NHH

INSTITUTT FOR FORETAKSØKONOMI

DEPARTMENT OF BUSINESS AND MANAGEMENT SCIENCE

FOR 36 2014

ISSN: 1500-4066

November 2014

Discussion paper

Optimal maintenance scheduling of local public purpose buildings

BY

Arnt O. Hopland AND **Sturla F. Kvamsdal**

Norges Handelshøyskole

NORWEGIAN SCHOOL OF ECONOMICS

Optimal maintenance scheduling of local public purpose buildings

Arnt O. Hopland
NoCeT and the Department of Business and Management Science, Norwegian School of Economics (NHH)
arnt.hopland@nhh.no

Sturla F. Kvamsdal*
SNF - Centre for Applied Research at the Norwegian School of Economics (NHH)
sturla.kvamsdal@nhh.no

Abstract
We formulate the maintenance scheduling decision as a dynamic optimization problem, subject to an accelerating decay. This approach offers a formal, yet intuitive, weighting of the trade-offs involved when deciding a maintenance schedule. The optimal maintenance schedule reflects the trade-off between the interest rate and the rate at which the decay accelerates. The prior reflects the alternative cost, since the money spent on maintenance could be saved and earn interests, while the latter reflects the cost of postponing maintenance. Importantly, it turns out that it is sub-optimal to have a cyclical maintenance schedule where the building is allowed to decay and then be intensively maintained before decaying again. Rather, local governments should focus the maintenance either early in the building's life span and eventually let it decay towards replacement/abandonment or first let it decay to a target level and then keep it there until replacement/abandonment. Which of the two is optimal depends on the trade-off between the alternative cost and the cost of postponing maintenance.

Keywords: Maintenance, local governments, policy, public sector

* Kvamsdal acknowledge financial support from the Research Council of Norway (project no. 234238/E40).

1 Introduction

Local government purpose buildings play a vital role in the production of welfare services, and vast resource is required to build and maintain them. Maintenance is in many cases not sufficient to maintain a high technical level, even in affluent countries. This has, for example, been thoroughly discussed in several public investigations in Norway (NOU, 2004; Riksrevisjonen,[1] 2004-2005; Arbeidstilsynet,[2] 2013). Despite the importance of purpose buildings and the widespread debate on how they are maintained, an analytical framework for optimal maintenance of public purpose buildings has not yet been well established. The contribution of this paper is to formulate the maintenance scheduling decision in a dynamic optimization framework. The conceptual framework we propose is widely used in studies of maintenance scheduling for machines (see e.g., Rapp, 1974; Pierskalla and Volker, 1976; Sethi and Thompson, 2000; Dogramaci and Freiman, 2004), but we are not aware of any studies that have used it for studying public purpose buildings.

Two mechanisms drawing in opposite directions are likely to be important for the maintenance decision. Discounting suggests that expenditures should be postponed as long as possible, because money that are saved will yield interests. However, the technical decay tends to accelerate, which may more than offset the interest benefits from postponing the expenditures. The intuition is straightforward. If a wall is not painted on a regular basis (relatively cheap), one may have to replace it much earlier (rather expensive) owing to rapid decay.

A complicating factor is that it at some point in time will be reasonable to reduce the maintenance activity for a building. There are at least two good reasons for this. First, demands for usability change over time; an old building can be perceived as out-dated and inadequate even though it is in excellent technical condition. Second, the need for purpose buildings changes over time. Consider, for example, a local government that owns and operates a school in a small village. Owing to an ageing population, the local government knows that there will be too few children left to keep the school running in ten years. The fact that purpose buildings at some point in time most likely will be replaced or abandoned regardless of their technical condition illustrates that identifying an optimal maintenance and replacement strategy is far from trivial.

We consider a local government that is to produce a welfare service in a purpose building (e.g., schooling or care for the elderly) as cost efficiently as possible. For simplicity, we assume the amount and quality of the service to be governed by central standards and thus to be exogenous to the model. We also assume other costs to be fixed, so that maintenance is the only decision variable. Importantly, public buildings must be above a threshold that defines an "acceptable condition" throughout the period it is used. The threshold can be thought of as the lowest condition where the service production remains unaffected by the building condition (since the production is assumed exogenous).

A model for optimal maintenance scheduling tries to answer how much maintenance effort to apply at different stages of a building's life cycle. Should the local governments keep the standards high initially, and then allow it to decay towards the end, or allow it to decay towards the threshold condition before maintaining it at that level throughout the remaining life-span? The solution of the mathematical model gives an optimal maintenance schedule that reflects the trade-off between the interest rate and how fast the decay accelerates. This result fits well with the intuitive reasoning above, since the interest rate makes spending in the future less costly than spending today, while the accelerating decay introduces a cost of postponing maintenance. Hence, the model weighs these arguments and shows how the optimal strategy varies with their relative importance.

The maintenance cost function plays a vital role in the model, and this can take various forms, depending on building type and cost structure in the local government. We will thus study two formulations that are widely used in economics; a linear and a quadratic cost function. A linear cost function gives a bang-bang solution, where the maintenance effort is either "on" or "off" and this formulation of the cost function thus describes the choice of timing in a clear-cut manner. It should be noted, though, that when maintenance is said to be "off", this must be interpreted on an aggregate level. Everyday actions in order to keep operations running (e.g., changing lightbulbs or repairing broken windows) will, of course, still take place. If we employ a quadratic cost function, the trade-off is also apparent, albeit less clear-cut. In this case the optimal maintenance effort either increases or decreases throughout the life-span of the building, depending on which of the factors in the trade-off that dominates.

Our findings are of importance for policy makers and facility managers, since they provide guidance for how maintenance should be planned. Efficient maintenance scheduling is important, since local governments have limited funds and need to "make every dime count" in order to provide good services to the public. If money is "wasted" on a sub-optimal maintenance strategy, there are fewer resources left to employ teachers or nurses and for purchase of necessary equipment.

Since different buildings will have different rates of accelerating decay, careful considerations must be made for each building as to whether it makes sense to schedule most of the maintenance effort towards the start or end of the period the building is in use. Importantly, the model clearly shows that maintenance effort should be focused either at the start or towards the end of the building's serving, rather than e.g., first letting it decay, then putting in an all-out-effort to improve it and allowing it to decay again. This result is interesting, since the findings in public investigations such as NOU (2004), Riksrevisjonen (2004-2005), and Arbeidstilsynet (2013) suggests that this cyclical strategy seem to be widely used. Our results thus suggest that local governments can benefit greatly from planning maintenance activity over the full life-span of facilities rather than planning maintenance activities over shorter time horizons.

