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## Discussion paper

# The equity premium in a production economy; A new perspective involving recursive utility

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# The equity premium in a production economy; A new perspective involving recursive utility

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## Abstract

We study a rational expectations' competitive equilibrium in a production economy, i.e., a system of prices at which firms' profit maximizing production decisions and individuals' preferred affordable consumption choices equate supply and demand in every market. We derive the equilibrium price of the firm and the equilibrium short term interest rate, the optimal per capita consumption in society, as well as the risk premium on equity. First a simple linear production technology with constant coefficients is studied, then a more general technology, and finally a general production economy with recursive utility is analyzed by the use of the stochastic maximum principle. While the two first models can not explain the empirics well using conventional preferences, the latter model is found to be much more promising in this regard. We also demonstrate a simple proof for the ICAPM.

*KEYWORDS: Equity risk premium, production economy, recursive utility, CAPM, CCAPM, ICAPM*

JEL-Code: G10, G12, D9, D51, D53, D90, E21

## 1 Introduction

The paper analyzes risk premiums and the interest rate in a production economy. As is well-known, rational expectations, a cornerstone of modern economics and finance, has been under attack for quite some time. Authors ask: Are prices too volatile relative to the information arriving in the market?

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Is the mean premium on equities over the risk-less rate too large? Is the real interest rate too low? Is the market's risk aversion too high?

Mehra and Prescott (1985) gave rise to these questions in their well-known paper, using a variation of Lucas's (1978) pure exchange economy with a Kydland and Prescott (1982) "calibration" exercise. They chose the parameters of the endowment process to match the sample mean, variance and the annual growth rate of per capita consumption in the years 1889-1978. The puzzle is that they were unable to find a plausible parameter pair of the utility discount rate and the relative risk aversion to match the sample mean of the annual real rate of interest and of the equity premium over the 90-year period.

The puzzle has been verified by many others, e.g., Hansen and Singleton (1983), Ferson (1983), Grossman, Melino, and Shiller (1987). Many theories have been suggested during the years to explain the puzzle, but to date there does not seem to be any consensus that the puzzles have been fully resolved by any single of the proposed explanations <sup>1</sup>.

We utilize a continuous-time setting, to take the full advantage of the analytic power of infinite dimensional analysis. A survey of work in the intersection between macroeconomics and finance in a discrete time setting is given in Cochrane (2005). Our model describes a production economy, where firms produce a single perishable consumption good, which can be used for consumption as well as for investment in production technologies. Prices are derived at which firms' profit maximizing production decisions and individuals' preferred affordable consumption choices equate supply and demand.

The firms' optimal production decisions are taken as given by the consumers, who observe what the firms' shares sell for. Actual dividends paid to the shareholders are irrelevant, as the firms' investment decisions are now fixed, in accordance with the Miller and Modigliani (1961) result. By national accounting, in equilibrium the representative agent holds one share of the firms, and consumes the aggregate output from the firms.

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<sup>1</sup>Constantinides (1990) introduced habit persistence in the preferences of the agents. Also Campbell and Cochrane (1999) and Haug (2001) used habit formation. Rietz (1988) introduced financial catastrophes, Barro (2005) developed this further, Aase (1993a-b) extended the standard model to allow for semimartingales containing jumps, Weil (1992) introduced non-diversifiable background risk, Heaton and Lucas (1996) introduce transaction costs, and Jouini and Napp (2006) consider pessimism and doubt with heterogeneous beliefs. There is a rather long list of other approaches aimed to solve the puzzles, among them are borrowing constraints (Constantinides et al. (2001)), taxes (Mc Grattan and Prescott (2003)), loss aversion (Benartzi and Thaler (1995)), survivorship bias (Brown, Goetzmann and Ross (1995)), and heavy tails and parameter uncertainty (Weitzmann (2007)).

The original goal of this paper was to shed some light on the asset pricing problems by including production. For this purpose we found it useful to start our approach with a neoclassical growth model in a continuous-time setting, along the lines of Cox, Ingersoll and Ross (1985a), and Duffie (2001).

The two first models considered use the conventional Eu-preferences by adding expected utility period for period, and discounting utility. We explore the possibilities of this model in explaining empirical facts. A two-factor model is derived for risk premiums, which seems promising. However, we proceed to formally prove that the consumption based capital asset pricing model holds also in the production model, which more or less settles this issue. The conventional model does not explain well empirical facts.

Expanding the set of technologies in a pure exchange economy to admit capital accumulation as in Brock (1979) or Donaldson and Mehra (1984) does not increase the set of joint equilibrium processes on consumption and asset prices. This is argued in Mehra and Prescott (2007)), and is consistent with our result. Therefore the resolution the asset pricing puzzles seems unrealistic in the conventional model.

As a consequence we turn to stochastic differential utility in continuous time. Here we analyze the resulting model using the stochastic maximum principle. This model gives a more convincing explanation of the data than the conventional model.

The version of recursive utility that we consider dates back to Epstein and Zin (1989-91), who developed a framework for generalized expected utility, which allows for the separation of risk aversion from the intertemporal elasticity of substitution in consumption. Weil (1989) claimed that recursive utility does not solve the puzzle. While he obtained a risk premium of the same order as the conventional model, his risk-free rate was around 20 – 25 per cent, which was much even higher than what Mehra and Prescott obtained. He termed this "the risk-free rate puzzle".

In Aase (2013) it is shown that by employing the market portfolio as a proxy for the wealth portfolio, the agent becomes rather impatient ( $\delta$  around 10 per cent). With a lower growth rate on the wealth portfolio the model fits the data rather convincingly. By attempting to fit the recursive model with a low value for the impatience rate and reasonable values for the other parameters, in a situation where the market portfolio is used as a proxy for wealth, this can give large values for the risk free interest rate, which explains the results of Weil (1989).

Recursive utility use the foundational work by Kreps and Porteus (1978) and Chew and Epstein (1991) of utility adapted to a dynamic context. A fundamental problem with the conventional model is that in a temporal context derived utility does not satisfy the substitution axiom, in which case

additivity in probability of utility is lost (e.g., Mossin (1969)). Then it does not help to add up expected utility across time.

In a continuous-time setting, Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994) elaborate the foundational work by Kreps and Porteus (1978) in dynamic models. Duffie and Epstein (1992a), is the continuous-time analogue of the model by Epstein and Zin (1989-91).

Aase (2014a-b) develop this model further and computes both risk premiums and the equilibrium interest rate in a pure exchange economy, using the stochastic maximum principle, and also include jumps in the continuous-time model. These models are calibrated to the data of Mehra and Prescott (1985), and provide reasonable values for the parameters of the utility function. In addition this model framework is likely to give many other insights that are difficult, or impossible, to obtain using the conventional model.

The paper is organized as follows: A neoclassical growth model is introduced in Section 2, and reinterpreted as a production economy. Equilibrium in this latter economy is defined, and established in Section 3. Section 4 calibrates the equilibrium to the historical data, and makes the connection to the standard exchange economy. Section 5 attempts a generalization, and a discussion appears in Section 6. Section 7 introduces stochastic differential utility in the production setting, and Section 8 gives an alternative application of the stochastic maximum principle. Here we give a short derivation of the ICAPM of Merton (1973). Section 9 concludes.

## 2 The first model

### 2.1 A growth model

As a motivation, consider first the following variant of the neoclassical growth model under certainty. Below we shall extend this model to include uncertainty, which is the situation we are interested in. An economy is developing over time in which  $K = K_t$  denotes the capital stock,  $c = c_t$  consumption and  $Z = Z_t$  net national product at time  $t$ , and where  $Z = \mu(K)$  denotes the production function. For each  $t$  we have the national accounting identity

$$\frac{dK_t}{dt} = \mu(K_t) - c_t$$

which means that production,  $\mu(K_t)$ , is divided between consumption,  $c_t$ , and investment,  $dK_t/dt$ . The problem is to find the optimal investment, or equivalently, the optimal consumption, that solves

$$\sup_{c \in \mathcal{C}} U(c) \tag{1}$$

where  $\mathcal{C}$  is the choice set, and  $U$  the central planner's utility function.

Uncertainty is introduced next via a probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ , where  $\Omega$  is the set of states,  $\mathcal{F}$  is the set of events on which the probability measure  $P$  is defined, and  $\mathcal{F}_t$  is the set of possible events that may occur by time  $t$ , often referred to as the "information available" at time  $t$ . On this probability space is defined a standard Brownian motion  $B$ , that is assumed to generate the information filtration  $\mathcal{F}_t$ . The dynamics of capital stock process  $K$  is assumed to follow a process of the form

$$dK_t = (\mu(K_t, l_t) - c_t)dt + \sigma(K_t, l_t)dB_t; \quad K_0 > 0,$$

where  $l$  is a vector of state variables, satisfying its own dynamic equation. In a Solow variant (no uncertainty)  $l$  is labor, and  $\mu$  is a Cobb-Douglas type function. Cox, Ingersoll and Ross (1985b) specified  $l$  to be a mean reverting diffusion process, a square root process, to capture cycles in the equilibrium interest rate. In this case  $\mu(K_t, l_t) = \mu_K K_t l_t$  and  $\sigma(K_t, l_t) = \sigma_K K_t l_t$ . As we are not primarily concerned with these issues in the following, we choose a linear production technology, and set  $l \equiv 1$ . Our model for the capital stock is

$$dK_t = (\mu_K K_t - c_t)dt + \sigma_K K_t dB_t; \quad K_0 > 0, \quad (2)$$

where  $\mu_K$  and  $\sigma_K$  are strictly positive scalars.<sup>2</sup>

The objective is to maximize utility subject to the dynamic constraints (2) when the utility function  $U$  is time additive expected utility. The felicity index is separable with a constant coefficient of relative risk aversion  $\gamma > 0$ ,  $\gamma \neq 1$ , and an impatience rate  $\delta \geq 0$ , i.e.,  $u(c, t) = \frac{1}{1-\gamma} c^{1-\gamma} e^{-\delta t}$ . With an infinite time horizon, the objective (1) can be written

$$\sup_{c \in \mathcal{C}} E \left[ \int_0^\infty u(c_t, t) dt \right]. \quad (3)$$

The first order conditions for this problem is given by the Bellman equation, which takes the form ( $x = K_t$ )

$$\sup_{c \in R_+} \left( \mathcal{D}^c J(x) - \delta J(x) + \frac{c^{1-\gamma}}{1-\gamma} \right) = 0 \quad (4)$$

for all  $x > 0$  where  $J(\cdot)$  is the indirect utility function and

$$\mathcal{D}^c J(x) = J_x(x)(\mu_K x - c) + \frac{1}{2} J_{xx}(x) \sigma_K^2 x^2.$$

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<sup>2</sup>The model (2) could, perhaps, be considered as an extension of Domar's growth model to include uncertainty.

The solution is given by

$$c(t) = \theta K(t) \quad \text{for all } t, \quad (5)$$

where the constant  $\theta$  is

$$\theta = \frac{1-\gamma}{\gamma} \left( \frac{\gamma}{2} \sigma_K^2 + \frac{\delta}{1-\gamma} - \mu_K \right). \quad (6)$$

The detailed derivations are carried out in Appendix 1. For  $\theta > 0$  the necessary transversality condition

$$\lim_{T \rightarrow \infty} E\{e^{-\delta T} |J(K_T^c)|\} = 0 \quad (7)$$

is satisfied for all initial values of the capital stock  $K_0 > 0$  and for all admissible  $c \in \mathcal{C}$ , see Duffie (2001), p 213. For the parameter ranges of interest, it can readily be verified that  $\theta \in (0, 1)$ . Accordingly the optimal consumption is a certain fraction of the capital stock. Notice that

$$\text{var}(c_t) = \theta^2 \text{var}(K_t^c) < \text{var}(K_t^c) \quad (8)$$

for all  $t$ , so the variance of the consumption at each time  $t$  is smaller than the variance of the capital stock at  $t$ . The implication of (5) is that the capital stock  $K^c(t)$  is lognormally distributed along the optimal consumption path, with dynamics

$$dK^c(t) = K^c(t)(\mu_K - \theta)dt + K^c(t)\sigma_K dB(t), \quad (9)$$

The conditional expected investment rate  $E_t("dK^c(t)/dt") = K^c(t)(\mu_K - \theta)$  for all  $t$ , where  $E_t$  signifies conditional expectation given the information set  $\mathcal{F}_t$  at time  $t$ .

A direct consequence of (5) is that the volatility of the consumption growth rate  $\sigma_c = \sigma_K$ , a fact we return to in Section 4.

