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CONTRIBUTIONS TO THE NORMATIVE THEORY OF TAXATION

by

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CONTENTS

INTRODUCTORY NOTE

The normative theory of taxation explores the implications of postulated social objectives for the choice of tax policy. Hence it derives recommendations which are conditional on the social preferences or welfare functions which are adopted. A model of the economy and a set of feasible taxes constitute the setting within which the social objectives can be pursued.

The theory can either adopt the optimum tax approach or the related tax reform approach. The purpose of optimum tax theory is to characterize the optimum choice of tax system, tax schedules and tax rates within the assumed economic, informational and political constraints regardless of the initial tax policy. The purpose of tax reform analysis is to assess (usually small) revisions of the tax policy from a specific starting-point. Both branches of normative tax theory are usually defined as being concerned with the structure of taxation rather than the tax level. The properties of the tax system do however have important implications for the assessment of tax-financed public spending and thus for the choice of tax level. Exploring such implications is therefore a natural extension of normative tax theory.

Normative tax theory is an application of second best welfare theory. By the second best assumption the full use of lump sum taxes is ruled out. The implications for economic efficiency are very significant. It has been a major achievement of economic theory to demonstrate that under certain conditions a competitive market economy, possibly supplemented by appropriate correcting interventions by the government, produces a socially efficient allocation. The key to the understanding of this result is the observation that under the appropriate conditions the agents of the economy do in fact individually bear the full social costs and receive the full social benefits of their actions. Hence incentives are such that private optimizing behaviour is in perfect harmony with social optimization.

A crucial assumption behind this outcome is that a complete set of lump sum taxes is available. In practice it is not. In particular, redistribution through lump sums requires that we are able to detect the exogenous characteristics of individuals on which lump sum taxes and transfers would have to be based. But in practice there is no device by which the true skills and other characteristics can be screened. In other words, the true initial endowments are hidden, and the assumption behind the basic theorems of welfare economics that redistribution through lump sums is possible, is not satisfied. When lump sum taxes are ruled out as impracticable and the tax instruments are income taxes, excise taxes, VAT, etc., private costs and benefits get distorted in the sense that they no longer reflect the social costs and benefits. Choices are then motivated by comparisons between private costs and benefits which may reverse the outcome of comparisons between social costs and benefits.

Ideally a commodity should be used to the extent that the marginal benefit from having more of it equals the marginal real cost of providing it measured as the potential benefit from other commodities foregone. When there is a tax on a commodity, there is a private incentive to use it to the extent that the benefit from having more of it is no less than the private cost which includes the tax, which in general is not a real cost but simply a transfer of spending power from the private to the public sector within society. An exception is of course externality-reflecting taxes. Thus private agents are encouraged to make choices that are not socially efficient in the first best sense. This is a major concern of normative tax theory along with the concern with income distribution.

It is important to distinguish between tax-induced changes in quantities and tax distortions of the allocation. No matter how taxes are moulded one does not escape from income effects. Even lump sum taxes well known to be compatible with first best efficiency change the allocation as compared to a no-tax situation. The concern is therefore with allocative distortions defined as changes deviating from the lump sum tax income effects. It is clearly desirable as such to choose a tax structure which minimizes the efficiency loss from tax distortions. On the other hand it is desirable to use the tax policy for distributional purposes.

Normative tax theory has derived a number of characteristics of optimum taxes and welfare-improving tax changes under various circumstances. The basic insight underlying most special results is that taxes should distort the real allocation as little as possible for a given distribution and within the constraints on feasible taxes. Translated into the price space this insight implies that prices strongly affecting real quantities through high price elasticities in absolute values should be relatively less distorted than prices having less effect on real quantities. And there must be a trade-off between conflicting concerns with distribution and efficiency.

There are several strands of research in normative tax theory. There is an important distinction between analysis focusing on efficiency and neglecting distributional concerns and analysis explicitly concerned with distribution. Within each broad category models can be classified according to a number of criteria. Models can focus on various markets and distortions or interactions between various markets and distortions. The focus may be on the labour market, the capital market, consumer goods or certain interactions between these markets. We can also label models according to the kinds of taxes that are analysed, such as income tax, commodity taxes, expenditure tax, or according to the technical forms of taxes, such as linear or general tax schedules. Most tax models choose assumptions by which they escape from dealing with tax-induced changes in equilibrium prices. Only rather few analyses are concerned with endogenous prices. In most tax models taxpayers are assumed to be individuals. Normative models of multiperson households as taxable units are rather rare.

The present study consists of seven separate papers which contribute to the normative theory of taxation. Strictly speaking one can argue that the first article entitled "Some important properties of the social marginal utility of income" is a paper in general welfare economics rather than tax theory in particular. It has, however, useful applications in normative tax theory. The paper contains an analysis of how the social marginal utilities of income assigned to different persons change in response to changes in prices, the provision of public goods and other parameters faced by the individual. The effect of a parameter change is interpreted as composed of a change in the real value of marginal income and a change in total real income (or utility level). It is implied that even if utility levels are kept constant, social marginal utilities of income remain unaltered only under special conditions, which are exposed and interpreted. Some of the formal results are known from before.

The main contributions are interpretations and applications.

The second contribution is the article "Which commodity taxes should supplement the income tax." The analysis takes as its point of departure a continuum of consumers economy in which an optimum non-linear income tax exists and is the only tax instrument in operation. Individuals face exogenous wage rates. The welfare effects of introducing small excise taxes to supplement the income tax are then explored. Essential in this context are changes in the tax distortions of work incentives. It is shown that a commodity should be taxes or subsidized depending on whether it is positively or negatively related to leisure in a sense which is precisely defined. The results are related to earlier contributions to the literature on direct versus indirect taxation.

The third paper is entitled "The choice of excise taxes when savings and labour decisions are distorted." The framework is a simple two period life-cycle model with identical individuals who work in period 1 and are retired in period 2. Initially there are optimally chosen uniform income and commodity tax rates, which lead to distortions in both the labour/leisure choice and the consumption/savings choice. The purpose is to demonstrate in an intuitively comprehensible way how differentiating commodity taxation by slightly increasing the tax on one consumption good can mitigate existing distortions. The sufficient conditions for a welfare improvement are similar to that derived by Corlett and Hague in the early fifties plus some conditions on average and marginal consumption propensities.

In optimum tax models income is usually assumed to be endogenous as the result of tax-affected labour supply or savings decisions. There may, however, be reasons for assuming that there is a mixture of endogenous and exogenous income. If the income tax cannot discriminate between the two kinds of income, we face the second best tax problem analysed in "The optimum taxation of mixed endogenous and exogenous income." There is a discrete distribution of individuals according to exogenous income and exogenous wage rates. The total income of an individual consists of exogenous income and endogenous labour income. An optimum linear income tax is analysed. Special attention is focused on how the composition of income may affect the optimum degree of income tax progressivity.

In the papers surveyed above the taxable units are individuals as has been the tradition in normative tax theory. The special problems involved in taxing families are then missed out. Such problems are dealt with in "Income taxation of two-person households." A model is constructed to analyse the tax treatment of secondary wage earners in two-person households. The households have different income opportunities, and potential secondary wage earners differ in their willingness to take a job. In a variety of numerical cases the optimum tax structure is computed allowing for income distribution and the tax distortions of the labour market participation of secondary wage earners. Special tax systems of the kinds actually in operation are analysed and compared.

General equilibrium effects of taxes on prices and wages are addressed in the paper entitled "Choice of occupation, tax incidence and piecemeal tax revision." A model is presented in which workers move between two different occupations in response to economic incentives which are distorted by a linear income tax. Prices and wages assume equilibrium values which are affected by the tax parameters. Incidence and welfare effects of small tax revisions are analysed within different variants of the basic model and with particular attention paid to the role of tax-induced wage and price changes. It is demonstrated that within the economic setting of the model one may neglect such wage and price effects in assessments of piecemeal tax revisions.

In the last article with the title "Evaluation of public projects under optimal taxation" we return to the standard model of a continuum of taxpaying individuals facing exogenous wage rates. In this paper the scope of analysis is extended to the implications of optimum taxation for cost-benefit analysis. The conventional cost-benefit criterion accepts or rejects public projects on the basis of the sum of unweighted net benefits. It can generally be blamed for neglecting distributional objectives and tax distortions. It turns out, however, that more commonly accepted social welfare criteria can be reduced to the conventional criterion in certain interesting cases. In this paper conditions are established under which the conventional cost-benefit criterion or a simple modification of it is valid as such a reduced form in the presence of distributional objectives and optimal second best taxation. Such results may help simplifying cost-benefit analysis.

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Some Important Properties of the Social Marginal Utility of Income

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Abstract

This paper contains an analysis of how the social marginal utilities of income assigned to different persons change in response to changes in prices, the provision of public goods and other parameters faced by the individual. The effect of a parameter change is interpreted as composed of a change in the real value of marginal income and a change in total real income (or utility level). It is implied that even if utility levels are kept constant, social marginal utilities of income remain unaltered only under special conditions, which are exposed and interpreted. The results are useful in applied welfare economics.

I. Introduction

The social marginal utility of income or the distributive weight assigned to an individual is a key concept in applied welfare economics concerned with income distribution. A very good basic introduction to this concept is found in Meade (1976). An excellent survey of various approaches to the concept is found in Stern (1977). In this paper, we analyze some important properties of the social marginal utility of income which have not received proper attention in the literature.

Most modern books in applied welfare economics, for instance cost-benefit analysis, make use of or at least refer to distributive weights.[1] But the discussion of circumstances which determine the distributive weights is rather scanty. In most cases the weights are simply treated as conditional on the value of some measure of real income. Often the weights are only assumed in principle to vary among population groups that are understood to be different with respect to economic well-being.

Economists sometimes appear to believe that if social preferences are egalitarian, a lower social marginal utility of income should always be assigned to a person on a higher level of utility than to a person on a lower level of utility. Similarly the social marginal utility of income should be the same if two individuals enjoy the same level of utility. But, in general, these

[1] A few references which may serve as examples are Brown & Jackson (1978), Layard (1974) and Pearce & Nash (1981).

are spurious conclusions. It is therefore important to consider more closely the relationship between the social marginal utility of income, the utility level and the conditions faced by the individual and perhaps also by other individuals. We begin by exploring the circumstances under which the social marginal utility of income is uniquely determined by the utility level(s). We then investigate how social marginal utilities of income are affected by changing the parameters of the economy (prices, provision of public goods, etc.). Some applications of the analysis are also discussed.

II. The Model

We consider a market economy with a public sector. Consumers are assumed to act in accordance with standard theory of consumer behavior. The utility of an individual is expressed by the indirect utility function

$$V(y, a),$$

where y is the exogenous income of the individual and a is a vector of other parameters. These parameters may be of different kinds, i.e., prices, public goods or other exogenous parameters which affect an individual's situation such as health status or job characteristics. Some of the parameters may vary across individuals, while others may be common parameters in all utility functions. Examples are provided by differing wage rates and uniform commodity prices.

We consider a population of individuals with uniform preferences. Homogeneous preferences represent a common assumption in much of the literature to which this analysis is relevant.

Social preferences are assumed to be represented by a Bergson-Samuelson welfare function

$$W(V^1, \ldots, V^N), \tag{1}$$

which is increasing in all arguments.

The question to which we address ourselves first is under what conditions relative distributive weights are determined solely by the utility levels of the individuals. Since only relative weights matter in economic analyses, we are not interested in absolute values. The social marginal utility of income of person i is

$$\omega^i = \frac{\partial W}{\partial V^i} \cdot \frac{\partial V^i}{\partial y^i}. \tag{2}$$

The relative weight is expressed as

$$\frac{\omega^i}{\omega^1} = \frac{(\partial W/\partial V^i)}{(\partial W/\partial V^1)} \frac{\partial V^i/\partial y^i}{\partial V^1/\partial y^1} \quad i = 2, \ldots, N \tag{3}$$

III. When are Relative Weights Uniquely Determined by Utility Levels?

Since the arguments of the W-function are all utility levels, the relative distributive weights ω^i/w^1 will only depend on utility levels when

$$\frac{\partial V}{\partial y} = f(V)\, g(c), \tag{4}$$

where c denotes the parameters of a which are common to all individuals. Expression (4) is a partial differential equation which can be solved to obtain the class of utility functions

$$V(g(c)\,y + h(a)). \tag{5}$$

Equation (5) is equivalent to

$$\frac{\dfrac{\partial V}{\partial a_i}}{\dfrac{\partial V}{\partial y}} = \frac{\partial h/\partial a_i}{g(c)} \quad \text{for all } i \text{ when } a_i \text{ is not a common parameter.} \tag{6a}$$

and

$$\frac{\dfrac{\partial V}{\partial c_j}}{\dfrac{\partial V}{\partial y}} = \frac{(\partial g/\partial c_j)\,y + \partial h/\partial c_j}{g(c)} \quad \text{for all } j. \tag{6b}$$

In order to interpret (6) it is useful to distinguish among four cases defined according to whether the parameter is common or individual and whether or not it is a price. When a_i is the price of a commodity, we know from Roy's identity that the quantity demanded is

$$x_i = -\frac{\dfrac{\partial V}{\partial a_i}}{\dfrac{\partial V}{\partial y}}. \tag{7}$$

Then (6) implies that the demand for a commodity is independent of income if the price is individual and a linear function of income if the price is common. The best example of an individual price is probably the price of leisure, which varies with the (after-tax) wage rate. In this case, (6) requires the income elasticity of leisure to be zero. A more intuitive interpretation of these results is easily provided.

