Equilibrium Selection in Hawk-Dove Games

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Abstract

In Hawk-Dove games with mulitiplicity of equilibria, we study which equilibria are selected using various equilibrium selection methods. Using a uniform price auction as an illustrative example, we apply the tracing procedure method of Harsanyi and Selten (1988), the robustness to strategic uncertainty method of Andersson, Argenton and Weibull (2014), and the quantal response method of McKelvey and Palfrey (1998) to predict which equilibrium is selected by the players and how changes to the various model parameters impact the selected equilibria.

KEYWORDS: Hawk-Dove games, equilibrium selection, tracing procedure method, robustness to strategic uncertainty method, quantal response method.
JEL codes: C72, C79, D44, D47.

1 Introduction

The Hawk-Dove game is one of the salient models used to study a wide range of questions in the economic literature as well as being one of the prominent models used in evolutionary biology. In the classical representation of the game, players compete by using two strategies - hawk and dove. In this case, the Hawk-Dove game has two possible Nash equilibria in pure strategies - one of the players behaves as hawk and the other as dove. As both players prefer the equilibrium in which they select the hawk strategy and the opposing player selects the dove strategy, which equilibrium were to emerge between two individuals in any given iteration of the game is ambiguous - either of the two pure strategies Nash equilibria could be played. To break the ambiguity associated with multiplicity of equilibria in games, numerous equilibrium selection methods have been proposed.

We analyze in detail the outcome of three different equilibrium selection methods applied to a Hawk-Dove game in which we extend the set of strategies, and consequently the

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set of possible Nash equilibria. In each of those equilibria, as in the basic two strategies Hawk-Dove game, one of the players behaves as a dove, the other as hawk, and both prefer the equilibrium in which the other player selects the dove strategy. We apply the tracing procedure method of Harsanyi and Selten (1988), the robustness to strategic uncertainty method of Andersson, Argenton and Weibull (2014), and the quantal response method of McKelvey and Palfrey (1998) to predict which equilibrium is selected by the players.

We frame the Hawk-Dove game as a uniform price auction in which two players with asymmetric production capacities compete to satisfy an inelastic demand. Having observed the demand, the players simultaneously and independently submit a bid for their entire production capacity, assumed to be lower than the total demand, i.e., each player faces a positive residual demand when it is dispatched last in the auction. The auctioneer establishes the maximum and the minimum bids that can be submitted in the auction. The player that submits the higher bid sets the price and satisfies the residual demand while the player that submits the lower bid is dispatched first and satisfies the total demand at the price set by the other player. In case of a tie, the players are dispatched in proportion to their production capacity. Therefore, the tie-breaking rule implemented determines that the game has the structure of a Hawk-Dove game.[1]

The game described above has multiple pure strategies Nash equilibria in which one of the players submits the maximum bid allowed by the auctioneer (dove strategy) and the opponent submits a bid that makes undercutting unprofitable (hawk strategy). As in the classic representation of the Hawk-Dove game, the players have opposing preferences for both sets of equilibria; each player prefers the set of equilibria in which the opposing player submits the maximum bid allowed by the auctioneer (dove strategy), since in that case the player dispatched first sells its entire production capacity at the maximum price allowed by the auctioneer.

The tracing procedure method proposed by Harsanyi and Selten (1988) selects the equilibrium in which the player with higher production capacity submits the maximum bid (dove strategy) and the player with lower production capacity submits the minimum bid allowed by the auctioneer (hawk strategy). This result is very intuitive, since the tracing procedure method is based on the idea that some equilibria are risky for some of the players and that players prefer to avoid risky equilibria. In particular, the equilibrium in which the player with lower production capacity submits the maximum allowed bid is risky for this player, as doing so exposes the player to a chance of being dispatched last when residual demand is very low. In contrast, the equilibrium in which the player with higher production capacity submits the maximum bid is less risky for this player, as residual demand is very high even when this player is dispatched last.

When the auctioneer increases the minimum bid that can be submitted in the auction, the tracing procedure method still selects the equilibrium in which the player with higher production capacity submits the maximum bid allowed. However, in this case, the tracing procedure method predicts that it is more difficult for the players to select that equilibrium. An increase in the minimum bid that the players can submit makes submitting low

[1] We provide a complete description of the uniform price auction in the model section. In that section, we also discuss the importance of the tie breaking rule to determine if the uniform price auction has the structure of a Hawk-Dove game or the structure of a Battle of the Sexes game.

bids more attractive for both players, and it becomes more difficult to coordinate in the equilibrium in which the player with higher production capacity submits the maximum bid.

With independence of suppliers' production capacities asymmetries, the robustness to strategic uncertainty method proposed by Andersson, Argenton and Weibull (2014) selects the two equilibria in which one of the players submits the maximum bid (dove strategy) and the other player submits the minimum bid allowed by the auctioneer (hawk strategy). The main idea of this method is that players face some uncertainty about the strategies played by their opponents. Consequently, if one of the players thinks that the opposing player will submit a high bid, its best strategy is to submit the minimum bid since this increases the probability of being dispatched first. In contrast, if one of the players thinks that the opposing player will submit a low bid, its best strategy is to submit the maximum bid, since this increases the probability of being dispatched last and satisfying the residual demand at the maximum bid. Therefore, the robustness to strategic method selects the two equilibria in which one player submits the maximum bid and the other submits the lower bid allowed by the auctioneer. An increase in the minimum bid does not change the equilibrium predicted by this method.

The quantal response method proposed by McKelvey and Palfrey (1998) predicts that the player with higher production capacity submits the maximum bid (dove strategy) and the player with lower production capacity submits the lower bids in the strategies set with higher probabilities (hawk strategy). In the quantal response method the players choose among the strategies in the game based on their relative expected payoff. The key idea is that when the players calculate their expected payoff, they make calculation errors according to some random process. Based on that random process, the players assign more probability to the strategies that give them a higher expected payoff. Therefore, the player with higher production capacity assigns more probability to the maximum bid since it faces a high residual demand and that bid has higher expected payoffs. In contrast, the player with lower production capacity assigns more probability to the lower bids in the strategies set, since those bids have higher expected payoffs.

An increase in the minimum bid does not change the strategy of the player with higher production capacity, since it still faces a high residual demand and does not change its strategy, but makes more attractive for the player with lower production capacity to submit lower bids, since for that player, the expected payoff associated to low bids increases. Therefore, an increase of the minimum bid allowed by the auctioneer facilitates the coordination in the equilibria of the game in which the player with higher production capacity submits the maximum bid and the player with lower production capacity submits the minimum bid.

In a similar setup as the one used in this paper, Boom (2008) applies the tracing procedure method proposed by Harsanyi and Selten (1988) to predict the equilibrium selected by the players in a uniform price auction. To apply the tracing procedure method, the author assumes that the players only play the strategies that appear in the Nash equilibria of the game. In contrast, when we apply the tracing procedure method, the robustness to strategic uncertainty and the quantal response methods we do not restrict the set of strategies that can be selected by the players except through the minimum and maximum bids which are allowed by the auctioneer. Moreover, we also study the importance of the market design in determining which equilibrium is selected by the players. When we apply the tracing procedure method, the equilibrium selected crucially depends on the realization of the demand. This contrasts with Boom (2008), where the equilibrium selected is always the same.

Due to the multiplicity of equilibria, the tracing procedure method plays a crucial role in the industrial organization literature that endogenize the emergence of a price leader in duopoly models. In particular, van Damme and Hurkens (1999) focus on a model with homogeneous products with linear demand, constant marginal cost, and with one firm being more efficient than the other. Using an endogenous timing game introduced by Hamilton and Slutksky (1990), the authors apply the tracing procedure method to show that the player with lower production costs emerge as a leader in a Stackelberg model with continuous set of strategies. van Damme and Hurkens (2004) focus on a model with price competition in a duopoly with differentiated substitutable products, linear and symmetric demand, and constant marginal cost. In contrast with the models of quantity competition however, when the players compete in prices, the leadership role is not the most preferred one. This result is in line with other models of price competition with capacity constraints (Deneckere and Kovenock, 1992; Canoy, 1996; Osborne and Pitchik, 1986).

Using a similar approach as in van Damme and Hurkens (1999), Sadanand and Sadanand (1996) analyze a Stackelberg model in which firms face demand uncertainty that is resolved before production begins in the second stage. They analyze an asymmetric model where one firm has a higher production capacity than the other and conclude that the high capacity production player emerges as a price leader. Spencer and Brander (1992) study a Stackelberg model with demand uncertainty and conclude that the better informed player emerge as Stackelberg leader. By extending the models of price competition with capacity constraints Deneckere and Kovenock (1992) develop a game theoretic framework to study the emergence of a price leader in a duopolistic price-setting game in which the players have capacity constraints and are allowed to choose the timing of their price announcements. Deneckere, Kovenock and Lee (1992) extend Varian's (1980) simultaneous price setting game and find that the player with a larger segment of loyal consumers becomes an endogenous price leader. Reinganum (1985) and Farrell and Shapiro (1988) also study the emergence of a price leader using industrial organization models. We extend the previous industrial organization literature by applying the tracing procedure method, the robustness to strategic uncertainty method, and the quantal response method to Hawk-Dove games.

The emergence of a price leader has also been studied in experimental settings. In particular, Cabrales, García-Fontes and Motta (2000) use a vertical product differentiation model with two asymmetric players first choosing qualities and then choosing prices in order to design an experiment with the structure of a Battle of the Sexes game. The tracing procedure method applied to their theoretical framework and their experimental results predict that the higher the degree of asymmetry of the game, the higher the predictive power of the tracing procedure method. These results are in line with our results, but in contrast with the experimental design proposed by Cabrales, García-Fontes and Motta (2000) where the players compete only by setting two prices, we provide a theoretical framework where the players compete using a continuous set of strategies.

By applying the same setting used in this paper, Bigoni, Blázquez and Le Coq (2021) extend the experimental literature that study the emergence of a price leader. In particular, the authors study which equilibria are selected by the players in a Hawk-Dove game, and how the structural parameters of the model (suppliers' production capacities asymmetries) and the market design parameter (minimum bid allowed by the auctioneer) affect the equilibrium selected by the players. The theoretical analysis in this paper helps to frame the experimental design, and helps to predict and understand the experimental results in that paper. Bigoni, Blázquez and Le Coq (2021) find more probable that the players coordinate in the equilibrium where the player with higher production capacity submits the maximum bid (dove strategy), and the player with lower production capacity submits the minimum bid allowed by the auctioneer (hawk strategy). This result is in line with the three equilibrium selection methods analyzed in this paper. The authors also find that an increase in the minimum bid allowed by the auctioneer makes the coordination in one of the equilibria slightly more difficult.[2] This result is in line with the predictions by using the tracing procedure method, but not with the ones by using the quantal response method.

We also contribute to the evolutionary biology literature. By applying the evolutionary stable strategy equilibrium proposed by Maynard Smith and Price (1973), Vega-Redondo (1996) and Vega-Redondo (2003) study the equilibrium in a Hawk-Dove game when different parameters are introduced in the payoff matrix. Cressman (2003) Friedman (1991) and Weibull (1995) provide a theoretical framework to study the Hawk-Dove game when the players of that game belong to a single population or when they belong to two disjoint populations. Benndorf, Martíne-Martínez and Normann (2016) and Oprea, Henwood and Friedman (2011) test those predictions using experiments in which the players can choose only among two possible strategies. Following a similar approach, Berninghaus, Ehrhart and Ott (2012) endogenize the formation of social networks in a Hawk-Dove game. We extend the previous analysis by introducing a larger set of strategies in the game.

The article proceeds as follows. Section 2 describes the set-up, the timing and the equilibrium of the game. Sections 3, 4 and 5 apply the tracing procedure method, the robustness to strategic uncertainty method and the quantal response method to the game and study the equilibrium selected by the players. Section 6 concludes the paper. The proofs are found in annex 1. In annex 2, we provide an example to illustrate the theoretical analysis. In annex 2, we also describe the algorithms that we apply in the computational simulations used in the paper.

2 Model

In this section, we present the set-up and the timing of a uniform price auction, and we characterize the equilibrium.

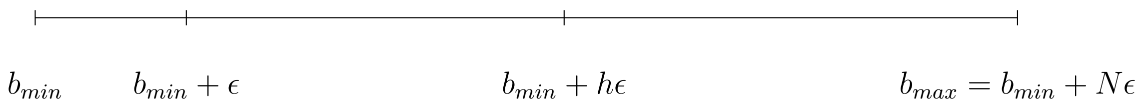
Set-up: There are two players, i and j, with production capacity k_i and k_j, where $k_i > k_j$. The level of demand, θ is perfectly inelastic. Moreover, $\theta \in [k_i, k_i + k_j]$, i.e., the demand

[2]It is important to notice that the experimental results in this point are not as strong as the ones that show that the supplier with higher production capacity submits the maximum bid (dove strategy), and the player with lower production capacity submits the minimum bid (hawk strategy).

is large enough to guarantee that both players face a positive residual demand. We introduce this assumption because when the demand is very low ($\theta < k_j$), both players have enough production capacity to satisfy the demand individually. In that case, the resulting Nash equilibrium is unique, and selection methods are no longer useful.

Timing: Having observed the realization of demand θ, each player simultaneously and independently submits a bid specifying the lowest price at which it is willing to supply up to its capacity, $b_i \in [b_{min}, b_{max}]$, $i = 1, 2$, where b_{min} and b_{max} are determined by the auctioneer. The players can only submit bids higher than or equal to b_{min} and lower than or equal to b_{max}.[3] The number of bids in that interval (N) is determined exogenously and it can be freely set. The minimum bid increment between one bid and the next is defined by $\epsilon = \dfrac{b_{max} - b_{min}}{N}$. Let h be an integer between 1 and N. The set of strategies is represented in figure 1.

Figure 1: Strategies set



Let $b \equiv (b_i, b_j)$ denote a bid profile. On basis of this profile, the auctioneer calls players into operation. The output allocated to player i (player j's output is symmetric), denoted by $q_i(b; \theta, k)$, is given by:[4]

$$q_i(b; \theta, k) = \begin{cases} k_i & \text{if } b_i < b_j \\ \dfrac{k_i \theta}{k_i + k_j} & \text{if } b_i = b_j \\ \theta - k_j & \text{if } b_i > b_j \end{cases} \tag{1}$$

When player i submits the lower bid, it sells its entire production capacity ($q_i = k_i$). When both players submit the same bid, the demand is split among them in proportion to their production capacity $\left(q_i = \dfrac{k_i \theta}{k_i + k_j}\right)$. When player i submits the higher bid, it satisfies the residual demand ($q_i = \theta - k_j$).

Finally, the payments are worked out by the auctioneer. When the auctioneer runs a uniform price auction, the price received by a player for any positive quantity dispatched by the auctioneer is equal to the higher offer price accepted in the auction. Hence, player i's payoffs (player j's payoffs are symmetric), denoted by $\pi_i(b; \theta, k)$, are given by:[5]

[3]The minimum bid in the auction (b_{min}) and the maximum bid (b_{max}) are determined by the auctioneer. The minimum bid guarantees a minimum profit for the players. The maximum bid represents the reservation price for the consumers of the good.

[4]It is important to emphasize that $q_i(b; \theta, k)$ is only valid under the assumptions that $\theta \in [k_j, k_i + k_j]$. When $\theta < k_j$, $q_i(b; \theta, k)$ is slightly different, since in this case both players have enough production capacity to satisfy the entire demand and the equilibrium is unique. For a complete analysis of the uniform price auction when the demand is low see Fabra, von der Fehr and Harbord (2006).

[5]As with $q_i(b; \theta, k)$, $\pi_i(b; \theta, k)$ is slightly different when the assumptions of the model are relaxed.

