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Options and Futures in the FishPool market: a brief analysis

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Abstract

Since 2006, FishPool ASA has been operating as a regulated market place for the trading of futures and options written on the spot price of fresh farmed salmon. The impressive increase of the trading volums experienced by this young has not been homogenous, leading to a well-developed market for futures contracts, while the options market still suffer significant liquidity problem. It is difficult to identify the reasons behind the different trend characterizing options and futures markets, but two main drivers can be identified. From one side, the lack of understanding among the market practitioners of the financial profile of the offered Asian American option contracts, on the other side the absence in the literature of a model able to completely describe the characteristics of this option contracts disincentive institutional investors and hedgers to get into a market they are not able to completely understand.

This Master Thesis investigates the main characteristics of the salmon market and the available derivatives pricing models in order to identify some of the reasons underlying the observed liquidity problem in the options market. In particular, after a brief literature review (Part II) and an empirical analysis of the salmon market (Part III), in Part IV (and in the Appendixes) I will re derive the pricing model proposed by Bjerksund (1991) for both futures and options contracts, underlying that while the available futures pricing formulas allow to efficiently manage trading and risk management strategies, the most common options pricing formulas rely on too strong assumption and thus are not able to well represent the real market structure.

Thus, in the conclusion it is suggested that FishPool ASA might reconsider the typology of the offered options contract, switching to plain vanilla derivatives that might allow to fast up to expansion of this still limited market.

I. Introduction

Since 2006, FishPool ASA has been operating as a regulated market place for the trading of futures and options written on the spot price of fresh farmed salmon. The fast growth of the volumes of futures contract traded in this market, for which Hirschleifer (1988), Bulte and Penning (1997), Dalton (2005) and Bergfjord (2007) provide possible explanations, has attracted the interest of several institutional investors who, by acting as speculators, have played a fundamental role in solving the thinness of the futures market. On the other hand, the options market still presents significantly high bid ask spread and low liquidity, both determined by the extremely low volumes traded in the market.

It is difficult to identify the reasons behind the different trend characterizing options and futures markets. A first explanation can be found in the lack of understanding among the market practitioners of the financial profile of the offered option contracts, a problem that appears to be common also in many other markets. Moreover, the only options traded in the FishPool market are American-Asian options, whose particular financial profile makes them less appealing to both hedgers and speculator who are often not familiar with this type of product. Nonetheless, the choice of a similar derivative can be justified in a young and small market. It is in fact commonly accepted that the use of average-value options allows to reduce the risk of price manipulation of the underlying asset, which appears to be particularly relevant in thin markets. In order to solve these problems, FishPool ASA is trying to involve new financial counterparties in the options market to increase the trading volumes. Despite their effort, poor results have been achieved so far. The absence in the literature of a model able to completely describe the characteristics of the offered Asian-American options and of the salmon market disincentives institutional investors to get into a market they are not able to completely understand. Thus, the availability of a theoretically solid pricing formula for these derivatives would allow to remove this major constraint FishPool ASA is facing to include new financial counterparties into the market. In this sense, some recent papers have provided different approaches for the computation of the price of these financial derivatives. In particular, Ewald (2011) has proposed a closed-form pricing formula for forward contracts and an approximate pricing formula for European option contracts written on fresh-catch wild salmon, while Ewald et al. (2014) has underlined the relevance of the Schwartz 97 two-factor model for fish farming, using a real option approach and adopting the Longstaff-Schwartz method to compute monetary values for lease and ownership of a model fish farm.

Unfortunately, the proposed pricing formulas discussed in these papers for both forward and option contracts relies on strong assumptions that do not well describe the salmon market. In particular, Ewald (2011) assumes that the population is exclusively wild and managed as an open access fishery, while the real market appears to be mainly driven by aquaculture and to be strictly regulated throughout a license system. Moreover, all the proposed pricing models refer to plain-vanilla European options, while, in the FishPool market, only American-Asian options are traded, thereby creating a gap between academic research and the situations faced by the market operators.

This Master Thesis investigates the main characteristics of the salmon market and the available derivatives pricing models in order to identify potential reason underlying the observed liquidity problem in the options market. In particular, after a brief literature review (Part II) and an empirical analysis of the salmon market (Part III), in Part IV (and in the Appendix) I will re derive the pricing model proposed by Bjerksund (1991) for both futures and options contracts. Finally, in Part V some conclusions will be offered.

II. Literature Review

The neoclassical theory of investment is based on the net present value (NPV) approach, which provides a simple decision rule based on the sign of the difference between the present value of the expected profits and the present value of the expected costs. In particular, if this value is positive, then the NPV rule suggests to proceed with the investment. Nevertheless this approach is widely used by market practitioners, it presents several limitation that should be taken into account when it comes to understanding the results provided by this method. First, the choice of the risk-adjusted discount rate for the specific investment appears to be critical. Even though its computation is generally based on the capital asset pricing model (CAPM), if we allow for uncertainty about future interest rate or if we account for the presence of embedded options affecting the overall risk profile of the investment (i.e. the option to delay the investment), then the determination of the discount rate appears to be far more complex, thereby raising the risk of relying on wrong assumptions. Additionally, the presence of embedded options can severely impact the expected value of the project. It is in fact sufficient to think about how the presence of an abandonment option can completely change the risk profile and the expected cash flow of a project. Finally, many other issues can severely affect the efficiency of this model: how should inflection or depreciation taxes be treated?

All these limitation are reflected in the implicit assumptions that the NPV approach relies on. In particular, this method assumes that if the investment is irreversible, the decision rule has to be structured as a now-or-never proposition: if the firm decides not to invest, it will not be able to do it in the future. On the other hand, if the investment is reversible, it is assumed that it can be undone and that the expenditures can be recovered at market conditions. However, not many investments respect these assumptions: the option to delay, in fact, generally represents one of the most important decisions and it appears critical to take it into account when valuing an investment. But this element undermines the theoretical foundation of the NPV approach, creating the necessity of a new valuation method.

In this direction, real options theory offers a different approach to project valuation: this method is based on the idea that a firm with an opportunity to invest is basically holding an American call option which provide the right but not the obligation to buy specific assets at a future time. The decision to exercise this option represents the choice to give up the opportunity of waiting for new information that might affect the desirability or the optimal investing timing of the project. This creates an opportunity cost that must be included as a part of the investment costs. In fact, different studies (i.e. Huchzermeier et al. (2001) Trigeorgis (1993) and Trigeorgis (1993)) have shown the relevance of these hidden costs, which appear to be highly sensitive to different sources of uncertainties, thus enlightening the reasons behind the low degree of accuracy provided by the approach suggested by the neoclassic investment theory.

While the NPV rule states that a project is profitable when the difference between the discounted revenues and discounted costs is positive, the real option approach modifies this decision rule in order to take into account the opportunity cost generated by the exercise of the investment option. In this sense, Dixit and Pindyck (1994) state that investment occurs when the difference between the discounted revenues and the sum of the discounted costs and the value of the option to delay is zero, or, in other words, when the marginal profit lost from waiting one more unit of time is equal to the marginal value derived by the reduced uncertainty obtained by waiting one more unit of time. This corrected decision rule appears to be consistent with the behaviors followed by the market practitioners[1], who generally delay investment until prices are sensibly above the long term average costs and stay in the business even though the prices level fell below it , in contrast with what stated by the NPV approach.

In the specific case of the aquaculture industry, the real option approach appears to be particularly relevant: the significant uncertainties surrounding investments in fish farm, generated by both financial and biological variables, may increase the value of the options embedded in the project, thereby making the NPV approach unsuitable to manage the risk of the investment and, therefore, to correctly evaluate it. With respect to the salmon industry, high volatility of both spot and futures prices and the significant uncertainty determined by different biologic and natural variables (i.e. the sea temperature and the biomass growth function) represent an important element that has to be taken into account when computing the value of a specific project, for which, therefore, it appears again preferable to rely on the real options approach rather than on the NPV approach. While an analysis of the characteristic and of the management of the uncertainty due to natural and biologic elements is not within the scope of this thesis, the attention will be mainly focused on the market risks.

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[1] For further details see, i.e., Summers (1987)

Salmon prices exhibit high level of week-to-week volatility, which severely impact harvesting decisions. In particular, their dynamics present substantial within-year calendar-related fluctuations. This trend are determined by two main sources. First, events such as Christmas and Easter significantly impact the demand side, determining these particular trends that cannot be differently explained. Secondly, salmon production is strongly dependent to biological factors. In example, weather and climatic conditions, such as water temperature, affect the biomass growth rate and, thus, harvesting decisions. Hence, also production costs present a seasonal pattern, leading to significant cost differences between the salmon ready for marketing, for example, in May and in October. In general, prices peaks occur between week 20 and 24. Prices start then decreasing, reaching the lowest level between week 45 and week 50. The difference between peaks and floors level are generally around 20%. All these analysis consider salmon as an aggregate product, called "Atlantic salmon", consistently with all the major indexes. Nonetheless, it is important to underline that prices for different size and types of salmon presents relevant differences, tending not to move synchronously.

The significant variability in future price levels severely impact both harvesting and investment decisions, representing one of the most relevant source of uncertainty surrounding the profitability of a model salmon farm. In this sense, Forsberg and Guttormsen (2006) analyzed before the establishment of the FishPool market how the presence of an efficient futures market provides further information that, by improving the decision making process, allow to achieve an higher expected value for a model salmon farm, ceteris paribus. According to the authors, fully informed farmers can in fact approximately triple their profits compared to those farmers basing their decisions on only historical prices or simple decision making models. Despite the fact that Forsberg and Guttormsen's analysis overestimates the value of these information, since the now existent futures market are not complete and the theoretical harvesting model used in the paper appears to be extremely simple and unable to fully appreciate the complexities of the salmon market, the proposed results are indicative of the high value hidden in the options embedded in the projects, providing further justifications to prefer the real option approach to the NPV method to evaluate model fish farm.

Since the establishment of the FishPool market, researchers have particularly focused on the definition of futures price and options premium in order to allow market practitioners to fully benefit from the further information provided by the market itself.

In his recent working paper, Ewald (2011) studies forwards and European call options written on the spot price of fresh catch wild salmon. The underlying is described as a non-storable renewable resource, that is managed as open access under perfect competition. In particular, salmon biomass growth is described as a stochastic logistic growth dynamic in which uncertainty is generated by both environmental, ecological and economic sources, featuring a carrying capacity and mean reversion. Ewald (2011) derives an inverse demand function for the market[2], in which a reciprocal relationship between the spot price and the harvested marketed resource is featured. From this relationship, a pricing model for futures contract written on the spot price is defined. In particular, the author shows that, at least in the described market, forward prices written on renewable resources do not follow a Geometric Brownian Motion (GMB) but a far more complex dynamic, since it exists a relationship between the spot price and the underlying of an Asian option. This link allows Ewald (2011) to propose also an approximate option pricing formula for a European call written on a renewable resource. The structure of the formula appears to be similar to the one derived by Black (1976), with the exception that the stochastic process describing the stock dynamics substitute the GMB, as shown, in example, consistently with the results shown in Ewald and Yang (2008) and Ewald and Wang (2010).

The market and the results presented in Ewald (2011) appears to be more representative of the American case, where the establishment of a new US based fish futures market is currently under discussion, as presented in Rohrlich (2010). In fact, the American salmon farming is less developed than the Norwegian one, since it has been facing a fierce opposition from various environmental groups during the last years. This situation can be observed by comparing the US wild catches of fresh salmon, approximately 340 thousand tons per annum, and the farmed salmon American production, approximately 17 thousand tons per annum. For these reasons, a futures market on fresh salmon in the US would likely be focused on wild catches only. Despite Ewald (2011) appears thus to be relevant for the American case, the strong assumptions on which the model relies are not representative of the Norwegian (and global) market structure, which appears to be strictly

[2] The inverse demand function is derived under the strong assumptions of identical and atomistic profit maximizer agents, acting as price taker, while the resource is managed a pure open access

regulated by a license system instead of being managed under a pure open access. Moreover, about 60% of the world's salmon production and all the commercially available Atlantic salmon is farmed, and, therefore, the problem of non-storability appears to be less prominent. Moreover, even if we consider the salmon production as mainly driven by wild catches instead of by aquaculture, describing the stock with such a complex dynamic adds structure to the problem, inserting a set of strong assumptions that do not necessarily realistically represent the market and that don't allow to identify a closed-form formula for the European call options, requiring to identify an approximate formula to overcome this problem.

These considerations lead to the necessity of taking into account also the convenience yield, which has been shown to play a significant role also in the case of non-storable resources in Lautier (2009).

The role of the convenience yield in the relationship between spot and futures price has been analyzed extensively in the academic literature, thus providing economic explanation for important phenomenon such as the backwardation, that can be defined as the situation in which futures price are lower than spot price[3] and for which the traditional asset pricing theory fails to identify a proper justification.

Taking these feature into account, the population dynamics have to be described in a different way than what proposed in Ewald (2011), considering, in particular, that the control variable for a profit maximizer farmer is not the quantity (biomass) harvested but the harvesting time, as generally described in the famous Faustmann's (1849) rotation problem. In particular, it is possible to assume that in this market both the spot price and the convenience yield follow a stochastic process. Following the Schwartz (1997) setting and these assumptions, it is possible to show that the spot price is a fundamental, but not unique, determinant of the price of future claims on a similar resource, justifying the preference for a two factor model for pricing financial and real assets written on the spot price of a storable commodity. Schwartz (1997) proposes analytical formula for pricing both futures and European option contracts, which are shown to perform well in valuing short term positions and to explain the intrinsic difference in price volatility between spot and futures price and the decreasing maturity pattern observed among the latter.

