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STUDIES IN THE THEORY OF INCOMPLETE MARKETS

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Preface.

In the preparation of the articles included in this dissertation I have benefitted from fruitful and stimulating discussion with my many colleagues at NHH, and I am particularly indebted to Karl Borch, Jan Mossin and Agnar Sandmo for valuable comments and advice. One of the articles included here was written in collaboration with Jacques H. Drèze at CORE during my stay there for the academic year 1974-75 and I am grateful to CORE for giving financial assistance for my visit there. Moreover, the collaboration with Drèze was a particularly stimulating intellectual experience which will hopefully have a lasting effect on my future research.

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by

Kåre P. Hagen

Contents.

Outline of the Study.

In the present study consisting of six separate papers we examine some aspects of prices and values in incomplete markets and the economic efficiency of profit- or value-related decision criteria for production decisions in incomplete market economies.

An economy with incomplete markets is an economy where the set of existing markets - the market structure - is effectively constraining the opportunity sets for consumers. The market structure will always be incomplete if markets do not exist for all commodities entering into the preference structure of consumers (as in the case of quantity rationing). More generally, the market structure will be incomplete if all points in the consumption set for consumers cannot be generated as linear combinations of market goods. This will generally be the case if market goods (objects of trade) differ from the ultimate objects of consumers' satisfaction (objects of preference) such that the set of market goods does not span the whole consumption set for consumers.

It may be instructive to give some examples of situations with incomplete markets:

(i) Commodity bundling and tie-in sales, i.e., cases where firms sell their goods in packages. Examples are numerous: sporting and cultural organizations offer season tickets, restaurants provide complete dinners consisting of many separate dishes, travel agencies offer inclusive tours, etc,... More generally, markets trading in composite goods will be incomplete if the set of composite goods does not span the commodity space.

(ii) In financial market theory under uncertainty one starts from the premise that consumer preferences over financial assets (objects of trade) are derived from preferences over certain basic characteristics of asset returns (objects of preference); e.g. their means and variances, or the asset returns under each state of nature, etc. If there are more different characteristics than assets with linearly independent return characteristics, the financial market structure is likely to be incomplete.

(iii) Allowing for variable product quality and product differentiation, it seems natural to assume that there are certain aspects, or characteristics, embodied in market goods entering into consumers' preferences so that a market good of a given quality may be considered as a package of such characteristics. The most extreme in this respect is the "hedonic" approach to consumer demand theory which is built on the premise that it is exclusively the characteristics of goods rather than the good itself which are the ultimate objects of consumer satisfaction. Thus, if there are more characteristics than market goods with linearly independent characteristics, the market structure will be incomplete in the sense that it will constrain the choice of consumers in the space of characteristics.

(iv) A problem related to the completeness of markets arises in the theory of local public goods. Consider a set of communities (or clubs) offering a certain package of local public goods to their members. If there were many communities with differentiated local public services to their members, an individual may obtain a vector of public goods close to his likings by sharing his time between several different communities ("voting by feet"). Clearly, the individual opportunity sets will be larger the larger is the number of communities with different public goods supply.

Incompleteness of markets will of course in most cases be due to increasing returns to scale relative to market size. In that respect, fixed marketing costs will be as important as indivisibilities in the production technologies. In some cases one may find that firms use commodity bundling and tie-in sales in order to extract consumers' surplus. In such cases fixed costs of merchandising may serve as barriers to entry conserving the incomplete market structure.

One has to be somewhat careful in defining the efficiency concept under an incomplete market structure. The market allocation of resources and market goods may be efficient relative to a given and possibly incomplete market structure. Hypothetically, one might, however, obtain an allocation being better in the Pareto sense if one could reorganize the market structure in such a way that consumers' range of choice were increased. Hence, one can define efficiency in a constrained - or second best - sense, i.e., relative to a given incomplete market structure. This will also be the approach taken in the present study, that is, we shall restrict the efficiency concept to the set of attainable allocations given by the nature of existing markets. Hence, the market mechanism will be said to allocate goods and resources efficiently if it can do as well as a central planner aiming at Pareto efficiency and being constrained to allocate goods and resources through existing markets. If fixed marketing costs and costs associated with organizing and running markets were explicitly considered, the second best allocation relative to an incomplete market structure may not be so far from the maximum maximorum after all.

When the incompleteness of the market structure is effectively constraining the opportunity set for consumers, the particular market structure becomes part of the infrastructure of the economy and decisions affecting the incomplete structure of markets will be in the nature of infrastructural decisions. In the present study we shall explore some economic aspects of such infrastructural decisions in different contexts.

The assumption of price-taking behaviour is basic to the efficiency of decentralized market behaviour. However, if production decisions affect the incompleteness of the market structure, the notion of price-taking market behaviour becomes somewhat blurred as production decisions may change consumers' opportunity sets even for given commodity prices. In that case a production decision will create market opportunities not previously available in the market and it is not clear what price-taking should mean in that context as the market valuation of these new opportunities will be known to the producer only after the decicion is made.

There is a close analogy between the presence of many close substitutes and the presence of many commodities relative to the number of characteristics. In particular, with as many linearly independent commodities as characteristics (complete markets) there would exist market opportunities representing a perfect substitute for any new commodity or new models of old commodities. In that case the market valuation of the corresponding perfect substitute could serve as the competitive price. In incomplete markets such perfect substitutes may not exist.

Indeed, one may look at incompleteness of markets as a market failure caused by non-convexities and increasing returns to scale in the production and marketing of goods. With economy-wide decreasing returns to scale there would exist a market for any good in positive demand so that the incompleteness of markets may be considered as just another example of the fact that increasing returns to scale choke off competition. It may, therefore, at the outset seem somewhat paradoxical to look for workable or meaningful definitions of competitive behaviour in incomplete markets as the very reason why markets are incomplete may be due to non-convexities and absence of perfect competition. This is also confirmed in the present study where we show that if firms' production decisions change the attainable set of market opportunities spanned by existing market goods, there does not in general seem to be any meaningful definition of price-taking behaviour for profit- or value-maximizing firms.

More specifically, the present study deals with some properties of market values in incomplete markets and with the efficiency of the market mechanism based on profit- or value-maximizing behaviour by firms. In the first two papers we examine the exchange of risky returns in competitive financial markets trading in risky securities. Risk is in this context defined in terms of uncertainty about the economic environment (nature). Assuming a finite and discrete state space a risky security is characterized by a return vector with components being the returns under each state of nature. The objects of preference are in this context state-contingent returns while the objects of trade are shares or claims to risky return vectors. Thus, if there are fewer securities with linearly independent returns than states of nature, the financial market structure is incomplete.

In this model firms' financial decisions will change the incomplete nature of the market structure - and hence be of an infrastructural kind - if there is thereby created securities with return vectors not previously obtainable as linear combinations of existing securities. In the first two papers we examine the possibilities for aggregation of risky securities in financial markets and we show that if the conditions for global aggregation are satisfied, financial decisions affecting the incomplete nature of financial markets will have no effect on equilibrium prices and values.

In the third paper we focus attention on the efficiency of value-related investment criteria under uncertainty, and we show that if financial markets are incomplete, there will in the general case be external effects associated with firms' investment decisions under uncertainty which will not be fully reflected in changes in firms' stock market values pointing to likely inefficiency of decentralized market behaviour based on stock market value maximization. These external effects will occur if firms' investment decisions change the feasible range of choice for portfolio returns spanned by the existing financial assets. We show, however, that if the conditions for global aggregation in financial markets are satisfied, these externalitites will be evaluated equally by all shareholders.

In that case shareholders will be unanimous as to the social value of a given investment and we can endow the firm with the preferences of an arbitrary shareholder and the market value of an investment computed at prices equal to the marginal rates of substitution for that shareholder, will capture the true social value of the investment.

The model of firms' investment under uncertainty lends itself quite naturally to the study of quality choice and product differentiation under certainty. Indeed, a unit of a commodity of a given quality may be considered as a vector of characteristics - or attributes - and we postulate that consumer preferences are basically defined on the set of such characteristics and that firms' production technologies allow for substitution between characteristics. This idea is introduced in paper 4 and elaborated further in the last two. In paper 4 and 5 we assume a linear relationship between bundles of market goods and bundles of characteristics while the last paper is extended to non-linear relationships.

Clearly, if there are many more characteristics than market goods, the market structure will be incomplete so that the number of commodities and their particular characteristics will effectively constrain consumers' range of choice in the characteristics space. Thus, if the characteristics composition of any one good is changed, this will in general change this common range of choice for all consumers so that decisions on product quality will have external effects which may not be fully reflected in firms' marginal revenues with respect to product qualities. The last three papers are devoted to the study of the efficiency of the market mechanism guided by the profit motive in determining product quality and product differentiation. It is shown that quality choices set under the pressure of market forces will in general not be in the consumers' interest and in paper 5 and 6 we derive sufficient conditions for profit-maximizing quality choices to satisfy the necessary conditions for Pareto efficiency under linear and non-linear relationships between market goods and characteristics, respectively.

ON THE ADDITIVITY OF MARKET VALUATION OF RISKY INCOME STREAMS.

In this paper we shall examine under what circumstances the equilibrium market value of a sum of risky income streams will be equal to the sum of the market values of the constituent risky income streams in a competitive stock market. Some authors in the field of business finance seem to believe that the stock market values of risky income streams will always be additive in the above sense regardless of the structure of preferences and the nature of expectations in the market. See [4], [6] and [7]. However, the additivity property of market values in a competitive stock market does not hold in general. More precisely, the additivity property only holds for a special class of preferences, the generality of which depends upon whether there exists a riskless income stream in the market and moreover, homogeneous expectations are necessary for the additivity of stock market values to obtain.

We shall use the following notation:

Ω = the set of possible states of the world

$\theta\varepsilon\Omega$ = a typical element of Ω.$\{\theta\}$ is a finite partition of the state space Ω and we assume that s is the finite dimension of Ω, i.e., s is the number of possible states of the world.

$\pi_i(\theta)$ = the discrete subjective probability density function defined on Ω for the i-th individual (i=1, ..., m), i.e., we allow for the possibility of heterogeneous expectations.

$X_j(\theta)$ = the risky income stream[1] j, (j=1, ...,n). We consider a one-period model such that the risky incomes are supposed to materialize at the end of the period. $X_j(\theta)$ is of course to be interpreted as the income of income stream j if state θ occurs so that

[1] The term "income stream" is not quite appropriate here since we are considering a one-period model. However, to generalize the results to income streams stretching over more than one period would be technically straight forward. The crucial thing is that the model is static in the sense that trade takes place once and for all at the market date.

each risky income will be a vector with at most s non-zero components. We assume that all such income vectors are linearly independent and the s-component vector $[X_j(1), \ldots, X_j(s)]$ will be denoted $X_j(\bar{\theta})$

$\mathcal{S}$ $\equiv$ $\{X_1(\bar{\theta}), \ldots, X_n(\bar{\theta})\}$ = the set of exogeneously given risky income streams.

$W_i(\theta)$ = the initial endowments of the i-th individual, i.e., the $W_i(\theta)$ represent the initial distribution of income prior to trade.

$q(\theta)$ = the price as of to-day for a unit of income in state θ.

$Z(\theta)$ $\equiv$ $\sum_{i=1}^{m} W_i(\theta)$ = the total social endowment in state θ.

V = the market valuation functional which is a mapping from the vector set $\mathcal{S}$ to the positive real line. That is $V(X_j(\bar{\theta})) \equiv V_j$ where V_j is the market value of income stream $X_j(\bar{\theta})$.

We shall first examine the additivity property of the market valuation functional V in an Arrow-Debreu market, i.e., a market where risky incomes are traded by exchange of Arrow-Debreu securities. We are interested in laying down necessary and sufficient conditions for V to be additive in terms of properties of the individual preferences along their demand schedules. After that we shall examine what kind of restrictions, if any, we have to impose on the preferences in order for the conditions for additivity to be met in a competitive stock market.

The objects of exchange in an Arrow-Debreu market are claims to income in different states of the world, i.e., state-contingent claims, the basic units of which, so-called Arrow-Debreu securities[1)], are defined such as to pay one unit of income to the holder if a particular state occurs and nothing otherwise. Clearly, there are as many Arrow-Debreu securities as there are states of the world.

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1) See for instance Arrow [1].

We consider a pure exchange economy so that the total amount traded must be equal to the total social endowment, i.e.,

$$\sum_{j=1}^{n} X_j(\theta) = \sum_{i=1}^{m} W_i(\theta), \qquad \theta\varepsilon\Omega$$

By definition of value the market value of a risky income stream in an Arrow-Debreu market is given by $V_j \equiv V(X_j(\bar{\theta})) \equiv \sum_{\theta} q(\theta)X_j(\theta)$.

Thus, a necessary and sufficient condition for the market valuation functional to be additive in an Arrow-Debreu market is obviously that the prices of income in different states be unchanged when income streams are added together or linearly combined in the market. We shall argue that this is trivially true in an Arrow-Debreu market.

We let $d_i(\theta)$ denote the i-th individual's demand for income in state θ. Each individual's demand pattern for income in different states is a vector with s components. We denote the set of all feasible demand vectors by T, i.e., T is the set of feasible trades in the market. The set of feasible trades is determined by the social endowment vector or more precisely, $T \equiv \{d(\theta)|d(\theta) \leqq Z(\theta),\ \theta\varepsilon\Omega\}$. The Arrow-Debreu securities defined as the s unit vectors $e(\bar{\theta})$ where $e(\bar{\theta})$ gives one unit of income if and only if state θ occurs, will clearly span the set of feasible trades so that any feasible trading point in T can be obtained as a linear combination of the Arrow-Debreu securities. Since the set of feasible trades only depends on the social endowment in each state, the individual opportunity sets will be independent of the operation of combining income streams in the market implying that demands and prices must remain unchanged.

Assuming that individuals rank risky income streams according to expected utility[1)], the individual demand functions are determined by

$$\text{Max} \sum_{\theta} \pi_i(\theta)U_i(d_i(\theta)) \qquad i=1, \ldots, m$$

[1)] The individual utility functions are assumed to be strictly quasi-concave so that the first-order conditions define a maximum.

subject to the budget constraints

$$\sum_{\theta} q(\theta)[d_i(\theta) - W_i(\theta)] = 0.$$

The first-order maximum conditions are given by

(1) $$\pi_i(\theta)U_i'(d_i(\theta)) = \lambda_i q(\theta), \text{ for all } \theta\varepsilon\Omega \text{ and for all } i$$

where λ_i is the Langrangian multiplier which of course has the usual interpretation of expected marginal utility of initial wealth.[1)]

Condition (1) can be restated in terms of marginal rates of substitution between income in different states, say r and s:

(2) $$\frac{\pi_i(r)U_i'(d_i(r))}{\pi_i(s)U_i'(d_i(s))} = \frac{q(r)}{q(s)}, \text{ for all } i.$$

(2) says that along the individual demand schedules the marginal rate of substitution between income in different states of the world be equal for all individuals and equal to the relative prices.

Regardless of the structure of preferences and the nature of expectations, the market valuation functional will have a very simple general form in an Arrow-Debreu market. To see this observe that $\sum_{\theta} q(\theta) \equiv q_0$ is to-day's price for a unit of certain income at the end of the period ($q_0 \equiv \frac{1}{1+r}$ where r is the rate of interest)[2)] so that $\lambda_i = \sum_{\theta} \pi_i(\theta)U_i'(d_i(\theta))/q_0$.

This gives

$$q(\theta) = \frac{q_0\pi_i(\theta)U_i'(d_i(\theta))}{\sum_{\theta}\pi_i(\theta)U_i'(d_i(\theta))} \quad \text{for all } i.$$

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1) We assume non-satiation so that $\lambda_i > 0$ for all i.

2) This means that the riskless income stream yielding 1+r in each state of the world is a numeraire good in this economy.

Define $M_i(\theta) \equiv \dfrac{U_i'(d_i(\theta))}{\sum_\theta \pi_i(\theta)U_i'(d_i(\theta))}$

Considering the risky income stream $X_j(\bar{\theta})$ its market value is defined by

$$V(X_j(\bar{\theta})) \equiv \sum_\theta q(\theta)X_j(\theta) = q_0 \sum_\theta \pi_i(\theta)M_i(\theta)X_j(\theta).$$

Observing that $\sum_\theta \pi_i(\theta)M_i(\theta) = 1$ and recalling $q_0 \equiv 1/(1+r)$ we have

$$V_j \equiv V(X_j(\bar{\theta})) = \frac{1}{1+r}\{E_i[X_j(\theta)] + Cov_i[M_i(\theta),\ X_j(\theta)]\} \quad \text{for all } i,$$

where E_i and Cov_i are the expectation and covariance operators for the i-th subjective probability distribution and where we have utilized the fact that $Cov(X,Y) = E[XY] - EXEY$. Consequently, in an Arrow-Debreu market the equilibrium value of a risky income stream can always be expressed as a function of its discounted expected value and a risk term depending on the covariance between the risky income and a market variable $M_i(\theta)$ in general different for different individuals.

When restricting exchange of risky income streams to other market structures than the Arrow-Debreu market, it is clear[1)] that a necessary and sufficient condition for market valuation of risky income streams to be additive under a given market structure, is that individual marginal rates of substitution between income in different states depend only on total income in each state and not on the particular pattern of risky income streams. That means that individual exchange optima must remain unaffected when risky income streams are linearly combined as long as total social endowment in each state remains unchanged. But this can only be true if the equilibrium allocation in an Arrow-Debreu market is attainable under the specific market structure given. Thus, for the market value to be additive in the sense that the

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[1)] Sufficiency is obvious and necessity follows from the fact that if individual demand prices for income in different states (for a given numeraire) depend on the pattern of risky income streams, these individual demand prices for state-contingent incomes would change when risky income streams are merged and hence market values would change too.

value of a sum of risky income streams equals the sum of the values of these incomes, the market must generate the same price and demand structure as would an Arrow-Debreu market for a given set of preferences and expectations.

We formalize this in a theorem which for the sake of reference will be called the Market Valuation Theorem:

> A necessary and sufficient condition for the market valuation functional to be additive is that the marginal rates of substitution between income in different states be the same for all individuals and remain equal to the relative prices.

This is indeed an aggregation property since we can always form aggregates if relative prices remain constant.

We shall now examine the properties of the market valuation functional V in a competitive stock market, i.e., a market where the exchange of state-contingent incomes is restricted to take place by means of trading in shares of the existing income streams. We let α_{ij} denote the proportion of income stream j held by the i-th individual. First of all we note that the objects of exchange in a stock market are not income in different states but shares of income patterns across the states of the world. By the same token we could think of the various income streams as composite goods where the constituent commodities are income in different states.

We make the rather realistic assumption that the dimension of the state space Ω is larger than the number of linearly independent income streams ($s > n$). But this means that the set S of income vectors $X_j(\bar{\theta})$ cannot possibly span the s-dimensional space T of feasible trades defined above. Consequently, some feasible trades in an Arrow-Debreu market will be infeasible in a stock market, i.e., there exists vectors $[d_i(1), \ldots, d_i(s)]$ which are inexpressible as a linear combination of the risky income streams.[1)] Restricting exchange of income streams to

[1)] If $s<n$, we see that a stock market would generate the same set of feasible trades as an Arrow-Debreu market and the two market structures would in fact be equivalent.

trading in stock markets may therefore mean a considerable narrowing down of the set of feasible trades, the extent of which depends upon the maximum number of linearly independent income vectors compared with the dimensionality of the state space. Therefore, there should be no à priori reason to believe that the additivity of the market valuation functional should be preserved in a stock market since merging income streams in the market reduces the set of feasible trades and consequently leaving less room for diversification of the individual portfolios.

We now pose the question under what conditions the market valuation functional will in fact be additive in a competitive stock market trading in shares of risky income streams. From our Market Valuation Theorem we derive

Corollary 1:

> If there are more states of the world than linearly independent income streams, the market valuation functional defined on the set of risky income streams traded in a competitive stock market will be additive if and only if there are homogeneous expectations in the market and the individual utility functions satisfy the following conditions[1)]

(i) If a riskless income stream does not exist,

$$U_i'(x) = x^{\frac{-1}{\gamma}} \qquad \text{for all } i$$

(ii) If a riskless income stream does exist

$$U_i'(x) = \begin{cases} e^{\frac{-1}{\beta_i}x} \\ (\beta_i + \gamma x)^{\frac{-1}{\gamma}} \end{cases} \qquad \text{for all } i$$

[1)] These utility functions are of course unique only up to increasing linear transformations.

Proof:

We know that the market valuation functional is additive if and only if the premises for the Market Valuation Theorem are satisfied. But that means that the optimality conditions (1) or (2) must be satisfied in the stock market. Consequently, the i-th individual's demand for income in state θ must be expressible as a linear combination of the available income streams, i.e.,

$$(3) \qquad d_i(\theta) = \sum_{j=1}^{n} \alpha_{ij} X_j(\theta) + m_i \quad \text{for all } \theta\varepsilon\Omega$$

where α_{ij} is the optimal proportion of the risky income stream j held by the i-th individual and m_i is the i-th individual's demand for riskless income (m_i and α_{ij} are of course independent of θ).

From the first-order conditions (1) we have that for the market valuation functional to be additive in the stock market it is necessary and sufficient that the utility functions satisfy

$$(4) \qquad \frac{1}{\lambda_i} \pi_i(\theta) U_i'(d_i(\theta)) = \frac{1}{\lambda_1} \pi_1(\theta) U_1'(d_1(\theta)) \quad \text{for all } \theta\varepsilon\Omega \text{ and for all } i$$

where $d_i(\theta)$ must be of the linear form (3) and where the λ_i are expected marginal utility of initial wealth. First of all we note that (3) and (4) cannot hold true unless $\pi_i(\theta)$ are the same for all i so that homogeneity of expectations is clearly necessary for the additivity property to hold true in the stock market. We can demonstrate this by means of a simple counter-example. To keep things as simple as possible, assume one risky income stream $X(\theta)$, one riskless income in the amount of M and three states of the world, θ=1, 2, 3. Moreover, assume there are two individuals and let individual 1 hold the share α of $X(\theta)$ and the amount m of the riskless income. The other is then holding $(1-\alpha)X(\theta)$ and M-m. Assume (3) and (4) hold for heterogeneous expectations. That means

$$\frac{1}{\lambda_1} \pi_1(\theta) U_1'(\alpha X(\theta) + m) = \frac{1}{\lambda_2} \pi_2(\theta) U_2'((1-\alpha)X(\theta) + M - m), \ \theta = 1,2,3.$$

This is a system of three equations in two variables α and m and consequently overdetermined so that it does not have any solution unless all equations by chance should intersect at the same point. This can always be avoided by a suitable choice of subjective probabilities. For if the above equations are satisfied for some class of utility functions and some set of probability distributions possibly different for the two individuals, then the above equations must be satisfied for any set of probability distributions for this particular class of utility functions since otherwise probabilities and preferences would not be separable. Consequently, we are free to choose any subjective probability distribution and let us assume that $\pi_1(1) > \pi_1(2) > \pi_1(3)$. Assume moreover that $X(1) < X(2) < X(3)$. Decreasing marginal utility implies that

$$\frac{1}{\lambda_1}\pi_1(1)U_1'(\alpha X(1)+m) > \frac{1}{\lambda_1}\pi_1(2)U_1'(\alpha X(2)+m) > \frac{1}{\lambda_1}\pi_1(3)U_1'(\alpha X(3)+m).$$

Choosing $\pi_2(1) = \pi_2(2) = \varepsilon/2$, $\pi_2(3) = 1-\varepsilon$, $0 < \varepsilon < 1$, then we will have for sufficiently small ε that

$$\frac{1-\varepsilon}{\lambda_2}U_2'((1-\alpha)X(3)+M-m)) > \frac{\varepsilon}{2\lambda_2}U_2'((1-\alpha)X(2)+M-m)$$

contradicting condition (4). This counter-example demonstrates the necessity of homogeneous expectations for (3) and (4) to hold true for all $\theta\varepsilon\Omega$. Consequently, condition (4) reduces to

(5) $$k_iU_i'(d_i(\theta)) = U_1'(d_1(\theta)) \text{ for all } \theta\varepsilon\Omega \text{ and for all } i$$

where the proportionality factor k_i is given by λ_1/λ_i.

In an Arrow-Debreu market prices and demand will only depend on total social endowment in each state and not on the particular pattern of risky income streams making up the total endowment. That means in particular that any zero-sum redistribution of income among risky income streams must leave individual demand and prices unchanged if total social endowment and the initial distribution of income remain unchanged. But this can generally be true if and only if α_{ij} are the same for all j. Hence, if the necessary conditions for exchange optima as given by (1) are to hold true in a stock market, the equilibrium demand vectors must be of the form

(6) $$d_i(\theta) = \alpha_iZ(\theta) + m_i \text{ for all } i \text{ and for all } \theta\varepsilon\Omega.$$

If the valuation functional is to be additive in the stock market, (5) and (6) must hold for any distribution of total social endowment and for any initial distribution of income. To generate the parameters of the utility functions satisfying (5) and (6) we differentiate these conditions with respect to total social endowment $Z(\theta)$ and with respect to the proportionality factor k_i since the Lagrangian multipliers will be continuously differentiable functions of initial wealth. This gives

$$\left.\begin{aligned} & k_i U_i''(d_i(\theta))\alpha_i = U_1''(d_1(\theta))\alpha_1 \quad \text{and} \\ & U_i'(d_i(\theta)) + k_i U_i''(d_i(\theta))[\alpha_{ii}Z(\theta) + m_{ii}] \\ & \qquad = U_1''(d_1(\theta))[\alpha_{1i}Z(\theta) + m_{1i}] \end{aligned}\right\} \text{ for all } \theta\varepsilon\Omega \text{ and for all } i.$$

where α_{rs} and m_{rs} denote partial derivatives of α_r and m_r with respect to k_s. Substituting from the first equation into the second, using the fact that $Z(\theta) = [d_i(\theta)-m_i]/\alpha_i$ and rearranging gives

$$(7) \qquad -\frac{U_i'(d_i(\theta))}{U_i''(d_i(\theta))} = \gamma_i d_i(\theta) + \beta_i$$

where

$$\gamma_i \equiv k_i \left(\frac{\alpha_{ii}}{\alpha_i} - \frac{\alpha_{1i}}{\alpha_1}\right)$$

$$\beta_i \equiv k_i\left[\left(\frac{\alpha_{1i}}{\alpha_1} - \frac{\alpha_{ii}}{\alpha_i}\right)m_i + m_{ii} - \frac{\alpha_i}{\alpha_1} m_{1i}\right]$$

If no riskless income stream exists in the market, the constant term β_i in (7) is clearly restricted to zero and except for a constant of integration the only solution of the differential equation (7) is given by

$$(8) \qquad U_i'(d_i(\theta)) = (d_i(\theta))^{-1/\gamma_i}$$

If a riskless income stream does exist, β_i will be unrestricted and (7) will have the following solutions

$$
(9) \qquad U_i'(d_i(\theta)) = \begin{cases} e^{-\frac{1}{\beta_i} d_i(\theta)} & \text{for } \gamma_i = 0 \\ (\beta_i + \gamma_i d_i(\theta))^{-1/\gamma_i} & \text{otherwise} \end{cases}
$$

This concludes the necessity part of the proof. As for sufficiency, assume that we have homogeneous expectations and that the utility functions belong to the class (8) or (9). It is then easy to verify that (5) and (6) are satisfied if and only if the β_i is the same for all i and this completes the proof.

Consequently, we see that if no riskless income stream exists, the only utility functions yielding additive valuation functionals in the stock market are the constant relative risk aversive utility functions.[1)] If a riskless income stream does exist, this class is enlarged to contain all the linear risk tolerance utility functions which include the constant relative risk aversive and the constant absolute risk aversive utility functions as special cases.[2)] Also, we observe, not surprisingly, that the utility functions listed in (i) and (ii) are those for which the stock market brings about an unconstrained Pareto-optimal allocation of risk.[3)] Homogeneity of expectations is, however, also necessary for this to hold.

What makes trouble for the additivity property of the market valuation functional to be preserved in the stock market is the fact that if the conditions (i) and (ii) of Corollary 1 are not satisfied and/or different individuals have different probability assessments for the various states of the world, the market will not in general equate the marginal rates of substitution between income in different states of the world for different individuals implying that relative prices for state-contingent income common to all individuals in the market do not exist. Individual exchange optima are in that case constrained in various ways by the particular pattern of the risky income streams and merging risky incomes will affect

[1)] The relative risk aversion function is defined as $-U''(x)x/U'(x)$.

[2)] The absolute risk aversion function is defined as $-U''(x)/U'(x)$ and the risk tolerance is defined as the inverse of the absolute risk aversion.

[3)] See i.a. Borch [2] and Mossin [6].

the individuals' relative demand prices for state-contingent income which indicates that aggregation is not possible. In that case there should be no reason to expect that the value of a sum of two income streams should be equal to the sum of their values unless they are perfectly correlated.

From (6) we have that for the class of preferences defined in Corollary 1, each individual will in equilibrium hold the same share of all risky income streams. If all risky income streams were added to form a mutual fund, the set of feasible trades would shrink down to scalar multiples of the social endowment vector $Z(\bar{\theta})$.

However, all individuals can obtain precisely the same pattern of income across the states of the world as before by holding α_i of the shares of the mutual fund. In this case shares of the various income streams can be considered as being perfectly complementary. To take an analogy from the market for consumer goods, the equilibrium prices for pairs of shoes should not change if right and left shoes were sold separately in a perfect market with no transaction costs.

Portfolios consisting of equal shares of all risky income streams may be called perfectly balanced portfolios. Clearly, the equilibrium portfolios will be perfectly balanced if and only if there are homogeneous expectations and the utility functions belong to the class given by (i) and (ii). Consequently, the market valuation functional will be additive if and only if the equilibrium portfolios are perfectly balanced. This result can be proved in a more direct way than Corollary 1 by just studying the effects on the individual opportunity sets from adding or splitting up risky income streams. Certainly, if adding or splitting up risky income streams in the market shall have no effect on their equilibrium prices, the individual demand vectors must be left unchanged.

Let us assume that two risky income streams are merged into one. That means that the set of feasible trades shrinks down to a hyperplane in the n-dimensional Euclidean space. If the individual demands shall be independent of merging two arbitrary income streams, the individual demand vectors must be contained in the intersection of all such hyperplanes formed by adding two arbitrary income streams. It is trivial to verify that the intersection of all such hyperplanes does only consist of vectors of the form $\lambda \sum_{j=1}^{n} X_j(\bar{\theta})$, where λ is some scalar. Hence the individual portfolios must be perfectly balanced. Certainly, the same applies if the set of feasible trades is mapped into a subspace of dimension $n - k + 1$ by merging an arbitrary number k of the original risky income streams into one.[1)]

As for splitting up the existing risky income streams into substreams, it is clear that if the splitting up were carried far enough, we would end up with as many linearly independent incomes as there are states of the world in which case the additivity property holds trivially in the stock market. Otherwise, the above reasoning still applies. That means if a risky income stream is split up into an arbitrary finite number of substreams, the sum of the equilibrium values of the substreams will equal the equilibrium value of the original stream if and only if the equilibrium portfolios are all perfectly balanced.

We may also define additivity of the market valuation functional in a more local sense by restricting it to a subset of the risky income streams. For example $V(\alpha X_j(\bar{\theta}) + \beta X_k(\bar{\theta})) = \alpha V(X_j(\bar{\theta})) + \beta V(X_k(\bar{\theta}))$ for arbitrary scalars α and β, if and only if the income streams j and k enter all individual equilibrium portfolios with equal shares. Again, if the market value is to be additive on the whole set of risky incomes, all the risky incomes must enter the individual equilibrium portfolios with <u>equal</u> shares, that is, perfectly balanced portfolios.

The property that all relevant market opportunities can be spanned by the riskless income and an aggregate of all risky income streams (a mutual fund) such that each individual can obtain his most preferred pattern of income over states of the world by holding a certain share of the aggregate risky income stream and the rest of his initial wealth in the riskless income, is known as the <u>market separation</u> property.

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1) A more general formulation would be to reorganize the original risky incomes into, say, m mutual funds ($m < n$) and the equilibrium value of the mutual funds would be equal to the equilibrium values of the constituent risky incomes (assets) if and only if the equilibrium portfolios are perfectly balanced.

Separation in the individual's portfolio choice means that the optimal portfolio decisions can be broken down into two stages: First one determines the optimal asset proportions of the risky portfolio (which will be independent of initial wealth) and then one allocates initial wealth to the riskless asset and the risky portfolio in an optimal way. Clearly, linear risk tolerance and homogeneous expectations are necessary and sufficient conditions for market separation to obtain.[1)] Also market separation implies separation in the individual portfolios. In order to see this, let δ_{ij} be the share of initial wealth W_i allocated to the risky portfolio, that individual i allocates to the j-th risky asset (j-th risky income stream). That is, $\delta_{ij} = \alpha_{ij}V_j/(W_i-m_i)$ where V_j is the market value of the j-th risky income (with the riskless income as numeraire). The asset ratios in individual i's risky portfolio are then given by $\delta_{ij}/\delta_{ik} = \alpha_{ij}V_j/\alpha_{ik}V_k$. Assuming market separation equilibrium portfolios must be perfectly balanced and hence $\delta_{ij}/\delta_{ik} = V_j/V_k$ for all i. That means that asset ratios in the risky portfolio must be equal to the market value ratios of the risky assets. Hence, asset proportions in the individual risky portfolios must be the same for all individuals and thus independent of initial wealth. That means also that in this case the price structure of risky assets must be independent of the distribution of initial wealth.

