The Structure of MMP-Elections

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Abstract

In MMP-elections (Mixed Member Proportional representation), a QP-ballot contains a *first-vote* for party Q's candidate in a single-seat constituency and a *second-vote* for a list of candidates from party P in one common tally. In *split* ballots P≠Q.

***Traditional accounting* does not record the ballot's *combination* of first- and second-vote; collecting them in separate ballot boxes will not change the result. In the case of Bundestag elections, the assembly size is out of control, with 111 (136) extra-ordinary list seats in 2017 (2021).**

***Faithful accounting* makes use of these combinations to stabilize the assembly size at some given norm (598 seats in the Bundestag), while still complying with MMP's *proportionality rule*.**

The Federal Constitutional Court emphasizes the principle of all voters' *equal influence* on the result. In 2017 and 2021 split QP-ballots often gave full support to two winners, but QQ-ballots only to one (Q=CSU). If two ballots are from the same constituency, faithful accounting, gives them equal influence.

Under traditional accounting, a fusion of the sister parties CDU/CSU to one party "C*U" would reduce the size to 667 (598) seats in 2017 (2021).

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THE STRUCTURE OF MMP-ELECTIONS

Introduction

Followng a *proportionality principle*, the Bundestag election 2017 achieved transparency: A column in Table 1 shows the results of elections in 299 single seat constituencies; CSU won 46 seats with their *first-votes*. Then 7 strong parties competed again for more seats with their *second-votes*. With 46 *"direct* seats*"* already won, CSU could not get more seats despite receiving 2,869688 second-votes:
How many seats should then SPD get with its 9,539381 second-votes?

The problem and its theoretical answer, i.e. 46 × 9539381/2869685 = 152.921... seats, is within reach in school classes early in their social science study. Here theory fits with the needed integer approximation: SPD got 153 seats, adding 94 "*list* seats" to its 59 direct seats won in the constituencies.

Similar calculations and addition give a theoretical 708.348... seais for the 7 parties, and the assembly got 709 seats.

Suppose our class next will analyse the 2013 election the same way. Five parties (CDU, SPD, Linke, Grüne, CSU) with 36,867417 second-votes competed for list seats. How many seats should they get when CSU got 45 direct seats and 3,243569 second-votes? *Mutatis mutandis,* the students calculate 45 × 36867417/3243569 = 511.484... seats. Knowing the law's "598 norm", they understand that 512 is not the final answer, that 598–512=86 more seats must be distributed, and that even CSU must get some list seats. The reasoning is correct, but the 2013 assembly got 631 seats, not 598: A complicated list seat allocation needed also 33 *extra-ordinary* list seats; thus, it violated the 598 norm.

Abandoning this complication improved transparency in 2017. In the next election, 2021, CSU won a direct seat in 45 constituencies and received 2,402826 second-votes. The same 7 parties received 42,360565

second-votes; by proportionality, the assembly theoretically should get 45×42360565/2402826 = 793.326... seats. However, the assembly got 735 seats, not 794. The reason is not obvious. A new law of 2020 kept 3 of CSU's direct seats outside the calculation. The reduction from 45 to 42 also reduces 793.326... to 740.438.... Pulling in the 3 "tax-free" seats again, the theory says 743.438... . A full explanation of the 735 is in (1.13).

Both the simplification in 2017 and the complification in 2021 made the assembly size closer to 598 than it would have been without any change. *"Faithful accounting"* is designed to satisfy a given norm for the assembly size while keeping the transparency obtained in 2017.

Faithful accounting makes use of ballot data that now are ignored: The report from the tally of first-votes in constituency C_k must be extended to include the numbers N(j,k) of voters who combine first-vote to the winner in C_k with second-vote to party P_j.

The natural principle of voters' equal influence on the outcome was emphasized by the German Federal Constitutional Court (2008). In many elections across the world, each ballot supports just one party list, and the principle is interpreted as proportionality. However, in Bundestag elections there are constituencies where the winner is supported by about 40 thousand voters. With about 40 million voters, equal influence then requires a Bundestag of about 1000 seats.

Excessive influence of a voter through the first-vote cannot be avoided without an intolerably high assembly size. Faithful accounting keeps an account for this excessive influence in each constituency where it occurs. Larger pluralities or majorities in the first-vote tallies are clearly of some importance. In elections with single-seat constituencies only, many voters choose among the *feasible* candidates, and candidates depend on a broad support (i.e. the Duvergerian mechanism). However, MMP's second-vote is a fallback security if the first-vote fails to win.

With no ballot change, a majority tally method *"W-U"* is designed to stimulate voters' interest in their constituency elections.

1. Traditional accounting

(1.1) ***Notation*** **In *mixed member elections* to a legislature, a member is elected either**

as winning a single-seat tally in a constituency C_k ($1 \le k \le c$), or
as getting a seat won by a list of candidates from a party P_j ($1 \le j \le p$).

Each ballot contains two votes;
a first-vote, *ErSt* (Erststimme) in the tally of the voter's constituency, whence the winner qualifies *directly* to the legislative assembly, and
a second-vote, *ZwSt* (Zweitstimme) for the *candidate list* of one party in a final ZwSt tally.

Usually, a direct seat is won through the Plurality method "first-past-the post", but various preferential methods may also be considered for a constituency election.

Usually, in order to qualify for the final ZwSt tally, P_j must

either, with its $z(j)$ ZwSt obtain a certain share of *all* z^* ZwSt,
or, with its ErSt win a certain number $\omega(j)$ of direct seats.

