The tyranny of non-aggregation versus the tyranny of aggregation in social choices: A real dilemma*

Marc Fleurbaey† and Bertil Tungodden‡

May 2007

Abstract

Can a trifle gain to sufficiently many well-off justify imposing a much larger sacrifice on the worst-off? We show that if one answers negatively to such a question, one is forced to accept the maximin principle and give full priority to the worst-off even when a trifle gain to the worst-off imposes a substantial sacrifice on arbitrarily many well-off. If one dislikes this consequence, one faces a real dilemma in choosing between the tyranny of aggregation and the tyranny of non-aggregation.

Keywords: aggregation, social choice, maximin, utilitarianism.

JEL Classification: D63, D71.

1 Introduction

The maximin principle of Rawls (1971) is widely considered implausible as a principle of justice, because it implies that we give absolute priority to the worst-off individual in all situations. Harsanyi (1975) provides the following example in support of his rejection of the maximin principle: 'For example, let us assume that society would consist of a large number of individuals, of whom one would be seriously retarded. Suppose that some extremely expensive treatment were to become available, which could very slightly improve the retarded individual's condition, but at such high costs that this treatment could be financed only if some of the most brilliant individuals were deprived of all higher education. The difference principle would require that the retarded individual should all the same receive this very expensive treatment at any event - no matter how

*Thanks to seminar participants at Queen Mary, University of London, Geir Asheim, Aanund Hylland, Marco Mariotti, Erwin Ooghe, and Agnar Sandmo for valuable comments. The usual disclaimer applies.

†CNRS and University Paris-Descartes, LSE and IDEP. Email: marc.fleurbaey@univ-paris5.fr.

‡Department of Economics, Norwegian School of Economics and Business Administration and Chr. Michelsen Institute (Bergen). Email: bertil.tungodden@nhh.no.

many people would have to be denied a higher education, and no matter how strongly they would desire to obtain one.' (p. 597). We may name this *the tyranny of non-aggregation*. On the basis of this kind of argument, economists and philosophers have been attracted by utilitarian and, more recently, "prioritarian" (i.e. generalized utilitarian) criteria (for a discussion of this literature, see Tungodden, 2003). However, these approaches have very counterintuitive implications. Scanlon (1998) provides an example: 'Suppose that Jones has suffered an accident in the transmitter room of a television station. Electrical equipment has fallen on his arm, and we cannot rescue him without turning off the transmitter for fifteen minutes. A World Cup match is in progress, watched by many people, and it will not be over for an hour. Jones's injury will not get any worse if we wait, but his hand has been mashed and he is receiving extremely painful electrical shocks. Should we rescue him now or wait until the match is over? Does the right thing to do depend on how many people are watching – whether it is one million or five million or a hundred million? It seems to me that we should not wait, no matter how many viewers there are...' (p. 235). Both utilitarian and prioritarian reasoning would support the conclusion that for a sufficiently large number of viewers, the right thing to do would be not to turn off the transmitter before the match is over. We may name this *the tyranny of aggregation*.

In this paper, we consider the possibility of avoiding both the tyranny of non-aggregation and the tyranny of aggregation. We provide an example of a continuous social ordering function that does so. Our main result, however, shows that all such examples violate a basic consistency requirement, and thus that there is no attractive solution to this dilemma.

Section 2 provides the basic framework, and we present the results in Section 3 and some concluding remarks in Section 4. In the appendix, we present an alternative formulation of our impossibility result.

2 Framework and basic axioms

Let $\mathbb{Z}_{++}$ be the set of positive integers, and also the set of potential individuals. A particular population is $N \subset \mathbb{Z}_{++}$, $N \neq \emptyset$. Let $\mathcal{N}$ be the set of non-empty finite subsets of $\mathbb{Z}_{++}$ with at least two elements. An individual is $i \in N$, and we use the notation $N - i = N \setminus \{i\}$. Let $|N|$ denote the cardinality of N.