[3] Some authors refer to backwardation as the case in which futures price are lower than the expected future spot price

As shown in Bjerksund (1991), a two factor model appears to be a natural generalization of the standard Black & Scholes (1973) model, as it adjusts for the case of an underlying asset paying a constant proportional dividend. From what is in my knowledge a similar model has never been applied for pricing fish derivatives, with the only exception of Ewald (2014). The author combines this approach with the classical literature on aquaculture to model the aggregate salmon farming production to derive the monetary value for lease or ownership of a model fish farm by following a real option approach and adopting the Longstaff – Schwartz method in the context of multiple state variable. In particular, Ewald (2014) derives an inverse demand function assuming that the supply side is characterized by the presence of many small profit maximizing farmers who uniquely choose the optimal harvesting time, while in the demand side a representative consumer chooses between farmed salmon and an alternative consumption good according to its utility function, assumed to be Cobb-Douglas type, and to its budget constraint.

By analyzing the functional form of the inverse demand function, Ewald (2014) replicates Schwartz (1997) results, thereby underlying the relevance of a two-factor model for pricing fish derivatives. The obtained formula is then used to represent future prices and to apply a real option approach to a model fish farm, computing its value in the case of a single rotation and of an infinite rotation problem. Even though Ewald (2014) justifies the application of a two-factor model for the salmon market, it still relies on the assumption that the options traded in the FishPool market are European type, and it does not provide significant results to analyze the impact that the different risk management strategies have on the monetary value of a lease or ownership of a model fish farm. Aside this limit, Ewald (2014) opens a new path for the analysis and pricing of fish derivatives, providing the basis for a better description of the salmon market and of both physical and financial investments that hedgers and speculators can realize in it.

Assumption and Notation

The salmon market is mainly driven by aquaculture and only secondarily by wild catches. For this reason, the following analysis is conformed to the classical approach described in Cacho (1997).

Salmons do not reproduce in the pens, determining that the number of salmons in each pen has to decrease over the time. In particular, by assuming that the mortality rate of salmons in the pens m(t) follows an adapted stochastic process on (Ω,P,F), the dynamic of the number of salmon can therefore be described at any point of time before harvesting as:

$$dn(t) = -m(t) * n(t)dt \tag{1}$$

At the same time, each survived salmon gains in weight over the time; this dynamic, called dw(t), is assumed to follow the following process:

$$dw(t) = [\Phi - \beta(t)]w(t)dt + \sigma_w w(t)dB(t) \tag{2}$$

where B(t) is a standard Brownian motion on (Ω,P,F) and $\beta(t)$ is an arbitrary stochastic process such that dw(t) is well defined. In other words, $\beta(t)$ can be interpreted as the weight saturation coefficient, introducing a mean reversion feature in this dynamic throughout the mean reversion level Φ, assumed to be constant.

From this setting it follows that the total biomass X(t) at any point of time has to be equal to

$$X(t) = n(t) * w(t) \tag{3}$$

And, therefore

$$dX(t) = [\Phi - \beta(t) - m(t)]X(t)dt + \sigma_w X(t)dB(t) \tag{4}$$

Even though in the salmon market the supply side has been historically fragmented, especially in Norway and in Scotland, many mergers and acquisitions have taken place in the last decade and this trend is expected to continue, leading to an oligopolistic market structure. For what in my knowledge, the oligopolistic aquaculture harvesting problem has not been discussed in the literature: for this reason, the supply side will be simplified and described similarly to the Ewald (2014) setting.

In particular, I will assume the presence of many homogeneous salmon farmers facing a limited market demand, from which it follows that it cannot be efficient for them to harvest all at the same time. Therefore, no unique harvesting time can be identified. By assuming that a portion v(t) of salmon farmers will harvest at any point of time and that each salmon farmer own the same percentage of the total biomass, the dynamic (4) can be adjusted according to the following equation:

$$dX(t) = [\Phi - \beta(t) - (m(t) + v(t))]X(t)dt + \sigma_w X(t)dB(t) \qquad (5)$$

From which it can be easily seen that the salmon supply in each infinitesimal interval of time dt will be $v(t)X(t)dt$.

On the demand side, I assume the presence of a representative consumer that has to choose between farmed salmon x(t) and an alternative consumption good y(t) according to a Cobb-Douglas type utility function. The consumer want to maximize its utility at each time t, according to the following optimization problem:

$$\begin{aligned} &Max\left(x(t)^{\alpha(t)}y(t)^{\gamma(t)}\right) \\ &subject\ to: \quad P(t)x(t) + y(t) = b(t) \end{aligned}$$

b(t) represents the costumer's budget constrain which, as in Ewald (2014), can vary stochastically, while P(t) represents the spot price of farmed salmon, while the price of the alternative consumption good is normalized to 1. The preference parameters α(t) and γ(t) sum to 1 and are assumed to follow a stochastic process, so that changes in the consumer preferences can be taken into account. This problem leads to a unique solution for the salmon consumption:

$$x(t) = \frac{\alpha(t)b(t)}{P(t)} \qquad (6)$$

Since in equilibrium demand equalizes supply, x(t) has to be equal to v(t)X(t): from this relation it is possible to derive the relative inverse demand function:

$$P(t) = \frac{\alpha(t)b(t)}{X(t)v(t)} \qquad (7)$$

Since P(t) represents the spot price, it can be interpreted as the FishPool Index. By following the Ewald (2014) simplification, I assume that:

$$\begin{aligned} &d\left(\frac{\alpha(t)b(t)}{v(t)}\right) = d\varepsilon(t) \\ &= \varepsilon(t)\big(\varphi(t)dt + \eta dW(t)\big) \qquad (8) \end{aligned}$$

Where W(t) is a Brownian motion correlated with B(t) according to the relationship $W(t)B(t) = \rho_D dt$. By applying the Ito-formula, it is therefore possible to show that:

$$dP(t) = P(t)(m(t) + v(t) + \sigma_w^2 - \eta\sigma_w\rho_D + \beta(t) + \varphi(t) - \theta)dt + P(t)\big(\eta dW(t) - \sigma_w dB(t)\big) \quad (9)$$

Finally, since the variance of $(\eta dW(t) - \sigma_w dB(t))$ is equal to $\eta^2 + \sigma_w^2 - 2\eta\sigma_w\rho_D$, the dynamic of the spot price can be rewritten as:

$$dP(t) = P(t)(m(t) + v(t) + \sigma_w^2 - \eta\sigma_w\rho_D + \beta(t) + \gamma(t) - \theta)dt + P(t)(\eta^2 + \sigma_w^2 - 2\eta\sigma_w\rho_D)dZ_1(t) \quad (10)$$

where $Z_1(t)$ is a standard Brownian motion. By defining now:

$$\mu = \sigma_w^2 - \eta\sigma_w\rho_D - \theta \quad (11)$$

$$\sigma_1 = \eta^2 + \sigma_w^2 - 2\eta\sigma_w\rho_D \quad (12)$$

It is possible to restate the (10) as the following process:

$$dP(t) = \big(\mu + m(t) + v(t) + \beta(t) + \gamma(t)\big)P(t)dt + \sigma_1 P(t)dZ_1(t) \quad (13)$$

While Ewald (2014) accept this dynamic to describe the spot price, it should be considered that the proposed process doesn't take into account the strong seasonality pattern that can be observed in the FishPool index, as shown in Part III. For this reason, the formula can be adjusted to incorporate this important feature. As shown in Appendix 1, the dynamic of the log spot price can therefore be restated as:

$$dL = \frac{d\mu_L(t)}{\vartheta} + \vartheta\left(\frac{\mu_L(t)}{\vartheta} - \ln P(t)\right)dt + \sigma_L dZ_L \quad (14)$$

$$d\mu_L(t) = \mu_{x,0} + \sum_{h=1}^{H}\left[\mu_{x,h,cos}\cos\left(\frac{2\pi h}{s}t\right) + \mu_{x,h,sin}\sin\left(\frac{2\pi h}{s}t\right)\right] \quad (15)$$

Where dL represents the dynamic of the log spot price ln[P(t)], $\vartheta > 0$ is the speed of mean reversion, $\frac{\mu_L}{\vartheta}$ is the long run mean; s indicates the number of observation per year, while $\mu_{x,h,cos}$ and $\mu_{x,h,sin}$ are the seasonality parameters and H determines the number of term in the sum, chosen equal to 2 according to the Akaike Information Criterion, AIC (see Y. Sakamoto et al. (1986)).

The convenience yield can be defined as the benefits derived from holding a physical asset in inventory, instead of owning a financial derivative, such as a futures or an option, written on the same commodity. In general terms, it is possible to argue that the convenience yield describes the market's expectation about future availability of the commodity, represented by the storage level. As previously described, the salmon market is mainly driven by aquaculture and, therefore, storage (the convenience yield) plays an important role for determining the value of future claims written on this resource. At least in first approximation, it is possible to define the convenience yield for the salmon market as:

$$\delta = -\big(m(t) + v(t) + \beta(t) + \gamma(t)\big) \quad (16)$$

Even though Ewald (2014) relies on this simple formula to argue that the dynamic of the convenience yield should follow a normal Ornestein-Uhlenbeck stochastic process, it has been observed that it seems more accurate to define the convenience yield as a far more complex process and to account for seasonality in its dynamic. As shown in Appendix 2, the convenience yield can therefore be better defined as

$$\delta(t) = \frac{ln \dfrac{\left[\left(P(0) + \left[\int_0^T \left(gf\left(\left(\theta - \beta(t)\right)w(t)dt + \sigma_w w(t)dB(t)\right)N(t)e^{rT}\right)dt\right]\right)e^{rT}\right]}{F_0}}{T} \quad (17)$$

Where F_0 is the price of a future delivery of the commodity in T, representing the time to maturity. It can be observed that the convenience is defined as the sum of the log spot price and of a complex dynamic dq(t), featuring at least some mean reversion. For T=1, the (17) can therefore be restated as:

$$\delta(t) = L(t) + q(t) \quad (18)$$

In particular, by following the same approach used in Appendix 1 it is possible to introduce seasonality in the dynamic dq(t), which can then be defined as

$$dq(t) = \frac{d\mu_q(t)}{k_q} + \left[k_q\left(\frac{\mu_q(t)}{k_q} - y(t)\right)\right]dt + \sigma_q dZ_q \tag{19}$$

$$d\mu_y(t) = \mu_{y,0} + \sum_{h=1}^{H}\left[\mu_{y,h,cos}\cos\left(\frac{2\pi h}{s}t\right) + \mu_{y,h,sin}\sin\left(\frac{2\pi h}{s}t\right)\right] \tag{20}$$

Where k > 0 is the speed of mean reversion and $\frac{\mu_y}{k_y}$ is the long run mean.

In equilibrium, it has to hold that the expected return to the commodity holder has to be equal to the risk free rate plus the relative market price of risk. Therefore, it can be written that:

$$E\left(\frac{dP(t)}{P(t)} + \delta(t)\right) = r + \lambda_L(t) = \frac{d\mu_L(t)}{\vartheta} + \left[\left(\mu_L(t) - \vartheta L(t)\right) + \left(q(t) + \vartheta L(t)\right)\right]dt \tag{21}$$

Where $\lambda_L(t)$ and $\lambda_\delta(t)$ are the relative risk premiums, which have to be necessarily defined as a periodical function of time:

$$\lambda_L(t) = \lambda_{L,0} + \sum_{h=1}^{H}\left[\lambda_{L,h,cos}\cos\left(\frac{2\pi h}{s}t\right) + \lambda_{L,h,sin}\sin\left(\frac{2\pi h}{s}t\right)\right] \tag{22}$$

$$\lambda_\delta(t) = \lambda_{\delta,0} + \sum_{h=1}^{H}\left[\lambda_{\delta,h,cos}\cos\left(\frac{2\pi h}{s}t\right) + \lambda_{\delta,h,sin}\sin\left(\frac{2\pi h}{s}t\right)\right] \tag{23}$$

It is now easy to observe that the risk neutral process for the log spot price dL(t)* and for the process dy(t)* can be written as:

$$dL(t) = \left[r - \left(q(t) + \vartheta L(t)\right)\right]dt + \sigma_L dZ_L{}^* \tag{24}$$

$$dq(t) = \left[\frac{1}{k_q}\frac{d\mu_q(t)}{dt} + k_q\left(\frac{\mu_q(t)}{k_q} - q(t)\right) - \lambda_q(t)\right]dt + \sigma_q dZ_q{}^{*} \quad (25)$$

Where $dZ_L{}^{*}$ and $dZ_y{}^{*}$ are two Wiener process taken under the equivalent martingale measure. This two dynamics represents the basic foundation that will be used for the derivation of the futures and options pricing formula for the Atlantic salmon and, generalizing, for commodities featuring strong seasonality pattern.

III. Empirical Analysis of the salmon market

Data

In this section I analyze prices registered in the FishPool market from 12/06/2006 to 12/06/2014, underlying the main features characterizing the salmon industry and the basis on which the derivatives pricing models have been derived. In particular, both weekly spot prices and daily futures prices for different maturities can be observed[4], while no public data about the convenience yield seems to be available. For this reason, it has been necessary to derive analytically the value of the convenience yield over the time, as shown in Appendix 2. Similarly to Schwartz (1997), I refer to the contract with the lowest time to maturity as F1, while the contract with the longest maturity as F28.