Because consumption goods and capital are interchangeable, the production technology may be interpreted in the context of either the model by Cox, Ingersoll and Ross (1985) (real investment opportunities) or the models by Hayashi (1982) and Abel and Eberly (1994) (profit maximizing representative firm). In the latter investment of the firm is  $I_t = K_t - c_t$ , and capital accumulation is given by

$$dK_t = (I_t - (1 - \mu_K)K_t)dt + \sigma_K K_t dB_t.$$

Hence, the model is equivalent to an adaptation of the model in Hayashi (1982) and Abel and Eberly (1994) in the sense that capital depreciation  $(1 - \mu_K)K_t dt - \sigma_K K_t dB_t$  is risky.

## 2.2 The production/exchange economy

We now reinterpret the description in Section 2.1 as a single firm that depletes its capital stock  $K_t$  at rate  $\delta_t \in \mathcal{Y}$ , where  $\mathcal{Y}$  is the production set, and that maximizes its share price  $S_t$ . The economy is populated with one agent having preferences specified by (1) and (3), and endowment one share of the firm.

Thus  $\delta$  is the optimal real output of the firm controlling the capital stock production process and maximizing its share price, provided  $\delta_t = c_t$  for all  $t$ , where  $c_t$  is given in (5).

The consumer ignores what the firm is trying to do and merely observes that the firm's common share sells for  $S_t$  and each share pays the dividend process  $\delta$  that the firm determines. The consumer is free to purchase any number of these shares, or to short-sell them, and can also borrow or lend at a short-rate process  $r$ . The price process of the riskfree asset is  $\theta_t$ , satisfying  $d\theta_t = r_t\theta_t dt$ . These are the only two securities available. The riskfree asset is supposed to be in zero net supply.

Let  $W_t$  be the consumer's wealth at time  $t$ , and  $n_t = (n_t^S, n_t^\theta)$  the number of stocks held in the risky asset and the riskfree asset, respectively, at time  $t$ . The agent's optimal consumption and investment strategy  $(c_t, n_t)$  satisfies

$$\sup_{(c,n) \in \mathcal{A}} E \left( \int_0^\infty \frac{1}{1-\gamma} c_t^{1-\gamma} e^{-\delta t} dt \right)$$

where the set  $\mathcal{A}$  signifies the set of permissible consumption processes  $c$  and trading strategies  $n$  that finances  $c$ . We use the following notation for the valuation functional:  $\Pi(c)$  is the value of the consumption stream  $c \in \mathcal{C}$ , where  $\Pi(\cdot)$  is defined by

$$\Pi(c) = \frac{1}{\pi_0} E \left\{ \int_0^\infty \pi_t c_t dt \right\}.$$

The state prices  $\pi$  *strictly supports* the allocation  $(c, \delta)$  provided

$$U(\tilde{c}) > U(c) \Rightarrow \Pi(\tilde{c}) > \Pi(c) \tag{10}$$

for all  $\tilde{c} \in \mathcal{C}$ , and

$$\Pi(\delta) \geq \Pi(\tilde{\delta}) \tag{11}$$

for all  $\tilde{\delta} \in \mathcal{Y}$ . Here the consumption choice set  $\mathcal{C}$  is equal to the production set  $\mathcal{Y}$ .

Also  $(c, \delta)$  is *budget constrained* by  $\pi$  if

$$\Pi(c) \leq \Pi(n^S \delta + n^\theta r). \tag{12}$$



Here (10) and (12) are the optimality conditions for the agent, given the state prices  $\pi$ . Condition (11) is market value maximization by the firm, given  $\pi$ . Because of strict monotonicity of the utility function, the budget constraint (12) holds with equality.

### 3 Equilibrium

Consider the economy  $\mathcal{E} = [(S, \theta), \pi, \delta, r, (c, n)]$ . A triple  $(c, \delta, \pi)$  is an *equilibrium* for  $\mathcal{E}$  provided  $(c, \delta)$  is a feasible allocation that is budget constrained and strictly supported by  $\pi$ .

In a representative agent economy this means that the optimal consumption  $c_t = \delta_t$  for all  $t \geq 0$ , and that the optimal strategy for the agent is to hold one share of the firm and no shares of the riskfree security for each  $t \geq 0$ .

In order to find an equilibrium for this economy, we start with the state price, which is given by the marginal utility at the optimal output, or  $\pi_t = u'(\delta_t, t)$ , where  $\delta_t = \theta K_t^{(\delta)}$  for any  $t$ . The state price  $\pi_t = e^{-\delta t}(\theta K_t^{(\delta)})^{-\gamma}$ , a geometric Brownian motion process, satisfies the dynamics

$$d\pi_t = -\pi_t(\gamma(\mu_K - \theta) + \delta - \frac{1}{2}\gamma(\gamma + 1)\sigma_K^2)dt - \gamma\pi_t\sigma_K dB_t. \quad (13)$$

This representation is instrumental in finding the equilibrium short term interest rate, as we do next.

#### 3.1 The interest rate

Our candidate for the equilibrium riskfree rate is  $r_t = -\frac{\mu_\pi(t)}{\pi_t}$ , where  $\mu_\pi(t)$  is the drift term in (13). It follows that

$$r_t = \delta + \gamma\mu_K - \frac{1}{2}\gamma(1 + \gamma)\sigma_K^2 - \gamma\theta \quad \text{for all } t, \quad (14)$$

i.e., the equilibrium interest rate is a constant. Recall the expression for the interest rate in a pure exchange economy

$$r_t^{ex} = \delta + \gamma\mu_c - \frac{1}{2}\gamma(1 + \gamma)\sigma_c^2, \quad (15)$$

where the parameter  $\mu_c$  is the conditional expected growth rate in aggregate consumption and  $\sigma_c$  is the corresponding volatility parameter. In the latter model aggregate consumption is exogenous, while in our model consumption

is endogenous. For these two models to be internally consistent, it must be the case that  $\sigma_c = \sigma_K$ .

We return to a comparison with the standard exchange economy in Section 4.2.

A closer examination of the expression (14) reveals that it can be written

$$r = \mu_K - \gamma\sigma_K^2, \quad (16)$$

i.e., as the marginal product of capital adjusted for uncertainty. A closer examination shows that  $r^{ex} = r$  (see Section 4.2).

If the conditional expected growth rate of the capital stock increases, the equilibrium interest rate will increase, which is the income effect. Faced with better prospects for the future, our consumer would like to consume more now, and hence borrow. Since this is impossible, the interest rate must increase to make the agent just indifferent to status quo.

(Equation (16) says that the interest rate equals the expected stock return minus the equity premium, as will become clear from the next section.)

### 3.2 The price of the firm's stock

We now turn to the candidate for the price process for the firm's shares. Given a dividend stream  $\delta_t$  from the firm and state prices  $\pi_t$ , the price  $S$  at time  $t$  equals

$$S_t = \frac{1}{\pi_t} E_t \left( \int_t^\infty \pi_s \delta_s ds \right). \quad (17)$$

By carrying out this computation, first we obtain by Fubini's theorem that

$$S_t = \theta K_t^{(\delta)} \int_t^\infty e^{-\delta(s-t)} E_t \left\{ \exp \left( (1-\gamma)(\mu_K - \theta - \frac{1}{2}\sigma_K^2)(s-t) + (1-\gamma)\sigma_K(B_s - B_t) \right) \right\} ds.$$

Next, by the moment generating function of the normal distribution we get

$$S_t = \theta K_t^{(\delta)} \int_t^\infty e^{[(1-\gamma)(\mu_K - \theta) - \frac{1}{2}\gamma(1-\gamma)\sigma_K^2 - \delta](s-t)} ds = \frac{\theta}{\alpha} K_t^{(\delta)},$$

where

$$\alpha = -[(1-\gamma)(\mu_K - \theta) - \frac{1}{2}\gamma(1-\gamma)\sigma_K^2 - \delta].$$

Finally it can be verified that  $\alpha = \theta$ , so the spot price is  $S_t = K_t^{(\delta)}$  for all  $t$ . As we have shown that  $K_t^{(\delta)}$  is lognormal when  $\delta = c$ , and  $c$  is given by (5), it follows that our candidate price process  $S_t$  is a geometric Brownian

motion process, where the conditional expected return on the capital gains are  $(\mu_S - \theta) = (\mu_K - \theta)$ , and the associated volatility  $\sigma_S = \sigma_K$ . This means, for example, that the securities market model is dynamically complete.

In the production model of Section 2 there are no adjustment costs, in which case it is known that Tobin's marginal Q is constant and equal to 1. This is consistent with  $S_t = K_t$ .

Recall when there are dividends, we adjust the price process for dividends and obtain the gains process  $G_t$ , sometimes called the adjusted price process, defined by

$$G_t = S_t + \int_0^t \delta_s ds \quad (18)$$

Using the above results the gains process is

$$dG_t = (\mu_K - \theta)S_t dt + \delta_t dt + \sigma_S S_t dB_t,$$

or, since  $\delta_t = \theta S_t$  we obtain

$$dG_t = \mu_S S_t dt + \sigma_S S_t dB_t. \quad (19)$$

The cumulative-return process  $R_t$  for this security is defined by  $dG_t = S_t dR_t$ , so that

$$dR_t = \mu_S dt + \sigma_S dB_t. \quad (20)$$

The process  $R_t$  takes into account both the capital gains and the dividends over the small time interval  $(t, t + dt]$ . This expression shows that  $R$  is a Brownian motion with drift. Because of this relation, we sometimes write  $\mu_R$  instead of  $\mu_S$ , and similarly  $\sigma_R$  instead of  $\sigma_S$ .

### 3.3 The optimal consumption and portfolio problem

Having a candidate for the price process of the firm's stock, we can now reformulate the consumer's optimal consumption and portfolio choice problem. The problem is to solve

$$\sup_{(c, \varphi)} E \left( \int_0^\infty \frac{1}{1 - \gamma} c_t^{1-\gamma} e^{-\delta t} dt \right)$$

subject to the dynamic budget constraint

$$dW_t = \left( W_t(\varphi_t(\mu_S - r_t) + r_t) - c_t \right) dt + W_t \varphi_t \sigma_S dB_t, \quad W_0 = S_0, \quad (21)$$

where  $W_t$  is the agent's wealth at time  $t$ , and  $\varphi_t = \frac{n_t^S G_t}{W_t}$  is the fraction of wealth held in the risky asset at time  $t$ .

In formulating the budget constraint (21) we have made use of the dynamics of the price process  $G_t$  that adjusts for dividends. This problem is now well suited for dynamic programming, and the Bellman equation is

$$\sup_{c, \varphi} \left\{ \mathcal{D}^{c, \varphi} J(w) - \delta J(w) + \frac{c^{(1-\gamma)}}{1-\gamma} \right\} = 0, \quad w > 0, \quad (22)$$

where ( $w = W_t$ )

$$\mathcal{D}^{c, \varphi} J(w) = J_w(w)(\varphi(\mu_S - r)w + rw - c) + \frac{1}{2}w^2\varphi^2\sigma_S^2 J_{ww}(w).$$

The first order condition in  $\varphi$  is

$$J_w(w)(\mu_S - r)w + w^2\varphi\sigma_S^2 J_{ww}(w) = 0 \quad \text{for all } w > 0,$$

which gives in terms of the dynamics of  $G$  that

$$\varphi_t = \left( - \frac{J_w(W_t)}{J_{ww}(W_t)W_t} \right) \frac{\mu_S - r}{\sigma_S^2}, \quad (23)$$

Here  $\varphi$  is proportional to the relative risk tolerance of the agent's indirect utility, increases with the risk premium ( $\mu_S - r$ ), and decreases as the volatility parameter  $\sigma_S$  increases, *ceteris paribus*.

Next we find the first order condition for optimization in the consumption variable  $c$ . From the Bellman equation it is seen to be

$$-J_w(w) + c^{-\gamma} = 0,$$

which implies that

$$c = (J_w(w))^{-\frac{1}{\gamma}}, \quad \text{or} \quad c_t = (J_w(W_t))^{-\frac{1}{\gamma}}$$

in terms of the random wealth process  $W_t$ . Notice how the consumption choice problem is separated from the investment problem. In Appendix 2 it is shown that the solution is

$$c_t = \eta W_t \quad (24)$$

where the constant  $\eta$  is

$$\eta = \left[ \frac{1-\gamma}{\gamma} \left( \frac{\delta}{1-\gamma} - r - \frac{1}{2} \frac{1}{\gamma} \frac{(\mu_S - r)^2}{\sigma_S^2} \right) \right]. \quad (25)$$

The agent optimally consumes a constant proportion of current wealth.