An allocation is $x = (x_i)_{i \in N} \in \mathbb{R}^N_+$, where x_i is i's utility. We assume that utilities are fully interpersonally comparable, which implies that no social orderings are excluded from the analysis because of informational constraints. We apply the notation $x_{-i} = (x_j)_{j \in N-i}$, $x_{-M} = (x_i)_{i \in N \setminus M}$. The subsets of worst-off and best-off individuals are defined as follows.

$$\begin{aligned} W(x) &= \left\{ i \in N \mid x_i = \min_{j \in N} x_j \right\}, \\ B(x) &= \left\{ i \in N \mid x_i = \max_{j \in N} x_j \right\}. \end{aligned}$$

A preordering is a reflexive and transitive binary relation. An ordering is a complete preordering. A social preordering (resp., ordering) function R defines a preordering (resp., ordering) R^N over $\mathbb{R}_+^N$ for every $N \in \mathcal{N}$. Let P^N denote the corresponding strict preference relation: $x\ P^N\ y$ if and only if $x\ R^N\ y$ and not $y\ R^N\ x$.

We now list basic requirements that will be imposed on social (pre)ordering functions. First, we have the standard Pareto principle.

Weak Pareto For all $N \in \mathcal{N}$, all $x, y \in \mathbb{R}_+^N$, if $x_i > y_i$ for all $i \in N$, then $x\ P^N\ y$.

In the analysis, we also apply the stronger version of the Pareto principle and a continuity requirement.

Strong Pareto For all $N \in \mathcal{N}$, all $x, y \in \mathbb{R}_+^N$, if $x_i \geq y_i$ for all $i \in N$, then $x\ R^N\ y$; if in addition there is $j \in N$ such that $x_j > y_j$, then $x\ P^N\ y$.

Continuity For all $N \in \mathcal{N}$, all $x \in \mathbb{R}_+^N$, the sets $\left\{y \in \mathbb{R}_+^N \mid y\ R^N\ x\right\}$ and $\left\{y \in \mathbb{R}_+^N \mid x\ R^N\ y\right\}$ are closed.

We also introduce the following basic consistency requirement, that says that removing someone who opposes an alternative, or adding someone who supports it, does not make it less attractive.

Reinforcement For all $N \in \mathcal{N}$, all $x, y \in \mathbb{R}_+^N$, all $i \in N$, if $N - i \in \mathcal{N}$ and $y_i > x_i$, then $x\ R^N\ y$ implies $x_{-i}\ R^{N-i}\ y_{-i}$, and $y_{-i}\ R^{N-i}\ x_{-i}$ implies $y\ R\ x$.

3 An impossibility theorem

The aim of the analysis is to study the possibility of avoiding the tyranny of aggregation and the tyranny of non-aggregation. Formally speaking, this implies that the social (pre)ordering function needs to satisfy the following two conditions.

First, Minimal Aggregation states that if all individuals, except one, gain sufficiently, then it is tolerable to impose a loss on the remaining individual if the loss is sufficiently small.

Minimal Aggregation For all $N \in \mathcal{N}$, all $y \in \mathbb{R}_+^N$, all $i \in N$, there exist $\alpha > \beta > 0$ such that for all $x \in \mathbb{R}_+^N$, if
(i) $y_i - x_i \leq \beta$;
(ii) for all $j \in N - i$, $x_j - y_j \geq \alpha$,
then $x\ R^N\ y$.

Second, Minimal Non-Aggregation states that if the worst-off gains, there is a sufficiently small loss that is tolerable for all the best-off, no matter how numerous they are.

Minimal Non-Aggregation For all $q, r \geq 0$, $\alpha > 0$, there is $0 < \beta < \alpha$ such that for all $N \in \mathcal{N}$, all $x, y \in \mathbb{R}_+^N$, all $i \in N$, if
(i) $W(x) = W(y) = \{i\}$, $y_i \leq q$ and $x_i - y_i \geq \alpha$;
(ii) for all $j \in N - i$ such that $x_j \neq y_j$, $j \in B(x) \cap B(y)$, $y_j \geq r$ and $y_j - x_j \leq \beta$,
then $x \; R^N \; y$.