Letting a_{ij} be the demand for risky asset j by individual i in terms of the numeraire, it is clear that asset shares in the risky portfolio will be independent of initial wealth if asset demand functions take the form

$$a_{ij} = \mu_{ij} + \eta_{ij}W_i$$

where μ_{ij} and η_{ij} are independent of W_i and $\mu_{ij}\eta_{ik} = \mu_{ik}\eta_{ij}$ for all j and k and as shown in [3], asset demand functions linear in initial wealth is also necessary for separation. In that case asset shares in the risky portfolios are given by

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[1)] For a more detailed discussion on market separation, see Cass and Stiglitz [3].

$$\delta_{ij} = \frac{a_{ij}}{\sum_k a_{ik}} = \frac{\eta_{ij}(\mu_{ij}/\eta_{ij}+W_i)}{\sum_k \eta_{ik}(\mu_{ik}/\eta_{ik}+W_i)} = \frac{\eta_{ij}}{\sum_k \eta_{ik}} = \frac{\mu_{ij}}{\sum_k \mu_{ik}}$$

Moreover from the fact that equilibrium portfolios must be perfectly balanced we have that

$$\frac{\delta_{ij}}{\delta_{ik}} = \frac{\eta_{ij}}{\eta_{ik}} = \frac{\mu_{ij}}{\mu_{ik}} = \frac{V_j}{V_k} \quad \text{for all } i$$

and hence the parameters μ_{ij} and η_{ij} must be independent of i. Thus, market separation implies that asset demand functions are of the form

(10) $$a_{ij} = \mu_j + \eta_j W_i$$

with $\mu_j\eta_k = \mu_k\eta_j$. Hence, if $\mu_j = 0$ for one j, it has to be zero for all j and the same must hold for η_j. Thus, linear asset demand functions of the form (10) is a necessary condition for market separation and hence also for additivity of stock market values and together with homogeneous expectations it can also be shown to be sufficient.[1)] We can then state necessary and sufficient conditions for additivity of stock market values in terms of properties of asset demand functions.

<u>Corollary 2:</u>

> If there are more states of the world than linearly independent income streams, it is necessary for additivity of stock market values that individual asset demand functions are linear in initial wealth and taken together with homogeneous expectations this is also sufficient.

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1) If $\eta_j = 0$ for all j, demand for risky assets will be completely wealth-inelastic and the utility function must exhibit constant absolute risk aversion. If $\mu_j = 0$ for all j, demand for risky assets has a unitary wealth elasticity and the utility function must exhibit constant relative risk aversion. In the general case it can be shown (see [3]) that asset demand functions linear in initial wealth implies that utility functions must exhibit linear risk tolerance.

As noted earlier, additivity of stock market values may be considered as an aggregation property and in particular, linear risk tolerance utility functions will imply homogeneous separation in the risky assets in which case we already know from conventional consumer theory that two-stage budgeting will be optimal which in the portfolio context is the same as portfolio separation. Also, the conditions of Corollary 2 correspond to the general conditions for aggregation over consumers in commodity markets, namely, linearity and equal slope of individual Engel curves.

Finally we would like to point out yet another implication of our Market Valuation Theorem:

Corollary 3:

> If there are more states of the world than securities with linearly independent returns, the market values of firms in a competitive market for stocks and bonds will be independent of their capital structures in case of default risk if and only if there are homogeneous expectations in the market and the conditions (i) and (ii) of Corollary 1 are satisfied.[1)]

Proof:

Consider gross earnings as being divided into two risky income streams, one going to the shareholders and the other to the bondholders. Corollary 1 implies that the total market value of a firm is independent of how the gross earnings are split up into the two substreams and the result is immediate.

The widely held opinion[2)] that in a perfect capital market the market values of firms are independent of their capital structures is therefore

1) This is the Modigliani-Miller theorem for the case of a non-zero default risk in each firm. See [5].

2) See i.a. Fama and Miller [4], Mossin [6] and Schall [7].

not correct in general. From a theoretical point of view the perfect capital market assumption is not sufficient to make market values of firms independent of how they have been financed. In fact, homogeneous expectations and condition (i) and (ii) of Corollary 1 are taken together both necessary and sufficient for the market values of firms to be independent of their capital structures in the case of risky debt.

The set of feasible trades in a market for stocks and risky bonds and thus the individual opportunity sets will be dependent on the financial arrangements in the various firms. It follows from this that in general the productive and financial decisions of a firm cannot be separated. More precisely, homogeneous expectations and condition (i) and (ii) of Corollary 1 are necessary and sufficient for separation of financial and productive decisions in the case of a non-zero default risk in each firm.[1)] To give an intuitive explanation for this, we recall the fact that if there are homogeneous expectations in the market and (i) and (ii) of Corollary 1 are satisfied, each individual will in equilibrium hold the same share of all the risky assets in the market (including risky bonds). Then it is obvious that the relevant individual opportunity sets must be independent of the specific capital structures in the various firms. That means that in this case each individual can obtain his preferred distribution of income over the states of the world for any set of financial arrangements so that changing the firms' capital structures will not change the individuals' demand for income in the various states of the world and consequently prices and values must remain unchanged. It follows from this that, however determined, the market value of a particular firm must be independent of its specific capital structure.

As is well known, if one imposes the restriction of a zero default risk in each firm for all relevant debt levels, the irrelevancy of capital structures for the market values of firms in perfect capital markets will hold quite universally without imposing any restrictions on the structure of preferences and the nature of expectations. This result can be given a very intuitive explanation within the context of the present paper.

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1) The separation theorem of Fama and Miller [4] is therefore not generally correct unless one imposes the quite restrictive conditions of Corollary 1.

With no default risk a fixed amount is sliced off from the gross earnings in each state of the world for each firm and distributed as a riskless income. We assume again that the vectors of gross earnings $X_j(\bar{\theta})$ are linearly independent. The maximum debt liability of firm j is given by $\underset{\theta\in\Omega}{\text{Min}}\{X_j(\theta)\}$. Assume that the j-th firm is increasing its debt level by Δd_j, that is, firm j is converting Δd_j of its stock into debt. That means that we are slicing off the additional amount $(1+r)\Delta d_j$ from each component of $X_j(\theta)$. (r is the rate of interest). This will not change the set of feasible trades in the market. So by a suitable readjustment of the individual portfolios the individuals can obtain precisely the same income distribution over the states of the world as before. For the i-th individual there is a unique way of reproducing his income distribution, namely by increasing his bond holdings by $\alpha_{ij}\Delta d_j$ while keeping the proportions he owns of the firms' stock unchanged (α_{ij} is the proportion individual i owns of the shares of the firm j). For the stock market to clear this requires that the individuals must reduce their share holdings in the firm j by the same amount, i.e., $\alpha_{ij}\Delta d_j$. Consequently, in the new market equilibrium the market value of firm j's stock has been reduced by $\sum_i \alpha_{ij}\Delta d_j = \Delta d_j$ while the market value of firm j's bonds has been increased by the same amount leaving the total market value of the firm unchanged. The same holds if the j-th firm converts debt into stock so that the above reasoning may be considered as a simple proof of the Modigliani-Miller theorem [5]. However, as has been shown in the present paper, if we relax the non-default risk assumption we have to impose quite severe restrictions on the individuals' market behaviour for the Modigliani-Miller theorem still to obtain.

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Default Risk, Homemade Leverage, and the Modigliani-Miller Theorem: Note

By KÅRE P. HAGEN*

In the March 1974 issue of this *Review* David Baron uses a stochastic dominance argument in an attempt to prove that the Modigliani-Miller (M-M) theorem is generally valid even in case of default risk. For this purpose he uses the familiar two-firm paradigm where both firms have identical probability distributions for gross returns, and one firm has some debt in its capital structure whereas the other firm is financed entirely by equity capital. If all equity investors in the levered firm also hold bonds in that firm or if all investors can borrow at the same nominal interest rate as firms, then he shows that in equilibrium both firms must have the same total market value.

Although there is nothing formally wrong in his arguments, Baron does not prove what he sets out to prove. What he proves seems to be the (fairly obvious) fact that at an equilibrium in a perfect capital market two firms with identical probability distributions for gross returns must have the same total market value. This does not imply, however, that the equilibrium value of a firm is independent of its capital structure. In fact, the common market value of the two firms will in general be dependent on the debt-equity ratio in the levered firm. To show this a simple counterexample will suffice.

I

The conventional one-period model where investors invest at the beginning of the period while returns materialize at the end is used with the following notations:

$X_j(\theta)$ = gross returns in firm j if state of the world θ obtains, $\theta \in \Omega$, where Ω is the state space

$R_j(\theta)$ = returns to equity in firm j in state θ

$B_j(\theta)$ = returns to bondholders in firm j in state θ

D_j = total debt liability in firm j (principal plus interest payments due at the end of the period)

α_{ij} = the fraction investor i holds of the shares of firm j

β_{ij} = the fraction investor i holds of the bonds issued by firm j

S_j = market value of firm j's equity

B_j = market value of firm j's bonds

$V_j \equiv S_j + B_j$ = total market value of firm j

$Y_i(\theta)$ = investor i's final wealth in state θ

W_i = investor i's initial wealth

$\pi_i(\theta)$ = investor i's probability assessments

$U_i(Y_i(\theta))$ = investor i's utility function (strictly concave)

Clearly,

$$R_j(\theta) = \text{Max}\,[0, X_j(\theta) - D_j]$$
$$B_j(\theta) = \text{Min}\,[D_j, X_j(\theta)]$$
$$\text{and } X_j(\theta) = R_j(\theta) + B_j(\theta) \qquad \text{for all } \theta \in \Omega$$

Two firms with identical gross return patterns over states of the world are assumed, i.e., $X(\theta) = X_j(\theta)$ for all $\theta \in \Omega$, $j = 1, 2$. Firm 1 is financed entirely by equity capital such that $V_1 \equiv S_1$, while the total debt liability in firm 2 is D_2. Short selling of stocks and bonds is generally possible. If $\beta_{ij} < 0$, investor i is issuing bonds with the same return characteristics as those issued by firm j. Moreover, riskless lending and borrowing are available to the investors at a zero interest rate. Each investor owns initially a fraction δ_i of each firm and investors are assumed to rank portfolios according to expected utility of final wealth.

Define $\hat{\Omega}(D_2) \equiv \{\theta \in \Omega \mid X(\theta) \geqq D_2\}$

and $\bar{\Omega}(D_2) \equiv \Omega \backslash \hat{\Omega}(D_2)$

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i.e., $\bar{\Omega}(D_2)$ are the states of the world in which firm 2 goes bankrupt.

Using the budget constraint, final wealth can be written as (1a) if $\theta\in\hat{\Omega}$ or (1b) if $\theta\in\bar{\Omega}$.

$$(1a)\quad Y_i(\theta)=W_i+\alpha_{i1}(X(\theta)-V_1)+\alpha_{i2}(X(\theta)-S_2)-\beta_{i2}B_2+(\beta_{i2}-\alpha_{i2})D_2$$

$$(1b)\quad Y_i(\theta)=W_i+\alpha_{i1}(X(\theta)-V_1)+\beta_{i2}(X(\theta)-B_2)-\alpha_{i2}S_2$$

where $W_i=\delta_i(V_1+V_2)$.

The first-order conditions for optimal portfolios for all i can be written as:

$$(2)\qquad V_1=\sum_{\theta\in\Omega}\omega_i(\theta)X(\theta)$$

$$(3)\qquad S_2=\sum_{\theta\in\hat{\Omega}(D_2)}\omega_i(\theta)(X(\theta)-D_2)$$

$$(4)\qquad B_2=\sum_{\theta\in\hat{\Omega}(D_2)}\omega_i(\theta)D_2+\sum_{\theta\in\bar{\Omega}(D_2)}\omega_i(\theta)X(\theta)$$

$$\text{where } \omega_i(\theta)\equiv\frac{\pi_i(\theta)U_i'(Y_i(\theta))}{\sum_{\theta\in\Omega}\pi_i(\theta)U_i'(Y_i(\theta))}$$

The conditions (2), (3), and (4) together with the market-clearing conditions $\Sigma_i\alpha_{ij}=1$, $\Sigma_i\beta_{ij}=1$ for all j, characterize an equilibrium in the capital market.

The term $\omega_i(\theta)$ is investor i's marginal rate of substitution between a unit return in state θ and a unit of riskless return. Hence, $\omega_i(\theta)$ can be interpreted as the implicit price in terms of the riskless asset investor i is willing to pay for a claim to a unit return contingent on state θ. As the price of a unit of riskless return is $\Sigma_{\theta\in\Omega}\omega_i(\theta)=1$, the riskless asset is a numeraire.

As can be seen from (2), (3), and (4), $V_2\equiv S_2+B_2=V_1$ so that the two firms will in equilibrium always have the same market value. But this common market value will in general depend on the debt level in the levered firm unless $\omega_i(\theta)$ is independent of D_2 for all θ and for all i. Hence we have the immediate result that the market value of the two firms is independent of the leverage in the levered firm if and only if $\omega_i(\theta)$ does not depend on D_2 for all θ and for all i.

If the market structure were complete, i.e., there are as many securities with linearly independent return patterns as states of the world, the return pattern on the optimal portfolios would be independent of the return structure of the individual securities in which case $\omega_i(\theta)$ would clearly be independent of D_2. Moreover, irrespective of the completeness of the market structure, if $\bar{\Omega}(D_2)=\phi$ for any relevant D_2 (no default risk), then $B_2=D_2$ from (4) so that D_2 drops out of (1) which again would imply $\omega_i(\theta)$ to be independent of D_2.

In general, with incomplete markets and default risk in the levered firm, $\omega_i(\theta)$ will be independent of D_2 if and only if $\beta_{i2}=\alpha_{i2}$ for all i as can be seen from (1). Hence, in an economy with incomplete markets for risk, a sufficient condition for firm values to be dependent on leverage is a nonzero default risk and that investors in equilibrium hold different fractions of the equity and bonds of the levered firm ($\alpha_{i2}\neq\beta_{i2}$).

II

A numerical example may be in order to illustrate the above point. Assume two investors each of whom owns initially 50 percent of each firm, i.e., $W_1=W_2=0.5V_1+0.5V_2$. The utility functions are assumed to be given by $U_1(Y_1)=Y_1-Y_1^2/730$ and $U_2(Y_2)=Y_2-Y_2^2/870$. First it is assumed that both firms are financed entirely by equity capital so that $V_j\equiv S_j$, $j=1, 2$. Riskless lending and borrowing at a zero interest rate are available for both investors. There are three different states of the world labelled 1, 2, and 3. It is postulated that the following data are given:

θ	1	2	3
$X_1(\theta)=X_2(\theta)$	150	200	300
$\pi_1(\theta)$	1/2	1/4	1/4
$\pi_2(\theta)$	1/4	1/4	1/2

It is fairly easy to verify that the equilibrium solution is given by $\alpha_{11}=\alpha_{12}=1/3$, $\alpha_{21}=\alpha_{22}=2/3$, $V_1=V_2=193.8$.

The above example is now modified such that the firm labelled 2 is financed both by equity and debt. In all other respects the relevant data are the same as above. It is assumed that $D_2=160$. Clearly D_2/B_2 will be the gross nominal interest rate on firm 2's bonds. Investors are assumed to be able to issue bonds on the same terms as firms. That means that investors can issue bonds with the same return characteristics as those of firm 2 at the same nominal interest rate. A personal bond issue takes place if $\beta_{i2}<0$. Moreover, riskless lending and borrowing at a zero interest rate are still available.

The total returns to shareholders and bondholders of the two firms will now be as follows:

θ	1	2	3
$X_1(\theta)=R_1(\theta)$	150	200	300
$R_2(\theta)$	0	40	140
$B_2(\theta)$	150	160	160
$X_2(\theta)$	150	200	300

As can be seen from the table, firm 2 goes bankrupt in state 1 so that there is a positive probability for bankruptcy for both investors. It is assumed again that each investor holds initially 50 percent of each firm, that is, 50 percent of the shares of firm 1 and 50 percent of the shares and bonds of firm 2.

After some tedious calculations, it is found that an equilibrium is characterized by

$$V_1 \equiv S_1 = 195$$
$$S_2 = 40$$
$$B_2 = 155$$
$$V_2 \equiv S_2 + B_2 = 195$$
$$\alpha_{11} = -6 - \beta_{12}$$
$$\alpha_{12} = (20 + 3\beta_{12})/3$$
$$\beta_{12} = \text{arbitrary}$$
$$\alpha_{21} = 7 + \beta_{12}$$
$$\alpha_{22} = -(17 + 3\beta_{12})/3$$
$$\beta_{22} = 1 - \beta_{12}$$

As can be seen, the optimal portfolios are not unique. This is due to the fact that there are four securities with different return patterns (including the riskless opportunity) and just three states of the world. Only three of these return patterns are linearly independent. Hence, the investors can obtain their most preferred pattern of final wealth over states of the world by combining these four securities in an infinite number of ways. Note, however, that for any choice of β_{12}, $\alpha_{i2}\neq\beta_{i2}$, $i=1, 2$. For example, for $\beta_{12}=-6$, the solution is:

$$\alpha_{11} = 0 \qquad \alpha_{21} = 1$$
$$\alpha_{12} = 2/3 \qquad \alpha_{22} = 1/3$$
$$\beta_{12} = -6 \qquad \beta_{22} = 7$$

The nominal interest rate on risky bonds, $(D_2/B_2-1)100$, is 100/31 percent. Investor 1 issues six times the amount of risky bonds issued by firm 2 with the same return characteristics at the same nominal interest rate. The individual transactions corresponding to the above portfolios are summarized below:

	Shareholdings		Borrowing (−)/ Lending (+)	
	Firm 1	Firm 2	Riskless	Risky
Investor 1	0	80/3	3295/3	−930
Investor 2	195	40/3	−3295/3	1085
Total	195	40	0	155

The notable point in the above example is that although $V_1=V_2=195$, this common value is different from the case where both firms were financed by equity capital only.[1]

III

Consider in general a one-period exchange economy with s states of the world and n firms with gross returns denoted by $X_j(\theta)$, $\theta\in\Omega$. Investor i's income pattern over states of the world at the end of the period is given by

$$(5) \qquad Y_i(\theta) = \sum_{j=1}^{n} \alpha_{ij}R_j(\theta) + \sum_{j=1}^{n} \beta_{ij}B_j(\theta) + rm_i, \quad \theta \in \Omega$$

where m_i denotes riskless lending or borrowing and r is one plus the riskless rate of interest. If $\beta_{ij}<0$, investor i is issuing risky bonds

[1] In fact, if the levered firm in the above example borrows at least as much as to make the default risk positive, the values of the two firms will be independent of the particular debt level in the levered firm because we then have as many linearly independent securities as states of the world (complete markets).

EFFICIENT CRITERIA FOR DECENTRALIZED INVESTMENT DECISIONS IN A STOCK MARKET ECONOMY [1)] [*)]

1. Introduction

In the present paper we shall examine the efficiency properties of the stock market allocation of investment in an economy with stock value maximizing firms and where the only source of uncertainty is the stochastic relationship between input and output, so-called technological uncertainty.

Clearly, efficient allocation of investment resources will depend on the possibilities in the economy for redistributing risk associated with production which in turn depends on the institutional structure of the capital market. As shown by Arrow (1953) and Debreu (1959), if there exist perfect markets for claims on incomes contingent on states of the world, i.e., complete markets, all the efficiency properties of the competitive economy under certainty will carry over to the uncertainty case.

In the absence of a complete set of markets - one market for each contingency - the market structure is said to be incomplete if the existing market opportunities do not span the market opportunities in the corresponding Arrow-Debreu economy. One such example which will be examined here is the stock market trading in risky securities, or claims to return patterns over states of the world. As is well-known, if there are fewer securities with linearly independent returns than states of the world, some feasible risk allocations in an Arrow-Debreu economy will be infeasible in the stock market economy. Hence, the stock market allocation of risk may be inefficient in the Arrow-Debreu sense.

As the stock market will in general provide an incomplete set of instruments for redistributing risk associated with production, efficiency in the Arrow-Debreu sense will not be a particularly useful criterion by which to evaluate

1) This paper is a substantially revised version of Discussion Paper 09/74 with the same title.

*) The author is indebted to Karl Borch and Jan Mossin for useful comments.

allocation of resources in such cases. It seems more useful to define allocative efficiency relative to the risk distributive instruments in fact available.

In the present paper we shall follow Diamond (1967) and define Pareto efficient allocations of investment resources in a stock market economy subject to the constraint that risky returns have to be distributed to consumers through a stock market trading in shares of risky firms. This is certainly a constrained optimum or a second-best allocation subject to the institutional structure of the capital market. It has, however, the advantage that we are comparing the market allocation with what is in fact attainable, i.e., we restrict the efficiency concept to the set of attainable allocations.

Diamond (1967) is the seminal paper on allocative efficiency of a decentralized stock market economy where it is shown that for a particular structure of technological uncertainty value maximization would guide competitive firms to choose efficient investment plans. This result was later criticized by Stiglitz (1972a), Jensen and Long (1972) and Fama (1972) who demonstrated in a mean-variance framework that stock value maximization would not lead to allocative efficiency in the investment market. These results lead some authors to question value maximization as a sound basis for firms' investments since it might not be consistent with the shareholders' interests. In particular, Ekern and Wilson (1974), Leland (1974a) and Radner (1974) claimed that value maximization might not be unanimously supported by a firm's shareholders.

In the present paper we shall argue that the fundamental obstacle to efficiency of the competitive market mechanism under technological uncertainty is the fact that investment decisions will in the general case have external effects in incomplete markets for which there do not exist well-defined market prices. In such cases there do not even seem to be any natural definition of competitive behaviour in the investment market. Moreover, we shall argue that in the presence of externalities shareholders will typically disagree on the desirability of any investment plan including the Pareto optimal one. On the other hand, when these externalities are absent or do not affect shareholders' optima, value maximization by competitive firms is well-defined and will lead to allocative efficiency and shareholders will always support value maximization which, of course, is what one should expect from conventional welfare theory.

The present paper is organized as follows. In section 2 we discuss some properties of exchange equilibria in complete and incomplete markets, and in section 3 we establish under what conditions value maximization will lead to Pareto efficiency in the investment market and in section 4 these results are discussed in some more detail within the mean-variance framework. Section 5 is devoted to the problem of finding under what conditions unanimity will prevail among the shareholders about the desirability of a firm's investment plan and under what conditions perceived and actual value maximization are in the shareholders' interests.

II. Market allocations of risky returns

We start out with a brief discussion on some well-known properties of the market allocation of risky returns. First some notation and assumptions. There are assumed to be J firms indexed j and S different states of the world indexed s and firm j is characterized by gross returns Z_{js} in state s which are supposed to materialize at the end of the period. There are H consumers indexed h and consumer h receives an income amounting to c_s^h on his portfolio in state s, and portfolio returns are assumed to be the only source of income. The set of income vectors $c^h = (c_1^h,\ldots,c_S^h)'$ are supposed to be completely ordered by expected utility $\sum_s p_s^h U^h(c_s^h) \equiv E^h(U^h)$, where p_s^h is consumer h's subjective probability for state s to obtain and U^h is h's utility function defined on the set of income patterns c^h. For each h, U^h is supposed to be strictly quasi-concave with $\partial U^h/\partial c_s^h \equiv U_s^h > 0$, $s=1,\ldots,S$.

If there were a complete set of markets, the feasible income space, or the set of attainable income vectors, would be an S-dimensional rectangle in the positive orthant of the S-dimensional Euclidean space bounded by the total returns in each state of the world. Letting π_s denote the price as of now for a unit income in state s and w_s^h initial endowment for each consumer h in state s, a pure exchange equilibrium in an economy with a complete set of markets is given by consumer optimum for each h,

$$\max_{c^h} \sum_s p_s^h U^h(c_s^h)$$

subject to

$$\sum_s \pi_s (c_s^h - w_s^h) = 0$$

and market clearing in each state,

$$\sum_h c_s^h = \sum_j Z_{js}, \qquad s=1,\ldots,S$$

First-order conditions for interior consumer optima are given by

$$(1) \qquad p_s^h U_s^h = \lambda^h \pi_s$$

where λ^h is expected marginal utility of initial wealth.

As usual for exchange economies without money, the market determines relative prices only and we normalize by fixing a price of 1 for a riskless return pattern yielding r units of income in each state. This is equivalent with fixing the riskless rate of interest at r - 1 per cent, i.e., $\sum_s \pi_s = 1/r$. With this normalization (1) can be rewritten as

$$(2) \qquad \frac{p_s^h U_s^h}{\sum_s p_s^h U_s^h r} = \pi_s$$

The left-hand side of (2) is the marginal rate of substitution between income in state s and the riskless income for consumer h - or alternatively, consumer h's (Marshallian) demand price for a unit of income in state s in terms of the numeraire. In equilibrium, the marginal rates of substitution are equated to the equilibrium prices for all consumers and hence the equilibrium allocation in an Arrow-Debreu market economy is Pareto-efficient in the unconstrained sense.

Turning to the stock market economy we assume that riskless lending and borrowing are available at the riskless rate of interest. We let $a^h = (a_1^h, \ldots, a_J^h)'$ denote the vector of shareholdings of consumer h with a_j^h denoting the number of shares held in firm j by consumer h. $X_j = (X_{j1}, \ldots, X_{jS})$ denotes the vector of state-dependent returns per share in firm j, i.e., $X_{js} \equiv Z_{js}/a_j$ where a_j is the total number of shares issued by firm j.

Letting m^h denote the amount of riskless bonds bought (issued, if negative) by consumer h, an attainable income pattern in a stock market economy will be given by

(3) $$c_s^h = rm^h + \sum_j a_j^h X_{js}$$

or in matrix notation

(3') $$(c^h)' = rm^h e' + (a^h)'X$$

where X is a (JxS)-matrix with rows consisting of the J return vectors X_j and e is the unity vector $(1,\ldots,1)'$ (transposed vectors denote row vectors).

The rows of the matrix X will be assumed to be linearly independent. Together with the strict quasi-concavity of U^h, this will ensure that expected utility $E^h(U^h)$ will be strictly quasi-concave in the decision variables m^h, a_j^h.

It is clear from (3') that the set of attainable income patterns in a stock market economy generates a linear subspace, the dimension of which is determined by the number of linearly independent rows of X. Consequently, if there are more states of the world than linearly independent return patterns, some attainable income patterns in an economy with a complete set of markets will not be attainable in a stock market economy in which case the stock market is said to be incomplete.

Letting the vector $v = (v_1,\ldots,v_J)'$ denote share prices, a stock market exchange - or financial - equilibrium is determined by consumer optima

$$\max_{m^h, a^h} \quad E^h(U^h)$$

subject to the budget constraints

$$m^h + \quad (a^h)'v = w_o^h \text{ (= initial wealth)},$$

and by the market clearing conditions

$$\sum_h m^h = 0, \quad \sum_h a_j^h = a_j, \quad j=1,\ldots,J.$$

Substituting the budget constraint into (3') and ignoring the fact that

a_j^h are integer variables, necessary conditions for interior[1] consumer optima are given by

(4) $$-rv_j \sum_s p_s^h U_s^h + \sum_s p_s^h U_s^h x_{js} = 0$$

If we define the (JxS+1)-matrix (-rv,X) as the matrix X augmented by the vector $(-rv_1,\ldots,-rv_J)'$ as the first column and the (1xS+1)-vector $\underline{u}^h$ with components $\underline{u}_o^h \equiv \sum_s p_s^h U_s^h$, $\underline{u}_s^h \equiv p_s^h U_s^h$, s=1,...,S, we can write the first-order conditions in matrix notation as

(4') $$\frac{\partial E^h(U^h)}{\partial a^h} = (-rv,X)\underline{u}^h = 0 \quad \text{(a column vector)}$$

We define the (S+1)x(S+1)-matrix $\underline{U}^h$ by

$$\underline{U}^h \equiv \begin{pmatrix} \sum_t\sum_s p_s^h U_{st}^h & \sum_s p_s^h U_{s1}^h & \cdots & \sum_s p_s^h U_{sS}^h \\ p_1^h \sum_t U_{1t}^h & p_1^h U_{11}^h & \cdots & p_1^h U_{1S}^h \\ \vdots & \vdots & & \vdots \\ p_S^h \sum_t U_{St}^h & p_S^h U_{S1}^h & \cdots & p_S^h U_{SS}^h \end{pmatrix}$$

where $U_{st}^h \equiv \partial^2 U^h / \partial c_s^h \partial c_t^h$.

By strict quasi-concavity of U^h and full row rank of X, the (JxJ)-matrix of second-order derivatives[2]

(5) $$\frac{\partial^2 E^h(U^h)}{\partial a^h\,(\partial a^h)'} = (-rv,X)\underline{U}^h(rv,X)'$$

is negative definite and the second-order maximum conditions are therefore satisfied.

If we define $\pi^h = (\pi_1^h,\ldots,\pi_S^h)$ as the vector of marginal rates of substitu-

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1) We shall in this paper ignore possible constraints on shortselling.

2) If x and y are column vectors, $\partial x/\partial y'$ will denote the matrix with entry $\partial x_k/\partial y_l$ in k-th row and the l-th column.

tion between incomes in the various states of the world and the numeraire asset (the riskless asset yielding r units of income in each state), i.e.

$$\pi_s^h \equiv \underline{u}_s^h/\underline{u}_o^h r \equiv p_s^h U_s^h / \sum_s p_s^h U_s^h \, r,$$

we have from (4) for each consumer

$$(6) \qquad \sum_s \pi_s^h X_{js} = v_j$$

or alternatively in matrix form from (4')

$$(6') \qquad X\pi^h = v$$

Clearly, $\left(-\dfrac{\partial a_j^h}{\partial m^h}\right)_{E^h(U^h)=\text{constant}} = \sum_s \pi_s^h X_{js}$ for all h.

Thus, condition (6) implies that at a stock market equilibrium, the individual marginal rates of substitution between shares are equated to their relative prices for all consumers and hence, the stock market allocation of shares, or claims to risky return patterns, is Pareto efficient. In general, however, condition (6) does not imply equalization of marginal rates of substitution between state-contingent income so that the stock market allocation may be a constrained optimum.

There will, however, be special cases where the stock market allocation is Pareto efficient in the unconstrained sense. Clearly, this will be the case if the equilibrium allocation in an Arrow-Debreu economy is attainable in a stock market economy. This will obtain if and only if at least one of the following two conditions are satisfied.

(i) There are as many securities with linearly independent return patterns as states of the world.

(ii) There are homogeneous expectations (p_s^h are the same for all h) and the consumers' utility functions exhibit the separation property.

Condition (i) is obvious since in that case the attainable income space in an Arrow-Debreu economy is spanned by the rows of X and the riskless income. Condition (ii) means that all consumers can obtain their most preferred income patterns by allocating their initial wealth to the riskless asset and a risky portfolio with asset proportions independent of initial wealth (portfolio separation) and with homogeneous expectations the asset proportions in the risky portfolio will be the same for all individuals. That implies that in equilibrium each consumer will hold the same share of the total amount of every risky asset and thus all risky assets can be aggregated or pooled into a mutual fund without changing the relevant individual opportunity sets and hence the incompleteness of the market structure will not constrain the consumer behaviour in the stock market. If all consumers are ranking portfolios according to expected utility, it can be shown[1)] that this aggregation property will obtain for any state-distribution of risky returns if and only if consumers have identical probability distributions over states of the world and their preferences exhibit linear risk tolerance with equal slope for all consumers. On the other hand, if we restrict probability distributions for risky returns to be jointly normal, portfolio separation, and hence the aggregation property, will obtain without any restrictions on preferences.

The results of the present paper are stated without any a priori restrictions on probability distributions for risky returns. It is, however, clear that all results which are dependent on utility functions exhibiting the separation property, will also be valid for arbitrary preferences if we restrict probability distributions for risky returns to be jointly normal.

The marginal rate of substitution π_s^h for consumer h - or his (Marshallian) demand price for income in state s - will be called consumer h's implicit price for income in state s. The implicit price for a unit of riskless income is $\sum_s \pi_s^h = 1/r$ for all h since the riskless asset yielding r units of income in each state is used as a numeraire. Clearly, if the different individual implicit prices for state-contingent income associated with a stock market equilibrium were imposed on consumers in an economy with a complete set of markets, the resulting equilibrium allocation in the Arrow-Debreu market would in fact coincide with the competitive stock market allocation if the distribution of initial wealth were the same in the two situations. This follows from the uniqueness of consumer optima as the budget constraints would be the same since at a stock market equilibrium the budget constraint can be written as $m^h + \sum_j a_j^h v_j = \sum_s \pi_s^h (rm^h + \sum_j a_j^h X_{js}) = \sum_s \pi_s^h c_s^h = w_o^h$, in view of (6).