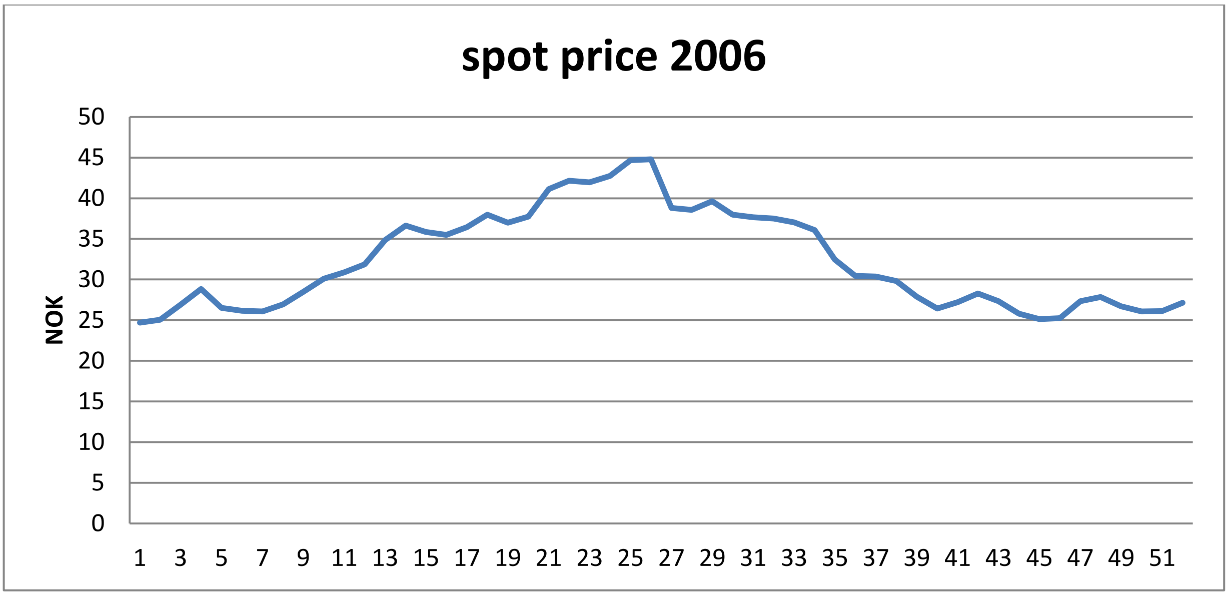
The whole sample of data is divided in 3 equally long periods characterized by different interest rate regime, represented by the 2-years average Norwegian Key Policy Rate[5], shown in table1. The length of each period has been chosen in order to be representative of the average rotation length of a salmon farm, from the juvenile phase to harvesting. Moreover, similarly to Ewald (2014), each period is further divided in 3 panels, Panel A, Panel B and Panel C, representing respectively a proxy for short-term, medium-term and long-term futures contracts. In particular, Panel A contains F1, F3, F5, F7 and F9; Panel B contains F12, F14, F16 F18 and F20; Panel C contains F24, F25, F26, F27 and F28.

Data Set	Time period	Interest Rate	Daily Observations
$Data_1$	12/06/2006 - 11/06/2008	4.22%	513
$Data_2$	12/06/2008 - 11/06/2010	2.87%	512
$Data_3$	12/06/2010 - 11/06/2012	2.00%	513
$Data_4$	12/06/2012 - 11/06/2014	1.50%	512

[4] http://fishpool.eu/iframe.aspx?iframe=forwardone.asp&pageId=45
[5] http://www.norges-bank.no/en/Monetary-policy/Key-policy-rate/Key-policy-rate-Monetary-policy-meetings-and-changes-in-the-key-policy-rate/

Data[1]

The one-year spot price dynamics present common features during the period 2006-2008. In particular, the minimum price always occurred between week 43 and week 45, while the maximum price was reached at the beginning of the summer. It is possible to observe a generally positive trend from week 1 until the peak is reached, mainly due to low production level. After the maximum is reached, a decline in the spot price level is observed, with the minimum level registered around week 44. The high temperatures characterizing this period cause, in fact, an higher production level, driving the observed decline. Finally, in the last period of the year, in particular the higher demand drives a new increase in prices. The chart below represents the price dynamic in 2006, which well represents the described trends.



The table below summarized the main features of the 3 analyzed years.

Year	Mean Price (NOK)	Standard Deviation (NOK)	Min-Max % difference
2006	32.36	6.04	43.87%
2007	25.74	2.52	32.21%
2008	26.36	2.01	23.90%

The convenience yield is computed assuming storage cost per unit equal to a constant proportion u of the spot price. Even though the convenience yield would be more properly defined as far more complex dynamics, this assumption significantly simplify the computation of the curve, without loss in generality. In fact, it might be argued that the increase in the feeding cost and decrease in the marginal net biomass growth over the time balance each other, leading to fairly constant storage costs.

The relevant parameters for the computation of u are resumed in the following table[6].

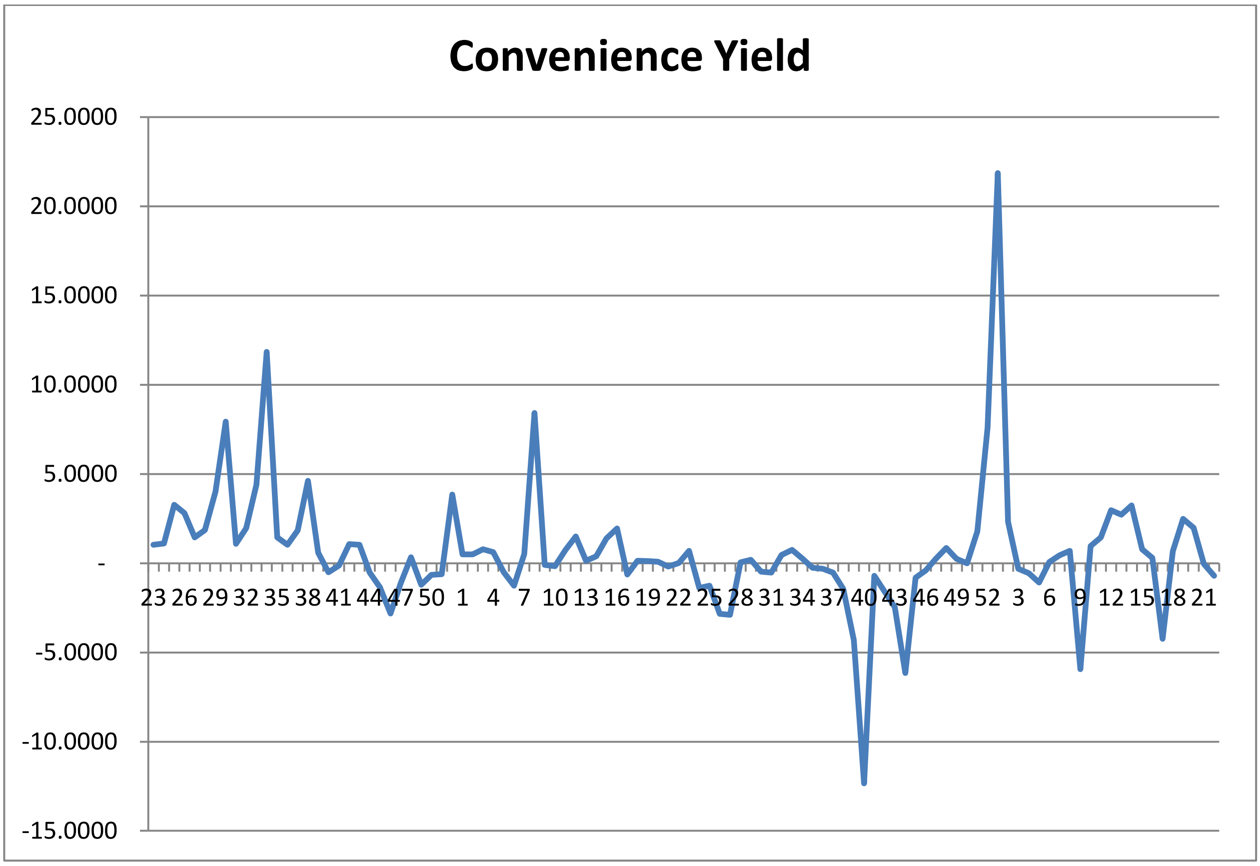
Parameters	Value
Conversion rate	1.2
Growth coefficient	0.5
Constant weekly feeding cost for a 4 Kg Salmon (NOK)	32

Table 1: Relevant parameters for storage cost (obtained from Marine Harvest (2012))

I use the futures contract with the shortest time-to-maturity (F1) to compute the convenience yield. In particular, since the spot price is observed weekly, while the futures price are registered daily, I use the weekly average futures price and the time to maturity registered during the Wednesday of the analyzed week.

The obtained convenience yield is chartered in the following graph.

[6] For further details about the computation of u, see Appendix 3

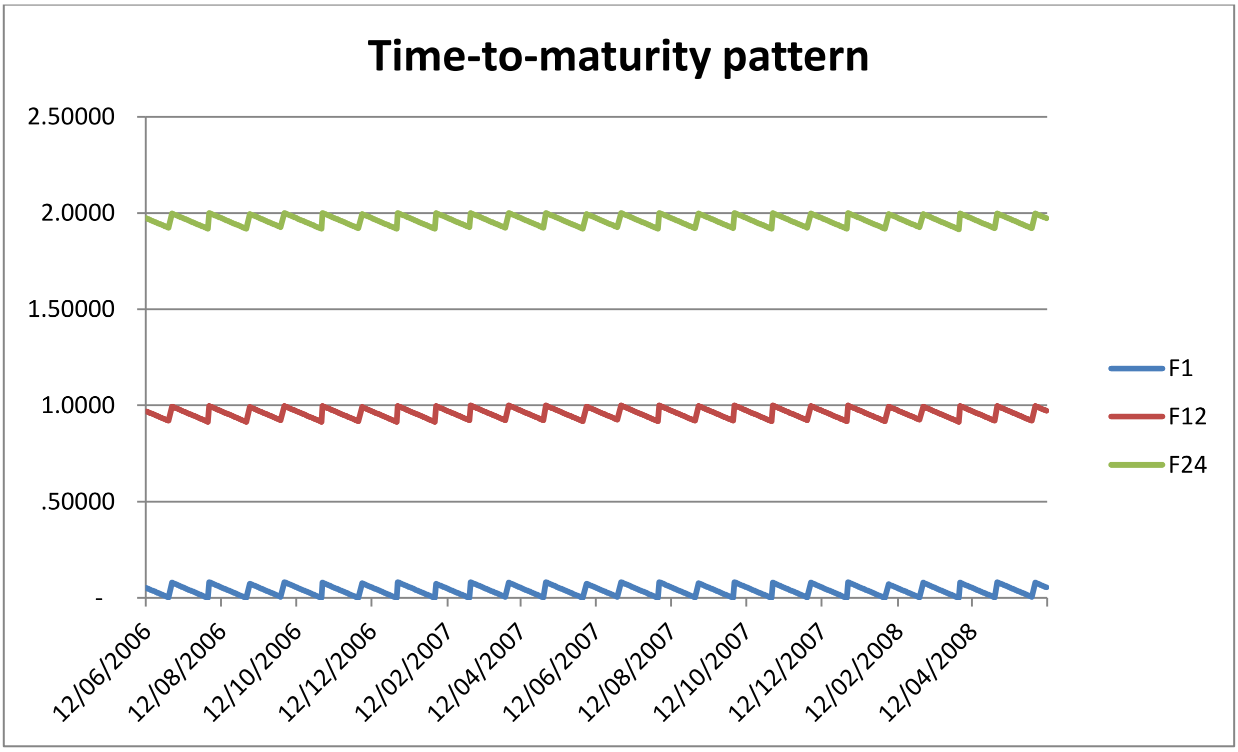


The obtain curve appears to be consistent with the characteristics of the salmon market. In fact, the convenience yield is implicitly related to storage levels and, for this particular case, to the production level. During the summer (week 40-46 approximately), production rate peak, implying that the physical availability of the commodity yields low value. Consistently with this considerations, the derived convenience yield reaches its minimum values both in 2006 and in 2007 between week 40 and 45. Analogously, during winter and spring the lower production rate implies an higher value for the physical detention of the commodity, determining the observed spikes.