To appreciate the weakness of this condition, let us emphasize that the admissible loss to the best-off may be arbitrarily small. Note also that this quantity may depend on the levels of the worst-off's and the best-off's utility. In particular, one can imagine that, for a given gain to the worst-off, the size of the admissible loss to the best-off is decreasing in the worst-off's utility and increasing in the best-off's utility.

The question we want to address is whether there exists a social (pre)ordering function that satisfies both conditions. As a preliminary result, we observe that there exist continuous and Paretian social ordering functions that avoid both the tyranny of aggregation and the tyranny of non-aggregation.

Proposition 1 *There exist social ordering functions that satisfy Strong Pareto, Continuity, Minimal Aggregation, and Minimal Non-Aggregation.*

Proof. The Geometric Gini social ordering functions satisfy all these conditions. They are defined as follows: For any $N \in \mathcal{N}$ and $x, y \in \mathbb{R}_+^N$, $x \; R^N \; y$ iff

$$\sum_{k=1}^{|N|} a^{|N|-k} x_{(k)} \geq \sum_{k=1}^{|N|} a^{|N|-k} y_{(k)},$$

where $x_{(k)}$ is the kth component by increasing order, and $a > 1$.

It is clear that every member of this family satisfies Strong Pareto, Continuity and Minimal Aggregation. Let us prove that it also satisfies Minimal Non-Aggregation. Consider any $a > 1$ and any $q, r \geq 0$, $\alpha > 0$. Let $\beta < \min\{\alpha, \alpha(a-1)\}$. Consider any $N \in \mathcal{N}$, $x, y \in \mathbb{R}_+^N$, and $i \in N$ such that:
(i) $W(x) = W(y) = \{i\}$, $y_i \leq q$ and $x_i - y_i \geq \alpha$;
(ii) for all $j \in N - i$ such that $x_j \neq y_j$, $j \in B(x) \cap B(y)$, $y_j \geq r$ and $y_j - x_j \leq \beta$.
We have to prove that $x \; R^N \; y$.

One has

$$\sum_{k=1}^{|N|} a^{|N|-k} x_{(k)} - \sum_{k=1}^{|N|} a^{|N|-k} y_{(k)} = a^{|N|-1} \left(x_{(1)} - y_{(1)} \right) + \sum_{k=2}^{|N|} a^{|N|-k} \left(x_{(k)} - y_{(k)} \right).$$

We know that $x_{(1)} - y_{(1)} \geq \alpha$ and for all $k = 2, ..., |N|$, $x_{(k)} - y_{(k)} \geq -\beta$. This implies

$$\begin{aligned} \sum_{k=1}^{|N|} a^{|N|-k} x_{(k)} - \sum_{k=1}^{|N|} a^{|N|-k} y_{(k)} \quad & \geq \\ a^{|N|-1}\alpha + \sum_{k=2}^{|N|} a^{|N|-k} (-\beta) \quad & > \quad a^{|N|-1}\alpha - \sum_{k=2}^{|N|} a^{|N|-k} \alpha (a-1). \end{aligned}$$

Now, for $a > 1$,

$$a^{|N|-1}\alpha - \sum_{k=2}^{|N|} a^{|N|-k}\alpha\,(a-1) \geq 0 \Leftrightarrow \sum_{k=2}^{|N|} a^{1-k} \leq \frac{1}{a-1},$$

and the latter inequality is true because

$$\sum_{k=2}^{|N|} a^{1-k} = \frac{1-a^{1-|N|}}{a-1}.$$

Therefore,

$$\sum_{k=1}^{|N|} a^{|N|-k} x_{(k)} - \sum_{k=1}^{|N|} a^{|N|-k} y_{(k)} \geq 0,$$

which means that $x\ R^N\ y$. ■

However, it turns out that the Geometric Gini does not satisfy our consistency requirement. To illustrate, consider the case where $a = 2$, with the allocations $x = (5, 5, 5)$, $y = (3, 6, 10)$, $z = (5, 5)$, $w = (3, 10)$. In this case, the Geometric Gini deems that x is better than y, whereas w is better than z. The only difference between the comparison of z and w and x and y is that we have removed a person who is better-off in y than in x. Because we consider x to be better than y, Reinforcement therefore requires that we should also consider z at least as good as w.