1) See i.a. Cass & Stiglitz (1970) and Mossin (1977) for a proof and more detailed discussion on separation.

Clearly, if the stock market allocation is Pareto efficient in the unconstrained sense, the implicit prices must be the same for all consumers and identical with the corresponding Arrow-Debreu market prices.

III. Efficiency criteria for firms' investment in a stock market economy.

Most models of stock market allocation of investment resources are of the two-period type where the consumer in the first period decides how much to consume in that period and allocates the rest of his resources to the various firms in exchange for shares and bonds which in turn determine his second-period consumption pattern over states of the world. We shall, however, stick to our one-period model and assume that first-period consumption decisions have already been made such that the total amount of resources available for investment is given. Suppressing first-period consumption decisions means simply that the riskless rate of interest cannot be endogeneously determined.

Some economists[1)] have argued that competitive firms should maximize stock market values as a criterion for optimal investment. That is, firms should use stock market prices when calculating the values of marginal return patterns on new investment. It is well known[2)] that under certainty and with perfect capital markets efficient allocation of investment requires that firms maximize present values regardless of the owners' preferences. The same must clearly hold true under uncertainty in an economy with a complete set of markets. It is, however, by no means obvious that this kind of separation of productive and distributive efficiency should carry over to the uncertainty case in a stock market economy with an incomplete set of markets. In fact we shall see that the concept of competitive behaviour becomes somewhat blurred when security markets are incomplete. Also, in that case firms' investment decisions will have certain external effects in the market pointing to likely inefficiency of decentralized market behaviour.

We assume throughout the paper that the Modigliani-Miller theorem[3)] holds

1) See i.a. Diamond (1967) and Mossin (1969), (1973).
2) See Fisher (1930).
3) See Miller and Modigliani (1958).

so that firms' equilibrium values are independent of how they finance their investments. If not, the market value of a firm would depend on the way it distributes its investment returns in the capital market as the set of attainable income patterns over states of the world might change with different financial policies of the firm displacing consumers' optima in the stock market. This would work very much like an externality in the market indicating inefficiency of the competitive market mechanism.

It is well known[1] that if firms borrow under default risk the Modigliani-Miller theorem will obtain under the same conditions as those under which the stock market allocation of risky returns is efficient in the unconstrained sense. Thus, strictly speaking, separation of financial and production decisions is legitimate only if the relevant individual opportunity sets are unaffected by the firms' financial decisions which will be the case with complete markets or if consumers have identical beliefs and utility functions exhibiting the separation property. If there is no default risk, the Modigliani-Miller theorem is of course always valid.

As the market values of firms are supposed to be independent of their capital structures, we can with no loss of generality assume that firms finance their investments by issuing new shares. Firms are supposed to maximize the market value of initial interests, i.e., $V_j - I_j$ with $V_j \equiv a_j v_j$ (or equivalently, the price per share for initial shareholders) where I_j denotes the amount of investment measured in terms of the numeraire and v_j is total market value divided by the number of outstanding shares.

If we let $Z_{js}(I_j) \equiv a_j X_{js}(I_j)$ denote firm j's total return in state s when the amount of investment is I_j, we have from (6) that the equilibrium value of firm j is given by

$$(7) \qquad V_j \equiv a_j v_j = \sum_s \pi_s^h a_j X_{js}(I_j) \equiv \sum_s \pi_s^h Z_{js}(I_j), \qquad h=1,\dots,H.$$

Firms are supposed to behave as price takers which in this context will be understood to mean that the firm takes the implicit prices for state-contingent income, π_s^h, as given and independent of its investment decisions. This corresponds to Leland's definition of a <u>completely competitive</u> capital market (Leland

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1) See i.a. Hagen (1976), Stiglitz (1969).

(1974b)) and in view of (6) it implies that the firm considers the price per share with a given return distribution as independent of its investment decisions and hence also of the number of outstanding shares, which seems to be a natural behavioural requirement in a competitive capital market. Also, the above definition of price-taking implies that the firm considers its market value to be linearly homogeneous in state-contingent returns. That means that if the return pattern of some firm is scaled by some scalar, the firm believes that its stock market value is scaled by the same scalar since this can be seen as scaling (by the given scalar) the amount sold of shares with a given return distribution.

With the above definition of price-taking, maximization of stock market value of initial interests yields the first-order conditions

$$(8) \qquad \sum_s \pi_s^h Z'_{js}(I_j) = 1 \text{ for all } h$$

and assuming that all feasible investment plans $(I_j, Z_{j1}(I_j), \ldots, Z_{jS}(I_j))$ belong to a closed and strictly convex production set in R^{S+1}, the second-order conditions for a maximum are also satisfied.

Condition (8) says that firm j should increase investment until the market value of the marginal return pattern for each consumer equals marginal investment cost (recalling that investment is a numeraire). The trouble with the investment rule (8) from an applied point of view is that it presupposes that the firm knows or is able to compute the implicit prices π_s^h. However, in the general case these prices are not revealed in the stock market or cannot be computed from stock market values unless the matrix X is quadratic in which case $\pi^h = X^{-1}v$ from (6').

Turning to efficiency and restricting return distribution rules to the linear form (3) (constrained efficiency), a Pareto efficient allocation can be determined as a solution to the following constrained maximization problem.

$$\max_{m^h, \{a_j^h\}, I_j} \sum_h \lambda^h \sum_s p_s^h U^h(c_s^h) \qquad (\lambda^h \text{ are arbitrary positive weights})$$

subject to the constraints

$$c_s^h = rm^h + \sum_j a_j^h X_{js}(I_j) \qquad s=1,\dots,S$$

$$\left.\begin{array}{lll} \sum_h m^h = 0 & & \\ \sum_h a_j^h = a_j & j=1,\dots,J & \\ \sum_j I_j = \sum_h \bar{I}^h & \text{($\bar{I}^h$ is the amount of investment resources supplied by consumer h - a given constant).} & \end{array}\right\} \text{Market clearing conditions.}$$

First of all we note that although the individual utility functions are strictly quasi-concave in a_j^h, they need not be jointly quasi-concave in a_j^h and I_j since the equation defining c_s^h need not be jointly concave in a_j^h and I_j. Hence, a global optimum may not be reached through partial marginal variations of the decision variables and the first-order maximum conditions may only define a local optimum[1] (excluding the possibility of a stationary point being a saddlepoint).

The first-order optimum conditions are given by

$$(9)\quad \left\{\begin{array}{llll} \text{a)} & \lambda^h \sum_s p_s^h U_s^h r & = \alpha & h=1,\dots,H \\ \text{b)} & \lambda^h \sum_s p_s^h U_s^h X_{js}(I_j) & = \beta_j & j=1,\dots,J \\ & & & h=1,\dots,H \\ \text{c)} & \sum_h \lambda^h \sum_s p_s^h U_s^h X'_{js}(I_j) a_j^h & = \gamma & j=1,\dots,J \end{array}\right.$$

where α, β_j and γ are the relevant Lagrangean multipliers associated with the market clearing constraints. Since the first-order conditions are linearly homogeneous in λ^h and the Lagrangean multipliers, we normalize[2] by choosing $\gamma/\alpha=1$. Thus, from (9a) and (9c) the 1.-order condition for (locally) efficient investment in the risky firm j can be written as

$$(10)\quad \sum_h a_j^h \sum_s \frac{p_s^h U_s^h X'_{js}(I_j)}{\sum_s p_s^h U_s^h r} \equiv \sum_h a_j^h \sum_s \pi_s^h X'_{js}(I_j) \equiv \sum_h \alpha_j^h \sum_s \pi_s^h Z'_{js}(I_j) = 1$$

where $\alpha_j^h \equiv a_j^h / a_j$ is the fraction of the total number of shares in firm j allocated to consumer h so that $\sum_h \alpha_j^h = 1$ for all j.

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1) See Drèze (1972) and (1974) for a detailed discussion on the problem of non-convexity in this context.

2) That means, we consider the particular Pareto efficient allocation for which $\gamma/\alpha=1$.

The term $\sum_s \pi_s^h X'_{js}(I_j)$ may be interpreted as consumer h's marginal rate of substitution between claims to the marginal risky return vector $X'_j(I_j)$ and claims to the riskless return yielding r units of income in each state (the numeraire). Hence, Pareto efficiency requires that the weighted sum over all consumers of the marginal rates of substitution between claims to $X'_j(I_j)$ and claims to returns on the riskless opportunity be equated to unity (marginal social investment cost) where the weights are the shares of the j-th risky return allocated to the various consumers.

It is easily verified that if the necessary conditions (8) for stock value maximization of initial interests are satisfied for all h, the market allocation of investment will satisfy the necessary conditions for constrained Pareto efficiency as given by (10). On the other hand, conditions (8) are not necessary for Pareto efficiency.

It is, however, not clear whether there for given π^h will exist any feasible investment plan $(I_j, Z_{j1}, \ldots, Z_{jS})$ maximizing $\sum_s \pi_s^h Z_{js} - I_j$ for all h. Indeed, if the technology allows for the possibility of adjusting production plans continuously so as to increase or decrease the returns conditional upon a certain state independently of those conditional upon other states (provided the investment level is suitably adjusted), then for any h, the value maximizing investment plan would be given by the point on the efficiency frontier in (S+1)-space having a tangent plane with the (S+1)-dimensional normal $(\pi^h, -1)$, and in view of strict convexity of production technologies, tangent planes with different normals would correspond to different points on the efficiency frontier (different investment plans). Hence, it is clear that if π^h are different for different h, there will not exist a common investment plan maximizing $\sum_s \pi_s^h Z_{js} - I_j$ over the production set for each h unless one imposes certain restrictions on substitution possibilities between output conditional upon different states. In that respect, the production technology specification implicit in (8) represents an extreme case in that it does not allow for any substitution between returns in different states.

In any case, the market investment rule as given by (8) is not meaningful or applicable for the individual firm to the extent that it presupposes that the firm knows the individual implicit prices π^h which are generally not available or computable from capital market data. Indeed, from (6') the implicit prices compatible with a stock market equilibrium will for each h lie in an affine subspace of dimension S-J. On the other hand, if the left-hand side of (8) were independent of individual utility characteristics (implicit prices) at a stock market equilibrium, the decentralized competitive market mechanism based on stock value maximization would be well-defined in the sense that the firm could check the optimality conditions (8) without having further information than that given by stock market equilibrium prices. Some such cases will be considered below.

As is well-known[1], if the marginal return pattern on a new investment in a particular firm is linearly dependent on return patterns of some subset of firms in the economy, then the individual shareholder's valuation of this marginal return pattern will be the same for all h at a stock market equilibrium. That is, if there exist technological coefficients $\gamma_{jk}(I_j)$ such that

$$(11) \qquad Z'_{js}(I_j) = \sum_k \gamma_{jk}(I_j)\, Z_k(I_k) \qquad\qquad s = 1,\ldots,S$$

then we have

$$(12) \qquad \sum_s \pi^h_s\, Z'_{js}(I_j) = \sum_k \gamma_{jk}(I_j)\, V_k \qquad\qquad \text{for all } h$$

by condition (7) for a stock market equilibrium. Thus, the competitive stock market value of the marginal return pattern (11) is simply the same linear combination of the stock market values of the firms on whose returns it is linearly dependent. This may seem as a natural extension of the notion of price-taking in capital asset markets, namely that claims to identical return patterns have the same price regardless of firms' investment levels. If firms' technologies satisfy condition (11), the market allocation rule (8) is well-defined in the sense that competitive and value maximizing firms only need to know the technological coefficients $\gamma_{jk}(I_j)$ and the stock market values of the firms for which $\gamma_{jk}(I_j) \neq 0$.

In terms of returns per share (11) implies that the marginal return vector on firm j's investment is contained in the row space of the matrix X and this technological property is therefore called <u>spanning</u> as the attainable set of portfolio return vectors after the investment is made, is spanned by the return vectors corresponding to the initial allocation of investment. Hence, if firms' stochastic technologies satisfy this spanning property,

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1) See i.a. Ekern and Wilson (1974) and Leland (1974a,b).

the diversification opportunities in the stock market will remain unchanged when investment resources are reallocated among firms.

The seminal study of efficiency implications of value maximizing firms in a stock market economy is that by Diamond (1967). He postulated a stochastic technology where the random component enters the firm's return function in a multiplicative manner. A stochastic technology of the Diamond type is thus given by

$$Z_{js}(I_j) = \psi_j(I_j)\varphi_{js}, \qquad s = 1,\dots,S; \quad \psi_j' > 0, \quad \psi_j'' < 0,$$

where φ_{js} is a stochastic variable independent of the investment level. Clearly,

$$Z_{js}'(I_j) = \psi_j'(I_j)\varphi_{js} = \frac{\psi_j'(I_j)}{\psi_j(I_j)} Z_{js}(I_j) \qquad s = 1,\dots,S$$

which satisfies the spanning property as defined by (11). Hence, in an economy with stochastic technologies of the Diamond type, the competitive stock value maximization rule (8) is well-defined in terms of stock market prices and Diamond's well-known result that competitive stock value maximization leads to Pareto efficiency in the investment market follows immediately from (8) and (10).

Turning to the general case where the return patterns on new investment are not obtainable as linear combinations of return patterns already present in the market, price-taking with respect to shares makes no sense as the shares of a firm after the investment is made will represent claims to a new and unique return pattern not previously present or attainable in the market.[1] In that case stock value maximizing firms would have to know the individual marginal rates of substitution π^h in order to choose investment plans consistent with Pareto efficiency. Indeed, the existence of a particular investment opportunity in firm j as characterized by the marginal return pattern $Z_{js}'(I_j)$ has in the non-spanning case the property of a public good as can be seen from (10) where the socially optimal trade-off between claims to the marginal risky return vector $Z_{js}'(I_j)$ and claims to the riskless return is given by a weighted average of the consumers' individual marginal rates of substitution with the consumers'

1) The problem is analogous to that of a firm under certainty with the quality of the firm's product changing with the scale of the firm.

fractional shareholdings in firm j as weights. The reason for this public good property of investment is that increased investment in firm j will in the non-spanning case have external effects in the sense that it will change the set of attainable trades or more precisely, the linear subspace of attainable income vectors spanned by the return vectors in the market. Hence, the feasible range of choice for portfolio return patterns will change for all shareholders, which will clearly be in the nature of an external effect. Moreover, if the incomplete nature of the market is effectively constraining shareholders' optima in the stock market, implicit prices will differ among shareholders and they will evaluate this change in the attainable set differently.[1] This is reflected by the fact that $\sum_s \pi_s^h Z'_{js}(I_j)$ may be different for different h at a Pareto optimum.

We might define some sort of quasi-equilibrium[2] by instructing firms to use a weighted average $\sum_h \alpha_j^h \pi_s^h$ of individual implicit prices when computing the value of marginal investment projects. The quasi-equilibrium allocation would of course satisfy the necessary conditions for Pareto efficiency. One has, however, still to face the problem of computing implicit prices as they are not revealed by market prices.

For the special case that shareholders have identical beliefs and preferences satisfying the separation property, implicit prices would be the same for all shareholders at a stock market equilibrium. In that case all shareholders would be in agreement as to the competitive stock market value of any investment plan in the sense that the <u>perceived</u> market value $\sum_s \pi_s^h Z'_{js}(I_j)$ (at given implicit prices) would be independent of h. In that case the investment plan satisfying condition (8) and hence also condition (10) for Pareto efficiency, could be chosen by a unanimous vote or alternatively, by associating the firm's prices for state-contingent returns with the implicit prices for an arbitrary shareholder - e.g. the manager.

We summarize this in a proposition

<u>Proposition 1</u>

Given completely competitive capital markets, if there exists a feasible

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1) See Drèze (1972) and (1974) for a detailed discussion on the public goods property of investment under uncertainty.

2) See Malinvaud (1969).

investment plan maximizing the stock market value of initial interests for each shareholder's implicit prices, then it will satisfy the necessary conditions for Pareto efficiency. The required price information for computing the competitive stock market value of a given investment plan will, however, generally not be provided by competitive stock market prices unless

(i) firms' stochastic technologies satisfy the spanning property[1)]

On the other hand, regardless of firms' technologies, all shareholders would at a stock market equilibrium be in agreement as to the competitive stock market value of any given investment plan if

(ii) shareholders have identical beliefs and utility functions exhibiting the separation property.

The fact that the firm needs information about shareholders' preferences other than that reflected by shareprices in order to choose a Pareto efficient investment, demonstrates that investment under uncertainty has external effects which are not explicitly priced in the capital market. In such cases the decentralized competitive market mechanism is not well-defined since the market behaviour of competitive and value maximizing firms is not uniquely determined in terms of technologies and market prices. One exception is, however, the case where a firm's technology satisfies the spanning property. In that case the competitive market value of an investment plan with a given return pattern would be given by the stock market value of a corresponding feasible portfolio with the same return pattern. If spanning does not obtain, there do not exist feasible market portfolios replicating the return pattern $Z'_{js}(I_j)$ on a new investment. However, with identical beliefs and utility functions exhibiting separation, all consumers would be in agreement as to the competitive (or perceived) market value of any given investment plan. Thus, in both these cases the unanimously agreed upon change in the competitive stock market value resulting from a given investment, will correctly reflect the social value of undertaking that investment and

1) The spanning property will always obtain if there are as many risky securities with linearly independent returns as states of the world.

Pareto efficiency calls for equating this social value to marginal investment cost (normalized at unity).

The reason why shareholders are in agreement on the competitive market (and social) value of investment plans in these cases is that in case of spanning the investment will have no external effects since diversification opportunities will remain unchanged, and in case of separation the external effects of investment will be evaluated equally by all shareholders as implicit prices will be the same. In the latter case changes in diversification opportunities will have no effect on the relevant opportunity sets for shareholders so that in equilibrium each shareholder will hold the same fraction of all risky assets and hence only total returns in each state will matter.

It seems fair to conclude that as to market allocations of investment under uncertainty the competitive benchmark based on stock value maximization does not seem to be well-defined unless spanning or separation obtains. In case of spanning the firm can rely on market prices in order to choose investment plans maximizing competitive stock market values, while in case of separation competitive value maximizing investment will depend on utility characteristics common to all shareholders which would in principle be easy to check. If the conditions for spanning or separation are not satisfied, it seems more fruitful to examine alternative equilibrium concepts such as quasi- or Lindahl equilibria as in Drèze (1974) or various kinds of voting equilibria as in Gevers (1974).

IV. The mean-variance example.

Much work on the efficiency properties of stock market allocations of investment has been within the mean-variance framework. Jensen and Long (1972), Stiglitz (1972a) and Fama (1972) have shown that if competitive firms behave such as to maximize the stock market value of the initial owners' interests, the equilibrium allocation of investment will be inefficient even in the constrained sense. This may seem surprising in view of proposition 1 and it will be shown here that it is due to an inadequate specification of price-taking behaviour. Indeed, if shareholders are ranking portfolios according to means and variances of returns, preferences exhibit separation and together with identical beliefs implicit prices must be the same for all shareholders and competitive value maximization at these prices will lead to Pareto efficiency.

We assume an economy with consumers ranking portfolios according to expected value of quadratic utility functions $U^h(x) = x - \kappa^h x^2$, $\kappa^h > 0$, and with identical probability assessments.

From the equilibrium value relation (7) we have

$$(13) \qquad V_j = a_j v_j = \frac{\sum_s p_s\{1 - 2\kappa^h(rm^h + \sum_k a_k^h X_{ks}(I_k)\}}{\sum_s p_s\{1 - 2\kappa^h(rm^h + \sum_k a_k^h X_{ks}(I_k)\}r} \; a_j X_{js}(I_j)$$

Factoring out $2\kappa^h$ and summing the numerator and denominator over h, using the market clearing conditions $\sum_h m^h = 0$ and $\sum_h a_k^h = a_k$ for all k, and setting $Z_{js}(I_j) \equiv a_j X_{js}(I_j)$ for all j, we get

$$(14) \qquad V_j = \frac{\sum_s p_s\{\sum_h \frac{1}{2\kappa^h} - \sum_k Z_{ks}(I_k)\}Z_{js}(I_j)}{\{\sum_h \frac{1}{2\kappa^h} - \sum_k \mu_k(I_k)\}r}$$

where $\mu_k(I_k)$ denotes expected total returns in firm k. Defining $\delta \equiv \dfrac{1}{\sum_h \frac{1}{2\kappa^h} - \sum_k \mu_k(I_k)}$ and $\mathrm{Cov}\,(Z_j(I_j), Z_k(I_k)) \equiv \sigma_{jk}(I_j, I_k)$ as the covariance between total returns in firm j and k (which will depend on the investment level in both firms), (14) can be rewritten as

$$(15) \qquad V_j = \frac{1}{r}\{\mu_j(I_j) - \delta\sum_k \sigma_{jk}(I_j, I_k)\}$$

where δ is usually interpreted as a risk discount factor.

In the mean-variance model one has usually defined price-taking with respect to the riskless rate of interest r and the risk discount factor δ[1]. With price-taking with respect to implicit prices firms would behave as if the market value were linearly homogenous in state-contingent returns, i.e., behave as if $V(\lambda Z) = \lambda V(Z)$ where $V(Z)$ is the market value of the return pattern $(Z_1, \ldots, Z_S)$ and λ is some scalar, and it is easy to show that this does not follow from price-taking with respect to r and δ. Scaling the return pattern Z_j of firm j by a scalar $\lambda > 1$ and assuming that the firm uses the valuation function (15) and takes r and δ as given, we get

$$V(\lambda Z_j) = \lambda V(Z_j) + \frac{\delta\lambda(1-\lambda)\sigma_{jj}}{r}$$

1) Stiglitz (1972a) quite explicitly states that in a mean-variance model we can think of the firm as "selling two commodities, mean and variance which have prices $1/r$ and $-\delta/r$ respectly".

where σ_{jj} is the own variance of the return pattern Z_j. By assumption the last term is negative such that price-taking with respect to r and δ implies that $V(\lambda Z_j) < \lambda V(Z_j)$.[1] Hence, from an efficiency point of view the firm would underestimate the increase in its market value from increasing its scale. If we assume that one of the firms is riskless, we can immediately conclude (as in Stiglitz (1972a)) that in the mean-variance framework maximization of stock market values taking r and δ as given will imply that too small amounts of investment will be allocated to risky firms as compared with a Pareto optimum.

Jensen and Long (1972), Stiglitz (1972a) and Fama (1972) argue that this inefficiency is due to a kind of externality in the capital market as increased investment in a particular firm will affect the riskiness of other firms through the covariance terms. It is, however, clear that this kind of inefficiency will arise also in the spanning case where all external effects from investments will vanish, so that the inefficiency of the market allocation based on stock value maximization as demonstrated in the works cited above, is due to an inadequate specification of price-taking behaviour. Indeed, what is priced in the stock market are shares - or claims to risky return patterns - rather than means and variances, and price-taking with respect to means and variances does not imply price-taking with respect to shares, namely, that shares with identical return distributions have the same price. This can be seen from rewriting (15) as

$$v_j = \frac{1}{r}\{ E(X_j) - \delta\sum_k a_k \operatorname{Cov}(X_j, X_k)\}$$

which implies $\frac{\partial v_j}{\partial a_j} = -\frac{\delta}{r} \operatorname{Var}(X_j) < 0$, where $E(X_j)$ and $\operatorname{Var}(X_j)$ are expected return and variance of returns per share, respectively. Hence, price-taking with respect to r and δ implies that the firm will act as if the price of shares with identical returns is a decreasing function of the number of outstanding shares which explains why stock value maximizing firms taking r and δ as given will underinvest from an efficiency point of view.[2]

As can be seen from (14), if firm j takes implicit prices as given, this implies that

1) Clearly, firms' investment under stochastic technologies of the Diamond type satisfies this scaling assumption with $\lambda = 1+\psi_j'/\psi_j$, and as is clear from proposition 1, if Z_j denotes pre-investment returns, efficiency calls for evaluating post-investment returns according to the proportionality rule $V((1+\psi_j'/\psi_j)Z_j) = (1+\psi_j'/\psi_j)V(Z_j)$.

2) This point is also noted in Nielsen (1977).

$$(16) \qquad \pi_s = p_s \frac{\{\sum_h \frac{1}{2\kappa^h} - \sum_k Z_{ks}(I_k)\}}{\{\sum_h \frac{1}{2\kappa^h} - \sum_k \mu_k(I_k)\}r} = p_s\{\sum_h \frac{1}{2\kappa^h} - \sum_k Z_{ks}(I_k)\}\frac{\delta}{r}$$

must be considered as independent of firm j's investment decisions. We may note that only aggregate returns in state s matter for the implicit price π_s which is a consequence of the separation property.

From condition (10) it follows that Pareto efficient investment in firm j is given by the condition

$$\sum_s p_s \frac{\{\sum_h \frac{1}{2\kappa^h} - \sum_k Z_{ks}(I_k)\}}{\{\sum_h \frac{1}{2\kappa^h} - \sum_k \mu_k(I_k)\}r} Z'_{js}(I_j) = 1$$

which can be rewritten as

$$(17) \qquad \frac{1}{r}\{\mu'_j(I_j) - \delta\sum_k \sigma'_{jk}(I_j, I_k)\} = 1$$

where δ is the risk discount factor defined above, $\mu'_j(I_j)$ is the expected marginal return on new investment in firm j and $Cov(Z'_j(I_j),\ Z_k(I_k)) \equiv \sigma'_{jk}(I_j, I_k)$ is the covariance between the marginal return pattern $Z'_{js}(I_j)$ and the total returns in firm k. Specifically, $\sigma'_{jj}(I_j, I_j)$ is the covariance between the marginal returns on new investment in firm j and the pre-investment returns in that firm. It is clear from (17) that in a mean-variance framework the social desirability of an investment project with a <u>given</u> marginal return pattern is independent of which firm undertakes the project. This follows from the more general fact that if separation obtains, only aggregate returns under each state will matter for social welfare and hence the social desirability of a given project is determined by its contribution to aggregate returns under each state.

Comparing (15) and (17) we see that efficiency requires that stock market value maximizing firms taking r and δ as given should valuate the marginal returns on a new investment project as if it were a separate firm with risk measured by $\sum_k \sigma'_{jk}(I_j, I_k)$ so that the firm should pay no attention to the own variance of the marginal return pattern $Z'_{js}(I_j)$.

The efficient investment rule (17) may be rewritten as

$$(18) \qquad \mu'_j(I_j) = r + \delta\sum_k \sigma'_{jk}(I_j, I_k)$$

Specifically, (18) implies that if the marginal returns on new investment in firm j are uncorrelated with the returns of other firms as well as with the pre-investment returns in firm j, the firm should increase investment until $\mu_j'(I_j) = r$, i.e., until the expected marginal return on new investment equals the riskless rate of interest[1].

The efficiency requirement that the firm should neglect the own variance of the marginal return pattern Z_{js}' when calculating the value of a marginal investment is due to the fact that we are considering infinitesimal changes in the investment level in firm j so that at the margin the own variance of marginal investment projects will vanish. Considering finite changes ΔI_j in the investment level and assuming that $Z_{js}(I_j+\Delta I_j) - Z_{js}(I_j) \approx Z_{js}'(I_j)\Delta I_j$ for small finite changes ΔI_j as a first-order approximation, we have from (15) (when r and δ are considered as constants)

$$(19) \qquad V_j(I_j+\Delta I_j) - V_j(I_j) = \frac{1}{r}\{\mu_j'\Delta I_j - \delta[\sum_k \mathrm{Cov}(Z_j', Z_k)\Delta I_j + \mathrm{Cov}(Z_j', Z_j)\Delta I_j + \mathrm{Var}(Z_j')(\Delta I_j)^2]\}$$

where the arguments I_k are omitted on the right-hand side of (19) for ease of notation.

For infinitesimal changes we get at the margin

$$(20) \qquad \frac{\partial V_j}{\partial I_j} = \lim_{\Delta I_j \to 0} \frac{V_j(I_j+\Delta I_j) - V_j(I_j)}{\Delta I_j} = \frac{1}{r}\{\mu_j' - \delta(\sum_k \mathrm{Cov}(Z_j', Z_k) + \mathrm{Cov}(Z_j', Z_j))\}$$

which is equal to the derivative of (15) with respect to I_j. If we assume that the finite change in investment ΔI_j takes the economy to a Pareto optimum in one step, the social value of the finite investment project with returns $Z_{js}'\Delta I_j$ is given by

$$(21) \qquad \sum_s p_s \frac{(\sum_h \frac{1}{2K^h} - \sum_k Z_{ks} - Z_{js}'\Delta I_j)}{\sum_h \frac{1}{2K^h} - \sum_k \mu_k - \mu_j'\Delta I_j} Z_{js}'\Delta I_j = \frac{1}{r}\{\mu_j'\Delta I_j - \delta'(\sum_k \mathrm{Cov}(Z_j', Z_k)\Delta I_j + \mathrm{Var}(Z_j')(\Delta I_j)^2)\}$$

where $\delta' \equiv 1/(\sum_h \frac{1}{2K^h} - \sum_k \mu_k - \mu_j'\Delta I_j)$

1) This result corresponds to the well-known investment rule by Arrow and Lind (1970) although they did derive it for an economy with a complete set of markets. This is, however, not very surprising because with quadratic preferences and homogeneous expectations the stock market allocation of risky returns will be Pareto efficient in the unconstrained sense.

Dividing (21) by ΔI_j and letting ΔI_j go to zero, the own variance term vanishes and we are left with (17). Comparing (17) and (20), and (19) and (21), respectively, we see that both in the infinitesimal and finite case the investment rule based on maximizing $V_j - I_j$ taking r and δ as given - henceforth referred to as conventional value maximization - places twice as much weight on the covariance between the firm's marginal return pattern and its pre-investment returns as compared to the efficiency rule. This implies also that the profitability of an investment project with a given return pattern will depend on which firm undertakes it which is clearly inconsistent with Pareto efficiency in case of separation.

In the following we shall run through a few examples known from the literature where we shall compare the efficiency rule (17) with the conventional value maximizing rule.

First we assume a stochastic technology of the Diamond type with independent returns across firms. Total returns in firm j are given by $Z_{js}(I_j) = \psi_j(I_j)\phi_{js}$. We denote the variance of the return pattern ϕ_{js} by $\sigma^2_{\phi_j}$.

The conventional value-maximization rule gives

$$\frac{1}{r}(\mu_j' - 2\delta\psi_j\psi_j'\sigma^2_{\phi_j}) = 1$$

while Pareto efficiency requires

$$\frac{1}{r}\{\mu_j' - \delta \mathrm{Cov}(\psi_j\phi_{js}, \psi_j'\phi_{js})\} = \frac{1}{r}(\mu_j' - \delta\psi_j\psi_j'\sigma^2_{\phi_j}) = 1$$

where we have omitted the argument I_j for ease of notation. Thus we see that the conventional value maximization rule places twice as much weight on the firm's variance as compared with the Pareto efficient rule.

We next assume dependent returns across firms and specifically, that returns in firm j can be written as

$$Z_{js}(I_j) = \mu_j(I_j) + g_j(I_j)\varepsilon_{js} + m_j(I_j)M_s$$

where ε_{js} is a random factor specific to the firm while M_s is a random market factor common to all firms ("The state of business"). We assume

$\sum_s p_s \varepsilon_{js} = \sum_s p_s M_s = 0,\ \ \sum_s p_s \varepsilon_{js}\varepsilon_{ks} = 0,\ \ \sum_s p_s \varepsilon_{js} M_s = 0,\ \ \sum_s p_s \varepsilon_{js}^2 = 1,$
$\sum_s p_s M_s^2 = 1.$

Plugging this into the market value function (15) we get

$$V_j = \frac{1}{r}\{\mu_j - \delta(g_j^2 + m_j^2 + m_j \sum_{k \neq j} m_k)\}$$

and the conventional value maximization rule gives

$$\frac{1}{r}\{\mu_j' - \delta(2g_j g_j' + 2m_j m_j' + m_j' \sum_{k \neq j} m_k)\} = 1$$

Clearly, $Z_{js}'(I_j) = \mu_j' + g_j' \varepsilon_{js} + m_j' M_s$ such that $\sigma_{jj}' = g_j g_j' + m_j m_j'$ and $\sigma_{jk}' = m_j' m_k$, $j \neq k$.

Then from (17), Pareto efficiency requires

$$\frac{1}{r}\{\mu_j' - \delta(g_j g_j' + m_j m_j' + m_j' \sum_{k \neq j} m_k)\} = 1$$

such that again the conventional value maximizing firm places twice as much weight on its own risk component σ_{jj}' compared to what it should.