Figure 2: Payoff matrix in a uniform price auction

b_i \ b_j	b_{min}	$\cdots$	$b_{min}+h\epsilon$	$\cdots$	$b_{min}+N\epsilon = b_{max}$
b_{min}	$\pi_i(b_{min}, b_{min}) =$ $b_{min}\dfrac{\theta k_i}{k_i+k_j}$ $\pi_j(b_{min}, b_{min}) =$ $b_{min}\dfrac{\theta k_j}{k_i+k_j}$	$\cdots$	$\pi_i(b_{min}, b_{min}+h\epsilon) =$ $(b_{min}+h\epsilon)k_i$ $\pi_j(b_{min}, b_{min}+h\epsilon) =$ $(b_{min}+h\epsilon)(\theta-k_i)$	$\cdots$	$\pi_i(b_{min}, b_{min}+N\epsilon) =$ $(b_{min}+N\epsilon)k_i$ $\pi_j(b_{min}, b_{min}+N\epsilon) =$ $b_{min}(\theta-k_i)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$b_{min}+h\epsilon$	$\pi_i(b_{min}+h\epsilon, b_{min}) =$ $(b_{min}+h\epsilon)(\theta-k_j)$ $\pi_j(b_{min}+h\epsilon, b_{min}) =$ $(b_{min}+h\epsilon)k_j$	$\cdots$	$\pi_i(b_{min}+h\epsilon, b_{min}+h\epsilon) =$ $(b_{min}+h\epsilon)\dfrac{\theta k_i}{k_i+k_j}$ $\pi_j(b_{min}+h\epsilon, b_{min}+h\epsilon) =$ $(b_{min}+h\epsilon)\dfrac{\theta k_j}{k_i+k_j}$	$\cdots$	$\pi_i(b_{min}+h\epsilon, b_{min}+N\epsilon) =$ $(b_{min}+N\epsilon)k_i$ $\pi_j(b_{min}+h\epsilon, b_{min}+N\epsilon) =$ $(b_{min}+h\epsilon)(\theta-k_i)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$b_{min}+N\epsilon =$ b_{max}	$\pi_i(b_{min}+N\epsilon, b_{min}) =$ $(b_{min}+N\epsilon)(\theta-k_j)$ $\pi_j(b_{min}+N\epsilon, b_{min}) =$ $(b_{min}+N\epsilon)k_j$	$\cdots$	$\pi_i(b_{min}+N\epsilon, b_{min}+h\epsilon) =$ $(b_{min}+N\epsilon)(\theta-k_j)$ $\pi_j(b_{min}+N\epsilon, b_{min}+h\epsilon) =$ $(b_{min}+N\epsilon)k_j$	$\cdots$	$\pi_i(b_{min}+N\epsilon, b_{min}+N\epsilon) =$ $(b_{min}+N\epsilon)\dfrac{\theta k_i}{k_i+k_j}$ $\pi_j(b_{min}+N\epsilon, b_{min}+N\epsilon) =$ $(b_{min}+N\epsilon)\dfrac{\theta k_j}{k_i+k_j}$

$$\pi_i(b;\theta,k) = \begin{cases} b_j k_i & \text{if } b_i < b_j \\ b_i \dfrac{k_i}{k_i+k_j}\theta & \text{if } b_i = b_j \\ b_i(\theta - k_j) & \text{if } b_i > b_j \end{cases} \tag{2}$$

When player i submits the lower bid, it sells its entire production capacity k_i, and player j sets the price b_j. Therefore, player i's payoffs are $\pi_i = b_j k_i$. These are the payoffs over the diagonal in figure 2. When players i and j submit the same bid, the payoff is split among them in proportion to their production capacity $\pi_i = b_i \dfrac{\theta k_i}{k_i+k_j}$.[6] These are the payoffs on the diagonal in figure 2. When player i submits the higher bid, it satisfies the residual demand $(\theta - k_j)$, and it sets the price b_i. Therefore, player i's payoffs are $\pi_i = b_i(\theta - k_j)$. These are the payoffs below the diagonal in figure 2.

[6]The tie breaking rule implemented in this game is crucial since it determines if the game is a Hawk-Dove or a Battle of the Sexes game. According to Cabrales, García-Fontes and Motta (2000), the Battle of the Sexes game defined in Luce and Raiffa (1957) and the Hawk-Dove game defined in Binmore (1992) are equivalent. However, the payoff matrix in Benndorf, Martínez-Martínez and Normann (2016) (left-hand panel, figure 3) and the one in Belleflamme and Peitz (2015) (right-hand panel, figure 3) show that those games are different. Moreover, Tirole and Fudenberg (1991) study the Hawk-Dove and the Battle of the Sexes games, but the matrix that they present to characterize the Hawk-Dove game does not coincide with the one in Benndorf, Martínez-Martínez and Normann (2016). In this paper, we assume that a game has the structure of a Hawk-Dove game when it follows the structure presented in Benndorf, Martínez-Martínez and Normann (2016). When the auction is discriminatory, the tie-breaking rule is also very important, but for different reasons. In that case, the tie-breaking rule is important to guarantee the existence of the equilibrium (Blázquez, 2018; Dasgupta and Maskin, 1986; Fabra, von der Fehr and Harbord, 2006; Osborne and Pitchik, 1986).

Figure 3: Generalized Hawk-Dove and Battle of the Sexes payoff matrices

Hawk-Dove game ($a > b > c > d$)

b_i \ b_j	H	D
H	d, d	a, c
D	c, a	b, b

Battle of the Sexes game ($a, b > c, d, e, f$)

b_i \ b_j	H	D
H	a, b	c, d
D	e, f	b, a

Equilibrium: The uniform price auction described above has multiplicity of pure strategies equilibria defined by:

$$b_i^* = b_{max},\, b_j^* = \frac{b_{max}(\theta - k_j)}{k_i} \;\forall i, j \tag{3}$$

In each of the equilibria defined in equation 3, one player submits the maximum bid (dove strategy), and the other submits a bid that makes undercutting unprofitable (hawk strategy). When the players are asymmetric in production capacity, the set of equilibria in which the player with higher production capacity submits the maximum bid is larger than the set of equilibria in which the player with lower production capacity submits the maximum bid. Those sets of equilibria are represented in the dark grey cells in figure 2. To provide a better understanding of the uniform price auction and the set of equilibria in that game, in annex 2 (figure 14), we provide an illustrative example taken from Bigoni, Blázquez and le Coq (2021).

The players have opposite preferences on both sets of equilibria. Both players prefer the set of equilibria in which the opposing player submits the maximum bid (dove strategy), since in that case the player that is dispatched first sells its entire production capacity at the highest possible price. It is possible that both players coordinate by submitting the maximum bid. In that case, the price perceived by the players is the maximum bid and the players split the profit in proportion to their production capacity. However, it is very difficult to coordinate on this pair of strategies, since both players have incentives to deviate and to sell their entire production capacity at the maximum bid.

In the next three sections, we apply the tracing procedure method proposed by Harsanyi and Selten (1988), the robustness to strategic uncertainty method proposed by Andersson, Argenton, and Weibull (2014), and the quantal response method proposed by McKelvey and Palfrey (1998) to analyze which of the described equilibria is played in the game.

3 Tracing procedure method

In this section, we present the tracing procedure method proposed by Harsanyi and Selten (1988). We study which equilibrium is selected by the tracing procedure method when we apply it to the uniform price auction presented in the model section. We also study the importance of the structural parameters of the model (production capacity and demand), and the market design parameter (the minimum bid allowed by the auctioneer) in determining the equilibrium selected by the tracing procedure method.

The tracing procedure method assumes that players' payoffs are a linear combination of the original payoff matrix and the expected payoff matrix based on players' beliefs:

$$\pi_i = t\pi_i(b_i, b_j) + (1-t)\pi_i(b_i, p_j), \tag{4}$$

In equation 4, $\pi_i(b_i, b_j)$ represents the original payoff matrix (figure 2), and $\pi_i(b_i, p_j)$ represents the expected payoff matrix based on players' beliefs. In $\pi_i(b_i, p_j)$, p_j is the probability that player j assigns to each strategy based on player i's beliefs. Therefore, when $t = 0$, players' payoffs are determined only by players' expected payoff based on their prior beliefs. When $t = 1$, players' payoffs are determined only by the original payoff matrix.

In general, at $t = 0$ the players choose a pair of strategies that is not an equilibrium of the original game. When t increases the players change their strategies. At some $t \in [0, 1]$, the players chose a pair of strategies that is an equilibrium of the original game (Harsanyi and Selten, 1988). That pair of strategies (b_i^*, b_j^*) will be the equilibrium selected by the tracing procedure method. Therefore, the key point in the tracing procedure method is to find the player that first deviates to a Nash equilibrium in the original game, and to find the parameter t for which the deviation occurs.

Lemma 1: When all players have uniform prior beliefs about the strategies played by other players, and when the demand is low, players maximize their expected payoff by submitting the minimum bid. When the demand is high, players maximize their expected payoff by submitting the maximum bid.

When player i submits a bid equal to $b_i = b_{min} + h\epsilon$, its expected payoffs are defined by:[7]

$$\pi_i(b_i = b_{min} + h\epsilon, p_j) \quad = \quad (b_{min} + h\epsilon)(\theta - k_j)\frac{h}{N} + \frac{2b_{min} + (N+h)\epsilon}{2}k_i\frac{N-h}{N} \tag{5}$$

The first term in equation 5, $\left((b_{min} + h\epsilon)(\theta - k_j)\dfrac{h}{N}\right)$, represents player i's expected payoff when it submits the higher bid. Player i submits the higher bid with probability $\left(\dfrac{h}{N}\right)$. In that case, player i sets the price in the auction $(b_{min} + h\epsilon)$ and it satisfies the residual demand $(\theta - k_j)$. The second term, $\left(\dfrac{2b_{min} + (N+h)\epsilon}{2}k_i\dfrac{N-h}{N}\right)$, represents player i's expected payoff when it submits the lower bid in the auction. Player i submits the lower bid with probability $\left(\dfrac{N-h}{N}\right)$. In that case, player j sets the price, which in expectation is $\left(\dfrac{2b_{min} + (N+h)\epsilon}{2}\right)$, and player i sells its entire production capacity (k_i).[8]

[7]The proof of lemma 1 is in the annex. However, in this section, we introduce parts of the proof, since that help to introduce key concepts that are useful to understand the intuition behind the results presented in this section.

[8]The expected payoff function, the first order conditions, the stationary points and the second order conditions used in lemma 1 are in annex 1. In this section, we introduce some of the equations in annex 1 to facilitate the understanding of lemma 1 and proposition 1.

The two terms in equation 5 determine player i's trade off between submitting a high and a low bid. On one hand, player i wants to increase its bid to maximize its payoff when it is dispatched last in the auction. On the other hand, player i wants to submit a low bid to increase the probability of being dispatched first in the auction. This trade off is determined by the structural parameters of the model (θ, k_i, k_j), and by the market design parameter of the model (b_{min}). To determine the strategy that maximizes players' expected payoff, it is necessary to work out equation 5's first order conditions, critical points and second order conditions.

The first order conditions of equation 5 are determined by:

$$\frac{\partial \pi_i(b_i = bmin + h\epsilon, p_j)}{\partial h} = \frac{1}{N}((\theta - k_j)(bmin + 2h\epsilon) - k_i(bmin + \epsilon h)) \tag{6}$$

The first term in equation 6, $\left(\frac{1}{N}(\theta - k_j)(b_{min} + 2h\epsilon)\right)$, is positive, which means that if the players are dispatched last in the auction they want to increase their bids as much as possible. The second term in equation 6, $(-k_i(b_{min} + \epsilon h))$, is negative, which means that if the players are dispatched first in the auction they want to submit as low bid as possible.

After rearranging equation 6 to write it as a function of the structural parameters, we obtain:

$$\frac{\partial \pi_i(b_i = b_{min} + h\epsilon, p_j)}{\partial h} = \frac{1}{N}(b_{min}(\theta - k_j - k_i) + h\epsilon(2\theta - 2k_j - k_i)) \tag{7}$$

The first term in equation 7 is always negative, and the second term is negative if $2\theta < 2k_j + k_i$. Therefore, if $2\theta < 2k_j + k_i$, $\frac{\partial \pi_i(b_i = b_{min} + h\epsilon, p_j)}{\partial h}$ is negative and player i maximizes its expected profit by submitting the minimum bid allowed by the auctioneer. In contrast, if $2\theta \geq 2k_j + k_i$, player i could maximize its expected profit at b_{min}, at a critical point, or at b_{max}. Therefore, it is necessary to work out the critical points and the second order conditions to know which strategy maximizes the expected payoff. The critical points are determined by:

$$\frac{\partial \pi_i(b_i = bmin + h\epsilon, p_j)}{\partial h} = 0 \Rightarrow h = \frac{-bmin(\theta - k_i - k_j)}{\epsilon(2\theta - 2k_i - k_j)} \tag{8}$$

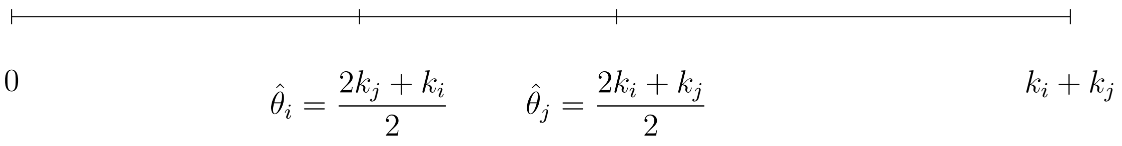
The second order conditions are determined by:

$$\frac{\partial \pi_i(b_i = bmin + h\epsilon, p_j)}{\partial h^2} = \epsilon(2\theta - 2k_j - k_i) \tag{9}$$

The second order conditions determine that the critical point is a global minimum. Therefore, the players maximize their expected payoffs by submitting the minimum bid or the maximum bid.

By using equation 7, we can establish the demand thresholds that determine if the players maximize their payoffs by submitting the minimum or the maximum bid. In particular, by using equation 7, we obtain player i's demand threshold $\left(\hat{\theta}_i = \frac{2k_j + k_i}{2}\right)$. If

Figure 4: Demand thresholds players i, j



we work out the first order conditions for player j, we obtain $\left(\hat{\theta}_j = \frac{2k_i + k_j}{2}\right)$. Given that in the model section, we assume that $k_i > k_j$, then $\hat{\theta}_i < \hat{\theta}_j$ (figure 4). These two thresholds determine the demand for which the players prefer to submit the minimum or the maximum bid. This result is formalized in proposition 1.

Proposition 1. When $t = 0$, the tracing procedure method selects one of the three possible types of equilibria.

i. Low demand: Both players submit a bid equal to the minimum bid allowed by the auctioneer.

ii. Intermediate demand: The player with higher production capacity submits the maximum bid, and the player with lower production capacity submits the minimum bid allowed by the auctioneer.

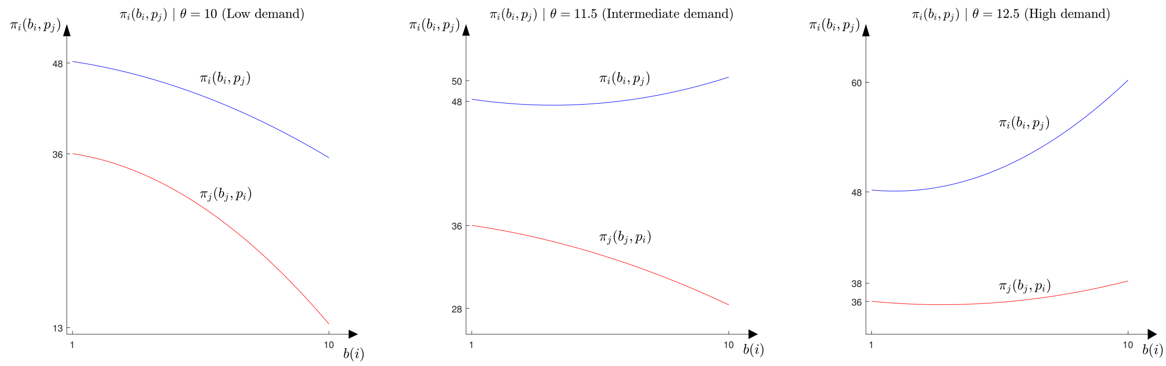
iii. High demand: Both players submit a bid equal to the maximum bid.

When $t = 0$, players' total payoffs are determined only by the expected payoffs. When the demand is low $\left(\theta < min\left\{\hat{\theta}_i, \hat{\theta}_j\right\}\right)$, players' residual demand is very low, and it is very risky for them to submit a high bid. Therefore, both players maximize their expected payoffs by submitting the minimum bid allowed by the auctioneer (left-hand panel, figure 5).[9] In this case, the equilibrium selected by the tracing procedure method is $b_i = b_j = b_{min}$. When the demand is intermediate ($\hat{\theta}_i \leq \theta \leq \hat{\theta}_j$), the player with higher production capacity faces a high residual demand and maximizes its expected payoff by submitting the maximum bid. In contrast, the player with lower production capacity faces a low residual demand and maximizes its expected payoff by submitting the minimum bid allowed by the auctioneer (central panel, figure 5). In that case, the equilibrium selected by the tracing procedure method is $b_i = b_{max}, b_j = b_{min}$. Finally, when the demand is high $\left(\theta > max\left\{\hat{\theta}_i, \hat{\theta}_j\right\}\right)$, both players face a high residual demand and they maximize their expected payoff by submitting the maximum bid allowed by the auctioneer (right-hand panel, figure 5). In that case, the equilibrium selected by the tracing procedure method is $b_i = b_j = b_{max}$.

It is easy to check that when $t = 0$ and the demand is intermediate, the tracing procedure method immediately selects one of the Nash equilibria in the original game. Therefore, it is not necessary to conduct further analysis. In contrast, when $t = 0$ and the demand is low or high, the players do not select one of the Nash equilibria in the initial game, and it is necessary to determine which player deviates first to a Nash equilibrium in the original game, and to find the parameter t for which that player deviates to the

[9] The examples in all the sections and in annex 2 are adaptations of the examples in Bigoni, Blázquez and Le Coq (2021).

