \overline{P}^{*(2)}_{\overline{0,A}}{}^{\beta_1} + \beta_2 \overline{E}^{(2)}_{0,A_vB} \overline{P}^{*(2)}_{\overline{0,A}}{}^{\beta_2} \tag{A-2.13}$$

$$\overline{C}^{(2)}_{0,A_vB} \overline{P}^{*(2)}_{\overline{0,B}}{}^{\beta_1} + \overline{E}^{(2)}_{0,A_vB} \overline{P}^{*(2)}_{\overline{0,B}}{}^{\beta_2} = \frac{D_B \overline{P}^{*(2)}_{\overline{0,B}}}{\gamma+\rho-\mu} - I_B$$

$$\beta_1 \overline{C}^{(2)}_{0,A_vB} \overline{P}^{*(2)}_{\overline{0,B}}{}^{\beta_1} + \beta_2 \overline{E}^{(2)}_{0,A_vB} \overline{P}^{*(2)}_{\overline{0,B}}{}^{\beta_2} = \frac{D_B \overline{P}^{*(2)}_{\overline{0,B}}}{\gamma+\rho-\mu} \tag{A-2.14}$$

3 Valuation of project A including all embedded options, and comparison with project B

3.1 Project A versus Project B when the expansion is certain: Scenario 3, $\mathbf{F}^{(3)}_{\mathbf{A_vB}}$

Investment decision: scenario 3, $F^{(3)}_{A_vB}(P)$

$$F^{(3)}_{A_vB}(P)=\begin{cases} A^{(3)}_{0,A}P^{\beta_1} & ,P<P^{*(3)}_{0,A} \\ V^{(3)}_{A,E}(P) & ,P^{*(3)}_{0,A}\le P\le P^{*\frac{(3)}{0,A}} \\ C^{(3)}_{0,A_vB}P^{\beta_1}+E^{(3)}_{0,A_vB}P^{\beta_2} & ,P^{*\frac{(3)}{0,A}}\le P\le P^{*\frac{(3)}{0,B}} \\ V^{(1)}_{B,0}(P) & ,P^{*\frac{(3)}{0,B}}\le P \end{cases} \tag{A-3.1}$$

In Equation (A-3.1), $A^{(3)}_{0,A}P^{\beta_1}$ and $P^{*(3)}_{0,A}$, are solved via value-matching and smooth-pasting between the first and second branch, while $P^{*\frac{(3)}{0,A}}$, $P^{*\frac{(3)}{0,B}}$, $C^{(3)}_{0,A_vB}P^{\beta_1}$ and $E^{(3)}_{0,A_vB}P^{\beta_2}$ are solved numerically via the second, third, and fourth branch. Value-matching and smooth-pasting between the second and third branch are presented in (A-3.2), and value-matching and smooth-pasting between the third and fourth branch are presented in (A-3.3).

$$\begin{gathered} \frac{D_A P^{*\frac{(3)}{0,A}}}{\gamma+\rho-\mu}-I_A+A^{(3)}_{A,E}P^{*\frac{(3)}{0,A}^{\beta_1}}=C^{(3)}_{0,A_vB}P^{*\frac{(3)}{0,A}^{\beta_1}}+E^{(3)}_{0,A_vB}P^{*\frac{(3)}{0,A}^{\beta_2}} \\ \frac{D_A P^{*\frac{(3)}{0,A}}}{\gamma+\rho-\mu}+\beta_1 A^{(3)}_{A,E}P^{*\frac{(3)}{0,A}^{\beta_1}}=\beta_1 C^{(3)}_{0,A_vB}P^{*\frac{(3)}{0,A}^{\beta_1}}+\beta_2 E^{(3)}_{0,A_vB}P^{*\frac{(3)}{0,A}^{\beta_2}} \end{gathered} \tag{A-3.2}$$

$$\begin{gathered} C^{(3)}_{0,A_vB}P^{*\frac{(3)}{0,B}^{\beta_1}}+E^{(3)}_{0,A_vB}P^{*\frac{(3)}{0,B}^{\beta_2}}=\frac{D_B P^{*\frac{(3)}{0,B}}}{\gamma+\rho-\mu}-I_B \\ \beta_1 C^{(3)}_{0,A_vB}P^{*\frac{(3)}{0,B}^{\beta_1}}+\beta_2 E^{(3)}_{0,A_vB}P^{*\frac{(3)}{0,B}^{\beta_2}}=\frac{D_B P^{*\frac{(3)}{0,B}}}{\gamma+\rho-\mu} \end{gathered} \tag{A-3.3}$$