The two examples above are adapted from Stiglitz (1972a) and we have gotten the same results as he did. We have, however, shown that if firms measure the riskiness of new investment according to $\sum_k \sigma_{jk}'$, efficiency requires that the firm should increase investment as long as the value of marginal investment returns exceeds marginal investment costs.

We conclude with the example analyzed by Jensen & Long (1972) and later by Merton & Subrahmanyam (1974). Assume an economy with n firms with stochastic return patterns Z_{js} which have means μ_j and covariances σ_{jk}. Moreover, there is an investment project with constant stochastic returns to scale available to all firms. We let ρ_s be the stochastic return per unit investment in the project and the expected return per unit investment is denoted $\bar{\rho}$. I_j denotes firm j's investment in the project which yields total returns $I_j \rho_s$ in state s. Since all firms are facing the same project, the distribution of ρ_s is independent of the firm which means that the returns on the project are perfectly correlated across firms. If firm j invests I_j in the project, its total returns will be $Z_{js} + I_j\rho_s$. Plugging this into the valuation function (15) we get

$$V_j = \frac{1}{r}\{\mu_j + I_j\bar{\rho} - \delta(\sum_k \sigma_{jk} + I_j \sum_k \sigma_{\rho k} + \sum_k I_k\sigma_{\rho j} + I_j \sum_k I_k\sigma_\rho^2)\} \tag{22}$$

where $\sigma_{\rho j}$ is the covariance between ρ and the pre-investment returns Z_{js} in firm j and $\sigma_\rho^2 \equiv \text{Var}(\rho)$.

If firm j takes investments of other firms as given and maximize $V_j - I_j$, we get the investment rule

$$\frac{1}{r}\{\bar{\rho} - \delta(\sum_k \sigma_{\rho k} + \sigma_{\rho j} + \sum_k I_k\sigma_\rho^2 + I_j\sigma_\rho^2)\} = 1.$$

The marginal returns Z'_{js} on new investment in firm j are ρ_s. From the efficiency rule (17) firm j should increase investment in the project up to the point where

$$\frac{1}{r}\{\bar{\rho} - \delta \sum_k \text{Cov}(\rho, Z_k + I_k\rho)\} = \frac{1}{r}\{\bar{\rho} - \delta(\sum_k \sigma_{\rho k} + \sum_k I_k\sigma_\rho^2)\} = 1$$

and this is indeed the Pareto efficiency rule as derived by Jensen & Long, and we see that it is different from the conventional value maximizing rule. The difference is again given by $\frac{\delta}{r}\text{Cov}(Z'_j, Z_j) = \frac{\delta}{r}\text{Cov}(\rho, Z_j + \rho I_j) = \frac{\delta}{r}(\sigma_{\rho j} + I_j\sigma_\rho^2)$.

As Merton & Subrahmanyam (1974) have pointed out, if firm j takes the aggregate investment in the project as given and independent of its own investment decisions, then the conventional value maximization rule will be efficient. This can be seen from differentiating (22) with respect to I_j under the constraint that $\partial\sum_k I_k/\partial I_j = 0.$[1] Merton & Subrahmanyam then conclude that a "correct" definition of price-taking should imply that firms consider aggregate investment in the project as given.

The reason why the conventional value maximizing rule gives Pareto efficient investment decisions if firms behave as if the aggregate investment in the project is fixed, follows from constant stochastic returns to scale and the fact that the returns on new investment are assumed to be perfectly correlated across firms. Assuming constant aggregate investment in the project implies

1) This corresponds to what Fama calls the "reaction principle" in Fama (1972).

in that case that the total returns in each state are fixed. Hence, the implicit prices for state-contingent income will remain unchanged since they will only depend on aggregate returns in each state in case of separation. Thus the capital market is completely competitive in the Leland sense and the result follows from our proposition 1.

In the general case where returns on new investment are not perfectly correlated across firms or with non-constant stochastic returns to scale, aggregate returns in state s will depend on the distribution of investment across firms and hence the implicit prices π_s will depend on the investment distribution too. In that case competitive behaviour with respect to r and δ together with assuming constant aggregate investment would not be sufficient for ensuring efficient investment decisions by value maximizing firms. Generally, efficiency in a mean-variance model requires that firms valuate marginal returns on new investment according to the valuation rule (17) or (21) in the infinitesimal and finite case, respectively.

V. Value maximization, Pareto efficiency and unanimity.

As shown in proposition 1, in the absence of external effects on the attainable set of income patterns (spanning) or if external effects from investment are evaluated equally by all shareholders (separation), stock value maximization taking implicit prices as given may be considered as well-defined at the level of the firm and will lead to local efficiency in the investment market.

On the other hand, Wilson (1972), Ekern and Wilson (1974) and Leland (1974a) have questioned the adequacy of value maximization on the ground that shareholders may not unanimously support value maximizing investment decisions and in fact, Ekern and Wilson (1974) show in a mean-variance context that all shareholders would disprove of the investment decisions of a conventional value maximizing firm.

This seems rather confusing in view of proposition 1 above and we argue here that this disparity of results is partly due to an inadequate specification of price-taking (conventional value maximization) and partly to some confusion over what is really meant by value maximization. Proposition 1 only establishes that if firms know the individual implicit prices and behave as if (believe or perceive) these prices are constant, stock value maximization at these prices will lead to local efficiency. But this does not imply that efficient investment decisions will maximize actual equilibrium values in the stock market and it is equally clear that actual stock market value maximization may not be in the shareholders' interest. This is so because a change in investment will have two equilibrium effects: a wealth effect due to the change in the market value of the firm and a consumption effect due to the change in the implicit prices for state-contingent income. These two effects may act in the same or opposite directions and in the latter case it is a priori impossible to say which one will dominate. It is, however, quite possible for the relative importance of these effects to differ among shareholders in such a way that some shareholders are made better off by a change in a firm's investment and others are made worse off[1].

The analogy with conventional welfare theory is perfect in this case: in the absence of externalities, profit maximization by price-taking firms will lead to Pareto efficiency. It may, however, not be in the owners' interest to maximize actual equilibrium profits if the firm is able to manipulate prices, i.e., has some monopoly power, since this will affect the owners' budget sets in the capacity of consumers (pecuniary externalities).

Returning to the mean-variance example for a while, we see from (16) that with price-taking with respect to r and δ the firm will act as if $\partial\pi_s/\partial I_j = -p_s\, Z'_{js}(I_j)\ \delta/r$ and the economy will not be completely competitive in the Leland sense. The true effect on implicit equilibrium prices from a partial change in the investment in firm j is

$$\partial\pi_s/\partial I_j = p_s\{-Z'_{js}(I_j) + \mu'_j(I_j)\ [\sum_h \frac{1}{2\kappa^h} - \sum_k Z_{ks}(I_k)]\delta\}\ \delta/r$$

so that conventional value maximization does not imply that the firm is maximizing actual stock market value, that is, takes full account of the effect partial changes in the investment level will have on implicit prices. For the special case that consumers have constant absolute risk aversion utility functions $U^h(x) = -e^{-\beta^h x}$ and jointly normally distributed risky returns, we have that[2] $E(U^h) = -\exp\left(-\beta^h E(c^h) + (\beta^h)^2\ Var(c^h)/2\right)$. In that case it is easily seen that equilibrium valuation functions take the form

1) This point is also observed in Hart (1977).

2) See Stiglitz (1972a).

$$V_j = \frac{1}{r}\{\mu_j(I_j) - \frac{1}{\sum_h \frac{1}{\beta^h}} \sum_k \sigma_{jk}(I_j, I_k)\}$$

so that the risk discount δ depends solely on the individual absolute risk aversions and hence it is constant. In that particular case, price-taking with respect to r and δ would be consistent with actual value maximization implying that a conventional value maximizing firm would take full account of its monopoly power in the capital market. Implicit prices are in this case given by $\pi_s = p_s U^h(c_s^h)/\sum_s p_s U^h(c_s^h) r$ and efficiency requires that firms behave as if these prices were given.

We shall now examine in some more detail the effect on the welfare level of an individual initial shareholder from a partial change in the investment level of firm j. The budget constraint for an initial shareholder h is given by

$$(23) \qquad m^h + \sum_j \alpha_j^h V_j = \bar{m}^h + \sum_j \bar{\alpha}_j^h (V_j - I_j) + \bar{I}^h \quad (= w_o^h)$$

where $\alpha_j^h \equiv a_j^h/a_j$, and barred variables denote initial values. Differentiating (23) with respect to I_j yields

$$(24) \qquad \frac{\partial m^h}{\partial I_j} + \sum_k \frac{\partial \alpha_k^h}{\partial I_j} V_k + \sum_k \alpha_k^h \frac{\partial V_k}{\partial I_j} = \sum_k \bar{\alpha}_k^h \frac{\partial V_k}{\partial I_j} - \bar{\alpha}_j^h$$

Moreover, we have

$$\frac{\partial c_s^h}{\partial I_j} = r \frac{\partial m^h}{\partial I_j} + \sum_k \frac{\partial \alpha_k^h}{\partial I_j} Z_{ks}(I_k) + \alpha_j^h Z'_{js}(I_j)$$

and substituting for $\partial m^h/\partial I_j$ from (24) yields

$$(25) \qquad \frac{\partial c_s^h}{\partial I_j} = \sum_k (\bar{\alpha}_k^h - \alpha_k^h) \frac{\partial V_k}{\partial I_j} r + \sum_k (Z_{ks}(I_k) - rV_k) \frac{\partial \alpha_k^h}{\partial I_j} + \alpha_j^h Z'_{js}(I_j) - \bar{\alpha}_j^h r$$

Hence, we have

$$(26) \qquad \frac{\partial E^h(U^h)}{\partial I_j} = \sum_s p_s^h U_s^h \{\sum_k (\bar{\alpha}_k^h - \alpha_k^h) \frac{\partial V_k}{\partial I_j} r + \alpha_j^h Z'_{js}(I_j) - \bar{\alpha}_j^h r\}$$

since all terms involving $\partial \alpha_k^h/\partial I_j$ vanish from the first-order conditions (4) for shareholders' optima. Rewriting (26) gives

$$(27)\qquad \frac{\partial E^h(U^h)/\partial I_j}{E^h(U^h_s)\,r} = \sum_k (\bar{\alpha}^h_k - \alpha^h_k)\frac{\partial V_k}{\partial I_j} + \sum_s \pi^h_s \alpha^h_j Z'_{js}(I_j) - \bar{\alpha}^h_j$$

where the left-hand side of (27) can be interpreted as the change in the welfare level of shareholder h measured in terms of the numeraire good.

From (7) we have

$$(28)\qquad \frac{\partial V_k}{\partial I_j} = \begin{cases} \sum_s \frac{\partial \pi^h_s}{\partial I_j} Z_{ks}(I_k) & k \neq j \\ \sum_s \frac{\partial \pi^h_s}{\partial I_j} Z_{js}(I_j) + \sum_s \pi^h_s Z'_{js}(I_j), & k = j \end{cases}$$

and inserting (28) into (27) yields

$$(29)\qquad \frac{\partial E^h(U^h)/\partial I_j}{E^h(U^h_s)\,r} = \sum_k (\bar{\alpha}^h_k - \alpha^h_k)\sum_s \frac{\partial \pi^h_s}{\partial I_j} Z_{ks}(I_k) + \bar{\alpha}^h_j\left(\sum_s \pi^h_s Z'_{js}(I_j) - 1\right)$$

The first term on the right-hand side of (29) is clearly the consumption effect from a partial increase in investment in firm j while the second term is the wealth effect on the welfare level of shareholder h, by condition (8).

We shall now examine in this partial equilibrium context under what condition (i) shareholders will all agree on the desirability of a new investment in firm j, and (ii) shareholders will unanimously support that the firm maximizes its stock market value. In the previous literature on this subject one has distinguished between two unanimity concepts. Ekern and Wilson (1974) and Leland (1974a) have studied <u>ex post</u> unanimity which means that unanimity with respect to a particular investment plan occurs when shareholders are currently holding portfolios which are optimal given current investment plans, i.e., the impact on the welfare level of shareholder h is evaluated at $\alpha^h_j = \bar{\alpha}^h_j$ for all j. Radner (1974) has discussed unanimity in the <u>ex ante</u> sense which means that ex ante unanimity occurs when shareholders holding initial equilibrium portfolios not necessarily optimal given current plans, all agree to a particular investment plan.

From (29) the following proposition is obvious.

Proposition 2.[1]

Considering a partial change in the investment level of firm j, we have that

i) Ex post unanimity will obtain if at least one of the following two conditions is satisfied

a) the firm's technology satisfies the spanning property

b) all shareholders have identical beliefs and utility functions exhibiting the separation property.

ii) Ex ante unanimity obtains if we in addition to condition a) or b) make the assumption that firms are believed to be unable to affect implicit prices (completely competitive capital markets).

Thus, we see that if firms' investment decisions have no external effects on the attainable set of income patterns or if such external effects are equally evaluated by shareholders, all shareholders will agree on the desirability of a given investment plan in the ex post sense, and if the capital market is completely competitive, also in the ex ante sense.

The last term on the right-hand side of (29) can in view of (7) and (8) be written as $\bar{\alpha}_j^h \frac{\partial}{\partial I_j}(V_j - I_j)_{\pi^h = \text{constant}}$, that is, the share of the increase in stock market value of initial interests in firm j accruing to initial shareholder h computed at h's (given) implicit prices. Thus we have

Proposition 3.

If initial[2] shareholders believe that firm j is unable to affect implicit prices and take investment levels in firms other than j as given, each initial shareholder will support maximization of stock market values of initial interests at his own implicit prices. As implicit prices may differ among initial shareholders, they may, however, not agree on what investment decision which will maximize stock market value of initial interests in firm j unless spanning or separation obtains.

In a recent paper Hart (1977) has shown by a "limit argument" that if one lays

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1) ia) is essentially Ekern and Wilson's result on ex post unanimity while ii) is Radner's result on ex ante unanimity.

2) It is easy to show that unanimity among initial shareholders will imply unanimity among ex post shareholders as well. However, as noted by Radner (1974), ex post shareholders would wish the firm maximizes total stock market value V_j which follows from the fact that for an ex post shareholder the budget constraint takes the form $m^h + \sum_j \alpha_j^h V_j = \bar{m}^h + \sum_j \bar{\alpha}_j^h V_j$.

down sufficient conditions for implicit prices to be independent of a firm's investment decisions, then all shareholders will be unanimous over the choice of a firm's investment decision whether or not the spanning condition holds and for arbitrary consumer preferences. At first glance, Hart's result may seem to contradict the above propositions, but the following argument will indicate that this conclusion may be too rash. If firms are unable to affect implicit equilibrium prices in a stock market economy where the consumer optima are effectively constrained by the lack of complete markets, this must imply that firms' investment decisions have negligible effects on the attainable set of state-contingent income and hence, "in the limit" the externality effect will vanish and ex ante unanimity will obtain.

If we take into account the general equilibrium effects a change in the investment level of firm j may have on the investments in firms other than j, we get by a way of reasoning parallel to that underlying (29), that the general equilibrium effect on the welfare level of initial shareholder h takes the form

$$(30)\qquad \frac{\partial E^h(U^h)/\partial I_j}{E^h(U_s^h)r} = \sum_k(\bar{\alpha}_k^h - \alpha_k^h)\sum_s\sum_\ell \frac{\partial \pi_s^h}{\partial I_\ell} Z_{ks}(I_k)\frac{\partial I_\ell}{\partial I_j} + \sum_\ell \bar{\alpha}_\ell^h\left(\sum_s \pi_s^h Z'_{\ell s}(I_\ell)-1\right)\frac{\partial I_\ell}{\partial I_j}$$

where $\partial I_\ell/\partial I_j$ is the general equilibrium effect a change in the investment level of firm j has on the investment level of firm ℓ. Hence, we see that if firms $\ell \neq j$ for which $\partial I_\ell/\partial I_j \neq 0$ are assumed to be completely competitive and at the investment levels maximizing stock market values of their initial interests at the implicit prices of all shareholders, proposition 2 and 3 remain valid also when general equilibrium effects of changed investment are considered. If firms other than j remain at an investment level not satisfying condition (8) for maximizing competitive stock market values of initial interests, proposition 2 on unanimity on firm j's investment plans may not be generally true when general equilibrium effects are taken into account. Ex ante and ex post unanimity will however, still obtain in a general equilibrium context if shareholders have identical beliefs and preferences exhibiting separation. In that case implicit equilibrium prices are the same for all h and moreover, for any h the shares $\bar{\alpha}_\ell^h$ will be the same for all ℓ if the initial allocation represents a stock market equilibrium and hence the last term on the right-hand side of (30) will be of the same sign for all shareholders h. As for proposition 3 on the desirability of stock value maximization, if condition (8) is systematically not satisfied for all firms, it leaves open the possibility that even if shareholders believe that firms are unable to affect implicit prices, an initial shareholder h of firm j will think he will be better off by an investment decision decreasing the perceived market value of initial interests in firm j if the perceived market value of his total initial interests is thereby increased.[1)]

If the investment levels in firms other than j are different from those maximizing the stock market values of initial interests at given implicit prices, this means that the marginal social value of increasing investment in those firms is different from marginal social cost of investment (being equal to one). This will be in the nature of a pecuniary external effect in the sense that the change in the stock market value of initial interests in firm j will for given implicit prices not even in the spanning case capture the net social value (in terms of the numeraire) of increased investment because of the reallocation effects in the investment market. The problem is analogous to that under certainty when some firms systematically deviate from marginal cost pricing. In that case changes in competitive profits of a particular firm may not capture changes in social welfare because of possible reallocative effects on firms deviating from marginal cost pricing; or to put it in more general terms partial welfare analysis is generally justified only if the rest of the economy is optimally adjusted.

For economists approaching the literature on capital market theory from conventional market and welfare theory, the relationship between Pareto efficiency and unanimity may seem somewhat obscure. Recalling that we are only considering interior optima in the stock market, it goes almost without saying that unanimity on a firm's investment decision is a sufficient condition for that investment to represent an improvement in the Pareto sense. On the other hand, unanimity is clearly not necessary for Pareto efficiency. In particular, in the presence of externalities (nonspanning) shareholders may valuate investment returns

1) Hart (1977) shows that this possibility is eliminated in "large" economies if for each j $\bar{a}_j^h > 0$ implies $a_j^{h'} > 0$ for some h', i.e., there are shareholders who buy shares in each firm in which h has an initial shareholding.

differently at an equilibrium in the stock market (no separation) and hence there is no reason to expect everybody to agree on a Pareto efficient investment plan since the capital market will not provide the individual shareholder with sufficient information for figuring out the true social value of investment. In such cases Pareto optima may not be attainable through unanimous decisions. One might therefore cast some doubt on the normative significance of the unanimity criterion as an alternative to value maximization in incomplete markets. On the one hand, given that the rest of the economy is optimally adjusted, then, if unanimity obtains, all shareholders will also support competitive stock value maximization when properly defined, and stock value maximization in a (completely) competitive capital market will lead to (local) Pareto efficiency. On the other hand, when the competitive market mechanism based on stock value maximization is likely to be ill-defined in terms of market prices, shareholders will typically disagree on Pareto efficient investment plans so that the unanimity criterion fails whenever competitive stock value maximization fails in guiding the economy to a Pareto optimum. In fact, in the presence of externalities, a Pareto optimum may not be achieved through decentralized means - a lesson which is well known from conventional welfare theory.

The above comments on the relationship between unanimity and competitive value maximization is based on the assumption that the rest of the economy is optimally adjusted. If this is not so, there is no presumption that firms should maximize competitive market values. Indeed, from (30) we see that if separation obtains in completely competitive capital markets, we can easily imagine situations in which all shareholders would be unanimous both ex ante and ex post that firm j should decrease the stock market value of its initial interests as long as $\sum_{\ell}(\sum_{s} \pi_s^h Z'_{\ell s}(I_{\ell}) - 1) \frac{\partial I_{\ell}}{\partial I_j} > 0$ where of course $\sum_{\ell} \partial I_{\ell}/\partial I_j = 0$. In such a case we are in a second best context as in the conventional economy under certainty where some firms deviate systematically from marginal cost pricing (monopoly pricing, taxation, etc.), and as is well known from second best theory, in that case competitive profit maximization may not be optimal in the controllable part of the economy either.

As noted earlier, although all shareholders may agree that firms behaving as price-takers with respect to implicit prices should maximize stock market values, this does not mean that maximizing actual stock market values would be in the shareholders' interests. We shall finally examine this problem in some more detail, that is, under what conditions would maximization of actual stock market values be conduicive to maximal shareholders' welfare. To answer this question in full generality would require that we were able to calculate the exact effects on the equilibrium prices π^h from adopting a particular investment plan in a given firm. Generally, this would be a formidable task which is presently beyond our means (at least mine). We therefore pose a more limited and hopefully manageable question.

Suppose that by a suitable reallocation of the investment resources among its investment projects firm j can undertake a costless change in the return pattern per outstanding share as given by $dX_{j\cdot} = (dX_{j1},\dots,dX_{jS})$. Then we ask under what conditions this costless change in the returns per share will increase excess demand for shares in firm j if and only if this change is in the interest of its shareholders. Clearly, maximization of excess demand would be consistent with actual market value maximization under most reasonable share price adjustment processes.

To find the social value of the infinitesimal change $dX_{j\cdot}$ of the j-th row of the return matrix X we first define dm^h as the compensating variation in riskless income that would compensate shareholder h for the change $dX_{j\cdot}$. Up to a first-order approximation this is given by

$$\sum_s p_s^h U_s^h r dm^h + \sum_s p_s^h U_s^h a_{j\cdot}^h dX_{js} = 0$$

or

$$-dm^h = \sum_s a_{j\cdot}^h \pi_s^h dX_{js}$$

Clearly, $-dm^h$ is the value of the infinitesimal change $dX_{j\cdot}$ for consumer h evaluated in terms of the riskless asset (numeraire). Total social value of that change is then given by

$$\text{(31)} \qquad \sum_h -dm^h = \sum_{hs}\sum a_{j\cdot}^h \pi_s^h dX_{js}$$

As $a_{j\cdot}^h$ and X_{js} enter the relation defining c_s^h in a bilinear form, the feasible set in the income space will be non-convex. Hence, (31) is a local measure of the social desirability of the change $dX_{j\cdot}$. That means that the absence of costless variations $dX_{j\cdot}$ generating positive social value as given by (31), is necessary and sufficient for local efficiency (excluding again the possibility of saddlepoints).

Our problem is therefore to explore under what conditions we have

$$(32) \qquad \sum_h\sum_s a_j^h \pi_s^h dX_{js} > 0 \quad \iff \quad \sum_h\sum_s \frac{\partial a_j^h}{\partial X_{js}} dX_{js} > 0$$

In order to examine the effect on excess demand from the infinitesimal variation dX_j we assume in addition to previous assumptions that shareholders have initial endowments of state-contingent income so that consumer h's endowments are given by the (S+1)-dimensional vector $w^h = (w_o^h, w_1^h, \ldots, w_S^h)'$ and taking the budget constraint into account, consumer h's income in state s is then given by

$$c_s^h = rw_o^h - r\sum_j a_j^h v_j + w_s^h + \sum_j a_j^h X_{js}$$

This does not change first-order conditions for interior consumer optima as given by (4') and restated below for the convenience of the reader.

$$(4') \qquad \frac{\partial E^h(U^h)}{\partial a^h} = (-rv, X)\underline{u}^h = 0 \qquad \text{(a column vector).}$$

Total differentiation of (4') with respect to da^h and $(-rdv_j, dX_j)$ yields[1)]

$$0 = \frac{\partial^2 E^h(U^h)}{\partial a^h(\partial a^h)'} da^h + \frac{\partial^2 E^h(U^h)}{\partial a^h(-r\partial v_j, \partial X_j)} (-rdv_j, dX_j)'$$

$$= (-rv, X)\underline{U}^h(-rv, X)'da^h + \{a_j^h(-rv, X)\underline{U}^h + \delta_j(\underline{u}^h)'\}(-rdv_j, dX_j)'$$

where δ_j is the Kronecker delta vector with the j-th component equal to one and all other components equal to zero. Rearranging gives

$$(33) \qquad \frac{\partial a^h}{(-r\partial v_j, \partial X_j)} = -\{(-rv, X)\underline{U}^h(-rv, X)'\}^{-1}\{a_j^h(-rv, X)\underline{U}^h + \delta_j(\underline{u}^h)'\}$$

Total differentiation of (4') with respect to da^h and dw^h gives

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1) For the matrix X the row vector X_j is understood to be the j-th row of X, X^k is its k-th column and X_{jk}^j its jk-entry.

$$0 = \frac{\partial^2 E^h(U^h)}{\partial a^h(\partial a^h)'}\ da^h + \frac{\partial^2 E^h(U^h)}{\partial a^h(\partial w^h)'}\ dw^h = (-rv,X)\underline{U}^h(-rv,X)'da^h +$$

$$+(-rv,X)\underline{U}^h(-rdw_o^h,dw_1^h,\ldots,dw_S^h)'$$

and rearranging yields

$$(34)\qquad \frac{\partial a^h}{(r\partial w_o^h,\ldots,\partial w_S^h)} = -\{(-rv,X)\underline{U}^h(-rv,X)'\}^{-1}(-rv,X)\underline{U}^h$$

Combining (33) and (34) we have

$$(35)\qquad \frac{\partial a^h}{(-r\partial v_j,\partial X_j)} - a_j^h\frac{\partial a^h}{(r\partial w_o^h,\ldots,\partial w_S^h)} = -\{(-rv,X)\underline{U}^h(-rv,X)'\}^{-1}\delta_j(\underline{u}^h)'$$

$$= -\{\left(\frac{\partial^2 E^h(U^h)}{\partial a^h(\partial a^h)'}\right)^{-1}\}^j(\underline{u}^h)'$$

(where the superscript j denotes the j-th column of the matrix)

This gives the ordinary Slutsky matrix for asset demand functions

$$(36)\qquad \frac{\partial a^h}{\partial v'} + (a^h)'\frac{\partial a^h}{\partial w_o^h} = \frac{\partial^2 E^h(U^h)}{\partial a^h(\partial a^h)'}\ \underline{u}_o^h r$$

with negative diagonal entries from the negative definiteness of the matrix of second-order derivatives as given by (5).

Looking at a particular element of (35) and using (36) we have

$$(37)\qquad \frac{\partial a_k^h}{\partial X_{js}} = a_j^h\frac{\partial a_k^h}{\partial w_s^h} - \{((-rv,X)\underline{U}^h(-rv,X)')^{-1}\}_{kj}\ \underline{u}_s^h$$

$$= a_j^h\frac{\partial a_k^h}{\partial w_s^h} - \left[\frac{\partial a_k^h}{\partial v_j} + a_j^h\frac{\partial a_k^h}{\partial w_o^h}\right]\pi_s^h$$

where $\pi_s^h \equiv \dfrac{p_s^h U_s^h}{\sum_s p_s^h U_s^h r} \equiv \underline{u}_s^h / \underline{u}_o^h r$ by previous definitions.

Thus, the total demand effect from the change of returns per share in firm j can be split up into a wealth effect being proportional to the effect of an increase in initial endowments of state-contingent income and a substitution effect being proportional to the net substitution effect of a decrease in the price per share in firm j.[1)]

It will be convenient to rewrite (37) in the form

$$(38)\qquad \frac{\partial a_k^h}{\partial X_{js}} = a_j^h \left(\frac{\partial a_k^h}{\partial w_s^h} - \frac{\partial a_k^h}{\partial w_o^h} \pi_s^h \right) - \frac{\partial a_k^h}{\partial v_j} \pi_s^h$$

where the first term on the right-hand side of (38) will be referred to as the general wealth effect on demand for shares in firm k from increasing returns per share in firm j in state s.

We define $\eta_j^h \equiv (\partial a_j^h/\partial v_j)(v_j/a_j^h)$ as the uncompensated price elasticity of demand for shares in firm j for shareholder h. Clearly, we have that the price elasticity of total demand for shares in firm j is given by $\eta_j = \sum_h a_j^h \eta_j^h / \sum_h a_j^h$.

We find the effect on aggregate excess demand for shares in firm j from the infinitesimal variation dX_j by aggregating (38) over h for k=j which after some simple manipulations gives

$$(39)\qquad \sum_{hs}\sum \frac{\partial a_j^h}{\partial X_{js}} dX_{js} = \sum_h a_j^h \sum_s \left(\frac{\partial a_j^h}{\partial w_s^h} - \frac{\partial a_j^h}{\partial w_o^h} \pi_s^h \right) dX_{js} - \frac{\eta_j}{v_j} \sum_h a_j^h \sum_s \pi_s^h dX_{js}$$

$$- \frac{1}{v_j} \sum_h a_j^h (\eta_j^h - \eta_j) \sum_s \pi_s^h dX_{js}$$

The third term on the right-hand side of (39) can be interpreted as a weighted sum (with weights $\frac{a_j}{v_j} dX_{js}$) of the covariances over consumers between the implicit prices π_s^h and the uncompensated price elasticities of demand η_j^h.

1) An analogous expression for the effect of a quality change on the consumer demand for commodities is derived in Drèze and Hagen (1978).

Comparing (39) with (32) we have the following result:

Proposition 4:

Given that shares in firm j are normal assets ($\eta_j < 0$), a sufficient condition for infinitesimal variations dX_j to generate positive excess demand for shares in firm j if and only if these variations are locally consistent with the interests of the shareholders, is that the general wealth effect associated with dX_j be zero and that the individual implicit prices be uncorrelated with price elasticities of individual demands.

We note that the covariance term will vanish at a stock market equilibrium if the firm's technology satisfies the spanning property or if the shareholders have identical beliefs and utility functions exhibiting separation. In case of spanning there must exist a non-zero vector $\alpha = (\alpha_1, \ldots, \alpha_J)'$ such that $dX_j = \alpha' X$ and hence $\sum_s \pi_s^h dX_{js} = \sum_s \pi_s^h \sum_k \alpha_k X_{ks} = \alpha' v$ and the result is obvious and in the latter case it follows from the observation that individual implicit prices must be the same for all shareholders at a stock market equilibrium.

This suggests that the inefficiency of actual market value maximization due to the external effects of firms' investment under uncertainty is reflected in the covariance term while the inefficiency caused by the effect on implicit prices is reflected in the general wealth effect.

In case of spanning the general wealth effect takes a particular simple form. To see this, we note that

$$(40) \qquad a_j^h \sum_s \left(\frac{\partial a_j^h}{\partial w_s^h} - \frac{\partial a_j^h}{\partial w_o^h} \pi_s^h \right) dX_{js} = a_j^h \frac{\partial a_j^h}{(r\partial w_o^h, \partial w_1^h, \ldots, \partial w_S^h)} (-r\pi, I)' dX_j' \quad ^{1)}$$

Substituting (40) into (34) using $dX_j' = X'\alpha$ by assumption and $(-r\pi, I)'X' = (-rv, X)'$ by (6), we have that

1) $(-r\pi, I)$ is the (SxS) identity matrix augmented by the vector $-r\pi$ as the first column.

$$(41)\qquad a_j^h \sum_s \left(\frac{\partial a_j^h}{\partial w_s^h} - \frac{\partial a_j^h}{\partial w_o^h} \pi_s^h \right) dX_{js} = -a_j^h \{(-rv,X)\underline{U}^h(-rv,X)'\}_j^{-1} x$$

$$x\{(-rv,X)\underline{U}^h(-rv,X)'\}\alpha = -a_j^h \alpha_j$$

and the general wealth effect vanishes if and only if $\alpha_j = 0$. Thus we have.

Proposition 5.

Under spanning and given that $\eta_j < 0$, a necessary and sufficient condition for maximization of excess demand for firm j's shares to be locally consistent with the shareholders' interests is that dX_j is contained in the row space of X other than row j.

Under spanning all shareholders will agree in the ex post sense on the desirability of a firm's investment plan and if they perceive that implicit prices do not change they will all support stock value maximization. From proposition 5 it follows, however, that actual value maximization will not be in the shareholders' interests unless the marginal return pattern associated with the investment plan can be obtained as a linear combination of existing shares in the market not involving the shares of the firm contemplating investment. In that case firm j's monopoly power in the stock market washes out as there exists a perfect substitute in the market for the marginal return pattern dX_j consisting of a linear combination of shares in firms other than firm j.

VI. Concluding remarks.

In the present paper we have explored the efficiency implications of value-related investment criteria for decentralized investment decisions under technological uncertainty and the general conclusion is negative in the sense that except for some special cases, the market allocation of investment effected through decentralized and incomplete capital markets is likely to be inefficient. This is so because investment decisions under uncertainty will generally have external effects for which there do not exist well-defined stock market prices and hence, there does not in the general case exist any clear or well-defined definition of competitive behaviour based on observable market characteristics in which case the competitive benchmark with respect to investments does not seem to be meaningful. Indeed, in the general case the choice of a Pareto efficient investment plan in a given firm requires explicit knowledge of consumers' preferences as one would expect in cases where firms' production plans have external effects not fully reflected through competitive profits at market prices.