The social value of a marginal income unit to a person depends partly on his utility level and possibly the utility levels of others and partly on the real value of a marginal income unit to this person. The more expensive the goods on which he tends to spend a marginal income unit, the lower the real value (or purchasing power) of his marginal income, no matter how badly or well off he is. He will derive less utility from marginal income than before. Expressions (6a) and (6b) imply that the marginal propensity to spend income on a commodity is zero or constant, respectively. If only one person faces a compensated price increase, the marginal real value of his income is lowered, and so is the social marginal utility of income, unless no marginal income is spent on the good which becomes more expensive, as implied by (6a). If a common price increases, the marginal real value of a person's income is depressed and more so, the higher his marginal propensity to spend money on that good. Relative social marginal utilities of income are left unchanged only if this marginal propensity is the same for everybody as implied by (6b).

Let us now assume that the parameter is some physical good, e.g. a public good. An extra unit of the good tends to add more to the marginal value of income, the more of a marginal income unit the person is prepared to give up in order to obtain the extra unit of the good. This amount is equal to the rise in marginal willingness to pay for the good as an additional income unit is obtained. Relative social marginal utilities of income are left unchanged only if this change in marginal willingness to pay is the same for everyone, as implied by (6b). If the good does not effect everybody, the common change in marginal willingness to pay as income rises must be zero, as implied by (6a).

Equations (6a) and (6b) are strict conditions. They imply that the social marginal utility of income will be uniquely related to the level of utility only under certain strong restrictions on the preferences of the individual himself. It is not left to some social authority to determine, on the basis of a Bergson-Samuelson welfare function, whether the (relative) social marginal utilities of income should be functions of utility levels only. It follows that two persons who enjoy the same utility level may have different social marginal utilities of income if they otherwise face different conditions.

IV. The Effects of Compensated Parameter Changes

Since we cannot expect relative social marginal utilities of income to depend solely on utility levels, it may be interesting to see how they are affected by compensated parameter changes. We should then examine the compensated derivative of ω with respect to some parameter a, denoted by ω_a^{comp}. For simplicity subscripts are omitted, although a particular individual and a particular element of the a-vector are in fact considered. Let us also introduce the notation:

$$\lambda = \partial V/\partial y. \tag{8}$$

Then

$$\omega = \frac{\partial W}{\partial V}\lambda. \tag{9}$$

Since all utility levels are kept constant by the compensation

$$\omega_a^{\text{comp}} = \frac{\partial W}{\partial V}\lambda_a^{\text{comp}}. \tag{10}$$

Let $E(a, V)$ be the expenditure function. At the individual equilibrium

$$E(a, V(a, y)) = y, \tag{11}$$

from which it follows that

$$\lambda = 1/E_V. \tag{12}$$

Then it can easily be seen that

$$\lambda_a^{\text{comp}} = -E_{aV}/E_V^2 = -\lambda^2 E_{aV}. \tag{13}$$

To find E_a, we fix the utility level at V^0 and write

$$V(a, E(a, V^0)) = V^0, \tag{14}$$

which implies that

$$E_a = -V_a/\lambda = -m(a, E(a, V)), \tag{15}$$

where m is simply the individual's marginal willingness to pay for a marginal rise in the relevant parameter. Hence

$$E_{aV} = -m_y E_V = -m_y/\lambda. \tag{16}$$

This result is substituted into (13) to obtain

$$\lambda_a^{\text{comp}} = \lambda m_y. \tag{17}$$

From (9) and (10) we then find that

$$\omega_a^{\text{comp}} = \omega m_y. \tag{18}$$

Since only relative marginal utilities of income matter, elasticities are more appropriate tools. We therefore define the compensated elasticity

$$\hat{\omega}_a^{\text{comp}} = a\omega_a^{\text{comp}}/\omega. \tag{19}$$

It can be seen immediately that

$$\hat{\omega}_a^{\text{comp}} = am_y. \tag{20}$$

Let us now interpret the marginal willingness to pay, m, more carefully. If the parameter we have considered is a price, say the price of good i, then $m = -x_i$, which is seen from (7). Let the price be denoted by p_i and the expenditure derivative by x_{iy}. Then (20) can be written as

$$\hat{\omega}_{p_i}^{\text{comp}} = -p_i x_{iy}, \tag{21}$$

which is minus the marginal propensity to spend income on good i. In order to exclude changes in relative social marginal utilities of income, the elasticity must be the same for everyone, which implies that marginal expenditure propensities must be the same. This result confirms our earlier findings as to when relative social marginal utilities of income are left unaltered when utility levels are unchanged. Expression (21) implies that if a price rises and everybody is compensated, the marginal real value of income and consequently the marginal utility of income are reduced more strongly for those who are more heavily inclined to spend marginal income on the good which becomes more expensive.

If the parameter considered above was the amount of a public good, say g, then m is the marginal willingness to pay for a marginal unit or, in other words, the Lindahl price of g:

$$\hat{\omega}_g^{\text{comp}} = gm_y. \tag{22}$$

If a wealthier person (in terms of y) tends to place a higher value on the public good, the value of his marginal income will rise as compared to that of a poorer person.

If the health status of a person improves but his income is reduced so as to leave him no better off, the marginal value of his income rises if his marginal willingness to pay for better health is positively income elastic.

V. The General Effects of Parameter Changes

Of course, most changes in the parameters of the economy are not compensated changes. So, to obtain more general results let us now set aside the compensation requirement. Our results thus far also prove useful in the more general context. We can now easily derive the effect of a change in a parameter a on λ:

$$\lambda = \lambda(a, E(a, V)). \tag{23}$$

Hence

$$\lambda_a = \lambda_a^{\text{comp}} + \lambda_y E_V V_a. \tag{24}$$

From (12) and (15) we see that $E_V V_a = m$, so that

$$\lambda_a = \lambda_a^{\text{comp}} + \lambda_y m. \tag{25}$$

Thus the effect of a on λ has been split into a compensated effect (which we have already studied) and an income effect. The marginal social utility of person i's income is in general

$$\omega^i = \frac{\partial W}{\partial V^i} \lambda^i. \tag{26}$$

Differentiating and using (25), we find that

$$\begin{aligned} \omega_a^i &= \omega_a^{i\text{comp}} + m^i \frac{\partial W}{\partial V^i} \lambda_y^i + \lambda^i \sum_j \frac{\partial^2 W}{\partial V^i \partial V^j} V_a^j \\ &= \omega_a^{i\text{comp}} + m^i \frac{\partial W}{\partial V^i} \lambda_y^i + \lambda^i \sum_j \frac{\partial^2 W}{\partial V^i \partial V^j} \lambda^j m^j \end{aligned} \tag{27}$$

where (15) has been recalled. From (26) we obtain

$$\omega_y^i = \frac{\partial W}{\partial V^i} \lambda_y^i + \lambda^i \frac{\partial^2 W}{\partial V^{i2}} \lambda^i. \tag{28}$$

Substituting into (27) we get

$$\omega_a^i = \omega_a^{i\text{comp}} + m^i \omega_y^i + \lambda^i \sum_{j \neq i} \frac{\partial^2 W}{\partial V^i \partial V^j} \lambda^j m^j. \tag{29}$$

In this quite general form, the formula is not very illuminating. But if a is an individual parameter the last term vanishes, since $m^j = 0$ for $j \neq i$, and (29) then simply reads

$$\omega_a^i = \omega_a^{i\text{comp}} + m^i \omega_y^i. \tag{30}$$

Omitting the superscript and invoking (18) we obtain

$$\omega_a = \omega m_y + m \omega_y. \tag{31}$$

A simple manipulation then leads to

$$\hat{\omega}_a = a m_y + \frac{am}{y} \hat{\omega}_y, \tag{32}$$

where $\hat{\omega}_a$ and $\hat{\omega}_y$ are elasticities with respect to a and y, respectively. Formula (29) may also be useful once a more specific welfare function is postulated. The special class of welfare functions most frequently used in applied welfare economics is probably the additive form

$$W = \sum_i V^i, \tag{33}$$

where the cardinalization of V^i is chosen by the government so as to reflect its distributional preferences properly. When preferences are uniform

$$W = \sum_i V(b^i, c, y^i), \tag{34}$$

where b^i is the vector of individual parameters apart from income. In the additive case $\omega = \lambda$, so that (25) can be used to study the effect on the social marginal utility of income. Employing (17), we can rewrite (25) as

$$\lambda_a = \lambda m_y + \lambda_y m. \tag{35}$$

A simple manipulation then leads to

$$\hat{\lambda}_a = am_y + \frac{am}{y}\hat{\lambda}_y, \tag{36}$$

where $\hat{\lambda}_a$ and $\hat{\lambda}_y$ are elasticities with respect to a and y, respectively.

Formulas (32) and (36) are quite similar. Expression (32) is valid when the changing parameter is individual, while (36) is valid when the welfare function is additive. In the latter case λ and ω coincide.

If the income of an individual rises, he becomes better off than a person who does not receive any more income but otherwise faces the same economic conditions. Thus an inequality will arise. If the social marginal utility of income of the person who becomes better off falls, the government can be said to show aversion towards inequality and more so, the greater the relative change in the social marginal utility of income. Hence the absolute value of $\hat{\omega}_y$ can be taken as a measure of (local) inequality aversion. Formulas (32) and (36) tell us that the social marginal utility of income will rise relatively more in response to a parameter change, the more strongly the marginal value of income increases, the greater the loss of income needed to give compensation and the greater the inequality aversion.

Suppose that a is a common price, say p_k. Let e_k denote the expenditure (or Engel) elasticity of good k and π_k the budget share of good k. Using these entities, we can write (36) as

$$\hat{\lambda}_{p_k} = \pi_k(e_k + \hat{\lambda}_y). \tag{37}$$

The social marginal utility of income will then rise or fall depending on whether the Engel elasticity is greater than or lower than the inequality aversion $(-\hat{\lambda}_y)$.[2]

VI. Remarks on Applications

The (relative) social marginal utilities of income are essential in all branches of applied welfare theory concerned with distributional effects. They are particularly crucial in optimum conditions which counterbalance the distributional effects and the effects on allocative efficiency of policy measures in terms of tax policy, income maintenance programs, investment projects, etc. A typical example is provided by the theory of optimum taxation. In this field, relative marginal utilities of income are used to characterize optimum tax schedules and rates.[3]

A model frequently studied in optimum tax theory describes a population of individuals who have the same preferences and differ only with respect to their efficiency as workers, which is in turn reflected by differing wage rates. In this model it is important to know how the social marginal utility of income varies with the wage rate, which in this instance is the only cause of variation in the marginal utility of income. If we assume that the marginal tax rate on income is constant (as it is in the case of linear tax schedules), we can consider the after-tax wage rate. Let this be denoted by z and work effort by h. Let p and x denote the vector of commodity prices and the consumption bundle, respectively. The budget constraint of an individual is then

$$-hz+px=y, \tag{38}$$

and the indirect utility function is

$$V(z,p,y). \tag{39}$$

Using the notation introduced earlier in the paper, we observe that when z changes

$$m=h \text{ and } m_y=h_y. \tag{40}$$

Since the wage rate is an individual parameter, we can make use of (31), from which it follows that

$$\omega_z=\omega h_y+h\omega_y. \tag{41}$$

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[2] Formula (37) is the same as formula (50) in Frisch (1959), but his derivation of it is different and far more complicated. The interpretation and application of the formula also differ.
[3] See e.g. Atkinson & Stiglitz (1980).