Figure 5: Expected payoffs $k_i = 8.7$, $k_j = 6.5$, $b_{min} = 1$, $b_{max} = 10$, $N = 110$



equilibrium. Proposition 2 formalizes this analysis.

Proposition 2. When the demand is low or intermediate, the tracing procedure method selects the equilibrium in which the player with higher production capacity submits the maximum bid, and the player with lower production capacity submits the minimum bid allowed by the auctioneer. When the demand is high, no equilibrium is selected by the tracing procedure method.

When $t > 0$, players' profits are determined not only by their expected payoffs, but also by the payoffs in the original game. These are worked out in equation 2 and represented in the payoff matrix in figure 2. Therefore, to understand which equilibrium is selected by the tracing procedure method when $t > 0$, it is necessary to study the payoff functions $\pi_i(b_i, b_j)$ and $\pi_i(b_i, p_j)$ for different demand realizations.

According to proposition 1, when $t = 0$ and the demand is low, both players submit the minimum bid allowed by the auctioneer (left-hand panel, figure 5). In this case, as we show in lemma 1, the expected payoff ($\pi_i(b_i, p_j)$) decreases in b_i. Therefore, the players maximize their expected payoff by submitting the lower bid allowed by the auctioneer, independent of the value of t. In contrast, in the original game, if one player submits the minimum bid allowed by the auctioneer, the other player maximizes its payoff by submitting the maximum bid. Given that when $t > 0$, the players assign more weight to the original payoff matrix, when t is large enough, one of the players has incentives to deviate by raising its bid.

When the players are asymmetric, the player with higher production capacity faces a high residual demand and it deviates first by submitting a higher bid. However, the player never deviates to the maximum bid since player i's expected payoff decreases with b_i, and when t is close to 0, the expected payoff has a lot of weight in player i's total payoff. Therefore, when t increases, but is still close to 0, player i deviates by submitting an intermediate bid (left-hand panel, figure 6).

Once player i raises its bid, player j has no incentives to deviate, since in that case, player j is better off for two reasons: First, by submitting the lower bid in the auction, player j maximizes its expected payoff. Second, when player i submits a bid higher than the minimum bid allowed by the auctioneer, and that bid is high enough, player j maximizes its payoff by submitting the lower bid allowed by the auctioneer, since in that case, the player sells its entire production capacity for a high price (equation 2). Therefore, when t increases, player i continues raising its bid until finally submitting the maximum bid. In that moment, both players select a pair of strategies that are an equilibrium in the original game, and that is the equilibrium selected by the tracing procedure method (left-hand panel, figure 6).

According to proposition 1, when $t = 0$ and when the demand is intermediate, the tracing procedure method selects the equilibrium in which player i submits the maximum bid and player j submits the minimum bid allowed by the auctioneer (central panel, figure 5). Given that the equilibrium selected by the tracing procedure method when $t = 0$ is an equilibrium in the original game, the players do not deviate from this equilibrium when $t > 0$.

Finally, based on the results in proposition 1, when $t = 0$, and the demand is high, both players submit the maximum bid. In this case, as we show in lemma 1, the expected payoff ($\pi_i(b_i, p_j)$) is concave and achieves its maximum when $b_i = b_{max}$ (right-hand panel, figure 5). Therefore, the players maximize their expected payoff by submitting the maximum bid, independent of the value of t. In contrast, in the original game, if one player submits the maximum bid, the other player maximizes its payoff by decreasing its bid. In that case, the player that submits the lower bid sells its entire production capacity at the price set by the opposing player (equation 2). When t increases, the players assign more weight to the original payoff matrix. Therefore, when t is large enough, one of the players has incentives to deviate by decreasing its bid. In contrast with the low demand case where the player that does not deviate is better off, when the demand is high, the player that does not deviates is worse off, since it is dispatched last and satisfies the residual demand. That situation triggers a price war in which both players undercut each other. The price war finishes when the bids are low enough, since in that case at least one player has incentives to satisfy the residual demand by submitting the maximum bid. However, once that one player increases its bid, the other player slightly undercuts the bid to be dispatched first in the auction at a higher price, and the price war begins again. Therefore, a pure strategies equilibrium does not exist when $t > 0$ and the tracing procedure method does not select any of the equilibria in the original game.[10]

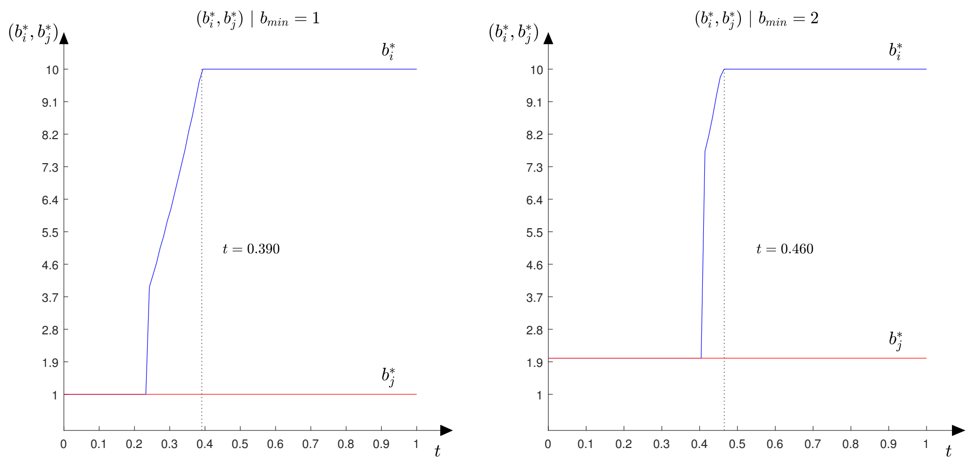
In proposition 2, we found a closed form solution to study the equilibrium that is selected by the tracing procedure method depending on the structural parameters of the model (production capacity and demand), and on the market design parameter (minimum bid allowed by the auctioneer). We conclude this section by using that closed form solution to study the impact that an increase in the minimum bid allowed by the auctioneer has on the equilibrium selected by the tracing procedure method.

Proposition 3. An increase in the minimum bid allowed by the auctioneer has two effects on the convergence to the equilibrium.

i. The parameter t for which the players deviate from the equilibrium when $t = 0$ increases.

[10] In annex 2, we provide an example to show that when the demand is high the tracing procedure method does not select any of the equilibria in the original game.

Figure 6: Tracing procedure method $k_i = 8.7$, $k_j = 6.5$, $\theta = 10$, $N = 110$



ii. The parameter t for which the players coordinate in one of the Nash equilibria of the original game increases.

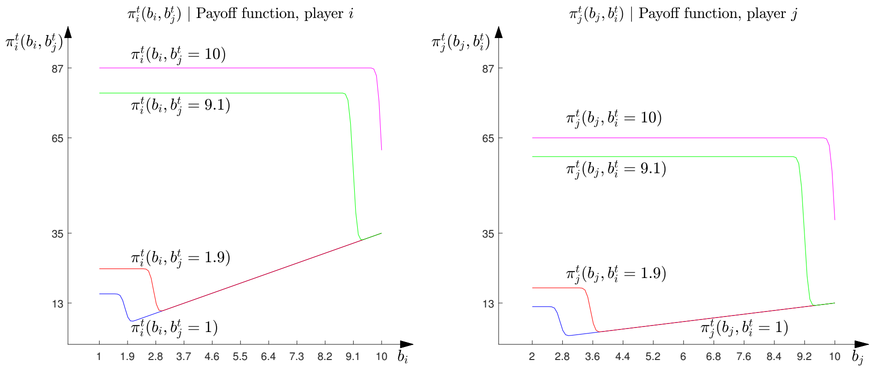
An increase in the minimum bid allowed by the auctioneer makes submitting a low bid more attractive, since in this case both players split the demand in proportion to their production capacity at a higher price. Therefore, the parameter t that makes them deviate from the equilibrium $b_i = b_{min}, b_j = b_{min}$ in the tracing procedure method when $t = 0$ increases.

Due to the increase in the minimum bid allowed by the auctioneer, the parameter t for which the players move from one equilibrium to the next in the tracing procedure method also increases. Therefore, the parameter t for which the players coordinate in one of the Nash equilibria of the original game increases. As can be observed in the left-hand side of figure 6, when the minimum bid allowed by the auctioneer is low, the parameter t for which the players coordinate in one of the equilibria of the game is $t = 0.390$. However, as can be observed in the right-hand panel of that figure, when the minimum bid allowed by the auctioneer increases to $b_{min} = 2$, the corresponding value of t is $t = 0.460$.

In the numerical simulations, we have also observed that the bid that makes players indifferent between submitting the minimum bid or deviate to a higher bid also increases. Therefore, when the players deviate from the original equilibrium in the tracing procedure method, they deviate by submitting a higher bid. As can be observed in the left-hand panel in figure 6, when the minimum bid allowed by the auctioneer is low $(b_{min=1})$, the player with higher production capacity deviates first by increasing its bid to 3.72. However, as can be observed in the right-hand panel of that figure, when the minimum bid allowed by the auctioneer increases to $b_{min} = 2$, the player with higher production capacity deviates by increasing its bid to 7.43.

The theoretical results in proposition 3 have important welfare implications, since an increase in the minimum bid makes coordinating in one of the equilibria of the game more difficult to the players. Therefore, the expected equilibrium price can be lower since the players do not coordinate in one of the equilibria in which the equilibrium price is equal to the maximum bid.

Figure 7: Players' payoff functions ($k_i = 8.7$, $k_j = 6.5$, $\theta = 10$, $b_{min} = 1$, $b_{max} = 10$, $N = 110$)



4 Robustness to strategic uncertainty method

In this section we present the robustness to strategic uncertainty method proposed by Andersson, Argenton and Weibull (2014). We study the equilibrium selected by this method when we apply it to the uniform price auction presented in the model section and we analyze the impact that the structural parameters (production capacity and demand) and the market design parameter (minimum bid allowed by the auctioneer) have in determining the equilibria that are selected.

The robustness to strategic uncertainty method proposes that players face some uncertainty about the strategies played by other players. Player i's uncertainty about player j's strategy is modeled as follows:

$$b_{ij}^t = b_j^t + t\epsilon_{i,j} \; \forall j \neq i, \tag{10}$$

where the random variables $\epsilon_{i,j} \sim \phi_{ij}$ are statistically independent.

Equation 10 can be interpreted as follows: player i thinks that player j will play strategy b_j plus some random perturbation. When the uncertainty parameter (t-parameter) goes to zero, the players do not face any uncertainty.

For $t > 0$, the random variable $b_{i,j}$ has probability density defined by:

$$f_{i,j}^t = \frac{1}{t}\phi_{i,j}\left(\frac{x - b_j^t}{t}\right)$$

The profit function is defined by:

$$\pi_i^t(b) = E\left[\pi(b_i, b_{-i}^t)\right] = \int \left[\pi_i(b_i, x_{-i}) f_{i,j}^t(b_j)\right] \partial x_{-i} \tag{11}$$

A pair of strategies (b_i^*, b_j^*) is a t-equilibrium of the game if b_i^* and b_j^* maximize 11. Therefore, to find the t-equilibrium of the game is enough to work out the best response functions and to find the intersection between them.

If we apply equation 11 to the payoff function in the uniform price auction model defined by equation 2, we obtain:

$$\begin{aligned}\pi_i^t(b_i, b_j^t) &= E\left[\pi(b_i, b_j^t)\right] = \int \left[\pi_i(b_i, x_{-i}) f_{i,j}^t(b_j)\right] \partial x_{-i} \\ &= b_j k_i \left[1 - \Phi_{i,j}\left(\frac{b_i - b_j}{t}\right)\right] + b_i(\theta - k_j)\left[\Phi_{i,j}\left(\frac{b_i - b_j}{t}\right)\right],\end{aligned} \tag{12}$$

where $\Phi_{i,j}$ is the cumulative distribution function of $\phi_{i,j}$. The first term in equation 12, $\left(b_j k_i \left[1 - \Phi_{i,j}\left(\frac{b_i - b_j}{t}\right)\right]\right)$, represents player i's expected payoff when it submits the lower bid in the auction. With probability $\left[1 - \Phi_{i,j}\left(\frac{b_i - b_j}{t}\right)\right]$ player i submits the lower bid in the auction. In that case, player j sets the price (b_j), and player i sells its entire production capacity (k_i). The second term in equation 12, $\left(b_i(\theta - k_j)\left[\Phi_{i,j}\left(\frac{b_i - b_j}{t}\right)\right]\right)$, represents supplier i's expected payoff when it submits the higher bid in the auction. With probability $\left[\Phi_{i,j}\left(\frac{b_i - b_j}{t}\right)\right]$ player i submits the higher bid in the auction. In that case, player i sets the price (b_i) and satisfies the residual demand $(\theta - k_j)$.

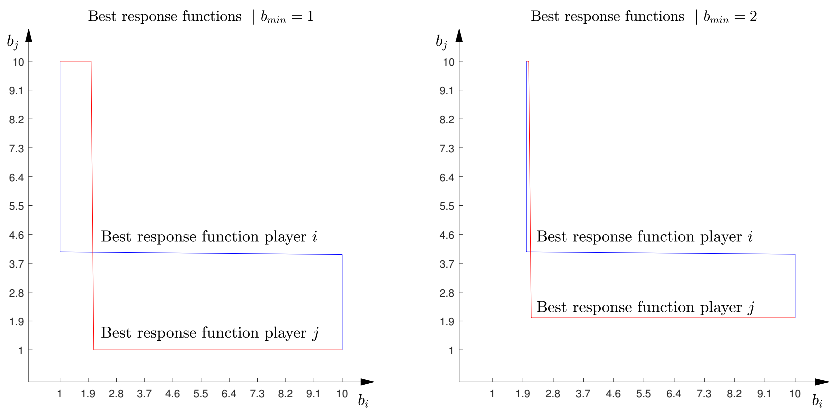
By using equation 12, it is easy to work out one player's expected payoff given the strategy of the other player. In particular, we set b_j and vary b_i between b_{min} and b_{max}. Knowing that the random variable $b_{i,j}$ has probability density defined by $f_{i,j}^t = \frac{1}{t}\phi_{i,j}\left(\frac{x - b_j^t}{t}\right)$, and if, as in Andersson, Argenton and Weibull (2014), we assume that $f_{i,j}^t \sim N(0,1)$, we work out player i' expected payoff $\pi_i^t(b_i, b_j^t)$, and we choose b_i that maximizes that payoff. Repeating that process for every $b_j \in [b_{min}, b_{max}]$, we work out player i's best response function. By using the same approach we work out player j's best response function. The intersection between both players' best response functions determines the equilibrium selected by the robustness to strategic uncertainty method.

To understand the best response functions, it is useful to work out the expected payoff for one player when we fix the strategy played by the other player. In the left-hand side of figure 7, we plot four of the expected payoff functions for player i. When player j sets a low bid, player i maximizes its expected payoff by submitting the maximum bid. In contrast, when player j sets a high bid, player i maximizes its expected payoff by submitting low bids. In both cases, as can be observed in figure 7, the players' expected payoff functions are concave and therefore, players maximize their expected payoff by submitting the minimum or the maximum bid, but never by submitting intermediate bids. Moreover, the players shift their best strategies around the bid that determines the threshold to work out the Nash equilibrium in the uniform price auction (equation 3).[11] For the parameters in the example, the bid that defines the threshold is between 3.7 and 4.6. As can be observed in the left-hand side of figure 7, when player j increases its bid, the bid that maximizes player i's payoff shift from 10 to 1.

The analysis of the players' expected payoff functions is useful to understand the equilibrium selected by the robustness to strategic uncertainty method. In figure 8, we plot

[11] The threshold that determine the Nash equilibrium in the uniform price auction is determined in the model section (equation 3), and it can be observed in the dark grey areas in figure 14

Figure 8: Players' best response functions ($k_i = 8.7$, $k_j = 6.5$, $\theta = 10$, $b_{max} = 10$, $N = 110$)



the players' best response functions. When one player submits a low bid, the best strategy for the other player is to submit a high bid. The players shift from low to high bids when the opponent player submits a bid around the threshold that determines the set of Nash equilibria (equation 3). The intersection of the best response functions selects two of the Nash equilibria in the original game. In each of these two equilibria, one player submits the maximum bid and the other player submits the minimum bid allowed by the auctioneer.