Under certain conditions as to market structure (complete set of markets) or on the stochastic technologies (spanning) the external effects associated with investment under uncertainty will vanish in which case all welfare-relevant consequences of an investment plan will be fully priced in the stock market and changes in stock market values at given stock prices will capture the social value of the plan. On the other hand, if preferences satisfy the separation property and shareholders have identical beliefs, all shareholders will evaluate the external effects associated with any given investment plan in the same way, and hence they will all agree on the social value of any given investment plan and the Pareto efficient plan could be chosen by a unanimous vote. Alternatively, the firm could choose the Pareto efficient plan through relying on the preferences of an arbitrary shareholder and formally the efficient plan would maximize the firm's stock market value of initial interests at the shareholder's given individual implicit prices for state-contingent income (being the same for all shareholders in that case).

It follows trivially that when decentralized value maximization by competitive firms is well-defined and leads to Pareto efficiency, value maximizing investment decisions will be unanimously approved by all shareholders. The converse conclusion does, however, not follow, that is a Pareto efficient allocation may not be obtained through the decentralized market mechanism with market value maximizing firms as there will in general be externalities associated with firms' investments in incomplete markets. In such cases the market will not provide adequate price and value signals for shareholders to compute the social value of investment and hence shareholders may not unanimously support Pareto efficient investment plans because of lack of relevant information except for the case with preferences exhibiting separation and with identical beliefs in which all will agree.

Most of the examples given in the literature assume implicitly away the price information problem in that they are within the mean-variance context and hence consumers' preferences exhibit separation. It is therefore argued here that the controversy between value maximization and shareholders' unanimity is in these cases caused by an inadequate specification of competitive behaviour in incomplete markets.

In the first place the conventional value maximizing rule as specified by i.a. Stiglitz (1972a), Jensen & Long (1972) and Fama (1972) defines price-taking with respect to the "wrong" prices as it implies a downward sloped demand schedule for shares which are the true objects of choice in incomplete markets. It is therefore not surprising that the market outcome in that case turns out to be inefficient and in such cases value maximization would clearly not be in the interest of all shareholders. Second, one must distinguish between value maximization in the perceived and in the actual sense. True competitive behaviour requires that producers do not take into account any relationships between prices and their own production decisions, i.e. , perceived value maximization. This is a well-known prerequisite for efficiency of market behaviour in general and should not be mixed up with the empirical question of whether or not firms <u>are</u> able to influence prices in the stock markets and <u>do</u> take this into account.

Some authors[1)] have stressed the fact that in the case firms' return patterns are linearly independent, each firm would have a monopoly on its own return pattern and one would expect this to affect firms' market behaviour. It has been verified here, however, that in case of spanning any monopoly power firm j may have will vanish if the returns on new investment is contained in the space spanned by the returns of firms other than j. In case of separation only aggregate returns in each state would matter for implicit prices so that a firm's monopoly power would in that case depend on its marginal contribution to aggregate returns in each state. In all other cases firms' investment under uncertainty generates external effects pointing to likely inefficiency of decentralized market behaviour. As the market in such cases will not provide adequate information for a proper evaluation of the social desirability of firms' investment plans, there does in such cases not exist any market alternative for aggregation of consumer preferences. In this respect, the allocative problems caused by the external effects of investment are in a fundamental way different from those caused by firms' ability to affect market prices which is clearly in the nature of pecuniary external effects.

1) See i.a. Mossin (1973).

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ON THE OPTIMALITY OF THE COMPETITIVE MARKET SYSTEM IN AN ECONOMY WITH PRODUCT DIFFERENTIATION*

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Summary

This paper deals with an economy where firms can vary the quality design of their products. By postulating that consumers have preference orderings over quality characteristics, Pareto optimal rules for production, distribution and quality design are derived. We then examine the price implications of a Pareto optimal range of quality designs and it is shown that, in general, commodity prices will not provide profit maximizing firms with the correct information for Pareto-optimal choices of quality design.

I. Introduction

The efficiency achieved in the allocation of resources through competitive markets is perhaps one of the most powerful results in economics.[1] For this reason a system of decentralized decision-making through competitive markets is often considered to be an ideal way of organizing economic acitivty.

Despite the completeness and intellectual beauty of competitive market theory, it certainly leaves many market phenomena of the real world unexplained. One of the most striking features of modern societies is the huge number of varieties of each basic commodity traded in the market. Traditionally, product differentiation has been regarded as one of the characteristics of monopolistic competition and consequently, as being outside the realm of the competitive model. Furthermore, since any deviation from competitive behavior will be detrimental from an efficiency point of view, one might be tempted to view product differentiation as an obstacle to efficiency. This line of reasoning, however, may be false. The notion of efficiency as used in economic theory is merely concerned with production and distribution of a given list of goods. However, when we allow for a large number of quality variants of each basic commodity, a qualitative dimension is introduced into the efficiency concept. In addition to whether the economy produces and distrib-

* The author is indebted to Agnar Sandmo for many useful comments.

[1] For an excellent survey of the competitive market model, see Essay I in Koopmans [2].

utes a given commodity in optimal amounts, the question may be raised as to whether the economy produces the right quality variants of each basic commodity. The latter aspect of efficiency may be called qualitative efficiency.

In this paper, we try to introduce possibilities for product differentiation in a competitive setting. In particular, we look for the existence of efficiency prices in such an economy. The concept of efficiency prices means that if consumers maximize utilities and firms maximize profits, taking these prices as given, the resulting equilibrium production and distribution of commodities will be efficient from a quantitative as well as a qualitative point of view.

Formally, product differentiation could be introduced into the market model by considering each possible quality variety of a given good as a separate commodity and consumer preferences could be defined over all conceivable varieties, including quality variants which are technically feasible but not yet present in the market. However, it might be difficult to conceive of consumers ranking both existing and possible, but non-existing, goods. Also, such a model could hardly explain why firms differentiate their products.

A more natural approach to the problem of quality choice, based on Lancaster [3] and [4], would be to postulate that the ultimate objects of utility are not commodities, but rather the characteristics or properties which commodities stand for in the eyes of the consumer. Consequently, a commodity can be represented by a set of quality indicators, one for each characteristic. A given commodity can thus be considered as a package of quality characteristics. Consumer preferences are defined over quality characteristics which will induce a preference ordering over the commodity space.

This is the approach used here. The main advantage of this approach in the present context is perhaps that it enables us to give a precise and operational definition of quality change of a given good.[1] A change in quality can be defined as a change in the amount of characteristics provided by a unit of the good. Product differentiation possibilities exist for a given firm if the firm can change the amount of characteristics contained in a unit of its product.

In this paper we focus particular attention on the qualitative aspects of efficiency, i.e. determination of criteria for a socially optimal degree of product differentiation and the price implications of an optimal quality pattern.

II. Exchange of Quality Characteristics through Competitive Commodity Markets

It is assumed that each commodity is defined by a finite number, s, of quality characteristics and, given the unit of measurement, these characteristics are perceived identically by all consumers. That is, the quality of a commodity is assumed to be of an objective nature and known to all consumers.

[1] A similar approach was also applied by Griliches [1] in an attempt to incorporate quality changes in price indexes.

Quality characteristics, rather than the commodity itself, are assumed to be the ultimate objects of consumer satisfaction. Consequently, consumer preferences are basically defined on the space generated by the quality characteristics, which will be called the quality space or for short, the Q-space.

If there were a market for each quality characteristic, which implies that each characteristic could be traded separately, this exchange economy would simply be a trivial reinterpretation of the traditional exchange economy where commodities are reinterpreted as characteristics. The set of feasible trades in such an economy, i.e. the set of obtainable quality vectors, to be denoted Ω, would be an s-dimensional rectangle in the s-dimensional Euclidian space bounded by the total amount of each characteristic available. Exchange equilibrium would be characterized by the marginal rates of substitution between any pair of characteristics being equal for all consumers and the equilibrium exchange would of course be efficient in the Pareto sense.

However, quality characteristics will generally not be available in pure form. They must be acquired through commodity markets where a particular commodity represents a package of quality characteristics in fixed proportions. Since, for the moment, we are studying an exchange economy, the number of commodities and the total amount of each commodity are fixed. Each commodity will be defined by a quality vector where the components are the amount of the various quality characteristics contained in a unit of that commodity.

Notations and assumptions:

$\bar{q}^i$ $=(q_1^i, ..., q_s^i)$ is the vector of characteristics purchased by consumer i, $i=1, ..., m$. All characteristics are assumed to be measured in non-negative terms. All superscripts in this paper refer to consumers and barred variables denote vectors.

$U^i(\bar{q}^i)$ = the utility function of consumer i defined on the Q-space. The functions $U^i(\cdot)$ are assumed to be strictly quasi-concave with strictly positive partial first-order derivatives. That is, we assume no satiation and that every quality characteristic is desirable.

$\bar{q}_c$ $=(q_{c1}, ..., q_{cs})$ is the vector in the Q-space representing a unit of commodity c, $c=1, ..., n$. All vectors $\bar{q}_c$ are assumed to be linearly independent, which means that there can at most be as many commodities as characteristics.

p_c = the price of a unit of commodity c, $c=1, ..., n$.

$\bar{x}^i$ $=(x_1^i, ..., x_n^i)$ is the commodity vector purchased by consumer i. It will generally be assumed that $\bar{x}^i \geqslant \bar{0}$ for all i.

$\bar{y}^i$ $=(y_1^i, ..., y_n^i)$ is the vector of initial holdings of commodities held by consumer i.

Assuming that quality vectors can be added linearly, the vectors of quality characteristics obtainable for consumer i in commodity markets are given by

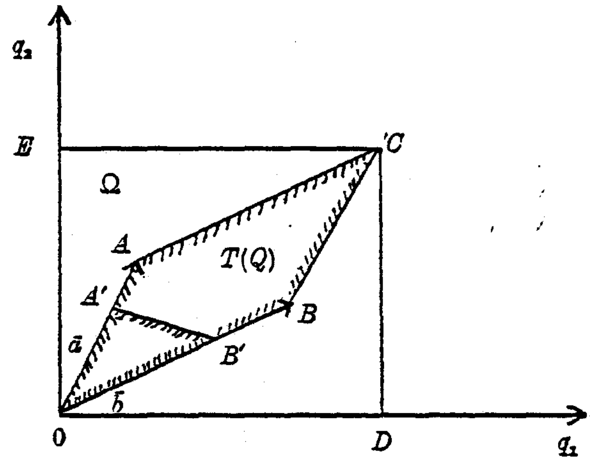

Fig. 1

$$\bar{q}^i = \sum_{c=1}^{n} x_c^i \bar{q}_c = \bar{x}^i Q$$

where Q is an $(n \times s)$ matrix where the rows are the n quality vectors. Consequently, the mapping from the commodity space (C-space) to the Q-space is linear. The matrix Q is not indexed, which indicates the unanimity of quality judgements. We let $T(Q)$ denote the n-dimensional subset of the n-dimensional polyhedral cone spanned by the row vectors of Q where the coefficients are restricted by the availability constraints $x_c^i \leqslant \sum_{i=1}^{m} y_c^i$, $c = 1, ..., n$. $T(Q)$ is clearly the set of quality vectors which can be obtained by trading in commodity markets and it is denoted the feasible trading set. Since we have assumed that no commodity can be purchased in negative quantities, the feasible trading set in the commodity markets will be a proper subset of the set of feasible trades obtainable with a complete set of quality markets (Ω). This applies unless the set of row vectors of the Q-matrix contains a basis for the Q-space of the form $(\lambda_1 \bar{e}_1, ..., \lambda_s \bar{e}_s)$, where $\bar{e}_j$ is the jth unit vector, which is a trivial case because each characteristic would then be available in pure form and the economy would be equivalent to the exchange of characteristics through a complete set of quality markets, one for each characteristic. Therefore, it is generally assumed that $T(Q)$ is a proper subset of Ω.

The situation is depicted in Fig. 1 where we have assumed two characteristics, q_1 and q_2, and two commodities whose units are represented by the vectors $\bar{a}$ and $\bar{b}$. The vectors A and B represent total amounts of commodity a and b, respectively.

If each quality component could be traded in separate markets, the feasible trading set would be given by the rectangle $ODCE$. However, when exchange of quality characteristics is restricted to taking place by means of an exchange of non-negative amounts of the two commodities, the feasible trading set is given by the shaded area $OBCA$.

For given commodity prices, the obtainable set of quality vectors for each individual, i.e. his opportunity set, is determined by his budget constraint $\sum_{c=1}^{n} p_c(x_c^i - y_c^i) = 0$. It is easy to show that the individual opportunity sets in the Q-space are closed, bounded and convex subsets of $T(Q)$. In particular, if there are at least as many characteristics as commodities, the budget plane in commodity space is transformed into an n-dimensional hyperplane in the Q-space. In Fig. 1, the individual opportunity set may be represented by the area $OB'A'$.

The preference ordering defined on the Q-space induces a preference ordering over C-space. This relationship is given by $U^i(\bar{x}^i Q) \equiv F^i(\bar{x}^i)$ where $F^i(\cdot)$ may be interpreted as the induced utility function on C-space for the ith individual. It is trivial to verify that if $U^i(\cdot)$ is strictly quasi-concave on the Q-space, then the induced function $F^i(\cdot)$ will be the same on the C-space, so that indifference surfaces on the C-space will have the usual convexity properties.

Equilibrium in commodity markets is determined by utility maximizing:

$$\left.\begin{array}{l} \text{Max } U^i(\bar{x}^i Q) \\ \text{subject to} \\ \sum_{c=1}^{n} p_c(x_c^i - y_c^i) = 0 \end{array}\right\} \quad i = 1, \ldots, m$$

Market clearing:

$$\sum_{i=1}^{m} (x_c^i - y_c^i) = 0 \quad c = 1, \ldots, n$$

Necessary conditions for maximum are given by[1]

$$\sum_{j=1}^{s} U_j^i(\bar{x}^i Q)\, q_{cj} = \lambda^i p_c \quad c = 1, \ldots, n \tag{1}$$

where $U_j^i(\cdot)$ denotes the partial derivative of $U^i(\bar{x}^i Q)$ with respect to the jth argument and the Lagrangean multiplier λ^i has the usual interpretation of marginal utility of initial wealth. (1) may be rewritten as:

$$\frac{\sum_{j=1}^{s} U_j^i(\bar{x}^i Q)\, q_{cj}}{\sum_{j=1}^{s} U_j^i(\bar{x}^i Q)\, q_{dj}} = \frac{p_c}{p_d} \quad i = 1, \ldots, m \tag{2}$$

[1] If there are more commodities than characteristics, $n > s$, the system of first-order conditions would be over-determined and for a solution to exist, $n-s$ of the first-order conditions must be expressible as linear combinations of the remaining ones. This implies that $n-s$ of the prices must be linear combinations of the s prices corresponding to the maximum number of linearly independent quality vectors.

The left-hand side of (2) can be interpreted as the marginal rate of substitution between commodity (quality vector) c and d and in equilibrium this is equal to the relative prices for all consumers. Hence, the equilibrium exchange of *commodities* is Pareto optimal. It can easily be seen that condition (2) will not necessarily imply equalization of marginal rates of substitution between quality characteristics over consumers, so that the allocation of quality characteristics effected through commodity markets will not be Pareto optimal in general. In this sense the competitive allocation of characteristics in commodity markets will be a constrained optimum or some kind of second-best allocation relative to the commodity market structure.

Rewriting condition (1) as

$$\frac{1}{\lambda^i}\sum_{j=1}^{s} U_j^i(\vec{x}^i Q)\, q_{cj} \equiv \sum_{j=1}^{s} \pi_j^i q_{cj} = p_c, \quad c=1,\ldots,n \tag{3}$$

the factor $\pi_j^i \equiv (1/\lambda^i)\, U_j^i(\vec{x}^i Q)$ may be interpreted as the implicit price per unit of characteristic j for consumer i evaluated at equilibrium in the commodity markets, i.e. π_j^i is the implicit price consumer i is willing to pay for a unit of characteristic j. Since the equilibrium exchange of commodities generally represents a constrained Pareto optimum in the Q-space, the implicit prices for characteristics will differ for different consumers.

However, if there are as many commodities as characteristics, $n=s$, and the purchase of commodities in negative quantities is permitted, the equilibrium exchange of commodities will represent an unconstrained Pareto-optimal exchange of characteristics since the set of commodities (quality vectors) will span the s-dimensional Q-space. In this case the implicit prices for characteristics, at an equilibrium in the commodity markets, would be the same for all consumers and could in principle be computed from (3) as $\vec{\pi}' = Q^{-1}\vec{p}'$ where[1] $\vec{\pi} = (\pi_1^i, \ldots, \pi_s^i)$ (the same for all i) and $\vec{p} = (p_1, \ldots, p_n)$.

In general, however, an equilibrium exchange of commodities represents a constrained Pareto optimum in the Q-space, in which case the implicit price vectors for characteristics will be in a subspace of dimension $s-n$. This implies that there will not be any one-to-one relationship between implicit prices for characteristics and equilibrium commodity prices.

As separate markets for each quality characteristic are not likely to exist in any economy, the equalization of marginal rates of substitution between quality characteristics is not a particularly useful criterion for exchange efficiency in commodity markets. In the analysis below, the concept of efficiency in commodity markets refers to efficiency in the sense of condition (2) since the unconstrained Pareto-optimal allocation is generally not obtainable under a commodity market structure.

[1] Prime denotes transposition.

III. Allocation of Resources and Commodities in a Production Economy with Fixed Product Differentiation

Each firm is assumed to produce a single commodity (quality vector) with a single input common to all firms. Thus, the firms' decisions on product differentiation are suppressed for the time being. All firms behave as price-takers in the commodity and input markets. Firm c produces commodity c, a unit of which is defined by the vector $\bar{q}_c = (q_{c1}, ..., q_{cs})$, $c = 1, ..., n$.

k_c = input of the common factor of production in the cth firm,
k_0 = total endowment of factor k,
r = price for a unit of factor k,
z_c = amount of commodity c produced by firm c, and
$z_c = \psi_c(k_c)$ is the production function of firm c. It is assumed that $\psi_c'(k_c) > 0$, $\psi_c''(k_c) < 0$.

Equilibrium in commodity and factor markets is determined by utility maximizing on the consumer side and profit maximizing by firms. Utility maximizing implies

$$\frac{\sum_{j=1}^{s} U_j^i(\bar{x}^i Q)\, q_{cj}}{\sum_{j=1}^{s} U_j^i(\bar{x}^i Q)\, q_{dj}} = \frac{p_c}{p_d} \quad i = 1, \ldots, m$$

Profit maximizing implies

$\psi_c'(k_c) p_c = r$ or $p_c = r/\psi_c'(k_c)$, $\quad c = 1, ..., n$

Finally, commodity and factor markets must clear:

$$\sum_{c=1}^{n} k_c = k_0 \quad \text{and} \quad \sum_{i=1}^{m} x_c^i = \psi_c(k_c) \quad c = 1, \ldots, n$$

By combining the maximum conditions above we get the competitive allocation rule

$$\frac{\sum_{j=1}^{s} U_j^i(\bar{x}^i Q)\, q_{cj}}{\sum_{j=1}^{s} U_j^i(\bar{x}^i Q)\, q_{dj}} = \frac{\psi_d'(k_d)}{\psi_c'(k_c)} \quad i = 1, \ldots, m \tag{4}$$

The left-hand side of (4) is consumer i's marginal rate of substitution between commodity c and d, while the right-hand side may be interpreted as the marginal rate of transformation in the production of commodity c and d. Hence, for fixed product differentiation, the competitive allocation rule is Pareto-optimal since the marginal rates of substitution between any pair of commodities are equalized for all consumers and equated to the marginal rates of transformation in production. Again, this result should be rather trivial, because when we restrict the efficiency concept to the commodity space, we are left with the standard competitive model.

IV. Optimal Allocation of Resources and Commodities in a Production Economy with Possibilities for Product Differentiation

We now make the rather realistic assumption that, within limits set by its technology, a firm can produce a certain range of quality vectors. The set of producible quality vectors for a particular firm is called the firm's production set. All producible quality vectors of a given firm are assumed to form a convex polyhedral cone in the Q-space. For fixed factor input, the maximum amount which the firm can produce of the various quality vectors in its production set is assumed to be given by a concave boundary surface in the Q-space. The boundary surface of a particular firm will indicate the substitution possibilities between the various quality characteristics producible in that firm with a given input level. We call this the firm's transformation surface. It is assumed that the transformation surface in firm c is given by the implicit function

$$h_c(z_c\bar{q}_c) = 0, \; z_c \geqslant 0, \; \bar{q}_c \in C_c, \quad c = 1, ..., n$$

where C_c is the convex polyhedral cone representing firm c's production set. Moreover, we make the simplifying assumption that the boundary function $h_c(z_c\bar{q}_c)$ is a homothetic function so that all information about the substitution possibilities is contained in the unit surface $h_c(\bar{q}_c) = 0$. This means that the transformation surfaces corresponding to different production scales will be radial expansions of the unit surface, i.e. the substitution possibilities will be independent of the level of input. This seemingly rigid assumption is motivated by the fact that it enables us to separate decisions as to production scale and product differentiation. Needless to say, these decisions will generally overlap. However, relaxing the assumption of homotheticity of the boundary functions $h_c(\cdot)$ would complicate the formulas without changing the qualitative structure of the results.

Since each commodity has a positive price, all efficient production choices must be located on the boundary surface for a profit-maximizing firm. To keep things simple, we retain the assumption that for technological reasons, each firm can only produce one commodity. This means that each firm can select only one vector from its production set. Thus, we have substitution possibilities in each firm *ex ante* but no substitution *ex post*. It is also assumed that any quality vector in the production set of a particular firm is produced

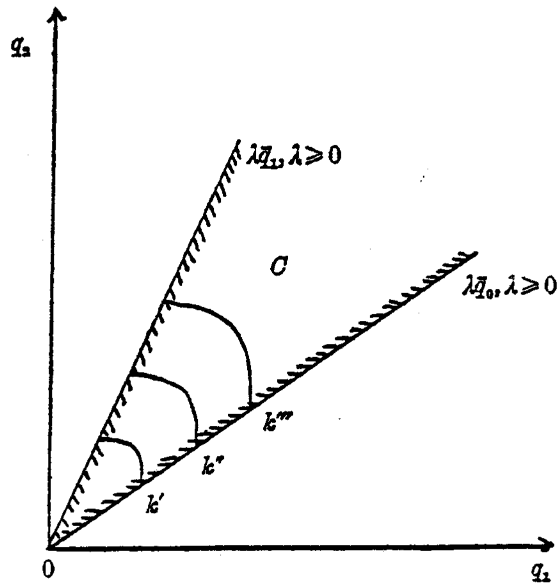

Fig. 2

on the basis of the same production function,[1] i.e. $z_c = \psi_c(k_c)$, where z_c is the amount produced of quality vector $q_c \in C_c$.

A typical production set is depicted in Fig. 2 for the case of a two-dimensional Q-space. We have drawn three different transformation curves, $h(\psi(k)\bar{q}) = 0$, corresponding to factor inputs $k''' > k'' > k'$. These curves are characterized by the fact that the derivative dq_2/dq_1 is constant along any quality ray in the cone C.

In order to avoid the problem of corner solutions, it is assumed that quality vectors on a facet of the convex polyhedral cones C_c will never be optimal. In the example in Fig. 2, this can be ensured by imposing that $dq_2/dq_1 = -\infty$ along the ray $\lambda\bar{q}_0$ and $dq_2/dq_1 = 0$ along $\lambda\bar{q}_1$.

Efficiency in an economy such as this raises two problems: (i) How should an optimal quality vector in each production set be selected? (ii) What are the optimal production and distribution of the set of quality vectors (commodities) selected under (i).

(ii) is the traditional efficiency problem in a production economy. Problem (i), however, introduces a qualitative aspect into the efficiency concept. Since each firm has, within certain limits, an *ex ante* choice between producing different quality variants of its product, the question arises as to how an optimal pattern of product differentiation should be determined for the whole economy.

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[1] This implies that variations in the quality design of the product do not require resources.

Pareto-optimal rules for production, distribution and product differentiation can be obtained by solving the following constrained maximum problem:

$$\text{Max} \sum_{i=1}^{m} \tau^i U^i(\bar{x}^i Q) \quad (\text{where } \tau^i, i=1,\dots,m, \text{ are arbitrary positive weights})$$

subject to

$\sum_{i=1}^{m} x_c^i = \psi_c(k_c), c=1,\dots,n$ (commodity availability constraints),

$\sum_{c=1}^{n} k_c = k_0$ (input availability constraint),

$h_c(\bar{q}_c) = 0, c=1,\dots,n$ (quality constraints).

Forming the Lagrangean, we have

$$L = \sum_{i=1}^{m} \tau^i U^i(\bar{x}^i Q) - \sum_{c=1}^{n} \mu_c \left(\sum_{i=1}^{m} x_c^i - \psi_c(k_c)\right) - \varrho\left(\sum_{c=1}^{n} k_c - k_0\right) - \sum_{c=1}^{n} \nu_c h_c(\bar{q}_c)$$

and the first-order conditions for an optimum are given by[1]

$$\tau^i \sum_{j=1}^{s} U_j^i(\bar{x}^i Q)\, q_{cj} = \mu_c \quad i=1,\dots,m; c=1,\dots,n \tag{5}$$

$$\mu_c \psi_c'(k_c) = \varrho \quad c=1,\dots,n \tag{6}$$

$$\sum_{i=1}^{m} \tau^i U_j^i(\bar{x}^i Q)\, x_c^i = \nu_c h_{cj}'(\bar{q}_c) \quad j=1,\dots,s; c=1,\dots,n \tag{7}$$

where $h_{cj}'(\cdot)$ denotes the partial derivative of $h_c(\cdot)$ with respect to the jth argument.

Combining (5) and (6) gives the efficient allocation rule

$$\frac{\sum_{j=1}^{s} U_j^i(\bar{x}^i Q)\, q_{cj}}{\sum_{j=1}^{s} U_j^i(\bar{x}^i Q)\, q_{dj}} = \frac{\mu_c}{\mu_d} = \frac{\varrho/\psi_c'(k_c)}{\varrho/\psi_d'(k_d)} = \frac{\psi_d'(k_d)}{\psi_c'(k_c)} \quad i=1,\dots,m \tag{8}$$

Condition (7) may be rewritten as:

$$\frac{\sum_{i=1}^{m} \tau^i U_j^i(\bar{x}^i Q)\, x_c^i}{\sum_{i=1}^{m} \tau^i U_k^i(\bar{x}^i Q)\, x_c^i} = \frac{h_{cj}'(\bar{q}_c)}{h_{ck}'(\bar{q}_c)} = -\frac{\partial q_{ck}}{\partial q_{cj}} \quad c=1,\dots,n \tag{9}$$

[1] Although the individual utility functions are strictly quasi-concave on the Q-space, and also strictly quasi-concave in the x_c^i for a given set of quality vectors, the $U^i(\cdot)$ will generally not be jointly quasi-concave in x_c^i and q_{cj}. Thus the first-order conditions are not sufficient for a global optimum.

Condition (8) is of course identical with the Pareto-optimal production rule for a production economy with fixed product differentiation. Condition (9), however, must be interpreted as a condition for Pareto-optimal product differentiation or qualitative efficiency. Generally, the left-hand side of (9) will depend explicitly on the distribution of x_c^i over consumers. Consequently, the Pareto-optimal trade-offs between quality characteristics may differ for different firms. A Pareto-optimal degree of product differentiation may even require that firms with identical technologies, i.e. identical production sets and substitution possibilities, produce different quality vectors.

Moreover, the optimal differentiation rule (9) is not equated to any parameter ratios of the problem and since optimal differentiation depends explicitly on the optimal exchange of the resulting commodities, this implies that exchange efficiency and qualitative efficiency cannot be separated. Thus, it seems that the separation of productive and distributive efficiency which is so fundamental to allocation theory cannot be extended to the case of product differentiation.

For an economy with product differentiation, Pareto-optimal production choices will generally not be efficient in the Q-space.[1] The reason for this follows from the fact that characteristics have to be distributed through commodity markets so that there may be a trade-off between the degree of product differentiation, i.e. the range of choice, and the amount of each characteristic produced. In other words, there may be a trade-off between the "width" and the "length" of the feasible trading set. This is illustrated in Figure 3, where two characteristics and two production units have been assumed. The allocation of inputs to firms is assumed to be optimal and the resulting transformation curves in the Q-space are given by AA'' and BB'' for the two firms, respectively. The transformation curve for the whole economy is given by CC'' and it is constructed by adding all points on AA'' and BB'' with equal slope. This is an efficiency frontier in the sense that for any given point located on this curve, more of one characteristic cannot be produced without producing less of the other. The *ex ante* feasible trading set is the collection of all (q_1, q_2)-combinations which can be obtained in the commodity market with the given state of technology. In Fig. 3, the *ex ante* feasible trading set is given by the area $OB''C''C'CA$.

We assume that the vector A' is selected from the production set of the first firm and B' from the production set of the second firm. Then the *ex post* feasible trading set is given by the shaded area $OB'C'A'$. It is clear that productive efficiency in Q-space requires that points on the two transformation curves with equal slope are selected, that is, if there were a market for each characteristic, only points on the transformation curve CC'' would be part

[1] Productive efficiency in the Q-space refers to an allocation of resources such that the total amount of one particular characteristic cannot be increased unless the total amount of some other characteristic is reduced.

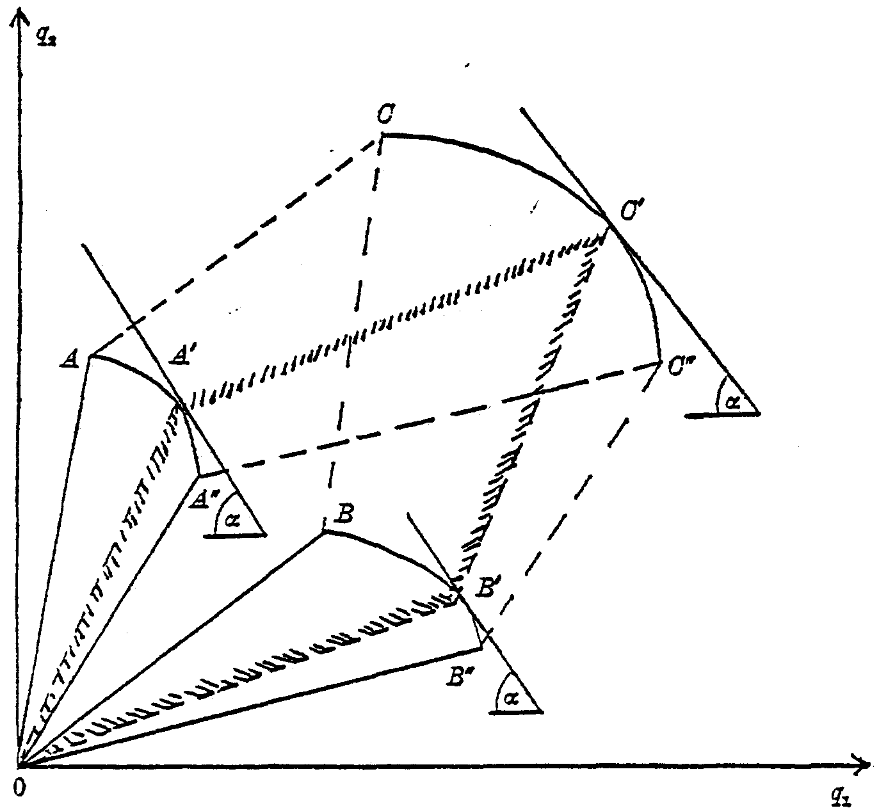

Fig. 4

of a Pareto-optimal solution. However, since characteristics are distributed by means of commodities, we may conceive of situations where it is optimal to select points on the two transformation curves with different slopes. This may be the case if consumer demands are very dispersed in the Q-space in the sense that some consumers have a very strong preference for characteristic q_1 while others have a very strong preference for q_2. The widest possible *ex post* trading set would be obtained by selecting points A and B'' from the two production sets, respectively. It can easily be seen that total production in this case will fall short of the efficiency frontier CC'', thus confirming our assertion that there will be a trade-off between efficiency in the production of characteristics and the range of choice for consumers.

We now turn to the question of whether a Pareto-optimal allocation satisfying conditions (8) and (9) can be effected through a decentralized competitive market system. As noted above, the problem of qualitative efficiency cannot in general be separated from that of distributive efficiency. Hence, there is no price system common to all economic agents which ensures that profit maximizing of competitive firms will lead to a Pareto-optimal pattern of product differentiation.