(In Bundestag elections, the criterion is either $z(j)/z^* \ge 0.05$, or $\omega(j) \ge 3$.)
Parties are here enumerated in order of their $z(j)$ so that, when r parties qualify, then P_j qualifies if $1 \le j \le r \le p$; see Table 1.

(1.2) ***Proportionality*** **In MMP (Mixed Member Proportional), one goal for the ZwSt tally is to distribute $\alpha(j)$ party *list seats* to P_j *in addition to* the $\omega(j)$ direct seats that P_j won through its ErSt, so that**

$\beta(j) = \alpha(j) + \omega(j)$ is *proportional* to $z(j)$, with ratio ρ:

(1.3) $$z(j) = \rho \times \beta(j), \quad i \le j \le r, \text{ i.e.}$$
$$\alpha(j) = z(j)/\rho - \omega(j)$$

In solutions $(\rho, \beta(1), \ldots, \beta(r))$ of (1.3), ρ is a real variable. The $\beta(j)$ are almost never integers. An *approximation method* distributes seats one-by-one to parties P_j, $1 \le j \le r$, with $z = z(1)+z(2)+\ldots+z(r)$ ZwSt. When P_j at a stage has $\beta(j) = s-1$ seats, it may contest for its s^{th} seat P_j-s with a "*contest number*",

(1.4) $$T(j) = s \times z/z(j)$$

The idea is that when P_j gets s seats, r parties together should get T(j).

After P_j gets its $(s-1)^{th}$ seat, it contests for its s^{th} seat with increased *Tribus value* T(j) in (1.4) as its next contest number: Increased T(j) from P_j usually lets some other parties get new seats before P_j. See Table 2.

Let seat distribution start from $\beta(j) = 0$, $1 \le j \le r$.
Let $\alpha(j) = \max[0, \beta(j) - \omega(j)]$, i.e. so that P_j gets all its $\omega(j)$ direct seats *re-distributed* first, and thence get only list seats.

From early stages, the approximation to (1.3) is good. While P_j has not got all $\omega(j)$ direct seats re-distributed, it still has *overhang* seats (i.e. more seats than proportionality entitles it to). Direct seats to some parties come interspersed between list seats to other parties. Thus, in Table 2, CSU-46 is the only direct seat, re-distributed as seat nr. 705.

(1.5) *Pivotality* **In (1.3), ρ appears as a *price* (in ZwSt *currency* per seat), which P_j is *committed to pay* for each of its $\beta(j)$ seats.**

In non-negative solutions $(\rho, \beta(1), \ldots, \beta(r))$ of (1.3), a lower price ρ defines a new solution with increased $\beta(j)$. *Raising* ρ to its maximum $\rho=\rho^*$, gives (at least) one 0-entry $\alpha(j)$ in the solution vector:

(1.6) **a *pivotal* party, P_{piv} with $\alpha(piv) = 0$ determines ρ^* by (1.3);**
(1.7) $$z(piv) = \rho^* \times \omega(piv)$$

Table 1 shows that CSU is pivotal in both elections.

(1.8) *Assembly size* **The ZwSt tally adds h party list seats to the c constituency winners; the *assembly size* is then h+c seats:**

$$h = \alpha(1) + \alpha(2) + \ldots + \alpha(r); \quad c = \omega(1) + \omega(2) + \ldots + \omega(r)$$

The size is determined by proportionality and restrictions on the number of list seats; Bundestag elections require h≥c. In 2017, h–c is just high enough to avoid direct seats in overhang: c=299, h=410; see Table 2.

(1.9) *Critical assembly size* **This is the smallest size that allows proportionality; it occurs for $\rho=\rho^*$: Aggregation over j in (1.3) and (1.7) give the critical size:**

$$h + c = z/\rho^* = z/[z(piv)/\omega(piv)] = \omega(piv) \times z/z(piv), \text{ and } \alpha(piv) = 0.$$

(1.10) P_{piv} **is seen as *pivotal for the critical size*, since, by (1.9), this size is found from data *specific* for party** P_{piv}**:**

P_{piv}**'s *ErSt success*, i.e.** ω**(piv) constituency winners, together with** P_{piv}**'s *relative ZwSt support*, i.e.** z**(piv)/**z**.**

In the Bundestag elections 2017 and 2021, r=7 parties qualified for the final ZwSt tally: SPD, CDU, Grüne, FDP, AfD, CSU, and Linke.

As long as the one-state party CSU (= P_7 **in 2017) is the largest in Bavaria, and wins most of the 46 direct seats, it is likely to remain pivotal. According to Table 1 (2017), the *critical size* was, by (1.9):**

(1.11) $\omega(CSU) \times z/z(CSU) = 46 \times 44189959 / 2869688 = 708.348...$

	BUNDESTAG 2017					BUNDESTAG 2021				
j	ErSt	cons	list	ZwSt	P_j	ErSt	cons	list	ZwSt	j
2	11,429231	59	94	9,539381	SPD	12,228363	121	85	11,949756	1
1	14,030751	185	15	12,447656	CDU	10,445571	98	53	8,770980	2
6	3,717922	1	66	4,158400	Grüne	6,465502	16	102	6,848215	3
4	3,249238	0	80	4,999449	FDP	4,040783	0	92	5,316698	4
3	5,317499	3	91	5,878115	AfD	4,694017	16	67	4,802097	5
7	3,255487	46	0	2,869688	CSU	2,787904	45	0	2,402826	6
5	3,966637	5	64	4,297270	Linke	2,306755	3	36	2,269993	7
∑	44,966765	299	410	44,189959		42,968900	299	435	42,360565	∑