More generally, it turns out that it is not possible to combine Reinforcement with our two minimal conditions, even if we relax the Pareto principle and drop the requirements of continuity and completeness. To see this, let us first establish the following lemma, which is also of some interest in itself as a characterization of the strict preference part of the maximin criterion.

Lemma 1 *If a social preordering function R satisfies Weak Pareto, Reinforcement and Minimal Non-Aggregation, then for all $N \in \mathcal{N}$ and all $x, y \in \mathbb{R}^N_+$, if $\min_{i\in N} x_i > \min_{i\in N} y_i$, then $x\ P^N\ y$.*

Proof. Consider any $N \in \mathcal{N}$ and $x, y \in \mathbb{R}^N_+$ such that $\min_{i\in N} x_i > \min_{i\in N} y_i$. We will now prove that $x\ P^N\ y$.

Step 1. If $\min_{i\in N} x_i > \max_{i\in N} y_i$, then $x\ P^N\ y$ follows from Weak Pareto. Hence, in the rest of the proof we assume that $\min_{i\in N} x_i \leq \max_{i\in N} y_i$, where i_0 refers to some person who has $y_{i_0} = \min_{i\in N} y_i$.

Step 2. Define y^*, x^*, x^{***} such that:

$$y^* > \max_{i\in N} y_i, \tag{1}$$

$$\min_{i\in N} x_i > x^{***} > x^* > \min_{i\in N} y_i, \tag{2}$$

and let $0 < \beta_2 < \alpha_2 < x^{***} - x^*$ be corresponding terms for the application of Minimal Non-Aggregation at $q = x^*$ and $r = x^{***}$. Then pick $x^{**} \in (x^*, x^{***})$ such that $x^{**} - x^* \geq \alpha_2$ and $x^{***} - x^{**} \leq \beta_2$.

Let $\gamma_2 = x^{***} - x^{**}$ and α_1 be such that:

$$0 < \alpha_1 < \frac{9}{10}\gamma_2.$$

It then follows that $x^{***} > x^{**} + \frac{\gamma_2}{10} + \alpha_1$. Pick some $0 < \beta_1 < \alpha_1$ to satisfy Minimal Non-Aggregation given α_1 and for $q = x^{**} + \frac{\gamma_2}{10}$ and $r = x^{**} + \frac{\gamma_2}{10} + \alpha_1$. It then follows straightforwardly that there exist $\gamma_1 \in (0, \beta_1)$ and $m \in \mathbb{Z}_{++}$ such that:

$$x^{***} > y^* - m\gamma_1 > x^{**} + \frac{\gamma_2}{10} + \alpha_1. \tag{3}$$

Let $M \in \mathcal{N}$ be such that $|M| = m$, $M \cap N = \emptyset$, and let $M_k = \{m(1), ..., m(k)\}$ denote the subset of M containing the first k members. Consider now the following allocations:

$$\begin{aligned} z^1 &= (\underbrace{y_{-i_0}}_{N-i_0}, \underbrace{x^{**} + \frac{\gamma_2}{20}}_{M_1}), \\ \bar{z}^1 &= (\underbrace{y^*, ..., y^*}_{N-i_0}, \underbrace{x^{**} + \frac{\gamma_2}{10}}_{M_1}), \\ \hat{z}^1 &= (\underbrace{y^* - \gamma_1, ..., y^* - \gamma_1}_{N-i_0}, \underbrace{x^{**} + \frac{\gamma_2}{10} + \alpha_1}_{M_1}). \end{aligned}$$