If, as is usually assumed, work effort is an inferior good, h_y is negative. If, in addition, the government is inequality averse, both terms in (41) are negative, and the social marginal utility of income is negatively related to the wage rate. The interpretation is the same as before. A higher wage rate implies that leisure becomes more expensive. If a person is inclined to spend some of his marginal income on leisure, the marginal purchasing power of income is reduced. This effect tends to reduce the marginal utility of income. On the other hand, a higher wage rate implies that the individual becomes better off and also for this reason only becomes entitled to a reduced social marginal utility of income, as the government is inequality averse. Hence the normal case is that the social marginal utility of income declines as the wage rate rises.

Another normal case is that an individual's pre- and post-tax wage income increases as the wage rate becomes higher. A situation whereby consumption of market goods is noninferior is sufficient to obtain this result. From the analysis above it then follows that the social marginal utility of income is negatively related to pre- and post-tax wage income.

Once optimality conditions have been established in tax theory or other relevant areas, it may be interesting to explore the comparative static effects of changing certain parameters of the economy. For instance, how do optimal tax rates change in response to changes in exogenous prices or the provision of certain public goods? In such comparative statics analyses, we obviously need to know the effects on social marginal utilities of income examined earlier in this paper.

The relationship between the social marginal utility of income and the wage rate discussed above confirms the conventional notion of this relationship. Let us now consider a case which may have a more unconventional outcome. Suppose that a transfer payment T is made from one population group to another. For simplicity each group is assumed to be homogeneous. The transfer payment is received as a lump sum. But it is financed by a proportional tax on the labor income of the other group. Prices and wage rates are assumed to be fixed. These assumptions make the case under consideration as simple as possible. Let $v(y+T, a)$ be the utility function of the recipients of the transfer payment, and let $V((l-t)w, Y, a)$ be the utility function of the taxpayers. The wage rate is denoted by w, the tax rate by t, and Y is exogenous income. Let h denote the labor supply of the taxpayers. It is assumed that h is determined in accordance with the standard textbook theory of household behavior. An additive welfare function is assumed so that

$$W = v + V. \tag{42}$$

The possible combinations of T and t are given by the constraint

$$T = twh. \tag{43}$$

Eliminating T by means of (43), we can write the welfare function with its arguments as

$$W = v(y+twh, a) + V((l-t)w, Y, a). \tag{44}$$

Let

$$\lambda = \partial v/\partial y, \quad \Lambda = \partial V/\partial Y. \tag{45}$$

The social optimization problem of the government is then to maximize W with respect to t. The first-order condition of this maximization is

$$W' = dW/dt = \lambda wh - \Lambda wh + \lambda twh_t = 0, \tag{46}$$

where $h_t = \partial h/\partial t$.

The second-order condition is

$$W'' < 0. \tag{47}$$

The recipients of the transfer payment are assumed to be worse off and to be assigned a higher marginal social weight than the taxpayers. We can rewrite (46) as

$$1 - \Lambda/\lambda = -th_t/h. \tag{48}$$

This is a standard trade-off between the distributional improvement and the loss of efficiency caused by the tax/transfer policy. We may note that the marginal effect of the tax rate on labor supply at the optimum is negative under our assumptions:

$$h_t < 0, \tag{49}$$

and the absolute value of the corresponding elasticity is less than unity:

$$-\hat{h}_t = -th_t/h < 1. \tag{50}$$

Let us now examine the effects of a shift in external circumstances represented by a change in one of the parameters of a (denoted by a to avoid subscripts). Differentiating (46), we get

$$\lambda_a wh - \Lambda_a wh + \lambda_a twh_t + W'' t_a = 0, \tag{51}$$

where, for simplicity, no direct effect on labor supply has been assumed. In order to simplify even more, assume that only the marginal utility of income of the worse-off group is affected. We then see that

$$t_a = \frac{-1}{W''}\lambda_a w(th_t+h). \tag{52}$$

Now it follows from (47) and (50) that the tax rate and the transfer payment will both fall if $\lambda_a<0$. The preceding analysis in this paper has shown that *even if the less well-off group becomes worse off, it may be given a lower social marginal utility of income* because the marginal real value of income is somehow reduced or, in other words, the ability or opportunity to derive satisfaction from marginal expenditure is reduced. The effect then arises that the transfer payment to the less well-off group is reduced when the recipients become worse off. Whether this will happen depends among other things on the degree of inequality aversion. The important conclusion is that comparative statics analysis of distribution and welfare policy within the conventional analytical framework requires more insight than simple observations of whether external shifts make various groups better or worse off.

VII. Concluding Remarks

It has been emphasized in this paper that the weight attached to a change in the income of a given individual in a Bergson-Samuelson type of social welfare function is in general not solely a function of utility levels. Even if all individuals have identical preferences and face identical external circumstances, the weights will depend on these circumstances. Thus, while weights will depend solely on utility levels, for a given set of circumstances, as soon as these circumstances change (e.g. a change in relative prices), the weights will change as well. Once some of the external circumstances are allowed to vary across individuals, the weights at any point in time will not solely be a function of utility levels. Indeed, two people with the same utility level may have different weights. Or, whether and in what direction they differ may depend on the external circumstances. One individual may have a higher weight than another, even if the latter has a lower utility level than the former. This fact was emphasized in principle by Sen (1973) in his criticism of the utilitarian approach.[4] Sen's main point was that when people are different, the approach of equating marginal utilities from income (as required in a first-best welfare optimum) does not, in general, amount to equating total utilities.

Another objective of this paper, which is at least as important as these general conclusions, has been to add to our understanding of the relations between the social marginal utilities of income, utility levels and external circumstances such as prices, provision of public goods and more personal

[4] We may note, however, that this fact is not limited to the utilitarian case, i.e. to the case of an additive welfare function.

characteristics. This has been accomplished by establishing the effects of changes in various parameters of the economy on social marginal utilities of income, distinguishing between compensated and real income effects and interpreting the implications. At the same time tools have been provided for comparative statics analysis of optimum welfare and distribution policy. This kind of analysis has been surprisingly rare. The formulas may also be useful for the purpose of revising distributive weights—once derived from the implicit trade-offs underlying actual decisions or some other source—in response to changing external circumstances.

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WHICH COMMODITY TAXES SHOULD SUPPLEMENT THE INCOME TAX?

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The analysis takes as its point of departure a continuum of consumers economy in which an optimum income tax exists and is the only tax instrument in operation. The welfare effects of introducing small excise taxes to supplement the income tax are then explored. Essential in this context are changes in the tax distortions of work incentives. It is shown that a commodity should be taxed or subsidized depending on whether it is positively or negatively related to leisure in a sense which is precisely defined. The results are related to earlier contributions to the literature on direct versus indirect taxation.

1. Introduction

The history of debates on the proper roles of direct and indirect taxation goes back at least to the days of Gladstone, as well described by Atkinson (1977). The prevailing political opinion of the balance between the two types of taxes has varied over time. At present the swing in a number of European countries seems to be in favour of reforms towards tax systems which rely more heavily on indirect taxation and less on income taxation. In view of this long economic-political record, it is not surprising that the choice between income tax and commodity taxes has also become an important subject in tax theory.

An early contribution to the understanding of this issue was Corlett and Hague (1953–54). Their main model considers a three-good economy, containing leisure and two taxed commodities. There is only one consumer (or a population of identical consumers). Labour is the only source of income. Producer prices are fixed. The government's revenue requirement is given. The starting point of the analysis is a situation in which the two commodities are taxed at uniform rates. The question which is analyzed is then how the government can raise welfare by slightly differentiating the tax rates. The answer which is derived is that the consumer good which is the stronger substitute for labour (complement with leisure) should be taxed at

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the higher rate. The degree of substitutability (or complementarity) can be measured by the compensated cross-elasticity with labour (leisure). It is intuitively easy to grasp the essence of this result. As expressed by Sandmo (1976): 'The economic rationale of this rule is clearly that since we have barred ourselves from taxing leisure, we can do it indirectly by taxing commodities that are complementary with liesure.'

As Corlett and Hague put it: 'the main analysis considers small changes in tax rates and does not indicate the size of the movements away from the initial equilibrium position needed to obtain an "optimum" system of taxation'. Thus, it may be considered as an early contribution to what is now known as tax reform analysis. But, as has been shown, the same result is valid at optimal taxation; see for example Sandmo (1976).

It is important to note, as was emphasized by Corlett and Hague, that taxation of the two consumer goods at uniform rates is equivalent to a proportional income tax. Deviation from uniform tax rates is therefore equivalent to the introduction of an excise tax in addition to a proportional income tax. In this sense the model is suitable for throwing light on the income versus commodity tax issue. Although the model is rather special, it may be argued that the insight obtained is rather basic.

Meade (1955) discussed the role of commodity taxes as a supplement to the income tax within a more general, but purely non-mathematical framework. He allows the income tax schedule to have a more general form, and the taxpayers may have unequal income. His approach is clearly described in his own words:

> We assume, therefore, that the revenue is being raised by a progressive income tax which, as explained on p. 47, introduces throughout the system a large rate of divergence between the value of the marginal product of effort and the marginal cost of that effort.... The question which we shall discuss is whether, given this situation, it would be desirable to turn to some extent from the direct taxation of income to the indirect taxation of particular goods and services as a means of raising revenue [Meade (1955, p. 112)].

His reasoning leads him to the conclusion that a welfare improvement would be obtained by making a small marginal change in the tax system which raises the price of those things which are jointly demanded with leisure, and lowers the price of those things which are good substitutes for leisure, provided that seriously adverse effects on the distribution of income are avoided. This is a result which is very close to that of Corlett and Hague. It is, however, less precise, as one might expect from a non-mathematical analysis. In particular the substitute concept is not precisely defined. Meade's analysis deals with the welfare effects of 'a small marginal change in the tax system'. Thus, his analysis may also be considered as an early contribution to tax reform theory.

A third important contribution to the literature on direct versus indirect taxation is Atkinson and Stiglitz (1976). In their analysis the taxpayers are assumed to have homogeneous preferences, but different wages. Optimality characteristics of simultaneous non-linear schedules for income and commodity taxes are derived. These characteristics are related to the properties of the taxpayers' common utility function. It turns out that whether a good is complementary with or a substitute for leisure in the Edgeworth sense (defined by the sign of the cross derivative of the utility function) is crucial in determining the excise tax to be imposed on it. The great merit of the paper was to show that if the utility function is weakly separable between labour and all goods taken together, then there is no need to employ indirect taxation in the optimum solution.

A fourth key paper to be mentioned in this field is Mirrlees (1976) which derived conditions for optimal mixed taxation consisting of optimal tax schedules and rates. Some details of this paper will be discussed in section 8.

The existing body of optimum tax literature has obviously a good deal to say about the optimal choice of indirect taxes in addition to the income tax. Yet economists who want to apply these theories, for instance as political advisers, do encounter a number of problems. One reason is that modern optimum tax results are often given in such a form that they are hard to convey to the layman on the political scene or elsewhere.[1] There is obviously a need for simpler characteristics of optimal tax policy. In older analyses of commodity taxes such as Corlett and Hague (1953–54) and Meade (1955) the key to understanding the role of commodity taxes is presented in terms of substitutability and complementarity between leisure and consumer goods. No doubt this approach has a strong intuitive appeal both to the expert and the layman. In recent and technically more complicated optimum tax theory the possible roles of substitutes and complements are much less exposed or even left completely in the dark, in my opinion at the expense of intuitive insight. This is also one reason why the connection between the various analyses included in the brief survey above is not easy to see, although one would suspect that they are closely related. In particular one would expect the simple implications of the older analyses to be embodied in some form in the more complicated results of modern theories. Further exploration of this subject therefore seems worthwhile.

The first purpose of this paper is to provide a mathematical and more precise analysis of the problem formulated and discussed by Meade (1955). In order to define the starting-point more precisely than was done by Meade, the shape of the initial income tax schedule is assumed to have been optimized by a welfare-maximizing government. The analysis will then examine how welfare can be raised even further by turning slightly from income taxation to the taxation of particular commodities. This will allow us

[1]An exposition aiming at a larger public is found in Atkinson (1977).

to review the Meade results in a precise manner. It also paves the way for the second task, which is to relate the Meade type results to those of Atkinson and Stiglitz (1976) and Mirrlees (1976). The third and closely related purpose is to focus attention on the roles of various relations between leisure and the demand for other goods, which I believe to be most helpful guides to the understanding of theoretical results in this field of tax analysis. In contrast to a lot of modern tax literature the technical analysis is going to be rather simple.