TABLE 1 In the cons columns, $\omega(j)$ **measures** P_j**'s ErSt success in winning direct seats. All 299 direct seats are accounted for in each election. Also a party** P_j **not qualified, i.e. r<j, may win 1 or 2 constituencies; in 2021 Linke qualified with 3. SSW got 1 special list seat outside this tally.**

(1.12) ***Approximation, 2017 data*** **A ZwSt tally distributes seats by proportionality. At each stage** P_j **here contests for its next,** s^{th} **seat** P_j**-s with its current T(j), see (1.4). The last 10 stages are in Table 2:**

700	701	702	703	704	705	706	707	708	709
CDU-198	SPD-152	CDU-199	AfD-94	FDP-80	CSU-46	SPD-153	Lin-69	CDU-200	Grü-67
702.913	704.121	706.462	706.665	707.117	708.348	708.753	709.545	710.013	711.987

TABLE 2 P_j **contests for its** s^{th} **seat** P_j**-s with its current *Tribus* value T(j) =** $s \times z/z(j)$**. The party with the smallest current Tribus gets the next seat.** P_j **is *entitled by proportionality*** **to** P_j**-s when all r parties together, with** z **ZwSt, have passed T(j) seats.**

Proportionality *"legitimizes"* CDU-198 *after* 198×z/z(CDU) ≈ 702.913, i.e. when 703 seats are distributed, SPD-152, CDU-199, and AfD-94 included.

***Critical size* is Tribus 708.348… of CSU-46, i.e. the *last direct seat* for the *pivotal party*. At 709, the *list seats* Lin-69, CDU-200, and Grü-67, finally *"compensates"* for the direct CSU-46, but are themselves left in a legal *overhang*. (Seats 710, 711, and 712 could "rescue" them, but would create new overhangs. The Tribus of CSU-46 tells when to stop W/S-L.**

However, the W/S-L approximation (Webster, 1832; Sainte-Laguë, 1910) is requested by law in Bundestag elections, and also deals out seats one-by-one. Party P_j contests for its s^{th} seat P_j-s with z(j)/[2s–1] as *contest number*. The inverses are linear in s, like the T(j).
However, the link to proportionality is not visible in Table 3 which is according to W/S-L, but obvious in Table 2 according to Tribus. [1]

The sequences are slightly different, but incidentally give the same seat distributions for the 2017 data. A 9-seat tail of *compensation seats* after CSU-46, with W/S-L contest numbers, just passes the 708.348… point:

700	701	702	703	704	705	706	707	708	709
CSU-46	CDU-198	SPD-152	FDP-80	AfD-94	Lin-69	CDU-199	SPD-153	Grü-67	CDU-200
31535	31513	31483	31443	31434	31367	31354	31277	31266	31197

TABLE 3 W/S-L gives another sequence than Tribus does (Table 2), but in this case they give the same seat distribution at turnoff.

Tribus values T(j)=s×z/z(j) have a traditional normative foundation and offer an alternative to W/S-L. In Table 2, P_j contests for its s^{th} seat P_j-s with its current T(j): From schooldays, generations understand the problem behind a Tribus b×c/a, either through equation a/b = c/x or verbally: If a voters get b seats, how many seats should c voters get?

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[1] ***d'Hondt's me*thod of 1872 distributes the P_j-s by decreasing z(j)/s; Tribus produces the same sequence from increasing T(j)=s×z/z(j). The history of apportionment methods used to determine the number of seats for each state in the US House of Representatives contains many ideas, going back to Jefferson in 1792 (Balinski and Young 2001).**
W/S-L and d'Hondt produce unending sequences in a simple way. With the distinction between direct and list seats, it is good for transparency to treat the stopping rule separately.

Regula de Tribus (Dreisatz, Rule of three) is a simple tool, but usehful to construct and analyze MMP-elections, structures of some complexity.

(1.13) ***Free overhang seats in 2021*** **The critical size in 2021 was a new record, 85 seats above the one from 2017. By (1.9) and Table 1,**

(1.14) $\omega(\text{CSU}) \times z/z(\text{CSU}) = 45 \times 42360565 / 2402826 = 793.326\ldots$ **seats**

A new law of October 2020 gave a smaller increase than (1.14) indicates. Its three "mechanisms to contain the Bundestag's size" were assessed by Behnke (2020), who expresses doubt (based on simulations) about their efficiency, and about their constitutionality.

The law abandoned the principle that all constituency seats must be paid with ZwSt. Thus, the pivotal CSU got its *commitment reduced* by a quota of 3 direct seats, *from its ErSt success* ω(CSU) = 45 down to 42.