It is not even quite clear what competitive behavior should mean in this context. For a given choice of quality, price-taking certainly requires that firms behave as if they believe that $\partial p_c/\partial z_c = 0$. However, it is not at all clear what price-taking with respect to quality variants should mean. A possible suggestion may be to take the implicit prices for characteristics as given, i.e. as independent of the decisions of the individual producers. In this case, firm c would perceive the relation between its choice of quality and its product price as given by $(\partial p_c/\partial q_{cj})\, dq_{cj} = \pi^i_j dq_{cj}$. The problem inherent in this approach is that the implicit prices for characteristics are not readily available in commodity markets. Moreover, for the case where $n < s$, the implicit prices will differ for different consumers so that the outcome would depend on whose implicit prices are being used.

For the special case where $n = s$, the implicit prices are uniquely determined by equilibrium commodity prices[1] and can in principle be computed from $\bar{\pi}' = Q^{-1}p'$. In this particular case, price-taking with respect to quality variants would be well-defined. It is also clear that a profit-maximizing quality variant in firm c would be characterized by the fact that the marginal rates of transformation between any two characteristics must be equal to their implicit price ratios, that is

$$\frac{U^i_j(\bar{x}^i Q)}{U^i_k(\bar{x}^i Q)} = \frac{\pi_j}{\pi_k} = \frac{h'_{cj}(\bar{q}_c)}{h'_{ck}(\bar{q}_c)} = -\frac{\partial q_{ck}}{\partial q_{cj}} \quad i = 1, \dots, m \tag{10}$$

In this case, the market differentiation rule (10) will be Pareto-optimal. This can be seen by noting that for $n = s$, the marginal rates of substitution between characteristics j and k will in optimum be the same for all consumers. This implies that the left-hand side of (9) reduces to π_j/π_k so that the Pareto-optimal rule and the market rule will be identical.

In the general case where $n < s$, competitive behavior with respect to quality decisions and hence competitive equilibrium are not well defined.

The problem of finding appropriate prices for evaluating the social desirability of different quality variants of a given commodity would also be encountered in a centrally planned economy. Assuming that a market planner imposes his own or someone else's implicit prices for quality characteristics and instructs the firms to behave as price-takers and use the imposed prices in their profit calculations, profit maximizing firms would select points on their transformation surfaces with equal slope. This will certainly be productively efficient in the Q-space but it may not result in a socially optimal quality range of commodities.

The reason for the general lack of optimal market rules for product differentiation is the basic asymmetry of the definition of goods on the consumption and production sides of the economy. Consumer preferences are basically

[1] We note that this may require that commodities can be purchased in negative amounts.

defined over quality characteristics whereas firms supply bundles of characteristics and it is the bundles which are priced in commodity markets. In the general case, however, commodity prices will not convey the correct information to firms about the social desirability of various quality choices. Nor will the implicit prices be of any help because they differ for different consumers, thus indicating that a given quality change will affect consumers in different ways. This amounts to a kind of structural inefficiency since the market does not provide the necessary information for optimal quality choices by firms.

In fact, the notion of quality in commodity markets has much the same properties as a public good. If a new and unique quality vector is produced, it will enlarge the range of choice for all consumers since it increases the dimension of the set of feasible trades in commodity markets. It is not the commodity in itself, but rather the *availability* of a unique quality variant which has the public good property. To illustrate this point, the unique beauty of, say, a Beethoven Symphony can be enjoyed by everyone and the pleasure one particular music lover derives from it is independent of the pleasure of others.[1] Once a particular symphony has been composed, however, the recordings of it have to be considered as ordinary private goods.

The effect of introducing a new quality variant into the market may also be regarded as analogous to the effects of technological progress in the classical model. Technological progress, or more specifically, positive shifts in the production functions will lower cost, thereby reducing prices, and then induce positive shifts in the individual budget constraints. Therefore, the problems of establishing efficient incentives for decentralized quality decisions are more or less the same as those involved in establishing efficient market incentives for inventions or the production of new knowledge in a wide sense.

As the consumer optimum in commodity markets may be a constrained optimum in the Q-space, i.e. it may be on the boundaries of the feasible trading set, the desirability of a marginal variation in the quality characteristics of a given commodity may be judged differently by different consumers. The correct measure of the gain in social welfare from varying the quality design of a given commodity should therefore be some weighted sum of individual marginal utilities. This is also confirmed by the Pareto-optimal differentiation rule (9) where a Pareto optimal quality pattern requires firm c to equate its marginal rate of transformation between any two characteristics to the ratio between weighted sums over all consumers of the individual marginal utilities for the two characteristics, with weights given by the distributional parameters τ^i and the amount each consumer gets of commodity c. This is a very natural requirement because if consumer i gets nothing of commodity c, his preferences should be given no weight in determining the optimal quality

[1] Of course, this presupposes that interpersonal preferences do not exist.

design of that commodity. For the polar case whereby commodity c is only consumed by consumer i, the preferences of that consumer alone should determine its optimal quality characteristics.

In the public good spirit, we can define some sort of pseudo-equilibrium by instructing firms to use a weighted sum of individual marginal utilities as prices for characteristics when calculating the profitability of various quality choices. This would certainly lead to an equilibrium allocation that would be efficient both from a quantitative and a qualitative point of view, i.e. it would satisfy conditions (8) and (9). As these prices are not reflected by the market, however, this would require complete knowledge of all individual preferences.

If firms use the same implicit prices for characteristics and behave competitively with respect to these prices, it may result in a too narrow quality structure of commodities. As noted in connection with Fig. 3, optimal quality decisions may involve a trade-off between quantitative efficiency in the Q-space and the range of choice, i.e. a trade-off between the "width" and the "length" of the feasible trading set. If this is to take place in a market context, different firms have to use different implicit prices for characteristics in their profit calculations. The efficiency properties of the market allocation are therefore crucially dependent on how firms perceive the relation between the quality design of a commodity and its market price, i.e. the derivatives $\partial p_c/\partial q_{cj}$. Thus it may seem natural to raise the question as to whether some kind of monopolistic competition among producers might be better in terms of efficiency, compared to a situation where all producers use the same implicit prices for characteristics and behave competitively with respect to these prices. It should perhaps be expected that if some technologically feasible and desirable quality vectors were not being produced, some producers would see this opportunity and plug the gap in the Q-space if this were profitable.

It is clear that if different firms face different demand schedules for their products, they would generally perceive the implicit prices for the quality characteristics of their products differently and thus select points on their transformation surfaces with different slopes. But it would be difficult to verify whether this would be in the direction of a Pareto optimum or not. If firms behave in a monopolistic way, however, they would probably take into account the downward slope of the demand curves in the commodity space, i.e. behave as if $\partial p_c/\partial z_c < 0$, and thus curtail production of a *given* quality vector below the efficiency level. This may more than outweigh the gains in other respects.

V. Concluding Remarks

In this paper we have tried to introduce product differentiation into a competitive setting and we have shown that, for given quality designs, a competitive economy will produce and distribute these commodities in efficient amounts. If producers also have to decide on the quality design of their products, a competitive equilibrium is defined only in a pseudo-sense.

Admittedly, the simplifying assumptions made as to consumer preferences and production technology are rather heroic. Certainly, the results of Section IV are critically dependent on the behavioral assumption that consumer preferences are basically defined over quality characteristics.[1] On the other hand, it is difficult to evaluate the importance of the assumption that firms can produce only one single quality variant in their production sets. However, as long as firms only produce a few of the technically feasible quality variants in their production sets, the main conclusions should remain valid.

In general, efficiency prices for characteristics will not exist in commodity markets, so that a system of decentralized quality choices by profit maximizing firms is not likely to produce a socially optimal quality pattern of goods. Prices in commodity markets simply do not perform the dual function of providing producers with necessary information for optimal decisions with regard to both production levels and quality choices.

The basic problem is that a Pareto-optimal range of quality designs requires profit maximizing firms to use different prices for characteristics when evaluating the profitability of various quality choices. This will normally be the case in an economy with some degree of monopolistic competition and it might well be the case that the price of having a well-differentiated quality pattern of goods in a market economy must be some loss in quantitative efficiency in the commodity space.[2] However, it is not easy to conceive of any market system striking the optimal balance between quality differentiation and productive efficiency in the Q-space. Since profits are not likely to be efficient incentives for decentralized quality decisions, we should perhaps look for more efficient ways of organizing economic activity.

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[1] The reader is referred to Lancaster and his pioneering works [3] and [4], which eloquently advocate this approach to consumer theory.

[2] That is, inefficiency due to the fact that firms will behave as if $\partial p_c/\partial z_c < 0$, so that even for *fixed* quality choices, the firms' supplies of commodities will be inefficient.

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CHOICE OF PRODUCT QUALITY: EQUILIBRIUM AND EFFICIENCY

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1. INTRODUCTION

1.1

THE PURPOSE OF THIS PAPER is to examine the role of prices and to assess the efficiency of the market mechanism in guiding production decisions with respect to both product quantities and the qualities of the products produced in economies with product differentiation.

A rigorous treatment of product differentiation requires an operational definition of product quality and quality changes. For this purpose we shall adopt the "hedonic" approach to consumer demand theory as developed by Lancaster [9]. This systematic reformulation of consumer demand theory rests on two fundamental assumptions:[2]

(i) All goods possess objective characteristics relevant to the choices people make among different collections of goods.

(ii) It is the characteristics, rather than goods, which are the ultimate objects of consumer satisfaction. The consumers are endowed with preferences for collections of characteristics and the preferences for goods are indirect or derived in the sense that goods are required only in order to produce characteristics.

This formulation permits a natural definition of product quality and quality changes: Given the units of measurement, the quality of a given commodity is objectively defined by the amount of the various characteristics embodied in a unit of that commodity. It is implicit in this formulation that all characteristics are quantitatively measurable and we shall assume that they can be varied continuously.

The model is closed by introducing objective technical relationships between quantities of commodities and the characteristics which they possess. The simplest, and most widely used, specification is the linear technology, fully described by a "consumption technology matrix" B, relating column vectors of commodities x to column vectors of characteristics z through $z = x'B$ where the jth row of B represents the amount of the various characteristics embodied in a unit of commodity j.

The linear technology specification will also be used here, although the assumption that characteristics can be added linearly may seem rather unrealistic

[1] The authors are indebted to Donald J. Roberts for helpful comments, which have contributed to an increase in the quality of the product.

[2] See Lancaster [9, p. 7].

in most cases. We believe, however, that this simplified approach will give us some insight into the efficiency of the market mechanism in economies with variable product qualities also under more general consumption technologies. In particular, sources of inefficiency detected in the linear case will remain present in the general case as well.

This model has also proved useful to account for quality changes in the construction of price indexes. In this literature the general presumption is that the multitude of models and varieties of a particular commodity can be comprehended in a much smaller number of characteristics or basic attributes so that most new models of commodities may be viewed as new combinations of old characteristics.[3]

The same approach has emerged naturally in the literature on investment decisions by firms in physical assets and decisions by consumers investing in financial assets. The literature on portfolio choices by consumers starts from the unquestioned premise that preferences among collections of financial assets are derived from underlying preferences among collections of characteristics. In the seminal works by Tobin [26] and Markowitz [15], only two characteristics are considered, namely mean and variance of the random return to a portfolio. A more general model, which is receiving increasing attention, describes the uncertainty associated with investment decisions in terms of physical uncertainty about the economic environment (nature). Following Arrow [1] and Debreu [3], one considers a finite set of alternative, mutually exclusive "states of nature" and one describes a financial asset by its return under each state. The characteristics in which consumers are interested are similarly the global returns of a portfolio under each state and these are naturally defined by a linear technology. Similarly, a physical asset is described by its return pattern over the states of nature so that physical investment can be viewed as a characteristics producing activity.

A sizeable literature deals with the allegedly realistic case where there are many more states of nature (characteristics) than financial assets (commodities) and one studies conditions under which competitive markets for financial assets (the stock market) provide adequate signals to guide the physical investment decisions by firms.[4]

Situations with fewer commodities than characteristics (i.e., incomplete markets) in the traditional framework of certainty[5] are of course due to economies of scale (relative to market size). The fixed costs of merchandising additional commodities are as relevant in this respect as economies of scale in production or indivisibilities in the commodities.

[3] See Griliches [8, p. 4].

[4] See, for instance, the "Symposium on the Optimality of Competitive Capital Markets" in the Spring 1974 issue of the *Bell Journal of Economics and Management Science* [7, 11, 17, 19].

[5] Under uncertainty, examples of incomplete markets are numerous also outside the capital markets model. Probabilities of shortage are an interesting case (especially for countries that do not produce oil), early recognized in connection with electricity supply. See Boiteux [2] and Drèze [4].

1.2

Except for recent contributions by Rosen [20], Leland [12], and Lancaster [9], which explore some aspects of optimal product differentiation in a context similar to ours, the problem of efficient supply of commodities with multiple hedonic characteristics has received little attention in the literature on consumer demand and hedonic price indexes. On the other hand, the properties of demand functions for commodities with multiple characteristics are hardly explored in the literature on competitive capital markets. The present paper will hopefully contribute to both streams of literature, by using some properties of demand functions in an attempt to assess the conditions under which prices in the commodity markets guide efficiently the choices of product qualities by firms motivated by profit incentives.[6]

The main issue to be investigated is the following. Consider a system of C characteristics and J commodities, C and J being *given*. Assume that commodities are produced by firms endowed with production sets that allow for substitution among characteristics in the production of commodities. *Will profit maximization lead firms to choose for each commodity the combination of characteristics which is most desirable from the standpoint of consumer welfare?*

In Section 2 below, we define the model. In Section 3, we discuss the problem of optimal product differentiation in a general equilibrium context and we derive necessary conditions for optimal product variety in terms of characteristics. In Section 4, we discuss profit maximizing choices of product quality under monopolistic and competitive pricing of commodities. Using some useful properties of demand functions for commodities with multiple hedonic characteristics, which are investigated in the Appendix, we give necessary and sufficient conditions for local optimality of the profit maximizing choices of product quality. In Section 5, we investigate briefly the possibility of computing prices for characteristics from commodity prices and we relate our results to the "unanimity theorem" in capital market theory.

We conclude this introduction by noting that we could easily invert the roles of input and outputs so as to apply the analysis in this paper to the problem of choosing working conditions, viewed as the characteristics of the input commodity labor. The results in Section 4 also provide an answer to the question whether profit maximization will lead capitalistic firms to adopt working conditions that are efficient from the viewpoint of workers' preferences. That question is discussed in a recent paper by Drèze [6].

2. THE MODEL

The model considered in this paper consists of H households, indexed $h = 1 \ldots H$, and J firms, indexed $j = 1 \ldots J$. Each firm j produces a single

[6] We shall not be concerned with the problem of existence of equilibria in economies with product differentiation in this paper. We may, however, refer to a recent paper by Mas-Colell [16] where the characteristics framework is used to introduce product differentiation in a pure exchange economy (with infinitely many traders). Mas-Colell proves that the set of equilibria is non-empty and equal to the core.

consumption good, also indexed by j, in quantity x_j. Each consumption good embodies some or all of C characteristics, indexed $c = 1 \dots C$, and b_{jc} measures the quantity of characteristic c embodied in one unit of commodity j. B denotes the $J \times C$ (variable) "technology matrix" with typical element b_{jc}. Each firm j uses a single input indexed zero, in quantity x_{jo}, and chooses a $(C+2)$-dimensional vector $y_j = (x_{j0}, x_j, b_{j1} \dots b_{jC})'$ in its production set $Y_j \subseteq R^{C+2}$. The interpretation of Y_j is discussed in a remark below.

The input is common to all firms and it is also used as a numeraire. In alternative interpretations of the model, the input may be labor, capital, foreign exchange, a composite commodity (money?) summarizing non-consumption goods, a good identified with a $C+1$st characteristic, etc.

To distinguish between changes in output levels x_j and changes in characteristics embodied per unit output, we need a definition of unit sizes of commodities in terms of characteristics. To that effect, we assume that every commodity j contains at least one *given* characteristic in positive amount,[7] to be indexed j', and we set $b_{jj'} = 1$ for size normalization. Hence, the characteristics of commodity j, b_{jc}, are measured per unit of characteristic j'. Of course, the index j' may be different for different j.

This normalization procedure may seem natural in some cases (as in cases where $b_{jj'}$ measures weight and commodity j is sold by weight), but in other cases it may seem rather artificial. Also, this procedure has the undesirable property that all characteristics are not treated symmetrically. Symmetry would call for normalizing unit sizes to the unit sphere or the unit simplex in the characteristics space. This would, however, render the analysis more cumbersome without changing the results.

Each household h has a consumption set Z^h in the space R^{C+1} of characteristics and the numeraire. The initial resources of household h are defined by a non-negative vector $w^h = (w_0^h, w_1^h \dots w_C^h)'$ in R_+^{C+1}. The reason for introducing initial quantities (possibly zero) of the characteristics will become clear later when we shall derive a Slutsky equation for quality changes (Appendix). The consumption of commodities by household h is denoted $x^h = (x_1^h \dots x_J^h)' \in R_+^J$. The purchase or sale of the numeraire by household h is x_0^h. The consumption of household h in the space of characteristics and the numeraire is

$$(0) \qquad z^h = (z_0^h, z_1^h \dots z_C^h)' \equiv w^h + (x_0^h, x^{h\prime}B)'$$

and is constrained to be an element of Z^h.

An *allocation* for this economy is a $(J+2H)$-tuple of vectors $\{y_j\}$, $\{x^h, z^h\}$, hereafter denoted a. An allocation a is *feasible* if it satisfies the following conditions:

$$(1) \qquad \forall j, \qquad y_j \in Y_j,$$

$$(2) \qquad \forall h, \qquad z^h \in Z^h,$$

[7] In the capital market context this amounts to assuming that there is at least one given state in which firm j never goes bankrupt.

(3) $\sum_j x_{jo} + \sum_h z_0^h \leq \sum_h w_0^h,$

(4) $\forall j, \quad \sum_h x_j^h - x_j \leq 0,$

(5) $\forall h, c, \quad z_c^h - \sum_j x_j^h b_{jc} \leq w_c^h.$

The set of feasible allocations is denoted A.

The interpretation of conditions (1)–(4) is standard: (1) technical feasibility in production, (2) consumers' survival (technical feasibility in consumption), (3) availability of input, (4) availability of output.

The set of conditions (5) (HC conditions altogether) is specific to the model with many characteristics; it relates the consumption of characteristics to the allocation of commodities. Technically, it implies free disposal of characteristics separately from commodities, which is sometimes unrealistic. Under the monotonicity of Assumption 1 this restriction is immaterial.

The model (1)–(5) is analogous to that used in the literature on production decisions under uncertainty, with x_{jo} denoting investment by firm j at time o and $x_j b_{jc}$ output of firm j at time 1 in state c; see, e.g., Dréze [5] or Radner [19].

The "single input" assumption is made here for simplicity and convenience. The generalization to many inputs (with fixed or variable characteristics) is immediate and does not modify the results. The "single output per firm" assumption is more important but entails no loss of generality. Like the single input simplification, it is standard in the literature on production under uncertainty, where it corresponds to equity financing. (Several commodities per firm would correspond to several, diversified, financial assets, say preferred stock, common stock, mortgaged bonds, etc....). In the general demand context, outputs of different firms must be treated as different commodities, since firms may combine the characteristics differently. With several outputs per firm, the results below remain valid at the local equilibrium level (the level adopted anyhow in this paper).

3. PRICES AND EFFICIENCY

A price system for the economy (1)–(5), using the input as a numeraire, is a vector $p = (p_1 \dots p_J)' \in R_+^J$. An income distribution is a vector $v = (v^1 \dots v^H)' \in R^H$. A feasible allocation a, a price system p, and an income distribution v define a budgeted allocation provided

(6) $\forall h, \quad x_0^h + p'x^h \leq v^h.$

Clearly $\Sigma_j (p_j x_j - x_{jo}) \geq \Sigma_h (x_0^h + p'x^h)$ and one could thus let the income distribution correspond to private ownership of the firms with $v^h = \Sigma_j \theta_j^h (p_j x_j - x_{jo})$, $\Sigma_h \theta_j^h = 1$, with θ_j^h denoting the fraction of firm j owned by household h.

The following assumptions are used throughout the paper:

ASSUMPTION 1: *$\forall h$, Z^h is closed, convex, and completely ordered by a quasi-concave, twice continuously differentiable utility function $U^h(z^h)$, with $\partial U^h/\partial z_0^h > 0$, $\partial U^h/\partial z_c^h > 0$ for all c and for all $z^h \in Z^h$.*

ASSUMPTION 2: *$\forall j$, Y_j is closed and convex.*

A *competitive products equilibrium relative to a given technology matrix B*, is a budgeted allocation (a, p, v) such that:

(7) $\forall h$, $U^h(\bar{z}^h) > U^h(z^h)$ implies $\hat{x}_0^h + p'\hat{x}^h > v^h$ for all $(\hat{x}_0^h, \hat{x}^h)$ such that $\bar{z}^h \leq \hat{z}_c^h \equiv w^h + (\hat{x}_0^h, \hat{x}^{h\prime}B)'$,

(8) $\forall j$, $p_j x_j - x_{j0} \geq p_j \bar{x}_j - \bar{x}_{j0}$ for all $(\bar{x}_{j0}, \bar{x}_j)$ such that $(\bar{x}_{j0}, \bar{x}_j, b_{j1} \dots b_{jC}) \in Y_j$.

Conditions (7) are a natural extension of the traditional concept of utility maximization, and say that if there exists a feasible bundle $\hat{z}^h$ which is strictly preferred by h to the equilibrium bundle z^h, then it must cost more,[3] while (8) says that y_j maximizes profits over Y_j. In this definition of a competitive products equilibrium the technology matrix is taken as given and firms are assumed to choose *quantities* of input and output so as to maximize profits at *given* prices.

The set of *Pareto optima* for the economy (1)–(5), denoted A^p, is defined by:

(9) $$A^p = \{a \mid a \in A, \nexists \hat{a} \in A, U^h(\hat{z}^h) \geq U^h(z^h) \forall h, \sum_h U^h(\hat{z}^h) > \sum_h U^h(z^h)\}.$$

In (9), the technology matrix is allowed to vary over the set of feasible allocations. It is instructive to consider a more restrictive concept. For a given technology matrix $\bar{B}$, define the set of feasible allocations with technology $\bar{B}$, denoted $A(\bar{B})$, by:

(10) $$A(\bar{B}) = \{a \in A \mid B = \bar{B}\} = \{a \in A \mid y_j = (x_{j0}, x_j, \bar{b}_{j1} \dots \bar{b}_{jC})', \quad j = 1 \dots J\}.$$

The set of *Pareto optima relative to* (the technology) $\bar{B}$, denoted $A^p(\bar{B})$, is defined by:

(11) $$A^p(\bar{B}) = \{a \mid a \in A(\bar{B}), \nexists \hat{a} \in A(\bar{B}), U^h(\hat{z}^h) \geq U^h(z^h) \forall h, \sum_h U^h(\hat{z}^h) > \sum_h U^h(z^h)\}.$$

The following results are classic:

PROPOSITION 1: *Under Assumptions* 1 *and* 2, *for every $\bar{B}$ such that $A(\bar{B}) \neq \varnothing$: (a) $A(\bar{B})$ is convex; (b) if $(a \in A(\bar{B}), p, v)$ define a competitive products equilib-*

[3] (7) could have been written alternatively as follows:

$$U(w^h + (\bar{x}_0^h, \bar{x}^{h\prime}B)') > U(w^h + (x_0^h, x^{h\prime}B)') \text{ implies } \bar{x}_0^h + p'\bar{x}^h > v^h.$$

rium relative to $\bar{B}$, then $a \in A^p(\bar{B})$; (c) if $a \in A^p(\bar{B})$, then there exist (p, v) such that (a, p, v) is a competitive products equilibrium relative to $\bar{B}$.

As an immediate corollary we have:

PROPOSITION 2: *Under Assumptions* 1 *and* 2, *if $a \in A^p$, there exist (p, v) such that (a, p, v) is a competitive products equilibrium.*

PROOF: Follows from Proposition 1(c) since B is kept fixed in the definition of a competitive products equilibrium. *Q.E.D.*

Of course, the above results are rather obvious because when the technology matrix B is kept fixed, we are back to the classical model with fixed quality of goods. The following proposition is perhaps less widely appreciated.[9]

PROPOSITION 3: *Assumptions* 1 *and* 2 *are not sufficient for either of the following two conditions to hold: (a) A is convex; (b) if (a, p, v) defines a competitive products equilibrium (relative to $\bar{B}$), then $a \in A^p$.*

PROOF: (a) follows from the bilinear form of constraint (5) in the definition of the set of feasible allocations.

(b) Assume $A^p \cap A^p(\bar{B}) = \varnothing$ (an example, satisfying Assumptions 1 and 2, is given below). By Proposition 1(c), any $a \in A^p(\bar{B})$ will provide the desired counterexample. *Q.E.D.*

EXAMPLE: Let $J = H = 2$, $c = 3$, $\Sigma_h w_0^h = 2$, $\sum_h w_c^h = 0$, $c = 1, 2, 3$, $Z^h = R_+^4$, $h = 1, 2$, and $Y_1 = \{y_1 | x_1 \leq x_{10} \leq 1, \ b_{11} = 1, \ b_{11} + b_{12} + b_{13} \leq 2\}$, $Y_2 = \{y_2 | x_2 \leq x_{20} \leq 1, b_{22} = 1, b_{21} + b_{22} + b_{23} \leq 2\}$.

Let the allocation a be as follows:

$$x_1 = x_{10} = 1, \quad b_{11} = b_{12} = 1, \quad b_{13} = 0, \quad x_1^1 = 1, \quad x_1^2 = 0;$$

$$x_2 = x_{20} = 1, \quad b_{21} = b_{22} = 1, \quad b_{23} = 0, \quad x_2^1 = 1, \quad x_2^2 = 0;$$

consequently, $z^1 = (0, 2, 2, 0)'$ and $z^2 = (0, 0, 0, 0)'$.

Let the allocation $\bar{a}$ be defined as follows:

$$\bar{x}_1 = \bar{x}_{10} = 1, \quad \bar{b}_{11} = \bar{b}_{13} = 1, \quad \bar{b}_{12} = 0, \quad \bar{x}_1^1 = 0, \quad \bar{x}_1^2 = 1;$$

$$\bar{x}_2 = \bar{x}_{20} = 1, \quad \bar{b}_{22} = \bar{b}_{23} = 1, \quad \bar{b}_{21} = 0, \quad \bar{x}_2^1 = 0, \quad \bar{x}_2^2 = 1.$$

[9] See also Lemma 2.3 in Drèze [5].

Consequently, $\bar{z}^1=(0,0,0,0)'$ and $\bar{z}^2=(0,1,1,2)'$. A does not contain the allocation $\hat{a}=(a+\bar{a})/2$. Indeed, let $\hat{y}_j=(y_j+\bar{y}_j)/2$, $j=1,2$; $\hat{x}^h=(x^h+\bar{x}^h)/2$, $h=1,2$; then $\hat{z}^1=(0,\frac{3}{4},\frac{3}{4},\frac{1}{2})'=\hat{z}^2$ and $\hat{z}^h\neq(z^h+\bar{z}^h)/2$, $h=1,2$. Conversely, let $\mathring{z}^h=(z^h+\bar{z}^h)/2$, so that $\mathring{z}^1=(0,1,1,0)'$ and $\mathring{z}^2=(0,\frac{1}{2},\frac{1}{2},1)'$. Although $\mathring{z}^1+\mathring{z}^2=\hat{z}^1+\hat{z}^2$, there do not exist $\mathring{x}^h$ and $\mathring{B}_j$ such that $\mathring{z}^h_c=\mathring{x}^h_1\mathring{b}_{1c}+\mathring{x}^h_2\mathring{b}_{2c}$, $h=1,2$; $c=1,2,3$.

In this example, two pairs of consumption vectors in the space of characteristics, $\{z^h\}$ and $\{\bar{z}^h\}$, are obtained by varying simultaneously the technology matrix and the consumption vectors in the commodity space. Intermediate consumption vectors in the space of characteristics are consistent with the constraints (4), but not with the constraints (5): The required total output of each characteristic is feasible but cannot be allocated properly among households, because there are only two goods to allocate three characteristics.

In the same example, let the utility functions of both households be $U^h=\frac{1}{4}z^h_0+z^h_1+\frac{1}{2}z^h_2+z^h_3$. The allocation a together with prices $p_1=p_2\ (=p_j)=1.5$ (say) and incomes such that $w^1_0+v^1=3$, $w^2_0+v^2=0$, define a competitive products equilibrium. This competitive products equilibrium is not efficient because characteristics 1 and 3 could be substituted one-to-one (in production) for characteristic 2, which is less preferred. That is, the technology matrix

$$B=\begin{bmatrix}110\\110\end{bmatrix}$$

associated with the allocation a is Pareto dominated by the technology matrix

$$\bar{B}=\begin{bmatrix}101\\011\end{bmatrix}$$

associated with allocation $\bar{a}$. Allocation $\bar{a}$ is Pareto optimal and sustained by prices $\bar{p}_2=\frac{3}{4}\bar{p}_1$, $8>\bar{p}_1>\frac{4}{3}$, and incomes $w^1_0+v^1=0$, $w^2_0+v^2=\frac{7}{4}\bar{p}_1$.

REMARK CONCERNING THE PRODUCTION SET Y_j: The convexity assumption on Y_j means non-increasing marginal returns both with respect to output levels x_j and characteristics per unit output b_{jc}.

In the above example Y_j is defined as the Cartesian product of a convex set in the space R^2 of (x_{j0}, x_j) and a convex set in the space R^C of $(b_{j1}\dots b_{jC})$. Thus the feasible combinations of characteristics are independent of input and output levels. It may be more realistic to assume that the total feasible quantity of a given characteristic is determined by the input level. In such a case the convexity assumption on Y_j is rather restrictive. For example, if the total feasible quantity of characteristic c were given by a bilinear constraint of the type $x_jb_{jc}\leq\beta_{jc}x_{j0}$, then Y_j would clearly not be convex in x_{j0}, x_j, and b_{jc}.

3.2

We are now ready to approach the central problem raised in this paper, namely Pareto efficiency in the supply of characteristics. In view of Proposition

3(a), sufficient conditions for Pareto optimality must be of a global nature since a local optimum is not necessarily an overall optimum, in the absence of convexity. No satisfactory approach to the global problem is currently available. There remains only to define necessary conditions for Pareto optima, that is, conditions for *Pareto stationary points.*[10]

To focus on the characteristics of firm j's product, let $\bar{a} \in A$ be a given allocation and define $F_j(\bar{a})$ by:

$$(12) \qquad F_j(\bar{a}) = \{a \mid a \in A, x^h = \bar{x}^h \ \forall h, \ y_k = \bar{y}_k \ \forall k \neq j, \ x_j = \bar{x}_j\}.$$

That is, $F_j(\bar{a})$ is the set of feasible allocations which can be obtained by varying the characteristics of commodity j and the consumptions of the numeraire for a given production level in firm j, for given production plans of firms other than j and for a given allocation of commodities over consumers.

The set of Pareto optima in $F_j(\bar{a})$, denoted $F_j^o(\bar{a})$, is defined in the usual way as:

$$(13) \qquad F_j^o(\bar{a}) = \{a \mid a \in F_j(\bar{a}), \nexists \hat{a} \in F_j(\bar{a}), U^h(\hat{z}^h) \geq U^h(z^h) \forall h, \sum_h U^h(\hat{z}^h) > \sum_h U^h(z^h)\}.$$

Accordingly, the quality of firm j's product as given by the vector of characteristics $B_j = (b_{j1} \dots b_{jC})'$ will be said to be a *quality optimum for firm j relative to* $\bar{a}$ if and only if $a \in F_j^o(\bar{a})$.

We can then define a Pareto stationary point as follows: An allocation $a \in A$ is a Pareto stationary point if and only if

(i) $\qquad a \in A^o(B)$,

(ii) $\qquad \forall j, \quad a \in F_j^o(a)$.

Hence, a is Pareto optimal relative to B, and B is such that for every j, B_j is a relative quality optimum given x_j, y_k, $k \neq j$, and $\{x^h\}$. Clearly, (i) and (ii) correspond to necessary conditions for a Pareto optimum. Given a Pareto-stationary point $a \in A$, there may exist feasible, *simultaneous*, marginal changes in $\{x^h\}$ and B such that the resulting allocation Pareto dominates a, which reflects the fact that a Pareto stationary point may be a saddlepoint.

We are here interested in the price implications of relative quality optima for firms. To that effect, we note that $F_j(\bar{a})$ is the set of feasible allocations for an economy, say $E_j(\bar{a})$, with preferences given by:[11]

$$(14) \qquad U^h\Big(z_0^h, \sum_{k \neq j} \bar{x}_k^h \bar{b}_{k1} + \bar{x}_j^h b_{j1} \dots \sum_{k \neq j} \bar{x}_k^h \bar{b}_{kC} + \bar{x}_j^h b_{jC}\Big) \qquad (h = 1 \dots H)$$

[10] We do not use the term "local Pareto optimum" here since a stationary point may be a saddlepoint and not a local maximum.