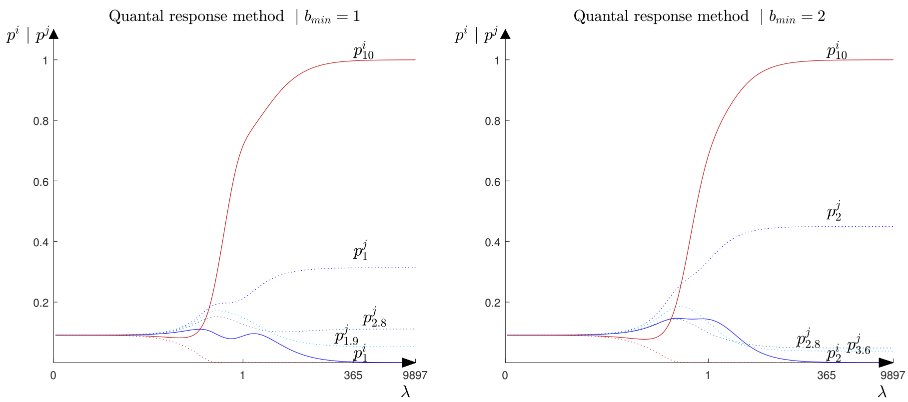
We conclude this section studying the impact that an increase in the minimum bid allowed by the auctioneer has on the equilibrium selected by the tracing procedure method. An increase in the minimum bid allowed by the auctioneer reduces the set of Nash equilibria (right-hand panel, figure 14). However, given that the maximum bid does not change, the threshold that defines the Nash equilibria also does not change (equation 3). When we apply the robustness to strategic uncertainty method to the game where the minimum bid is higher, we observe that an increase in the minimum bid shrinks the best response functions, but the threshold around which the players shift from high to low bids given the other player strategies does not change. Moreover, the equilibrium selected by the robustness to strategic uncertainty method also does not change. As in the case when the minimum bid allowed by the auctioneer is lower, the robustness to strategic uncertainty method selects the equilibria in which one player submits the minimum bid and the other player submits the maximum bid (right-hand panel, figure 8).

5 Quantal response method

As in the previous sections, we present the quantal response method and we study the equilibrium selected by this method when we apply it to the uniform price auction presented in the model section.

The quantal response method proposed by McKelvey and Palfrey (1998) assumes that the players choose among the strategies in the game based on their relative expected payoff. The key idea is that when the players calculate their expected payoff, they make calculation errors according to some random process. Based on that random process, the players assign more probability to the strategies that give then a higher expected

Figure 9: Quantal response method ($k_i = 8.7$, $k_j = 6.5$, $\theta = 10$, $b_{max} = 10$, $N = 11$)



payoff. The Nash equilibrium in the quantal response method is the set of probabilities for which none of the players wants to deviate. Formally, the Nash equilibrium in the quantal response method is defined as follows: Given $\{\lambda_1, \lambda_2, ...\}$ a sequence such that $\lim_{t\longrightarrow\infty} \lambda_t = \infty$, and $\{p_1, p_2, ...\}$ a corresponding sequence with $p_t \in \pi^*(\lambda_t)$ for all t, such that $\lim_{t\longrightarrow\infty} p_t = p^*$, then p^* is a Nash equilibrium.

In their seminal paper, McKelvey and Palfrey (1998) use the logistic quantal response function. That specific function is a particular parametric class of quantal response functions that has a long tradition in the study of individual choice behaviour. The logit equilibrium correspondence is the correspondence $\pi^* : \Re \longrightarrow 2^{\triangle}$ given by:

$$\pi^*(\lambda) = \left\{ \pi \in \triangle : \pi_{ij} = \frac{e^{\lambda \overline{u}_{ij}(\pi)}}{\sum_{k=1}^{J_i} e^{\lambda \overline{u}_{ik}(\pi)}} \forall i, j \right\}, \tag{13}$$

where the term in the numerator $\left(e^{\lambda \overline{u}_{ij}(\pi)}\right)$ is one of the players' expected payoff when it selects strategy i and the oppose player selects strategy j. The term in the denominator $\left(\sum_{k=1}^{J_i} e^{\lambda \overline{u}_{ik}(\pi)}\right)$ is the sum of one of the players' expected payoff when it selects strategy i and the oppose player selects all the strategies in its strategies set. Therefore, by using equation 13 each player assigns more probability to the strategies that give it higher expected payoff.

When we apply the quantal response method to the uniform price auction presented in the model section we observe that the player with higher production capacity (player i) plays the maximum bid with a probability close to one. In contrast, the player with lower production capacity (player j) assigns higher probabilities to the lower bids (left-hand panel, figure 9).

The equilibrium selected by the quantal response method is in line with the equilibrium selected by the tracing and the robustness to strategic uncertainty methods. Moreover, the pattern that appears in the equilibrium selected by the quantal response method is very similar to the pattern that appears in the other two methods, since in the three methods the players tend to select extreme strategies. In particular, the player with higher production capacity submits the maximum bid and the player with lower production capacity submits the lower bids in the strategies support with higher probabilities. Moreover, the quantal response method shows similarities to the tracing procedure method, since the quantal response method defines a unique selection from the set of Nash equilibrium by "tracing" the graph of the logit equilibrium correspondence beginning at the centroid of the strategy simplex (the unique solution when $\lambda = 0$) and continuing for larger and larger values of λ.

We conclude this section by analyzing the effect that an increase in the minimum bid allowed by the auctioneer has on the equilibrium selected by the quantal response method. After an increase in the minimum bid allowed by the auctioneer, player i assigns probability one to the maximum bid allowed by the auctioneer and player j assigns higher probabilities to the lower bids in the strategies' support (left-hand panel, figure 9). An increase in the minimum bid does not change the strategy of the player with higher production capacity, since it still faces a high residual demand and does not change its strategy, but makes more attractive for the player with lower production capacity to submit lower bids, since for that player, the expected payoff associated to low bids increases. Therefore, an increase in the minimum bid allowed in the auction facilitates the coordination in the equilibrium of the game where the player with higher production capacity submits the maximum bid allowed by the auctioneer. This contrasts with the results that we obtain when we apply the tracing procedure method, where an increase in the minimum bid allowed by the auctioneer reduces the possibilities that the players coordinate in that equilibrium.[12]

6 Conclusion

We study a uniform price auction with a continues set of strategies that has the structure of a Hawk-Dove game with multiplicity of Nash equilibria. In each of those equilibria, one of the players submits the maximum bid (dove strategy) and the other player submits a bid that makes undercutting unprofitable (hawk strategy). We apply the tracing procedure method (Harsanyi and Selten, 1988), the robustness to strategic uncertainty method (Andersson, Argenton and Weibull, 2014) and the quantal response method (McKelvey and Palfrey, 1998) to predict which equilibrium is selected by the players. We also study the impact that the structural parameters of the model (demand and players' production capacities) and the market design parameter (minimum bid allowed by the auctioneer) have on the equilibrium selected by the players.

The tracing procedure method proposed by Harsanyi and Selten (1988) selects the equilibrium in which the player with higher production capacity submits the maximum bid (dove strategy) and the player with lower production capacity submits the minimum bid allowed by the auctioneer (hawk strategy). When the auctioneer increases the minimum bid allowed in the auction, the equilibrium selected by the players does not change, but the coordination in that equilibrium is more difficult.

With independence of suppliers' production capacities, the robustness to strategic uncertainty method proposed by Andersson, Argenton and Weibull (2014) selects the two

[12]It is important to remark, that this result is only true when the players are asymmetric. When the players are symmetric an increase in the minimum bid increases the possibilities of miscoordination, since in that case, both players assigns higher probabilities to the lower bid and that complicates the coordination in one of the equilibria of the game. For a detailed analysis of the symmetric case see Bigoni, Blazquez and le Coq (2018).

equilibria in which one of the players submits the maximum bid (dove strategy) and the other player submits the minimum bid allowed by the auctioneer (hawk strategy). When the auctioneer increases the lower bid allowed in the auction, the equilibria selected by the players do not change.

The quantal response method proposed by McKelvey and Palfrey (1998) predicts that the player with higher production capacity submits the maximum bid (dove strategy) and the player with lower production capacity submits the lower bids in the strategies set with higher probabilities (hawk strategy). An increase in the minimum bid allowed by the auctioneer does not change the strategy of the player with higher production capacity, but increases the probability that the low production capacity player assigns to the minimum bid, and the coordination in one of the equilibria of the game is easier.

This paper contributes to frame the experimental design, and helps to predict and understand the experimental results in Bigoni, Blázquez and Le Coq (2021). In that paper, the authors conduct an experimental analysis of a Hawk-Dove game similar to the one presented in this paper. The authors find more probable that the players coordinate in the equilibrium where the player with higher production capacity submits the maximum bid (dove strategy), and the player with lower production capacity submits the minimum bid allowed by the auctioneer (hawk strategy). This result is in line with the three equilibrium selection methods analyzed in this paper. The authors also find that an increase in the minimum bid allowed by the auctioneer makes the coordination in one of the equilibria slightly more difficult. This result is in line with the predictions in Harsanyi and Selten (1988), but not with the ones in McKelvey and Palfrey (1998).

The theoretical results that we present in this paper contribute to the industrial organization and the trade literature that endogenize the emergence of a price leader in duopoly models. It also provides a theoretical framework to study evolutionary biology problems when the players can choose among a continue set of strategies.

We frame the theoretical analysis as a uniform price auction with the structure of a Hawk-Dove game with a continuous set of strategies. Therefore, the theoretical framework extend the literature of Hawk-Dove games where the players can choose only among two different strategies.

The theoretical analysis that we have developed also gives us the opportunity to compare the equilibrium played in a uniform price auction with the one played in a discriminatory price auction. Therefore, the theoretical framework that we develop could be useful to compare the results among different types of auctions (uniform vs. discriminatory price auctions).

Annex 1

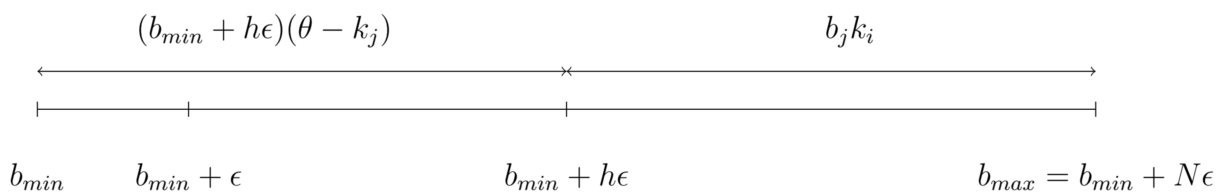
Lemma 1.

In lemma 1, we study the relation between the structural parameters of the model (demand and production capacities) and players' expected payoff. When player i submit a bid equal to $b_i = b_{min} + h\epsilon$, its expected payoff is defined by $\pi_i(b_i = b_{min} + h\epsilon, p_j)$. Where b_{min} is the minimum bid allowed by the auctioneer, h is an integer number, $\epsilon = \dfrac{b_{max} - b_{min}}{N}$ is the increase between one bid and the next one, and can be as small as we want by increasing N.[13] N is the total number of bids in the strategies set. The set of strategies is represented in figure 10. The probability that player j assigns to each bid in the strategies set is defined by p_j. If the probability that each player assign to each bid (p_j) follows a uniform distribution, the close form solution of the expected payoff function for player i when it submits a bid $b_i = b_{min} + h\epsilon$ is worked out as follows:

$$
\begin{aligned}
\pi_i(b_i = b_{min} + h\epsilon, p_j) &= \int_{b_{min}}^{b_{min}+h\epsilon} (b_{min} + h\epsilon)(\theta - k_j) f(b_j)\partial b_j + \int_{b_{min}+h\epsilon}^{b_{min}+N\epsilon} b_j k_i f(b_j)\partial b_j \\
&= (b_{min} + h\epsilon)(\theta - k_j)\frac{(b_{min} + h\epsilon) - b_{min}}{(b_{min} + N\epsilon) - b_{min}} + \\
&\quad \frac{(b_{min} + N\epsilon) + (b_{min} + h\epsilon)}{2} k_i \frac{(b_{min} + N\epsilon) - (b_{min} + h\epsilon)}{(b_{min} + N\epsilon) - b_{min}} \\
&= (b_{min} + h\epsilon)(\theta - k_j)\frac{h}{N} + \frac{2b_{min} + (N+h)\epsilon}{2} k_i \frac{N-h}{N} \qquad (14)
\end{aligned}
$$

The first term in equation 14 represents player i' expected payoff when it submits the higher bid in the auction. With probability $\left(\dfrac{(b_{min} + h\epsilon) - b_{min}}{(b_{min} + N\epsilon) - b_{min}}\right)$ player i submits the higher bid in the auction. In that case, player i sets the equilibrium price $(b_{min} + h\epsilon)$ and satisfies the residual demand $(\theta - k_j)$. The second term in equation 14 represents player i's expected payoff when player j submits the higher bid in the auction. With probability $\left(\dfrac{(b_{min} + N\epsilon) - (b_{min} + h\epsilon)}{(b_{min} + N\epsilon) - b_{min}}\right)$ player j submits the higher bid in the auction. In that case, player j sets the equilibrium price that in expectation is equal to $\dfrac{(b_{min} + N\epsilon) + (b_{min} + h\epsilon)}{2}$, and player i sells its entire production capacity (k_i).

Figure 10: Strategies set, and payoffs



To work out the bid that maximize the expected profit, it is necessary to work out equation 14's first order conditions.

$$
\begin{aligned}
\frac{\partial \pi_i(b_i = b_{min} + h\epsilon, p_j)}{\partial h} &= \frac{(\theta - k_j)}{N}(h\epsilon + (b_{min} + h\epsilon)) + \\
&\quad \frac{k_i}{2N}(\epsilon(N-h) - (2b_{min} + (N+h)\epsilon)) \\
&= \frac{1}{N}((\theta - k_j)(b_{min} + 2h\epsilon) - k_i(b_{min} + \epsilon h)) \qquad (15)
\end{aligned}
$$

[13]We assume that N is large. That guarantees the set of strategies is continuous, i.e., the probability of a tie is nil. By doing that we can work out easily player i's expected payoff given player j's probability distribution function.

Where the first term in equation 15 $\left(\frac{1}{N}((\theta - k_j)(b_{min} + 2h\epsilon)\right)$ is positive, which means that when the player satisfies the residual demand, it increases its payoffs by increasing its bid. By the contrary, the second term in equation 15 $(k_i(b_{min} + \epsilon h))$ is negative, which means that when the player satisfies the total demand it prefers to decreases its bid to increase its chances to be dispatched first in the auction.

After rearranging equation 15 to write it as a function of the structural parameters, we obtain:

$$\frac{\partial \pi_i(b_i = b_{min} + h\epsilon, p_j)}{\partial h} = \frac{1}{N}(b_{min}(\theta - k_j - k_i) + h\epsilon(2\theta - 2k_j - k_i)) \tag{16}$$

The first term in equation 16 is always negative, and the second term is negative if $2\theta < 2k_j + k_i$. Therefore, if $2\theta < 2k_j + k_i$, $\frac{\partial \pi_i(b_i = b_{min} + h\epsilon, p_j)}{\partial h}$ is negative and player i maximizes its expected profit by submitting the lower bid in the auction. In contrast, if $2\theta \geq 2k_j + k_i$, player i could maximize its expected payoff at b_{min}, at a critical point, or at b_{max}. Therefore, it is necessary to work out the critical points and the second derivative to know which strategy maximizes the expected payoff.

To work out the critical points, it is enough to equalize equation 16 to zero. Therefore,

$$\begin{aligned} \frac{\partial \pi_i(b_i = b_{min} + h\epsilon, p_j)}{\partial h} &= 0 \Leftrightarrow \frac{1}{N}(b_{min}(\theta - k_j - k_i) + h\epsilon(2\theta - 2k_j - k_i)) = 0 \Leftrightarrow \\ h &= \frac{-b_{min}(\theta - k_j - k_i)}{\epsilon(2\theta - 2k_j - k_i)}. \end{aligned} \tag{17}$$

To know if the critical point is a maximum or a minimum, it is necessary to work out the second derivative.

$$\frac{\partial \pi_i(b_i = b_{min} + h\epsilon, p_j)}{\partial h^2} = \epsilon(2\theta - 2k_i - k_j) \tag{18}$$

Given that we are studying the case in which $2\theta > 2k_j + k_i$, the second term in equation 16 is positive, then equation 18 is also positive. Therefore, the expected payoff is a concave function, and the critical point defined by equation 17 is a global minimum. Hence, player i maximizes it profits by submitting the lower or the higher bid allowed by the auctioneer, but not by submitting a bid equal to the critical point $b_{min} + \epsilon h = b_{min} + \epsilon \frac{-b_{min}(\theta - k_2 - k_1)}{\epsilon(2\theta - 2k_2 - k_1)}$.