Returning to the optimal investment policy, it is seen to be

$$\varphi_t = \frac{1}{\gamma} \frac{\mu_S - r}{\sigma_S^2}, \quad (26)$$

i.e., the relative risk tolerance of the indirect utility function is the same as the relative risk tolerance of the felicity index. The optimal investment ratio is of the same form as the classical solution in the no-dividend case, known to follow when the price process is lognormal (e.g., Mossin (1968), Samuleson (1969), Merton (1971)<sup>3</sup>). The result is the same as when the price  $S$  is cum-dividends.

Since we have only one consumer in our model, he is interpreted as the representative agent in the context of equilibrium. For the above to be an equilibrium, it must be the case that the price  $S_t$  of the firm and the interest rate  $r_t$  are both set at each time  $t$  such that the agent's fraction of wealth in the risky asset is always equal to 1, or  $\varphi_t = 1$  for all  $t$ . From (26) it follows that in equilibrium it must be the case that

$$\mu_S - r = \gamma\sigma_S^2. \quad (27)$$

The above investment strategy is only feasible if the dividends  $\delta$  from the firm equals the optimal consumption  $c$  derived in (24). This is indeed the case: By comparing  $c$  to the optimal consumption in (5), derived in the centralized economy of Section 2.1, we can show that equating these two expressions is equivalent to the equilibrium relation (27). In other words

$$\delta_t = \theta K_t = \eta W_t = c_t \text{ for all } t \Leftrightarrow \mu_S - r = \gamma\sigma_S^2. \quad (28)$$

Thus taking the output from the single firm  $\delta_t$  to be equal to the optimal consumption  $c_t$  in the centralized economy of Section 2.1, we have shown that this is also equal to the optimal consumption of the representative agent, denoted  $c_t$  as well, in the decentralized economy, provided that (27) holds.

Returning to (27) and recalling that our candidate for the riskfree rate is

$$r = \gamma(\mu_K - \theta) + \delta - \frac{1}{2}\gamma(1 + \gamma)\sigma_K^2,$$

it follows that

$$\mu_S = \gamma(\mu_K + \sigma_S^2 - \theta) + \delta - \frac{1}{2}\gamma(1 + \gamma)\sigma_K^2$$

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<sup>3</sup>Notice that these references do not deal with equilibrium; the prices of the risky assets are given exogenously.

Inserting for  $\theta$  from (6) we obtain that  $\mu_S = \mu_K$ , which is consistent with our earlier conjecture for the stock price.

Notice that the wealth of the representative agent can always be found by a prospective point of view as

$$W_t = \frac{1}{\pi_t} E_t \left( \int_t^\infty \pi_s \delta_s ds \right),$$

which by (82) means that  $W_t = S_t$  for all  $t$ .

What remains to be verified for an equilibrium to be satisfied is profit maximization at the state prices  $\pi_t$ . To this end, recall that the securities market is dynamically complete. This means that the dynamic optimization problem in Section 2.1 is equivalent to the following "static" problem

$$\sup_{\tilde{\delta}} U(\tilde{\delta}) \quad \text{subject to} \quad \Pi(\tilde{\delta}) \leq w,$$

where  $w = S_0 \cdot 1 = K_0$ , and  $\Pi(\tilde{\delta}) = \frac{1}{\pi_0} E \{ \int_0^\infty \tilde{\delta}_t \pi_t dt \}$ . Since we have shown that the solution  $\delta$  to this problem satisfies  $\Pi(\delta) = S_0$ , the problem can be written

$$\sup_{\tilde{\delta}} U(\tilde{\delta}) \quad \text{subject to} \quad \Pi(\tilde{\delta}) \leq \Pi(\delta),$$

or,

$$U(\tilde{\delta}) \leq U(\delta) \Leftrightarrow \Pi(\tilde{\delta}) \leq \Pi(\delta) \quad \text{for any } \tilde{\delta} \in \mathcal{Y},$$

which shows that the requirement (11) holds, i.e., the optimal output  $\delta$  from the firm maximizes profits at prices  $\pi$ .

## 4 Comparisons and calibrations

In this section we first relate our results to the corresponding results of the pure exchange economy, that is most commonly employed in the present setting. Then we calibrate our model to the data used by Mehra and Prescott (1985), and as expected, we recover the equity premium puzzle. In doing so, we interpret our firm as the US production economy, and the risk premium of the risky asset as the equity premium.

### 4.1 The connection to the CCAPM

One result of our analysis is that the optimal consumption  $c_t = \theta K_t$ , which means that the optimal consumption has the dynamics

$$dc_t = (\mu_K - \theta)c_t dt + \sigma_K c_t dB_t, \tag{29}$$

or the growth rate in consumption can be expressed as follows

$$\frac{dc_t}{c_t} = (\mu_K - \theta)dt + \sigma_K dB_t. \quad (30)$$

We define the growth rate of the per capita real consumption by  $C$ , or  $dc_t = c_t dC_t$ , so that

$$dC_t = \mu_C dt + \sigma_C dB_t,$$

where  $\mu_C = \mu_c$ ,  $\sigma_C = \sigma_c$ . Recall the corresponding expression for the cumulative-return process  $R_t$  of the firm in (20). Using this, the consumption based CAPM has the following form

$$\mu_R - r = \gamma \sigma_{R,C} \quad (31)$$

in the pure exchange economy - where  $\sigma_R = \sigma_S$ . Here  $\sigma_{R,C}$  is the covariance rate between  $R$  and  $C$ . From the equation for the consumption growth in (30), we see that the risk premium in (31) can be written

$$\mu_R - r = \gamma \sigma_{S,K}. \quad (32)$$

Note that this is consistent with our result (27), since  $\sigma_S = \sigma_K$  because  $S_t = K_t^{(\delta)}$ , diffusion invariance, so the instantaneous correlation coefficient is unity, and the equality  $\mu_R = \mu_S$ .

The linear relationship  $c_t = \theta S_t$  between consumption and equity has as a consequence that (8) holds, or  $\text{var}(c_t) = \theta^2 \text{var}(S_t) < \text{var}(S_t)$ , since  $\theta \in (0, 1)$ . Thus very different levels of variances of equity and consumption are allowed. However, as we have demonstrated, the linear relationship leads to the same percentage-wise changes in consumption and equity, so the values for parameters  $\sigma_R$  and  $\sigma_C$  are the same. As we shall see, this is not consistent with the data.

## 4.2 A numerical calibration exercise

In Table 1 we present the key summary statistics of the data in Mehra and Prescott (1985), of the real annual return data related to the S&P-500, denoted by  $M$ , as well as for the annualized consumption data, denoted  $c$ , and the government bills, denoted  $b$ <sup>4</sup>.

Since our development is in continuous time, we have carried out standard adjustments for continuous-time compounding, from discrete-time compounding. This gives, e.g., the estimate  $\hat{\kappa}_{Mc} = .4033$  for the instantaneous correlation coefficient  $\kappa(t)$ <sup>5</sup>.

<sup>4</sup>There are of course newer data by now, but these retain the same basic features. If our model can explain the data in Table 1, it can explain any of the newer sets as well.

<sup>5</sup>The full data set was provided by Professor Rajnish Mehra.

	Expectation	Standard dev.	Covariances
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{Mc} = .002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	

Table 1: Key US-data for the time period 1889-1978. Continuous-time compounding.

Using these summary data for the volatility of equity and the market portfolio  $M$ , we have an estimate of 0.1584 for the parameter  $\sigma_S$ . Taking the CCAPM (31) as the starting point, from the data of Table 1 we obtain an estimate of the relative risk aversion  $\hat{\gamma} = 26.37$ , which is considered implausible. This is the equity premium puzzle.

From the expression (32) it may appear that we can get a reasonable risk aversion in isolation when using .1584 as an estimate of  $\sigma_R$  and  $\sigma_K$ , but the model also tells us that  $\sigma_C = \sigma_R = \sigma_K$ , and using the estimate .0355 for  $\sigma_C$  the result is not plausible, in fact much worse than above. This is another way of expressing this puzzle.

Returning to the equilibrium interest rate, when aggregate consumption is taken as exogenously given, and, moreover, is lognormal as in (29), the equilibrium interest rate for our CRRA consumer is known to have the form

$$r_t = \delta + \gamma\mu_c - \frac{1}{2}\gamma(1 + \gamma)\sigma_c^2 \quad (33)$$

in the canonical model, as was remarked in (15). Since the growth rate in aggregate consumption is  $\mu_c = (\mu_K - \theta)$  and the volatility of the growth rate of consumption is  $\sigma_c = \sigma_K$ , it follows that (33) can be written

$$r_t = \delta + \gamma(\mu_K - \theta) - \frac{1}{2}\gamma(1 + \gamma)\sigma_K^2,$$

which is seen to be the the same as our expression (14) for the equilibrium short term interest rate in the production economy. Accordingly our results are consistent with those of the standard pure exchange economy. Using (33), with the above value of  $\gamma$  and the estimate of  $\sigma_c$  in Table 1, we obtain an estimate of the impatience rate  $\hat{\delta} = -.015$ .

### 4.3 Summary for the linear model

The simple linear production and exchange economy considered does not solve the puzzles, but yields some insights that will be of value in later



sections. In all the major aspects this production model is at the same level of complexity as the standard Lucas (1978)-model, so we should expect the same level of explanatory power from either of these two approaches.

The consumers' investment problems can be separated from the optimal consumption choices, and as a consequence the consumers' behavior in the financial market can be explained from financial market data alone, so long as national accounting is satisfied (the budget constraint must hold).

As in Sargent and Hansen (1999), we could imagine that investors somehow do not thrust the model (since it is so simple), and this added uncertainty leads them to require a higher compensation for risk bearing. These authors analyze this type of problem in a model of habit formation. Instead, we consider recursive utility in Section 7, where this type of construction is not needed to explain the observed equity premium.

In order to address this issue of  $\sigma_C = \sigma_R$  in the simple model, we next turn to a more general model. In particular we consider from now a  $d$ -dimensional, standard Brownian motion, where  $d > 1$ , so that covariance rates can be written as inner products, i.e.,  $\sigma_{C,R}(t) = \sum_{i=1}^d \sigma_{C,i}(t)\sigma_{R,i}(t)$  where the individual terms are not constants, but adapted stochastic (ergodic) processes satisfying standard conditions. We start with the Markovian case.

## 5 The general set up

The model we present here is in the same spirit as the one of sections 2 and 3, and will have the advantage that it overcomes the weakness of the linear model, since it allows  $\sigma_C \neq \sigma_R$ .

First, there exists one production good, which is also the consumption good. This good may be consumed or invested in two technologies. One is risk-free, the other consists of the capital stock  $K$  satisfying the dynamics

$$dK_t = (K_t\mu_K(K_t, l_t) - c_t)dt + K_t\sigma_K(K_t, l_t)dB_t, \quad (34)$$

where  $l$  is a state variable satisfying its own dynamics

$$dl_t = l_t\mu_l(l_t)dt + l_t\sigma_l(l_t)dB_t. \quad (35)$$

The term  $\mu_K$  may be nonlinear. We also allow the various drift and diffusion terms to be functions, not merely constants as in the first section. Thus we depart from the convenient log-normality universe of the first section.

If we interpret  $l$  as labor, it is clear that the utility function  $u$  must depend upon leisure, so that  $u = u(c_t, l_t)$  at time  $t$ , where the utility function

is decreasing in its second variable. At first the agent is not allowed to use the risk-free technology. The problem of the agent is then the following

$$\max_{(c,l) \in \mathcal{C}} E \left\{ \int_0^\infty u(c_t, l_t) e^{-\delta t} dt \right\} \quad (36)$$

subject to the wealth dynamics

$$dW_t = (W_t \mu_K(W_t, l_t) - c_t) dt + W_t \sigma_K(W_t, l_t) dB_t.$$

The Brownian motion may be augmented by one independent component corresponding to the factor  $l$ . It is here assumed that the agent invests everything in the production technology. The Bellman equation for this problem is

$$\sup_{c,l} \{ \mathcal{D}^c J(w, l) - \delta J(w, l) + u(c, l) \},$$

where

$$\begin{aligned} \mathcal{D}^c J(w, l) = & J_w(w, l)(\mu_K(w, l)w - c) + J_l(w, l)l\mu_l(l) + \\ & \frac{1}{2} J_{ww}(w, l)w^2 \sigma_K(w, l)\sigma_K(w, l) + \frac{1}{2} J_{l,l}(w, l)l^2 \sigma_l(l)\sigma_l(l) + \\ & J_{wl}(w, l)w \sigma_K(w, l)\sigma_l(l)l. \end{aligned} \quad (37)$$

Assuming an interior solution, the first order condition in the consumption variable  $c$  is,

$$-J_w(w, l) + u_c(c, l) = 0. \quad (38)$$

Further, assuming that the marginal utility  $u_c$  is invertible in its first variable, and that the indirect utility function  $J$  is well defined and sufficiently smooth, the optimal consumption is given by

$$c^*(t) = u_c^{-1}(J_w(W_t, l_t), l_t). \quad (39)$$

## 5.1 The equilibrium real interest rate

As in CIR (1985a), we may first introduce riskless borrowing and lending, and second a securities market. Considering the first, in equilibrium the representative agent is just indifferent to holding the riskfree asset, so the short term equilibrium interest rate  $r$  is determined from the constraint that the agent invests everything in the risky technology.