[11] See Drèze [5] for a similar approach to the theory of investment under uncertainty.

and with constraints on production and distribution given by

$$\sum_h z_0^h + x_{j0} \leq \sum_h w_0^h - \sum_{k \neq j} \bar{x}_{k0}, \tag{15}$$

$$(x_{j0}, \bar{x}_j, b_{j1} \ldots b_{jC})' \in Y_j, \tag{16}$$

$$z^h \in Z^h \qquad \forall h. \tag{17}$$

With this reformulation, the economy $E_j(\bar{a})$ can be considered as formally equivalent to an economy with one private good z_o^h and C local, or regional, public goods $b_{j1} \ldots b_{jC}$, where b_{jc} is the amount of public good c produced per unit of time in region (club) j and where production decisions in regions other than j are given. x_j^h may be interpreted as the time spent by consumer h in region j so that $x_j^h b_{jc}$ can be considered as the amount of public good c consumed by consumer h in region j.

We define $\pi_c^h = \text{def}\, (\partial U^h/\partial z_c^h)/(\partial U^h/\partial z_0^h) \forall c, h$. π_c^h is household h's marginal rate of substitution between characteristic c and the numeraire, that is, h's implicit price for characteristic c in terms of the numeraire. We can then define a *Lindahl equilibrium*[12] for the economy $E_j(\bar{a})$ by an allocation $a \in F_j(\bar{a})$ and a set of implicit individual price vectors for characteristics, $\pi^h \in R_+^C$, such that:

$$\forall h, \hat{U}^h > U^h \quad \text{implies} \quad \hat{z}_0^h + \sum_c \pi_c^h \bar{x}_j^h \hat{b}_{jc} > z_0^h + \sum_c \pi_c^h \bar{x}_j^h b_{jc}; \tag{18}$$

$$(x_{jo}, B_j) \quad \text{maximizes} \quad \left\{ \sum_c b_{jc} \sum_h \bar{x}_j^h \pi_c^h - x_{jo} \right\} \quad \text{on} \quad Y_j. \tag{19}$$

In (18) $\hat{U}^h$ and U^h are the utility functions, as defined by (14), evaluated at $(\hat{z}_0^h, \hat{B}_j)$ and (z_0^h, B_j), respectively. From monotonicity of the utility functions we may set:

$$z_c^h = \sum_{k \neq j} \bar{x}_k^h \bar{b}_{kc} + \bar{x}_j^h b_{jc}, \qquad \forall c, h.$$

The conditions (18) in the definition of a Lindahl equilibrium can then be rewritten as:

$$\forall h, \ U^h(\hat{z}^h) > U^h(z^h) \quad \text{implies} \quad \hat{z}_0^h + \sum_c \pi_c^h \hat{z}_c^h > z_0^h + \sum_c \pi_c^h z_c^h. \tag{20}$$

From the literature on economies with public goods it is well-known (see, e.g., Milleron [**18**]) that under Assumptions 1 and 2, every Lindahl equilibrium for $E_j(\bar{a})$ is an element of $F_j^P(\bar{a})$, and with every allocation $a \in F_j^P(\bar{a})$, one can associate implicit prices $\pi^h \in R_+^C$, $h = 1, \ldots, H$, such that $\{a, \pi^h\}$ is a Lindahl equilibrium for $E_j(\bar{a})$.

We summarize this in a proposition.

[12] Sometimes called a "pseudo-equilibrium", as in Malinvaud [14]. The name of Lindahl has the mnemonic advantage of being associated with public goods.

PROPOSITION 4: *Under Assumptions* 1 *and* 2, *for any* $a \in A$, B_j *is a quality optimum for firm* j *relative to* $\bar{a}$ *if and only if the conditions* (19) *and* (20) *are satisfied.*

We now combine the concept of a products equilibrium for given product qualities (for given B) with that of Lindahl equilibria for the conomies $E_j(a)$, $j = 1 \ldots J$. Accordingly, an allocation $a \in A$ will be said to be a *competitive Lindahl equilibrium* if and only if there exist a commodity price vector $p \in R^J_+$ and a set of individual price vectors $\pi^h \in R^C_+$, with the input as a numeraire, such that: (i) (a, p, v) is a competitive products equilibrium relative to B; (ii) $\forall j$, $(a, \pi^h, h = 1 \ldots H)$ is a Lindahl equilibrium for the economy $E_j(a)$.

As $a \in A$ is a Lindahl equilibrium allocation for the economy $E_j(a)$ if and only if B_j is a quality optimum for firm j relative to a, we have the immediate result:

PROPOSITION 5: *Under Assumptions* 1 *and* 2 (a) *a competitive Lindahl equilibrium is a Pareto stationary point*; (b) *with every Pareto stationary point* $a \in A$, *one can associate* (p, v) *and* π^h, $h = 1 \ldots H$, *such that* a *is a competitive Lindahl equilibrium.*

For applications in Section 4, it will be convenient to allow for continuous substitution between characteristics in the production sets. To that effect we need the following strengthening of Assumption 2:

ASSUMPTION 2 BIS: $\forall j$, Y_j *is convex with relative interior points and defined by* $\phi_j(y_j) \leq 0$, $y_j \geq 0$, *where* ϕ_j *is twice continuously differentiable with* $\partial\phi_j/\partial x_{jo} < 0$, $\partial\phi_j/\partial x_j > 0$, $\partial\phi_j/\partial b_{jc} > 0$ $\forall c \neq j'$. *Moreover,* $x_j > 0$ *implies* $x_{jo} > 0$, *and* $\forall r \in R_+$, $\{y_j | \phi_j(y_j) \leq 0,\ x_{jo} \leq r\}$ *is compact.*

In that case we have the following corollary to Proposition 5.

COROLLARY 5.1:

$$(21) \qquad \forall j, \qquad \sum_h \pi_c^h x_j^h \leq \partial x_{jo}/\partial b_{jc}, \qquad \Big(\sum_h \pi_c^h x_j^h - \partial x_{jo}/\partial b_{jc}\Big) b_{jc} = 0, \qquad c \neq j',$$

are necessary conditions for Pareto optimal product quality in terms of characteristics (Pareto stationary point).[13]

PROOF: Conditions (21) are the Kuhn-Tucker necessary and sufficient[14] conditions for maximizing the maximand in (19) with respect to x_{jo}, B_j over the convex set

$$\{y_j | \phi_j(x_{jo}, x_j, b_{j1} \ldots b_{jC}) \leq 0, \quad b_{jc} \geq 0, \quad \forall c\}. \qquad Q.E.D.$$

[13] The necessary conditions (21) characterize Pareto stationary quality choices for any given quantities and may thus be understood as necessary conditions for second-best optimality (under monopoly pricing for instance).

[14] Under Assumption 2 bis, the constraint qualification is automatically satisfied.

The conditions (21) are a natural extension of the "Samuelson conditions" for Pareto optimal production of public goods (Samuelson [21]) and can be interpreted as saying that if it is at all optimal to include characteristic c in product j, the amount of characteristic c per unit output should be increased until marginal social value in terms of the numeraire equals marginal cost, where marginal social value is the sum of the individual evaluations of the marginal change in characteristic c.

The public goods aspect of product quality is due to the fact that the *feasible range of choice* in the space of characteristics is determined by the row space of B so that (i) the same range of choice is imposed on *all* consumers and (ii) if the jth row of B is changed, this common range of choice will in general be modified.

4. PROFIT MAXIMIZATION AND CONSUMERS' WELFARE

4.1

In this section we shall investigate to what extent profits will provide adequate signals for efficient choices of products qualities by firms guided by the profit motive. In order to do so, we must define more precisely profit maximization under variable product quality.

Traditionally, product differentiation and quality choice have been studied in the context of monopolistic competition. Clearly, the monopoly power of every single firm is limited by the existence of substitutability relations among commodities and there is a close connection between the presence of many close substitutes and the presence of many commodities relative to the number of characteristics. Indeed, if there were as many commodities as characteristics, then there could exist perfect substitutes in the market for any new combination of characteristics. On the other hand, if commodity j does not lie in the row space of B, it seems reasonable that firm j perceives that it has some market power. The firm can influence the demand for its product through manipulating both price and product quality in terms of characteristics.

We assume that monopolistic firms are facing continuously differentiable demand functions of the form

$$x_j = x_j(p, B) \qquad (j = 1 \dots J), \tag{22}$$

and that they use price and product quality (in terms of characteristics) as decision variables.[15]

Accordingly, an allocation $a \in A$ will be said to be a *monopolistic Nash equilibrium* if and only if p and B are such that $\forall j$, $p_j x_j(p, B) - x_{jo} \geqslant \bar{p}_j x_j(\bar{p}, \bar{B}) - \bar{x}_{jo}$ for all $\bar{p}$, $\bar{B}$, $\bar{x}_{jo}$ such that $\bar{p}_k = p_k$, $\bar{B}_k = B_k$, $\forall k \neq j$, and $(\bar{x}_{jo}, x_j(\bar{p}, \bar{B}), \bar{B}_j) \in Y_j$.

[15] The firm's opportunity set will remain convex in terms of x_{jo}, p_j, B_j if the demand function is convex in p_j, B_j. For our purposes, it is convenient to use the demand function $x_j(p, B)$ rather than the inverse demand function $p_j(x, B)$.

Under the differentiability Assumption 2 bis, the necessary[16] conditions for monopolistic profits maxima are given by

$$(23)\qquad p_j\left(\frac{1}{\eta_j}+1\right)=\frac{\partial x_{jo}}{\partial x_j},$$

$$(24)\qquad \left(p_j-\frac{\partial x_{jo}}{\partial x_j}\right)\frac{\partial x_j}{\partial b_{jc}}\leq\frac{\partial x_{jo}}{\partial b_{jc}},\qquad \left[\left(p_j-\frac{\partial x_{jo}}{\partial x_j}\right)\frac{\partial x_j}{\partial b_{jc}}-\frac{\partial x_{jo}}{\partial b_{jc}}\right]b_{jc}=0,\qquad \forall c\neq j',$$

where η_j in (23) is the price elasticity of the demand for commodity j. Combining (23) and (24), we get

$$(25)\qquad -\frac{p_j}{\eta_j}\frac{\partial x_j}{\partial b_{jc}}\leq\frac{\partial x_{jo}}{\partial b_{jc}},\qquad \left[\frac{p_j}{\eta_j}\frac{\partial x_j}{\partial b_{jc}}+\frac{\partial x_{jo}}{\partial b_{jc}}\right]b_{jc}=0,\qquad \forall c\neq j'.$$

4.2

It is not clear how one should define a competitive equilibrium when firms choose simultaneously the quantity and the qualitative characteristics of their output. To the best of our knowledge, no clear definition has been given in the literature to date. The difficulty is the following. On the one hand we wish to assume that firms act as price takers and, for given quality, choose the output which maximizes profits at given prices. Thus, we wish to assume that firms act as if prices were insensitive to their *quantity* decisions. On the other hand, we do not wish to assume that firms act as if prices were insensitive to their *quality* decisions. Indeed, if that were the case, each firm would produce the quality variant in its production set with the lowest production cost. Thus, one must assume that firms are aware of a relationship between price and quality. If there were as many commodities as characteristics, then each characteristic could have its implicit market price, and firms could be assumed to maximize profits at given prices for the characteristics. More generally, we will assume here (for lack of a more convincing alternative) that firms know how their quality choices affect the price at which a given output can be sold, and choose quality so as to maximize profits at given output. This leads to the definition of a *competitive profits equilibrium*; this term is meant to indicate that prices, but not necessarily quality choices, are "competitive". An allocation $a\in A$ is a competitive profits equilibrium if and only if p and B are such that $\forall j$,

(i) $\quad p_jx_j-x_{jo}\geq p_j\bar{x}_j-\bar{x}_{jo}\quad$ for all $\quad(\bar{x}_{jo},\bar{x}_j,B_j)\in Y_j$,

(ii) $\quad p_jx_j-x_{jo}\geq \bar{p}_jx_j-\bar{x}_{jo},\quad$ for all $\quad(\bar{x}_{jo},x_j,\bar{B}_j)\in Y_j\quad$ and $\quad\bar{p}_j$

such that $x_j=x_j(\bar{p},\bar{B}),\quad \bar{p}_k=p_k,\quad \bar{B}_k=B_k,\quad k\neq j.$

Condition (i) says that x_j maximizes firm j's profits over Y_j for *given price* and *given product quality*, while (ii) says that for *given output level* x_j, B_j and p_j are such that firm j's competitive profits are maximized.

[16] Sufficiency would require monopoly profits to be strictly concave in price and commodity characteristics.

Under Assumption 2 bis, the necessary conditions for maximization of competitive profits with respect to product quality can be written as:

$$(26) \qquad \frac{\partial x_{i0}}{\partial b_{ic}} db_{ic} \geqslant x_i \, dp_i, \qquad \forall c \neq j',$$

for all db_{ic}, dp_i such that

$$(27) \qquad \frac{\partial x_i}{\partial b_{ic}} db_{ic} + \frac{\partial x_i}{\partial p_i} dp_i = 0.$$

Solving for dp_i in (27) and substituting into (26) gives

$$(28) \qquad -\frac{p_i}{\eta_i}\frac{\partial x_i}{\partial b_{ic}} \leqslant \frac{\partial x_{i0}}{\partial b_{ic}} \qquad \forall c \neq j',$$

where (28) must hold as an equality for all c such that $b_{ic} > 0$.

Comparing (25) and (28), one sees that the necessary conditions for competitive profits equilibria are formally the same as those for monopolistic Nash equilibria, except for the fact that at any given product quality, a monopolist will charge a higher price and operate at a lower output level than a competitive firm. Hence, the monopolist extracts the monopoly profits through monopoly pricing. As to choices of product quality, he will behave as a competitive firm maximizing competitive profits. This does *not* imply that a monopolist will choose the same product quality as would an otherwise identical competitive firm. Indeed, prices and output levels will be different, so that marginal costs with respect to characteristics may also be different.

4.3

We shall now investigate under what circumstances the necessary conditions (21) for Pareto optimal choices of product quality are compatible with conditions (25) and (28) for profit maximization under monopolistic and competitive pricing of commodities, respectively.

Profit maximizing choices of product qualities will clearly depend on the properties of demand functions for commodities with multiple hedonic characteristics. To investigate some of these properties, we need two additional assumptions to ensure single-valued demand functions.

ASSUMPTION 1 BIS: *U^h is strictly quasi-concave for all h.*

ASUMPTION 3: *B has full row rank.*[17]

As shown in the Appendix (A8), under Assumptions 1 bis and 3, the effect on household h's demand for commodity j of an increase in the quantity of characteristic c contained in one unit of x_j can be written as

$$(29) \qquad \frac{\partial x_j^h}{\partial b_{jc}} = x_j^h \frac{\partial x_j^h}{\partial w_c^h} - \left(\frac{\partial x_j^h}{\partial p_j} + x_j^h \frac{\partial x_j^h}{\partial w_0^h}\right)\pi_c^h.$$

[17] Assumption 3 ensures that for fixed commodity characteristics, the utility functions are strictly quasi-concave in terms of (x_0^h, x^h).

(29) is in the nature of a Slutsky equation for quality changes and decomposes the demand effect of a quality change into an *income effect* $x_j^h\, \partial x_j^h/\partial w_c^h$ and a *substitution effect.* The income effect is *specific to characteristic c*, being proportional to the effect of an increase in the initial endowment of characteristic c.[18] The substitution effect is *specific to commodity j*, being proportional to the substitution effect of a decrease in the price of commodity j and hence always positive. The factors of proportionality relate the units of b_{jc} to those of w_c^h, and those of p_j, respectively.

In the sequel it will be convenient to use an alternative statement of (29):

$$\text{(30)} \qquad \frac{\partial x_j^h}{\partial b_{jc}} = x_j^h\left(\frac{\partial x_j^h}{\partial w_c^h} - \frac{\partial x_j^h}{\partial w_0^h}\pi_c^h\right) - \frac{\partial x_j^h}{\partial p_j}\pi_c^h$$

expressing the effect of an increase in b_{jc} as proportional to the effect of a decrease in p_j up to a correction for the possible differences between two income effects. These two income variations cancel each other in terms of utility, but not necessarily in terms of demand for commodity j. For later reference, the first term on the right-hand side of (30) will be called the *specific income effect* with respect to characteristic c.

The sign of the specific income effect will depend on how effective commodity j is in providing characteristic c, and on whether it is a superior or inferior good. Assume that $\partial x_j^h/\partial w_0^h > 0$. Then, if commodity j were purchased primarily to procure characteristic c, one would expect $\partial x_j^h/\partial w_c^h < 0$. Conversely, if commodity j were richer in most characteristics other than c, one would expect $\partial x_j^h/\partial w_c^h > 0$ and even $\partial x_j^h/\partial w_c^h > \pi_c^h \partial x_j^h/\partial w_0^h$. Loosely speaking, one would expect the specific income effect to be negative (positive) when commodity j is more (less) effective than the average basket in procuring characteristic c.

Aggregating (30) over households and substituting into (25) and (28), we get after some simple manipulations

$$\text{(31)} \qquad -\frac{p_j}{\eta_j}\sum_h x_j^h\left(\frac{\partial x_j^h}{\partial w_c^h} - \frac{\partial x_j^h}{\partial w_0^h}\pi_c^h\right) + \sum_h x_j^h \pi_c^h + \frac{1}{\eta_j}\sum_h x_j^h(\eta_j^h - \eta_j)\pi_c^h \leq \frac{\partial x_{jo}}{\partial b_{jc}}, \qquad \forall c \neq j',$$

where η_j^h is the price elasticity of household h's demand for commodity j. Observing that $\eta_j = \Sigma_h x_j^h \eta_j^h / \Sigma_h x_j^h$, i.e., a weighted average over households of the price elasticities of the individual demand functions, the last term on the left-hand side of (31) can be interpreted as x_j/η_j times the covariance over households between the marginal rates of substitution, π_c^h, and the individual price elasticities η_j^h.

In view of Corollary 5.1, conditions (31) provide necessary and sufficient conditions for efficiency of quality choices guided by profit motives, when efficiency must here be understood in the sense of Pareto stationarity. It is left for the reader to decide whether he wants to regard these necessary conditions as

[18] If characteristic c of commodity j is changed by db_{jc}, consumer h will enjoy an increase in his endowment of characteristic c by $dw_c^h = x_j^h\, db_{jc}$.

quite restrictive,[19] pointing to likely inefficiencies of the market mechanism, even when competitive, or whether he wants to regard the sufficient conditions as apt to be approximately satisfied,[20] pointing to the opposite conclusion.

THEOREM 1: *In order for profit maximizing firms to choose a Pareto stationary amount of characteristic c per unit of output*:

(*i*) *it is sufficient that the specific income effect vanishes, for each individual, and that the individual marginal rates of substitution* π_c^h *be uncorrelated with the individual price elasticities*;

(*ii*) *it is necessary that the covariance between the individual price elasticities and marginal rates of substitution be equal to the average specific income effect, weighted by the expenditures* $p_j x_j^h$.

Let $dB_j = (db_{j1} \dots db_{jC})'$ denote a vector of marginal variations of the characteristics embodied in commodity j. From Theorem 1, we have the following corollaries.

COROLLARY 1.1: *A sufficient condition under which the marginal quality variation* dB_j *will increase the* (*monopoly or competitive*) *profits of firm* j *only if this variation is consistent with the interests of the consumers, is that the variation* dB_j *lie in the space of rows of* B *other than row* j.

PROOF: Suppose dB_j is in the row space of B. Then there exists a vector $\alpha = (\alpha_1 \dots \alpha_J)'$ such that $dB_j = \alpha' B$. In that case $\sum_c \pi_c^h db_{jc} = \alpha' B \pi^h = \alpha' p$, by (A2) (Appendix), so that the covariance term vanishes. Moreover, from (A5),

$$x_j^h \sum_c \left(\frac{\partial x_j^h}{\partial w_c^h} - \frac{\partial x_j^h}{\partial w_0^h} \pi_c^h \right) db_{jc}$$

$$= -x_j^h \{([-p, B]\underline{U}[-p, B]')^{-1}\}_j [-p, B]\underline{U}[-\pi^h, I]B'\alpha$$

$$= -x_j^h \delta_j' \alpha = -x_j^h \alpha_j$$

since $[-\pi^h, I]B' = [-p, B]'$ in view of (A2).

Hence, if $\alpha_j = 0$, then

$$-\frac{p_j}{\eta_j} \sum_c \frac{\partial x_j}{\partial b_{jc}} db_{jc} \geq \sum_c \frac{\partial x_{j0}}{\partial b_{jc}} db_{jc} \Leftrightarrow \sum_c \sum_h x_j^h \pi_c^h db_{jc} \geq \sum_c \frac{\partial x_{j0}}{\partial b_{jc}} db_{jc}. \qquad Q.E.D.$$

COROLLARY 1.2: *If,* $\forall j$, *all feasible variations* dB_j *are contained in the space of rows of* B *other than row* j, *then the competitive profits equilibrium is a Pareto stationary point.*

[19] Very restrictive assumptions on individual preferences are required to guarantee that specific income effects vanish (for instance linearity in both numeraire and characteristics).

[20] Some may claim that specific income effects are in the nature of second-order effects and may therefore be ignored as a first approximation.

PROOF: Under the conditions stated in the corollary, the competitive profits equilibrium is a competitive Lindahl equilibrium. *Q.E.D.*

REMARKS: (i) Under monopolistic pricing of commodities, Theorem 1 must be understood to be relative to a given output level. That is, *given the output level* x_j, monopolist j will choose a Pareto stationary amount of characteristic c per unit output if he is equating marginal social value to marginal cost with respect to characteristics.

(ii) The efficiency of profit maximizing choices of product quality has been explored in a partial equilibrium context by Spence [23, 24] and Sheshinski [22] among others. Their main point is that profit maximizing firms will respond to the market's ***marginal*** valuation of product quality while the ***average*** valuation of product quality (at the margin) is relevant for welfare judgements and they present necessary conditions for efficiency of market choices of product quality in terms of properties of the inverse demand function. Theorem 1 deals with the same problem in a general equilibrium framework and presents necessary conditions for market efficiency with respect to product quality in terms of properties of the individual and aggregate demand functions.

(iii) There is quite an extensive literature on product durability under different market structures which is closely related to the problem of quality choice and product differentiation. In that literature some authors have argued that a monopoly is likely to supply a good of a lower quality than would a competitive firm with the same cost structure,[21] while others have argued that monopoly exploitation is best achieved by manipulating price rather than product quality.[22] The preceding analysis reveals that market imperfections in the quantitative supply of commodities can be separated from market imperfections in the choices of product qualities. As for choices of product quality, both monopolistic and price-taking firms will take into account the demand effects of different product designs and if the marginal variations in the quality of a given product do not lie in the space of rows of B other than the row corresponding to the product concerned, an element of market imperfection enters the picture and profit maximizing choices of product quality will not be conducive to maximal consumer welfare.

5. COMPUTABLE LINDAHL EQUILIBRIA AND CONSUMERS' UNANIMITY

5.1

As stated in Proposition 5, Pareto stationary choices of product qualities require knowledge of the implicit prices for characteristics for all households and these prices are typically not revealed in the market. One might, however, ask whether these prices can be computed from available market data such as commodity prices and the technology matrix.

[21] See Levhari and Srinivasan [13].

[22] See Swan [25].

From the necessary conditions (A2) for interior consumer optima in the commodity space we have that equilibrium commodity prices and implicit prices for characteristics are related by $p = B\pi^h$. Hence, if B were non-singular, the implicit prices could be computed from $\pi^h = B^{-1}p$ and they would in equilibrium be the same for all consumers. In that case, the firms could check the necessary conditions (21) for Pareto optimality if they cared to do so.

More generally, let $dB_j = (db_{j1} \dots db_{jC})'$ be in the row space of B. In that case, $\Sigma_c \pi_c^h \, db_{jc} = \alpha' B\pi^h = \alpha' p$ for all h, so that the marginal social value of the quality variation dB_j could be computed from $\Sigma_c \Sigma_h x_j^h \pi_c^h \, db_{jc} = x_j \alpha' p$ which only requires knowledge of commodity prices and the technological relationship $\alpha = (\alpha_1 \dots \alpha_J)'$. Stated alternatively, one would in this case only need to know the implicit prices of *one* (arbitrary) consumer in order to check the necessary conditions (21).

In the general case where dB_j does not lie in the row space of B, consumers will evaluate the marginal variation dB_j differently and these individual evaluations can no longer be obtained from available market data (commodity prices and the technology matrix). This reveals that if variations of product quality change the range of choice for consumers in the characteristics space, there will *not* exist any natural market substitute for aggregation of consumer preferences.

5.2

The case where quality changes lie in the row space of B has received substantial attention in the literature on competitive capital markets (Ekern and Wilson [7], Leland [12] and Radner [19]). One has there emphasized the so-called "unanimity theorem" stating that all consumers agree in their evaluation $\Sigma_c \pi_c^h \, db_{jc} = \alpha' p$ of the proposed marginal variation in the jth row of B. It has been verified here that this common evaluation does not have the same sign as the resulting effect on firms' profits under either monopolistic or competitive pricing of commodities, unless the specific income effect vanishes. Consumers' unanimity in evaluating a proposed variation is not always consistent with profit maximization, a point already emphasized in Ekern and Wilson [7] and Leland [12]. The interpretation given there is that consumers' unanimity departs from profit maximization when commodity markets are not perfectly competitive. However, market imperfections in the *qualitative supply of characteristics*, created by the specific income effect, are distinct from market imperfections in the *quantitative supply of commodities* and the inefficiency of profit maximizing choices of product quality arises both under monopolistic and competitive pricing of commodities. Moreover, profit maximizing choices of *product qualities* would be consistent with the interests of the consumers both under monopolistic and competitive supply of commodities if the conditions of Theorem 1 are fulfilled, a case which has not been considered in the literature referred to above.

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APPENDIX

We shall here derive the Slutsky equation (29) for an arbitrary household h whose superscript is omitted for ease of notation.

First we introduce some notation. If M is any matrix, M_k will denote its kth row and M^l its lth column. If x and y are column vectors, $\partial x/\partial y'$ will denote the matrix with element $\partial x_k/\partial y_l$ in the kth row and the lth column. δ_k denotes the Kronecker delta vector whose kth element is equal to one, all other elements being equal to zero.

Starting from the twice differentiable, strictly quasi-concave utility function $U(z)=U(z_0, z_1 \dots z_C)$, we shall denote by $\underline{u}$ the $(C+1)$-dimensional vector $(\partial U/\partial z_0, \partial U/\partial z_1 \dots \partial U/\partial z_C)'$ and by $\underline{U}$ the $(C+1)\times(C+1)$ matrix with elements $\partial^2 U/\partial z_k\, \partial z_l$, $k, l=0, 1 \dots C$; the $C+1$ rows and columns of $\underline{U}$ will be numbered from O to C (thus, $\underline{U_0}$ is the first row of $\underline{U}$, $\underline{U_k}$ its $k+1$st row now with entries $\partial^2 U/\partial z_k\, \partial z_l$, $l=0, 1 \dots C$).

Using (0) and (6), but ignoring the distinction between w_0 and v, and denoting by ${}_0w$ the C-dimensional vector $(w_1 \dots w_C)'$, the utility function may be written as

$$\text{(A1)} \qquad U(z)=U(w_0-p'x, {}_0w'+x'B)=U(x|w, p, B),$$

a formulation that incorporates both the budget constraint (6) and the technological relationships between commodities and characteristics (5). At an interior solution the first order conditions are

$$\text{(A2)} \qquad \frac{\partial U}{\partial x}=[-p, B]\underline{u}=0$$

where $[-p, B]$ is the B-matrix augmented by the negative of the price vector as the first column. The $(J\times J)$ matrix of second order derivatives of U with respect to x (for given w, p, B)

$$\text{(A3)} \qquad \frac{\partial^2 U}{\partial x\, \partial x'}=[-p, B]\underline{U}[-p, B]'$$

is negative definite by Assumptions 1 bis and 3. The second-order conditions are thus satisfied.

Differentiation of (A2) yields

$$\begin{aligned} 0&=\frac{\partial^2 U}{\partial x\, \partial x'}\,dx+\frac{\partial^2 U}{\partial x \partial[-p, B]_i}(-dp_i, dB_i)' \\ &=[-p, B]\underline{U}[-p, B]'\,dx+\{x_i[-p, B]\underline{U}+\delta_i\underline{u}'\}(-dp_i, dB_i)' \end{aligned}$$

and hence,

$$\text{(A4)} \qquad \frac{\partial x}{\partial[-p, B]_i}=-\{[-p, B]\underline{U}[-p, B]'\}^{-1}\{x_i[-p, B]\underline{U}+\delta_i\underline{u}'\},$$

$$0=\frac{\partial^2 U}{\partial x\, \partial x'}\,dx+\frac{\partial^2 U}{\partial x\, \partial w'}\,dw=[-p, B]\underline{U}[-p, B]'\,dx+[-p, B]\underline{U}\,dw,$$

and hence,

$$\text{(A5)} \qquad \frac{\partial x}{\partial w'}=-\{[-p, B]\underline{U}[-p, B]'\}^{-1}[-p, B]\underline{U}.$$

The invariance of the demand functions with respect to monotonic transformations of the utility functions is readily verified. Indeed, let $V=F(U)$. Then $\partial V/\partial z=F'(\partial U/\partial z)$, $(\partial^2 V/\partial z\, \partial z')=F'(\partial^2 U/\partial z\, \partial z')+F''(\partial U/\partial z)(\partial U/\partial z')$ where F' and F'' denote derivatives of V with respect to U. Then $\partial V/\partial x=F'(\partial U/\partial x)=0$, by (A2), and

$$\frac{\partial^2 V}{\partial x\, \partial x'}=[-p, B]\left(F'\frac{\partial^2 U}{\partial z\, \partial z'}+F''\frac{\partial U}{\partial z}\frac{\partial U}{\partial z'}\right)[-p, B]'=F'\frac{\partial^2 U}{\partial x\, \partial x'},$$

by (A3), since $F''(\partial U/\partial z)(\partial U/\partial z')\,[-p, B]'=0$, by (A2). Then, in (A4) and (A5) the scalar F' cancels out between the inverse matrix and its multiplicand.

Upon combining (A4) and (A5), we get:

$$\text{(A6)} \qquad \frac{\partial x}{\partial[-p, B]_l} - x_l \frac{\partial x}{\partial w'} = -\{[-p, B]\underline{U}[-p, B]'\}^{-1}\delta_l \underline{u}'$$

$$= -\left\{\left(\frac{\partial^2 U}{\partial x\, \partial x'}\right)^{-1}\right\}^l \underline{u}',$$

$$\text{(A7)} \qquad \frac{\partial x}{\partial p'} + \frac{\partial x}{\partial w_0} x' = \left(\frac{\partial^2 U}{\partial x\, \partial x'}\right)^{-1} \underline{u}_0,$$

where $\underline{u}_0$ has a "marginal utility of income" interpretation. Looking at a particular element of (A6), we can write:

$$\text{(A8)} \qquad \frac{\partial x_k}{\partial b_{lc}} = x_l \frac{\partial x_k}{\partial w_c} - \{([-p, B]\underline{U}[-p, B]')^{-1}\}_{kl} \underline{u}_c$$

$$= x_l \frac{\partial x_k}{\partial w_c} - \left(\frac{\partial x_k}{\partial p_l} + x_l \frac{\partial x_k}{\partial w_0}\right)\pi_c \qquad (k = 1 \ldots J)$$

where $\pi_c = \underline{u}_c / \underline{u}_0$.

Under the assumption that B is a square matrix and hence non-singular by Assumption 3, we can take another step, writing from (A2) and (A5):

$$\text{(A9)} \qquad \pi = B^{-1}p, \qquad \frac{\partial x}{\partial w'}[-\pi, I]' = -B'^{-1} = \frac{\partial x}{\partial_0 w'} - \frac{\partial x}{\partial w_0}\pi',$$

$$\text{(A10)} \qquad \frac{\partial x_k}{\partial b_{lc}} = -x_l (B^{-1})_{ck} - \frac{\partial x_k}{\partial p_l}(B^{-1})_c p.$$

Finally, if B were an identity matrix, each commodity being identified with a single characteristic, then:

$$\text{(A11)} \qquad \frac{\partial x_k}{\partial b_{lc}} = \begin{cases} -x_l - \dfrac{\partial x_k}{\partial p_l} p_k, & c = k, \\[2ex] -\dfrac{\partial x_k}{\partial p_l} p_c, & c \neq k. \end{cases}$$

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ENVIRONMENTAL EFFECTS ON CONSUMER DEMAND AND SOME ASPECTS OF THE PROBLEM OF OPTIMAL ENVIRONMENTAL DESIGN WITH SPECIAL REFERENCE TO THE CHOICE OF PRODUCT QUALITY.