By (1) and Weak Pareto,

$$\bar{z}^1 \; P^{N-i_0 \cup M_1} \; z^1.$$

By Minimal Non-Aggregation,

$$\hat{z}^1 \; R^{N-i_0 \cup M_1} \; \bar{z}^1$$

(because there is a worst-off who remains the worst-off and gains α_1 in $\hat{z}^1$ compared to $\bar{z}^1$ and all the people who lose are best-off in both alternatives and lose γ_1 which is less than β_1; recall that α_1, β_1 have been chosen to satisfy Minimal Non-Aggregation for $q = x^{**} + \frac{\gamma_2}{10}$ and $r = x^{**} + \frac{\gamma_2}{10} + \alpha_1$ –by (3) the latter is less than the best-off's utility in $\bar{z}^1$, i.e., y^*). Hence, by transitivity,

$$\hat{z}^1 \; P^{N-i_0 \cup M_1} \; z^1.$$

Step 3. Consider now the sequence $z^t, \bar{z}^t, \hat{z}^t$, for all $t = 2, ..., m$, defined as follows:

$$\begin{aligned}
z^t &= (\underbrace{y_{-i_0}}_{N-i_0}, \underbrace{x^{**}+\frac{\gamma_2}{20}, x^{**}+\frac{\gamma_2}{20},...,x^{**}+\frac{\gamma_2}{20}}_{M_t}),\\
\bar{z}^t &= (\underbrace{y^*-(t-1)\gamma_1,...,y^*-(t-1)\gamma_1}_{N-i_0}, \underbrace{\hat{z}^{t-1}_{m(1)},...,\hat{z}^{t-1}_{m(t-1)}, x^{**}+\frac{\gamma_2}{10+\frac{t-1}{m}}}_{M_t}),\\
\hat{z}^t &= (\underbrace{y^*-t\gamma_1,...,y^*-t\gamma_1}_{N-i_0}, \underbrace{\bar{z}^t_{m(1)},...,\bar{z}^t_{m(t-1)}, \bar{z}^t_{m(t)}+\alpha_1}_{M_t}),
\end{aligned}$$

In particular, this definition implies that everyone in M_t is indifferent between $\hat{z}^t$ and $\bar{z}^t$, except for the worst-off who gains α_1 :

$$\begin{aligned}
\hat{z}^t_{m(1)} &= \hat{z}^{t-1}_{m(1)} = \ldots = \hat{z}^1_{m(1)} = x^{**}+\frac{\gamma_2}{10}+\alpha_1 = \bar{z}^t_{m(1)},\\
&\vdots\\
\hat{z}^t_{m(t-1)} &= \hat{z}^{t-1}_{m(t-1)} = x^{**}+\frac{\gamma_2}{10+\frac{t-2}{m}}+\alpha_1 = \bar{z}^t_{m(t-1)},\\
\hat{z}^t_{m(t)} &= x^{**}+\frac{\gamma_2}{10+\frac{t-1}{m}}+\alpha_1 > \bar{z}^t_{m(t)}.
\end{aligned}$$

By Minimal Non-Aggregation, for all $t=2,...,m$,

$$\hat{z}^t \ R^{N-i_0\cup M_t} \ \bar{z}^t$$

(because there is a worst-off who remains the worst-off and gains α_1 in $\hat{z}^t$ compared to $\bar{z}^t$ and all the people who lose are best-off in both alternatives, by (3), and lose γ_1 which is less than β_1; recall that α_1, β_1 have been chosen to satisfy Minimal Non-Aggregation for $q = x^{**}+\frac{\gamma_2}{10}$ and $r = x^{**}+\frac{\gamma_2}{10}+\alpha_1$ –the former is greater than the worst-off's utility in $\bar{z}^t$, i.e., $x^{**}+\frac{\gamma_2}{10+\frac{t-1}{m}}$, and by (3) the latter is less than the best-off's utility in $\bar{z}^t$, i.e., $y^*-(t-1)\gamma_1$).