We keep the model simple and highly stylized on purpose. In particular, we do not go into details about how maintenance should be implemented and how different types of maintenance may affect the decay in various ways, but restrain ourselves to discussing maintenance scheduling on a strategic and aggregated level. Our intention with this paper is thus not to answer all questions regarding the optimal maintenance strategy, but to provide a formal, yet intuitive, framework for further analysis of maintenance scheduling. The model can be extended and generalized in several interesting directions, and thus has the potential to paint a more realistic picture of the optimal maintenance strategy. In addition to deriving the base model, we will present several suggestions for further research.

2 Background

The public reports mentioned in the introduction have also sparked an academic debate. A central point is that carrying a maintenance backlog is essentially the same as running a deficit, in the sense that expenditures are postponed. Borge and Hopland (2012) argue that this is owing to myopic politicians who are unable to make long-run prioritizations, and thus favor other expenditures that are more visible to voters in the short-run. This argument implies that the maintenance backlog at least to some extent is owing to irrational behavior from the policy makers. Their theoretical predictions are also backed up by results from an empirical investigation. Hopland (2015) argues, however, that Goodspeed's (2002) model where local governments use deficits strategically to extract additional funds (bailouts) from the central government can also explain maintenance backlogs. He thus argues that decay of public buildings is not necessarily a consequence of myopia or irrationality, but can be owing to rational (and rather cynical) behavior. Importantly, Borge and Hopland (2012) and Hopland (2015) only attempt to explain why one might see deviations from some optimal level of maintenance, without attempting to define what is optimal. The present paper's discussion of an optimal maintenance strategy thus complements both of these studies.

Other studies look into the organizational framework, and suggest that differences in organizational structure and competence are vital factors for the successfulness of local government facility management. Haugen (2003) reviews the historical development of local government facility management and conclude that an increasing share of the local governments centralizes the facility management, and more services are contracted out to external companies. The driving force behind this development is a wish for a more professional and competent facility management. Hopland (2014b) looks more closely into the local governments' choice of structure for the facility management, i.e., centralized or decentralized. He argues that even though most local governments have chosen to centralize the facility management it is not given that "one size fits all", but that characteristics of the local government will decide which is best. Moreover, his empirical analysis shows that local governments actually seem to choose the structure of their facility management at least partly based on such characteristics. The focus in these studies are on describing institutional settings that might stimulate good maintenance, while the optimal strategy is left undecided. Hence, the present paper also complements this literature.

A concern is that poor buildings may affect service production. In particular, education has received much attention, and some suggest that improving environmental conditions may bring gains (Earthman, 2002, Buckley et al., 2005; Mendell and Heath, 2005). However, recent empirical studies in affluent countries find very little effect from building conditions and investment in school facilities on student achievement (Hopland 2012; 2013; Cellini et al., 2010). Hopland (2012) suggests that school buildings in affluent countries are "too good to matter", i.e., that the difference between "poor" and "good" buildings is not big enough to affect achievement.[3] This implies that the production of services is not affected by the technical condition of purpose buildings as long as the condition is above a threshold.

Another strand of the literature studies how the design of school buildings relates to changes in pedagogical practices (see, e.g., Bjurström 2004; Dudek, 2000; OECD, 2001). Importantly, users may perceive schools in excellent technical condition as poor if the design is not adapted to the pedagogic ideas of the time. Thus, one must also take the usability of facilities into account (see, e.g., Hansen et al., 2012; Lindahl et al., 2012). Health care facilities have also received quite some attention. An important emphasis in this literature is that it is important to take an integrated approach to facility management, and see maintenance activity in relation to the core activities that take place in the facilities. Recent contributions include a series of papers by Lavy and Shohet (2007; 2009; 2010).

3 A dynamic model for optimal maintenance scheduling

Central assumptions

Before deriving the model, we clarify our central assumptions, all based on lessons from the literature discussed above. A common denominator for many of the assumptions is that the model describes a deterministic situation where all relevant information is available to the decision makers. It is possible to extend the model by loosening these assumptions, but this is beyond the scope of the present paper and left for future research.

Since the problem at hand is fundamentally similar regardless of which type of public purpose building we study, we stick to a general and simple formulation where the local government is responsible for the production of a welfare service, produced within a single purpose building. The demanded amount and quality of the service production is exogenous to the model, and can be thought of as decided by the central government. This is a reasonable assumption in many countries, such as Norway. As discussed in the literature review, purpose buildings seem to serve their purpose well, as long as the building condition is above a lower limit for acceptable conditions (Hopland, 2012). We thus require that the building condition stays above a threshold that secures the exogenous demands for the public service. In order to keep things simple, we assume that the threshold is known and constant.

As mentioned in the introduction, one of the major problems with low maintenance is that it creates a maintenance backlog and that accelerating decay gives that this is expensive to cover (see e.g., Hopland, 2015). For simplicity, we assume that the local government knows how the building will decay over time, under different maintenance schedules. This is a rather mild assumption, given that local governments have access to engineers who can give good advice on these issues.

Owing to changing needs and different demands for usability (see Hansen et al., 2012; Lindahl et al., 2012 for discussions of usability), buildings are eventually taken out of service or replaced regardless of the technical condition. We assume that the date of abandonment or replacement is determined by the local government (i.e., the horizon is known and fixed), such that the horizon can be taken into account when scheduling the maintenance.

As a further simplification, we do not consider a resale option in this model. This is admittedly a strong assumption, but it is not entirely without justification. First, the resale market for such buildings is quite limited. Second, since maintenance of purpose buildings should be related to the service production (Lavy and Shohet, 2007; 2009; 2010), it is not obvious that all maintenance activity will increase the resale value. The reason is that the new owners are not likely to use the building for the same purposes, but rather re-build it, e.g., as apartments. Since a change of purpose will require substantial rebuilds of the interior, such investors are probably only concerned with the technical condition of fundamentals such as the roof and outer walls. Hence, we choose to omit the real estate market from our simple base model.