It may be useful to give a preliminary idea of the approach to be followed as it deviates somewhat from the standard analysis. The behaviour of the consumer/taxpayer is assumed to be ordinary maximizing behaviour both in the commodity and labour market. But for analytical reasons it is useful to treat it here as a two-stage optimization whereby the demand for consumption goods is optimized for a given supply of labour in the first stage, and the supply of labour is optimized in the second stage taking into account the relations between commodity demand and labour supply established in the first stage. This approach will enable us to extend the tradition of making use of relations between the demand for various goods and labour supply in throwing light on the choice of excise taxes. The income tax will be treated in an analytically simple manner by applying a tax function with a shift parameter which allows us to carry out a shift in the whole tax schedule to accompany the introduction of an excise tax.

The main assumptions underlying the analysis to follow are presented in section 2. Individual behaviour is described in section 3, and section 4 briefly presents the optimum income tax. The analysis of marginal commodity taxes in section 5 is the main part of the current paper. Section 6 discusses the roles of substitutes and complements. Sections 7 and 8 provide comparisons with the results of Atkinson and Stiglitz and those of Mirrlees. Section 9 takes a closer look at the treatment of leisure goods. Section 10 presents some concluding remarks and also draws attention to some of the limitations of the preceding analysis.

2. Main assumptions

The analysis will be based on a number of assumptions which have become common in modern optimum tax literature (including the above references). A one-period model (or timeless economy) is considered. This is important because we have then barred ourselves from discussing effects on savings behaviour which might be interesting. Individuals are assumed to have identical preferences on consumption bundles and work effort (leisure). Work is the only source of income apart from possible government transfers.

The wage rate of each person is exogenous and determined by his ability. It is then convenient to consider the ability level and the wage rate as equal.

There is a continuum of individuals distributed by ability (wage rate). The distribution is characterized by the density function $f(a)$, where a is the ability level which is taken to be positive. Producer prices are given. There is no tax evasion.

Each person chooses his work effort and consumption bundle optimally taking his own ability, the prices and the tax policy as given. This individual behaviour, which is analysed in the next section, is taken as given by the government when designing its tax policy. The revenue requirement is given. An additive welfare function is used as welfare criterion.

3. Individual behaviour

We study an individual who does an amount h of work at a given wage rate, a (reflecting his ability). His gross income is:

$$I = ah. \tag{1}$$

He faces an income tax schedule $T(I)$, so that his disposable income becomes:

$$y = I - T(I) = ah - T(ah). \tag{2}$$

This income is spent on n consumer goods in quantities $x_1, \ldots, x_n$ at prices $P_1, \ldots, P_n$. Let x and P denote the consumption vector and price vector, respectively. The scalar product of the two vectors is written as Px. Preferences are described by the utility function $u(x,h)$. The individual is assumed to maximize u as a price-taker subject to his budget constraint. As suggested above, it will prove useful to conceive of this maximization as being carried out in two stages. First the work effort, h, is treated as fixed, and u is maximized with respect to x. We establish the Lagrange expression:

$$L = u(x,h) - \omega(Px - y), \tag{3}$$

and derive the well-known necessary first-order conditions:

$$\frac{\partial L}{\partial x_i} = u_i(x,h) - \omega P_i = 0, \qquad i = 1, \ldots, n. \tag{4}$$

$$Px - y = 0. \tag{5}$$

Partial derivatives are indicated by appropriate subscripts. These conditions define a special kind of demand functions:

$$x_i(P, y, h), \qquad i = 1, \ldots, n. \tag{6}$$

Adopting the terminology of Pollak (1969), we may call them conditional demand functions since they express the demand for consumer goods conditional upon the value of h. $\partial x_i/\partial h$ is the marginal effect on the demand for good i of an increase in work effort when disposable income, y, is somehow kept constant. The corresponding conditional indirect utility function may be written as:

$$v(P, y, h) = u(x(P, y, h), h), \tag{7}$$

where $x(\cdot)$ is a vector function.

We know from duality theory that:

$$v_i = -\omega x_i, \qquad i = 1, \ldots, n, \tag{8}$$

$$v_y = \omega, \tag{9}$$

$$v_h = u_h. \tag{10}$$

Subscripts y and h indicate partial derivatives with respect to these arguments, respectively. The second stage of the optimization is to maximize (7) with respect to h taking into account that y is a function of h. Provided that the income tax function is differentiable, the first-order condition for an interior optimum is:

$$v' = \frac{\mathrm{d}v}{\mathrm{d}h} = v_y(P, y, h)a(1 - T'(ah)) + v_h(P, y, h) = 0, \tag{11}$$

where the marginal income tax is denoted by $T' = \mathrm{d}T/\mathrm{d}I$. The differentiability assumption will be discussed in more detail below.

The second-order condition is:

$$v'' < 0. \tag{12}$$

For given prices and tax schedule h, x, I, v and ω become functions of the wage rate or ability parameter, a. We denote the (unconditional) indirect utility function by $V(a)$.

4. The optimal income tax

We shall explore the effect on welfare of switching slightly away from pure income taxation to the combined taxation of income and some commodity. As it seems natural to exhaust the opportunities for welfare improvements within the original system before introducing a new tax instrument, we shall assume that the initial income tax has got an optimal design. We can then benefit from making use of the optimality characteristics. In particular it will allow us make use of the envelope properties.

The optimal income tax has been analysed in a number of papers [see, for example, Mirrlees (1971, 1976, 1977)], and we shall not go into details in the present context. It is not the purpose of this paper to extend the analysis of the pure income tax. On the contrary, we shall make assumptions about the optimal tax schedule (differentiability, etc.) which mean that we have to be somewhat modest about the generality of the analysis. The analysis of the optimal income tax is a complicated piece of mathematical economics. The optimization problem is usually formulated as an optimum control problem. But, as emphasized by Mirrlees (1977), it is hard to tell, because of the special nature of the problem, when the optimum is characterized by the standard first-order conditions usually found in the literature. In particular, it may be dubious to represent the individual optimization simply by the first-order conditions of that problem in the social optimization. It is not the aim of this paper to take up these mathematical problems which apply to a wider class of optimum tax problems than the one presented here.

The analysis will be based on a number of crucial differentiability assumptions without which the analysis becomes much more complicated. First, the tax function itself is assumed to be differentiable. This may not be true in general, as pointed out by Mirrlees (1971). As we shall see, the assumption has got important implications. The budget set corresponding to a particular tax schedule and the consumption points chosen by different individuals are often illustrated in an I, y-diagram. For a given wage (ability), a, indifference curves can be drawn on this diagram to illustrate the trade-off between gross income and net income and hence the consumption–leisure trade-off of an a-individual. One such indifference curve is shown in fig. 1. We shall adopt the common assumption that an individual with a higher ability has a flatter indifference curve through any given point (I, y) than an individual with a lower ability. [See Seade (1982).] With this assumption it is obvious that individuals on different ability levels can have the same consumption point only if this is a corner point of the budget set. With corners ruled out by the differentiability assumption, gross income becomes a strictly increasing function of the individual wage-rate except for possible wage-rates at which no labour is supplied. Let $I(a)$ denote this relationship. [For more details see Seade (1982).]

In the following analysis we shall make use of shifts in the tax function. Introducing a shift parameter S we obtain an income tax function $T(I, S)$. It is common to assume that the economic variables are differentiable functions of the ability parameter a. [See, for example, Mirrlees (1976).] In the current analysis it is an essential assumption that the economic variables are differentiable with respect both to a and S. These assumptions are to some extent related. In general one of the problems encountered in tax optimization (in particular with a finite population) is that some consumers may be left indifferent among widely different consumption bundles. If this were the case, a small change in a might lead to discontinuous jumps in consumption points. In that case a small change in the tax schedule is also likely to produce discrete shifts of consumption points.

The case is illustrated in fig. 1. It shows the budget curve (B–B) resulting from a particular tax schedule and the indifference curve (I'–I') for a person of ability a'. As the figure has been drawn, this person is indifferent between point P and point Q. With the usual assumption that people with higher ability have flatter indifference curves, people with a greater than a' will be to the right of Q and people with lower a than a' will be to the left of P. Thus, there will be a discrete jump. If the a'-person is initially at P, and a marginal shift in the budget curve takes place which makes it slightly less favourable at and around P, the person will move his consumption point discretely to Q. Such discrete shifts are not permitted in the current analysis. Differentiability is crucial.

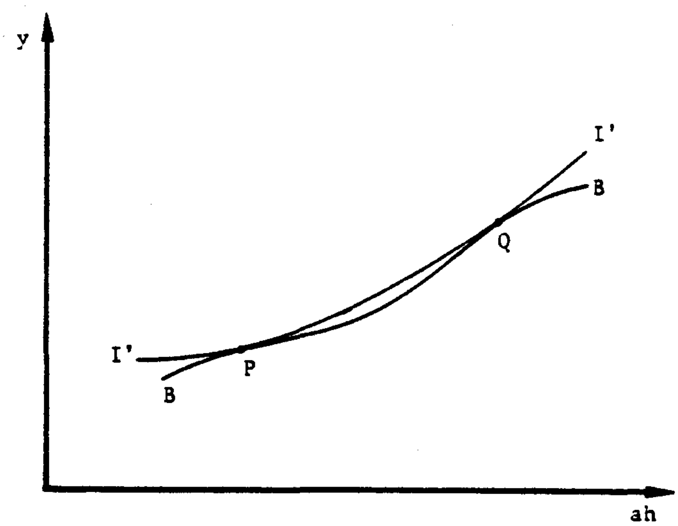

Fig. 1.

Let us now turn to our characterization of the optimal income tax. We start out by considering the situation in which a general income tax is the only tax instrument of the government. When designing its tax policy the government must take into account the whole population of individuals, each following the behaviour described above, and the ability distribution $f(a)$. A total tax revenue amounting to T^0 is required. The government must then choose its tax policy within the budget constraint

$$\int T(I(a))f(a)\mathrm{d}a=T^0, \tag{13}$$

where the size of the population has been normalized at unity.[2] The shape of the tax schedule is chosen so as to maximize

$$W=\int V(a)f(a)\mathrm{d}a \tag{14}$$

subject to (13).

It is not the concern of this paper to characterize in detail the optimum shape of the income tax function. A rather compact characterization will do for our purpose. Let us therefore assume that the optimal shape of the income tax function has been determined up to a number of parametric shifts. The last part of the optimization can then be carried out by means of usual parametric optimization. In order to do this we make use of the shift parameter S in the income tax function, $T(I,S)$. A shift is generated by changing S. It is denoted by $T_S=\partial T/\partial S$. Let us also assign the shadow price μ to the tax revenue constraint (13). The tax function can then be optimized with respect to S by means of the standard Lagrange expression

$$L=\int V(a,S)f(a)\mathrm{d}a+\mu\left(\int T(ah,S)f(a)\mathrm{d}a-T^0\right). \tag{15}$$

The first order condition can then be expressed as

$$\frac{\mathrm{d}L}{\mathrm{d}S}=-\int \omega T_S f\,\mathrm{d}a+\mu\int T_S f\,\mathrm{d}a+\mu\int T' a h_S f\,\mathrm{d}a=0, \tag{16}$$

where we have used the fact that $\partial V/\partial S=-\omega T_S$, which is easily established by applying the envelope theorem to (7). The second and third term of the left hand side express the resulting change in tax revenue evaluated by means of μ. The Lagrange multiplier has the usual interpretation:

$$\mu=-\partial W/\partial T^0 \quad \text{evaluated at the optimum.} \tag{17}$$

At the optimum an arbitrary marginal shift in the tax function must neither lower nor raise welfare, otherwise a shift could always be devised which would increase welfare, and it follows that the initial schedule would not have been optimal.

In order to be able to differentiate h with respect to S, as done in expression (16), our differentiability assumptions are obviously essential. As discussed above, a situation such as the one depicted in fig. 1 might lead to discrete jumps in the consumption point in response to a marginal shift in the tax schedule. It is important that such cases have been ruled out.

[2]For simplicity integration limits are omitted.

A shift parameter, S, can obviously be used to express any shift in the tax function from the optimal one. To see this let $T^*(I)$ be the optimal tax function, and let $F(I)$ be some arbitrary function of I. The income tax function can generally be written as $T(I,S)=T^*(I)+SF(I)$. The optimum value of S is obviously zero, and a small change in S will generate a marginal shift, of which the first-order effect on welfare is zero.

5. Marginal commodity taxes

Let $p_1,\dots,p_n$ denote the fixed producer prices per unit of $x_1,\dots,x_n$, respectively, and let p denote the corresponding price vector. Suppose that commodity taxes may be levied as taxes $t_1,\dots,t_n$ per unit of $x_1,\dots,x_n$, respectively. Let t denote the corresponding vector. Negative taxes are allowed, which means that commodities may be subsidized.