The equilibrium interest  $r$  may either be less or greater than  $\mu_K$ , the expected return on optimally invested wealth. Although investment in the production process exposes an individual to uncertainty about the output

received, it may also allow him to hedge against the risk of less favorable changes in technology. An individual investing only in locally riskless lending would be unprotected against this latter risk. This is, for example, the case with the individual in the first part of the paper, when the riskless rate is

$$r = \mu_K - \gamma\sigma_K^2,$$

which does not take into account the covariance between wealth and the capital stock. In general, either effect may dominate.

As noted in the first part, the spot rate can be determined from the state price deflator  $\pi$  as follows

$$r_t = -\mu_\pi(t)/\pi_t, \quad (40)$$

where the state price deflator is  $\pi_t = u_c(c^*(t), l_t^*)e^{-\delta t} = J_w(W_t, l_t)e^{-\delta t}$ . In terms of the dynamics for the quantity  $J_w(W_t, l_t)$ , by Ito's formula we then get the following dynamics of  $\pi$

$$d\pi_t = \mu_\pi(t)dt + e^{-\delta t}(J_{ww}(W_t, l_t)W_t\sigma_K(l_t) + J_{wl}(W_t, l_t)l_t\sigma_l(l_t))dB_t \quad (41)$$

where the drift term  $\mu_\pi$  is the following

$$\begin{aligned} \mu_\pi(t) = & -\delta\pi_t + e^{-\delta t}(J_{ww}(W_t, l_t)(W_t\mu_K(W_t, l_t) - c^*(t)) \\ & + J_{wl}(W_t, l_t)(l_t\mu_l(l_t) + \frac{1}{2}J_{www}(W_t, l_t)W_t^2\sigma_K(W_t, l_t)\sigma_K(l_t) \\ & + J_{wvl}(W_t, l_t)W_tl_t\sigma_K(W_t, l_t)\sigma_l(l_t) \\ & + \frac{1}{2}J_{wll}(W_t, l_t)l_t^2\sigma_l(l_t)\sigma_l(l_t)). \end{aligned} \quad (42)$$

From this it follows that the equilibrium short rate is

$$\begin{aligned} r_t = & \delta + \left(\frac{-J_{ww}W_t}{J_w}\right)\left(\mu_K(W_t, l_t) - \frac{u_c^{-1}(J_w, l)}{W_t}\right) + \left(\frac{-J_{wl}l_t}{J_w}\right)\mu_l(l_t) \\ & + \frac{1}{2}\left(\frac{-J_{www}W_t^2}{J_w}\sigma_K(W_t, l_t)\sigma_K(W_t, l_t)\right) + \frac{1}{2}\left(\frac{-J_{wll}l_t^2}{J_w}\sigma_l(l_t)\sigma_l(l_t)\right) \\ & + \left(\frac{-J_{wvl}W_tl_t}{J_w}\sigma_K(W_t, l_t)\sigma_l(l_t)\right), \end{aligned} \quad (43)$$

for all  $t \geq 0$ . This may be compared to equation (14) for the corresponding linear technology, which is

$$r_t = \delta + \gamma(\mu_K - \theta) - \frac{1}{2}\gamma(1 + \gamma)\sigma_K^2.$$

In the above the term  $\frac{u_c^{-1}(J_w, l)}{W_t} = \frac{c_t^*}{W_t} = \theta$  = the consumption to wealth ratio in the linear model, the next term has no counterpart in this model, the fourth term on the right hand side of (43) corresponds to last term above, while the last two terms have no counterparts in the simpler model.

## 5.2 The price of the firm's stock

Next we introduce a securities market. The setting and notation are the same as in Section 3.3. The equilibrium price process of the firm is denoted by  $S_t$  and is given by equation (82) with the state price  $\pi$  satisfying the dynamic equation (41), and the dividends  $\delta(t) = c^*(t)$ , the latter given in (39). The gains process  $G_t$ , the price process adjusted for dividends, has the representation

$$dG_t = \mu_G(S_t, l_t)dt + \sigma_G(S_t, l_t)dB_t,$$

where the wealth  $W_t$  depends on the optimal dividends and labor given in (39). Defining the cumulative-return process  $R$  of this security by  $dG_t = S_t dR_t$ , we may write

$$dR_t = \mu_R(S_t, l_t)dt + \sigma_R(S_t, l_t)dB_t,$$

where  $\mu_R(S_t, l_t) = \frac{1}{S_t}\mu_G(S_t, l_t)$  and  $\sigma_R(S_t, l_t) = \frac{1}{S_t}\sigma_G(S_t, l_t)$ , assuming  $S_t > 0$  a.s. for all  $t$ . Furthermore  $\mu_R(S_t, l_t) = \mu_K(K_t, l_t)$ ,  $\sigma_R(S_t, l_t) = \sigma_K(K_t, l_t)$  and  $S = K$ .

Finally we let the agent trade freely in the capital market consisting of the firm's shares and the riskfree asset.

## 5.3 The optimal consumption and portfolio problem

The consumer/investor is initially endowed with one share of the firm, and solves the problem

$$\sup_{c, l, \varphi} E \left\{ \int_0^\infty e^{-\delta t} u(c_t, l_t) dt \right\},$$

subject to the dynamic wealth constraint

$$dW_t = (W_t[\varphi_t(\mu_R(S_t, l_t) - r_t) + r_t] - c_t)dt + W_t\varphi_t\sigma_R(S_t, l_t)dB_t,$$

where  $W_0 = S_0$ . Here the wealth  $W_t$  depends on the optimal consumption  $c$  and labor  $l$ . The associated Bellman equation is

$$\sup_{c, l, \varphi} \left\{ \mathcal{D}^{c, \varphi} J(w, l) - \delta J(w, l) + u(c, l) \right\} = 0, \quad w > 0,$$

where

$$\begin{aligned} \mathcal{D}^{c, \varphi} J(w, l) = & J_w(w, l)(\varphi(\mu_R(w, l) - r_t)w + r_t w - c) + J_k(w, l)l\mu_l \\ & + \frac{1}{2}J_{ww}(w, l)w^2\varphi^2\sigma_R(w, l)\sigma_R(w, l) + \frac{1}{2}J_{l,l}(w, l)l^2\sigma_l\sigma_l \\ & + J_{wl}(w, l)wl\varphi\sigma_R(w, l)\sigma_l. \end{aligned}$$

The first order condition in  $\varphi$  is

$$J_{ww}(w, l)w^2\sigma_R(w, l)\sigma_R(w, l)\varphi + J_w(w, l)(\mu_R(w, l) - r_t)w + J_{wl}(w, k)wl\sigma_R(w, l)\sigma_l(l) = 0.$$

This gives for the optimal demand of the risky asset

$$W_t\varphi_t = \left( -\frac{J_w(W_t, l_t)}{J_{ww}(W_t, l_t)} \right) \left( \frac{\mu_R(W_t, l_t, t) - r_t}{\sigma_R(t)\sigma_R(t)} \right) + \left( -\frac{J_{wl}(W_t, l_t)l_t}{J_{ww}(W_t, l_t)} \right) \left( \frac{\sigma_R(t)\sigma_l(t)}{\sigma_R(t)\sigma_R(t)} \right). \quad (44)$$

The demand function is seen to have two components: The first one is the usual demand function for a risky asset, similar to the one encountered by a single-period mean-variance maximizer. This is what an investor can relate to when he only has access to the financial market. For the linear model this is the only term that appears in the demand function, as can be seen from (23). In this respect the time continuous model with the linear production technology has much in common with the widely taught, one-period mean-variance model.

The last term reflects the investor's demand for the risky asset to hedge against unfavorable shifts in the investment opportunity set, here represented by the variable  $l$ . This term is the hedging demand, available when the investor also uses information about the production (labor) part of the economy. For the linear model of the first part, this hedging component is not present. The special issue here is that labor, or leisure, is a decision variable determined by the agent according to his or her preferences. This determination we have left out in the above derivation, just assuming that  $l_t$  is optimally set at each time  $t$ .

## 5.4 The risk premium

The representative agent is initially endowed with one share of the firm, in which case the market clearing condition is  $\varphi_t = 1$  a.s. for all  $t$ , so the risk free asset is in zero net supply. From the expression (44) we get the equilibrium risk premium

$$\mu_R(t) - r_t = \left( -\frac{J_{ww}(W_t, l_t)W_t}{J_w(W_t, l_t)} \right) \sigma_R(t)\sigma_R(t) + \left( -\frac{J_{wl}(W_t, l_t)l_t}{J_w(W_t, l_t)} \right) \sigma_R(t)\sigma_l(t). \quad (45)$$

Comparing with the simple model of the first part, we see from (27) that the second term on the right-hand side in the above expression is missing. For investors who only focus on the stock market, this may appear to give a reasonable risk premium. However, it does not fit the data, since the model also implies that  $\sigma_c = \sigma_R$ . The second term on the right hand side appears in our framework because of the inclusion of the "state variable"  $l$ .

Considering the expression in (45), could it be, for example, that the first term on the right hand side is approximately equal to the relative risk aversion  $\gamma$ , times the variance rate of the return, and that the last term is small compared to the first term, such that  $\mu_R - r \approx \gamma\sigma_R^2$ ? If this were the case, this model would give a reasonable equity premium. That this is not so, will be explained in the next section.

As a preparation of this, we first seek an interpretation of the terms of the risk premium in (45). In doing so, we find the dynamics of the quantity  $e^{-\delta t}u_c(c_t^*, l_t)$ , and compare this to the dynamics of the state price deflator  $\pi_t$  given in (41). By diffusion invariance and the envelope theorem, it follows that

$$u_{cc}(c_t^*, l_t)c_W^* = J_{ww}(W_t, l_t) \quad \text{and} \quad u_{cc}(c_t^*, l_t)c_l^* = J_{wl}(W_t, l_t)$$

where  $c_W^*$  is the partial derivative of  $c^*$  with respect to wealth, and  $c_l^*$  is the partial derivative of  $c^*$  with respect to the state variable  $l$ . Using this, the risk premium can be represented in the following convenient form

$$\mu_R(S_t, l_t) - r_t = \left( - \frac{u_{cc}(c_t^*, l_t)c_t^*}{u_c(c_t^*, l_t)} \right) \left( el_W(c_t^*)\sigma_R(S_t, l_t)\sigma_R(S_t, l_t) + el_l(c_t^*)\sigma_R(S_t, l_t)\sigma_l(l_t) \right), \quad (46)$$

where  $el_W(c_t^*) = \frac{c_W^*W_t}{c_t^*}$ , and  $el_l(c_t^*) = \frac{c_l^*l_t}{c_t^*}$  are the partial consumption elasticities with respect to wealth and leisure, respectively.

Similarly, the equilibrium demand for the risky asset is given by

$$\varphi_t = \left( - \frac{u_c(c_t^*, l_t)}{u_{cc}(c_t^*, l_t)c_t^*} \right) \frac{1}{el_W(c_t^*)} \frac{\mu_R - r}{\sigma_R\sigma_R} - \frac{el_l(c_t^*)}{el_W(c_t^*)} \frac{\sigma_R\sigma_l}{\sigma_R\sigma_R}. \quad (47)$$

The first term is seen to be the classical one in standard finance in the case when  $el_W(c_t^*) = 1$ , that is known to be the only term in the pure demand theory (Mossin (1968), Samuelson (1969), Merton (1971)). The last term is the hedging demand related to  $l$ .

The fraction  $\frac{el_l(c_t^*)}{el_W(c_t^*)}$  is a marginal substitution ratio between  $l$  and  $W$ .