Model and analysis

In this section, we formalize the intuition presented above and formulate it as a dynamic optimization problem where a local government decides the maintenance strategy for a new purpose building. Maintenance and decay takes place in continuous time, similar to studies of optimal maintenance and replacement of machines (see e.g., Rapp, 1974; Pierskalla and Volker, 1976; Sethi and Thompson, 2000; Dogramaci and Freiman, 2004).

The local government's objective is to maximize production q less the maintenance cost $c(u)$ and other production costs $\bar{c}$ (e.g., wages)

$$max \int_{0}^{T} e^{-rt}(q - c(u) - \bar{c})dt \tag{1}$$

As discussed above, q is constant, and we also assume this to be the case for $\bar{c}$. This reduces the complexity of the analysis substantially, since constants have no effect on the maximization problem. We maximize the objective function with respect to the maintenance u. $c(u)$ is the cost function for maintenance, and we will explore both a linear and quadratic specification. *r* is the real interest rate and *t* is time (*T* denotes the horizon). The solution must obey

$$(i)\ \dot{x} = f(x,u), \qquad (ii)\ x(0) = x_0, \qquad (iii)\ x_0 \geq x \geq \bar{x} \tag{2}$$

where *x* is the condition of the building and (*i*) is a differential equation that describes how the building condition develops over time. (*ii*) defines the initial condition of the building condition, and (*iii*) tells that the building condition at any time must be better than the lower threshold required to secure the exogenously given service production, while it cannot be better than the new condition. Maintenance cannot be negative and also has an upper limit, owing to budget and technical constraints as well as political priorities

$$u_{max} \geq u \geq 0 \tag{3}$$

The evolution of the building condition is fully described as

$$f(x,u) = u - m_1 - m_2(x_0 - x) \tag{4}$$

There is an ever-present downward drift at rate m_1, i.e., standard wear and tear as the building is used and gets older. In addition, there is an accelerating drift that is scaled by m_2 and proportional to the distance to the initial condition x_0. In other words; the further the condition is from the initial condition, the more rapidly the building decays. Maintenance activity (i.e., $u > 0$) has a positive impact on the building condition.

In order for our study to be of any empirical interest, we must assume that the life-cycle of the building is of such length that maintenance is required, i.e., that *T* is larger than the time it takes the condition to decay to $\bar{x}$ under no maintenance. Otherwise, we would reach the trivial and empirically irrelevant solution that $u = 0$ for all t is the only solution of the optimization problem. Mathematically, the assumption is

$$T > \frac{1}{m_2} \ln\left(\frac{1}{m_1}(m_1 + m_2(x_0 - \bar{x}))\right) \tag{5}$$

In other words, with $u = 0$ over the entire period, the condition $x \geq \bar{x}$ is not satisfied when (5) holds and thus all feasible solutions demand maintenance at some point.

An important point is that since the building is to be replaced or abandoned at time *T*, it will in a cost minimizing setting be a pure waste to maintain the building to such extent that the condition is above the lower threshold at the end of the life-cycle. Hence, we assume the state variable to be at its lower limit at T, $x(T) = \bar{x}$.

In order to secure a flexible model formulation, we allow the local government not only to perform preventive maintenance, but also to improve the condition of the building, i.e.,

$$u_{max} > m_1 + m_2 x_0 \tag{6}$$

This means that the local government can in principle restore the building back to "good as new" condition (keeping in mind that the condition cannot be better than x_0).

While the structure of the problem in (1)-(4) is relatively simple, the constraints on the state (here: building conditions) and control (here: maintenance) variables in (2) and (3) add substantial complexity to the problem. While constraints on the control is not problematic in optimization problems like this, constraints on the state require careful treatment. To deal with the state constraints in (2) we thus use the indirect adjoining method (see, e.g., Sethi and Thompson, 2000).

We find it instructive to introduce the optimization problem step-wise. Hence, we start out by solving the problem as if there are no constraints on the building condition and rather introduce the constraints below. In particular, the constraint-free solution is often relevant on intervals where no constraints are binding (free intervals). The Hamiltonian is given by

$$H(x, u, \lambda) = -e^{-rt} c(u) + \lambda \left(u - m_1 - m_2 (x_0 - x) \right) \tag{7}$$

λ is the costate variable. We have

$$H_u = -e^{-rt} c_u + \lambda \tag{8}$$

where the subscript denotes the partial derivative. The costate along the optimal path is governed by

$$\dot{\lambda} = -H_x = -m_2 \lambda \tag{9}$$

which has the solution

$$\lambda(t) = \lambda_0 e^{-m_2 t} \tag{10}$$

The costate is in economics commonly referred to as the shadow price, and in our case it illustrates the cost (benefit), in terms of increased (reduced) decay, from reduced (increased) maintenance. Interpreted in terms of capital theory, the building condition x is a form of capital, and maintenance adds value to the capital stock. The shadow price is interpreted as the return on investment in x, and from (10) we see that the return decay with rate m_2. Before proceeding, it is useful to assume a specific function for the maintenance cost function. In the remainder of the paper, we will discuss two cases, a linear and a quadratic functional form.

Linear cost function

We start out by a simple linear specification of the maintenance cost function

$$c(u) = pu \tag{11}$$

where p is the real unit price of maintenance at any point in time. In the case with linear cost function, the optimal control is of bang-bang type. This means that, with no state constraints, we will have $u = u_{max}$ if (8) is positive and $u = 0$ if (8) is negative. (8) can only be zero over an interval of time in the special case where the interest rate is equal to the parameter for decay acceleration, i.e.,

$$\lambda_0 e^{-m_2 t} = e^{-rt} p \Rightarrow r = m_2$$

In this special case, (1) is invariant to any feasible maintenance schedule because the return on maintenance exactly offsets alternative costs. Nevertheless, $r = m_2$ is unlikely to hold exactly.