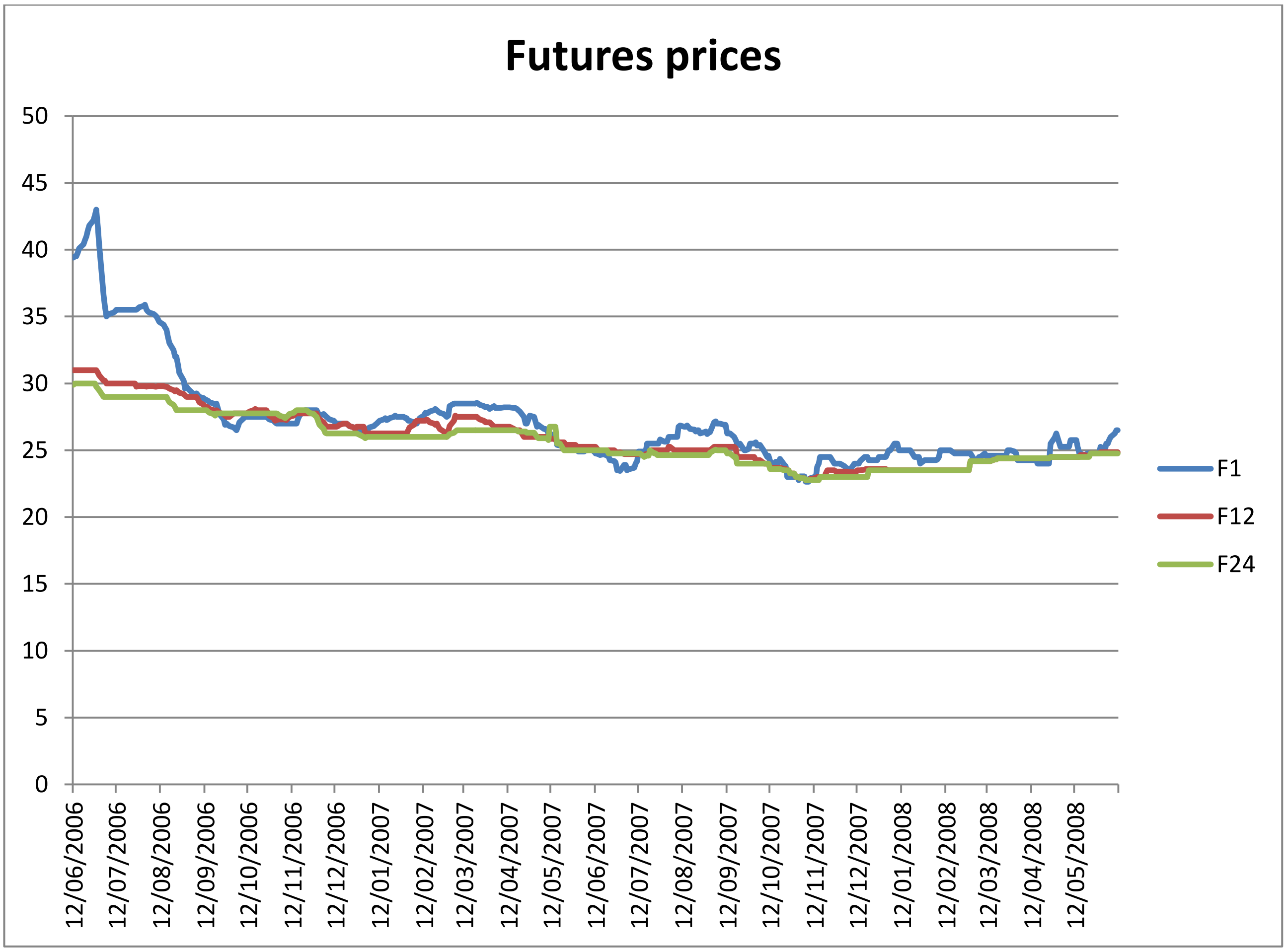
The following table briefly summarizes the main features of the convenience yield.

Year	Mean (NOK)	Standard deviation (NOK)	Min-Max % difference
2006	1.652	2.918384267	124%
2007	- 0.244	2.748954579	247%
2008	1.364	5.067862537	127%

Futures prices for several maturities are daily registered on the FishPool market. The following chart present the time-to-maturity pattern of 3 representative futures contract (F1, Panel A, F12, Panel B, F24, Panel C). The observed values, consistently with the analysis proposed in Ewald (2013), fluctuate but remain within a narrow range during the sample period. This pattern is common to all the contracts registered on the market.



Futures prices show an high standard deviation, presenting a coefficient of variation of approximately 10% in the short term (Panel A, F1), 8% in the medium term (Panel B, F12), 7% in the long term (Panel C, F24). In the following chart three representative futures contract are chartered. It is possible to observe that they present the common seasonal features, reaching the minimum yearly value during the summer and the peak at the end or at the beginning of the period.



The following table briefly resume the statistical characteristics of the observed futures prices.

Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F1	27.08	3.72	0.0406	0.0241	513
F3	26.84	2.98	0.2075	0.0241	513
F5	26.61	2.53	0.3743	0.0242	513
F7	26.34	2.17	0.5412	0.0242	513
F9	26.18	2.14	0.7081	0.0242	513

Table 2: Panel A, short term futures prices

Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F12	25.93	2.11	0.9585	0.0241	513
F14	25.80	2.05	1.1254	0.0242	513
F16	25.73	1.99	1.2923	0.0242	513
F18	25.64	1.90	1.4592	0.0242	513
F20	25.62	1.88	1.6260	0.0242	513

Table 3: Panel B, medium term futures prices

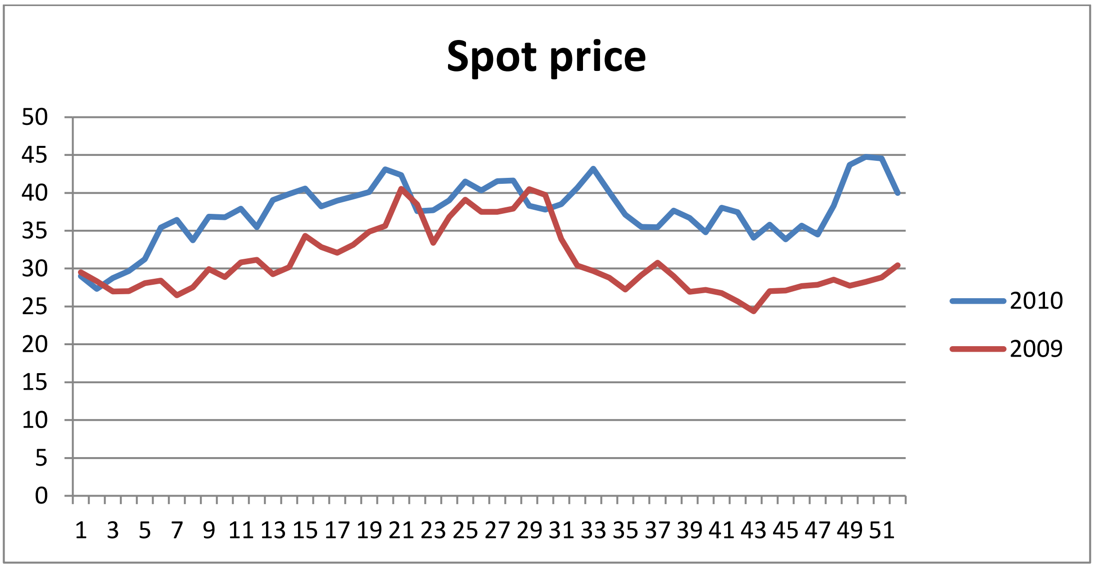
Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F24					

	25.63	1.89	1.9595	0.0240	513
F25	25.63	1.89	2.0429	0.0241	513
F26	25.63	1.89	2.1262	0.0242	513
F27	25.63	1.89	2.2095	0.0241	513
F28	25.63	1.89	2.2928	0.0242	513

Table 4: Panel C, long term futures prices

Data$_2$

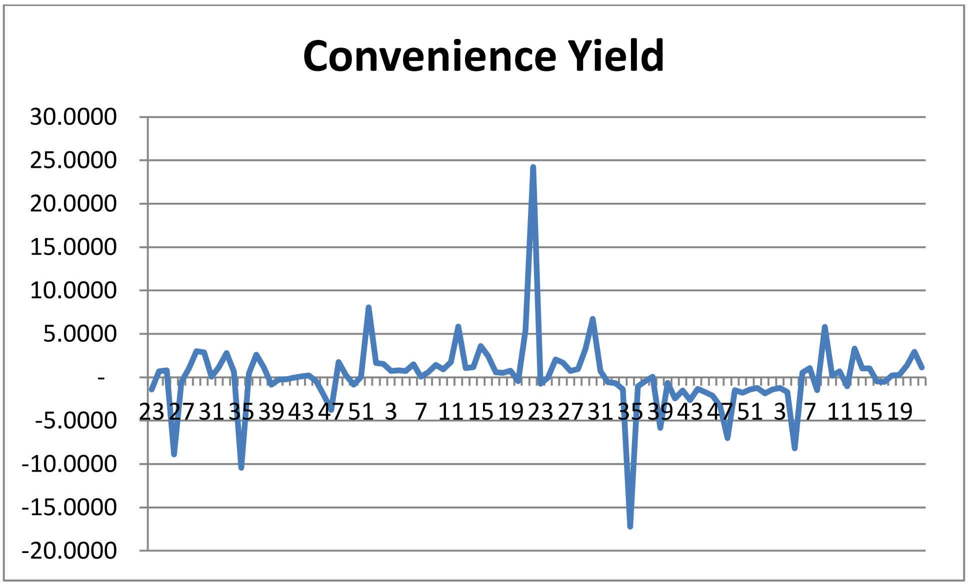
The main features observed in the period 2006-2008 also characterized years 2009 and 2010. Between week 41 and 45 relative minimums are reached, while price peaks occur between weeks 27 and 32. In particular, it is possible to observe a positive trend between week 1 and, approximately, week 30, when the maximum is reached, followed by a significant decline in the spot price level. Finally, in the last period of the year a new rise in prices is observed. The following chart represents the spot price dynamic in 2009 and 2010.



The table below summarized the main features of the 2 analyzed years.

Year	Mean Price (NOK)	Standard Deviation (NOK)	Min-Max % difference
2009	30.97	30.97	39.83%
2010	37.62	3.91	39.01%

The convenience yield is computed following the same procedure and the using the same parameters presented in the section Data$_1$. The resulting convenience yield is chartered in the following graph.



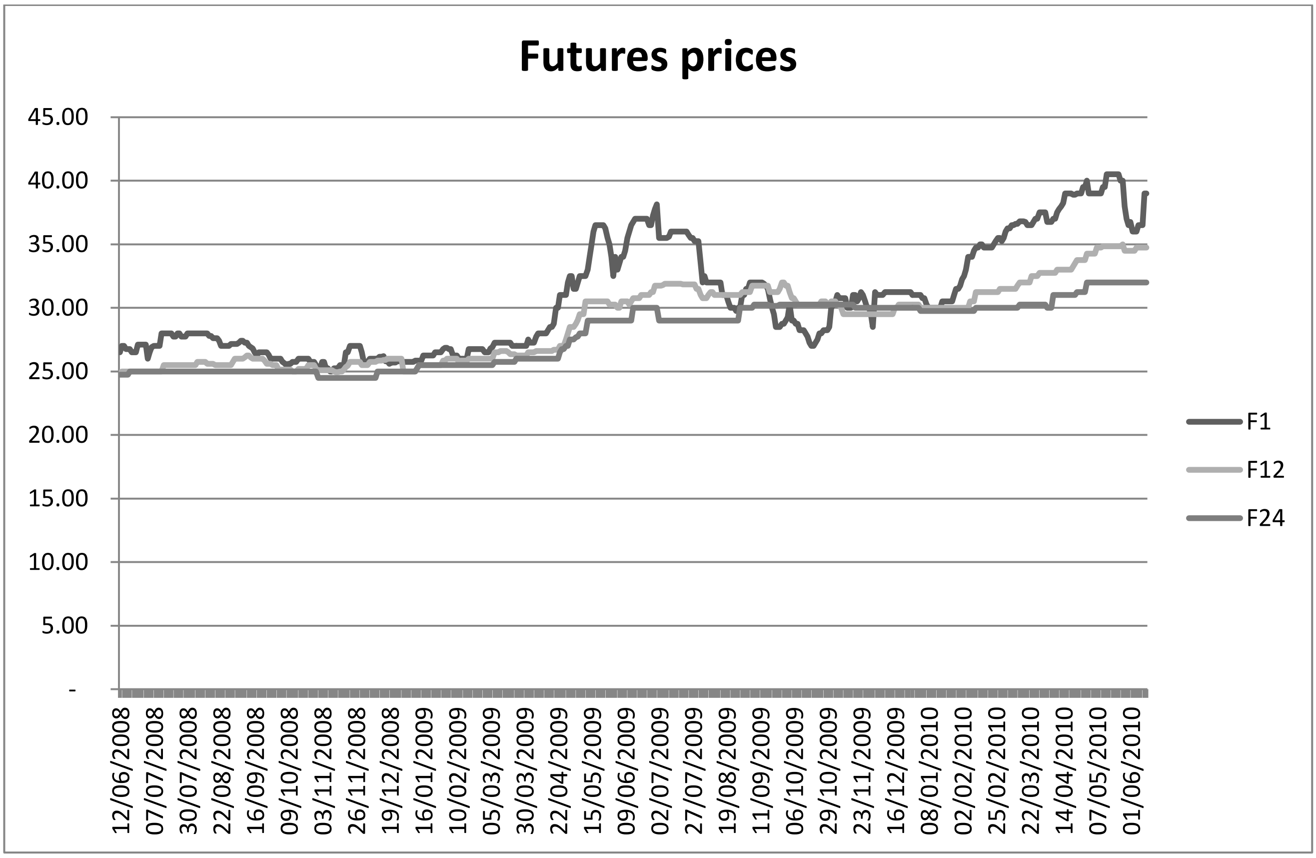
The obtain curve appears to be consistent with the characteristics of the salmon market. During the warmer seasons (week 35-47 approximately), the high production rate yields low values for the physical detention of the commodity, which presents a particular minimum in week 35, 2009. Analogously, during winter and spring, the lower production rate generates shortfall in the availability of the commodity, generating the observed peaks of the convenience yield.

The following table briefly summarizes the main features of the convenience yield.

Year	Mean (NOK)	Standard deviation (NOK)	Min-Max % difference
2008	- 0.066	3.303	446%
2009	0.320	3.307	171%
2010	0.075	3.317	241%

The registered futures prices present the same time-to-maturity patterns described in the section data[1], fluctuating within a narrow range during the whole sample period. This pattern is common to all the contracts registered on the market.

Futures prices show an high standard deviation, presenting a coefficient of variation of approximately 12% in the short term (Panel A, F1), 10% in the medium term (Panel B, F12), 9% in the long term (Panel C, F24). In the following chart three representative futures contract are chartered. It is possible to appreciate the low liquidity affecting the two-years futures contract, while in particular the futures contract with the shortest maturity present the already presented seasonal features.