We are now prepared to consider the introduction of marginal excise taxes to supplement the optimal income tax. What commodities should then be (positively) taxed, left untaxed or subsized, respectively?

It should be noted at this stage that proportional taxation of all commodities is obviously equivalent to a proportional income tax. So this possibility is already covered by assuming the existence of an optimal initial income tax. The question we are asking is therefore: What commodities should be taxed if differentiated indirect taxes may be imposed?

In order to deal with commodity taxes, we must write the tax revenue constraint as

$$\int T(ah,S)f\,\mathrm{d}a+\int txf\,\mathrm{d}a-T^0=0. \tag{18}$$

The special case $t_1=t_2=\dots=t_n=0$ takes us back to (13) and the results of the preceding section, which we now take as our point of departure.[3] The imposition of marginal unit tax rates $t_1,\dots,t_n$ can now be analysed by applying the envelope theorem to the Lagrange expression:

$$L=\int v(P,y,h)f\,\mathrm{d}a+\mu\left(\int T(ah,S)f\,\mathrm{d}a+\int txf\,\mathrm{d}a-T^0\right). \tag{19}$$

Making use of (8) and bearing in mind that $P_i=p_i+t_i$ and $t_i=0$ for all i, we find that:

$$\frac{\partial W}{\partial t_i}=-\int \omega x_i f\,\mathrm{d}a+\mu\int x_i f\,\mathrm{d}a+\mu\int T'ah_{t_i}f\,\mathrm{d}a,\quad \forall i, \tag{20}$$

where $h_{t_i}=\partial h/\partial t_i$.

[3]The exposition is simplified by omitting the arguments of the functions where no confusion is likely to arise. The reader should bear in mind that $h=h(a,S,t)$, $y=ah-T(ah,S)$, $x_i=x_i(a,S,t)$, and $f=f(a)$.

Let us now pick one good, say good 1, for further consideration. How a small excise tax on good 1 would affect social welfare depends on the sign of

$$\frac{\partial W}{\partial t_1} = -\int \omega x_1 f \,\mathrm{d}a + \mu \int x_1 f \,\mathrm{d}a + \mu \int T' a h_{t_1} f \,\mathrm{d}a. \tag{21}$$

Eq. (21) expresses the welfare effect of levying a marginal commodity tax on commodity 1 without changing total tax revenue. The first term of (21) captures the immediate effect of the tax burden imposed by the new tax while the second and third term together capture the effect of the ensuing tax changes which are required to keep total tax revenue unaltered.

There is not much to say on the basis of (21). Further manipulation is obviously necessary to be able to arrive at policy recommendations. The first thing we do is to define a marginal shift in the income tax function of which the immediate effect is to impose a tax increase x_1 on each taxpayer. Formally:

$$T_S = x_1. \tag{22}$$

This analytical trick is a crucial point which may require a more detailed explanation. Since the income tax is a function of gross income, and the function must be the same for everybody, it is only admissible to define such a shift if x_1 can be expressed as a function of gross income alone. Taking the initial income tax function as given, and recalling that preferences are uniform across individuals, the individual decision variables ultimately become functions of the wage-rate only. We can therefore express x_1 as a function $x_1(a)$ which is the initial relationship between x_1 and a.[4] Moreover, as we have seen already, gross income is a strictly increasing function of a which can be inverted so that a becomes a function of I. Inserting this relationship into $x_1(a)$ we obtain a function $x_1^*(I)$ which is exactly the kind of relationship which allows us to write eq. (22). We can now write $T(I,S) = T^*(I) + Sx_1^*(I)$. Hence, the shift is well defined for all I.

Our differentiability assumptions are crucial at this stage. If the tax schedule were kinked, there would be people with different wage-rates and different consumption bundles earning the same income. Then there would not be a unique value of x_1 associated with each value of I, and (22) would not be a meaningful definition of a shift in the income tax function.

The reader should not be confused by the fact that $I = ah$ is a variable in the tax function, while we also write h as a function of S. A change in S changes the tax associated with each value of $I = ah$. This is a quite ordinary shift. But, since the individual will normally respond to the shift in the tax

[4] x_i is used as a function symbol both in $x_i(P, y, h)$ and $x_i(a)$ since this is not likely to cause any confusion.

schedule by changing his consumption point, the chosen value of h (or l) depends on S. The actual change in the tax paid by an individual is the combined effect of a shift in the tax function and a movement along the tax schedule. This is analogous to a shift in an ordinary partial demand function which implies that we can write the price as a function of the shift parameter.

Since (16) is true for any marginal shift in the tax function from the initial optimum, it is also valid for the shift defined by (22). Hence, if we substitute x_1 for T_S in (16):

$$-\int \omega x_1 f\, \mathrm{d}a + \mu \int x_1 f\, \mathrm{d}a + \mu \int T' a h_S f\, \mathrm{d}a = 0,$$

or:

$$-\int \omega x_1 f\, \mathrm{d}a + \mu \int x_1 f\, \mathrm{d}a = -\mu \int T' a h_S f\, \mathrm{d}a. \tag{23}$$

Inserting this expression into (21) we obtain:

$$\frac{\partial W}{\partial t_1} = \mu \int T' a (h_{t_1} - h_S) f\, \mathrm{d}a. \tag{24}$$

Now the complexity of the formula has been reduced, and we can approach the problem of signing. In general $(h_{t_1} - h_S)$ can be of any sign, and the question is whether positive terms outweigh the negative ones or vice versa. A more interesting approach is to ask whether there are classes of utility functions or demand patterns which ensure sufficiently unambiguous reactions across individuals to guarantee a unique sign. Indeed, we know of one such case already from the weak separability result of Atkinson and Stiglitz (1976).

Since μ, T', a and f are all positive, what we want to explore is the sign of $(h_{t_1} - h_S)$. The effect of t_1 on h is obtained from the optimality condition (11):

$$v_y(p+t, ah - T(ah, S), h)\, a(1 - T'(ah, S)) + v_h(p+t, ah - T(ah, S), h) = 0.$$

Differentiating with respect to t_1 we find:

$$v'' h_{t_1} + a(1 - T') v_{y1} + v_{h1} = 0,$$

and hence:

$$h_{t_1} = \frac{1}{v''}\left(\frac{v_h}{v_y} v_{y1} - v_{h1}\right), \tag{25}$$

where (11) has again been applied. h_S is found in a similar way by differentiating (11):

$$v''h_S+a(1-T')v_{yy}(-T_S)+v_ya\left(-\frac{\partial T'}{\partial S}\right)-v_{hy}T_S=0. \tag{26}$$

A number of substitutions can be made by means of the relations:[5]

$$T_S=x_1=-\frac{v_1}{v_y}, \qquad \frac{\partial T'}{\partial S}=\frac{\partial T_S}{\partial I}=\frac{\partial x_1}{\partial I}=-\frac{\partial(v_1/v_y)}{\partial I}. \tag{27}$$

We find that:

$$h_S=\frac{1}{v''}\left[\frac{v_h}{v_y}v_{yy}\frac{v_1}{v_y}-v_{hy}\frac{v_1}{v_y}-v_ya\frac{\partial}{\partial I}\left(\frac{v_1}{v_y}\right)\right]. \tag{28}$$

Since y and h are functions of a, and, as we have seen, a can be expressed as $a(I)$, we can write:

$$x_1(P,y,h)=x_1(P,y(I),h(I))=-v_1(P,y(I),h(I))/v_y(P,y(I),h(I)),$$

which hold at the initial equilibrium where $S=t_1=0$.[6] Hence we find that:

$$a\frac{\partial x_1}{\partial I}=-a\frac{\partial}{\partial I}\left(\frac{v_1}{v_y}\right)=-a\frac{\partial}{\partial y}\left(\frac{v_1}{v_y}\right)\frac{\partial y}{\partial I}+a\frac{\partial x_1(P,y,h)}{\partial h}\frac{\partial h}{\partial I}$$

$$=-\left(\frac{v_{1y}}{v_y}-\frac{v_1v_{yy}}{v_y^2}\right)(1-T')a+\frac{\partial x_1(P,y,h)}{\partial h}a\frac{\partial h}{\partial I},$$

and due to (11):

$$-a\frac{\partial}{\partial I}\left(\frac{v_1}{v_y}\right)=\left(\frac{v_{1y}}{v_y}-\frac{v_1v_{yy}}{v_y^2}\right)\frac{v_h}{v_y}+\frac{\partial x_1(P,y,h)}{\partial h}a\frac{\partial h}{\partial I}. \tag{29}$$

This result simplifies (28) so that:

$$h_S=\frac{1}{v''}\left(\frac{v_{1y}v_h}{v_y}-\frac{v_{hy}v_1}{v_y}+v_y\frac{\partial x_1(P,y,h)}{\partial h}a\frac{\partial h}{\partial I}\right). \tag{30}$$

Applying (25) and (30):

$$h_{t_1}-h_S=\frac{1}{-v''}\left(v_{h1}-\frac{v_{hy}v_1}{v_y}+v_y\frac{\partial x_1(P,y,h)}{\partial h}a\frac{\partial h}{\partial I}\right). \tag{31}$$

[5]Recall that $T(I,S)=T^*(I)+Sx_1^*(I)$. Hence, $T'(I,S)=\partial T^*/\partial I+S\partial x_1^*/\partial I$, and $\partial T'/\partial S=\partial x_1^*/\partial I$.

[6]These parameters have been suppressed. However, we shall use the partial derivative $\partial h/\partial I$ to denote $\partial h(I,S,t_1)/\partial I$ evaluated at $S=t_1=0$.

We know that $x_1 = -v_1/v_y$. Differentiating with respect to h, we obtain:

$$\frac{\partial x_1(P,y,h)}{\partial h} = -\frac{1}{v_y}\left(v_{h1} - \frac{v_{hy}v_1}{v_y}\right),$$

which is evaluated at the initial consumption point. Inserting this result into (31) we obtain:

$$h_{t_1} - h_S = \frac{v_y}{v''}\frac{\partial x_1(P,y,h)}{\partial h}\left(1 - a\frac{\partial h}{\partial I}\right). \tag{32}$$

Under the assumptions which have been made, a and I are positively related. By definition $I = ah$, which implies that $1 - a\partial h/\partial I = h\mathrm{d}a/\mathrm{d}I > 0$.

From the second-order condition of the individual optimum we know that $v'' < 0$. It then follows from (24) and (32) that:

$$\begin{aligned} &\frac{\partial x_1}{\partial h} > 0, \quad \text{for all } a \Rightarrow \frac{\partial W}{\partial t_1} < 0, \\ &\frac{\partial x_1}{\partial h} = 0, \quad \text{for all } a \Rightarrow \frac{\partial W}{\partial t_1} = 0, \\ &\frac{\partial x_1}{\partial h} < 0, \quad \text{for all } a \Rightarrow \frac{\partial W}{\partial t_1} > 0. \end{aligned} \tag{33}$$

Analogous results may, of course, be derived for $x_2, \ldots, x_n$. Thus, there are certain demand patterns which uniquely determine whether a small tax or subsidy on a commodity should be recommended. The partial derivative of the conditional demand function, $\partial x_1(P,y,h)/\partial h$, expresses the effect on the demand for x_1 when an individual who is initially optimally adjusted is forced to work a little more without any change in disposable income. A detailed discussion of conditional demand functions is found in Pollak (1969). Adopting his terminology, we say that

(a) if $\partial x_i/\partial h > 0$, x_i is positively related to h,
(b) if $\partial x_i/\partial h = 0$, x_i is unrelated to h, and
(c) if $\partial x_i/\partial h < 0$, x_i is negatively related to h.

We can now state:

Proposition I. Starting from a situation in which an optimal income tax is the only tax, a welfare gain is achieved by imposing a (positive) marginal excise tax on a commodity which is negatively related to labour and by introducing a marginal subsidy on a commodity which is positively related to labour.

In other words, commodities one typically buys more (less) of if more leisure is obtained without any loss of income, are candidates to be taxed (subsidized). This is a very simple result, although it may not be quite as simple to determine in practice what kind of relationship holds true for a particular commodity.

Proposition 1 gives a more precise meaning to the results of Meade (1955). A precise meaning has been assigned to Meade's references to 'substitutes for leisure' and 'things which are jointly demanded with leisure'. The relevant relations between the demand for various goods and the demand for leisure (or labour supply) are those precisely defined by Pollak (1969) as stated above. In a rough sense these relations are a kind of complement and substitute concepts, which are akin to but *not* identical to the usual Hicksian concepts used by Corlett and Hague. With this background it is interesting to explore further the various relations between leisure and the demand for the various commodities, which we now do.