Importantly, with $r \neq m_2$, (8) switches sign only once because when we insert from (10) and (11) it is monotonic in t. and the optimal control has at most one shift from its minimum to its maximum value or vice versa. It can be shown that if $r > m_2$, we have $u = 0$ initially and until the switch time; if $r < m_2$, we have $u = u_{max}$ initially and until the switch time. In both cases, the maintenance level is at the same level (u_{max}), but in the first case, all maintenance effort is carried out at the beginning of the building's life time, while in the latter case, all effort is carried out at the end of the building's life time. That the optimal solution hinges on the relative strength between r and m_2 has some intuition to it. Whereas r measures how fast future maintenance costs wear down, m_2 measures how fast the decay-process accelerates, and thus how fast future maintenance needs increases. If future maintenance costs wears down faster than the maintenance needs increase ($r > m_2$), the maintenance effort will be postponed for as long as possible; in the opposite case ($r < m_2$), the maintenance will be carried out as soon as possible.

The solution above presents important intuition, but does not fulfill the requirements from (2). In order to pursue the linear model with state constraints properly, we need to introduce the Lagrangian given by

$$L(x, u, \lambda, \eta_1, \eta_2) = H(x, u, \lambda) + \eta_1 f(x, u) - \eta_2 f(x, u) \tag{12}$$

where η_1 and η_2 are multipliers for the constraints $h_1 = x - \bar{x} \geq 0$ and $h_2 = x_0 - x \geq 0$. In (12), η_1 and η_2 are multiplied with $\dot{h}_1 = f(x, u)$ and $\dot{h}_2 = -f(x, u)$, respectively. $H(x, u, \lambda)$

is the Hamiltonian as presented in (7). In (12), we have omitted the constraints on the control variable to avoid excessive notation; the control constraints (3) are readily imposed directly.

Necessary conditions for an optimal solution include maximization of the Hamiltonian and the Lagrangian. Thus, on any free interval where the state constraints do not bind, the optimal control is the bang-bang solution, as discussed above, i.e., $u = 0$ or $u = u_{max}$. The Hamiltonian should be continuous along the optimal path. The equation governing the costate along the optimal path is

$$\dot{\lambda} = -L_x = -H_x + m_2\eta_1 - m_2\eta_2 \tag{13}$$

There are complementary slackness conditions on the state constraint multipliers:

$$\begin{aligned} &\eta_1 \geq 0, \qquad \eta_1(x - \bar{x}) = 0, \qquad \dot{\eta_1} \leq 0 \\ &\eta_2 \geq 0, \qquad \eta_2(x_0 - x) = 0, \qquad \dot{\eta_2} \leq 0 \end{aligned} \tag{14}$$

Finally, we have jump conditions on the costate variable at times τ_{bound} when a state constraint becomes binding

$$\lim_{\epsilon\to 0} \lambda(\tau_{bound} - \epsilon) = \lim_{\epsilon\to 0} \lambda(\tau_{bound} + \epsilon) + \zeta_i(\tau_{bound}) \frac{\partial h_i(\tau_{bound})}{\partial x} \tag{15}$$

where h_i denotes constraint i, and ζ_i is a jump parameter with complementary slackness at times τ_{bound}. Sethi and Thompson (2000) provide a more comprehensive treatment.

The derivative of h_1 in (15) is 1, and of h_2 is -1. Hence, when constraint i becomes binding, (15) simplifies to

$$\lim_{\epsilon\to 0} \lambda(\tau_{bound} - \epsilon) = \lim_{\epsilon\to 0} \lambda(\tau_{bound} + \epsilon) \pm \zeta_i(\tau_{bound}) \tag{16}$$

(16) imply that the costate is discontinuous at times τ_{bound}. However, the jump parameters ensure that the Hamiltonian remain continuous if the optimal control shifts discontinuously when constraints become binding (at τ_{bound}). In problems without state constraints (and thus, without jump parameters), the optimal control can only shift in a discontinuous fashion if $H_u = 0$.

The building condition can at a given time be either in the interior where no constraints are binding and the optimal maintenance level is either $u = 0$ or $u = u_{max}$, or at a boundary where one of the constraints binds. However, the maximization problem along the constraints is also linear in maintenance, and optimal maintenance is still of bang-bang type. If the upper constraint is binding, we have $f(x_0, u) \leq 0$ which implies $u \leq m_1 < u_{max}$, i.e., the local government can either hold the condition constant at "good as new level" or let it start to decay. Alternatives for the bang-bang maintenance are then $u = 0$ or $u = m_1$. If the lower constraint is binding, we have $f(\bar{x}, u) \geq 0$ which implies $u \geq m_1 + m_2(x_0 - \bar{x}) > 0$, i.e., the local government at least has to maintain the building to such extent that the condition is constant at the lower bound. Alternatives for the bang-bang maintenance are then $u = m_1 + m_2(x_0 - \bar{x})$ or $u = u_{max}$.

We proceed as follows. First, we assume that there is exactly one control shift (like in the constraint free solution) and characterize this solution. Next, we show that additional shifts are sub-optimal. At $t = 0$, the upper constraint ($x = x_0$) is active. The initial maintenance level is thus either $u = m_1$ (we call it alternative a) or $u = 0$ (alternative b). In alternative a, assuming there is exactly one maintenance shift, we will have $u = m_1$ until the switch time t_a, when the maintenance shifts to $u = 0$. After the shift, the building condition descends to the given terminal state $x(T) = \bar{x}$. In alternative b, we will have $u = 0$ until $x(t) = \bar{x}$, which characterizes the switch time t_b when the maintenance shifts to $u = m_1 + m_2(x_0 - \bar{x})$. Note that in both alternatives, we have $u = 0$ for equally long periods (the time it takes the state to descend from the upper to the lower constraint under $u = 0$). We denote the length of this time interval τ

$$\tau = \frac{1}{m_2} \ln\left(\frac{1}{m_1}\big(m_1 + m_2(x_0 - \bar{x})\big)\right) \tag{17}$$

In order to simplify notation in the remaining discussion, we define

$$\begin{aligned} u_a &= m_1 \\ u_b &= m_1 + m_2(x_0 - \bar{x}) \end{aligned} \tag{18}$$