Pick $t\in\{2,...,m\}$. If $\hat{z}^{t-1} \ P^{N-i_0\cup M_{t-1}} \ z^{t-1}$, then by Reinforcement,

$$\bar{z}^t \ R^{N-i_0\cup M_t} \ z^t$$

(because $\bar{z}^t_i = \hat{z}^{t-1}_i$ and $z^t_i = z^{t-1}_i$ for all $i \in N-i_0\cup M_{t-1}$, and $\bar{z}^t_{m(t)} > z^t_{m(t)}$). Hence, by transitivity,

$$\hat{z}^t \ R^{N-i_0\cup M_t} \ z^t.$$

By Step 2, a recursive argument applies and therefore this holds true for all $t=2,...,m$. In particular, one has:

$$\hat{z}^m \ R^{N-i_0\cup M_m} \ z^m.$$

Step 4. Consider

$$\tilde{z}^m = (\underbrace{x^{***},...,x^{***}}_{N-i_0\cup M_m}).$$

By (3) and Weak Pareto,

$$\tilde{z}^m \ P^{N-i_0 \cup M_m} \ \hat{z}^m.$$

Hence, by Step 3 and transitivity,

$$\tilde{z}^m \ P^{N-i_0 \cup M_m} \ z^m.$$

Step 5. Consider:

$$\begin{aligned} \tilde{w}^m &= (\underbrace{x^*}_{i_0}, \underbrace{x^{***}, ..., x^{***}}_{N-i_0 \cup M_m}), \\ w^m &= (\underbrace{y_{i_0}}_{i_0}, \underbrace{y_{-i_0}}_{N-i_0}, \underbrace{x^{**} + \frac{\gamma_2}{20}, ..., x^{**} + \frac{\gamma_2}{20}}_{M_m}). \end{aligned}$$

By Step 4 and Reinforcement,

$$\tilde{w}^m \ R^{N \cup M_m} \ w^m$$

(because $\tilde{w}_i^m = \tilde{z}_i^m$ and $w_i^m = z_i^m$ for all $i \in N - i_0 \cup M_m$, and $\tilde{w}_{i_0}^m > w_{i_0}^m$).

Step 6. Let:

$$\frac{19}{20}\gamma_2 < \delta < \gamma_2.$$

Recall that $\gamma_2 = x^{***} - x^{**}$. One then has,

$$x^{**} < x^{***} - \delta < x^{**} + \frac{\gamma_2}{20}. \tag{4}$$

Consider:

$$z^* = (\underbrace{x^{**}}_{i_0}, \underbrace{x^{***} - \delta, ..., x^{***} - \delta}_{N-i_0 \cup M_m}).$$

By Minimal Non-Aggregation,

$$z^* \ R^{N \cup M_m} \ \tilde{w}^m$$

(because there is a worst-off who remains the worst-off, by (4), and gains $x^{**} - x^* \geq \alpha_2$ in z^* compared to $\tilde{w}^m$ and all the people who lose are best-off in both alternatives and lose $\delta < \gamma_2 = x^{***} - x^{**} \leq \beta_2$; recall that α_2, β_2 have been chosen to satisfy Minimal Non-Aggregation for $q = x^*$ and $r = x^{***}$).

Hence, by Step 5 and transitivity,

$$z^* \ R^{N \cup M_m} \ w^m.$$

Step 7. By Step 6, (4) and Reinforcement,

$$z^*_{-M_m} \ R^N \ w^m_{-M_m}.$$

By the fact that $\min_{i \in N} x_i > x^{***} > x^{**}$ and Weak Pareto,

$$x \ P^N \ z^*_{-M_m}.$$

Hence, by transitivity,

$$x \; P^N \; w^m_{-M_m}.$$

The result follows from the fact that $w^m_{-M_m} = y$. ■

We can now establish our main result.