Thus, CSU keep 3 seats in overhang; with ω(CSU) = 42, critical size is:

(1.15) $\omega(\text{CSU}) \times z/z(\text{CSU}) = 42 \times 42360565/2402826 = 740.438\ldots$ **seats.**

W/S-L lets CSU-42 come as seat 731, but CSU's ZwSt resource is small for its commitment. Ten compensatory list seats reduce the price; they just cross the new critical mark from (1.15):

731	732	733	734	735	736	737	738	739	740	741
CSU-42	CDU-152	SPD-207	Grü-119	SPD-208	CDU-153	AfD-84	FDP-93	Lin-40	SPD-209	CDU-154
28950	28947	28934	28895	28795	28757	28755	28739	28734	28656	28570

TABLE 4 **Despite the mandatory W/S-L, the official result in 2021 was assembly size 735 seats. The main explanation is that the proportionality rule was abandoned: After re-distribution of CSU-42 at seat 731, the new Tribus 740.438… of (1.15) called for 10 more compensatory seats in order to legitimize CSU as entitled to CSU-42. They were not distributed; see Table 1. Thus, CSU kept *four* overhang seats: CSU-45, CSU-44, CSU-43 (not re-distributed), *and* CSU-42 (re-distributed as seat 731 but not paid, i.e. *not fully legitimized* by compensatory seats).** [2]

[2] **CSU-41 was paid at seat 723. After seat 731, the assembly had 731+3 seats, i.e. closer to 741 than to 723. Another "mechanism" is reduction of the number of constituencies, favored by Weinmann & Grotz (2020).**

SSW (Südschleswigscher Wählerverband) won 1 special seat representing a Danish and a Frisian minority: The assembly got (730+1)+3+1 seats.

(1.16) ***List seat distribution after re-distribution*** **After P_j has received all its ω(j) *direct* seats from W/S-L (i.e. in *re-distribution*), it contests for its g^{th} *list* seat with contest number**

(1.17) $$z(j)/\{2\times[\omega(j)+g] - 1\} = z(j)/\{2\omega(j) + [2\times g - 1]\},\ g \geq 1$$

Tables 2, 3, and 4 are stretches from a long sequence where direct seats already won come interspersed between list seats. One may use the second version in (1.17) with g>0 only and get a shorter sequence with the list seats only. This is natural under *faithful accounting*, which replaces the ω(j) by commitments that are not integers.

(1.18) ***A small pivotal party*** **"Sister parties" CDU and CSU work as one party in the Bundestag, but CSU runs for election only in Bavaria and CDU runs only in the 15 other states. The political structure in Bavaria is not what it would be in a state similar to a downscaled nation.**

CSU has a particularly small ZwSt supply to pay for commitments. CDU however, receives ZwSt from many constituencies where other parties won in 2021 and reduced CDU's commitment. Thus CDU in 2021 had an abundant ZwSt supply to pay for CDU-winners in the ErSt tallies.

Counterfactually, imagine that sister parties CDU/CSU run as one party C*U in 2017 and 2021, with 2017 rules: C*U gets all direct seats and ZwSt that Table 1 shows for CDU and CSU. C*U is clearly pivotal:

In 2017, C*U gets 12447656 + 2869688 = 15317344 ZwSt, and
185 + 46 = 231 direct seats. By (1.9), critical size is

$$231 \times 44189959 / 15317344 = 666.426\ldots \text{ seats, see (1.11);}$$

this is down from 708.348... seats in the real election; see also Table 5.

In 2021, C*U gets 8770980 + 2402826 = 11173806 ZwSt, and
98 + 45 = 143 direct seats. By (1.9), critical size is

143 × 42360565 / 11173806 = 542.122... seats;

this is down from 793.326... seats in the real election; see (1.14).

With critical size 542.122... seats, W/S-L follows up with 55 more *list* seats until the 598 ordinary seats are filled. Not being for compensation, 14 or 15 of them come to C*U, and some of them to Bavaria. Pivotal status is, at best, a mixed blessing for a small party.

However, critical sizes 666.426... and 542.122... for a pivotal C*U in consecutive elections, where the same six parties qualify for the final ZwSt tally, indicate that high volatility will be a significant feature even if only a nationwide party can obtain pivotal status.

With faithful accounting, the assembly size is pre-determined, e.g. at h=c=299, and the critical size will be smaller than this. Thus, C*U and CDU/CSU give different critical sizes but the same assembly size.

2. Faithful accounting

The first theme, in (2.1-2), is information faithful accounting needs from the ErSt tally in constituency C_k: numbers N(j,k) of voters in $\Lambda(P_j)$ (i.e. with ZwSt to party P_j) who give ErSt to the *winner*, and their sum E(k). ξ(j) measures the ErSt success of $\Lambda(P_j)$; $f = \xi(r+1)+\ldots+\xi(p)$ is the ErSt success of voters with ZwSt to a party that did not qualify for list seats.

For an assembly with h list seats and c direct seats, $q = h+c-f$ must be paid by the z voters in those $\Lambda(P_j)$ that contest for list seats. See (2.3-6).

The purchasing power is q/z seats/ZwSt. A voter with ErSt to the winner in C_k brings to $\Lambda(P_j)$ a success 1/E(k) and a commitment $\min[1/E(k), q/z]$. The commitment is limited so that the ballot does not increase $\Lambda(P_j)$'s commitment more than its purchasing power.

Consequences of the definition of commitment are in (2.6-10); C_{153} and C_{216} are illustrating examples from 2021.

The full unpaid ErSt success is σ in (2.11-12). A majority method in the ErSt tallies will raise the E(k) and reduce σ. The W-U method is a simple possibility suggested for consideration; see (2.13-14).

(2.1) *Notation* **$\Lambda(P_j)$ is the set of $z(j)$ voters who support party P_j with their ZwSt ($1 \le j \le p$); let N(j,k) be the number of them with ErSt to the winner in C_k. The constituency seat was won with a plurality of E(k) ErSt, here expressed by aggregation over parties:**

$$E(k) = N(1,k) + N(2,k) + \ldots + N(p,k).$$

Each of these E(k) voters wins one share, 1/E(k) of a seat, and

$N(j,k)/E(k)$ is a measure of $\Lambda(P_j)$'s (local) *success* in C_k.