Note that (18) states unambiguously that the required maintenance activity is lower if maintenance takes place early in a building's life span than late (i.e., $u_a < u_b$), which is a consequence of the accelerating decay. If we insert from (18) in (17), we arrive at the useful expression

$$\tau = \frac{1}{m_2} \ln \frac{u_b}{u_a} \tag{19}$$

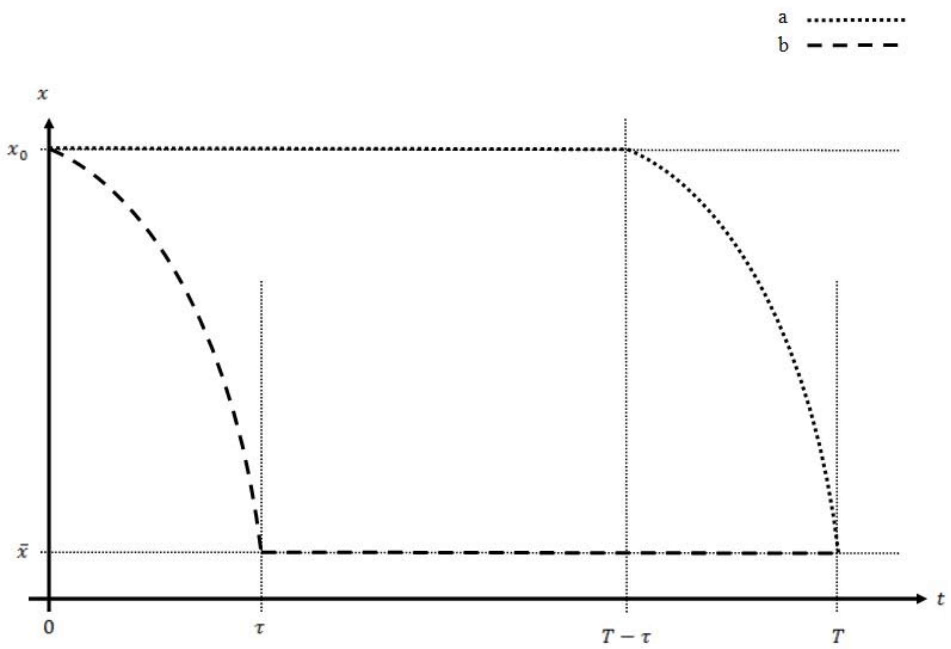

Figure 1: The figure illustrates alternative paths a and b in state-time space.

In order to determine which of the alternatives a and b that is optimal, we compare the value of (1) along the alternative paths. Note that the constants are omitted from the discussion as these do not affect the solution. Alternative paths a and b are illustrated in figure 1, with x along the vertical axis and t along the horizontal axis. The values along the path of the alternatives are given by

$$\begin{aligned} V_a &= \int_0^{T-\tau} -e^{-rt} p u_a \, dt + \int_{T-\tau}^{T} -e^{-rt} p u_0 \, dt = \frac{p}{r} u_a \left(e^{-r(T-\tau)} - 1\right) \\ V_b &= \int_0^{\tau} -e^{-rt} p u_0 \, dt + \int_{\tau}^{T} -e^{-rt} p u_b dt = \frac{p}{r} u_b (e^{-rT} - e^{-r\tau}) \end{aligned} \tag{20}$$

where $u_0 = 0$. Alternative a is optimal if the following difference is positive:

$$V_a - V_b = \frac{p}{r}\left(e^{-r(T-\tau)} - 1\right)(u_a - u_b e^{-r\tau}) \tag{21}$$

The first term is negative; the sign thus depends on $u_a - u_b e^{-r\tau}$. We know that $u_a < u_b$, but also have to take into account the discount factor $e^{-r\tau}$, since postponing expenditures offer the possibility to earn interest on the money. We use (19) and have

$$e^{-r\tau} = \left(\frac{u_a}{u_b}\right)^{r/m_2} \tag{22}$$

that yields

$$u_a - u_b e^{-r\tau} = u_a - u_b \left(\frac{u_a}{u_b}\right)^{r/m_2} = u_a \left(1 - \left(\frac{u_a}{u_b}\right)^{r/m_2 - 1}\right) \qquad (23)$$

From (23), it is clear that the difference $V_a - V_b$ is zero if $r = m_2$. If $r > m_2$, the last exponent $({}^{r}/_{m_2} - 1)$ is positive, and with $u_b > u_a > 0$, (23) is positive and (21) is negative. Hence, $r > m_2$ yields $V_a < V_b$. In a similar fashion, if $r < m_2$, $V_a > V_b$. If the interest rate wears down future maintenance costs faster than the accelerating decay increases maintenance needs (i.e., $r > m_2$), maintaining now is more costly than to postpone maintenance, and we have $u = 0$ initially and until the switch time (path b). In the opposite case ($r < m_2$), maintaining now is less costly than to postpone maintenance, and we have $u = u_a$ initially and until the switch time. The intuition is the same as when no constraints were imposed.

Until now, we have considered a single-shift policy, and described the solution under this restriction. Importantly, the model allows the local government to shift the maintenance level as many times as it desires. However, we will now show that the single-shift policy is in fact the optimal strategy.

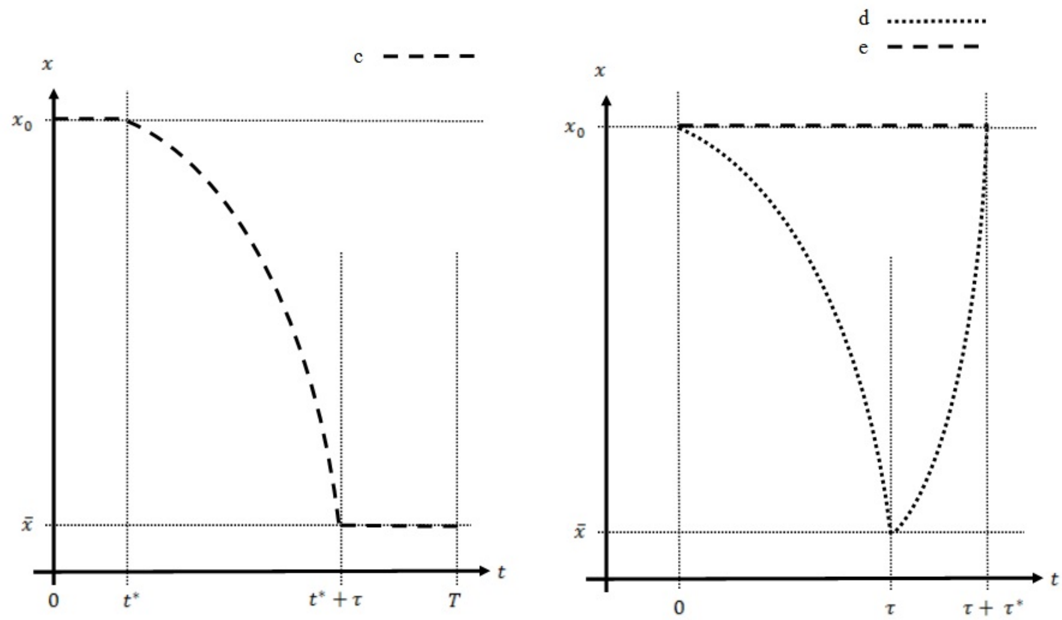

Figure 2: Illustration of alternative paths c (left panel) and d and e (right panel) in time-state space.