6. The roles of substitutes, complements and quasi-separability

It may be interesting to relate the results of this paper to the conventional substitute and complement concepts. This may be helpful when trying to trace the effects of working time on demand, and it may reveal to what extent the results of the simple Corlett–Hague model carry over to the more complicated model of mixed taxation studied in the current paper and in modern tax theory.

The simplest procedure is to establish directly the relationship between the Pollak concepts and the Hicksian concepts. Let us consider the conditional demand function,

$$x_i(P, ah - T(ah), h), \tag{34}$$

at the unconditional equilibrium; that is where h is optimally adjusted to the price and tax parameters. Let m denote the marginal wage rate:

$$m = a(1 - T'). \tag{35}$$

The effect of a change in m on x_i is found by differentiating (34):

$$\frac{\partial x_i}{\partial m} = \frac{\partial x_i}{\partial y} m \frac{\partial h}{\partial m} + \frac{\partial x_i}{\partial h}\frac{\partial h}{\partial m}.$$

Hence:

$$\frac{\partial x_i}{\partial h} = \frac{\partial x_i/\partial m}{\partial h/\partial m} - m\frac{\partial x_i}{\partial y}. \tag{36}$$

It is well known that the labour supply responds positively to a higher marginal wage rate so that $\partial h/\partial m>0$. For convenience let us rule out inferior goods so that $\partial x_i/\partial y>0$. Then we see that if x_i and h are substitutes, $\partial x_i/\partial m<0$, then x_i is negatively related to h in the Pollak sense. x_i is negatively related to h if they are 'weak' complements, and positively related to h if they are 'strong' complements. Thus, we find that *a complement with leisure (substitute for labour) should be taxed and so should a sufficiently weak substitute for leisure, while a strong enough substitute for leisure should be subsidized* from our tax reform point of view. Likewise the demand for a complement with or sufficiently weak substitute for leisure should be discouraged in the Mirrlees sense at the full optimum, while the demand for a strong enough substitute for leisure should be encouraged (see section 8 below). We may note that the effects pointed out by Corlett and Hague do play a crucial role, even in the present context.

An alternative way to relate our results to the roles of substitutes and complements may bring out more clearly how the various effects arise. It starts out considering the crucial difference $(h_{t_i}-h_S)$ of formula (24). Each term may be decomposed into a compensated effect and an income effect:

$$h_{t_i}-h_S=h_{t_i}|_u+\text{income effect}-h_S|_u-\text{income effect}$$

$$=h_{t_i}|_u-h_S|_u, \tag{37}$$

since the shift is designed in such a way that the income effects are equal and cancel out. From the definition $m=a(1-T')$ it follows that:

$$\frac{\partial m}{\partial S}=-a\frac{\partial T'}{\partial S}. \tag{38}$$

Making use of the assumption that $T_S=x_i$, we find that:

$$\frac{\partial T'}{\partial S}=\frac{\partial T_S}{\partial I}=\frac{\partial x_i}{\partial I}. \tag{39}$$

Eqs. (38) and (39) together imply that:

$$\frac{\partial m}{\partial S}=-a\frac{\partial x_i}{\partial I}. \tag{40}$$

Hence we can derive:

$$h_S|_u=\frac{\partial h}{\partial m}\frac{\partial m}{\partial S}=-\frac{\partial h}{\partial m}a\frac{\partial x_i}{\partial I}. \tag{41}$$

We insert this result into (37) and obtain:

$$h_{t_i} - h_S = \left.\frac{\partial h}{\partial t_i}\right|_u + \frac{\partial h}{\partial m} a \frac{\partial x_i}{\partial I}, \tag{42}$$

where the sign of the compensated derivative $(\partial h/\partial t_i)|_u$ defines whether h and x_i are substitutes or complements. The presence of $\partial x_i/\partial I$ in (42) shows that a sort of income effect is important. It should be noted that this effect arises only because it determines the change in the marginal after-tax wage-rate which in turn has a pure substitution effect. The greater the inequality aversion of the government, the steeper will the initial income tax schedule tend to be. If a tax is levied on a commodity of which high-income people will buy far more than low-income people, the steepness of the income tax schedule can be reduced significantly without changing the overall distributional profile of the tax policy. Hence, the marginal income tax is reduced with favourable effects on efficiency. This is exactly what is expressed by the second term of (42).

If the choice of commodity taxes is approached in terms of ordinary Hicksian substitute and complement concepts, one must also allow for the income effect on commodity demand discussed above, which in general may be positive or negative. In this sense Meade was right in making his qualification about the distributional impact of a commodity tax. If sufficiently adverse (from an egalitarian point of view) it may, as noted by Meade, dominate the otherwise advantageous effect of a tax on a complement with leisure.

The role of the income effect may be related to the tax results obtained in the case of quasi-separability [see Deaton (1981)]. Goods i and j are said to be quasi-separable from leisure (work) if the marginal rate of substitution between good i and good j is independent of leisure (work) along an indifference curve, if the consumer is compensated for the change in leisure (work) by a proportional change in the vector of all goods (including leisure). An important implication of quasi-separability is that compensated changes in the wage-rate affect commodity demands proportionately. Since a sole change in the marginal wage-rate is equivalent to a compensated change in the wage-rate, we have that in the case of quasi-separability:

$$\partial x_i/\partial m = \delta x_i, \tag{43}$$

where δ is independent of i. We can then reformulate (36) to obtain:

$$\frac{\partial x_i/\partial h}{x_i} = \frac{\delta}{\partial h/\partial m} - \frac{m}{y}\frac{\partial x_i}{\partial y}\frac{y}{x_i}. \tag{44}$$

The message derived from this formula is that commodity taxes should be levied on commodities with sufficiently high income elasticities, as may be the case for a typical luxury. This result bears a certain resemblance to the finding that progressive (non-linear) commodity taxation is desirable under quasi-separability [Deaton (1981, p. 1249)]. This means that the tax-rate should be higher the more luxurious one finds a good, in the sense that it is more highly valued by people on high utility levels than by people on less favourable indifference curves. It also seems interesting to relate the results of the present analysis to those of Atkinson and Stiglitz (1976) and Mirrlees (1976), as will be done in the following two sections.

7. Comparison with Atkinson and Stiglitz

From Atkinson and Stiglitz (1976) we know that commodity taxes are always non-optimal regardless of how the population is composed if the utility function is of the weakly separable form $u(\phi(x),h)$ so that $\partial(u_i/u_1)/\partial h=0$ for all i. It is easy to verify that this is equivalent to $\partial x_i/\partial h=0$ for all i, as indeed it should be if our results are correct. This equivalence sheds new light on the Atkinson–Stiglitz result.

The optimality conditions for x when h is given are:

$$\frac{u_j}{u_1}=\frac{P_j}{P_1}, \quad \forall j, \tag{45}$$

$$\frac{y}{P_1}=\frac{1}{P_1}Px. \tag{46}$$

Let us introduce the notation $x_j'=\partial x_j/\partial h$. Differentiating the equation system (45) and (46) we obtain the system:

$$\begin{aligned}
&\left(\frac{\partial}{\partial x_1}\frac{u_2}{u_1}\right)x_1'+\left(\frac{\partial}{\partial x_2}\frac{u_2}{u_1}\right)x_2'+\ldots+\left(\frac{\partial}{\partial x_n}\frac{u_2}{u_1}\right)x_n'=-\frac{\partial}{\partial h}\frac{u_2}{u_1},\\
&\left(\frac{\partial}{\partial x_1}\frac{u_3}{u_1}\right)x_1'\ldots=-\frac{\partial}{\partial h}\frac{u_3}{u_1},\\
&\quad\vdots\\
&\left(\frac{\partial}{\partial x_1}\frac{u_n}{u_1}\right)x_1'\ldots=-\frac{\partial}{\partial h}\frac{u_n}{u_1},\\
&x_1'+\frac{u_2}{u_1}x_2'+\ldots+\frac{u_n}{u_1}x_n'=0.
\end{aligned} \tag{47}$$

We immediately see that:

$$\frac{\partial(u_j/u_1)}{\partial h}=0, \text{ for all } j \Leftrightarrow \frac{\partial x_j}{\partial h}=0, \text{ for all } j. \tag{48}$$

Eq. (48) allows us to state:

Proposition 2. The demand pattern $\partial x_j/\partial h=0$, for all j, obtains if and only if the utility function belongs to the class of weakly separable utility functions $u(\phi(x),h)$.

For later application it is useful to solve the equation system of the two-good case using commodity 2 as the numeraire:

$$\frac{\partial x_1}{\partial h}=\frac{\partial(u_1/u_2)}{\partial h}\frac{1}{D}, \tag{49}$$

$$\frac{\partial x_2}{\partial h}=-\frac{\partial(u_1/u_2)}{\partial h}\frac{u_1}{u_2}\frac{1}{D}, \tag{50}$$

where

$$D=-\begin{vmatrix} \dfrac{\partial}{\partial x_1}\dfrac{u_1}{u_2} & \dfrac{\partial}{\partial x_2}\dfrac{u_1}{u_2} \\ \dfrac{u_1}{u_2} & 1 \end{vmatrix}.$$

As usual, u is assumed to be strictly quasi-concave in x_1 and x_2, which implies that $D>0$.

8. Comparison with Mirrlees

Mirrlees (1976) established conditions for optimal mixed taxation consisting of income and commodity taxation. His analysis is presented in a very general form. Let us now consider the case in which simultaneous optimization of a non-linear income tax and excise tax rates on the various commodities takes place.

Sticking to the notation of the current paper, we can write the demand functions used by Mirrlees as:

$$x_i(P,y,I,a) \tag{51}$$

(with P, I and a replacing q, z and n, respectively). Let us define

$$\left.\frac{dx_i}{dP_k}\right|_u = x_{ik}^c, \tag{52}$$

which denotes the compensated derivative of x_i with respect to P_k.

In our notation the following necessary optimum conditions derived by Mirrlees can be established:

$$e_i \equiv \int \sum_k x_{ik}^c t_k f(a)\,\mathrm{d}a = -\int v(a) \frac{\partial x_i(P, y, I, a)}{\partial a}\,\mathrm{d}a, \qquad i = 1, \ldots, n, \tag{53}$$

where $v(a) > 0$ [Mirrlees (1976, eq. (9.86))].

Adopting Mirrlees' own interpretation, e_i is a measure of the extent to which commodity taxes encourage consumption of the different commodities. Then $\partial x_i/\partial a > 0$ implies that the consumption of commodity i should be discouraged, while $\partial x_i/\partial a < 0$ implies that it should be encouraged.

The results of Mirrlees apply to the full optimum, while the analysis of the preceding sections of this paper is concerned with the introduction of small excise taxes to supplement the income tax. However, the results are closely related to each other. To see this, rewrite (51) in the following way:

$$\begin{aligned} x_i(P, y, I, a) &\equiv x_i(P, y, I/a) \\ &\equiv x_i(P, y, ah/a) \equiv x_i(P, y, h), \end{aligned} \tag{54}$$

which is identical to the conditional demand function (6). We immediately see that:

$$\frac{\partial x_i(P, y, I, a)}{\partial a} = -\frac{\partial x_i(P, y, h)}{\partial h} \cdot \frac{h}{a}. \tag{55}$$

We can therefore say that if x_i is negatively related to h, the consumption of commodity i should be discouraged by the optimum excise taxes, while if x_i is positively related to h, the demand for commodity i should be encouraged by the optimum excise taxes.

Condition (53) was explained by Mirrlees (1976, p. 347) in the following words: 'This surprisingly simple criterion says that commodity taxes should bear more heavily on the commodities high-a individuals have relatively strongest tastes for.' We have seen that a good is negatively related to labour if a reduction in hours worked, holding income constant, results in an increase in demand for the good. Such a reduction in hours worked with constant income can be achieved by increasing ability. It is in this sense that a good that is negatively related to labour is a good for which people of higher abilities have 'strongest tastes'.

9. Leisure goods and the Becker–Lancaster approach

In much of the literature on excise taxes and tax-distorted labour supply a special importance is attached to leisure goods. It is typical that after showing that there is no need to employ differentiated indirect taxation in the presence of a weakly separable utility function, Atkinson and Stiglitz (1980) make the following comment: '...and it is quite possible that this separability requirement may not in practice be met, for example, in the case of leisure goods'. This comment leads to no further conclusion. But it seems to be a popular belief among many economists that taxation of leisure goods should be recommended. It is therefore interesting to discuss whether this is an implication of the tax theory.