Using the above elasticities, the short term interest rate in (43) can be written

$$r_t = \delta + \left( - \frac{u_{cc}(c_t^*, l_t) c_t^*}{u_c(c_t^*, l_t)} \right) \left\{ (el_W(c_t^*) + el_l(c_t^*)) \mu_l(l_t) - el_W(c_t^*) \frac{u_c^{-1}(J_W, l_t)}{W_t} \right\} + \dots \quad (48)$$

where we have omitted the higher order terms.

As with the linear model, since we do not have any adjustment costs, Tobin's marginal Q is again be constant and equal to 1, and the stock price simply equals the capital stock, which we have employed above. This simplifies the risk premium to contain only one factor. What this factor looks like, will be derived next.

## 5.5 The consumption based capital asset pricing model

Returning to the risk premium in (46), we want to explore in what sense it is different from the risk premium obtained in the linear production model. For example, if  $el_W(c_t^*) = el_K(c_t^*) \approx \frac{1}{2}$ , these two risk premiums would yield approximately the same numerical results, provided  $\sigma_R = \sigma_K$ . Recall that we now operate with a nonlinear production technology, so, in particular it is no longer true that the optimal consumption is proportional to wealth. It turns out that also in the model of this section, the risk premium can be expressed as

$$\mu_R(t) - r_t = \left( - \frac{u_{cc}(c_t^*, l_t) c_t^*}{u_c(c_t^*, l_t)} \right) \sigma_C(t) \sigma_R(t), \quad (49)$$

i.e., the CCAPM holds true also here. The simplest way to demonstrate this is to find the dynamics of  $c_t^*$  using the representation in (39), which is  $c^*(t) = u_c^{-1}(J_w(W_t, l_t), l_t)$ . By Itô's lemma we get

$$dc_t^* = \mu_{c^*}(t) dt + \left( \frac{J_{ww}(W_t, l_t)}{u_{cc}(c_t^*, l_t)} \right) \sigma_W dB_t + \left( \frac{J_{wl}(W_t, l_t)}{u_{cc}(c_t^*, l_t)} \right) l_t \sigma_l dB_t + \frac{\partial}{\partial l} u_c^{-1}(J_w(W_t, l_t), l_t) l_t \sigma_l dB_t, \quad (50)$$

where the function  $u_c^{-1}(\cdot, l)$  inverts  $u_c(\cdot, l)$ , meaning that  $u_c^{-1}(u_c(x, l), l) = x$  for all  $(x, l)$ . From the first order condition in consumption given in (38) we have that  $J_w(W_t, l_t) = u_c(c_t, l_t)$ . This implies that

$$\frac{\partial}{\partial l} u_c^{-1}(u_c(c_t^*, l_t), l_t) = 0 \quad \text{for all values of } c_t^* \text{ and } l_t \text{ a.s.}$$

From this the volatility  $\sigma_C(t)$  of the consumption growths is

$$\sigma_C(t) = \left( \frac{J_{ww}(W_t, l_t)}{u_{cc}(c_t^*, l_t)c_t^*} \right) \sigma_W + \left( \frac{J_{wk}(W_t, l_t)l_t}{u_{cc}(c_t^*, l_t)c_t^*} \right) \sigma_l.$$

Accordingly is

$$\begin{aligned} \left( - \frac{u_{cc}(c_t^*, l_t)c_t^*}{u_c(c_t^*, l_t)} \right) \sigma_C(t) \sigma_R(t) &= \left( - \frac{J_{ww}(W_t, l_t)W_t}{J_w(W_t, l_t)} \right) \sigma_R \sigma_R \\ &+ \left( - \frac{J_{wk}(W_t, l_t)l_t}{J_w(W_t, l_t)} \right) \sigma_R \sigma_l = \mu_R(t) - r_t, \end{aligned} \quad (51)$$

where we have used the first order conditions in (38) once more, and the expression for the risk premium in (45) accounts for the last equality. That is, the CCAPM holds in this particular form.

Similarly, using the connection  $\pi_t = u_c(c_t^*, l_t)e^{-\delta t}$  between the state price and the marginal utility of consumption, it follows from the relation  $r_t = -\mu_\pi(t)/\pi(t)$  that the spot interest rate of this section can be written

$$\begin{aligned} r_t = \delta + \left( - \frac{u_{cc}(c_t^*, l_t)c_t^*}{u_c(c_t^*, l_t)} \right) \mu_C(t) \\ - \frac{1}{2} \left( - \frac{u_{ccc}(c_t^*, l_t)c_t^*}{u_{cc}(c_t^*, l_t)} \right) \left( - \frac{u_{cc}(c_t^*, l_t)c_t^*}{u_c(c_t^*, l_t)} \right) \sigma_C(t) \cdot \sigma_C(t), \end{aligned} \quad (52)$$

which reduces to (33) under the assumptions of Section 3. Thus this model is fairly robust.

As mentioned, Mehra and Prescott (2008) explain that expanding the set of technologies in a pure exchange economy to admit capital accumulation and production does not increase the set of joint equilibrium processes on consumption and asset prices. The results of this section are in accordance with this general observation, and can be considered as a formal proof of this result.

With a relative risk aversion of  $\gamma = 3$  and a subjective rate of  $\delta = 0.01$  for the representative consumer, the present model demands a risk premium of 0.68% and a short term interest rate of around 5.7% for the consumption and equity moment estimates in Table 1, i.e., far from the estimates in Table 1, which are 5.98% and .80% respectively.

## 6 Discussion of the conventional model

The result of the last section shows that the "real" economy matters also in the agent's portfolio choice problem: By merely using the local mean-variance theory, consistent with the market based CAPM, the agent will



seemingly look at the financial market in isolation, and with a relative risk aversion close to two, an equity risk premium of around six percent seem to emerge from the observed market volatility of equity. In its turn this leads to an equilibrium short rate of around one per cent, both numbers consistent with the estimates in Table 1. However, the model tells us that this is not consistent: It is the consumption-based CAPM that is valid when the representative agent has the conventional preferences represented by additive and separable von Neumann-Morgenstern expected utility, and this leads to the asset pricing puzzles discussed in this paper. This model also gives a much too large value for the risk-free interest rate. This clearly points in the direction of a new approach.

Platen and Heath (2006) propose a “benchmark approach” to the pricing of risky assets, in which the deflated price processes  $S_t\pi_t$  are supermartingales instead of martingales as in the standard theory. In particular, Platen (2011) presents a graph of the density process  $\xi_t = \pi_t \cdot \exp(\int_0^t r(u)du)$  related to the S&P500-index during the period from 1920 to 2010. Without giving any formal statistical analysis, it seems like this filtered estimate falls with time, supporting a supermartingale interpretation for  $\xi$ . This may indicate that there is no equivalent martingale measure, which in its turn implies “well-behaved” arbitrage (e.g., Duffie (2001)). Abstracting from the no-arbitrage issue, this gives an indication of a smaller interest rate  $r_t$  than predicted by the formula (52), but also a smaller equity premium than predicted by the formula (49). In order for a larger equity premium to follow, the density process must instead be a submartingale, showing how difficult it is to explain both these puzzles at once. Thus, a martingale interpretation of the density process seems like a happy compromise after all, since this is the only object that is both a supermartingale and a submartingale.

Kimball et.al. (2008) indicate a value of the relative risk aversion around 8, based on responses to hypothetical income gambles in the Health and Retirement Study, a large-scale survey in the USA. With the data in Table 1, this gives a low equity premium, and a risk-free rate that is much too large. Due to heterogeneity they also obtain an estimate of the relative risk tolerance of .206 (which is not  $1/\gamma$  due to Jensen’s inequality in their model). Using this expression in the formula for the risk-free interest rate and thus interpreting this in the “best way possible”, we only obtain a moderate improvement in the interest rate.

Turning to recursive utility, we demonstrate in the last section of the paper how the main issues discussed above can be resolved.

## 7 Recursive utility

### 7.1 The production economy

In this section we give a brief outline of the theory of risk premiums when the agent has recursive utility. We also derive the equilibrium interest rate. This is built on the treatment in Aase (2014a). We will, among other things, use the stochastic maximum principle, which we now short sketch: We are given a stochastic control problem for a system of coupled forward-backward stochastic differential equations (FBSDEs): The *forward* system is

$$\begin{cases} dX_t = \mu_X(t, X_t, V_t, Z_t, u_t)dt + \sigma_X(t, X_t, V_t, Z_t, u_t)dB_t \\ X_0 = x \in \mathbb{R} \end{cases} \quad (53)$$

Here  $u_t$  is a control variable, which in our case is  $(c_t, l_t)$ . The *backward* SDE is given by the recursive utility specification (see Duffie and Epstein (1992a,b))

$$\begin{cases} dV_t = -g(V_t, X_t, Z_t, u_t) dt + Z_t dB_t \\ V_T = 0, \end{cases} \quad (54)$$

where  $T$  is the finite horizon. The backward equation is dictated by the performance functional (= utility function) such that  $U(c) = V_0$ , which for recursive utility takes the form

$$U(u) = E\left(\int_0^T \tilde{f}(X_t, V_t, Z_t, u_t) dt\right), \quad (55)$$

for  $u \in \mathcal{A}$  where  $\mathcal{A}$  is a set of admissible  $\mathcal{F}_t$ -predictable controls. The problem is find  $u^* \in \mathcal{A}$  such that

$$\sup_{u \in \mathcal{A}} U(u) = U(u^*). \quad (56)$$

For recursive utility  $V_t$  is future utility at each time  $t$ , and  $U(c) = V_0$ . The function  $\tilde{f}(X_t, V_t, Z_t, u_t) = f(c_t, l_t, V_t) - \frac{1}{2}A(V_t)Z_t'Z_t$ , where  $f$  corresponds to a felicity index and  $A$  penalizes for risk aversion (recall the Arrow-Pratt approximation to the certainty equivalent of a mean zero random variable). The pair  $(f, A)$  is called an aggregator. The important novelty here is that consumption substitution in a deterministic model is contained in the  $f$ -term, separated from risk aversion measured by the  $A$ -term. This separation turns out to be important in a *temporal* context, meaning in models where the consumer consumes more than once, and there is uncertainty between points in time where consumption takes place (see e.g., Mossin (1969)).

Below we use the following specification for  $f$  and  $A$ :

$$f(c, v) = \frac{\delta}{1 - \rho} \frac{c^{1-\rho} - v^{1-\rho}}{v^{-\rho}} \quad \text{and} \quad A(v) = \frac{\gamma}{v}. \quad (57)$$

The function  $f$  is of a CES type, in which case the choice of functions fall in the Kreps and Porteus (1978)-family. The only new parameter here relative to the first part of the paper is  $\rho$ , called the time preference. It measures the agents's aversion to consumption variations across time in a deterministic model. The elasticity of intertemporal substitution in consumption is  $\psi = 1/\rho$ , referred to as the EIS-parameter. In the conventional model  $\rho = \gamma$ , here we allow  $\gamma$  to be different from  $\rho$ . The parameters  $\delta$  and  $\gamma$  have the same interpretations as in the first part of the paper, so that  $\gamma$  measures the agents aversion at each time to variations in consumption the next period due to the different states of the world than may occur. Clearly  $\gamma$  and  $\rho$  are related to different properties of an individual's attitudes.

The Hamiltonian for the system is given by

$$H(t) := H(X_t, V_t, Z_t, Y_t, p_t, q_t, u_t) = \tilde{f}(X_t, V_t, Z_t, u_t) + g(V_t, X_t, Z_t, u_t)Y_t + \mu_X(t, X_t, V_t, Z_t, u_t)p_t + \sigma_X(t, X_t, V_t, Z_t, u_t)q_t. \quad (58)$$

where the adjoint variables are  $Y_t$ ,  $p_t$ , and  $q_t$  (see, for example, Pontryagin (1972), Bismut (1978), Kushner (1972), Bensoussan (1983), Øksendal and Sulem (2013), or Peng (1990)).<sup>6</sup>

The associated forward/backward systems for the adjoint processes  $Y_t, p_t$ , and  $q_t$  are

$$\begin{cases} dY_t = \frac{\partial H}{\partial v}(t) dt + \frac{\partial H}{\partial z}(t) dB_t \\ Y_0 = 1, \end{cases} \quad (59)$$

and

$$\begin{cases} dp_t = -\frac{\partial H}{\partial x}(t) dt + q_t dB_t \\ p_T = 0, \end{cases} \quad (60)$$

Under certain conditions (see e.g., Øksendal and Sulem (2014)) the  $u_t$  that maximizes  $H(t)$  for each  $t$  solves problem (56).