We start out by considering the case with two switches, with $u = u_a$ up to some time t^*, $u = 0$ from t^* to $t^* + \tau$, and $u = u_b$ from $t^* + \tau$ to. The case is presented in Figure 2 (left panel). The value along the path (which we label alternative c) is given by

$$V_c = \int_0^{t^*} -e^{-rt} p u_a dt + \int_{t^*}^{t^*+\tau} -e^{-rt} p u_0 \, dt + \int_{t^*+\tau}^{T} -e^{-rt} p u_b dt \tag{24}$$

The derivate of (24) with respect to t^* is zero if $r = m_2$ can otherwise have either sign depending on the parametrization. That is, the value along path c either increases or decreases with t^* throughout, and the optimal t^* is thus either $t^* = 0$ (alternative a) or $t^* = T$ (alternative b).

Next, we look at the case with multiple shifts. In alternative a, the building condition follow the upper boundary with $\dot{x} = 0$ from $t = 0$ to $t = T - \tau$. Consider an alternative path (alternative d) where the policy at a given time (t_d) shifts to $u = 0$, whence the condition decays to $x = x_d$. The policy shifts again, to $u = u_{max}$ (the only option if x_d is inner[4]), and the condition improves to $x = x_0$. In what follows, we assume $x_d = \bar{x}$ to avoid unnecessary complications, but the argument holds for any x_d. We further assume $t_d = 0$ without loss of generality. The path of alternative d is illustrated in Figure 2, which also shows the part of the alternative a path that we will compare alternative d to. The relevant part of the alternative a path is for the occasion labeled alternative e.

In alternative d, the policy $u = u_{max}$ will last as long as it takes the condition to improve from $\bar{x}$ to x_0. We denote the interval τ^* and have

$$\tau^* = \frac{1}{m_2} \ln \left(\frac{u_{max} - u_a}{u_{max} - u_b} \right) \tag{25}$$

Consider the difference of the values along the paths:

$$V_d - V_e = \int_0^{\tau} -e^{-rt} p u_0 dt + \int_{\tau}^{\tau+\tau^*} -e^{-rt} p u_{max}\, dt - \int_0^{\tau+\tau^*} -e^{-rt} p u_a dt$$

$$= \frac{p}{r}\Big((u_{max} - u_a)e^{-r(\tau+\tau^*)} - (u_{max}e^{-r\tau} - u_a)\Big) \tag{26}$$

$$= \frac{p}{r}\left(u_a + \left(\frac{u_a}{u_b}\right)^{r/m_2}\left((u_{max} - u_a)^{1-r/m_2}(u_{max} - u_b)^{r/m_2} - u_{max}\right)\right)$$

In the last expression of (26), we have substituted from (22) and (25). We see that $V_d = V_e$ if $r = m_2$. Primarily, we are interested in (26) with $r < m_2$. The reason is that it is in this case the local government will start out with positive maintenance and move along the upper boundary for the condition initially. Note that (26) is zero with $u_a = u_b$. From (18), we have $u_b > u_a$. The derivative of (26) with respect to u_b is negative under $r < m_2$ and $u_b > u_a > 0$. Because (26) decreases monotonically along u_b from zero at $u_a = u_b$, we conclude that $V_d < V_e$. Hence, in-between shifts along path a is sub-optimal.

In alternative b, the building condition follows the lower boundary with $\dot{x} = 0$ from $t = \tau$ to $t = T$. Similar to the discussion of alternatives d and e, consider the alternative path (alternative f) where the policy at a given time shifts to $u = u_{max}$ and remains at u_{max} until $x = x_0$. The policy shifts again to $u = 0$ and the building condition decays back to the lower boundary ($x = \bar{x}$). We are primarily interested in alternative f under $r > m_2$ (under which alternative b is optimal). We compare f with the relevant part of alternative b, for the occasion labeled alternative g. Difference in value along the paths is given by:

$$V_f - V_g = \int_0^{\tau^*} -e^{-rt} p u_{max} dt + \int_{\tau^*}^{\tau^*+\tau} -e^{-rt} p u_0\, dt - \int_0^{\tau^*+\tau} -e^{-rt} p u_b dt \tag{27}$$

From an argument similar to that following (26), we can show that alternative g is optimal. Hence, in-between shifts along is sub-optimal also in this case.[5]

We are now ready to sum up our results for the linear model. An important finding is that cyclical maintenance is sub-optimal, and that maintenance effort should rather be focused either at the start or towards the end of the building's serving. The solution depends on the relationship between the interest rate and the rate at which the state accelerates away from the initial state. Thus, we have two potential solutions. If $r > m_2$, we have $u = 0$ initially and until the switch time; if $r < m_2$, we have $u = u_{max}$ initially and until the switch time. The result is intuitive, because the interest rate r reflects the cost of spending now rather than later, while the parameter for decay acceleration m_2 gives the cost of postponing maintenance rather than maintaining now. Whereas the interest rate is easy to observe empirically, assessing the size of the parameter m_2 is more difficult, since different buildings can have varying acceleration rate. Hence, planners need to consider this relationship when scheduling the maintenance for each building.