Theorem 1 *No social preordering function satisfies Weak Pareto, Reinforcement, Minimal Non-Aggregation and Minimal Aggregation.*

Proof. This directly follows from the fact that the maximin property obtained in Lemma 1 is incompatible with Minimal Aggregation. ■

The theorem shows that it is not possible to avoid both the tyranny of aggregation and the tyranny of non-aggregation in social choices. Note that all four conditions are needed in order to establish the impossibility result, as illustrated by General Indifference (violating Weak Pareto), Geometric Gini (violating Reinforcement), Utilitarianism (violating Minimal Non-Aggregation), and Maximin (violating Minimal Aggregation).[1] A variant of the result, which relies on another consistency condition (Replication Invariance) and a slightly stronger version of Minimal Non-Aggregation, is presented in the appendix.

4 Concluding remarks

The main result of this paper implies that there is a real dilemma in social choices. No consistent criterion avoids both the tyranny of aggregation and the tyranny of non-aggregation. Given that we find both the tyranny of aggregation and the tyranny of non-aggregation to be disturbing, we believe that one should be cautious when criticizing maximin, (generalized) utilitarianism or any other social ordering on the basis of how they perform in extreme cases. The assessment of the various possible social ordering functions should be more comprehensive and, maybe, more focused on cases that are directly relevant to actual policy issues.

Appendix

There is a variant of the impossibility result where Reinforcement is replaced by the requirement that the preordering is invariant to the scaling of the population. Let kN denote a k-replica of N, and kx the corresponding replica of an allocation x.

Replication Invariance For all $N \in \mathcal{N}$, all $x, y \in \mathbb{R}^N_+$, all $k \in \mathbb{Z}_{++}$, $x \; R^N \; y$ iff $kx \; R^{kN} \; ky$.

[1] The examples are defined as follows, where the quantifiers "For all $N \in \mathcal{N}$ and all $x, y \in \mathbb{R}^N_+$ " apply to each of them. General Indifference: $x \; I^N \; y$. Utilitarianism: $x \; R^N \; y$ iff $\sum_i x_i \geq \sum_i y_i$. Maximin: $x \; R^N \; y$ iff $\min_i x_i \geq \min_i y_i$.

We first note that there exist social ordering functions that satisfy Replication Invariance in combination with the other three conditions of our theorem.

Proposition 2 *There exist social ordering functions that satisfy Strong Pareto, Replication Invariance, Minimal Aggregation, and Minimal Non-Aggregation.*

Proof. Let $\bar{x} = \frac{1}{|N|} \sum_{i \in N} x_i$. The following social ordering function satisfies all the conditions: For all $N \in \mathcal{N}$ and $x, y \in \mathbb{R}^N_+$, $x \; R^N \; y$ iff $\min_{i \in N} x_i + \bar{x} \geq \min_{i \in N} y_i + \bar{y}$. ■

However, it turns out that the impossibility reemerges if we slightly strengthen Minimal Non-Aggregation. The strengthened version allows for the possibility that there may be more than one worst-off person in y, which implies that the worst-off person in y may no longer be the worst-off person in x. However, it is still required that it is not among the best-off in x.

Minimal Non-Aggregation* For all $q, r \geq 0$, $\alpha > 0$, there is $0 < \beta < \alpha$ such that for all $N \in \mathcal{N}$, all $x, y \in \mathbb{R}^N_+$, all $i \in N$, if
(i) $i \in W(y) \setminus B(x)$, $y_i \leq q$ and $x_i - y_i \geq \alpha$;
(ii) for all $j \in N - i$ such that $x_j \neq y_j$, $j \in B(x) \cap B(y)$, $y_j \geq r$ and $y_j - x_j \leq \beta$,
then $x \; R^N \; y$.

We can now establish the following lemma.