## 7.2 The pure exchange economy

For recursive utility it will be convenient to first consider the pure exchange economy. We return to including labor later. We then reformulate the problem from one that is tailor made for dynamic programming, to one in which

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<sup>6</sup>In the Hamiltonian we have omitted Tobin's Q, since it is equal to 1 in equilibrium for the same reason as before (the shadow price of one unit of investment is worth one unit of account).

the stochastic maximum principle is more appropriate. Instead of maximizing utility subject to a dynamic constraint on wealth  $W_t$ , as in (36), a slightly more general formulation is the following: The representative agent's problem is to solve

$$\sup_{c \in L} U(c)$$

subject to the budget constraint

$$E \left\{ \int_0^T c_t \pi_t dt \right\} \leq E \left\{ \int_0^T e_t \pi_t dt \right\}.$$

Here future utility at time  $t$ ,  $V_t = V_t^c$  and  $(V_t, Z_t)$  is the solution of the backward stochastic differential equation (BSDE)

$$\begin{cases} dV_t = -\tilde{f}(c_t, V_t, Z(t)) dt + Z(t) dB_t \\ V_T = 0. \end{cases} \quad (61)$$

so that the function  $g$  of the drift term of equation (54) is given by  $g = \tilde{f}$  for recursive utility.

Existence and uniqueness of solutions of this BSDE is treated in the general literature on this subject, see e.g., Theorem 2.5 in Øksendal and Sulem (2013), or Hu and Peng (1995). For the particular quadratic BSDE (61) existence and uniqueness follows from Duffie and Lions (1992).

Since  $U(c) = V_0$ , for  $\alpha > 0$  we define the Lagrangian

$$\mathcal{L}(c; \alpha) = V_0 - \alpha E \left( \int_0^T \pi_t (c_t - e_t) dt \right).$$

In order to find the first order condition for the representative consumer's problem, we use Kuhn-Tucker and the stochastic maximum principle. Suppose for each  $\alpha > 0$  we can find an optimal  $c_t^{(\alpha)}$  such that

$$\sup_c \mathcal{L}(c; \alpha) = \mathcal{L}(c^{(\alpha)}; \alpha)$$

without constraints. Next, suppose we can find  $\alpha^*$  such that the budget constraint is satisfied with equality

$$E \left\{ \int_0^T c_t^{(\alpha^*)} \pi_t dt \right\} = E \left\{ \int_0^T e_t \pi_t dt \right\}.$$

Then the optimal consumption  $c_t^*$  is given by

$$c^* := c^{(\alpha^*)}.$$

To see this, notice that for all  $c$  we have

$$V_0(c^{(\alpha^*)}) = V_0(c^{(\alpha^*)}) - \alpha^* E\left(\int_0^T \pi_t(c_t^{(\alpha^*)} - e_t) dt\right) =$$

$$\mathcal{L}(c^{(\alpha^*)}, \alpha^*) \geq \mathcal{L}(c, \alpha^*) = V_0(c) - \alpha^* E\left(\int_0^T \pi_t(c_t - e_t) dt\right) \geq V_0(c).$$

In other words, the representative agent's original problem is solved by maximizing the Lagrangian without constraints, and to solve this problem we propose to use the stochastic maximum principle. The Hamiltonian for this problem is

$$H(c, v, z, y) = -\alpha \pi_t(c_t - e_t) + y_t \tilde{f}(c_t, v_t, z_t), \quad (62)$$

where  $y_t$  is the adjoint variable. The conditions for an optimal solution to the stochastic maximum principle are the same as those related to the BSDE (61) (see Duffie and Lions (1992)). The FBSDE system consists of

$$dX(t) = 0; \quad X(0) = 0$$

and

$$\begin{cases} dY_t = Y(t) \left( \frac{\partial \tilde{f}}{\partial v}(c_t, V_t, Z(t)) dt + \frac{\partial \tilde{f}}{\partial z}(c_t, V_t, Z(t)) dB_t \right) \\ Y_0 = 1. \end{cases} \quad (63)$$

is the adjoint equation, which follows from (59).

If  $c^*$  is optimal we therefore have

$$Y_t = \exp\left(\int_0^t \left\{ \frac{\partial \tilde{f}}{\partial v}(c_s^*, V_s, Z(s)) - \frac{1}{2} \left( \frac{\partial \tilde{f}}{\partial z}(c_s^*, V_s, Z(s)) \right)^2 \right\} ds + \int_0^t \frac{\partial \tilde{f}}{\partial z}(s, c_s^*, V_s, Z(s)) dB(s)\right) \quad a.s. \quad (64)$$

$$\alpha \pi_t = Y(t) \frac{\partial \tilde{f}}{\partial c}(c_t^*, V(t), Z(t)) \quad a.s. \quad \text{for all } t \in [0, T]. \quad (65)$$

Notice that the state price deflator  $\pi_t$  at time  $t$  depends, through the adjoint variable  $Y_t$ , on the entire optimal paths  $(c_s, V_s, Z_s)$  for  $0 \leq s \leq t$ .

When  $\gamma = \rho$  then  $Y_t = e^{-\delta t}$  for the aggregator of the conventional model, so the state price deflator is a Markov process, the utility is additive and dynamic programming may be appropriate.

For the representative agent equilibrium the optimal consumption process is the given aggregate consumption  $c$  in society, and for this consumption process the utility  $V_t$  at time  $t$  is optimal.

Starting from the first order condition (65), and differentiating this equation along the optimal path, by Ito's lemma the dynamic equation for the adjoint variable  $\pi_t$  is

$$d\pi_t = f_c(c_t, V_t) dY_t + Y_t df_c(c_t, V_t) + dY_t df_c(c_t, V_t). \quad (66)$$

where  $f_c$  signifies the partial derivative with respect to  $c$ . The variable  $\pi$  can be interpreted as the state price deflator, i.e., as Arrow-Debreu state prices in units of probability. Thus "marginal utility equals price" takes the form of  $\pi_t = Y_t f_c(t, c_t, V_t)$  for each time  $t \in [0, T]$  as of time 0, so the adjoint variable  $Y_t$  is part of the agent's marginal utility at time  $t$ , the shadow price on one "utilon".

### 7.3 The equity premium and the interest rate

Denoting the dynamics of the state price deflator by

$$d\pi_t = \mu_\pi(t) dt + \sigma_\pi(t) dB_t, \quad (67)$$

from (66) and (57) we obtain the drift and the diffusion terms of  $\pi_t$  as

$$\begin{aligned} \mu_\pi(t) = \pi_t & \left( -\delta - \rho\mu_c(t) + \frac{1}{2}\rho(\rho+1)\sigma'_c(t)\sigma_c(t) \right. \\ & \left. + \rho(\gamma-\rho)\sigma'_c(t)\sigma_V(t) + \frac{1}{2}(\gamma-\rho)(1-\rho)\sigma'_V(t)\sigma_V(t) \right) \end{aligned} \quad (68)$$

and

$$\sigma_\pi(t) = -\pi_t(\rho\sigma_c(t) + (\gamma-\rho)\sigma_V(t)) \quad (69)$$

respectively.

Notice in particular that  $\pi_t$  is *not* a Markov process since  $\mu_\pi(t)$  and  $\sigma_\pi(t)$  depend on  $\pi_t$ , and the latter variable depends on consumption and utility from time zero to time  $t$ , as can be seen from the FOC (88), and the expression for the adjoint variable  $Y$  in (64).

The risk premium of any risky security with return process  $R$  is given by

$$\mu_R(t) - r_t = -\frac{1}{\pi_t} \sigma_\pi(t) \sigma_R(t). \quad (70)$$

It follows immediately from (69) and (70) that the formula for the risk premium of any risky security  $R$  is

$$\mu_R(t) - r_t = \rho\sigma_c(t)\sigma_R(t) + (\gamma-\rho)\sigma_V(t)\sigma_R(t). \quad (71)$$

This is a basic result for risk premiums. When  $\gamma \neq \rho$  there is an additional term in the risk premium, compared to the conventional model. Unlike the situation in (45), where we also encountered two factors, these will not collapse to the consumption based CAPM unless  $\sigma_V(t)$  is zero. From the theory of FBSDE we know that a unique non-zero solution  $(V, \sigma_V)$  exists.

It remains to determine the characteristics of "prices" from the primitives of the model, which we indicate below. Here this involves determining  $\sigma_M(t)$  from  $\sigma_V(t)$  and  $\rho$  (preferences), and  $\sigma_c(t)$  (aggregate consumption). As before the equilibrium short-term, real interest rate  $r_t$  is given by the formula

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}. \quad (72)$$

In order to find an expression for  $r_t$  in terms of the primitives of the model, we do as before and use (68). This gives the following formula

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\rho(\rho + 1)\sigma'_c(t)\sigma_c(t) - \rho(\gamma - \rho)\sigma_{cV}(t) - \frac{1}{2}(\gamma - \rho)(1 - \rho)\sigma'_V(t)\sigma_V(t). \quad (73)$$

This is a basic result for the equilibrium short rate. The potential for these two relationships to solve the puzzles should be apparent. We do not need to assume that the various moments appearing in these two expressions are constants. Thus we do not have the problem encountered in the first model of the paper. We return to a discussion below.

#### 7.4 The determination of the volatility process $\sigma_W(t)$ from the primitives of the economy

We proceed to connect the volatility of the growth rate of the wealth portfolio  $\sigma_W(t)$  to primitives of the model, which involve the term  $\sigma_V(t) = Z_t/V_t$ . In doing so, the latter term is linked to observable quantities in the market that can, at least in principle, be estimated from market data.

To this end, we use that the utility function  $U$  is homogeneous of degree one in consumption, together with market clearing in the financial market. Recall that the wealth of the agent at any time  $t$  is given by

$$W_t = \frac{1}{\pi_t} E_t \left( \int_t^T \pi_s c_s^* ds \right), \quad (74)$$

where  $c^*$  is optimal consumption. By the first order condition the gradient  $\nabla U(c^*; c^*)$  of  $U$  at  $c^*$  in the direction of  $c^*$  is a linear functional, and hence,

by the Riesz representation theorem, it is given by

$$\nabla U(c^*; c^*) = E\left(\int_0^T \pi_t c_t^* dt\right) = W_0 \pi_0 \quad (75)$$

where  $W_0$  is the wealth of the representative agent at time zero, and the last equality follows from (74) for  $t = 0$ <sup>7</sup>. By homogeneity of  $U$  it follows that  $\nabla U(c^*; c^*) = U(c^*) = \pi_0 W_0$ .

Let  $V_t = V^{(c_t^*)}$  denote future utility at the optimal consumption for our representation at any time  $t \in (0, T]$ . Since also  $V_t$  is homogeneous of degree one and continuously differentiable, by Riesz' representation theorem and the dominated convergence theorem, the same type of basic relationship holds here for the associated directional derivatives at time  $t$ , i.e.,

$$\nabla V_t(c^*; c^*) = E_t\left(\int_t^T \pi_s^{(t)} c_s^* ds\right) = V_t$$

where  $\pi_s^{(t)}$  for  $s \geq t$  is its Riesz representation, here the state price deflator at time  $s \geq t$ , conditional on time  $t$  information.

In the conventional, additive Eu-model, where  $Y_t = e^{-\delta t}$ ,  $\pi_s$  is the state price at time  $s$  as of time 0, and  $\pi_s^{(t)} = \pi_s / Y_t$ ,  $s \geq t$ , is the state price at time  $s$  as of time  $t$ .

In the same manner it follows, e.g., by results in Skiadas (2009a), that the the same relationship holds also here.

The financial market of this section has the same structure as in the two previous sections, except we do not need any Markov structure. Assuming all assets, including labor income, to be represented in the market portfolio (some interpreted as "shadow" assets), in equilibrium it is then the case that  $\varphi_t' \cdot \sigma(t) = \sigma_W(t)$ , and  $\sigma_W(t)$  is the volatility of the growth rate of the wealth portfolio. It follows that

$$V_t Y_t = \pi_t W_t. \quad (76)$$

Using Ito's lemma on both sides of this equality, and considering only the variance rates, we obtain

$$\sigma_W(t) = (1 - \rho)\sigma_V(t) + \rho\sigma_c(t). \quad (77)$$

This is where wealth is determined in terms of the primitives of the model, which are the preferences, here represented by  $\sigma_V(t)$  and  $\rho$ , and aggregate

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<sup>7</sup>Notice this has nothing to do with "normalizing" recursive utility, but is just the part of first order conditions for directional derivatives. Recall it is the directional derivative of the Lagrangian, not of  $U$ , that is zero in all feasible directions.



consumption, here  $\sigma_c(t)$ . The volatility of the market portfolio is a linear sum of the volatility of future utility and the volatility of the growth rate of aggregate consumption.