3.2 Project A versus Project B when the expansion is uncertain: Scenario 3, $\overline{\mathbf{F}}^{(\mathbf{3})}_{\mathbf{A_vB}}(\mathbf{P})$

Modelling the uncertainty effect on project A: Scenario 3, $\overline{V}^{(3)}_{A,E}(P)$

$$\overline{V}^{(3)}_{A,E}(P) = (D_A P dt - \rho I_A dt) + (1-\rho dt)\lambda dt \varepsilon\left[F^{(3)}_{A,E}(P+dP)\right] + (1-\rho dt)(1-\lambda dt)\varepsilon\left[\overline{V}^{(3)}_{A,E}(P+dP)\right] \tag{A-3.4}$$

Expanding the stochastic part on the right hand side of (A-3.4) using Itôs Lemma, the equation evolves into (A-3.5)

$$\begin{aligned}\overline{V}^{(3)}_{A,E}(P) &= D_A P - \rho I_A \\ &+(1-(\rho+\lambda)dt)\left[\overline{V}^{(3)}_{A,E}(P)dt + \frac{1}{2}\sigma^2 P^2 \overline{V}^{(3)''}_{A,E}(P)dt + \mu P \overline{V}^{(3)'}_{A,E}(P)dt\right] \\ &+(1-\rho dt)\lambda dt\left[F^{(3)}_{A,E}(P)dt + \frac{1}{2}\sigma^2 P^2 F^{(3)''}_{A,E}(P)dt + \mu P F^{(3)''}_{A,E}(P)dt\right]\end{aligned} \tag{A-3.5}$$

Reorganizing and simplifying (A-3.5), the differential equation that describes $\overline{V}^{(3)}_{A,E}(P)$ is presented in (A-3.6)

$$\frac{1}{2}\sigma^2 P^2 \overline{V}^{(3)''}_{A,E}(P) + \mu P \overline{V}^{(3)'}_{A,E}(P) - (\rho+\lambda)\overline{V}^{(3)}_{A,E}(P) + D_A P - \rho I_A + \lambda F^{(3)}_{A,E}(P) = 0 \tag{A-3.6}$$

As $F^{(3)}_{A,E}(P)$ has two solutions, when $P < P^{*(3)}_{A,E}$ and $P^{*(3)}_{A,E} \leq P$, the solution to $\overline{V}^{(3)}_{A,E}(P)$ can be written as:

$$\begin{cases}\frac{1}{2}\sigma^2 P^2 \overline{V}^{(3)''}_{A,E}(P) + \mu P \overline{V}^{(3)'}_{A,E}(P) - (\rho+\lambda)\overline{V}^{(3)}_{A,E}(P) \\ \qquad + \dfrac{D_A P}{\gamma+\rho-\mu} - I_A + \lambda A^{(3)}_{A,E} P^{\beta_1} = 0 & , P < P^{*(3)}_{A,E} \\ \frac{1}{2}\sigma^2 P^2 \overline{V}^{(3)''}_{A,E}(P) + \mu P \overline{V}^{(3)'}_{A,E}(P) - (\rho+\lambda)\overline{V}^{(3)}_{A,E}(P) \\ \qquad + D_A P - I_A + \dfrac{\lambda D_E}{\gamma+\rho-\mu} - \lambda I_E + \lambda A^{(3)}_{E,B} P^{\beta_1} = 0 & , P^{*(3)}_{A,E} \leq P\end{cases} \tag{A-3.7}$$

Using the framework presented in (A-1.4) – (A-1.12), the solution for $\overline{V}^{(3)}_{A,E}(P)$ is given in (A-3.8). Notice, when following the same reasoning as in (A-1.17), the terms containing negative exponents, β_2 and δ_2 can be rule out in the top part of (A-3.7) and the term containing the positive exponent, δ_1 in the bottom part of (A-3.7) can be ruled out.