Figure 11: Demand thresholds players i, j

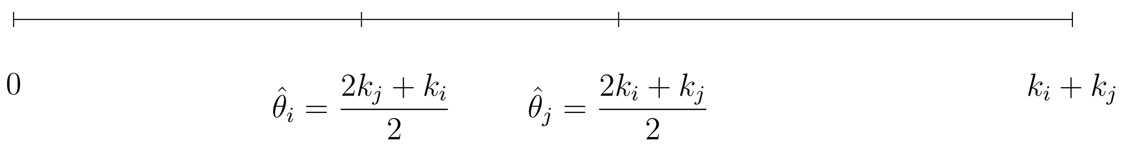


Figure 11 summarize the analysis derived from the first and the second order conditions (equations 16, 17 and 18). When the demand is low $\left(0 < \theta \leq \frac{2k_j + k_i}{2}\right)$, both players maximize their expected payoff by submitting the minimum bid allowed by the auctioneer (left-hand panel, figure 5). When the demand is intermediate $\left(\frac{2k_j + k_i}{2} < \theta \leq \frac{2k_i + k_j}{2}\right)$, player i maximizes it payoff by submitting the maximum bid allowed by the auctioneer and player j maximizes it payoff by submitting the minimum bid allowed by the auctioneer (central panel, figure 5). When the demand is high $\left(\frac{2k_i + k_j}{2} < \theta \leq k_i + k_j\right)$, both players maximize their expected payoff by submitting the maximum bid allowed by the auctioneer (right-hand panel, figure 5). $\square$

Proposition 1. When $t = 0$ players' total payoff are determined only by the expected payoff. Therefore, the proof of proposition 1 is straight forward using the results of lemma 1. $\square$

Proposition 2. Proposition 2 analyzes the equilibrium selected by the tracing procedure method when the demand is low (i), intermediate (ii), or high (iii). We analyze each case.

Case i: Low demand ($2\theta \leq 2k_j + k_i$). We prove that when the demand is low, the tracing procedure method selects the equilibrium in which the player with higher production capacity submits the maximum bid allowed by the auctioneer and the player with lower production capacity submits the minimum bid allowed by the auctioneer.

In proposition 1, we show that when $t = 0$, the tracing procedure method selects the equilibrium in which both players submit the minimum bid in the auction. However, that pair of strategies is not an equilibrium in the original game. Therefore, the key point in the tracing procedure method is to find the pair of strategies for each value of t until the players select a pair of strategies that are an equilibrium in the original game.

Within the proof, we use the next notation:

$t_{i,1} : (b_{min}, b_{min}) \rightarrow (b_{min} + h\epsilon, b_{min})$. $t_{i,1}$ solves equation 4 for player i, when it increases its bid from b_{min} to $b_{min} + h\epsilon$ while player j's bid is b_{min} (figure 12).
$t_{i,2} : (b_{min} + h\epsilon, b_{min}) \rightarrow (b_{min} + H\epsilon, b_{min})$. $t_{i,1}$ solves equation 4 for player i, when it increases its bid from $b_{min} + h\epsilon$ to $b_{min} + H\epsilon$ while player j's bid is b_{min} (figure 12).
$t_{j,1} : (b_{min}, b_{min}) \rightarrow (b_{min}, b_{min} + h\epsilon)$. $t_{j,1}$ solves equation 4 for player j, when it increases its bid from b_{min} to $b_{min} + h\epsilon$ while player i's bid is b_{min} (figure 12).
$t_{j,2} : (b_{min}, b_{min} + h\epsilon) \rightarrow (b_{min}, b_{min} + H\epsilon)$. $t_{j,2}$ solves equation 4 for player j, when it increases its bid from $b_{min} + h\epsilon$ to $b_{min} + H\epsilon$ while player i's bid is b_{min} (figure 12).
$t_{j,3} : (b_{min} + h\epsilon, b_{min} + h\epsilon) \rightarrow (b_{min} + h\epsilon, b_{min} + H\epsilon)$. $t_{j,3}$ solves equation 4 for player j, when it increases its bid from $b_{min} + h\epsilon$ to $b_{min} + H\epsilon$ while player i's bid is $b_{min} + h\epsilon$ (figure 12).
$t_{j,4} : (b_{min} + h\epsilon, b_{min}) \rightarrow (b_{min} + h\epsilon, b_{min} + H\epsilon)$. $t_{j,4}$ solves equation 4 for player j, when it increases its bid from b_{min} to $b_{min} + H\epsilon$ while player i's bid is $b_{min} + h\epsilon$ (figure 12).

The proof consists of two steps. In step one, we prove that the player with higher production capacity (player i) deviates first by increasing its bid from b_{min} to $b_{min} + h\epsilon$, i.e., $t_{i,1} < t_{j,1}$. In step two, we prove that once that player i deviates by increasing its bid from b_{min} to $b_{min} + h\epsilon$, it deviates first again by increasing its bid from $b_{min} + h\epsilon$ to $b_{min} + H\epsilon$, i.e., $t_{i,2} < t_{j,4}$. Prove step two directly is difficult, and we prove it in three different steps: $t_{i,2} < t_{j,2} < t_{j,3} < t_{j,4}$. Below, we present a sketch of the four steps of the proof, and we explain the intuition behind each step.

In step one, we prove that the player with higher production capacity (player i) deviates first from the pair of strategies ($b_i = b_{min}, b_j = b_{min}$) by increasing its bid, i.e., we prove that $t_{i,1} < t_{j,1}$, where $t_{i,1}$ solves $t_{i,1}\pi_i(b_{min}, b_{min}) + (1 - t_{i,1})\pi_i(b_{min}, p_j) = t_{i,1}\pi_i(b_{min} + h\epsilon, b_{min}) + (1 - t_{i,1})\pi_i(b_{min} + h\epsilon, p_j)$, and $t_{j,1}$ solves $t_{j,1}\pi_j(b_{min}, b_{min}) + (1 - t_{j,1})\pi_j(p_i, b_{min}) = t_{j,1}\pi_j(b_{min}, b_{min} + h\epsilon) + (1 - t_{j,1})\pi_j(p_i, b_{min} + h\epsilon)$ ($t_{i,1}$ and $t_{j,1}$ are represented in figure 12). Step one proves that the player with higher production capacity adopts a "dove" strategy in the game where players' payoff functions are defined by the tracing procedure method profits equation 4.

In the rest of the proof (steps two, three and four), we need to show that once the player with higher production capacity increases its bid, the player with lower production capacity never deviates. The intuition is simple, once that in the first step, the player with higher production capacity increases its bid by adopting a "dove" strategy, the player that adopts a "hawk" strategy is better off, and never deviates from the strategy in which it submits the minimum bid. Therefore, the player with higher production capacity continues increasing its bid until the players select the pair of strategies ($b_i = b_{max}, b_j = b_{min}$). That will be the equilibrium in the original game selected by the tracing procedure method. Prove that result directly is complex, since it is necessary to show that $t_{i,2} < t_{j,4}$ (figure 12). Therefore, we prove it in three different steps. In step two, we prove that $t_{i,2} < t_{j,2}$. In step three, we prove that $t_{j,2} < t_{j,3}$. In step four, we prove that $t_{j,3} < t_{j,4}$. Putting together steps two, three and four, we obtain $t_{i,2} < t_{j,2} < t_{j,3} < t_{j,4}$.

Figure 12: Proposition 2. Case i: Low demand ($2\theta \leq 2k_j + k_i$)

$b_i \backslash b_j$	b_{min}	$\cdots$	$b_{min}+h\epsilon$	$\cdots$	$b_{min}+H\epsilon$	$\cdots$
b_{min}	$t_i\pi_i(b_{min},b_{min})+$ $(1-t_i)\pi_i(b_{min},p_j)$ $t_j\pi_j(b_{min},b_{min})+$ $(1-t_j)\pi_j(p_i,b_{min})$	$\cdots$ $\xrightarrow[t_{j,1}]{}$	$\pi_j(b_{min},b_{min}+h\epsilon)+$ $(1-t_j)\pi_j(p_i,b_{min}+h\epsilon)$	$\cdots$ $\xrightarrow[t_{j,2}]{}$	$t_j\pi_j(b_{min},b_{min}+H\epsilon)=$ $(1-t_j)\pi_j(p_i,b_{min}+H\epsilon)$	$\cdots$
$\cdots$	$\cdots \downarrow t_{i,1}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$b_{min}+h\epsilon$	$t_i\pi_i(b_{min}+h\epsilon,b_{min})+$ $(1-t_i)\pi_i(b_{min}+h\epsilon,p_j)$ $t_j\pi_j(b_{min}+h\epsilon,b_{min})+$ $(1-t_j)\pi_j(p_i,b_{min}+h\epsilon)$	$\cdots$ $t_{j,4}$	$t_j\pi_j(b_{min}+h\epsilon,b_{min}+h\epsilon)+$ $(1-t_j)\pi_j(p_i,b_{min}+h\epsilon)$	$\cdots$ $\xrightarrow[t_{j,3}]{}$	$t_j\pi_j(b_{min}+h\epsilon,b_{min}+H\epsilon)+$ $(1-t_j)\pi_j(p_i,b_{min}+H\epsilon)$	$\cdots$
$\cdots$	$\cdots \downarrow t_{i,2}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$b_{min}+H\epsilon$	$t_i\pi_i(b_{min}+H\epsilon,b_{min})+$ $(1-t_i)\pi(b_{min}+H\epsilon,p_j)$	$\cdots$		$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

In step two, we prove that $t_{i,2} < t_{j,2}$, where $t_{i,2}$ solves $t_{i,2}\pi_i(b_{min}+h\epsilon, b_{min}) + (1-t_{i,2})\pi_i(b_{min}+h\epsilon, p_j) = t_{i,2}\pi_i(b_{min}+H\epsilon, b_{min}) + (1-t_{i,2})\pi_i(b_{min}+H\epsilon, p_j)$, and $t_{j,2}$ solves $t_{j,2}\pi_j(b_{min}, b_{min}+h\epsilon) + (1-t_{j,2})\pi_j(p_i, b_{min}+h\epsilon) = t_{j,2}\pi_j(b_{min}, b_{min}+H\epsilon) + (1-t_{j,2})\pi_j(p_i, b_{min}+H\epsilon)$, where $h < H$ ($t_{i,2}$ and $t_{j,2}$ are represented in figure 12).

In step three, we prove that $t_{j,2} < t_{j,3}$, where $t_{j,3}$ solves $t_{j,3}\pi_j(b_{min}+h\epsilon, b_{min}+h\epsilon) + (1-t_{j,3})\pi_j(p_i, b_{min}+h\epsilon) = t_{j,3}\pi_j(b_{min}+h\epsilon, b_{min}+H\epsilon) + (1-t_{j,3})\pi_j(p_i, b_{min}+H\epsilon)$ ($t_{j,3}$ is represented in figure 12).

In step four, we prove that $t_{j,3} < t_{j,4}$, where $t_{j,4}$ solves $t_{j,4}\pi_j(b_{min}+h\epsilon, b_{min}) + (1-t_{j,4})\pi_j(p_i, b_{min}) = t_{j,4}\pi_j(b_{min}+h\epsilon, b_{min}+H\epsilon) + (1-t_{j,4})\pi_j(p_i, b_{min}+H\epsilon)$ ($t_{j,4}$ is represented in figure 12).

In the rest of the proof, we explain in detail each of the four steps. In step one, we prove that $t_{i,1} < t_{j,1}$, where $t_{i,1}$ solves $t_{i,1}\pi_i(b_{min}, b_{min}) + (1-t_{i,1})\pi_i(b_{min}, p_j) = t_{i,1}\pi_i(b_{min}+h\epsilon, b_{min}) + (1-t_{i,1})\pi_i(b_{min}+h\epsilon, p_j)$.

$$t_{i,1}\pi_i(b_{min}, b_{min}) + (1-t_{i,1})\pi_i(b_{min}, p_j) =$$
$$t_{i,1} b_{min}\frac{k_i\theta}{k_i+k_j} + (1-t_{i,1})\frac{2b_{min}+N\epsilon}{2}k_i\frac{N}{N}, \text{ and}$$

$$t_{i,1}\pi_i(b_{min}+h\epsilon, b_{min}) + (1-t_{i,1})\pi_i(b_{min}+h\epsilon, p_j) =$$
$$t_{i,1}(b_{min}+h\epsilon)(\theta-k_j) + (1-t_{i,1})\left[(b_{min}+h\epsilon)(\theta-k_j)\frac{h}{N} + \frac{2b_{min}+(N+h)\epsilon}{2}k_i\frac{N-h}{N}\right].$$

Therefore, $t_{i,1}$ is defined implicitly by:

$$t_{i,1} b_{min}\frac{k_i\theta}{k_i+k_j} + (1-t_{i,1})\frac{2b_{min}+N\epsilon}{2}k_i\frac{N}{N} =$$
$$t_{i,1}(b_{min}+h\epsilon)(\theta-k_j) + (1-t_{i,1})\left[(b_{min}+h\epsilon)(\theta-k_j)\frac{h}{N} + \frac{2b_{min}+(N+h)\epsilon}{2}k_i\frac{N-h}{N}\right] \quad (19)$$

And explicitly by the close form solution:

$$t_{i,1} = \frac{n_{i,1}(\cdot)}{d_{i,1}(\cdot)}, \text{ where}$$

$$\begin{aligned}
n_{i,1}(\cdot) &= (b_{min} + h\epsilon)(\theta - k_j)\frac{h}{N} + \frac{2b_{min} + (N+h)\epsilon}{2}k_i\frac{N-h}{N} - \frac{2b_{min} + N\epsilon}{2}k_i\frac{N}{N} \\
d_{i,1}(\cdot) &= b_{min}\frac{k_i - \theta}{k_i + k_j} - (b_{min} + h\epsilon)(\theta - k_j) + n_{i,1}(\cdot)
\end{aligned}$$

By doing some algebra in $n_{i,1}(\cdot)$, we obtain:

$$\begin{aligned}
(b_{min} + h\epsilon)(\theta - k_j)\frac{h}{N} &= (b_{min} + h\epsilon)(2\theta - 2k_j)\frac{h}{2N} \\
\frac{2b_{min} + (N+h)\epsilon}{2}k_i\frac{N-h}{N} &= (2b_{min} + N\epsilon)k_i\frac{N-h}{2N} + h\epsilon k_i\frac{N-h}{2N} \\
\frac{2b_{min} + N\epsilon}{2}k_i\frac{N}{N} &= \frac{b_{min} + b_{min} + (N-h)\epsilon + h\epsilon}{2}k_i\frac{(N-h)+h}{N} \\
&= (b_{min} + h\epsilon)k_i\frac{h}{2N} + (b_{min} + h\epsilon)k_i\frac{N-h}{2N} + \ldots \\
&\quad (b_{min} + (N-h)\epsilon)k_i\frac{h}{2N} + (b_{min} + (N-h)\epsilon)k_i\frac{N-h}{2N} \\
&= (b_{min} + h\epsilon)k_i\frac{h}{2N} + (2b_{min} + N\epsilon)k_i\frac{N-h}{2N} + (b_{min} + (N-h)\epsilon)k_i\frac{h}{2N}
\end{aligned}$$

Therefore,

$$\begin{aligned}
n_{i,1}(\cdot) &= \left((b_{min} + h\epsilon)(2\theta - 2k_j)\frac{h}{2N} + (2b_{min} + N\epsilon)k_i\frac{N-h}{2N} + h\epsilon k_i\frac{N-h}{2N}\right) - \\
&\quad \left((b_{min} + h\epsilon)k_i\frac{h}{2N} + (2b_{min} + N\epsilon)k_i\frac{N-h}{2N} + (b_{min} + (N-h)\epsilon)k_i\frac{h}{2N}\right) \\
&= \left((b_{min} + h\epsilon)(2\theta - 2k_j - k_i)\frac{h}{2N} + h\epsilon k_i\frac{N-h}{2N}\right) - \\
&\quad \left(b_{min}k_i\frac{h}{2N} + (N-h)\epsilon k_i\frac{h}{2N}\right) \\
&= (b_{min} + h\epsilon)(2\theta - 2k_j - k_i)\frac{h}{2N} - b_{min}k_i\frac{h}{2N}, \\
d_{i,1}(\cdot) &= \left(b_{min}\frac{k_i\theta}{k_i + k_j} - (b_{min} + h\epsilon)(\theta - k_j)\right) + \left((b_{min} + h\epsilon)(2\theta - 2k_j - k_i)\frac{h}{2N} - b_{min}k_i\frac{h}{2N}\right), \text{ and} \\
t_{i,1} &= \frac{(b_{min} + h\epsilon)(2\theta - 2k_j - k_i)\frac{h}{2N} - b_{min}k_i\frac{h}{2N}}{\left(b_{min}\frac{k_i\theta}{k_i + k_j} - (b_{min} + h\epsilon)(\theta - k_j)\right) + \left((b_{min} + h\epsilon)(2\theta - 2k_j - k_i)\frac{h}{2N} - b_{min}k_i\frac{h}{2N}\right)} \qquad (20)
\end{aligned}$$

In equation 20, $n_{i,1}(\cdot) \leq 0$, since the we are studying the low demand case $(2\theta \leq 2k_j + k_i)$; and $d_{i,1}(\cdot) \leq 0$, since $2\theta \leq 2k_j + k_i$, and $b_{min}\frac{k_i\theta}{k_i + k_j} \leq (b_{min} + h\epsilon)(\theta - k_j)$. The last inequality holds to guarantee that equation 19 is satisfied: By proposition 1, we know that when the demand is low, the expected payoff is decreasing, i.e., $(1 - t_{i,1})\pi_i(b_{min}, p_j) = (1 - t_{i,1})\frac{2b_{min} + N\epsilon}{2}k_i\frac{N}{N} \geq (1 - t_{i,1})\pi_i(b_{min} + h\epsilon, p_j) = (b_{min} + h\epsilon)(\theta - k_j)\frac{h}{N} + \frac{2b_{min} + (N+h)\epsilon}{2}k_i\frac{N-h}{N}$. Therefore, to guarantee that equation 19 is satisfied, it is necessary that $t_{i,1}\pi_i(b_{min}, b_{min}) = t_{i,1}b_{min}\frac{k_i\theta}{k_i + k_j} \leq t_{i,1}\pi_i(b_{min} + h\epsilon, b_{min}) = t_{i,1}(b_{min+h\epsilon})(\theta - k_j)$. Therefore, $b_{min}\frac{k_i\theta}{k_i + k_j} \leq (b_{min} + h\epsilon)(\theta - k_j)$, and that guarantees that $d_{i,1}(\cdot) \leq 0$. Moreover, $d_{i,1}(\cdot) \leq$

$n_{i,1}(\cdot) \leq 0$, and that guarantees that $0 < t_{i,1} \leq 1$.