This relationship can now be used to express  $\sigma_V(t)$  in terms of the other two volatilities as

$$\sigma_V(t) = \frac{1}{1-\rho}(\sigma_W(t) - \rho\sigma_c(t)). \quad (78)$$

Inserting the expression (78) into (71) and (73) we obtain the risk premiums as

$$\mu_R(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho} \sigma'_c(t)\sigma_R(t) + \frac{\gamma-\rho}{1-\rho} \sigma'_W(t)\sigma_R(t), \quad (79)$$

and the short rate as

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2} \frac{\rho(1-\gamma\rho)}{1-\rho} \sigma_c(t)'\sigma_c(t) + \frac{1}{2} \frac{\rho-\gamma}{1-\rho} \sigma'_W(t)\sigma_W(t) \quad (80)$$

respectively.

The same expression for the risk premium was derived by Duffie and Epstein (1992a) based on an ordinally equivalent utility function with aggregator  $(h, 0)$ , where  $h$  is related to  $f$ . Their derivation was based on dynamic programming, assuming the volatilities involved to be constants. The expression for the real interest rate was derived in Aase (2014a).

## 7.5 Calibrations of the recursive model

This model calibrates to the data summarized in Table 1 much more convincingly than the conventional Eu-model.

As an illustration, consider the value of  $\delta = .03$ . Calibrating the two above equations to the data, assuming the market portfolio is a proxy for the wealth portfolio, gives two nonlinear equations in two unknowns,  $\gamma$  and  $\rho$ . The solution is  $\gamma = 1.74$  and  $\rho = 0.48$ .

The above calibration seems fairly reasonable. Recalling the background for the data however, we can not really expect that a single agent can convincingly explain these data in isolation. In the period considered, for example, only a small fraction (8-9%) of the population owned stock.

A heterogeneous model with two agents, one who do not invest in stocks, would be more appropriate in the present setting, see e.g., Guvenen (2009) using a discrete time model, and Aase (2014b) in a continuous-time setting.

In this paper we have by and large assumed that the market portfolio can be used as a proxy for the wealth portfolio. Suppose we can view exogenous income streams as dividends of some shadow asset, in which case our model is valid if the market portfolio is expanded to include the new asset. However, if

the latter is not traded, then the return to the wealth portfolio is not readily observable or estimable from available data. Still we should be able to get a pretty good impression of how the two models compare.

As an example, assume  $\sigma_W(t) = .10$ , the correlation coefficient  $\kappa_{W,R} = .80$ , and  $\kappa_{c,W} = .40$ . In this situation, when  $\delta = .02$  we obtain  $\gamma = 2.11$  and  $\rho = .74$ , which seem rather plausible (see e.g., Aase (2013) where the Epstein-Zin discrete-time model is calibrated to the US-data under similar assumptions, or Aase (2014a) where the continuous-time model of the Duffie-Epstein type is calibrated. The continuous-time model including jump dynamics is developed and calibrated in Aase (2015).)

Consider now the CAPM: As observed in the first part of the paper, the CAPM in the setting of the conventional model did not fit the data summarized in Table 1. With recursive utility this turns out to be different. When  $\rho = 0$  we have perfect substitutability of consumption across time. In this case the model reduces to

$$\mu_R(t) - r_t = \gamma \sigma_{W,R}(t), \quad r_t = \delta - \frac{\gamma}{2} \sigma'_W(t) \sigma_W(t). \quad (81)$$

Recall the original CAPM developed by Mossin (1966). It is a one-period model, but the present specialization is based on a dynamic and consistent framework with recursive utility. In the one-period model the interest rate has no particular meaning, since there is no consumption transfers across time for the individuals in the model. Here we have an interest rate determined in equilibrium.

The risk premium given above is that of the ordinary CAPM-type, while the interest rate is of course new. This version of the model corresponds to "neutrality" of consumption transfers. Also, from the expression for the interest rate we notice that the short rate decreases in the presence of increasing market uncertainty. Solving these two non-linear equations consistent with the data of Table 1, again assuming the market portfolio as a proxy for the wealth portfolio, we obtain  $\gamma = 2.38$  and  $\delta = .038$ . In the conventional model this simply gives risk neutrality, i.e.,  $\gamma = \rho = 0$ , so this model gives a risk premium of zero, and a short rate of  $r = \delta$ .

There is a large empirical literature on testing the CAPM, starting with Fama and MacBeth (1973), implicitly assuming the model is valid period by period. This is of course wrong. This literature takes as a criterion the model's explanatory power related to future returns, given this model has been applied in the past. The results from this rather long literature does not seem convincing for the CAPM.

However, recent research indicates a much better performance for this model (see e.g., Berk (1997), and Berk and Binsbergen (2012)).

## 8 Including labor

The problem is the same as in the previous section, but now we include labor with dynamic equation

$$dl_t = l_t \mu_l(t) + l_t \sigma_l(t) dB_t$$

where  $\mu_l(t)$  and  $\sigma_l(t)$  are not functions of  $l_t$ . Using the stochastic maximum principle, interpreting  $l_t$  as the  $X_t$  process, the performance functional is

$$\mathcal{L}(c; \alpha) = V_0 - \alpha E \left( \int_0^T \pi_t (c_t - e_t) dt \right).$$

The Hamiltonian for this problem is

$$H(c_t, v_t, z_t, l_t, y_t, p_t, q_t) = -\alpha \pi_t (c_t - e_t) + Y_t \tilde{f}(c_t, v_t, z_t, l_t) + l_t \mu_l(t) p_t + l_t \sigma_l(t) q_t$$

The new equation is now

$$dp_t = -\frac{\partial H}{\partial l}(t) + q_t dB_t,$$

$p_T = 0$ , corresponding to (60), where

$$\frac{\partial H}{\partial l}(t) = Y_t \frac{\partial f}{\partial l}(c_t, V_t, l_t) + p_t \mu_l(t) + q_t \sigma_l(t).$$

Since we have assumed that labor is also a decision variable, by the maximum principle the first order condition in  $l$  is given by

$$Y_t \frac{\partial f}{\partial l}(c_t, V_t, l_t) + p_t \mu_l(t) + q_t \sigma_l(t) = 0$$

for all  $0 \leq t \leq T$ . As before the first order condition in consumption is  $\alpha \pi_t = \tilde{f}_c Y_t$ , where both  $\tilde{f}$  and the adjoint variable  $Y_t$  depend on labor  $l_t$ .

As a consequence, the drift of the adjoint variable  $p_t$ , interpreted as wages, is zero, so that

$$dp_t = q_t dB_t$$

implying that, with a standard integrability constraint on the volatility process  $q_t$ , wages in real terms is a martingale. By the first order condition in  $l$   $q$  may be written

$$q_t = -\left( Y_t \frac{\partial f}{\partial l}(c_t, V_t, l_t) + p_t \mu_l(t) \right) (\sigma_l(t) \sigma_l'(t))^{-1} \sigma_l(t).$$

Recalling from (45) that the risk premium has included a term signifying the covariance rate between the risky asset under consideration and labor, an analogue of this fact may be seen to follow from the present approach by observing that the endowment process  $e_t$  must include labor, so that the value of the current endowment  $e_t$  can be written as  $\pi_t e_t = p_t l_t + \pi_t e_t^-$  where  $e_t^-$  is the part of the endowment process that does not include labor.

In the Lucas (fruit-) economy  $c_t = e_t$  in every period, which implies that the volatility of aggregate consumption can be written in terms of the volatilities of  $l$ ,  $\pi$ ,  $p$  and  $e^-$ . In particular is

$$dc_t = de_t = \frac{p_t}{\pi_t} dl_t + l_t d\left(\frac{p_t}{\pi_t}\right) + de_t^-$$

Thus the volatility of aggregate consumption can be written

$$\sigma_c(t) = \frac{p_t}{\pi_t} \sigma_l(t) + l_t \left(\frac{1}{\pi_t}\right) q_t + \frac{p_t}{\pi_t} \left(-\frac{1}{\pi_t} \sigma_\pi(t)\right) + \sigma_e^-(t)$$

using Ito's lemma. Recalling that  $\left(-\frac{1}{\pi_t} \sigma_\pi(t)\right) \sigma_R(t) = \mu_R(t) - r_t$  for any risky asset with volatility  $\sigma_R(t)$  and return rate  $\mu_R(t)$ , we obtain that

$$\mu_R(t) - r_t = \frac{\pi_t}{p_t} (\sigma_c(t) \sigma_R(t) - \sigma_e^-(t) \sigma_R(t)) - \frac{l_t}{p_t} q_t \sigma_R(t) - \sigma_l(t) \sigma_R(t)$$

where  $q_t$  is expressed in terms of  $\sigma_l(t)$  as above. To obtain further simplifications and insights, the felicity function  $f$  must be specified.

We leave this subject for now, and demonstrate how the intertemporal capital asset pricing model (ICAPM) may readily be derived using our present methodology.

## 8.1 The ICAPM

We may derive the intertemporal capital asset pricing model using the stochastic maximum principle. Generally one can not assumed that all income is investment income. Let us first treat labor income as dividends of some shadow asset, in which case our model is valid if the market portfolio is expanded to include the new asset. In reality the latter is not traded, which we here ignore. Referring to the model of the first part of the paper, we then have two underlying processes  $(S, l)$  with dynamics

$$\begin{pmatrix} dS_t \\ dl_t \end{pmatrix} = \begin{pmatrix} S_t \mu_S(t) \\ l_t \mu_l(t) \end{pmatrix} dt + \begin{pmatrix} S_t \sigma_{S,1}(t), S_t \sigma_{S,2}(t) \\ l_t \sigma_{l,1}(t), l_t \sigma_{l,2}(t) \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix} \quad (82)$$

i.e., the diffusion terms  $\sigma_S(t)$  and  $\sigma_l(t)$  are both of dimension  $1 \times 2$ . These processes are not assumed to be Markov processes. The state variable  $l$  as

in (35). As in Section 5.3 there is a utility function to be maximized subject to a wealth constraint, here referred to as a forward stochastic differential equation (FSDE)

$$\begin{cases} dW_t = (W_t[\varphi'_t(\nu(t) + r_t) - c_t]dt + W_t\varphi'_t\sigma(t)dB_t \\ W_0 = w. \end{cases} \quad (83)$$

Here  $\nu(t)$  is the vector of expected rates of return of the risky assets and labor income in (82) in excess of the riskless instantaneous return  $r_t$ , and  $\sigma(t)$  is the  $2 \times 2$  matrix of diffusion coefficients of the two stochastic processes, normalized by their respective values, so that  $\sigma(t)'\sigma(t)$  is an instantaneous covariance matrix. Prime means transpose of a vector. The state process  $X_t = W_t$ , i.e., defined by the wealth process  $W$ , in the setup for the stochastic maximum principle in Section 7.1

The backward equation is the as before and given by (54), so we still use recursive utility, where

$$U(c_t) = E\left(\int_0^T \tilde{f}(c_t, V_t, Z_t) dt\right). \quad (84)$$

The objective is to find equilibrium risk premiums, where  $c$  is the decision variables and  $V$  and  $Z$  are parts of the primitives of the model. The Hamiltonian of the system is given by

$$H(W_t, V_t, Z_t, \tilde{Y}_t, \pi_t, q_t, c_t) = \tilde{f}(c_t, V_t, Z_t)\tilde{Y}_t + (W_t[\varphi'_t(\nu(t) + r_t) - c_t]\pi_t + W_t\varphi'_t\sigma(t)q_t). \quad (85)$$

where the adjoint variables are  $\tilde{Y}_t$ ,  $\pi_t$ , and  $q_t$ . Here  $\tilde{Y}_t = Y_t + 1$ , where  $Y_t$  is the same as in Section 7.