$$\overline{V}_{A,E}^{(3)}(P) = \begin{cases} V_{A,0}^{(1)}(P) + A_{A,E}^{(3)}P^{\beta_1} + \underline{A}_{A,E}^{(3)}P^{\delta_1} & ,P < P^{*(3)}_{A,E} \\ \dfrac{P[\lambda D_E + (\gamma+\rho-\mu)D_A]}{(\rho+\lambda-\mu)(\gamma+\rho-\mu)} & \\ \quad -\dfrac{\lambda I_E}{\rho+\lambda} - I_A + A_{E,B}^{(3)}P^{\beta_1} + \underline{B}_{A,E}^{(3)}P^{\delta_2} & ,P^{*(3)}_{A,E} \le P \end{cases} \tag{A-3.8}$$

Investment decision: Scenario 3, $\overline{F}_{A_vB}^{(3)}(P)$

$$\overline{F}_{A_vB}^{(3)}(P) = \begin{cases} \overline{A}_{0,A}^{(3)}P^{\beta_1} & ,P < \overline{P}^{*(3)}_{0,A} \\ \overline{V}_{A,E}^{(3)}(P) & ,\overline{P}^{*(3)}_{0,A} \le P \le \overline{P}^{*(3)}_{\underline{0,A}} \\ \overline{C}_{0,A_vB}^{(3)}P^{\beta_1} + \overline{E}_{0,A_vB}^{(3)}P^{\beta_2} & ,\overline{P}^{*(3)}_{\underline{0,A}} \le P \le \overline{P}^{*(3)}_{\underline{0,B}} \\ V_{B,0}^{(1)}(P) & ,\overline{P}^{*(3)}_{\underline{0,B}} \le P \end{cases} \tag{A-3.9}$$

In Equation (A-3.9), $\overline{A}_{0,A}^{(3)}P^{\beta_1}$ and $\overline{P}^{*(3)}_{0,A}$, are solved via value-matching and smooth-pasting between the first and second branch, while $\overline{P}^{*(3)}_{\underline{0,A}}$, $\overline{P}^{*(3)}_{\underline{0,B}}$, $\overline{C}_{A_vB}^{(3)}P^{\beta_1}$ and $\overline{E}_{A_vB}^{(3)}P^{\beta_2}$ are solved numerically via the second, third, and fourth branch. Value-matching and smooth-pasting between the second and third branch are presented in (A-3.10), and value-matching and smooth-pasting between the third and fourth branch are presented in (A-3.11).

$$\begin{aligned} \frac{D_A\overline{P}^{*(3)}_{\underline{0,A}}}{\gamma+\rho-\mu} - I_A + A_{A,E}^{(3)}\overline{P}^{*(3)\,\beta_1}_{\underline{0,A}} + \underline{A}_{A,E}^{(3)}\overline{P}^{*(3)\,\delta_1}_{\underline{0,A}} &= \overline{C}_{A_vB}^{(3)}\overline{P}^{*(3)\,\beta_1}_{\underline{0,A}} + \overline{E}_{A_vB}^{(3)}\overline{P}^{*(3)\,\beta_2}_{\underline{0,A}} \\ \frac{D_A\overline{P}^{*(3)}_{\underline{0,A}}}{\gamma+\rho-\mu} + \beta_1 A_{A,E}^{(3)}\overline{P}^{*(3)\,\beta_1}_{\underline{0,A}} + \delta_1\underline{A}_{A,E}^{(3)}\overline{P}^{*(3)\,\delta_1}_{\underline{0,A}} &= \beta_1\overline{C}_{A_vB}^{(3)}\overline{P}^{*(3)\,\beta_1}_{\underline{0,A}} + \beta_2\overline{E}_{A_vB}^{(3)}\overline{P}^{*(3)\,\beta_2}_{\underline{0,A}} \end{aligned} \tag{A-3.10}$$

$$\begin{aligned} \overline{C}_{A_vB}^{(3)}\overline{P}^{*(3)\,\beta_1}_{\underline{0,B}} + \overline{E}_{A_vB}^{(3)}\overline{P}^{*(3)\,\beta_2}_{\underline{0,B}} &= \frac{D_B\overline{P}^{*(3)}_{\underline{0,B}}}{\gamma+\rho-\mu} - I_B \\ \beta_1\overline{C}_{A_vB}^{(3)}\overline{P}^{*(3)\,\beta_1}_{\underline{0,B}} + \beta_2\overline{E}_{A_vB}^{(3)}\overline{P}^{*(3)\,\beta_2}_{\underline{0,B}} &= \frac{D_B\overline{P}^{*(3)}_{\underline{0,B}}}{\gamma+\rho-\mu} \end{aligned} \tag{A-3.11}$$

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