By using equation 20, we can prove that $t_{i,1} \leq t_{j,1}$. In particular, given that $d_{i,1}(\cdot) \leq n_{i,1}(\cdot) \leq 0 \; \forall i, j$, it is enough to show that $t_{i,1} \leq t_{j,1}$ if $n_{i,1}(\cdot) \geq n_{j,1}(\cdot)$, and if $d_{i,1}(\cdot) \leq d_{j,1}(\cdot)$.

First, we prove that $n_{i,1}(\cdot) \geq n_{j,1}(\cdot)$:

$$\begin{aligned} n_{i,1}(\cdot) - n_{j,1}(\cdot) &= (b_{min} + h\epsilon)\frac{h}{2N}(k_i - k_j) - b_{min}\frac{h}{2N}(k_i - k_j) \\ &= (k_i - k_j)\frac{h}{2N}(b_{min} + h\epsilon - b_{min}) = \frac{k_i - k_j}{2N}\epsilon h^2 \geq 0, \end{aligned} \tag{21}$$

where the inequality holds since $k_i \geq k_j$.

Second, we prove that $d_{i,1}(\cdot) \leq d_{j,1}(\cdot)$:

$$\begin{aligned} d_{i,1}(\cdot) - d_{j,1}(\cdot) &= \frac{b_{min}\theta}{k_i + k_j}(k_i - k_j) - (b_{min} + h\epsilon)(\theta - k_j - \theta + k_i) + \frac{k_i - k_j}{2N}h^2\epsilon \\ &= (k_i - k_j)\left(\frac{b_{min}\theta}{k_i + k_j} - (b_{min} + h\epsilon)\right) + \frac{k_i - k_j}{2N}h^2\epsilon \\ &= \frac{k_i - k_j}{2N(k_i + k_j)}\left(2Nb_{min}\theta + h^2\epsilon(k_i + k_j) - 2N(k_i + k_j)b_{min} - 2N(k_i + k_j)h\epsilon\right) \\ &= \frac{k_i - k_j}{2N(k_i + k_j)}\left(2Nb_{min}(\theta - (k_i + k_j)) + h\epsilon(k_i + k_j)(h - 2N)\right) \leq 0, \end{aligned} \tag{22}$$

where the inequality holds since,

$$\begin{aligned} \frac{k_i - k_j}{2N(k_i + k_j)} &\geq 0, \text{ since } k_i \geq k_j; \\ 2Nb_{min}(\theta - (k_i + k_j)) &\leq 0, \text{ since } \theta \leq (k_i + k_j); \text{ and,} \\ h\epsilon(k_i + k_j)(h - 2N) &\leq 0, \text{ since } h \leq N. \end{aligned}$$

Therefore, $t_{i,1} < t_{j,1}$.

In step two, we prove $t_{i,2} < t_{j,2}$, where $t_{i,2}$ solves $t_{i,2}\pi_i(b_{min} + h\epsilon, b_{min}) + (1 - t_{i,2})\pi_i(b_{min} + h\epsilon, p_j) = t_{i,2}\pi_i(b_{min} + H\epsilon, b_{min}) + (1 - t_{i,2})\pi_i(b_{min} + H\epsilon, p_j)$.

$$\begin{aligned} &t_{i,2}\pi_i(b_{min} + h\epsilon, b_{min}) + (1 - t_{i,2})\pi_i(b_{min} + h\epsilon, p_j) = \\ &t_{i,2}(b_{min} + h\epsilon)(\theta - k_j) + (1 - t_{i,2})\left((b_{min} + h\epsilon)(\theta - k_j)\frac{h}{N} + \frac{2b_{min} + (N + h)\epsilon}{2}k_i\frac{N - h}{N}\right), \text{ and} \end{aligned}$$

$$\begin{aligned} &t_{i,2}\pi_i(b_{min} + H\epsilon, b_{min}) + (1 - t_{i,2})\pi_i(b_{min} + H\epsilon, p_j) = \\ &t_{i,2}(b_{min} + H\epsilon)(\theta - k_j) + (1 - t_{i,2})\left((b_{min} + H\epsilon)(\theta - k_j)\frac{H}{N} + \frac{2b_{min} + (N + H)\epsilon}{2}k_i\frac{N - H}{N}\right), \text{ where } h \leq H. \end{aligned}$$

Therefore, $t_{i,2}$ is defined implicitly by:

$$\begin{aligned} &t_{i,2}(b_{min} + h\epsilon)(\theta - k_j) + (1 - t_{i,2})\left((b_{min} + h\epsilon)(\theta - k_j)\frac{h}{N} + \frac{2b_{min} + (N + h)\epsilon}{2}k_i\frac{N - h}{N}\right) = \\ &t_{i,2}(b_{min} + H\epsilon)(\theta - k_j) + (1 - t_{i,2})\left((b_{min} + H\epsilon)(\theta - k_j)\frac{H}{N} + \frac{2b_{min} + (N + H)\epsilon}{2}k_i\frac{N - H}{N}\right) \end{aligned} \tag{23}$$

And explicitly by the close form solution:

$$t_{i,2} = \frac{n_{i,2}(\cdot)}{d_{i,2}(\cdot)}, \text{ where}$$

$$\begin{aligned}
n_{i,2}(\cdot) &= \left((b_{min} + H\epsilon)(\theta - k_j)\frac{H}{N} + \frac{2b_{min} + (N+H)\epsilon}{2}k_i\frac{N-H}{N}\right) - \\
&\quad \left((b_{min} + h\epsilon)(\theta - k_j)\frac{h}{N} + \frac{2b_{min} + (N+h)\epsilon}{2}k_i\frac{N-h}{N}\right) \\
d_{i,2}(\cdot) &= ((b_{min} + h\epsilon)(\theta - k_j)) - ((b_{min} + H\epsilon)(\theta - k_j)) + n_{i,2}(\cdot)
\end{aligned}$$

By doing some algebra in $n_{i,2}(\cdot)$ and $d_{i,2}(\cdot)$, we obtain:

$$\begin{aligned}
n_{i,2}(\cdot) &= \frac{\theta - k_j}{N}\left(b_{min}(H-h) + \epsilon(H^2 - h^2)\right) + \frac{k_i}{2N}\left(2b_{min}(N - H - N + h) + \epsilon(N^2 - H^2 - N^2 + h^2)\right) \\
&= \frac{\theta - k_j}{N}\left(b_{min}(H-h) + \epsilon(H-h)(H+h)\right) + \frac{k_i}{2N}\left(2b_{min}(h-H) + \epsilon(h-H)(h+H)\right) \\
&= \frac{\theta - k_j}{N}\left((H-h)(b_{min} + \epsilon(H+h))\right) + \frac{k_i}{2N}\left((h-H)(2b_{min} + \epsilon(h+H))\right) \\
&= \frac{2\theta - 2k_j}{2N}\left((H-h)(b_{min} + \epsilon(H+h))\right) - \frac{k_i}{2N}\left((H-h)(b_{min} + \epsilon(h+H))\right) - \frac{k_i}{2N}\left((H-h)b_{min}\right) \\
&= \frac{2\theta - 2k_j - k_i}{2N}\left((H-h)(b_{min} + \epsilon(H+h))\right) - \frac{k_i}{2N}\left((H-h)b_{min}\right), \\
d_{i,2}(\cdot) &= ((\theta - k_j)\epsilon(h-H)) + \frac{2\theta - 2k_j - k_i}{2N}\left((H-h)(b_{min} + \epsilon(H+h))\right) - \frac{k_i}{2N}\left((H-h)b_{min}\right), \text{ and} \\
t_{i,2} &= \frac{\frac{2\theta - 2k_j - k_i}{2N}\left((H-h)(b_{min} + \epsilon(H+h))\right) - \frac{k_i}{2N}\left((H-h)b_{min}\right)}{((\theta - k_j)\epsilon(h-H)) + \frac{2\theta - 2k_j - k_i}{2N}\left((H-h)(b_{min} + \epsilon(H+h))\right) - \frac{k_i}{2N}\left((H-h)b_{min}\right)} \qquad (24)
\end{aligned}$$

In equation 24, $n_{i,2}(\cdot) \leq 0$, since the we are studying the low demand case ($2\theta \leq 2k_j + k_i$) and $h \leq H$. And $d_{i,1}(\cdot) \leq 0$, since ($2\theta \leq 2k_j + k_i$), $h \leq H$ and $k_j \leq \theta$. Moreover, $d_{i,2}(\cdot) \leq n_{i,2}(\cdot) \leq 0$, and that guarantees that $0 < t_{i,2} \leq 1$.

By using equation 24, we can prove that $t_{i,2} \leq t_{j,2}$. In particular, given that $d_{i,2}(\cdot) \leq n_{i,2}(\cdot) \leq 0 \; \forall i, j$, it is enough to show that $n_{i,2}(\cdot) \geq n_{j,2}(\cdot)$, and that $d_{i,2}(\cdot) \leq d_{j,2}(\cdot)$.

First, we prove that $n_{i,2}(\cdot) \geq n_{j,2}(\cdot)$:

$$\begin{aligned}
n_{i,2}(\cdot) - n_{j,2}(\cdot) &= \frac{2\theta - 2k_j - k_i - 2\theta + 2k_i + k_j}{2N}\left(b_{min}(H-h) + \epsilon(H^2 - h^2)\right) - \left(\frac{b_{min}(H-h)}{2N}(k_i - k_j)\right) \\
&= \frac{k_i - k_j}{2N}\left(b_{min}(H-h) - b_{min}(H-h) + \epsilon(H^2 - h^2)\right) = \frac{k_i - k_j}{2N}\epsilon(H^2 - h^2) \geq 0, \qquad (25)
\end{aligned}$$

where the inequality holds since $k_i \geq k_j$ and $H \geq h$.

Second, we prove that $d_{i,2}(\cdot) \leq d_{j,2}(\cdot)$:

$$\begin{aligned}
d_{i,2}(\cdot) - d_{j,2}(\cdot) &= \epsilon(h-H)(\theta - k_j - \theta + k_i) + \frac{k_i - k_j}{2N}\epsilon(H^2 - h^2) \\
&= \frac{k_i - k_j}{2N}\left(2\epsilon N(h-H) + \epsilon(H-h)(H+h)\right) \\
&= \frac{k_i - k_j}{2N}\left(\epsilon(H-h)(H+h) - 2\epsilon N(H-h)\right) \\
&= \frac{k_i - k_j}{2N}\left(\epsilon(H-h)((H+h) - 2N)\right) \leq 0, \qquad (26)
\end{aligned}$$

where the inequality holds since,

$$\begin{aligned}
\frac{k_i - k_j}{2N} &\geq 0, \text{ since } k_i \geq k_j;\\
(H - h) &\geq 0, \text{ since } H \geq h; \text{ and,}\\
(H + h) - 2N &\leq 0, \text{ since } h < N,\ H \leq N, \text{ then } (H + h) \leq 2N.
\end{aligned}$$

Therefore, $t_{i,2} < t_{j,2}$.

In step three, we prove that $t_{j,2} < t_{j,3}$, where $t_{j,3}$ solves:

$$\begin{gathered}
t_{j,2}\pi_j(b_{min}, b_{min} + h\epsilon) + (1 - t_{j,2})\pi_j(p_i, b_{min} + h\epsilon) = \\
t_{j,2}\pi_j(b_{min}, b_{min} + H\epsilon) + (1 - t_{j,2})\pi_j(p_i, b_{min} + H\epsilon),
\end{gathered}$$

and $t_{j,3}$ solves:

$$\begin{gathered}
t_{j,3}\pi_j(b_{min} + h\epsilon, b_{min} + h\epsilon) + (1 - t_{j,3})\pi_j(p_i, b_{min} + h\epsilon) = \\
t_{j,3}\pi_j(b_{min} + h\epsilon, b_{min} + H\epsilon) + (1 - t_{j,3})\pi_j(p_i, b_{min} + H\epsilon).
\end{gathered}$$

Therefore, the close form solutions for $t_{j,2}$, and $t_{j,3}$ are:

$$\begin{aligned}
t_{j,2} &= \frac{n_{j,2}(\cdot)}{d_{j,2}(\cdot)} = \frac{\pi_j(p_i, b_{min} + H\epsilon) - \pi_j(p_i, b_{min} + h\epsilon)}{(\pi_j(b_{min}, b_{min} + h\epsilon) - \pi_j(b_{min}, b_{min} + H\epsilon)) + n_{j,2}(\cdot)}\\
t_{j,3} &= \frac{n_{j,3}(\cdot)}{d_{j,3}(\cdot)} = \frac{\pi_j(p_i, b_{min} + H\epsilon) - \pi_j(p_i, b_{min} + h\epsilon)}{(\pi_j(b_{min} + h\epsilon, b_{min} + h\epsilon) - \pi_j(b_{min} + h\epsilon, b_{min} + H\epsilon)) + n_{j,3}(\cdot)}.
\end{aligned}$$

Given that $n_{j,2}(\cdot) = n_{j,3}(\cdot)$, $t_{j,2} < t_{j,3}$ if $d_{j,2}(\cdot) < d_{j,3}(\cdot)$. We prove that $d_{j,2}(\cdot) < d_{j,3}(\cdot)$:

$$\begin{aligned}
d_{j,2}(\cdot) < d_{j,3}(\cdot) \iff\ & \pi_j(b_{min}, b_{min} + h\epsilon) - \pi_j(b_{min}, b_{min} + H\epsilon) < \\
& \pi_j(b_{min} + h\epsilon, b_{min} + h\epsilon) - \pi_j(b_{min} + h\epsilon, b_{min} + H\epsilon)\\
\iff\ & (b_{min} + h\epsilon)(\theta - k_i) - (b_{min} + H\epsilon)(\theta - k_i) < \\
& (b_{min} + h\epsilon)\frac{\theta k_j}{k_i + k_j} - (b_{min} + H\epsilon)(\theta - k_i)\\
\iff\ & \frac{b_{min} + h\epsilon}{k_i + k_j}((\theta k_j) - (k_i + k_j)(\theta - k_i)) = \frac{b_{min} + h\epsilon}{k_i + k_j}(\theta k_j - (k_i\theta - k_i^2 + k_j\theta - k_ik_j))\\
\iff\ & \frac{b_{min} + h\epsilon}{k_i + k_j}(k_i(k_i + k_j - \theta)) > 0,
\end{aligned}$$

where the inequality holds since $k_i + k_j \geq \theta$. Therefore, $t_{j,2} < t_{j,3}$.

In step four, we prove that $t_{j,3} < t_{j,4}$, where $t_{j,3}$ solves:

$$\begin{gathered}
t_{j,3}\pi_j(b_{min} + h\epsilon, b_{min} + h\epsilon) + (1 - t_{j,3})\pi_j(p_i, b_{min} + h\epsilon) = \\
t_{j,3}\pi_j(b_{min} + h\epsilon, b_{min} + H\epsilon) + (1 - t_{j,3})\pi_j(p_i, b_{min} + H\epsilon)
\end{gathered} \tag{27}$$

and $t_{j,4}$ solves:

$$\begin{gathered}
t_{j,4}\pi_j(b_{min} + h\epsilon, b_{min}) + (1 - t_{j,4})\pi_j(p_i, b_{min}) = \\
t_{j,4}\pi_j(b_{min} + h\epsilon, b_{min} + H\epsilon) + (1 - t_{j,4})\pi_j(p_i, b_{min} + H\epsilon)
\end{gathered} \tag{28}$$

By comparing equations 27 and 28, we can conclude that:

First, the right-hand side in equations 27 and 28 is the same for all $t \in [0,1]$.