By another aggregation, over constituencies, ξ(j) measures the (full) ErSt *success* for $\Lambda(P_j)$:

$$\xi(j) = N(j,1)/E(1) + N(j,2)/E(2) + \ldots + N(j,c)/E(c) \text{ seat shares.}$$

All constituency seats are now accounted for, i.e.

$$\xi(1) + \xi(2) + \ldots + \xi(p) = c \text{ seats.}$$

ξ(j) measures the ErSt success of $\Lambda(P_j)$, like ω(j) measures that of P_j.

(2.2) ***ZwSt failure but ErSt success*** **The ErSt success belongs to *"supporters clubs"* $\Lambda(P_j)$ with ZwSt to P_j, but some did not qualify for the final ZwSt tally ($r < j \le p$). Their ErSt success amounts to f seats;**

$$f = \xi(r+1) + \dots + \xi(p)$$

Thus, under faithful accounting, the full ErSt success which influences the distribution of list seats in the ZwSt tally is $c - f$.

The f-value requires N(j, k)-values and more work in the ErSt tallies. [3]

For a crude estimate of f, notice that in the 2017 election, *at least* 44966765 – 44198959 = 767806 ErSt came from $\Lambda(P_j)$ with $r < j \le p$. In the constituencies, 44966765 ErSt were counted; by proportionality 767806 ErSt should be expected to win f seats, where

$$f \approx 299 \times 767806/44966765 \approx 5.1 \text{ direct seats.}$$

Some entries in Table 1 are obviously meant for P_j and not for $\Lambda(P_j)$. Two examples from 2017 concern FDP and CSU:

- **Λ(CSU)'s ErSt success in 2017 is much less than 46: CSU won *all direct seats*; 3255487 ErSt support *winners*, but Λ(CSU) has 2869688 members.**
- **FDP had 0 success but members of Λ(FDP) with ErSt to a winner (many from CDU/CSU) won shares adding up to a significant success.**

(2.3) ***Individual success and commitment*** **The cons columns in Table 1 have success/commitment accounts for the P_j, $1 \le j \le r$. Under faithful accounting, each account is split in two: one for the success $\xi(j)$ and one for the commitment of the supporters club $\Lambda(P_j)$. The ZwSt tally distributes $q = h+c-f$ seats to the P_j, through its supporters in $\Lambda(P_j)$, $1 \le j \le r$. The price is z/q ZwSt per seat, i.e.**

the purchasing power is q/z seats per ZwSt.

If a ballot carries success $1/E(k)$ as *an increment to the commitment account* for $\Lambda(P_j)$, and if also $q/z < 1/E(k)$, then the extra ZwSt for P_j *does not compensate for the increased commitment* on $\Lambda(P_j)$'s account.

Obviously, $E(k) < z/q$ is likely to occur in many constituencies, and then

[3] N(j,k) is ballot information ignored with traditional accounting; one might as well collect ErSt and ZwSt in separate ballot boxes. The tallies in 2017 and 2021 would still have been the same as in Table 1.

each ballot with ErSt to the winner in C_k *harms the party it supports with its ZwSt.* Called *Negatives Stimmgewicht* (negative vote weight), this was in 2008 declared unconstitutional by the Federal Constitutional Court (Bundesverfassungsgericht). The mechanism behind it was more complicated than the one described here. For a party involved, it usually made a difference of ± 1 seat.

(2.4) ***Pre-determined assembly size*** **Faithful accounting is based on a *chosen value* for q in (2.3), i.e. a number of seats to be distributed.**

Definition: **A ballot which carries an ErSt success 1/E(k) from C_k, *carries a commitment* min[1/E(k), q/z] to the account of its $\Lambda(P_j)$.**

Thus, if E(k) > z/q, a successful ballot from C_k covers a commitment 1/E(k), and brings a *surplus* q/z – 1/E(k) to buy its P_j a part of a list seat. This surplus in each of E(k) successful ballots from C_k helps their various supporters clubs $\Lambda(P_j)$ to buy fractions of list seats for P_j, amounting to

(2.5) $E(k) \times [q/z - 1/E(k)] = E(k) \times q/z - 1$ **list seat fractions.**

Similarly, if E(k) < z/q, a successful ballot from C_k covers commitment q/z, leaving a part 1/E(k) – q/z of a direct seat as *unaccounted* success. For E(k) successful voters from C_k, the success *not to be paid* is,

(2.6) $E(k) \times [1/E(k) - q/z] = 1 - E(k) \times q/z$ **of one *direct seat.***

A ballot with ErSt to the winner of C_k increases both the purchasing power and the commitment of its $\Lambda(P_j)$. If E(k) < z/q, the two increments are equal; if E(k) > z/q, the purchasing power increases most. Therefore,

- **no ballot has Negatives Stimmgewicht;**
- **when q seats are distributed, the critical size is already passed.**

(2.7) ***Example*** C_{153} **(Leipzig II) is in 2021 won by Linke's candidate with a plurality E(153) = 40927 ErSt (22.81%). With q = 598 – *f* in the Definition from (2.4) and with the estimate in (2.2),**

$$\min[1/E(153), 592.9/42360565] \approx \min [1/40927, 1/71446].$$

Successful voters from C_{153} leave, by (2.6), an unaccounted share

1 – 40927/71446 of a direct seat, i.e. as *unpaid*.