The following table briefly resume the statistical characteristics of the observed futures prices.

Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F1	30.67	4.35	0.0407	0.0240	519
F3	30.19	3.70	0.2073	0.0241	519
F5	29.71	3.27	0.3740	0.0241	519
F7	29.25	3.01	0.5407	0.0242	519
F9	28.99	3.04	0.7073	0.0242	519

Table 5: Panel A, short term futures prices

Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F12	28.86	3.04	0.9573	0.0241	519
F14	28.69	2.93	1.1240	0.0241	519
F16	28.44	2.78	1.2907	0.0241	519
F18	28.34	2.76	1.4573	0.0242	519
F20	28.22	2.68	1.6240	0.0242	519

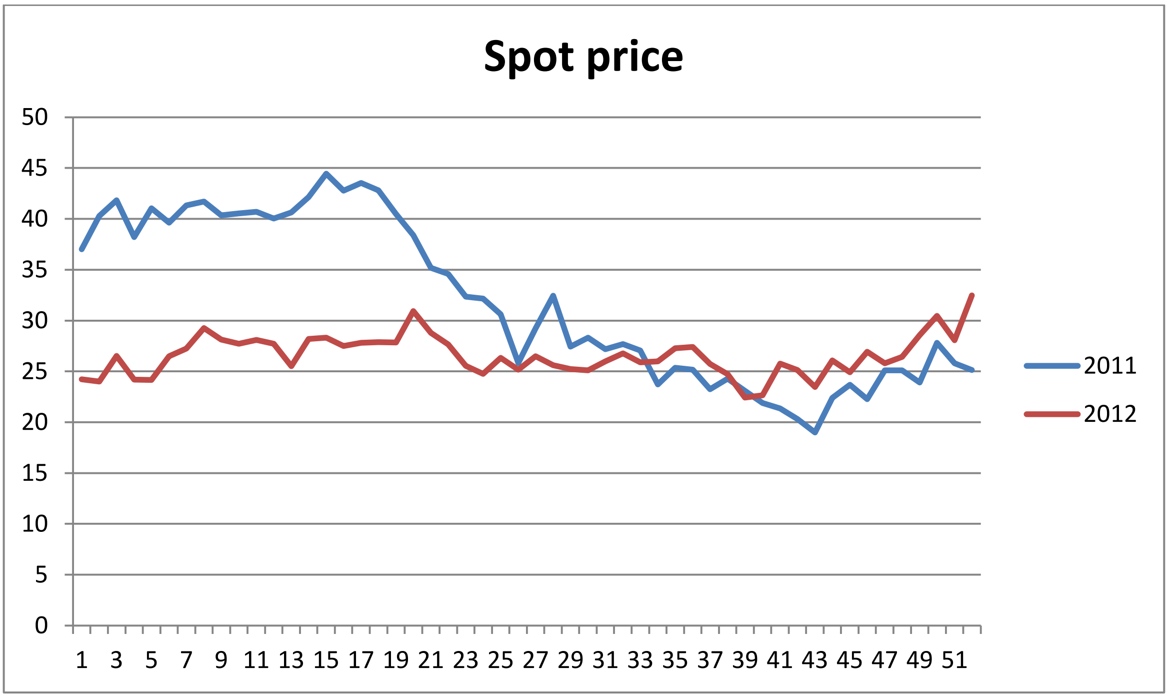
Table 6: Panel B, medium term futures prices

Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F24	27.90	2.52	1.9577	0.0241	519
F25	27.85	2.50	2.0411	0.0241	519
F26	27.82	2.52	2.1246	0.0242	519
F27	27.77	2.49	2.2081	0.0241	519
F28	27.72	2.45	2.2915	0.0242	519

Table 7: Panel C, long term futures prices

Data3

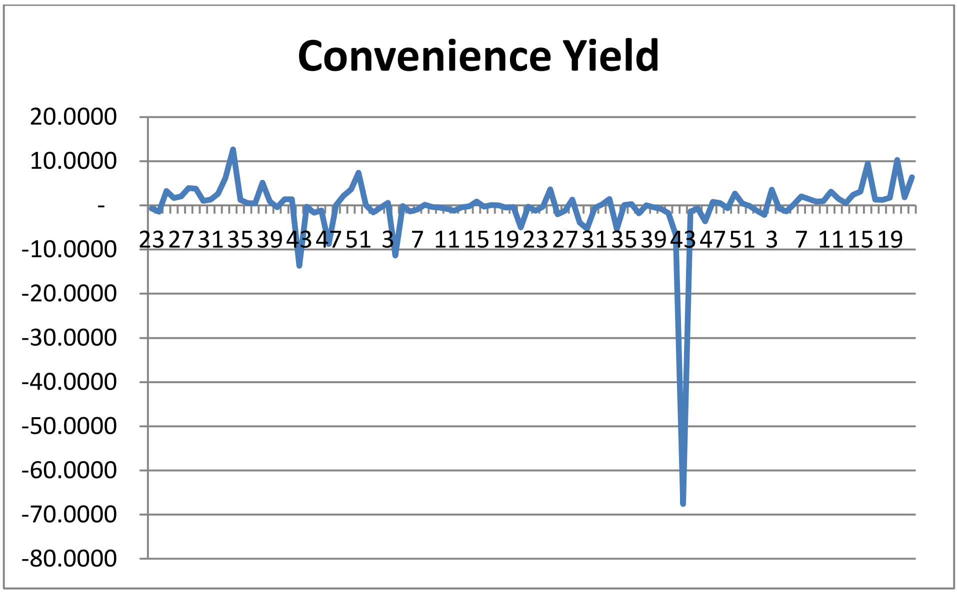
The previously described seasonal features characterize the dynamic of the spot price in the years 2011 and 2012. In the first part of the year a positive trend drives the spot price to reach a maximum between week 18 and 23. It then drops to its minimum during the warmer period of the year, between week 40 and 44. The spot price finally rises again until the end of the analyzed period. The following chart represents the described dynamic.



The table below summarized the main features of the 2 analyzed years.

Year	Mean Price (NOK)	Standard Deviation (NOK)	Min-Max % difference
2011	31.86	7.95	57.27%
2012	26.57	1.98	30.94%

The convenience yield is computed following the same procedure and the same parameters presented in the section $Data_1$. The following graph presents the obtained convenience yield.



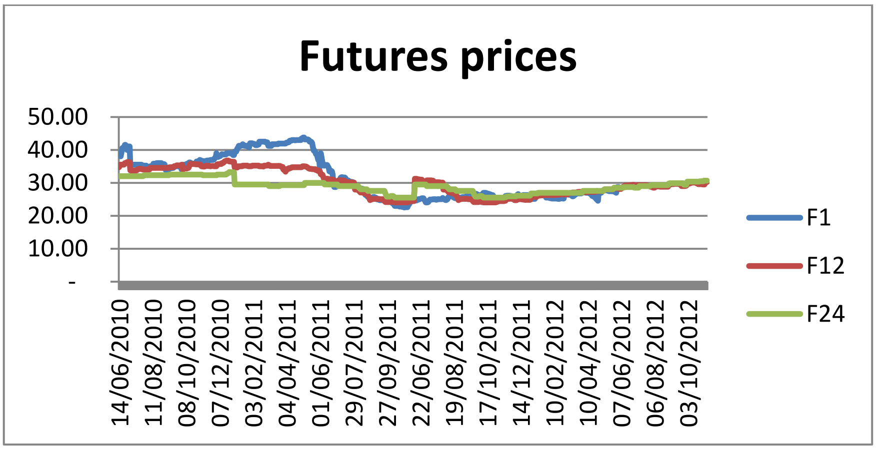
The convenience yield presents again the common seasonal features, peaking during the colder months and reaching its minimum values during spring and summer.

The following table briefly summarizes the main features of the convenience yield.

Year	Mean (NOK)	Standard deviation (NOK)	Min-Max % difference
2010	1.153	4.514	68%
2011	- 2.277	4.530	124%
2012	2.104	4.516	268%

The registered futures present the same time-to-maturity patterns described in the section $data_1$.

Futures prices feature high volatility, presenting a coefficient of variation of approximately 15% in the short term (Panel A, F1), 10% in the medium term (Panel B, F12), 6% in the long term (Panel C, F24). The graph below represents the futures price dynamics of the three considered contracts. It is again possible to appreciate the low liquidity affecting the two-years futures contract, while the contracts with shorter maturity present the already described seasonal trends.



Futures prices
50.00
40.00
30.00
20.00
10.00
-
14/06/2010
11/08/2010
08/10/2010
07/12/2010
03/02/2011
04/04/2011
01/06/2011
29/07/2011
27/09/2011
22/06/2011
19/08/2011
17/10/2011
14/12/2011
10/02/2012
10/04/2012
07/06/2012
06/08/2012
03/10/2012
F1
F12
F24

The following table briefly resume the statistical characteristics of the observed futures prices.

Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F1	35.19	6.22	0.0401	0.0242	371
F3	34.70	5.71	0.2071	0.0243	371
F5	34.27	5.24	0.3738	0.0242	371
F7	33.74	4.70	0.5404	0.0243	371
F9	33.27	4.26	0.7070	0.0244	371

Table 8: Panel A, short term futures prices

Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F12	32.46	4.02	0.9580	0.0243	371
F14	31.70	3.67	1.1252	0.0243	371
F16	31.64	3.14	1.2924	0.0242	371
F18	31.59	2.74	1.4591	0.0243	371
F20	31.27	2.58	1.6257	0.0242	371

Table 9: Panel B, medium term futures prices

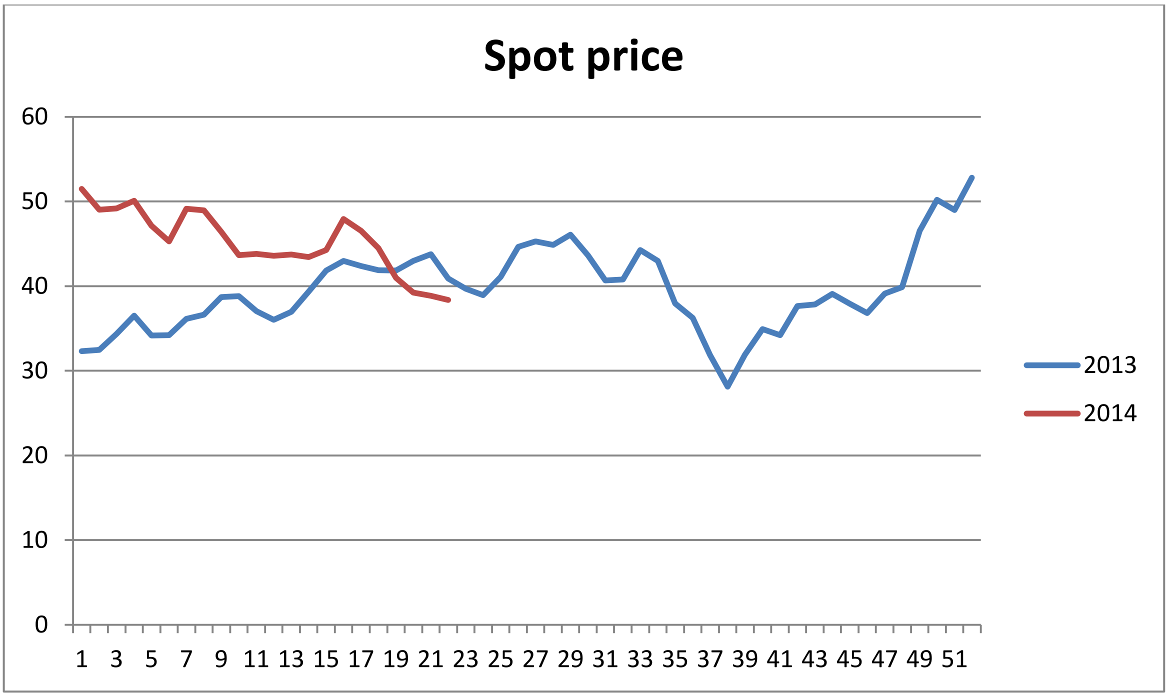
Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F24	30.03	2.19	1.9593	0.0242	371
F25	29.84	2.08	2.0429	0.0242	371
F26	29.72	1.94	2.1263	0.0242	371
F27	29.64	1.70	2.2098	0.0243	371
F28	29.47	1.57	2.2932	0.0242	371

Table 10: Panel C, long term futures prices

Data4

2013 presents the common pattern observed in the previous sections. Particularly interesting appears to be the spot price dynamic in the first half of 2014, during which, instead of the initial positive, or at least flat trend, a significant decline can be observed. This particular pattern derived from two main causes. First, particularly high price levels have been registered during 2013 due to strong demand increase and relatively low production rates. The persistency of this condition has weakened the demand growth, already affected by the economic crises affecting global markets. Secondly, in 2014 a particularly favorable weather has been experienced, boosting the production rate also in the generally colder months and leading to the observed decline.