The associated forward/backward systems for the adjoint processes  $\tilde{Y}_t$ ,  $\pi_t$  and  $q_t$  are

$$\begin{cases} d\tilde{Y}_t = \frac{\partial H}{\partial v}(t) dt + \frac{\partial H}{\partial z}(t) dB_t \\ \tilde{Y}_0 = \tilde{y}_0 > 0, \end{cases} \quad (86)$$

and

$$\begin{cases} d\pi_t = -\frac{\partial H}{\partial w}(t) dt + q_t dB_t \\ \pi_T = 0. \end{cases} \quad (87)$$

The first order condition in consumption  $c$  is  $\frac{\partial H}{\partial c} = 0$ , which means that

$$\pi_t = \tilde{Y}_t \frac{\partial}{\partial c} f(c_t, V_t, l_t), \quad (88)$$

and the equation for the adjoint variable  $\tilde{Y}$  is

$$\begin{cases} d\tilde{Y}_t = \tilde{Y}(t) \left( \frac{\partial \tilde{f}}{\partial v}(t, \tilde{c}_t, V_t, l_t, Z(t)) dt + \frac{\partial \tilde{f}}{\partial z}(t, \tilde{c}_t, V_t, l_t, Z(t)) dB_t \right) \\ \tilde{Y}_0 = \tilde{y}_0. \end{cases} \quad (89)$$

Hence the dynamics of  $\tilde{Y}$  is the same as the dynamics of  $Y$  of the last sections. This gives the same results as before regarding the recursive utility-based risk premiums and the real rate.

The equation for the state-price deflator  $\pi_t$  is

$$d\pi_t = -([\varphi'(t)\nu(t) + r_t]\pi_t + \varphi'(t)\sigma_t q_t)dt + q_t dB_t$$

The first order condition in the portfolio weights  $\varphi$  is  $\frac{\partial H}{\partial \varphi} = 0$ , which implies that

$$\nu_t \pi_t = -\sigma_t q_t. \quad (90)$$

In equilibrium the agent holds the asset, and receives the income from labor, so the optimal  $\varphi_t^* = (1, 1)$ . Using this we obtain

$$d\pi_t = -r_t \pi_t dt + q_t dB_t,$$

which has the right drift term ( $r_t = -\mu_\pi(t)/\pi_t$ ). Notice that this approach takes the equilibrium short rate  $r_t$  as given. From the condition (90) we get

$$-q_t/\pi_t = (\sigma'(t)\sigma(t))^{-1} \sigma'(t)\nu(t) \quad (91)$$

assuming the matrix  $(\sigma'(t)\sigma(t))$  is invertible. Since  $-(\sigma_\pi(t)/\pi_t)\sigma_R(t) = \mu_R(t) - r_t$  for any risky asset  $R$  with return rate  $\mu_R(t)$  and volatility  $\sigma_R(t)$ , we then have the following

$$\mu_R(t) - r_t = (\sigma'(t)\sigma(t))^{-1} \sigma'(t)\nu(t)\sigma_R(t). \quad (92)$$

Splitting up into components, this can be written

$$\begin{aligned} \mu_R(t) - r = (\sigma'(t)\sigma(t))^{-1} & \left( (\sigma_{S,1}(t)\sigma_{R,1}(t) + \sigma_{S,2}(t)\sigma_{R,2}(t))(\mu_S - r_t) \right. \\ & \left. + (\sigma_{l,1}(t)\sigma_{R,1}(t) + \sigma_{l,2}(t)\sigma_{R,2}(t))(\mu_l - r_t) \right) \end{aligned}$$

or

$$\mu_R(t) - r_t = \beta_S(t)(\mu_S - r_t) + \beta_l(t)(\mu_l - r_t) \quad (93)$$

where

$$\beta_S(t) = (\sigma'(t)\sigma(t))^{-1} \sigma'_S(t)\sigma_R(t)$$

and

$$\beta_l(t) = (\sigma'(t)\sigma(t))^{-1}\sigma'_l(t)\sigma_R(t)$$

which gives us a two factor model as before, but here with the coefficients represented by the familiar "betas".

The ICAPM can now be obtained as follows: Interpret the labor component as another risky asset, call them  $S_1$  and  $S_2$ , and omit labor from the analysis. Going back to equation (90), multiply both sides by the optimal  $\varphi^*(t)$  in this situation, which corresponds to the value weighted market portfolio. Then

$$\varphi^*(t)\nu_t\pi_t = -\varphi^*(t)\sigma_tq_t$$

This reduces to  $(\mu_M(t) - r_t)\pi(t) = -\sigma_M(t)q_t$  where  $M$  refers to the market portfolio. In equilibrium the agent holds precisely the value weighted market portfolio, so that  $\varphi^*(t)\sigma(t) = \sigma_M(t)$  and  $\varphi^*(t)\nu(t) = \mu_M(t) - r_t$ . This results in

$$\mu_R(t) - r_t = \beta_R(t)(\mu_M - r_t) \tag{94}$$

where

$$\beta_R(t) = (\sigma'_M(t)\sigma_M(t))^{-1}\sigma'_M\sigma_R(t).$$

This is the ICAPM. Our derivation shows that this model is valid also when the underlying processes are not of the Markov type.

This model is considerably more difficult to derive using the dynamic programming approach, which lies behind the original development of Merton (1973).<sup>8</sup> This model is in other words also true for recursive utility, but has a very different flavor from the version in (81), which is a full equilibrium model with an associated equilibrium interest rate. The latter can be used to tell us something about the people that populate the economy, a property which the model (94) lacks.

## 9 Conclusions

We have presented three different, but related production economies, one more elaborate than the other, which when viewed together, may shed some light on the the equity premium and the real interest rate of the last century. The standard mean-variance investment analysis is a guide that most investors understand; the elegant trade-off between expected return and 'risk'. As we have pointed out, this strategy is too simplistic when the 'real' economy is taken into account; the approach is not consistent with market clearing.

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<sup>8</sup>It is hard to find the exact derivation of this result in Merton's papers, since the papers tend to cross-refer to each other.

Even in the generality of the resulting production model, the classical approach with agents having additive and separable von-Neumann-Morgenstern expected utility functions leads to the CCAPM, which is unable to fit the data of Prescott and Mehra (1985), and other similar data sets from around the world.

Using stochastic differentiable utility in a production economy instead, the calibrations of the resulting model is demonstrated to be much more promising. Here we use a version based on the Kreps-Porteus class of utility functions. This model calibrates well to market and consumption data, with two 'factors' in the expression for the risk premiums. Our analysis also indicates that the one-factor CAPM model in a dynamic setting, with an associated equilibrium interest rate, may fit the data surprisingly well considering the model's simplicity. The basic usefulness of the CAPM is also supported by recent empirical literature.

We use the stochastic maximum principle when analyzing recursive utility. With this technique we derive both new results, and prove old ones, in particular a short proof for the ICAMP.

## Appendix 1

### Solution of the Bellman equation in the centralized economy

The conjectured solution of the Bellman equation (4) of Section 2.1 is ( $x = K_t$ )

$$J(x) = A \frac{1}{1-\gamma} x^{1-\gamma}, \quad J_x(x) = Ax^{-\gamma}, \quad J_{xx}(x) = -\gamma Ax^{-\gamma-1},$$

where  $A$  is some constant. Maximization in the Bellman equation gives

$$-J_x(x) + c^{-\gamma} = 0,$$

which implies that

$$c = (J_x(x))^{-\frac{1}{\gamma}},$$

or, in terms of the underlying random process, here the capital stock  $K$ , the optimal consumption takes the form

$$c_t = A^{-\frac{1}{\gamma}} K_t \quad \text{for all } t.$$



Inserting this conjecture into the Bellman equation reveals that our guess is successful in that the equation separates:

$$x^{-\gamma+1} \left( A(\mu_K - A^{-\frac{1}{\gamma}}) - \frac{\gamma}{2} A \sigma_K^2 - \frac{\delta A}{1-\gamma} + \frac{1}{1-\gamma} (A^{-\frac{1}{\gamma}})^{1-\gamma} \right) = 0$$

for all  $x > 0$ . Accordingly the constant  $A$  must satisfy the equation

$$\frac{\gamma}{1-\gamma} A^{\frac{\gamma-1}{\gamma}} + A \left( \mu_K - \frac{\gamma}{2} \sigma_K^2 - \frac{\delta}{1-\gamma} \right) = 0.$$

One solution is  $A = 0$ , which gives infinite consumption, and is thus not feasible. Dividing through by  $A$  we get

$$A = \left[ \frac{1-\gamma}{\gamma} \left( \frac{\gamma}{2} \sigma_K^2 + \frac{\delta}{1-\gamma} - \mu_K \right) \right]^{-\gamma}.$$

It follows that the optimal consumption is given by (5) as  $c_t = A^{-\frac{1}{\gamma}} K_t = \theta K_t$ , where  $\theta$  is given in (6).

An application of The Verification Theorem reveals that our conjectured solution solves the problem.

As for the transversality condition, we have to verify that

$$\lim_{T \rightarrow \infty} E \{ e^{-\delta T} |J(K_T^c)| \} = 0$$

As a consequence of what we just have shown, the capital stock  $K_t^{(c)}$  satisfies the following dynamics along the optimal consumption path:

$$dK_t^{(c)} = K_t^{(c)} (\mu_K - \theta) dt + K_t^{(c)} \sigma_K dB_t$$

which means that

$$(K_T^{(c)})^{(1-\gamma)} = K_0^{(1-\gamma)} \exp \left\{ (1-\gamma) (\mu_K - \theta - \frac{1}{2} \sigma_K^2) T + (1-\gamma) \sigma_K B_T \right\}.$$

Using the moment generating function of the normal probability distribution, we obtain that

$$E((K_T^{(c)})^{(1-\gamma)}) = K_0^{(1-\gamma)} e^{[(1-\gamma)(\mu_K - \theta) - \frac{1}{2} \gamma (1-\gamma) \sigma_K^2] T}.$$

From this it follows that the transversality condition is satisfied provided

$$(1-\gamma)(\mu_K - \theta) - \frac{1}{2} \gamma (1-\gamma) \sigma_K^2 - \delta < 0.$$

Denoting by

$$\alpha = -[(1-\gamma)(\mu_K - \theta) - \frac{1}{2} \gamma (1-\gamma) \sigma_K^2 - \delta],$$

it can be checked that  $\alpha = \theta$ . In other words, the transversality condition is satisfied if  $-\alpha < 0$ , which is equivalent to  $\theta > 0$ , as claimed in Section 2.1.

## Appendix 2

### Solution of the Bellman equation in the decentralized economy

The conjectured solution of the Bellman equation (22) of Section 3.3 is ( $w = W_t$ )

$$J(w) = B \frac{1}{1-\gamma} w^{1-\gamma}, \quad J_w(w) = B w^{-\gamma}, \quad J_{ww}(w) = -\gamma B w^{-\gamma-1},$$

where  $B$  is some constant. Using (23) this conjecture immediately leads to

$$\varphi = \frac{1}{\gamma} \frac{\mu_S - r}{\sigma_S^2},$$

which is (26). Next we find the first order condition for optimization in the variable  $c$ . From the Bellman equation it is seen to be

$$-J_w(w) + c^{-\gamma} = 0,$$

which implies that

$$c = (J_w(w))^{-\frac{1}{\gamma}}.$$

By our conjecture this means that in terms of the underlying stochastic process, here the agent's wealth, the optimal consumption takes the form

$$c_t = B^{-\frac{1}{\gamma}} W_t \quad \text{for all } t.$$

Inserting our candidate optimal portfolio rule and optimal consumption into the Bellman equation, we get the following

$$w^{-\gamma+1} \left[ B \left( \frac{1}{\gamma} \frac{(\mu_S - r)^2}{\sigma_S^2} + r - B^{-\frac{1}{\gamma}} \right) - \frac{1}{2} B \frac{1}{\gamma} \frac{(\mu_S - r)^2}{\sigma_S^2} - \frac{\delta B}{1-\gamma} + \frac{1}{1-\gamma} (B^{-\frac{1}{\gamma}})^{(1-\gamma)} \right] = 0.$$

Notice that this equation separates, indicating that our conjecture is promising. Since the constant  $B > 0$ , it is determined as

$$B = \left[ \frac{1-\gamma}{\gamma} \left( \frac{\delta}{1-\gamma} - r - \frac{1}{2} \frac{1}{\gamma} \frac{(\mu_S - r)^2}{\sigma_S^2} \right) \right]^{-\gamma}.$$

From this the optimal consumption is

$$c_t = (J_w(W_t))^{-\frac{1}{\gamma}} = (B W_t^{-\gamma})^{-\frac{1}{\gamma}} = \left[ \frac{1-\gamma}{\gamma} \left( \frac{\delta}{1-\gamma} - r - \frac{1}{2} \frac{1}{\gamma} \frac{(\mu_S - r)^2}{\sigma_S^2} \right) \right] W_t,$$

which is the solution (24) - (25) given in Section 3.3. Again we use The Verification Theorem to confirm that our conjectured solution solves the problem.

Finally, the transversality condition must be checked, and it holds provided  $\eta > 0$ , where the equilibrium restriction  $\mu_S - r = \gamma\sigma_S^2$  has been utilized.

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