Second, $\pi_j(b_{min}+h\epsilon, b_{min}) = (b_{min}+h\epsilon)k_i \geq (b_{min}+h\epsilon)\dfrac{\theta k_j}{k_i+k_j} = \pi_j(b_{min}+h\epsilon, b_{min}+h\epsilon)$, and by lemma 1, we know that $\pi_j(p_i, b_{min}) \geq \pi_j(p_i, b_{min}+h\epsilon)$. Therefore, the left-hand side in equation 28 is larger than the left-hand side in equation 27 for all $t \in [0,1]$. Hence, when $b_i = (b_{min}+h\epsilon)$ player j is better off by submitting $b_j = b_{min}$, than by submitting $b_j = b_{min}+h\epsilon$, and the parameter $t_{j,4}$ that makes player j deviates from $b_j = b_{min}$ to $b_j = b_{min}+H\epsilon$ is always larger than the parameter $t_{j,3}$ that makes player j deviates from $b_j = b_{min}+h\epsilon$ to $b_j = b_{min}+H\epsilon$, i.e., $t_{j,3} < t_{j,4}$.

To summarize, when the demand is low ($2\theta \leq 2k_j + k_i$), the player with higher production capacity (player i) deviates first from the pair of strategies $b_i = b_j = b_{min}$ by increasing its bid ("dove" strategy), i.e., $t_{i,1} < t_{j,1}$. Once that player i adopts a "dove" strategy by increasing its bid to $b_i = b_{min}+h\epsilon$, player j is better of by adopting a "hawk" strategy, and it never deviates from $b_j = b_{min}$, i.e., $t_{i,2} < t_{j,2} < t_{j,3} < t_{j,4}$. Therefore, player i continues increasing its bid until it submits a bid equal to the maximum bid allowed by the auctioneer. Hence, when the demand is low, the tracing procedure method selects the equilibrium in with the player with higher production capacity submits the maximum bid ($b_i = b_{max}$), and the player with lower production capacity submits the minimum bid ($b_j = b_{min}$).

Case *ii*: Intermediate demand ($2k_j + k_i < 2\theta \leq 2k_i + k_j$).

According with lemma 1, when the demand is intermediate ($2k_j + k_i < 2\theta \leq 2k_i + k_j$), the player with higher production capacity maximizes its expected payoff by submitting the maximum bid ($b_i = b_{max}$), and the player with lower production capacity maximizes it expected payoff by submitting the minimum bid ($b_j = b_{min}$). Therefore, when the demand is intermediate, the tracing procedure method selects the equilibrium in which the player with higher production capacity submits the maximum bid ($b_i = b_{max}$) and the player with lower production capacity submits the minimum bid ($b_j = b_{min}$) at $t = 0$, and the players do not deviate from that equilibrium when t increases. Therefore, when the demand is intermediate, the equilibrium selected by the tracing procedure method is ($b_i = b_{max}, b_j = b_{min}$).

Case *iii*: High demand ($2k_i + k_j < \theta$).

When the demand is high, the tracing procedure method does not select any of the equilibria of the original game. We illustrate this point with the example in figure 13. As we prove in lemma 1, when the demand is high the players maximize their expected payoff by submitting the maximum bid (left-hand panel, figure 5). Therefore, when $t = 0$, the unique Nash equilibrium is ($b_i = 10, b_j = 10$) (left-hand panel, figure 13). However, that is not an equilibrium in the original payoff matrix of the game (left-hand panel, figure 14). Therefore, when t increases one of the players has incentives to deviate (central panel, figure 14). When t increases, the player with lower production capacity deviates by undercutting the player with higher production capacity, and the Nash equilibrium is ($b_i = 10, b_j = 9.1$), which is not an equilibrium in the original payoff matrix of the game. It is important to notice, that the player that does not deviates is worse off, since it submits the higher bid and satisfies the residual demand. Therefore, as soon as t increases a little, it decreases its bid to be dispatched first in the auction, but that trigger a price war and the other player also wants to undercut. When the bid is too low, it is not profitable any more to undercut and the player with higher production capacity increases its bid, but in that case the player with lower production capacity increases its bid, but still undercutting the other player, since in that case it is dispatched first in the auction and at the same time it increases the expected payoff, since when the demand is high, the expected payoff is an increasing function. This cycle in the strategies is summarized in the right-hand panel in figure 13.[14] Therefore, when the demand is high the tracing procedure method does not select any of the equilibria of the original game. $\square$

Proposition 3.

i. The parameter t for which the players deviate from the equilibrium when $t = 0$ increases.

[14] It is important to notize that the matrices in figure 5 are rounded to the nearest integer number. This is the reason because in the matrix in the right-hand panel in figure 13 we can not observe the cycle that we describe in this paragraph, but as soon as we introduce decimals the cycle that we describe appears.

Figure 13: Payoff matrix in a uniform price auction. $k_i = 8.7$, $k_j = 6.5$, $\theta = 12.5$, $b_{min} = 1$, $b_{max} = 10$, $N = 11$

b_i \ b_j	1	1.9	2.8	3.7	4.6	5.5	6.4	7.3	8.2	9.1	10
1	36 / 48	36 / 48	36 / 48	36 / 48	36 / 48	36 / 48	37 / 48	37 / 48	38 / 48	39 / 48	39 / 48
1.9	36 / 48	36 / 48	36 / 48	36 / 48	36 / 48	36 / 48	37 / 48	37 / 48	38 / 48	39 / 48	39 / 48
2.8	36 / 48	36 / 48	36 / 48	36 / 48	36 / 48	36 / 48	37 / 48	37 / 48	38 / 48	39 / 48	39 / 48
3.7	36 / 49	36 / 49	36 / 49	36 / 49	36 / 49	36 / 49	37 / 49	37 / 49	38 / 49	39 / 49	38 / 49
4.6	36 / 50	36 / 50	36 / 50	36 / 50	36 / 50	36 / 50	37 / 50	37 / 50	38 / 50	39 / 50	39 / 50
5.5	36 / 51	36 / 51	36 / 51	36 / 51	36 / 51	36 / 51	37 / 51	37 / 51	38 / 51	39 / 51	39 / 51
6.4	36 / 52	36 / 52	36 / 52	36 / 52	36 / 52	36 / 52	37 / 52	37 / 52	38 / 52	39 / 52	39 / 52
7.3	36 / 54	36 / 54	36 / 54	36 / 54	36 / 54	36 / 54	37 / 54	37 / 54	38 / 54	39 / 54	39 / 54
8.2	36 / 56	36 / 56	36 / 56	36 / 56	36 / 56	36 / 56	37 / 56	37 / 56	38 / 56	39 / 56	39 / 56
9.1	36 / 59	36 / 59	36 / 59	36 / 59	36 / 59	36 / 59	37 / 59	37 / 59	38 / 59	39 / 59	39 / 59
10	36 / 61	36 / 61	36 / 61	36 / 61	36 / 61	36 / 61	37 / 61	37 / 61	38 / 61	39 / 61	39 / 61

b_i \ b_j	1	1.9	2.8	3.7	4.6	5.5	6.4	7.3	8.2	9.1	10
1	34 / 45	34 / 46	34 / 46	34 / 47	35 / 47	35 / 48	36 / 48	37 / 49	38 / 49	38 / 50	39 / 50
1.9	34 / 46	34 / 46	34 / 46	34 / 47	35 / 47	35 / 48	36 / 48	37 / 49	38 / 49	38 / 50	39 / 50
2.8	35 / 46	35 / 46	34 / 46	34 / 47	35 / 48	35 / 48	36 / 49	37 / 49	38 / 50	38 / 50	39 / 51
3.7	35 / 47	35 / 47	35 / 47	35 / 47	35 / 48	35 / 49	36 / 49	37 / 50	38 / 50	38 / 51	39 / 51
4.6	35 / 48	35 / 48	35 / 48	35 / 48	35 / 49	35 / 50	36 / 50	37 / 51	38 / 51	38 / 52	39 / 52
5.5	36 / 50	36 / 50	36 / 50	36 / 50	36 / 50	36 / 50	36 / 51	37 / 52	38 / 52	38 / 53	39 / 53
6.4	36 / 52	36 / 52	36 / 52	36 / 52	36 / 52	37 / 52	37 / 52	37 / 53	38 / 54	38 / 54	39 / 55
7.3	36 / 54	36 / 54	36 / 54	37 / 54	37 / 54	37 / 54	37 / 54	37 / 54	38 / 55	38 / 56	39 / 56
8.2	37 / 56	37 / 56	37 / 56	37 / 56	37 / 56	37 / 56	38 / 56	38 / 56	38 / 56	38 / 58	39 / 58
9.1	37 / 58	37 / 58	37 / 58	37 / 58	38 / 58	38 / 58	38 / 58	39 / 58	39 / 58	39 / 59	39 / 60
10	38 / 61	37 / 61	38 / 61	38 / 61	38 / 61	38 / 61	39 / 61	39 / 61	40 / 61	40 / 61	40 / 62

b_i \ b_j	1	1.9	2.8	3.7	4.6	5.5	6.4	7.3	8.2	9.1	10
1	30 / 40	30 / 41	31 / 43	31 / 45	32 / 46	33 / 48	34 / 49	35 / 51	37 / 52	38 / 54	39 / 56
1.9	31 / 40	30 / 41	31 / 43	31 / 45	32 / 46	33 / 48	34 / 49	35 / 51	37 / 53	38 / 54	39 / 56
2.8	32 / 42	32 / 42	32 / 43	31 / 45	32 / 47	33 / 48	34 / 50	35 / 51	37 / 53	38 / 54	39 / 56
3.7	33 / 44	33 / 44	33 / 44	33 / 44	32 / 47	33 / 49	34 / 50	35 / 52	37 / 53	38 / 55	39 / 57
4.6	34 / 45	34 / 45	35 / 45	35 / 45	34 / 46	33 / 49	34 / 51	35 / 53	37 / 54	38 / 56	39 / 57
5.5	36 / 47	36 / 47	36 / 47	36 / 47	36 / 47	35 / 49	34 / 52	35 / 54	37 / 55	38 / 57	39 / 58
6.4	37 / 50	37 / 50	37 / 50	37 / 50	37 / 50	37 / 50	36 / 51	35 / 55	37 / 56	38 / 58	39 / 59
7.3	38 / 52	38 / 52	38 / 52	38 / 52	38 / 52	39 / 52	39 / 52	38 / 54	37 / 58	38 / 59	39 / 61
8.2	39 / 55	39 / 55	39 / 55	39 / 55	40 / 55	40 / 55	40 / 55	41 / 55	39 / 57	38 / 61	39 / 62
9.1	40 / 58	40 / 58	40 / 58	41 / 58	41 / 58	41 / 58	41 / 58	42 / 58	42 / 58	41 / 60	39 / 64
10	42 / 61	42 / 61	42 / 61	42 / 61	42 / 61	42 / 61	42 / 61	43 / 61	43 / 61	44 / 61	42 / 63

In equation 20, we work out the parameter t that makes players deviate from the equilibrium $(b_i = b_{min}, b_j = b_{min})$ when $t = 0$. To prove proposition 3.i., it is enough to show that $\frac{\partial t_{i,1}}{\partial b_{min}} > 0$, i.e., when b_{min} increases, the players stay longer at the equilibrium $(b_i = b_{min}, b_j = b_{min})$ when $t = 0$.

$$t_{i,1} = \frac{(b_{min} + h\epsilon)(2\theta - 2k_j - k_i)\frac{h}{2N} - b_{min}k_i\frac{h}{2N}}{\left(b_{min}\frac{k_i\theta}{k_i + k_j} - (b_{min} + h\epsilon)(\theta - k_j)\right) + \left((b_{min} + h\epsilon)(2\theta - 2k_j - k_i)\frac{h}{2N} - b_{min}k_i\frac{h}{2N}\right)}$$

where,

$$\begin{aligned} n_{i,1}(\cdot) &= (b_{min} + h\epsilon)(2\theta - 2k_j - k_i)\frac{h}{2N} - b_{min}k_i\frac{h}{2N}, \\ d_{i,1}(\cdot) &= \left(b_{min}\frac{k_i\theta}{k_i + k_j} - (b_{min} + h\epsilon)(\theta - k_j)\right) + \left((b_{min} + h\epsilon)(2\theta - 2k_j - k_i)\frac{h}{2N} - b_{min}k_i\frac{h}{2N}\right), \text{ and} \end{aligned}$$

$$\frac{\partial t_{i,1}}{\partial b_{min}} = \frac{\frac{\partial n_{i,1}(\cdot)}{\partial b_{min}} d_{i,1}(\cdot) - \frac{\partial d_{i,1}(\cdot)}{\partial b_{min}} n_{i,1}(\cdot)}{\left(\frac{\partial d_{i,1}(\cdot)}{\partial b_{min}}\right)^2} \tag{29}$$

In equation 29, the denominator is always positive. Therefore, we only need to prove that $\frac{\partial n_{i,1}(\cdot)}{\partial b_{min}} d_{i,1}(\cdot) - \frac{\partial d_{i,1}(\cdot)}{\partial b_{min}} n_{i,1}(\cdot) > 0$. That expression is equal to:

$$\frac{\partial n_{i,1}(\cdot)}{\partial b_{min}}\left(\left(b_{min}\frac{k_i\theta}{k_i + k_j} - (b_{min} + h\epsilon)(\theta - k_j)\right) + n_{i,1}(\cdot)\right) - \left(\frac{k_i\theta}{k_i + k_j} - (\theta - k_j) + \frac{\partial n_{i,1}(\cdot)}{\partial b_{min}}\right) n_{i,1}(\cdot) \tag{30}$$

In equation 30, the term $\frac{\partial n_{i,1}(\cdot)}{\partial b_{min}} n_{i,1}(\cdot)$ appears with positive and negative sign. Moreover, $\frac{\partial n_{i,1}(\cdot)}{\partial b_{min}} = (2\theta - 2k_j - k_i)\frac{h}{2N} - k_i\frac{h}{2N}$. Taking this into account, equation 30 can be simplified to:

$$\begin{aligned}
\frac{\partial n_{i,1}(\cdot)}{\partial b_{min}} d_{i,1}(\cdot) - \frac{\partial d_{i,1}(\cdot)}{\partial b_{min}} n_{i,1}(\cdot) &= \left((2\theta - 2k_j - k_i) \frac{h}{2N} (b_{min}) \frac{k_i \theta}{k_i + k_j} \right) + \left(\frac{k_i h}{2N} (b_{min} + h\epsilon)(\theta - k_j) \right) \\
&\quad - \left((2\theta - 2k_j - k_i) \frac{h}{2N} (b_{min} + h\epsilon) \frac{k_i \theta}{k_i + k_j} \right) - \left(\frac{k_i h}{2N} (b_{min})(\theta - k_j) \right) \\
&= h\epsilon \frac{k_i h}{2N} \left((\theta - k_j) - \frac{\theta}{k_i + k_j} (2\theta - 2k_j - k_i) \right) > 0 \qquad (31)
\end{aligned}$$

Where the inequality holds, since $(\theta - k_j) > 0$, and $(2\theta - 2k_j - k_i) < 0$ (low demand). Therefore, $\frac{\partial t_{i,1}}{\partial b_{min}} > 0$, and the players stay longer at the equilibrium $(b_i = b_{min}, b_j = b_{min})$ when $t = 0$. The intuition of this result is simple, when the minimum bid increases, the equilibrium $(b_i = b_{min}, b_j = b_{min})$ is more attractive, and the players stay at that equilibrium longer.

ii. The parameter t for which the players coordinate in one of the Nash equilibria of the original game increases.

The results presented in propositions 1 and 2 are also valid when the minimum bid allowed by the auctioneer increases $(b_{MIN} > b_{min})$. Therefore, the equilibrium selected by the players is the one in which the larger player submits the maximum bid $(b_i = b_{max})$, and the player with lower production capacity submits the minimum bid $(b_j = b_{MIN})$. We focus our analysis on the low demand case, since, as we prove in proposition 2, when the demand is intermediate, the players coordinate in the equilibrium $(b_i = b_{max}, b_j = b_{min}(b_{MIN}))$ when $t = 0$, and when the demand is high, the tracing procedure method do not select any equilibrium.