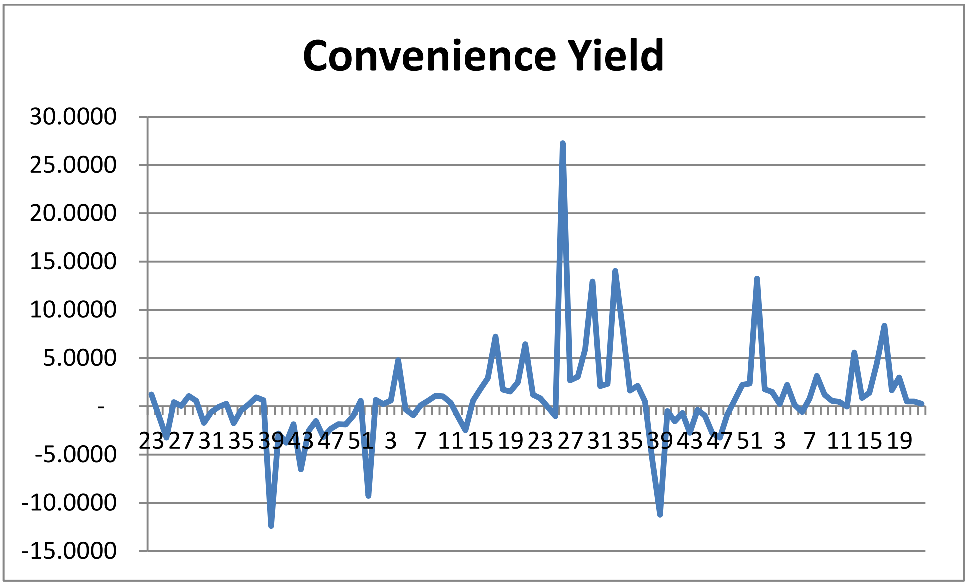
The 2013 and 2014 spot prices are represented in the following chart.



The table below summarized the main features of the 2 analyzed years.

Year	Mean Price (NOK)	Standard Deviation (NOK)	Min-Max % difference
2013	39.56	4.95	46.73%
2014	40.01	6.00	39.05%

The usual procedure is used to compute the convenience yield for the analyzed period. The results are shown in the following chart.

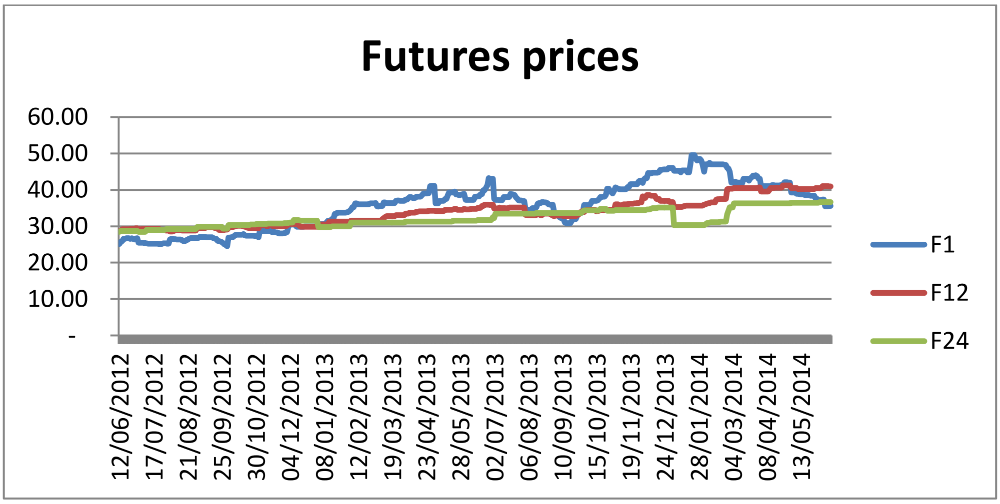


The derived convenience yield presents peaks during the colder seasons and reaches its minimum values during the summer, consistently with the characteristics of the salmon market.

In particular, the following table briefly resumes the main features of the convenience yield.

Year	Mean (NOK)	Standard deviation (NOK)	Min-Max % difference
2012	- 1.788	3.035	1098%
2013	1.943	3.016	141%
2014	1.730	3.036	107%

The observed futures prices, which present the same time-to-maturity patterns described in the section data[1], feature high volatility, present a coefficient of variation of approximately 14% in the short term (Panel A, F1), 10% in the medium term (Panel B, F12), 7% in the long term (Panel C, F24). The three representative futures are chartered in the following graph. The contracts with long time-to-maturity are characterized by low liquidity, while it is possible to identify again the seasonal feature of the futures prices, especially in the dynamics of the contract F1.



The following table briefly resume the statistical characteristics of the observed futures prices.

Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F1	35.76	6.49	0.0405	0.0241	523
F3	34.83	5.07	0.2071	0.0242	523
F5	34.52	4.37	0.3738	0.0242	523
F7	34.33	4.00	0.5405	0.0242	523
F9	34.31	4.08	0.7071	0.0242	523

Table 11: Panel A, short term futures prices

Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F12	33.87	3.73	0.9571	0.0240	523
F14	33.47	3.21	1.1238	0.0242	523
F16	32.93	2.76	1.2905	0.0242	523
F18	32.86	2.64	1.4571	0.0242	523
F20	32.82	2.71	1.6238	0.0242	523

Table 12: Panel B, medium term futures prices

Contract	Mean Price	Standard Deviation	Maturity in years	Standard deviation in years	Observation
F24	32.23	2.39	1.9575	0.0241	523
F25	32.00	2.40	2.0410	0.0241	523
F26	31.84	2.38	2.1244	0.0242	523
F27	31.66	2.28	2.2078	0.0242	523
F28	31.51	2.20	2.2913	0.0242	523

Table 13: Panel C, long term futures prices

IV. Modelling Fish derivatives

The convenience yield assumes a fundamental role for the dynamic of the salmon futures price and thus it appears necessary to take it into account aside the spot price when valuating "salmon derivatives". A one-factor pricing model relies, in fact, on the strong assumptions of constant interest rate and convenience yield, leading to two main consequences. From one side, volatilities of the computed futures prices have to be equal to the variability of the spot price, which appears to be not consistent with the historical observations. Moreover, it can be shown that with a one-factor model the distribution of the future spot price taken under the equivalent martingale measure has a variance increasing with no boundaries as the considered time horizon increases. For these two strong consequences, a one-factor model, for which a derivation is proposed in Appendix 4, appears to be unsatisfactory to price salmon derivatives, leading to a preference for a two-factor model.

In particular, Bjerksund 1991 and Schwartz 1997 represent the fundamental bases on which the modern pricing formulas have been derived. The two authors use similar settings, proposing a model for which a general derivation is proposed in Appendix 5.

Bjerksund 1991 represents one of the most relevant early study on the relevance of the convenience yield in pricing contingent claims on a commodity. By questioning the appropriateness of assuming a non-stochastic and constant convenience yield rate, the author adopts the Gibson-Schwartz 1990 assumptions on the economy to derive analytical solutions to price both futures contracts and European call options written on the commodity. In particular, the spot price is assumed to follow a geometrical Brownian motion and the instantaneous net marginal convenience yield rate on the commodity is described by an Ornstein-Uhlenbeck process. The risk free interest rate r is constant through time and it exists a constant market price per unit of convenience yield risks. Finally, the usual assumption of perfect frictionless markets and no-arbitrage conditions are assumed to hold. Further specification are provided in Appendix 5. Bjerksund 1991's results are a generalization of the standard Black&Scholes model for the case in which the underlying asset pays constant proportional dividend. The author provides therefore an useful benchmarks that can be used to approximate the price of a complex contingent claim for which no closed form solutions are known. Finally, whether the equivalent martingale measure, under which contingent claims prices can be described as theirs discounted expected future payoffs, is known, simulation techniques can be used to approximate the current market value of European style derivatives assets, as discussed in Boyle 1977.

Schwartz 1997 shows that the pricing and hedging of any commodity contingent claims depends critically on the assumed stochastic process for the underlying commodity. The author extends the Gibson-Schwartz 1990 framework by taking into account mean reversion for the commodity prices. In particular, three models are presented and compared in terms of their ability to price existing futures contract and real assets. The first model is a simple one-factor model in which the logarithm of the spot price is assumed to follow a mean reverting process of the Ornstein-Uhlenbeck type. The second model takes into account the role of the convenience yield of the commodity, which is assumed to follow a mean reverting process as well. Finally, the third model is a three-factors model in which also the instantaneous interest rate is assumed to follow a mean reverting process, as suggested in Vasicek 1977. For all the three proposed models, closed form solutions for the prices of futures an forward contracts are derived. Schwartz 1997 shows that the choice of the model has significant implications with respect to the volatility of futures returns and to the behavior of long term futures. In particular, the author points out the one-factor model implies that the volatility of futures returns converge to zero over the time and that futures prices necessarily converge to a fixed value, as maturity increases. Differently, the second and the third model imply that futures volatility will still decrease, but converge to a fixed value different from zero. Moreover, the term structure of futures prices tend to turn upward and to converge to a fixed rate of growth, even in the case in which it is initially in a condition of initial strong backwardation. Empirical evidences from the oil and copper markets show that these second properties are more desirable to price commodity prices. Finally, Schwartz 1997 shows that the real option approach tends to induce investment on natural resources at a too high price when the selected process does not account for mean reversion in prices.

As shown in Appendix 5, the value of a futures contract with time to maturity T-τ can be compute as:

$$F(T-\tau\,; S\,;\ \delta) = e^{r(T-\tau)} V_\tau[S(T)] \tag{26}$$

Where $V_\tau[S(T)]$ represents the value of a future delivery under Bjerksund 1991 setting.

Analogously, Theorem 2 in Appendix 5 shows that the premium of an European Call Option written on the commodity with exercise prize K and time to maturity T-τ is equal to:

$$
\begin{aligned}
V_\tau[(S(T)-K)^+] \\
&= e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2} S(\tau) N\left(\frac{\ln(S(\tau)/K) + R(T-\tau) + \hat{\mu} + \hat{\sigma}^2}{\hat{\sigma}}\right) \\
&- e^{-r(T-\tau)} K N\left(\frac{\ln\left(\frac{S(\tau)}{K}\right) + R(T-\tau) + \hat{\mu}}{\hat{\sigma}}\right) \qquad (27)
\end{aligned}
$$

Edwald (2014) shows that the application of a two-factor pricing model to the case of the salmon market appears to be efficient, leading to a limited margin of error at least in the case of short time to maturity.

On the other side, the necessity of relying on extremely strong assumptions to develop a simple options pricing model determines a theoretical mismatch between the propose model and the observed market structure. For this reasons, for what is in my knowledge any efficiency test has never been computed on the salmon option market, reflecting the low appetite shown by both risk manager and speculators in using the offered American Asian Options for their purposes. This condition opens important research spaces that I might be investigated in the future.

V. Conclusion

The significantly high price volatility observed in the salmon market brought to the creation in 2006 of a regulated market place for futures and options written on the spot price of fresh farmed salmon. The impressive growth registered in the trading volumes in the FishPool market, mainly explained by the relevant value that “predictability” can generate for salmon farmers, has not been homogenous, but almost exclusively driven by futures contracts. The FishPool options market still presents, in fact, high bid ask spread and low liquidity, both determined by the extremely low volumes traded in the market.

A first explanation for this structural condition can be found in the lack of understanding among market practitioners of the financial profile of the offered option contracts. The only options traded in the FishPool market are, in fact, American-Asian options, whose particular financial profile appears to be extremely difficult to be managed and understood by both hedgers and speculators. Aside this lack of understanding, which appears to be common in several other markets, the absence in the literature of a model able to completely describe the characteristics of the offered Asian-American options and of the salmon market disincentives institutional investors to get into a market they are not able to completely understand. Thus, the availability of a theoretically solid pricing formula for these derivatives would allow to remove this major constraint FishPool ASA is facing to include new financial counterparties into the market.

To move to a simpler typology of option contracts, such as a plain vanilla Amercan or European Options type, might thus fast up the development of this market, bypassing the observed gap in the literature and reducing a form of psychological repulsion shown by market practitioners, scared by the complexity of American-Asian Options.

Appendix 1: Introduction of seasonality and mean reversion in the salmon spot price

Moving from the famous Schwartz (1997) model, Ewald (2014) assumes that the dynamic of the spot price behaves as a simple geometric Brownian motion:

$$dP(t) = \left(\mu + m(t) + v(t) + \beta(t) + \gamma(t)\right)P(t)dt + \sigma_1 P(t)dZ_1(t) \qquad (28)$$

In order to introduce mean reversion and seasonality in this process, I follow the approach proposed by Jin, Lance, Hart and Hayes (2010), which generalizes the Schwartz model in order to account for mean reversion and seasonality. In order to simply the process let's define $\delta = -\left(m(t) + v(t) + \beta(t) + \gamma(t)\right)$.

δ represents a first approximation of the net convenience yield (which will be better defined in Appendix 2), which can be read as an aggregation of four unrelated processes featuring mean reversion. For this reason, as claimed in Ewald (2014), it is possible to assume that δ(t) follows an Ornstein-Uhlenbeck process, at least in approximation:

$$d\delta(t) = \left(k(\alpha - \delta(t))\right)dt + \sigma_2 dZ_2(t) \qquad (29)$$

By further assuming that the two stochastic processes Z_1 and Z_2 are related and, in particular, that $dZ_1 dZ_2 = \rho_{1,2}dt$ and by defining L=ln[P(t)], it is possible to apply the Ito's Lemma to obtain the stochastic process

$$dL = \mu_L dt + \sigma_L dZ_L \qquad (30)$$

The holder of the commodity obtains an expected return equal to the relative price change $\frac{dS}{S} = dL$ plus the convenience yield. In equilibrium this expected return has to be equal to the risk free rate r plus the relative risk premium, called λ_1. In other words, in equilibrium the following has to hold:

$$\mu - \delta + \delta = r + \lambda_1 \qquad (31)$$

From this condition it follows that the relative risk-neutral processes are:

$$dL = (r - \delta)dt + \sigma_L dZ_L^*$$

$$d\delta(t) = (k(\alpha - \delta(t)) - \lambda_2)dt + \sigma_2 dZ_2^*$$

Where λ_2 represents the market price for the risk associated to the dynamic of the net convenience yield and Z_1^* and Z_2^* are the two Wiener process taken under the equivalent martingale measure.