1. Introduction

Apart from prices and incomes, consumer demand functions will clearly depend on the consumer's environment. Broadly speaking, the consumer's environment may be considered as being made up of all welfare-relevant phenomena which are beyond the control of the individual consumer. More specifically, in the present paper the consumer's environment will be defined in such a way as to consist of all non-market goods affecting the consumer's welfare, i.e., goods which are not exchangeable through commodity markets. The supply of public goods and various types of externalities may clearly be considered as part of the consumer's environment. Preallocated goods and goods subject to straight rationing will also fall under this category.[1)] Even certain aspects of market goods which the consumers value but are beyond the control of the individual consumer such as product quality, may be thought of as environmental parameters determining the environment under which consumption of market goods takes place.

The economic consequences of changing the environment are not priced in the market. The market responses to changes in the environment are indirect in the sense that they will change the consumers' demand pattern for market goods. In some cases the producer can control certain aspects of the environment under which the consumption of his products takes place. In that case it seems natural to ask whether the producer's profits will adequately reflect the consumers' evaluation of those aspects of the environmental design which are under the producer's control such that profit maximization will guide the producer to choose an environmental design in the best interests of the consumers.

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1) See Pollak [7].

Effects of environmental changes on consumption behaviour and some aspects of optimal design of the environment will be discussed in the present paper. The problem of optimal environmental design will be discussed in the context of optimal choice of product quality. For this purpose we shall adopt the consumption activity approach to consumer demand theory as formulated by Lancaster [4] and Muth [6]. This approach assumes that market goods are inputs into consumption activities producing more basic or final goods over which preferences are defined. These final goods are in some contexts called characteristics (see Lancaster [4]) and are supposed to be the ultimate objects of consumer satisfaction. Thus, preferences over market goods are indirect or derived from preferences over final goods and properties of the household consumption technology.

Efficiency of market choices of product quality has been studied by Spence [9, 10] and Sheshinski [8] in a partial equilibrium context and by Drèze and Hagen [2] in a general equilibrium context but with a linear consumption technology. Spence and Sheshinski have emphasized that average valuation of product quality is relevant from an efficiency point of view while profit maximizing firms will respond to the market's marginal valuation of product quality. Drèze and Hagen have stressed that whatever the pricing of commodities, efficiency will generally not obtain if a profit maximizing producer explores his demand curve through his choice of product quality. In the present paper the results of Spence and Sheshinski and those of Drèze and Hagen will be reconciled in a general equilibrium framework and we show that in the Lancaster-Muth framework, deviation between consumers' average and marginal valuations of product quality may occur if the consumption technologies are non-linear.

2. Effects of environmental changes on consumer demand.

We shall assume a finite number J of market goods indexed j and C different final goods indexed c. Each consumer is endowed with a consumption technology which relates feasible output vectors of final goods to input vectors of market goods. It will be assumed that each final good is producible in one and only one activity and by obvious notation final good c is produced in activity c.

It is assumed that market goods are all joint inputs into the production of final goods.[1] We let x_j denote the amount the consumer[2] uses of market good j and z_c his consumption of final good c.

For a given household production technology the production of final goods will depend on the input vector of market goods and the environmental setting under which the production takes place. We shall assume that the environment can be completely described by a finite-dimensional vector of environmental parameters. We do not loose in generality by assuming that there is a one-to-one correspondence between environmental parameters and consumption activities and the environmental parameter(s) pertaining to the production of final good c will be denoted by E_c which will here be taken to be a scalar. More general cases where some environmental parameters affect a subset of consumption activities, possibly different subsets for different consumers, would not present any further difficulty.

We let φ_c denote the consumer's production function for final good c. Thus, we have

$$(1) \qquad z_c = \varphi_c(x_1, \ldots, x_J, E_c) \qquad c=1,\ldots,C.$$

Hence, from the standpoint of the individual consumer the environmental parameters will be in the nature of public goods or bads according as increases in the quantities representing these parameters will cause positive or negative shifts in the household production functions.

It will be assumed that there exists a market good which is present in final form. This commodity is indexed zero and used as a numeraire good.

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[1] Extending the analysis to the case where the inputs of market goods are specific to activities would be straight-forward.

[2] In the present section devoted to comparative statics analysis of the individual consumer, the consumer index is omitted.

The consumer's ranking of bundles of final goods is represented by a quasi-concave utility function

$$(2) \qquad U = U(z_0, z_1, \ldots, z_C)$$

where z_c is given by (1). It is assumed that $\partial U/\partial z_c \equiv U_c > 0$, $c = 0, 1, \ldots, C$.

For a given environment, (2) induces a preference ordering over market commodity bundles $(z_0, x_1, \ldots, x_J)$ through the derived utility function

$$(3) \qquad W(z_0, x_1, \ldots, x_J) \equiv U(z_0, \varphi_1(x_1, \ldots, x_J, E_1), \ldots, \varphi_C(x_1, \ldots, x_J, E_C))$$

If the environment changes, the consumer's preferences over market commodities will change since his consumption technology is changed.

For reasons to become clear below, we assume that the consumer has initial stocks of the numeraire good and final goods given by the endowment vector [1] $\omega = (\omega_0, \omega_1, \ldots, \omega_C)'$ where the initial stocks of final goods may be thought of as given by initial inputs of market goods. The budget constraint is given by $\sum_j p_j x_j + z_0 = \omega_0$ and substituting into (2) the consumer optimum is determined by maximizing

$$U(\omega_0 - \sum_j p_j x_j, \ \omega_1 + \varphi_1(x, E_1), \ \ldots, \ \omega_C + \varphi_C(x, E_C))$$

where $x = (x_1, \ldots, x_J)'$. This expression contains both the budget constraint and the technological relationships between market goods and final goods.[2]

The first-order conditions for an interior optimum are

$$(4) \qquad \frac{\partial U}{\partial x_j} = -U_0 p_j + \sum_c U_c \frac{\partial \varphi_c}{\partial x_j} = 0 \qquad\qquad j = 1, \ldots, J.$$

[1] In this paper untransposed vectors mean column vectors.

[2] The initial stocks of final goods may be viewed as additive shift parameters in the household production functions.

The (J x J)-matrix $\frac{\partial^2 U}{\partial x \partial x'}$ of second-order derivatives with typical entry $\partial^2 U/\partial x_j \partial x_k$ in the j-th row and k-th column will be assumed to be negative definite so that the second-order maximum conditions are satisfied. With $U(\cdot)$ quasi-concave and strictly increasing in all final goods, it is sufficient for $U(\cdot)$ to be strictly quasi-concave in the decision variables $(z_0, x_1, \ldots, x_J)$ that the household production functions $\varphi_c(x, E_c)$ are strictly concave in the inputs of market goods.

We may rewrite (4) as

$$(5) \qquad p_j = \sum_c \pi_c \frac{\partial \varphi_c}{\partial x_j} \qquad j = 1, \ldots, J$$

where $\pi_c \equiv U_c/U_0$ is the consumer's marginal rate of substitution between final good c and the numeraire, that is, the consumer's implicit price for final good c in terms of the numeraire. Hence, at the consumer optimum, the consumer's demand price for market good j equals the sum of the values of the marginal products of market good j in the production of final goods valuated at the consumer's implicit prices for final goods.

Differentiating the k-th equation in (4) with respect to p_j gives

$$(6) \qquad \frac{\partial^2 U}{\partial x_k \partial p_j} = U_{00} p_k x_j - \sum_c U_{c0} \frac{\partial \varphi_c}{\partial x_k} x_j - \delta_{kj} U_0, \qquad k=1,\ldots,J, \quad j=1,\ldots,J$$

where U_{cd} denotes the second-order partial derivative $\partial^2 U/\partial z_c \partial z_d$, c, d = 0,1, ..., C, and δ_{kj} denotes the Kronecker delta. Differentiating (4) with respect to E_d, ω_0 and ω_d, respectively, gives

$$(7) \qquad \frac{\partial^2 U}{\partial x_k \partial E_d} = -U_{0d} p_k \frac{\partial \varphi_d}{\partial E_d} + \sum_c U_{cd} \frac{\partial \varphi_c}{\partial x_k} \frac{\partial \varphi_d}{\partial E_d} + U_d \frac{\partial^2 \varphi_d}{\partial x_k \partial E_d}, \qquad k=1,\ldots,J$$

$$(8) \qquad \frac{\partial^2 U}{\partial x_k \partial \omega_0} = -U_{00} p_k + \sum_c U_{c0} \frac{\partial \varphi_c}{\partial x_k} \qquad k=1,\ldots,J$$

$$(9) \qquad \frac{\partial^2 U}{\partial x_k \partial \omega_d} = -U_{0d} p_k + \sum_c U_{cd} \frac{\partial \varphi_c}{\partial x_k} \qquad k=1,\ldots,J$$

Taking the total differential of (4) with respect to the input vector $x = (x_1, \ldots, x_J)'$ and the price p_j gives

(10) $$\frac{\partial^2 U}{\partial x \partial x'} dx + \frac{\partial^2 U}{\partial x \partial p_j} dp_j = 0$$

where $dx = (dx_1, \ldots, dx_J)'$. Solving (10) with respect to $\frac{\partial x}{\partial p_j} \equiv (\frac{\partial x_1}{\partial p_j}, \ldots, \frac{\partial x_J}{\partial p_j})'$ gives

(11) $$\frac{\partial x}{\partial p_j} = - (\frac{\partial^2 U}{\partial x \partial x'})^{-1} \frac{\partial^2 U}{\partial x \partial p_j}$$

where the inverse matrix exists by the negative definiteness assumption. The k-th component of the column vector $\partial x/\partial p_j$ is given by

(12) $$\frac{\partial x_k}{\partial p_j} = - \sum_t (\frac{\partial^2 U}{\partial x_k \partial x_t})^{-1} \frac{\partial^2 U}{\partial x_t \partial p_j}$$

The (J x J)-matrix of price derivatives of the consumption vector x is consequently given by

(13) $$\frac{\partial x}{\partial p'} = - (\frac{\partial^2 U}{\partial x \partial x'})^{-1} \frac{\partial^2 U}{\partial x \partial p'}$$

Differentiating (4) with respect to x and ω_0 and solving for $\partial x/\partial \omega_0$ yields

(14) $$\frac{\partial x}{\partial \omega_0} = - (\frac{\partial^2 U}{\partial x \partial x'})^{-1} \frac{\partial^2 U}{\partial x \partial \omega_0}$$

where the k-th component of (14) is

(15) $$\frac{\partial x_k}{\partial \omega_0} = - \sum_t (\frac{\partial^2 U}{\partial x_k \partial x_t})^{-1} \frac{\partial^2 U}{\partial x_t \partial \omega_0}$$

Combining (12) and (15) and using (6) and (8) gives

$$(16) \qquad \frac{\partial x_k}{\partial p_j} + \frac{\partial x_k}{\partial \omega_0} x_j = -\sum_t \left(\frac{\partial^2 U}{\partial x_k \partial x_t}\right)^{-1} \left(\frac{\partial^2 U}{\partial x_t \partial p_j} + \frac{\partial^2 U}{\partial x_t \partial \omega_0} x_j\right)$$

$$= \sum_t \left(\frac{\partial^2 U}{\partial x_k \partial x_t}\right)^{-1} \delta_{tj} U_0 = \left(\frac{\partial^2 U}{\partial x_k \partial x_j}\right)^{-1} U_0$$

In matrix form we then have

$$(17) \qquad \frac{\partial x}{\partial p'} + \frac{\partial x}{\partial \omega_0} x' = \left(\frac{\partial^2 U}{\partial x \partial x'}\right)^{-1} U_0$$

The left-hand side of (17) is the Slutsky substitution matrix $(\partial x/\partial p')_{U=\text{const.}}$ with negative diagonal entries by the negative definiteness of $\partial^2 U/\partial x \partial x'$.

Taking the total differential of (4) with respect to x and E_d and then with respect to x and ω_d and solving gives

$$(18) \qquad \frac{\partial x_k}{\partial E_d} = -\sum_t \left(\frac{\partial^2 U}{\partial x_k \partial x_t}\right)^{-1} \frac{\partial^2 U}{\partial x_t \partial E_d}$$

$$(19) \qquad \frac{\partial x_k}{\partial \omega_d} = -\sum_t \left(\frac{\partial^2 U}{\partial x_k \partial x_t}\right)^{-1} \frac{\partial^2 U}{\partial x_t \partial \omega_d}$$

Combining (18) and (19) and using (7) and (9) gives

$$(20) \qquad \frac{\partial x_k}{\partial E_d} - \frac{\partial x_k}{\partial \omega_d}\frac{\partial \varphi_d}{\partial E_d} = -\sum_t \left(\frac{\partial^2 U}{\partial x_k \partial x_t}\right)^{-1} \left(\frac{\partial^2 U}{\partial x_t \partial E_d} - \frac{\partial^2 U}{\partial x_t \partial \omega_d}\frac{\partial \varphi_d}{\partial E_d}\right)$$

$$= -\sum_t \left(\frac{\partial^2 U}{\partial x_k \partial x_t}\right)^{-1} U_d \frac{\partial^2 \varphi_d}{\partial x_t \partial E_d} .$$

From (16) we have

$$(21) \qquad \left(\frac{\partial^2 U}{\partial x_k \partial x_t}\right)^{-1} = \frac{1}{U_0}\left(\frac{\partial x_k}{\partial p_t} + \frac{\partial x_k}{\partial \omega_0} x_t\right)$$

and substituting into (20) gives

$$(22)\qquad \frac{\partial x_k}{\partial E_d} = \frac{\partial x_k}{\partial \omega_d}\frac{\partial \varphi_d}{\partial E_d} - \sum_t \left(\frac{\partial x_k}{\partial p_t} + \frac{\partial x_k}{\partial \omega_0} x_t\right)\pi_d \frac{\partial^2 \varphi_d}{\partial x_t \partial E_d}$$

Equation (22) is in the nature of a Slutsky equation[1)] for the effects on demand for commodity k caused by a change in the environmental parameter E_d. To see this, we observe that for given implicit prices of final goods, i.e., omitting second-order terms involving $\partial\pi_c/\partial E_d$, $\pi_d \partial^2\varphi_d/\partial x_t \partial E_d$ represents the change in the consumer's demand price for commodity t caused by a change in E_d. Hence, the last term on the right-hand side of (22) is in the nature of a substitution effect from the (first-order) changes in demand prices caused by the change in E_d, while the first term has the interpretation of an income effect and is specific to final good d.

With independent market commodities and omitting the income effect specific to final good d, we see that market good k and non-market good d will be substitutes (complements) according as an increase in E_d will decrease (increase) the marginal productivity of market good k in the production of final good d which is reminiscent of the cardinal Edgeworth-Pareto definition of substitutability and complementarity.[2)]

For later use it will be useful to rewrite the equation (22) as

$$(23)\qquad \frac{\partial x_k}{\partial E_d} = \left(\frac{\partial x_k}{\partial \omega_d}\frac{\partial \varphi_d}{\partial E_d} - \sum_t \frac{\partial x_k}{\partial \omega_0} x_t\, \pi_d \frac{\partial^2 \varphi_d}{\partial x_t \partial E_d}\right) - \sum_t \frac{\partial x_k}{\partial p_t}\pi_d \frac{\partial^2 \varphi_d}{\partial x_t \partial E_d}$$

Hence, up to a correction term, the effect on the demand for commodity k caused by a change in the environmental parameter E_d, will be equal to a weighted sum of the uncompensated price derivatives of demand for good k with weights given by the (first-order) changes in the consumer demand prices.

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1) See [2] for a similar result in the case of a linear consumption technology.

2) See Chipman [1].

The correction term is a difference between two income effects: The first one is caused by the fact that an increase in E_d will increase the consumer's initial endowment of final good d, and the second one by the fact that a change in demand prices will change the (perceived) value of the consumer's initial endowments. This correction term, which for later reference will be called the specific income effect, will vanish in terms of utility but not necessarily in terms of consumer demand.

3. Choice of product quality.

The consumption activity framework lends itself naturally to the analysis of firms' decisions on product quality and problems related to efficient choices of product quality. Indeed, from the standpoint of the individual consumer the quality of market goods will be in the nature of environmental characteristics determining the relationships between quantities of inputs of market goods and the output of final goods produced.

To get a simple parameterization of product quality, we assume that the quality of a unit of a given commodity is objectively determined by a finite set of product attributes (the product design). Without loss in generality we assume that there is a one-to-one relationship between final goods and product attributes and the quality of product j can then be represented by a vector $E_j = (E_{j1}, \ldots, E_{jC})$ where E_{jc} is the amount of attribute c embodied in a unit of commodity j.

The household's production of final good c is then given by the input of market goods into consumption activity c and the amount of attribute c embodied per unit of the various inputs. This production relation is assumed to be given by

$$(24) \qquad z_c^h = \varphi_c^h(x_1^h, \ldots, x_J^h, E_{1c}, \ldots, E_{Jc}) \qquad c = 1, \ldots, C$$

where the superscript h refers to consumer h. We may note that the "hedonic" characteristics approach of Lancaster [4] with a linear consumption technology is a special case of (24) where $z_c^h = \sum_j x_j^h E_{jc}$.

Commodity j is produced by firm j in the quantity of x_j. We assume that the numeraire good is a single common input to all firms and x_{j0} denotes the amount of the numeraire good used by firm j (this input may for example be labor).

The attributes of commodity j can be controlled by firm j and the firm selects a production plan $(x_{j0}, x_j, E_{j1}, \ldots, E_{jC})$ in its production set $Y_j \subseteq R_+^{C+2}$. We assume that the efficiency frontier of Y_j is given by the implicit production function

$$\psi_j(x_{j0}, x_j, E_{j1}, \ldots, E_{jC}) = 0 \qquad j = 1, \ldots, J.$$

A feasible allocation in such an economy is a set of consumption vectors (z_0^h, x^h) and a set of production plans $(x_{j0}, x_j, E_{j1}, \ldots, E_{jC})$ satisfying

$$\text{(25)} \qquad \sum_h z_0^h + \sum_j x_{j0} \leqq \sum_h \omega_0^h$$

$$\text{(26)} \qquad \sum_h x_j^h \leqq x_j, \qquad j = 1, \ldots, J$$

$$\text{(27)} \qquad \psi_j(x_{j0}, x_j, E_{j1}, \ldots, E_{jC}) = 0 \qquad j = 1, \ldots, J,$$

A Pareto optimal allocation is a feasible allocation maximizing

$$\sum_h \lambda^h U^h(z_0^h, z_1^h, \ldots, z_C^h), \qquad \lambda^h \geqq 0 \text{ for all } h$$

where $z_c^h = \omega_c^h + \varphi_c^h(x^h, E_{1c}, \ldots, E_{Jc})$.

The first-order maximum conditions can be written as[1)]

1) From strict monotonicity of the utility functions condition (25) and (26) will hold as strict equalities at a Pareto optimum. Also, we have assumed that possible sign restrictions on commodities and attributes are not binding.

$$\sum_c \pi_c^h \frac{\partial \varphi_c^h}{\partial x_j^h} = \frac{\partial x_{j0}}{\partial x_j} \qquad \text{for all } h \text{ and } j \tag{28}$$

$$\sum_h \pi_d^h \frac{\partial \varphi_d^h}{\partial E_{jd}} = \frac{\partial x_{j0}}{\partial E_{jd}} \qquad \text{for all } d \text{ and } j. \tag{29}$$

The condition (28) determines optimal output level of commodity j for given product attributes while (29) determines optimal product attributes for given output level.

We shall now examine whether profit maximization will lead firms to choose production plans, and in particular, product attributes, which maximize consumers' welfare. From (5) and (28) we see immediately that marginal cost pricing of commodities is necessary for the market choices of product <u>quantities</u> to be efficient.

As to profit maximizing choices of product <u>quality</u>, it must be the case that for any quantity supplied, the firm believes that it can influence the price for its product through manipulating the product quality. Otherwise, firms' quality decisions would be independent of the demand side in the sense that for any output level they would choose product qualities such as to minimize costs which does not seem to be a very plausible assumption for an economy with product differentiation.

We assume that attribute d of firm j's product is changed whilst the quantity supplied is kept unchanged. Then, ceteris paribus, the firm will expect the price for its product to change such that[1)]

$$\frac{\partial x_j}{\partial p_j}\frac{\partial p_j}{\partial E_{jd}} + \frac{\partial x_j}{\partial E_{jd}} = 0 \quad \text{and hence,} \quad \frac{\partial p_j}{\partial E_{jd}} = -\frac{1}{\partial x_j/\partial p_j}\frac{\partial x_j}{\partial E_{jd}}.$$

Thus, marginal revenue with respect to product quality for <u>given quantity</u> will be

$$x_j \frac{\partial p_j}{\partial E_{jd}} = -\frac{x_j}{\partial x_j/\partial p_j}\frac{\partial x_j}{\partial E_{jd}} \qquad \text{for all } d. \tag{30}$$

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1) With this behavioural assumption the resulting equilibrium will be of the Nash type.

We write the specific income effect in (23) as

$$I_{jd}^{h} \equiv \frac{\partial x_j^h}{\partial \omega_d^h} \frac{\partial \varphi_d^h}{\partial E_{jd}} - \sum_t \frac{\partial x_j^h}{\partial \omega_0^h} x_t^h \pi_d^h \frac{\partial^2 \varphi_d^h}{\partial x_t^h \partial E_{jd}}$$

and for given implicit prices, we write the (first-order) change in consumer h's demand price for commodity t caused by marginal changes in attribute d of commodity j as

$$\frac{\partial p_t^h}{\partial E_{jd}} \equiv \pi_d^h \frac{\partial^2 \varphi_d^h}{\partial x_t^h \partial E_{jd}}$$

Substituting (23) into (30) and using these definitions, firm j's marginal revenue with respect to E_{jd} for <u>given</u> x_j, can after some manipulations be written as

$$\text{(31)} \qquad x_j \frac{\partial p_j}{\partial E_{jd}} = \sum_h \frac{\partial \varphi_d^h}{\partial E_{jd}} \pi_d^h - \frac{p_j}{\eta_j} \sum_h I_{jd}^h + \sum_h x_j^h \left(\frac{\partial p_j^h}{\partial E_{jd}} - \frac{1}{x_j^h} \frac{\partial \varphi_d^h}{\partial E_{jd}} \pi_d^h\right)$$

$$+ \frac{1}{\eta_j} \sum_h x_j^h (\eta_j^h - \eta_j) \frac{\partial p_j^h}{\partial E_{jd}} + x_j \frac{\partial p_j}{\partial x_j} \sum_h \sum_{t \neq j} \frac{\partial x_j^h}{\partial p_t} \frac{\partial p_t^h}{\partial E_{jd}}$$

where η_j^h is the own price elasticity of consumer h's demand.

Comparing (29) and (31) we see that for given product quantity, firms' marginal revenue with respect to a change in product quality may deviate from the marginal social value of this change because of

(i) the specific income effects on consumer demand from quality changes (the second term on the right-hand side (RHS) of (31)),

(ii) for some consumers the marginal value of a quality change as given by the (first-order) change in the demand price for commodity j may be different from the true average value to the consumer of this change in product quality evaluated per unit of commodity j consumed (the third term on the RHS of (31)),

(iii) an aggregation effect which in view of $\eta_j = \sum_h x_j^h \eta_j^h / \sum_h x_j^h$, takes the form of a covariance over consumers between individual price elasticities of demand and changes in individual demand prices (the fourth term on the RHS of (31)),

(iv) an indirect effect which is due to the fact that a change in the attributes of commodity j may change the marginal productivities of commodities other than j in the production of final goods and hence consumers' demand prices for those commodities may change which in turn may affect the demand and equilibrium price for commodity j (the last term on the RHS of (31)).

Clearly, for any given x_j, a profit-maximizing firm will equate marginal revenue with respect to product quality to marginal cost $\partial x_{j0}/\partial E_{jd}$. Thus, we have the following result:

<u>Proposition 1.</u>

A sufficient condition for a profit-maximizing firm to choose a locally[1)] efficient product design is that the aggregate specific income effect, the aggregation effect (covariance term) and the indirect effect all vanish and that for each consumer, his marginal valuation of a quality change as expressed through the (first-order) change in his demand price, is equal to the true average value of this change to the consumer per unit consumed.

Since, the right-hand side of (31) expresses marginal revenue of product quality for <u>any given</u> x_j, the above sources of inefficiency of market choices of product quality will be present both under monopolistic and marginal cost pricing of commodities. The actual product qualities chosen under these two market regimes will however be different since prices and marginal costs will differ.

With a linear (Lancasterian) consumption technology, $\partial^2\varphi_d^h/\partial x_t^h \partial E_{jd} = \delta_{jt}$ (Kronecker delta) and moreover, $\partial\varphi_d^h/\partial E_{jd} = x_j^h$. Hence, we have the

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1) Local efficiency means here a stationary point satisfying the first-order conditions for a Pareto optimum.

following immediate result:

Proposition 2.

> A linear consumption technology is a sufficient condition for the indirect effect to vanish and for the consumer's marginal valuation of a quality change (for given implicit prices) to equal the true average value to the consumer of that change.

Thus, in the linear case the only sources of inefficiency of profit maximizing choices of product quality will be the specific income effects and the aggregation effect as shown by Drèze and Hagen [2]. However, with non-linear consumption technologies two additional inefficiency terms may enter the picture because marginal and average valuations of product quality may differ for some consumers and because of the indirect effect caused by the fact that a quality change of product j may affect the marginal productivities of commodities other than j in the production of final goods which again may affect the equilibrium price of product j.

The reason why marginal valuations of a quality change of product j as expressed through the (first-order) changes in demand prices p_j^h, may differ from the true average values to the consumers of that change under a non-linear consumption technology, is partly due to the non-linearity in itself and partly to an externality effect caused by the fact that marginal productivities of commodities other than j may change and hence part of the market's marginal valuation of a quality change of product j may spill over to other producers and will therefore not be captured by firm j.

To make the above point clear, it may be appropriate to look at a particular example. Assume an economy with a single differentiated commodity and a single quality attribute. There is a single consumer with a utility function $U(z_0, z_1) = z_0 + \pi_1 z_1$, where $\pi_1 > 0$ is the constant implicit price for final good z_1. The consumption technology is given by $z_1 = \omega_1 + (x_1 E_1)^\gamma$, $0 < \gamma < 1$, where $E_1 > 0$ is the quality index. In this example income, aggregation and indirect effects clearly vanish so that the only source of inefficiency in the market choice of product quality will be the difference between average and marginal valuations due to non-linearity of the consumption technology. To see this we observe that utility maximization implies the inverse demand function $p_1 = \pi_1 \gamma x_1^{\gamma-1} E_1^{\gamma}$. Thus the consumer's marginal valuation of a quality change is given by the change in his demand price $\partial p_1/\partial E_1 = \pi_1 \gamma^2 (x_1 E_1)^{\gamma-1}$ while the true average value to the consumer of this change is $\frac{1}{x_1} \pi_1 \partial \varphi_1/\partial E_1 = \pi_1 \gamma (x_1 E_1)^{\gamma-1}$ so that the marginal valuation will in this particular case be smaller than the average value causing the marginal revenue to be smaller than the marginal social value and hence the market will undersupply the product quality for any given price p_1.

We now extend the above example slightly by assuming two differentiated commodities supplied in quantities x_1 and x_2 with quality indexes E_1 and E_2. The household production function is now assumed to be $\varphi_1(x_1, x_2, E_1, E_2) = (x_1 E_1)^{\gamma} + (x_2 E_2 E_1)^{\gamma}$, $0 < \gamma < 1$; in other respects the example remains unchanged. The two inverse demand functions are given by $p_1 = \pi_1 \gamma x_1^{\gamma-1} E_1^{\gamma}$ and $p_2 = \pi_1 \gamma x_2^{\gamma-1} (E_2 E_1)^{\gamma}$, respectively. Clearly, income and aggregation effects still vanish and so does the indirect effect since $\partial x_1/\partial p_2 = 0$. For any given x_1, marginal revenue for firm 1 with respect to the quality of the first commodity is $x_1 \partial p_1/\partial E_1 = \pi_1 \gamma^2 x_1^{\gamma} E_1^{\gamma-1}$ while the marginal social value of this change is $\pi_1 \partial \varphi_1/\partial E_1 = \pi_1 \gamma [x_1^{\gamma} E_1^{\gamma-1} + (x_2 E_2)^{\gamma} E_1^{\gamma-1}]$. The difference between marginal revenue and marginal social value is here partly caused by an externality effect as an increase in the quality of the first commodity will make the second commodity more productive in producing z_1 and the marginal social value of this externality is given by $\pi_1 \gamma (x_2 E_2)^{\gamma} E_1^{\gamma-1}$. If the two commodities were produced by the same producer, the externality effect would have been internalized. Marginal revenue with respect to E_1 would in that case be $x_1 \partial p_1/\partial E_1 + x_2 \partial p_2/\partial E_1 = \pi_1 \gamma^2 [x_1^{\gamma} E_1^{\gamma-1} + (x_2 E_2)^{\gamma} E_1^{\gamma-1}]$ and the deviation between marginal revenue and marginal social value has now the same explanation as in the former example.

We finally alter the above example to allow for an indirect effect and to that effect we take the household production function to be $\varphi_1(x_1, x_2, E_1, E_2) = (x_1 E_1)^{\gamma_1} (x_2 E_2)^{\gamma_2}$, $\gamma_1, \gamma_2 > 0$, $\gamma_1 + \gamma_2 < 1$.

This implies the inverse demand functions $p_1 = \pi_1 \gamma_1 x_1^{\gamma_1 - 1} E_1^{\gamma_1} (x_2 E_2)^{\gamma_2}$ and $p_2 = \pi_1 \gamma_2 x_2^{\gamma_2 - 1} (x_1 E_1)^{\gamma_1} E_2^{\gamma_2}$. As income and aggregation effects vanish, we have from (31) that for given x_1, the marginal revenue with respect to product quality for producer 1 is given by

$$x_1\frac{\partial p_1}{\partial E_1} = \pi_1\gamma_1^2 x_1^{\gamma_1} E_1^{\gamma_1-1}(x_2E_2)^{\gamma_2} + x_1\frac{\partial p_1}{\partial x_1}\frac{\partial x_1}{\partial p_2}\pi_1\frac{\partial^2\varphi_1}{\partial x_2\partial E_1}$$

while the marginal social value of this change is

$$\pi_1\frac{\partial\varphi_1}{\partial E_1} = \pi_1\gamma_1 x_1^{\gamma_1} E_1^{\gamma_1-1}(x_2E_2)^{\gamma_2}$$

The difference between marginal revenue and marginal social value is here partly explained by the indirect effect since $\partial x_1/\partial p_2 \neq 0$ as can be seen from the inverse demand functions.

In two recent contributions Spence [9] and Sheshinski [8] have examined the efficiency of market choices of product quality in a partial equilibrium context based on a consumer surplus measure of social benefits (and hence assuming away income effects) and they conclude that the market choice of product quality is likely to be biased away from the social optimum because of possible differences between marginal and average valuations of product quality by consumers. It follows, however, from (31) that (even in the case of vanishing income effects) if individual demand elasticities and changes in individual demand prices differ among consumers, then an aggregation effect may enter the picture as a separate reason why firms' marginal revenues with respect to product quality do not equal the marginal social benefits. This aggregation bias in the market substitute for aggregation of consumer preferences is due to the fact that in their decisions on product quality firms respond to commodity prices rather than implicit prices for final goods. Furthermore, if there are many differentiated commodities, a change in the design of commodity j may reduce or increase the applicability of commodities other than j and apart from the externality effect of this which is reflected in the difference between changes in demand prices and true average values, it may affect the demand and equilibrium price for commodity j which will be another reason why marginal revenue and marginal social value differ.

4. Concluding remarks.

In the present paper we have suggested the consumption activity approach to consumer demand theory in studying effects on consumer demand from changes in the consumers' environment. The problem of optimal design of the environment was discussed in the context of choosing product quality and it was shown that profit-maximizing choices of product quality are generally not in the interests of the consumers and this conclusion is valid both under marginal cost pricing and mark up pricing of market goods. Depending on the sign and magnitude of the various terms on the right-hand side of (31), quality choices under the pressure of market forces may lead to both underprovision and overprovision of product attributes as compared with a social optimum.

Although the above proposition is stated in terms of product quality, it clearly applies to all cases in which some aspects of the consumers' environment can be controlled by the producer. Location and spatial competition is one such example which comes to mind, and as is well known, it was already intimated by Hotelling [3] that profit maximization may not bring about the socially optimal localization pattern of firms.

To the extent that the environment is controllable, optimal environmental design will generally require aggregation of consumer preferences. We may note, however, that with a sufficiently differentiated supply of market goods, or more precisely, if $J \geqq C$, then the planner only needs to know the household consumption technologies. The implicit prices for final goods, π_c^h, can in this case be inferred from market prices and demand patterns for market goods. If all consumers have access to the same consumption technology, these implicit prices will in equilibrium be the same for all consumers and if, in addition, this technology were linear, competitive behaviour with respect to these implicit prices would in this special case lead profit maximizing firms to choose environmental parameters in such a way that the necessary conditions for Pareto optimality were satisfied.[1] In general, however, a public goods problem must be solved for efficiency to obtain.

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[1] See Leland [5].

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