Lemma 2 *If a social preordering function R satisfies Weak Pareto, Replication Invariance and Minimal Non-Aggregation*, then for all $N \in \mathcal{N}$ and all $x, y \in \mathbb{R}^N_+$, if $\min_{i \in N} x_i > \min_{i \in N} y_i$, then $x \; P^N \; y$.*

Proof. Consider any $N \in \mathcal{N}$ and $x, y \in \mathbb{R}^N_+$ such that $\min_{i \in N} x_i > \min_{i \in N} y_i$. We will now prove that $x \; P^N \; y$.

Step 1. If $\min_{i \in N} x_i > \max_{i \in N} y_i$, then $x \; P^N \; y$ follows from Weak Pareto. Hence, in the rest of the proof, we assume that $\min_{i \in N} x_i \leq \max_{i \in N} y_i$, with i_0 referring to some person who has $y_{i_0} = \min_{i \in N} y_i$.

Step 2. Choose x^*, x^{**}, y^* such that:

$$\min_{i \in N} x_i > x^{**} > x^* > \min_{i \in N} y_i,$$

and $y^* > \max_{i \in N} y_i$. Define $\alpha = x^{**} - x^*$ and let β be the corresponding term for Minimal Non-Aggregation* at $q = x^*$ and $r = x^{**}$. Let $0 < \gamma < \beta$ and $m \in \mathbb{Z}_{++}$ be such that:

$$\min_{i \in N} x_i > y^* - m\gamma > x^{**}. \qquad (5)$$

Finally, define the allocations $z^0, z^1, ..., z^m$ for the replicated population mN as follows:

$$\begin{aligned}
z^0 &= (\underbrace{x^*, ..., x^*}_{m\{i_0\}}, \underbrace{y^*, ..., y^*}_{m(N-i_0)}),\\
z^1 &= (\underbrace{x^{**}, x^*, ..., x^*}_{m\{i_0\}}, \underbrace{y^* - \gamma, ..., y^* - \gamma}_{m(N-i_0)}),\\
z^2 &= (\underbrace{x^{**}, x^{**}, x^*, ..., x^*}_{m\{i_0\}}, \underbrace{y^* - 2\gamma, ..., y^* - 2\gamma}_{m(N-i_0)}),\\
&\vdots\\
z^m &= (\underbrace{x^{**}, ..., x^{**}}_{m\{i_0\}}, \underbrace{y^* - m\gamma, ..., y^* - m\gamma}_{m(N-i_0)}).
\end{aligned}$$

By Minimal Non-Aggregation*, for all $t = 1, ..., m$,

$$z^t \; R^{mN} \; z^{t-1}$$

(because there is a worst-off in z^{t-1} who gains $x^{**} - x^* = \alpha$ in z^t and all the people who lose are best-off in both alternatives, by (5), and lose $\gamma < \beta$; recall that α, β have been chosen to satisfy Minimal Non-Aggregation for $q = x^*$ and $r = x^{**}$ –the latter is less than the best-off's utility, i.e., $y^* - (t-1)\gamma$).

By transitivity,

$$z^m \; R^{mN} \; z^0.$$

By (5) and Weak Pareto,

$$mx \; P^{mN} \; z^m \text{ and } z^0 \; P^{mN} \; my.$$

By transitivity,

$$mx \; P^{mN} \; my.$$

By Replication Invariance,

$$x \; P^N \; y.$$

■

Hence, we have another version of our impossibility result.

Theorem 2 *No social preordering function R satisfies Weak Pareto, Replication Invariance, Minimal Non-Aggregation* and Minimal Aggregation.*

References

[1] Harsanyi, J. C. (1975) "Can the Maximin Principle Serve as a Basis for Morality? A Critique of John Rawls's Theory". *The American Political Science Review* 69: 594-606.

[2] Rawls, J. *A* (1971) *Theory of Justice.* Harvard University Press.

[3] Scanlon, T. (1998) *What We Owe Each Other.* Harvard University Press.

[4] Tungodden, B. (2003) "The Value of Equality". *Economics and Philosophy* 19(1): 1 - 44.