In order to introduce seasonality, let's first allows the spot prices to feature mean reversion to a long term mean throughout the process

$$dP(t) = (\mu - \vartheta \ln P(t))P(t)dt + \sigma_1 P(t)dZ_1 \qquad (32)$$

By applying the Ito's Lemma it is possible to show that, under this new setting, also the log spot price follows a mean reverting process of the Ornstein-Uhlenbeck type:

$$dL = (\mu_L - \vartheta \ln P(t))dt + \sigma_L dZ_L \qquad (33)$$

Where $\vartheta > 0$ is the speed of mean reversion and $\frac{\mu_L}{\vartheta}$ is the speed long run mean. It is now possible to relax the assumption for which all the parameters are constant throughout the year, representing this periodicity as a truncated Fourier series, see i.e. van der Hoeven (2004). In particular, it is possible to redefine the dynamic of the log spot price as a periodical deterministic function of time, such as:

$$dL = \frac{d\mu_L(t)}{\vartheta} + \vartheta\left(\frac{\mu_L(t)}{\vartheta} - \ln P(t)\right)dt + \sigma_L dZ_L \qquad (34)$$

$$d\mu_L(t) = \mu_{x,0} + \sum_{h=1}^{H}\left[\mu_{x,h,cos}\cos\left(\frac{2\pi h}{s}t\right) + \mu_{x,h,sin}\sin\left(\frac{2\pi h}{s}t\right)\right] \qquad (35)$$

Where s indicates the number of observation per year, $\mu_{x,h,cos}$ and $\mu_{x,h,sin}$ represents the seasonality parameters, while H determines the number of term in the sum, chosen equal to 2 according to the Akaike Information Criterion, AIC (see Y. Sakamoto, M. Ishiguro, and G. Kitagawa. Dordrecht (1986).

Appendix 2: Deriving the convenience yield for the salmon market

By defining the net present value of the storage costs as D, the convenience yield δ is defined such that[7]:

$$Fe^{\delta T} = (S_0 + D)e^{rT}$$

Where:

F_0 = The futures price with maturity in T

S_0 = The spot price

From which the convenience yield can be stated as:

$$\delta(t) = \frac{ln\frac{[(S_0 + D)e^{rT}]}{F_0}}{T} \qquad (36)$$

In particular, in the special case in which the storage costs per unit are a constant proportion u of the spot price, it is possible to rewrite the previous equations as:

$$F_0 e^{\delta T} = S_0 e^{(r+u)T}$$

$$\delta(t) = \frac{ln\frac{[S_0 e^{(r+u)T}]}{F_0}}{T} \qquad (37)$$

In the specific case of the salmon market, it is possible to assume that storage costs are represented by the feeding costs that the farmer would have to bear in the case he/she decides to delay harvesting. In other word, D would be equal to the NPV of the total cost for feeding the biomass from today to "maturity", T.

$$D = \int_0^T (c_t e^{rT})dt$$

Where c_t represents the total cost for feeding faced by the fish farmer in t. If we assume the feeding costs to be constant over the time, it is now possible to state that:

[7] "Options, Futures and other derivatives", John C. Hull, Eight Edition, p.119

$$D = c * \int_0^T (e^{rT})dt$$

$$D = \frac{c * (1 - e^{-rT})}{r}$$

In order to relax the assumption of constant feeding costs over the time, it is necessary to describe the function c(t). In particular, by defining the conversation ratio, f, as the measure of the salmons' efficiency in converting food in an increase in the biomass and assuming it to be constant over the time, c_t can be defined as:

$$c_t = gfw'(t)N(t) \quad (38)$$

Where:

g = price for fish feed, assumed constant over the time;
w'(t) = the marginal growth in the biomass of one salmon, in the case harvesting time is delayed;
N(t) = the number of salmon in the pen in t.

Since salmons don't reproduce in the pens, N(t) is decreasing over the time, and, in particular, its rate of change can be described at any time before harvesting as:

$$dN(t) = -m(t) * N(t)$$

Where m(t) represent the mortality rate.

While the number of salmon is decreasing, their weight increases over the time. In order to describe this dynamic, we assumed that the average marginal increase in the weight of one fish follow the process proposed in Ewald 2014:

$$w'(t) = \big(\theta - \beta(t)\big)w(t)dt + \sigma_w w(t)dB(t)$$

Where B(t) represents a standard Brownian motion, while β(t) is an arbitrary stochastic process, representing the "weight saturation", chosen such that the above process is well define and introducing a mean reversion feature toward the mean reversion level θ, assumed to be constant.

It is now possible to define the NPV of the storage costs as:

$$D = \int_0^T \left(gf\left((\theta - \beta(t))w(t)dt + \sigma_w w(t)dB(t) \right) N(t)e^{rT} \right) dt \qquad (39)$$

Under this setting, and assuming the convenience yield to be a not constant proportion of the spot price, y can be defined as follow:

$$\delta(t) = \frac{ln \frac{\left[\left(S_0 + \left[\int_0^T \left(gf\left((\theta - \beta(t))w(t)dt + \sigma_w w(t)dB(t) \right) N(t)e^{rT} \right) dt \right] \right) e^{rT} \right]}{F_0}}{T} \qquad (40)$$

From which it is possible, after the estimation of all the required parameters, to compute the convenience yield at any point of time.

Moreover, since δ includes some mean reverting feature, it appears to be fair to state that, at least in approximation, the dynamic of the convenience yield follows an Ornstein-Uhlenbeck process, such as:

$$d\delta(t) = \left(k(\alpha - \delta(t))\right)dt + \sigma dZ(t)$$

Where:

α = the long range mean to which $\delta(t)$ tends to revert;

k = mean reversion speed;

σ = the volatility term;

dZ(t) = it represents the increment of a standard Brownian motion, Z(t).

It is now possible to adjust the convenience yield to introduce seasonality in its dynamic by following the same approach used it Appendix 1.

In particular, by noticing that the convenience yield can be described as the sum of the log spot price and of a complex process y(t), which feature at least some mean reversion, it is possible to rewrite δ(t) as

$$\delta(t) = L(t) + y(t)$$

Where y(t) can be adapted to feature seasonality, by approximating its process to

$$d\delta(t) = \frac{d\mu_\delta(t)}{k} + k\left(\frac{\mu_\delta(t)}{k} - \delta(t)\right)dt + \sigma_\delta dZ_\delta \qquad (40)$$

$$d\mu_{\delta}(t) = \mu_{\delta,0} + \sum_{h=1}^{H}\left[\mu_{\delta,h,cos}\cos\left(\frac{2\pi h}{s}t\right) + \mu_{\delta,h,sin}\sin\left(\frac{2\pi h}{s}t\right)\right] \qquad (41)$$

Appendix 3: Approximation of the storage cost per unit as a constant proportion of the spot price

In Appendix 2 I show that in the case of storage cost per unit as a constant proportion u of the spot price the convenience yield is defined as:

$$\delta(t) = \frac{ln\frac{[S_0 e^{(r+u)T}]}{F_0}}{T}$$

By defining u as the ratio between the feeding costs faced by the farmer in the case he/she decides to delay harvesting by one period and the spot price, by inserting equation (39) it is possible to state:

$$u = \frac{D}{S_0} = \frac{\frac{gfw'(t)N(t) * (1 - e^{-rT})}{r}}{S_0} \quad (41)$$

This formula assumes constant feeding costs and a stochastic biomass growth. Since feeding costs are generally increasing, while, at the same time, the biomass growth is decreasing, I assume that the two effects compensate. In particular, I thus assume that both feeding costs and the biomass growth are constant. By normalizing the population size N(t) to 1, the (41) can now be rewritten as:

$$u = \frac{\frac{gfw * (1 - e^{-rT})}{r}}{S_0} \quad (42)$$

Where w represent the constant biomass growth coefficient.

Appendix 4: The Schwartz 1997 one-factor model

The model assumes that the logarithm of the commodity spot price follows a mean reverting process of the Ornstein-Uhlenbeck type. Hence, we can describe it as the stochastic process:

$$dS = k(\mu - lnS)Sdt + \sigma Sdz$$

Where k > 0 describes the degree of mean reversion to the long run mean log price μ , σ > 0 represents the volatility of the commodity spot price and dz is an increment to a standard Brownian motion.

By defining X = lnS, it is possible to show that Ito's Lemma implies that the log price can be described as another Ornstein-Uhlenbeck stochastic process.
In fact, since:

$$\frac{d(lnS)}{dS} = \frac{1}{S}, \qquad \frac{d^2(lnS)}{dS^2} = -\frac{1}{S^2}, \qquad \frac{d(lnS)}{dt} = 0$$

It follows that the process followed by X is

$$dX = \left[\frac{1}{S}k(\mu - \frac{\sigma^2}{2k} - X)S\right]dt + \frac{1}{S}\sigma Sdz$$

By defining:

$$\alpha = \mu - \frac{\sigma^2}{2k}$$

It is possible to rewrite the process as:

$$dX = [k(\alpha - X)]dt + \sigma dz$$

Where α represents the long run mean log price.

In this model, the log of the spot price plays the role of an underlying state variable upon which it is possible to write contingent claim. Therefore, assuming that the risk-free interest rate r is constant and by assuming perfect frictionless markets and absence of arbitrage opportunities, from the equivalent martingale theory[8] it is possible to describe the current value of any contingent claim at maturity T and pay-off described as:

[8] See Aaese (1988)

$$Y(T) \equiv Y(T, S(T), X(T))$$

By the formula

$$B_t[Y(T)] = e^{-r(T-t)} E_T^*[Y(T)]$$

where $E_T^*[Y(T)]$ represents the expectation taken under the equivalent martingale probability measure. Under these settings, the dynamic of the Ornstein-Uhlenbeck process under the equivalent martingale measure becomes:

$$dX = [k(\alpha^* - X)]dt + \sigma dz^*$$

Where $\alpha^* = \alpha - \lambda$, with λ, assumed constant over the time, representing the market price of risk, and dz^* is the increment to a standard Brownian motion taken under the equivalent martingale measure.

It is possible to observe that the conditional distribution of X at any time τ, taken under the equivalent martingale measure, is normal with:

$$E_0[X(\tau)] = e^{-k\tau} X(0) + (1 - e^{-k\tau})\alpha^*$$

$$Var_0[X(\tau)] = \frac{\sigma^2}{2k}(1 - e^{-k\tau})$$

Since X = lnS , it follows that the commodity spot price at time τ is log-normally distributed under the equivalent martingale measure with these same parameters. Assuming constant risk free interest rate and risk neutrality in the market, under the equivalent martingale measure the no-arbitrage condition requires that the futures (or forward) price of the commodity with maturity τ is equal to its expected spot price at time τ. Hence, from the properties of the log-normal distribution, the following has to hold:

$$F(S, \tau) = exp\left[E_0[X(\tau)] + \frac{1}{2} Var_0[X(\tau)]\right]$$

By substitution, we can rewrite then the futures (or forward) price with delivery in τ as:

$$F(S, \tau) = exp\left[e^{-k\tau} lnS + (1 - e^{-k\tau})\alpha^* + \frac{\sigma^2}{4k}(1 - e^{-k\tau})\right]$$

Or in log-form:

$$\ln[F(S,\tau)] = e^{-k\tau} lnS + (1 - e^{-k\tau})\alpha^* + \frac{\sigma^2}{4k}(1 - e^{-k\tau}) \qquad (43)$$

Which satisfy the no-arbitrage condition set by the partial differential equation:

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + k(\mu - \lambda - lnS)SF_s + F_\tau = 0$$

With terminal boundary F(S,0) = S .

Appendix 5:The Bjerksund 1991 two-factor model

This model assume the spot price of the commodity S(t) to follow a geometrical Brownian motion, such as:

$$dS(t) = \mu S(t)dt + \eta S(t)dz$$

Where μ represents the drift term, η is the volatility term and dz is the increment of a standard Brownian motion z(t).

While in the previous 1-factor model the convenience yield was assumed to be non-stochastic, under this setting the instantaneous net marginal convenience yield rate δ(t) on the commodity can be described by an Ornstein-Uhlenbeck process:

$$\mathrm{d}\delta(\mathrm{t}) = k\big(\alpha - \delta(\mathrm{t})\big)dt + \sigma dw$$

Where α represents the long-run mean to which tends to revert, k > 0 indicates the degree of mean reversion of the model, σ is the volatility term and dw is the increment of a standard Brownian motion w(t) correlated to z(t),

$$dzdw = \rho dt$$

Furthermore, it is assumed that the risk free interest rate is constant over the time and that a constant market price λ per unit of convenience yield risk exists, aside the usual assumption of market perfection and absence of arbitrage opportunities.

By assuming that the market value of any contingent yield on the commodity $B(S, \delta, \tau)$ is twice continuously differentiable both in S and in δ it is possible to apply Ito's Lemma to describe its dynamic as follows:

$$dB = dS + B_\delta d\delta + B_\tau d\tau + \frac{1}{2}B_{SS}(dS)^2 + \frac{1}{2}B_{\delta\delta}(d\delta)^2 + B_{S\delta}dSd\delta$$

Since:

$$B_S dS = \mu S B_S d\tau + \eta S B_S dz$$

$$B_{SS}(dS)^2 = \eta^2 S^2 B_{SS} d\tau$$

$$B_\delta d\delta = k\big(\alpha - \delta(\tau)\big)B_\delta d\tau + \sigma B_\delta dw$$

$B_{\delta\delta}(d\delta)^2 = \sigma^2 B_{\delta\delta} d\tau$

$dSd\delta = \eta\sigma S(\tau)\rho d\tau$

The instantaneous change in value of the contingent claim can be written as:

$$dB = \left[\mu S B_S + k\big(\alpha - \delta(\tau)\big)B_\delta + B_\tau + \frac{1}{2}\eta^2 S^2 B_{SS} + \frac{1}{2}\sigma^2 B_{\delta\delta} + \eta\sigma S\rho B_{S\delta}\right] d\tau + \eta S B_S dz + \sigma B_\delta dw$$

From which it is possible to show that the price of a contingent claim must satisfy the following P.D.E. in order not to allow for arbitrage opportunities:

$$\frac{1}{2}\eta^2 S^2 B_{SS} + \frac{1}{2}\sigma^2 B_{\delta\delta} + (r - \delta)SB_S + k(\alpha - \delta(\tau) - \lambda\sigma)B_\delta + B_\tau + \eta\sigma S\rho B_{S\delta} = rB$$

Where r is the risk free interest rate.