An approach to the relationship between working time (or leisure) and market goods which may be useful for this purpose, is the Becker–Lancaster approach [Becker (1965) and Lancaster (1966)]. The key idea of this approach is that the basic goods enjoyed by a consumer are produced in the household by means of own effort and commodities purchased in the market. Recreation may be one such basic good which is produced by means of a certain input of time (leisure) and leisure goods.

We shall consider a simple case where utility is derived from two basic goods, of which one is recreation and the other is simply a market commodity. The enjoyment of the latter is assumed not to be time consuming. Let z denote the amount of recreation. Leisure, r, is the amount of time spent on recreation. Commodity 1 is now interpreted as the market good used as an input in the production of z. The household production of recreation is described by the technology function:

$$z(x_1, r). \tag{56}$$

Partial derivatives are denoted by z_x and z_r and are assumed to be positive. x_1 may consist of sporting equipment, travelling, tickets for concerts or amusement parks, etc.

The relationship between r and h is given by the time budget:

$$r + h = k = \text{constant}. \tag{57}$$

The utility function is now written as:

$$u(z(x_1, r), x_2, h) = u(z(x_1, k-h), x_2, h), \tag{58}$$

which is basically a function of x_1, x_2 and h, as before. Note that h enters twice, once because of its opportunity cost through the time budget and once because work may be disliked or enjoyed as such. The following notation is used:

$$u_z=\frac{\partial u}{\partial z},\qquad u_i=\frac{\partial u}{\partial x_i},\qquad i=1,2;$$

$$z_{xr}=\frac{\partial^2 z}{\partial x_1 \partial r}.$$

As was discussed in section 7, a small excise tax on x_1 should be recommended or not depending on the sign of $\partial(u_1/u_2)/\partial h$. We easily find that:

$$\frac{u_1}{u_2}=\frac{u_z z_x}{u_2}. \tag{59}$$

Then the following expression is obtained:

$$\frac{\partial(u_1/u_2)}{\partial h}=-z_{xr}\frac{u_z}{u_2}-\frac{\partial(u_z/u_2)}{\partial z}z_x z_r+\frac{\partial(u_z/u_2)}{\partial h}z_x, \tag{60}$$

where both effects of h are allowed for. Let us consider the terms in reversed order. The last term expresses how the marginal valuation of recreation changes with work effort. It seems natural to assume that this effect is non-negative. Probably recreation is more highly valued the harder a person works. For a given labour supply $\partial(u_z/u_2)/\partial z<0$ if x_2 is a normal good, which seems to be a natural assumption, especially at this level of aggregation. If x_1 and r are technically independent, $z_{xr}=0$, or technical substitutes, $z_{xr}<0$, the first term on the right-hand side is non-negative, and the whole expression becomes positive. If x_1 and r are technical complements, $z_{xr}>0$, the overall sign is positive if x_1 and r are sufficiently weak complements, and negative if they are sufficiently strong complements in a technological sense. *Thus, a small excise tax on x_1 should be recommended only if x_1 and leisure are strong technical complements. Otherwise a small subsidy should be recommended.*

It follows that it is not necessarily appropriate to recommend taxation of leisure goods, even if one is willing to make the tempting assumption that they are technical complements with leisure within a Becker–Lancaster framework. The notion behind the belief that leisure goods should be taxed is presumably that by making leisure goods more expensive, recreation is made less attractive and so is leisure time. The weak point in this argument is that even if recreation becomes less attractive compared to other goods, leisure goods may be substituted by leisure time in the production of recreation, and the labour supply may shrink. Recreation may simply be made more time-intensive and less commodity-intensive. As an extreme example a long cheap cottage holiday may be substituted for a short service-intensive luxury cruise.

We may explore this matter further after adding a few more simplifying assumptions which reduce the efforts needed to bring out the main point. Utility is assumed to be a function of z and x_2 only, which means that there is no utility or disutility from work effort per se. This utility function is assumed to be homothetic. The technology function z is assumed to be homogeneous of degree one. The respective marginal rates of substitution can be expressed as:

$$\frac{u_z}{u_2}=\alpha\left(\frac{x_2}{z}\right) \tag{61}$$

and

$$\frac{z_x}{z_r}=\beta\left(\frac{r}{x_1}\right), \tag{62}$$

where α and β are both increasing functions.

The elasticities of these functions are denoted by $\hat{\alpha}$ and $\hat{\beta}$. We also define the elasticities of substitution:

$$\sigma_u=1/\hat{\alpha} \tag{63}$$

and

$$\sigma_z=1/\hat{\beta}. \tag{64}$$

In order to focus on demand properties, taxes are left on one side. The budget constraint of the consumer then becomes:

$$P_1x_1+P_2x_2=ah. \tag{65}$$

The optimum choice of the consumer is determined by eq. (65):

$$\alpha\left(\frac{x_2}{z}\right)=\frac{P_1}{P_2z_x}, \tag{66}$$

$$\beta\left(\frac{r}{x_1}\right)=\frac{P_1}{a}, \tag{67}$$

and

$$z(x_1,r)=z. \tag{68}$$

In order to study compensated effects, utility is assumed to be constant:

$$u(z, x_2) = \text{constant}. \tag{69}$$

Let ε_1 and ε_2 denote the partial elasticities of z. Also, let $\hat{z}_x$ denote the partial elasticity of z_x with respect to x_1. Since z_x is homogeneous of degree zero, the partial elasticity of z_x with respect to r is $-\hat{z}_x$. Let a caret (ˆ) over x_1, x_2, z and r denote the compensated elasticity with respect to P_1. The elasticities are derived by means of the equation system (66)–(69). We find that:

$$\hat{x}_2 - \hat{z} = \sigma_u + \sigma_u \hat{z}_x(\hat{r} - \hat{x}_1), \tag{70}$$

$$\hat{r} - \hat{x}_1 = \sigma_z, \tag{71}$$

$$\varepsilon_1 \hat{x}_1 + \varepsilon_2 \hat{r} = \hat{z}, \tag{72}$$

and

$$\frac{u_z z}{u}\hat{z} + \frac{u_2 x_2}{u}\hat{x}_2 = 0,$$

which is equivalent to:

$$\frac{P_1 z}{P_2 x_2 z_x}\hat{z} + \hat{x}_2 = 0. \tag{73}$$

From (71) and (72) we get:

$$\hat{r} = \varepsilon_1 \sigma_z + \hat{z}. \tag{74}$$

We see that even if the effect of a higher P_1 is to make z less attractive, $\hat{z} < 0$, r may still rise because of the positive substitution term $\varepsilon_1 \sigma_z$. From (70), (71), and (73):

$$\hat{z} = \frac{-1}{\gamma}(\sigma_u + \sigma_u \sigma_z \hat{z}_x), \tag{75}$$

where

$$\gamma = 1 + \frac{P_1 z}{P_2 x_2 z_x}.$$

Combining (74) and (75), we get:

$$\frac{\hat{r}}{\sigma_u} = \left(\frac{\varepsilon_1}{\sigma_u} + \frac{-\hat{z}_x}{\gamma}\right)\sigma_z - \frac{1}{\gamma}. \tag{76}$$

$\hat{z}_x$ is assumed to be negative so that the term in parentheses is positive. Hence, we see that the substitution properties of the z-function are crucial in determining the sign of $\hat{r}$.

If σ_z is big enough so that substitution possibilities are good, $\hat{r}$ will become positive, because a higher P_1 motivates a large-scale substitution of x_1 by r. Thus, x_1 and leisure become substitutes, even though they are used jointly to produce recreation. If σ_z is small enough so that substitution possibilities dwindle, $\hat{r}$ obviously becomes negative, and x_1 and r are complements.

Substitution between leisure time and leisure goods is a complication which must be carefully allowed for when excise taxes on leisure goods are considered. This is not to argue that leisure goods should be left untaxed. A number of leisure goods are probably not easily substituted by leisure time. And even if a commodity is a substitute for leisure, we know from section 6 that it may be desirable to tax this commodity if the substitution property is moderate. Hence, there is good reason to believe that many leisure goods should be taxed. But a general prescription to this effect may easily prove too hasty. Which specific commodities show such demand properties that they should be taxed (subsidized) is, of course, an empirical question which is not going to be pursued any further in this paper.

10. Conclusion

Within a tax-theoretical framework and starting from a situation in which an optimal income tax is the only tax, this paper has analysed which commodities should be taxed (subsidized) if small excise taxes may be levied and the income tax adjusted to keep total tax revenue unchanged. The answer is simple. A commodity should be taxed (subsidized) if it is positively (negatively) related to leisure in the Pollak sense, which means that more (less) of the good is consumed if more leisure is obtained at constant income. Furthermore, it was shown that the result of Atkinson and Stiglitz, that no excise taxes should be levied if the utility function exhibits weak separability between leisure and all other goods together, simply means that if all commodities are unrelated to leisure in the Pollak sense, no commodity should be taxed (or subsidized). It was also demonstrated that the sign of the same relation determines whether the consumption of a commodity should be encouraged or discouraged in the Mirrlees sense when the income tax and excise taxes are all set optimally.[7] The analysis has provided a link between the older analyses by Corlett and Hague and by Meade and modern optimum tax theory on the roles of direct and indirect taxation.

The analysis has been confined to circumstances where all essential functions are differentiable. This approach, shared with a number of other analyses in this field, has allowed the use of ordinary differential calculus.

[7]See Mirrlees (1976, p. 347).

Although of limited generality, it seems that differentiable cases can contribute to our insight into the optimum choice of commodity taxes to supplement the income tax, and thus have a fair claim for interest.

The partial nature of the analysis should be noted. The choice between direct and indirect taxation has a number of interesting aspects, of which the effect on labour incentives, focused on above, is only one. The argument for excise taxes most firmly rooted in economic theory is, of course, the externality argument for Pigouvian taxes. This aspect is well known and hardly needs elaboration. It has therefore been suppressed in the present context. Two aspects have received little attention in the theoretical literature so far. One is the choice of excise taxes when savings decisions are distorted by income taxation, and the other is the implications of tax evasion for the choice between direct and indirect taxation. Further exploration of these aspects within the framework of optimum tax theory or tax reform theory would certainly be worthwhile in order to broaden the basis of economic advice in this practically and politically important field.

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THE CHOICE OF EXCISE TAXES WHEN SAVINGS AND LABOUR DECISIONS ARE DISTORTED

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The framework is a simple two period life-cycle model with identical individuals who work in period 1 and are retired in period 2. Initially there are optimally chosen uniform income and commodity tax rates, which lead to distortions in both the labour/leisure choice and the consumption/savings choice. The purpose is to demonstrate in an intuitively comprehensible way how differentiating commodity taxation by slightly increasing the tax on one consumption good can mitigate existing distortions. The sufficient conditions for a welfare improvement are similar to that of Corlett and Hague plus some conditions on average and marginal consumption propensities.

1. Introduction

It is well known within tax theory that an income tax and a general tax on consumption (for instance a general purchase tax, VAT or expenditure tax) both distort the marginal trade-off between work and leisure by driving a wedge between the wage rate paid by the employer and the wage rate received by the employee. It is also known that excise taxes on particular commodities may help to mitigate this distortion, and thus may be recommended despite the fact that they tend to distort relative commodity prices. In an early contribution Corlett and Hague (1953–54) showed that by taxing a commodity which is complementary with leisure, one can indirectly make the enjoyment of leisure relatively more expensive, and hence to some extent counteract the opposite effect of a tax on income or general consumption. [For a modern exposition of this result see Sandmo (1976).] The same basic idea was discussed in Meade (1955). A central reference to modern tax reform analysis is Dixit (1975), which derives a number of results, including a generalization of the Corlett and Hague result. These analyses are all carried out within the framework of a timeless or one-period model.

Another well-known fact in tax theory is that an income tax distorts the trade-off between consumption and savings (or present and future consump-

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tion), while a general tax on consumption is neutral in this respect. This recognition has led to several studies of the relative merits of taxes on income and consumption when savings and labour supply distortions are both allowed for. [See, for instance, Atkinson and Sandmo (1980) and King (1980).]

The purpose of this paper is to approach the question of choosing excise taxes on particular commodities when the income tax distortion of savings behaviour is taken into account in addition to the wage distortion. As a starting-point for extending the analysis in that direction, the optimal balance between taxation of income and consumption in general will be assumed rather than analysed. The more precise question to be taken up is under what conditions the introduction of a small excise tax on a particular commodity should be recommended. Hence the analysis may be regarded as an exercise in tax reform theory.