The value $t_{b_{min}}$ for which the players select the equilibrium $(b_i = b_{max}, b_j = b_{min})$ when the minimum bid allowed by the auctioneer is b_{min} is defined implicitly by:

$$\begin{aligned}
t_{b_{min}} \pi_i(b_{min} + (N-1)\epsilon_{b_{min}}, b_{min}) + (1 - t_{b_{min}}) \pi_i(b_{min} + (N-1)\epsilon_{b_{min}}, p_j) &= \\
t_{b_{min}} \pi_i(b_{min} + (N)\epsilon_{b_{min}}, b_{min}) + (1 - t_{b_{min}}) \pi_i(b_{min} + (N)\epsilon_{b_{min}}, p_j)&,
\end{aligned}$$

and explicitly by:

$$t_{b_{min}} = \frac{\frac{2\theta - 2k_j - k_i}{2N}(b_{min} + (2N-1)\epsilon_{b_{min}}) - \frac{k_i}{2N} b_{min}}{-\epsilon_{b_{min}}(\theta - k_j) + \frac{2\theta - 2k_j - k_i}{2N}(b_{min} + (2N-1)\epsilon_{b_{min}}) - \frac{k_i}{2N} b_{min}} \qquad (32)$$

The value $t_{b_{MIN}}$ for which the players select the equilibrium $(b_i = b_{max}, , b_j = b_{MIN})$ when the minimum bid allowed by the auctioneer is b_{MIN} is defined explicitly by:

$$t_{b_{MIN}} = \frac{\frac{2\theta - 2k_j - k_i}{2N}(b_{MIN} + (2N-1)\epsilon_{b_{MIN}}) - \frac{k_i}{2N} b_{MIN}}{-\epsilon_{b_{MIN}}(\theta - k_j) + \frac{2\theta - 2k_j - k_i}{2N}(b_{MIN} + (2N-1)\epsilon_{b_{MIN}}) - \frac{k_i}{2N} b_{MIN}} \qquad (33)$$

The value of $\epsilon_{b_{min}}$ has been defined in lemma 1 as $\epsilon_{b_{min}} = \frac{b_{max} - b_{min}}{N}$, where b_{max} is the maximum bid, b_{min} is the minimum bid allowed by the auctioneer, and N is the number of bids in the strategies support. By using lemma 1, we also define $\epsilon_{b_{MIN}} = \frac{b_{max} - b_{MIN}}{N}$. If the difference between b_{MIN} and b_{min} is not too big, and if N is large enough, then $\epsilon_{b_{min}} \simeq \epsilon_{b_{MIN}}$, and equations 32 and 33 can be simplified to be compared:

$$t_{b_{min}} = \frac{\frac{2\theta - 2k_j - k_i}{2N}(b_{min} + (2N-1)\epsilon) - \frac{k_i}{2N} b_{min}}{-\epsilon(\theta - k_j) + \frac{2\theta - 2k_j - k_i}{2N}(b_{min} + (2N-1)\epsilon) - \frac{k_i}{2N} b_{min}}$$

$$t_{b_{MIN}} = \frac{\frac{2\theta - 2k_j - k_i}{2N}(b_{MIN} + (2N-1)\epsilon) - \frac{k_i}{2N}b_{MIN}}{-\epsilon(\theta - k_j) + \frac{2\theta - 2k_j - k_i}{2N}(b_{MIN} + (2N-1)\epsilon) - \frac{k_i}{2N}b_{MIN}}$$

For further reference, we introduce:

$$\begin{aligned}
a &= \frac{2\theta - 2k_j - k_i}{2N}(b_{min} + (2N-1)\epsilon) - \frac{k_i}{2N}b_{min} \leq 0, \text{ since } 2\theta \leq 2k_j + k_i \\
A &= \frac{2\theta - 2k_j - k_i}{2N}(b_{MIN} + (2N-1)\epsilon) - \frac{k_i}{2N}b_{MIN} \leq 0, \text{ since } 2\theta \leq 2k_j + k_i \\
b &= -\epsilon(\theta - k_j) \leq 0, \text{ since } \theta \leq k_j
\end{aligned}$$

Therefore,

$$t_{b_{min}} = \frac{a}{b+a} \leq \frac{A}{b+A} = t_{b_{MIN}} \iff a(b+A) \leq A(b+a) \iff ab + aA \leq Ab + aA \iff ab \leq Ab$$

Since $a \leq 0$, $A \leq 0$, and $b \leq 0$, then $ab \geq 0$ and $Ab \geq 0$. Therefore, $ab \leq Ab \iff a \geq A$:

$$a - A = \frac{2\theta - 2k_j - k_i}{2N}(b_{min} - b_{MIN}) - \frac{k_i}{2N}(b_{min} - b_{MIN}) \geq 0 \tag{34}$$

The inequality in 34 holds since $2\theta \leq 2k_j + k_i$ and $b_{min} \leq b_{MIN}$. Therefore, $t_{b_{min}} \leq t_{b_{MIN}}$. □

Annex 2

In this section, we introduce an example to illustrate the theoretical results presented in the paper. We work out the tracing procedure method by using the theoretical predictions in lemma 1, and propositions 1 and 2. We complement that analysis by using another three different methods to work out the tracing procedure method, and we show that the results are the same. We also detail the algorithm used to work out the equilibrium selected by the robustness to strategic uncertainty method. To work out the quantal response method, we apply equation 13.

To illustrate the theoretical results, in figure 14 we work out players' payoff using equation 2. As can be observed in that figure, for the parameters defined in this game, $k_i = 8.7$, $k_j = 6.5$, $\theta = 10$, $b_{min} = 1$, $b_{max} = 10$, $N = 11$, there exists six equilibria in pure strategies. In four of those equilibria, the player with higher production capacity, player i, submits the maximum bid, and player j submits a bid that makes undercutting unprofitable. In the other two equilibria, it is the player with lower production capacity, player j, the one that submits the maximum bid.

As can be observed in the payoff matrix, each player prefers to play one of the equilibria in which the other player submits the maximum bid allowed by the auctioneer. In this annex, we focus our analysis on the equilibrium selected by the tracing procedure method, the robustness to strategic uncertainty method and the quantal response method.For a detailed review of the equilibrium selected using behavioural economics methods, and for a complete analysis of the equilibrium selected by using a lab experiment, see Bigoni, Blázquez and le Coq (2018).

Tracing procedure method.

In the example that we use in this section, we increase the number of bids that the players can play from $N = 11$ to $N = 110$. We do that because in the equations used in lemma 1, and propositions 1, 2 and 3, we assume that the expected payoff function (equation 14) used by the players is continuous. Therefore, when the number of bids is low, the results using the raw information from the payoff matrices and the close form solutions differ slightly. In contrast, when the number of bids is large enough (N=110), the results are the same.

Figure 14: Payoff matrix in a uniform price auction. $k_i = 8.7$, $k_j = 6.5$, $\theta = 10$, $b_{max} = 10$, $N = 11$

$b_{min} = 1$

$b_i \backslash b_j$	1	1.9	2.8	3.7	4.6	5.5	6.4	7.3	8.2	9.1	10
1	4 6	2 17	4 24	5 32	6 40	7 48	8 56	9 64	11 71	12 79	13 87
1.9	12 7	8 11	4 24	5 32	6 40	7 48	8 56	9 64	11 71	12 79	13 87
2.8	18 10	18 10	12 16	5 32	6 40	7 48	8 56	9 64	11 71	12 79	13 87
3.7	24 13	24 13	24 13	16 21	6 40	7 48	8 56	9 64	11 71	12 79	13 87
4.6	30 16	30 16	30 16	30 16	20 26	7 48	8 56	9 64	11 71	12 79	13 87
5.5	36 19	36 19	36 19	36 19	36 19	24 31	8 56	9 64	11 71	12 79	13 87
6.4	42 22	42 22	42 22	42 22	42 22	42 22	27 37	9 64	11 71	12 79	13 87
7.3	47 26	47 26	47 26	47 26	47 26	47 26	47 26	31 42	11 71	12 79	13 87
8.2	53 29	53 29	53 29	53 29	53 29	53 29	53 29	53 29	35 47	12 79	13 87
9.1	59 32	59 32	59 32	59 32	59 32	59 32	59 32	59 32	59 32	39 52	13 87
10	65 35	65 35	65 35	65 35	65 35	65 35	65 35	65 35	65 35	65 35	43 57

$b_{min} = 2$

$b_i \backslash b_j$	2	2.8	3.6	4.4	5.2	6	6.8	7.6	8.4	9.2	10
2	9 11	4 24	5 31	6 38	7 45	8 52	9 59	10 66	11 73	12 80	13 87
2.8	18 10	12 16	5 31	6 38	7 45	8 52	9 59	10 66	11 73	12 80	13 87
3.6	23 13	23 13	15 21	6 38	7 45	8 52	9 59	10 66	11 73	12 80	13 87
4.4	29 15	29 15	29 15	19 25	7 45	8 52	9 59	10 66	11 73	12 80	13 87
5.2	34 18	34 18	34 18	34 18	22 30	8 52	9 59	10 66	11 73	12 80	13 87
6	39 21	39 21	39 21	39 21	39 21	26 34	9 59	10 66	11 73	12 80	13 87
6.8	44 24	44 24	44 24	44 24	44 24	44 24	29 39	10 66	11 73	12 80	13 87
7.6	49 27	49 27	49 27	49 27	49 27	49 27	49 27	33 43	11 73	12 80	13 87
8.4	55 29	55 29	55 29	55 29	55 29	55 29	55 29	55 29	36 48	12 80	13 87
9.2	60 32	60 32	60 32	60 32	60 32	60 32	60 32	60 32	60 32	39 53	13 87
10	65 35	65 35	65 35	65 35	65 35	65 35	65 35	65 35	65 35	65 35	43 57

To work out the graph presented in figure 6, we use four different approaches.

In the first approach, we use the theoretical results presented in propositions 1 and 2. According with those results, when $t = 0$ both players submit minimum bid. When t increases, the player with higher production capacity, player i, deviates first. By using this theoretical prediction, we fix player j's strategy assuming that it always plays the minimum bid. Then, we use equation 4 to work out player i's payoff, and for each t, we work out player i's best strategy.

In the second approach, we work out the payoff matrix for each t using the payoff function of the original game defined in equation 2 and the expected payoff defined in equation 14. Then, we develop an algorithm to work out the Nash equilibrium for each $t \in [0, 1]$. The value of t obtained using the previous algorithm is summarized in the column t_1 in table 1. In that column, we work out the value t for which the players select the equilibrium (b_i^*, b_j^*) defined by the first two columns in table 1. Moreover, the pair of strategies (b_i^*, b_j^*) for each t is the same for approaches 1 and 2. That information is summarized in figure 6.

In the third approach we find the value t_i by solving implicitly equation 35. To solve that equation, we use the routine "fminsearch" in Matlab. Those results are in the column t_2 in table 1. Finally, in the fourth approach, we use the close form solutions in equations 20 and 24. Those results are summarized in the column t_3 in table 1.

$$t_i \pi_i(b_i, b_j) + (1 - t_i)\pi_i(b_i, p_j) = t_i \pi_i(b_i + \epsilon h, b_j) + (1 - t_i)\pi_i(b_i + \epsilon h, p_j) \tag{35}$$

As can be observed in that table, the value of t that we obtain using the four different approaches is almost the same.

Robustness to strategic uncertainty method.

To find the equilibrium selected by the robustness to strategic uncertainty method, it is necessary to work out the best response functions for each player. In this section, we explain the algorithm that we use to work out those functions.

First, since we know that $\phi_{i,j}\left(\dfrac{b_i - b_j}{t}\right)$ follows a normal distribution $(\mathrm{N}(0, 1))$, we create 10000 random values drawn from a $(\mathrm{N}(0, 1))$.

Second, we set a value of $b_j \in [b_{min}, b_{max}]$. For that particular value, we set a value of $b_i \in [b_{min}, b_{max}]$,

Table 1: t parameter using different methods. $k_i = 8.7$, $k_j = 6.5$, $\theta = 10$, $b_{min} = 1$, $b_{max} = 10$, $N = 110$, $\epsilon = \dfrac{b_{max} - b_{min}}{N} = 0.0826$

b_i^*	b_j^*	h	H	t_1 (Nash equilibrium)	t_2 (fminsearch)	t_3 (equations 20, 24)
3.72	1	34	35	0.2330	0.2356	0.2409
4.63	1	45	46	0.261	0.2647	0.2679
5.54	1	56	57	0.2870	0.2901	0.2931
6.44	1	67	68	0.31	0.3138	0.3165
7.35	1	78	79	0.3330	0.3359	0.3385
8.26	1	89	90	0.3540	0.3566	0.3591
9.17	1	100	101	0.3740	0.3762	0.3784
10	1	109	110	0.39	0.3929	0.3934

and for each pair (b_i, b_j), we work out the value $\dfrac{b_i - b_j}{t}$.

Third, we compare the value $\dfrac{b_i - b_j}{t}$ with each of the 10000 points extracted from a the normal distribution (N(0, 1)) worked out in the first step. We count the values that are lower and higher than $\dfrac{b_i - b_j}{t}$. Those values are the cumulative distribution values $\left[1 - \Phi_{i,j}\left(\dfrac{b_i - b_j}{t}\right)\right]$ and $\left[\Phi_{i,j}\left(\dfrac{b_i - b_j}{t}\right)\right]$ in equation 12.

Fourth, with the values b_i, b_j, $\left[1 - \Phi_{i,j}\left(\dfrac{b_i - b_j}{t}\right)\right]$, $\left[\Phi_{i,j}\left(\dfrac{b_i - b_j}{t}\right)\right]$, and knowing if $b_i \leq b_j$ or if $b_i > b_j$, we can use equation 12 to work out the expected value for every $b_i \in [b_{min}, b_{max}]$ given a specific b_j.

Fifth, given a specific b_j, we find the value $b_i \in [b_{min}, b_{max}]$ that maximizes player i's expected value (figure 7).

Sixth, we repeat the process for every $b_j \in [b_{min}, b_{max}]$ and we obtain player i's best response function (figure 8).

Seventh, we repeat steps one to six to obtain player j's best response function.

Quantal response method.

We describe in detail how we calculate the quantile response equilibria presented in section 5 of the main text.

Define an increasing sequence of λ parameters $\{\lambda_i\}_{i=0}^n$ such that $\lambda_0 \simeq 0$ and initialize $i = 0$.[15] Then,

1. For i, set λ to λ_i.
2. Given λ_i, define a system of equations given by $\pi(x; \lambda_i) - x = 0$.
3. Define optimization starting parameters

$$x_0 = \begin{cases} (1, ..., 1)/s & \text{if } i = 0 \\ \pi^*(\lambda_{i-1}) & \text{otherwise} \end{cases}$$

   where s is the number of pure strategies. Using x_0, solve the system of equations given in step 2 to find $x^* = \pi^*(\lambda_i)$.[16]

[15] Choosing λ_0 to be roughly zero helps find good starting values for subsequent values of λ as we have that when $\lambda = 0$, each strategy is played with equal probability.

[16] We use Matlab's non-linear equation solver *fsolve*, with function and x tolerances both set to $10e-10$.

4. If $i < n$, set $i = i + 1$ and go to step 1.[17]

One issue in numerically solving the system of equations given in step 2, is that as $\lambda_i \to \infty$, we also have that $\sum_{k=1}^{J_i} exp\left\{\lambda \overline{u}_{ik}(\pi)\right\} \to \infty$. Even for relatively small values of λ, this can cause a numerical overflow. This despite the probability $\pi^*(\lambda_i)$ being well-defined and bounded. To overcome this issue, we calculate the probabilities $\pi(\lambda)$ using an alternative representation of the *LogSumExp* function.

First, we have that for an n-dimensional vector, the *LogSumExp* function is defined as

$$lse(x) = ln\left(\sum_{i}^{n} exp(x_i)\right)$$

Gao and Pavel (2018) note that this function can equivalently be written as

$$lse(x) = x^* + ln\left(\sum_{i}^{n} exp(x_i - x^*)\right)$$

where $x^* = max\left\{x_1, ..., x_n\right\}$.

From before, we have

$$\pi_{ij} = \frac{exp\left\{\lambda \overline{u}_{ij}(\pi)\right\}}{\sum_{k=1}^{J_i} exp\left\{\lambda \overline{u}_{ik}(\pi)\right\}}$$

Assuming that the probability of each strategy is strictly greater than zero, $\pi_{ij} > 0$, we can rewrite this probability as

$$exp\left\{ln(\pi_{ij})\right\} = exp\left\{\lambda \overline{u}_{ij}(\pi) - ln\left(\sum_{k=1}^{J_i} exp\left\{\lambda \overline{u}_{ik}(\pi)\right\}\right)\right\}$$

and replacing $ln\left(\sum_{k=1}^{J_i} exp\left\{\lambda \overline{u}_{ik}(\pi)\right\}\right)$ with $u^* + ln\left(\sum_{k=1}^{J_i} exp(\lambda \overline{u}_{ik}(\pi) - u^*)\right)$, we have that

$$exp\left\{ln(\pi_{ij})\right\} = exp\left\{\lambda \overline{u}_{ij}(\pi) - \left(u^* + ln\left(\sum_{k=1}^{J_i} exp(\lambda \overline{u}_{ik}(\pi) - u^*)\right)\right)\right\}$$

where $u^* = max\left\{\lambda \overline{u}_{i1}(\pi), ..., \overline{u}_{iJ_i}(\pi)\right\}$. Using this form, we avoid numerical overflow and are able to find solutions $x^* = \pi^*(\lambda_i)$ for large values of λ.

[17] An alternative termination criteria is to look at the distance between $\pi^*(\lambda_i)$ and $\pi^*(\lambda_{i-1})$. In our application, we define our sequence $\{\lambda_i\}_{i=0}^{n}$ such that the λ_i are increasing exponentially rather than linearly and choose therefore to terminate at a large predefined λ.

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