(2.8) ***Example*** **C_{216} (Ingolstadt) is in 2021 won by CSU's candidate with a plurality E(216) = 83663 ErSt (44.93%). With q = 592.9,**

$$\min[1/E(216), 592.9/42360565] \approx \min [1/83663, 1/71446].$$

Successful voters from C_{216} allow, by (2.5), their $\Lambda(P_j)$s to purchase a total of 83663/71446 – 1 of a list seat.

(2.9) ***Faithful accounting and seat distribution*** **As a measure of *ErSt success*, P_j's $\omega(j)$ is replaced by $\Lambda(P_j)$'s $\xi(j)$, which is used to calculate *f* in (2.2). As a measure of *commitment* P_j's $\omega(j)$ is replaced by $\Lambda(P_j)$'s commitment $\psi[\Lambda(P_j)]$, an aggregate over the C_k; see (2.1) and (2.4):**

(2.10) $$\psi[\Lambda(P_j)] = \sum_k N(j,k) \times \min[1/E(k), q/z], \quad 1 \le k \le c.$$ [4]

Generally, these new commitments are not integers; it is natural to use the W/S-L version of (1.7) and distribute list seats only, g≥1.

(2.11) ***Small E(k)*** **Aggregating the unaccounted fractions of direct seats, see(2.6), over the C_k gives the total, σ, of direct seat shares that are kept outside the commitment accounts in order to avoid a new kind of Negatives Stimmgewicht:**

(2.12) $$\sigma = \sum_k \max [1 - E(k) \times q/z, 0]$$

Critical size is reached before q seats have been distributed. Preferably, q should be set low enough to keep critical size below the norm (598 seats in the Bundestag). Voters with ErSt to the winner of a C_k where E(k)<c/q influence the outcome more than proportionality ideally should allow. E(k) may be small due to small population, to low turnout, but also to three or more front runners with about equal support.

E(k) may be increased by a majority method, like 2-day elections (a "primary" and a "general" part, i.e. a final between primary winner and runner-up) or an elimination method (e.g. Instant Runoff Voting). One alternative is an "instant 2-day election" W-U with the same ErSt and the same winner as the present first-past-the-post method:

[4] Negatives Stimmgewicht would still disappear in ballots from C_k if list seats were distributed until "break-even", i.e. E(k)=z/q. With data from (2.7), C_{153}, the solution is q≈1035 seats.

(2.13) The "W-U" majority method: **This requires three numbers of ErSt: w, u, and a for winner W, runner-up U, and All others together, respectively. Ballots with ErSt to candidates *not* in {W,U}**

count as half an ErSt for W and half an ErSt for U.

This is *"symmetrizing"*, a common principle for treating indifference in ranked choice voting. The w=E(k) plurality is joined by a half-voters. An

- **ErSt to W carries commitment min{2/[2E(k)+a], q/z}, see (2.4);**
- **ErSt to U carries 0 commitment;**
- **ErSt to Any other carries commitment min{1/[2E(k)+a], q/z}.**

(2.14) In Example (2.7), where W=Linke, U=Grüne,

w = E(153) = 40938, u = 32995, a = 105526; and after use of W-U,
w + a/2 = 93701 count for W; u + a/2 = 85758 count for U.

W-U is a *majority* method since all ballots or half-ballots count for W or U. It is likely (but not certain) that E(k)+a/2 passes the "benchmark" z/q in (2.4); in C_{153} 40938 voters then bring a tiny surplus to their $\Lambda(P_j)$. Moreover, W-U reduces σ only in C_k where E(k) < z/q, see (2.12); one may consider using it only in such constituencies.

(2.15) Voters' adapation to W-U **The W-U method is an incentive to act in the constituency election like many voters do in ordinary single-seat elections based on first-past-the-post: They support a *feasible* candidate, often one of two front runners. Voters who support W win "gold", i.e. one seat, and often with some surplus for their chosen Λ(P). But "silver" to those with their original ErSt to U will also be attractive, as their ballot does not carry any commitment to its Λ(P).**

An idea for voters with ZwSt to party P is *strategic voting*, giving ErSt to party Q with the *expected* runner-up candidate, in order to join Λ(P) with a ZwSt carrying 0 commitment. But, most likely, the idea will be commonly known and therefore too risky: Q's candidate may get the W-role instead of the U-role, and the ZwSt loses most of its value for P.

3. Background and discussion

(3.1) ***Proportionality and concentration*** **The history of MMP starts in West-Germany in the late 1940s. Those who made the new electoral system remembered the political fragmentation of the Weimar Reichstag under a proportionality rule alone. They wanted proportionality together with concentration around parties with broad local support.**

Split ballots **(i.e. with ErSt and ZwSt to different parties) are allowed from 1953. Some voters find it natural to give ErSt to party Q with the most acceptable among the front running candidates and ZwSt to their best liked party P, even when P≠Q. Larger plurality E(k) improves the winner's legitimacy as the representative of C_k.**

The common first-past-the-post voting method is much critizised, but defended e.g. by Dowding and Van Hees (2008). Anyway, Duverger's mechanism should strengthen the concentration part of MMP. Moreover, the impact of the ballot's ZwSt compensates for a failed ErSt.

Unfortunately, traditional accounting ignores the ballot's combination of of P and Q. Waste of this information has two serious side effects.