By defining claim on a future delivery of the commodity as

$V_\tau[S(T)] = e^{-r(T-\tau)}E_\tau^*[S(T)]$ (xx)

Where E_τ^* represents the expectation taken under the equivalent martingale probability measure, and rewriting the dynamic of the spot price in the equivalent form

$$S(T) = S(\tau)e^{\left\{\left(\mu - \frac{1}{2}\eta^2\right)(T-\tau) + \eta\int_\tau^T dz(s)ds\right\}}$$

With T > τ , it is possible to express the net present value of the future pay-off as:

$$e^{-r(T-\tau)}S(T) = S(\tau)e^{\left(\mu - r - \frac{1}{2}\eta^2\right)(T-\tau) + \int_\tau^T dz(s)ds}$$

By considering now that under this setting the relation between the true probability measure and the martingale probability measure is given by:+

$$dz(\tau) = dz^*(\tau) - (\lambda')d\tau$$

$$dw(\tau) = dw^*(\tau) - \lambda d\tau$$

Where $\lambda' = \frac{\mu + \pi\delta(\tau) - r}{\eta}$ represent the market price per unity of the commodity risk[9].

By expressing the cumulative convenience yield rate as:

[9] See Gibson & Schwartz 1990

$$X(\tau) = \int_{\tau}^{T} \delta(s)ds$$

$$= \int_{\tau}^{T} k\big(\alpha - \delta(s)\big)ds + \int_{\tau}^{T} \sigma dw(s)ds = k\alpha(T-\tau)$$

$$- k\int_{\tau}^{T} \delta(s)ds + \sigma\int_{\tau}^{T} dw(s)ds$$

From which it follows that:

$$\delta(T) - \delta(\tau) = \int_{\tau}^{T} d\delta(s)ds$$

Which implies:

$$X(T) - X(\tau) = \int_{\tau}^{T} \delta(s)ds$$

From which it is possible to state that:

$$\delta(T) - \delta(\tau) = k\alpha(T-\tau) - k\big(X(T) - X(\tau)\big) + \sigma\int_{\tau}^{T} dw(s)ds$$

By rewriting the dynamic of the convenience yield in the alternative form

$$\delta(T) = e^{-k(T-\tau)} + \big(1 - e^{-k(T-\tau)}\big)\alpha + \sigma e^{-kT}\int_{\tau}^{T} e^{ks}dw(s)ds$$

It is finally possible to write:

$$X(T) = X(\tau) + \frac{1}{k}\bigg[\big(1 - e^{-k(T-\tau)}\big)(\delta(\tau) - \alpha) + \alpha(T-\tau)$$

$$+ \sigma\int_{\tau}^{T} dw(s)ds - \sigma e^{-kT}\int_{\tau}^{T} e^{ks}dw(s)ds\bigg]$$

In order to express $\eta\int_{\tau}^{T} dz(\mathrm{s})\mathrm{ds}$ by the two processes $w^*(\tau)$ and $z^*(\tau)$, it is possible to combine the relation between the true probability measure and the martingale probability measure of the process z(s) with the function representing the cumulative yield rate to state that:

$$\int_{\tau}^{T} dz\,(\mathrm{s})\mathrm{ds} = \int_{\tau}^{T} dz^*\,(\mathrm{s})\mathrm{ds} - \frac{\mu - r}{\eta}(T-\tau) - \frac{1}{\eta}[X(T) - X(\tau)]$$

By substituting the previously determined value of X(T) and of X(τ) and rearranging the equation it is now possible to write:

$$\eta \int_{\tau}^{T} dz\,(\mathrm{s})\mathrm{ds} = -\left(\mu - r + \alpha - \frac{1}{k}\alpha\lambda\right)(T-\tau) + \frac{1}{k}\left(\alpha - \delta(\tau) - \frac{1}{k}\sigma\lambda\right)\left(1 - e^{-k(T-\tau)}\right)$$
$$+\eta \int_{\tau}^{T} dz^{*}(s)ds - \frac{1}{k}\sigma \int_{\tau}^{T} dw^{*}(s)ds + \frac{1}{k}e^{-kT}\sigma \int_{\tau}^{T} e^{ks}dw^{*}(s)ds$$

By inserting this value in the discounted future payoff function, it is possible to restate it as:

$$e^{-r(T-\tau)}S(T)$$
$$= S(\tau)e^{\left\{-\left(\frac{1}{2}\eta^2+\alpha-\frac{1}{k}\sigma\lambda\right)(T-\tau)+\eta\int_{\tau}^{T}dz^{*}(s)ds-\frac{1}{k}\sigma\int_{\tau}^{T}dw^{*}(s)ds+\frac{1}{k}\left[\left(\left(\alpha-\delta(\tau)-\frac{1}{k}\sigma\lambda\right)\left(1-e^{-k(T-\tau)}\right)\right)+\left(\sigma e^{-kT}\int_{\tau}^{T}e^{-ks}dw^{*}(s)ds\right)\right]\right\}}$$

By calling the exponent above q*, it follows from stochastic calculus that the expected value of the exponent can be defined as:

$$\hat{\mu} = E_{\tau}^{*}[q^{*}] - \left(\frac{1}{2}\eta^2 + \alpha - \frac{1}{k}\sigma\lambda\right)(T-\tau) + \frac{1}{k}\left(\left(\alpha - \delta(\tau) - \frac{1}{k}\sigma\lambda\right)\left(1 - e^{-k(T-\tau)}\right)\right)$$

Moreover, since it can be shown that:

$$E_{\tau}^{*}\left[\left(\eta\int_{\tau}^{T} dz^{*}(s)ds\right)^{2}\right] = \eta^{2}(T-\tau)$$

$$E_{\tau}^{*}\left[\left(\frac{1}{k}\sigma\int_{\tau}^{T} dw^{*}(s)ds\right)^{2}\right] = \left(\frac{1}{k}\right)^{2}\sigma^{2}(T-\tau)$$

$$E_{\tau}^{*}\left[\left(\frac{1}{k}\sigma e^{-kT}\int_{\tau}^{T} e^{-ks}dw^{*}(s)ds\right)^{2}\right] = \left(\frac{1}{k}\right)^{2}\frac{\sigma^{2}}{2k}\left(1 - \left(1 - e^{-k(T-\tau)}\right)^{2}\right)$$

$$E_{\tau}^{*}\left[\left(\eta\int_{\tau}^{T} dz^{*}(s)ds\right)\left(\frac{1}{k}\sigma\int_{\tau}^{T} dw^{*}(s)ds\right)\right] = \frac{1}{k}\sigma\eta\rho(T-\tau)$$

$$E_{\tau}^{*}\left[\left(\eta\int_{\tau}^{T} dz^{*}(s)ds\right)\left(\frac{1}{k}\sigma e^{-kT}\int_{\tau}^{T} e^{-ks}dw^{*}(s)ds\right)\right] = \left(\frac{1}{k}\right)^{2}\sigma\eta\rho\left(1 - e^{-k(T-\tau)}\right)$$

$$E_{\tau}^{*}\left[\left(\frac{1}{k}\sigma\int_{\tau}^{T} dw^{*}(s)ds\right)\left(\frac{1}{k}\sigma e^{-kT}\int_{\tau}^{T} e^{-ks}dw^{*}(s)ds\right)\right] = \left(\frac{1}{k}\right)^{3}\sigma^{2}\left(1 - e^{-k(T-\tau)}\right)$$

It follows that the variance of q* can be expressed as:

$$\hat{\sigma} = E_\tau^*[(q^*)^2] - (E_\tau^*[q^*])^2 \left(\eta^2 - 2\frac{1}{k}\sigma\eta\rho + \left(\frac{1}{k}\right)^2 \frac{\sigma^2}{2k}\right)(T-\tau) + 2\left(\left(\frac{1}{k}\right)^2 \sigma\eta\rho - \left(\frac{1}{k}\right)^3 \sigma^2\right)\left(1 - e^{-k(T-\tau)}\right) + \left(1 - \left(1 - e^{-k(T-\tau)}\right)^2\right)\left(\frac{1}{k}\right)^2 \frac{\sigma^2}{2k}$$

The exponent q* is normally distributed, which implies that the discounted future payoff is log-normally distribute. The value of the contingent claim can, hence, be written as:

$$V_\tau^*[S(T)] = E_\tau^*\left[e^{-r(T-\tau)}S(T)\right] = S(\tau)e^{\left(\hat{\mu}+\frac{1}{2}\hat{\sigma}^2\right)} \qquad (44)$$

Which leads to the following statement:

Theorem 1: The value of a future delivery

The current value (at date τ) of a claim on a future delivery of the commodity on the future date T is

$$V_\tau[S(T)] = S(\tau)e^{\left\{\left[-\alpha+\frac{1}{k}(\sigma\lambda-\eta\sigma\rho)+\frac{1}{2}\left(\frac{1}{k}\right)^2\sigma^2\right](T-\tau)-\frac{1}{k}\left[\delta(\tau)-\alpha+\frac{1}{k}(\sigma\lambda-\eta\sigma\rho)+\left(\frac{1}{k}\right)^2\sigma^2\right]\left(1-e^{-k(T-\tau)}\right)+\frac{1}{2}\left(\frac{1}{k}\right)^2\frac{\sigma^2}{2k}\left(1-\left(1-e^{-k(T-\tau)}\right)^2\right)\right\}}$$

Which satisfies the PDE

$$\frac{1}{2}\eta^2 S^2 B_{SS} + \frac{1}{2}\sigma^2 B_{\delta\delta} + (r-\delta)SB_S + k(\alpha - \delta(\tau) - \lambda\sigma)B_\delta + B_\tau + \eta\sigma S\rho B_{S\delta} - rB = 0$$

Futures price

In order to avoid the presence of risk-free arbitrage opportunities, the futures price F is determined by the relation:

$$V_\tau[S(T) - F] = 0$$

From which it follows that the futures price on a contract written on a commodity with maturity (T-τ) is defined as:

$$F(T-\tau\,; S\,;\ \delta) = e^{r(T-\tau)}V_\tau[S(T)]$$

It is possible to note that for k sufficiently high the stochastic property of δ(τ) will vanish and that:

$$\lim_{k\to\infty} V_\tau[S(T)] = e^{-\alpha(T-\tau)}S(\tau)$$

Which leads to the common case of a constant proportional convenience yield rate δ(τ) = α , translating the futures price formula in

$$F = e^{(r-\alpha)(T-\tau)}$$

The premium of an European call option

I consider now an European call option written on the commodity, with time to maturity T-τ and exercice prize K. By applying the evaluation formula (44), it is possible to state:

$$V_\tau[(S(T)-K)^+] = e^{-r(T-\tau)}E_\tau^*[(S(T)-K)^+]$$

Which can be express as:

$$V_\tau[(S(T)-K)^+] = E_\tau^*[\left(S(T)e^{q^*} - e^{-r(T-\tau)}K\right)^+]$$

By recalling that q* is normally distributed, with expected value $\hat{\mu}$ and $\hat{\sigma}^2$, the standard results for the Black and Scholes formula leads to:

$$V_\tau[(S(T)-K)^+] = e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}S(\tau)N\left(\frac{\ln(S(\tau)/K) + R(T-\tau) + \hat{\mu} + \hat{\sigma}^2}{\hat{\sigma}}\right) - e^{-r(T-\tau)}KN\left(\frac{\ln(S(\tau)/K) + R(T-\tau) + \hat{\mu}}{\hat{\sigma}}\right)$$

Finally, by inserting the value of a future delivery we obtain the premium of an European call option written on the commodity.

Theorem 2: European Call Option

The premium of an European call option with exercise price K and time to maturity T-τ is

$$V_\tau[(S(T)-K)^+] = V_\tau[S(T)]N|d_1| - e^{-r(T-\tau)}KN|d_2|$$

Where N(.) indicates a standard cumulative distribution function and

$$d_1 = \left(\frac{\ln(S(\tau)/K) + R(T-\tau) + \hat{\mu} + \hat{\sigma}^2}{\hat{\sigma}}\right)$$

$$d_2 = \left(\frac{\ln(S(\tau)/K) + R(T-\tau) + \hat{\mu}}{\hat{\sigma}}\right)$$

$$\hat{\sigma}^2 = \left(\left(\eta^2 - 2\frac{1}{k}\sigma\eta\rho + \frac{\sigma^2}{k^2}\right)(T-\tau) + 2\left(\left(\frac{1}{k}\right)^2 \sigma\eta\rho - \left(\frac{1}{k}\right)^3 \sigma^2\right)\left(1 - e^{-k(T-\tau)}\right) + \left(1 - \left(1 - e^{-k(T-\tau)}\right)^2\right)\left(\frac{1}{k}\right)^2 \frac{\sigma^2}{2k}\right)$$

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