Quadratic cost function

We next turn the attention to the solution of the free optimization problem with a quadratic cost function

$$c(u) = pu^2 \tag{28}$$

Maximization of the Hamiltonian (requires (8) to be zero) yields the optimal control on the form

$$u(t) = \frac{\lambda_0}{2p} e^{(r-m_2)t} \tag{29}$$

If we insert (29) into the equation for $\dot{x}$ (see (2) and (4)), we obtain an expression for the optimal state path:

$$x(t) = x_0 + \frac{m_1}{m_2} + \frac{\lambda_0}{2p(r-2m_2)} e^{(r-m_2)t} + ke^{m_2 t} \tag{30}$$

where k is an integration constant. The two unknowns λ_0 and k can be determined from the boundary conditions on x. From $x(0) = x_0$ we have

$$k = \frac{\lambda_0}{2p(2m_2 - r)} - \frac{m_1}{m_2} \tag{31}$$

From $x(T) = \bar{x}$, we then have

$$\lambda_0 = 2p(2m_2 - r)\left(x_0 - \bar{x} + \frac{m_1}{m_2}(1 - e^{m_2 T})\right)\left(e^{(r-m_2)T} - e^{m_2 T}\right)^{-1} \tag{32}$$

Note that if $r > m_2$, $u(t)$ increases over time; if $r < m_2$, $u(t)$ decreases over time; and if $r = m_2$, $u(t)$ is constant. While the nature of the optimal maintenance schedule in (29) is significantly different from the optimal schedule in the linear model, they do share some common features. They depend critically on the relationship between r and m_2, and the dependence is similar in that when, say, the interest rate is comparatively large, most of the maintenance effort occurs late. It turns out that for many reasonable parameterizations of the problem, the solution in (29) and (30) abides the constraints (2) on the state variable.[6] We do not pursue a formal analysis of the quadratic problem any further because the main intuitive lesson from the maximization problem is embodied in the above solution. However, a numerical example of the solution to the quadratic problem is presented in figure 3.

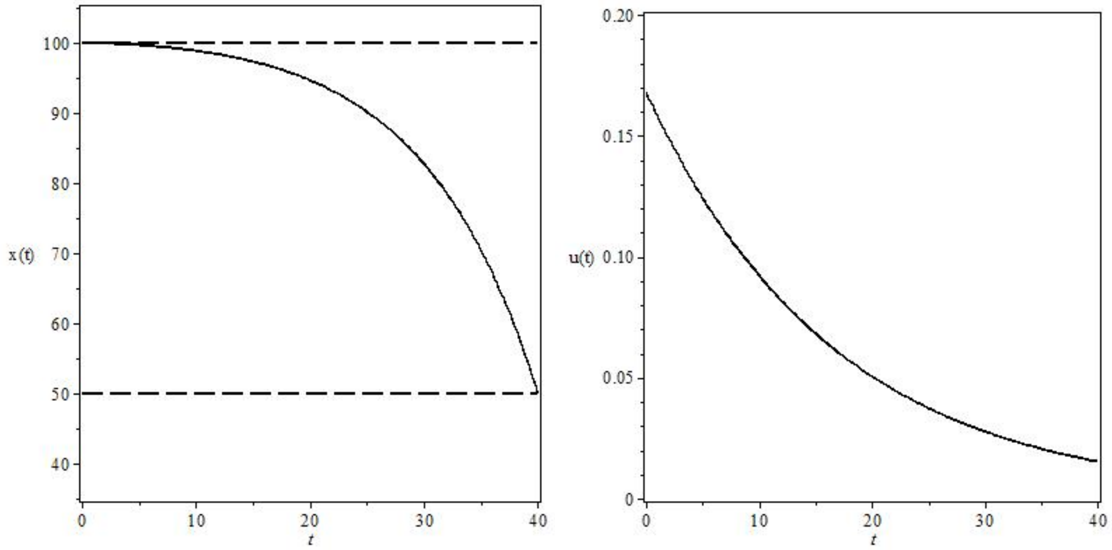

Figure 3: Numerical example of (29) and (30). The left panel shows the path in the time-state space; the right panel shows the optimal maintenance schedule. Parameter values are: $x_0 = 100,\ \bar{x} = 50,\ r = 0.04,\ m_1 = 0.2, m_2 = 0.1, T = 40, p_0 = 1.$

In the example in figure 3, $r < m_2$ and $u(t)$ decreases over time, see the figure notes for details. The right panel in the figure show $u(t)$. The left panel show the resulting path in the time-state space (solid curve); the panel also shows the constraints on the state variable; x_0 and $\bar{x}$ (dashed curves). In this particular example, the condition stays relatively close to x_0 for a long while, but as the amount of maintenance gets small, the accelerating decay takes hold and the condition falls quickly towards $\bar{x}$ when T approaches.

4 Conclusions

The deeds of maintenance seem to have hunted mankind since ancient times (Hoffer, 1969, p. 21). In this paper we have studied the optimal scheduling of maintenance activities for local government purpose buildings, using mathematical tools for dynamic optimization. The conceptual framework proposed is widely used in studies of maintenance scheduling for machines (see e.g., Rapp, 1974; Pierskalla and Volker, 1976; Sethi and Thompson, 2000; Dogramaci and Freiman, 2004). We find that the optimal strategy depends on a trade-off between the real interest rate, and the rate of accelerating decay. The prior makes spending in the future less costly than spending today, while the latter introduces a cost of postponing maintenance.

Our findings are of importance for policy makers and facility managers, because they provide clear guidance for how maintenance should be planned. Since different buildings will have different rates of accelerating decay, careful considerations must be made for each building as to whether it makes sense to place most of the maintenance effort towards the start or end of the period the building is in use. Importantly, the model shows that policies with multiple shifts are sub-optimal. Our results thus suggest that local governments can benefit from planning maintenance activity over the full life-span of facilities rather than planning maintenance activities over shorter time horizons.

The model we use is kept simple in order to establish a basic intuition for the central mechanisms. Further research should expand this model in several directions in order to gain further insights. A first suggestion would be to introduce uncertainty about the time horizon, the decay rates, and the effect of maintenance activity. Even though good planning can reduce these uncertainties, it is obviously difficult to remove them altogether. Facilities may be kept in service longer than originally planned, or may become obsolete earlier than expected owing to, e.g., technology shocks. Further, it would be interesting to loosen the assumption about a fixed lower limit for the condition. In reality, there can be different "levels of ambition" that local governments can choose from, making this boundary less obvious and potentially endogenous. This is similar to introducing an endogenous production as a function of the condition. Finally, one may introduce a market for used buildings, allowing the local government to sell the building. All of these suggestions may help to get a more realistic model and give valuable insights.

List of references

Arbeidstilsynet (2013), *Inneklima i norske skoler. Hovedfunn 2011-2012*.