Most conspicuous in 2017 is the effect of voters with ErSt to the pivotal CSU, who joined Λ(FDP), say (ZwSt to a former coalition partner which, with 4.765% of the ZwSt, had failed to qualify for list seats in 2013).

A structural feature of the Bavarian political landscape is the difference between ErSt and ZwSt received by CSU: More than 380 000 *split ballots* had ErSt to CSU in both elections of Table 1: If 380 000 left Λ(CSU), they reduced CSU's ZwSt-supply *z*(CSU) in 2017, from 3249688 to 2869688, and raised the critical size from 625.517... to 708.348... seats. [5]

[5] In 2017 CSU had 385799 more ErST than ZwSt; in 2021 the difference was 385078. The relatively large loss of ZwSt from 2017 to 2021 was about equally important, and critical size became 793.326… ; see (1.14). Also z(CDU) was reduced when voters with ErSt to CDU split the ballot and joined Λ(FDP) or other supporters clubs, but CDU was far from taking over the role as pivotal party in 2017: CSU was pivotal all the way.

The new law of 2020 allowed CSU 3 direct seats *in overhang* (i.e. not paid with ZwSt). Applied *counterfactually* on 2017 data, the law gives critical size 662.151... when ω(CSU) drops from 46 to 43 direct seats in (1.11).

There was no obvious need to rescue FDP in 2021, but the experience from 2017 made many voters aware that the supporters club Λ(CSU) would be too small to give CSU list seats, and *z*(CSU) decreased again. C_{217} (München-Süd) was lost to Grüne in 2021: this reduced commitment ω(CSU) just enough to keep critical size below 800 seats; see (1.14).

In 2021 three direct seats (CSU-45, CSU-44, CSU-43) are allowed in overhang (i.e. *unpaid*); by Table 4 however, 10 compensatory seats are needed to legitimize CSU-42, passing the new critical size in (1.15). Also these 10 seats were waived, and so W/S-L was turned off after seat 731.

The other side effect is *doubled influence*: Two voters, A and B give their ErSt to the same constituency winner from the pivotal party CSU. A stays in Λ(CSU), contributing one ZwSt to pay CSU's commitment. Moving from Λ(CSU) to Λ(FDP) say, B runs away from the bill, but pays one ZwSt for a share in one more list seat to FDP. Thus, A is in the electoral basis for *one* seat winner; B for *two* seat winners.
Germany's Federal Constitutional Court, in a ruling of July 3rd 2008 on Negatives Stimmgewicht, states in para 92, *obiter dictum,* a much broader principle of *equal influence*. [6]

Faithful accounting lets commitment follow each ballot with successful ErSt, and is here considered as a remedy against these two effects.

(3.2) ***Two balance norms*** **One norm is to have a pre-determined number h of list seats (h=c in the Bundestag). Another norm is the proportionality in (1.3): The assembly must at least have critical size. Traditional accounting gives voters with ErSt to a pivotal party with**

[6] **Aus dem Grundsatz der Wahlgleichheit folgt für das Wahlgesetz, dass die Stimme eines jeden Wahlberechtigten grundsätzlich den gleichen Zählwert und die gleiche rechtliche Erfolgschance haben muss. Alle Wähler sollen mit der Stimme, die sie abgeben, den gleichen Einfluss auf das Wahlergebnis haben.**

small ZwSt supply, an incentive to split the ballots instead of "wasting" a ZwSt: Then seats are distributed until proportionality at a large h–c. The votes from Bundestag elections of 2013, 2017, and 2021, all treated with the simple, transparent rules of 2017, give critical sizes that illustrate this. CSU is pivotal in all. Critical sizes are shown also with CDU/CSU as an imagined united union C*U, which will then be pivotal:

	2013	2017	2021
CSU pivotal	511.484...	708.348...	793.326...
C*U pivotal	478.970...	666.426...	542.122...

TABLE 5 Six critical sizes in three Bundestag elections:
In 3 cases, after critical size is passed, W/S-L distributes 86, 119, and 55 *ordinary list* seats, while keeping proportionality, in order to satisfy the h=c norm (512+86 = 479+119 = 543+55 = 598): All parties get list seats.
In the 3 other cases, critical size is reached by distributing 111, 196, and 69 *extra-ordinary list* seats (709–111 = 794–196 = 667–69 = 598), thereby violating the h=c norm. The pivotal party gets no list seats.
Critical size gets clearly smaller when C*U gets the pivotal role after CSU, but under faithful accounting *both* will be below the chosen assembly size, e.g. 2c = c+h; q=c+h–*f* seats are then distributed, see (2.4).

However, in the real election of 2013, W/S-L went on with 33 *extra-ordinary* list seats for another reason: Complicated rules for allocating each party's list seats (across the 16 member states) required more of them. The Bundestag got 631 seats. Used counterfactually on 2013 data, the amendment of 2017 gives the 2013 Bundestag 598 seats; and it must have prevented a 2017 Bundestag far above 709 seats.

Even so, the reduced ZwSt for the pivotal CSU, from 3243569 to 2869688, caused a record size in 2017. It was linked to ballot splitting with ZwSt support to push FDP above 5% again and qualify for list seats.

In 2021 CSU lost even more ZwSt, see Table 1. With 2017 rules the Bundestag would get 794 seats. However with new and complifying ad-hoc rules, W/S-L was stopped at 731 seats. A C*U would have been a more efficient remedy: Of the 598–543 = 55 last list seats also C*U would get its proportional share, and some of them even in Bavaria.

(3.3) ***Proportionality and equal influence*** **Faithful accounting makes a situation similar to the three cases of Table 5 with critical size below 598; proportionality will always be obtained without violation of the h=c norm.**

Under faithful accounting, commitments are not based on the ErSt success for the party P_j, but based on the ErSt success 1/E(k) for members of $\Lambda(P_j)$. Therefore faithful accounting is also a step towards equal influence of all voters. The need to avoid Negatives Stimmgewicht imposes a smaller commitment than 1/E(k), as defined in (2.4).

In (2.12), σ shows the amount of ErSt success not accounted in the new Table 1. Increasing the low E(k) will include more ErSt success in the commitments; see (2.4). Without changing the voting rules, W-U is intended to achieve this by stimulating voters' interest in how to use their ErSt. One may consider using W-U only if E(k)<z/q.

(3.4) ***Robustness of MMP*** **Through all changes of rules and despite flaws, the German MMP variatons show an essential robustness: When it was difficult to establish a government with parliamentary basis on the left or on the right, then there was a basis for a Grand Coalition, CDU/CSU and SPD. Representatives elected directly from their own constituency, preferably with a broad electoral basis, have a special legitimacy when they support that solution.**

Proportionality and concentration together give the theme *"ideology vs. pragmatism"* a place in political discourse, on level with the ubiquitous *"left vs. right"*. Despite a lopsided distribution of 410 list seats in 2017 (435 in 2021) vs. 299 direct seats, CDU/CSU and SPD have a majority including 290 (264) of the 299 direct seats.

(3.5) ***Other MMP-variations*** **Regarding the German experience as positive, some other countries consider, use, or have used their own variations on the MMP-theme. Their experiences however, have not always been good (Linhart & al, 2019).**

Mixed member methods may be of the parallel kind, i.e. there is no connection like (1.3) between constituency and party list tallies: Parties with a sub-proportional share of constituency seats cannot reach a proportional assembly share.

Hungary's single-vote variation of MMP transfers, from the first to the second tally, "surplus votes" for the winner and votes "wasted" on losing candidates (Csato, 2016).

(3.6) ***Public perception of MMP*** **What voters know about the properties of their election methods, what they should understand, and what they would like to understand, are themes that raise special questions when the method is MMP.**

One theme is the (lack of) control of the assembly size. The extra-ordinary list seats in 2017 (709–598=111) and 2021 (734–598=136), see Table 1, did not even exist before the election.
The very notion of candidacy hinges on the existence of a seat. Lists of "candidates" for seats that are not known to exist, raise problems of motivation, conceptuality and legitimacy. In the words of Hettlage (2018): ***Ohne Kandidat, kein Mandat.***

Lawmakers recognize the importance of a stable assembly size. Should they understand, e.g., the effect if the sister parties had fused to C*U; see Table 5 and (1.18)?

Simple proportionality checks which voters can do themselves are good for transparency. In this respect, the 2017 election may be the best ever: CSU's ErSt success is 46 seats, and Table 2 shows how long it takes before CSU with 2869688 ZwSt can honor its 46 seat commitment.

In Table 1, a piece of good luck extends the transparency: CSU won 46 direct seats out of 46 possible. ***Each*** **of the 3255487 ErSt for a CSU-*candidate* did support a *winner*; thus, 3255487 voters had ErSt-success, but only 2869688 voters provided ZwSt to pay the 46 seat commitment. Visibly, there is an accounting problem.**

Perhaps the motto of Jankowski & al (2020) points to another approach than a collection of ad-hoc changes: *"Keep it simple!"*

A common belief is that list seats are more important than direct seats, and ZwSt more important than ErSt. Obviously, c<h is an *accepted* result (when *proportionality* requires more than c list seats), but h<c is *not accepted* (when it is *the h=c norm* that requires more than h *list* seats).

Alone, these facts hardly explain the belief. The belief may exist because it is often expressed; basic facts contradict it: Of β(j) = α(j)+ω(j) seats to P_j, see (1.4), ω(j) are for *persons* elected *by ErSt only*, and then α(j) is determined *by ErSt and ZwSt together*.
Moreover, both ErSt and ZwSt support a party, just through different mechanisms. Some list seats to P_j are dubbed *"compensatory"*, as if P_j had a legitimate claim for redress. Unfortunately, media link party representation to α(j), stating that ZwSt are more important than ErSt. [7]

Behnke (2015) and Linhart & al (2020) consider other sides of public understanding of versions with traditional accounting.

(3.7) ***Adaptation to faithful accounting*** **Faithful accounting does not change the voting rules. Voters should learn that the influence of their ZwSt is reduced if their ErSt supports a winner, and reduced more if the ErSt wins a large share (e.g. 1/50000) than a small (e.g. 1/100000).**

(3.8) ***MMP in the world*** **Reynolds & al (2005) survey the use of various families of methods in elections for legislatures (as of 2004):**

91 "Plurality/Majority"; 72 "Proportionality"; 30 "Mixed systems".

Of the 30 mixed, 21 were "Parallel"; only 9 were "MMP". Experiencing strategic ballot-splitting, Italy and Albania had abandoned MMP by 2006. (Mudambi and Navarra 2004; OSCE 2005). For future introductions of MMP, the perception of how it works in Germany is likely to be essential.

[7] An example from a public institution: Mit der Zweitstimme werden demgegenüber Parteien gewählt. Die Zweitstimme entscheidet über die Zusammensetzung des Bundestages. Sie ist daher eindeutig wichtiger als die Erststimme.

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