Bjurström, P. (2004), *Att förstå skolbyggnader*, Doctoral thesis 2004:2 at KTH Arkitekturskolan.

Borge, L.-E., and Hopland, A.O., 2012, "Maintenance and building conditions in Norwegian local governments: Economic and Political determinants". Chapter 1 in Hopland, A.O. (2012) *Facility management, student achievement and fiscal adjustment*. Doctoral thesis 2012:266 at NTNU, Norwegian University of Science and Technology.

Buckley, J., Schneider, M., Shang, Y. (2005), "Fix It and They Might Stay: School Facility Quality and Teacher Retention in Washington D.C", *The Teachers College Record*, 107, 1107-1123.

Cellini, S. R., Ferreira, F., and Rothstein, J. (2010), "The Value of School Facility Investments: Evidence from a Dynamic Regression Discontinuity Design", *Quarterly Journal of Economics*, Vol. 125, pp. 215-261.

Dogramaci, A., Fraiman, N.M. (2004), "Replacement decisions with maintenance under uncertainty: An imbedded optimal control model", Operations Research, Vol. 52, No. 5, pp. 785-794.

Dudek, M. (2000), *Architecture of Schools, the New Learning Environment*, Oxford: Architectural Press.

Earthman, G.I. (2002), "School Facility Conditions and Student Academic Achievement", UCLA's Institute for Democracy, Education and Access (IDEA) Paper no. wws-rr008-1002.

Goodspeed, T.J., (2002), “Bailouts in a Federation” *International Tax and Public Finance*, 9, 409- 421.

Hansen, G.K., Blakstad, S.H., and Olsson, N.O.E., (2012), “Usability Reviewed: Summing up Norwegian Research on Usability”, in Jensen, P.A and Nielsen, S.B. (eds.) (2012) *Facilities Management Research in the Nordic Countries*. Polyteknisk Forlag pp. 173-186.

Haugen, T. 2003. Contracting out Property and Facility Services in Norwegian Municipalities. *Nordic Journal of Surveying and Real Estate Research - Special Series*, 1, 52-70.

Hoffer, E. (1969), *Working and thinking on the waterfront: A journal, June 1958 to May 1959*, New York: Harper & Row

Hopland, A.O. (2012), “School building conditions and student achievements: Norwegian evidence”. Chapter 2 in Hopland, A.O. (2012) *Facility management, student achievement and fiscal adjustment*. Doctoral thesis 2012:266 at NTNU, Norwegian University of Science and Technology.

Hopland, A.O. (2013), “School facilities and student achievement in industrial countries: Evidence from the TIMSS”, *International Education Studies*, Vol. 6 No. 3, 162-171

Hopland, A.O. (2015), “Can game theory explain poor maintenance of regional government facilities?”, *Facilities*, Vol 33, No. 3/4, forthcoming.

Hopland, A.O. (2014a), “How related are technical and subjective measures of building conditions? The case of Norwegian Public Schools”, *Facilities*, Vol. 32, No. 5/6, pp. 295-306.

Hopland, A.O. (2014b), “One size fits all? Facility management in Norwegian local governments”, *Nordic Journal of Surveying and Real Estate Research*, Vol. 10, No. 1, pp. 7-22.

Lavy, S., and Shohet, I.M., (2007), “On the effect of service life conditions on the maintenance of healthcare facilities”, *Construction Management and Economics*, Vol. 25, No. 10, pp. 1087-1098.

Lavy, S., and Shohet, I.M., (2009), “Integrated healthcare facilities maintenance management model: case studies”, *Facilities*, Vol. 27, No. 1/2, pp. 5-20.

Lavy, S., and Shohet, I.M., (2010), “Performance-based facility management – an integrated approach”, *International Journal of Facility Management*, Vol. 1, No. 1, pp. 1-14.

Lindahl, G., Nenonen, S., Hansen, G.K., and Blakstad, S.H, (2012), “Facilities in Use – Nordic Studies about Usability of Workplaces”, in Jensen, P.A and Nielsen, S.B. (eds.) (2012) *Facilities Management Research in the Nordic Countries*. Polyteknisk Forlag pp. 187-199.

Mendell, M. J., and Heath, G. A. (2005), “Do indoor pollutants and thermal conditions in schools influence student performance? A critical review of the literature”, *Indoor Air*, Vol. 15, pp. 27-52.

NOU (2004), *Velholdte bygninger gir mer til alle* Report 2004: 22, Government Administration Services, Oslo.

OECD (2001), *Designs for learning. 55 Exemplary Educational Facilities. Education and Skills*, Paris: OECD Publications Service. PEB (Programme on Educational Buildings, OECD).

Pierskalla, W. P. and Voelker, J. A. (1976), A survey of maintenance models: The control and surveillance of deteriorating systems. *Naval Research Logistics*, Vol. 23 pp. 353–388.

Rapp, B. (1974), *Models for Optimal Investment and Maintenance Decisions*, Stockholm and New York: Almqvist & Wiksell and John Wiley & Sons.

Riksrevisjonen (2004-2005), *Dokument nr. 3:13 (2004-2005) Riksrevisjonens undersøkelse av kommunenes ansvar for skolebygninger*.

Sethi, S.P., Thompson, G.L. (2000), *Optimal Control Theory – Applications to Management Science and Economics*, New York: Springer.

[1] The Office of the Auditor General of Norway.

[2] The Norwegian Labor Inspection Authority

[3] Hopland (2014a) studies the correlation between building condition and student satisfaction with the facilities. The two are correlated, but far from one-to-one. This can partly explain why school buildings do not seem to affect achievement.

[4] If x_d is not inner, but rather $x_d = \overline{x}$, u can alternatively shift to u_b and yield path c, previously found to be sub-optimal compared to either path a or b.

[5] $V_f < V_g$ can be established from taking the derivative of the right hand side of (27) with respect to u_a.

[6] If $r < m_2$ and $u(t)$ decreases over time, the constraints are broken if $\dot{x}|_{t=0} > 0 \iff \lambda_0 > 2pm_1$; the latter expression can be reduced to a condition $T_{lim} > T$. In the particular example presented in figure 3, we have $T_{lim} \approx 42$. Similar conditions for breaking the constraints can be derived in the opposite case where $r > m_2$ and the control increases over time.