The analysis will focus on pure efficiency aspects. Hence only one individual (or a population of identical individuals) is considered. To simplify the analysis, only two periods are considered. The person works, consumes and saves from one period to the next. In these activities he is faced with a two-period budget constraint, which is affected by the tax policy. The model can be interpreted in two ways. It can be considered as a simple extension of the one-period model needed to cope with intertemporal problems, while a fixed population is assumed. Or one may assume that in the background there are in fact overlapping generations, but we shall consider only one representative generation. This would be more in line with the model of Atkinson and Sandmo (1980) and King (1980). If all generations are assumed to be equally endowed, it also seems natural to require that they should be treated equally, so that the tax policy should be the same in all periods. In any case this may be a convenient simplification.

We shall assume that a person supplies labour only in the first period. The nearest interpretation is, of course, that the person is retired in the second period. This assumption is in line with the approach of Atkinson and Sandmo (1980) and King (1980). It is a useful simplification. In particular it makes it easier to relate the results of the two-period model to the one-period results of Corlett and Hague (1953–54) or Dixit (1975). The important extension of the current analysis is the introduction of savings, and additional complications in other respects are rather avoided.

We consider a simple tax system. First, there is a proportional income tax with the same tax rate in both periods. The tax rate is denoted by t. Second, there is a proportional tax on consumption, which we may think of as a general purchase tax. The tax rate is constant over time. It is denoted by τ. This is equivalent to a situation with two taxes on wage and interest. An important and well-known property of a purchase tax is that it is non-distortive with respect to the trade-off between savings and consumption provided that the tax rate is constant over time. This is a property which distinguishes the consumption tax from an income tax. The existence both of an income tax and a general purchase tax allows us to consider the effects of differentiating an originally uniform commodity tax when savings are distorted by an income tax. In order to differentiate the indirect taxation the introduction of an excise tax on one commodity, say commodity 1, is considered. The tax rate is denoted by τ_1, and is assumed to be the same in both periods. For simplicity, and without important consequences, prices are assumed to be constant over time. Section 2 presents an analysis of the individual behaviour. The tax policy is described in more detail in section 3. The main analysis of differentiating the indirect taxation is developed in section 4. Some brief concluding remarks are presented in section 5. There is an appendix for deriving mathematical results.

2. Individual behaviour

In general there are many consumption goods. The reform to be considered is to increase the tax on one consumption good consumed in both periods. Let x_i denote the quantity of this commodity consumed in period i. All other commodities may as well be lumped together and treated as one composite good. Let y_i denote its quantity in period i. Let h be the amount of labour supplied, s be the amount of savings in period 1, and i be the (pre-tax) interest rate. The pre-tax wage rate and producer prices are assumed to be constant, and are all set equal to unity by proper normalizations. The interest rate is also taken to be constant.

The assumptions are in line with those made in much of the optimum tax literature, and simplify the analysis a lot. Different justifications might be provided. A simple one is to assume that there are two commodities being produced under constant returns to scale with commodity prices given from the world market. As is well known, factor prices are then determined independently of factor supply. One might assume that savings take place at a given interest rate in an external financial market, and marginal productivities of labour being constant. Or more complicated stories might be told, for instance making use of golden rule assumptions, as in King (1980).

The preference ordering of an individual is represented by a utility function which is assumed to be additively separable between periods. This kind of assumption is fairly common and allows us to obtain more striking results. We write the utility function as

$$U_1(x_1, y_1, h) + U_2(x_2, y_2). \tag{1}$$

It is convenient to refer to U_1 and U_2 as the utility functions of period 1 and period 2, respectively. The individual is faced with the budget constraints:

$$(1+\tau)(x_1+y_1)+(1+\tau)\tau_1 x_1+s-(1-t)h=0 \tag{2}$$

and

$$(1+\tau)(x_2+y_2)+(1+\tau)\tau_1 x_2-(1+i-it)s=0. \tag{3}$$

We have assumed that the general purchase tax is levied on an excise tax inclusive price. We might use (2) or (3) to eliminate s and only consider explicitly the choices between different commodities consumed at the same or at different dates. We shall, however, choose to keep s as an explicit variable.

The individual is assumed to maximize (1) with respect to the consumption bundle, labour supply and amount of savings subject to (2) and (3) taking prices, interest rate and tax parameters as given. We treat this optimization as carried out in two stages. In the first stage U_1 is maximized subject to (2) and U_2 is maximized subject to (3) for a given arbitrary amount of savings. These standard textbook optimizations lead to ordinary demand and supply functions and the indirect utility functions corresponding to U_1 and U_2, respectively.

If we divide (2) and (3) by $(1-t)$, we can introduce the new variables,

$$q=(1+\tau)/(1-t) \tag{4}$$

and

$$q_1=(1+\tau)(1+\tau_1)/(1-t), \tag{5}$$

which can be interpreted as consumer prices of y_i and x_i, respectively. We also define:

$$I_1=-s/(1-t) \tag{6}$$

and

$$I_2=(1+i-it)s/(1-t). \tag{7}$$

For a given s, I_1 and I_2 can be interpreted as exogenous incomes received in the respective periods. We can then write the indirect utility functions as

$$V_1(q_1,q,I_1) \tag{8}$$

and

$$V_2(q_1, q, I_2). \tag{9}$$

Let

$$\lambda_i = \partial V_i / \partial I_i, \quad i = 1, 2. \tag{10}$$

The labour supply function is written as

$$h(q_1, q, I_1). \tag{11}$$

The final optimization takes place by maximizing $V = V_1 + V_2$ with respect to s. The necessary first- and second-order conditions are

$$V' = \mathrm{d}V/\mathrm{d}s = -\lambda_1/(1-t) + \lambda_2(1+i-it)/(1-t) = 0 \tag{12}$$

and

$$V'' = \mathrm{d}^2 V/\mathrm{d}s^2 < 0. \tag{13}$$

3. The tax policy

Taxes are set by the government so as to satisfy a given revenue requirement. The government may lend or borrow at the given interest rate so that the revenue requirement is given in terms of the present value of the tax proceeds, T, or its second-period equivalent, $(1+i)T$. To simplify the notation we define:

$$c_i = y_i + x_i, \quad i = 1, 2. \tag{14}$$

We can then express the revenue requirement as

$$R \equiv (1+i)(\tau c_1 + (1+\tau)\tau_1 x_1 + th) + \tau c_2 + (1+\tau)\tau_1 x_2 + tis = (1+i)T, \tag{15}$$

where T is given exogenously. Initially $\tau_1 = 0$. t and τ are assumed to be chosen so as to maximize V subject to the constraint (15) and subject to the individual optimizing s. The optimization is carried out by using a Lagrange function:

$$L = V + \mu(R - (1+i)T), \tag{16}$$

where μ is the shadow price of the revenue constraint. We shall not deal with the optimum taxes in detail. We shall only make use of the first-order conditions,

$$\frac{\partial L^*}{\partial t}=\frac{\partial L^*}{\partial \tau}=0, \tag{17}$$

where the asterisks indicate that the derivative is evaluated at the optimum. These conditions imply that at the initial optimum:

$$\frac{\partial R}{\partial t}\bigg/\frac{\partial V}{\partial t}=\frac{\partial R}{\partial \tau}\bigg/\frac{\partial V}{\partial \tau}. \tag{18}$$

4. Introduction of a small excise tax

Let us now consider the marginal welfare effect of introducing an excise tax on commodity 1. We consider a tax reform which in general involve changes in all three tax parameters represented by the vector $(\mathrm{d}t, \mathrm{d}\tau, \mathrm{d}\tau_1)$. The tax reform is welfare-improving if $\mathrm{d}V \geqq 0$, and $\mathrm{d}R \geqq 0$ with at least one strict inequality.[1] It is convenient to choose a reform such that $\mathrm{d}V=0$ which implies that we must have $\mathrm{d}R>0$ in order to obtain a welfare improvement. By slightly increasing τ_1 from zero initially, we shall normally get $\mathrm{d}V<0$ and $\mathrm{d}R>0$ keeping the original tax parameters fixed. In order to restore the utility level, changes $\mathrm{d}t$ and $\mathrm{d}\tau$ must be implemented. Several changes are then possible. State in mathematical terms, the surface $\mathrm{d}V=0$ is of dimension two in the three-dimensional tax space, and several curves on this surface may be considered. But since t and τ have been optimized initially, it makes no difference which particular curve (vector $\mathrm{d}t$, $\mathrm{d}\tau$) we select as far as the first-order effect on revenues is considered. To see this, remember that by the optimality assumption $\mathrm{d}R/\mathrm{d}V$ must be the same for any small deviation $(\mathrm{d}t, \mathrm{d}\tau)$ from the initial situation, as implied by (18). The welfare change $\mathrm{d}V$ resulting from a small movement $(\mathrm{d}t, \mathrm{d}\tau)$ is determined by being required to offset the partial effect on welfare of imposing a small tax $\mathrm{d}\tau_1$. Hence $\mathrm{d}V$ is given, and since $\mathrm{d}R/\mathrm{d}V$ is given and the same in any direction $(\mathrm{d}t, \mathrm{d}\tau)$, $\mathrm{d}R$ is also determined independently of the choice of direction. This is, of course, an envelope property of the original situation. Hence the net effect on tax revenue, that we are concerned with, is independent of the particular choice of path on the surface where the net welfare effect of changing all three tax parameters is zero.

For analytical reasons we choose a path of tax reform represented by the vector $(\mathrm{d}t, \mathrm{d}\tau, \mathrm{d}\tau_1)=(t', \tau', 1)\,\mathrm{d}\tau_1$, such that $\mathrm{d}V_1=\mathrm{d}V_2=0$, keeping savings constant. These conditions determine the compensating variations t' and τ'.

[1]For a general introduction, see Atkinson and Stiglitz (1980, p. 382).

One verifies immediately that a change in savings has no first-order effect on utility since savings are optimized in the initial period.

It follows that the introduction of a small excise tax will be welfare-improving if the reform defined above has a positive total effect on tax revenue, $\mathrm{d}R/\mathrm{d}\tau_1 > 0$. In order to analyse this effect, we have to compute the vector of tax reform $(t', \tau', 1)$ from the conditions $\mathrm{d}V_1/\mathrm{d}\tau_1 = \mathrm{d}V_2/\mathrm{d}\tau_1 = 0$. This is a straightforward computation which has been done in the appendix. We find that

$$\tau' = \frac{(1-t)(1+\tau)ish(x_1/h - x_2/is)}{s(c_1+c_2)} \tag{19}$$

and

$$t' = \frac{(1-t)(1+\tau)c_1c_2(x_2/c_2 - x_1/c_1)}{s(c_1+c_2)}. \tag{20}$$

The general purchase tax rate will rise or fall depending on whether the consumption of commodity 1 as a share of current income (from labour or capital) is higher or lower in period 1 than in period 2. The income tax rate will rise or fall depending on whether the expenditure share of commodity 1 is higher or lower in period 2 than in period 1.

Making use of these results, as shown in the appendix, we find that the total effect on tax revenue can be expressed as

$$\frac{\mathrm{d}R}{\mathrm{d}\tau_1} = (1+i)\frac{\tau+t}{1+\tau}\frac{\mathrm{d}h}{\mathrm{d}\tau_1} + i\frac{t}{1+\tau}\frac{\mathrm{d}s}{\mathrm{d}\tau_1}, \tag{21}$$

where all derivatives with respect to τ_1 are total derivatives which also take into account the compensating variations in τ and t. Whether the tax revenue will increase depends on the responses of labour supply and savings. This is not surprising since these are the decisions which are distorted in the first place. We may note that if $t=0$ savings decisions are not distorted and only the labour supply effect remains as long as the purchase tax is present. The same thing is obviously true if the interest rate is zero.

In order to find the effects on labour supply and savings, we need to make use of the price changes, q_1' and q', derived in the appendix:

$$q_1' = \frac{1+\tau}{1-t}\cdot\frac{y_1+y_2}{c_1+c_2}. \tag{22}$$

We see that the price of commodity 1 increases and relatively more the greater the share of commodity y in total consumption over the two periods.

$$q' = -\frac{1+\tau}{1-t} \cdot \frac{x_1+x_2}{c_1+c_2}. \tag{23}$$

We see that the price of y decreases. The relative fall is greater the greater the share of commodity 1 in total consumption over the two periods.

Let us then analyse the effect on labour supply. Since the price changes which take place are compensated changes, we can make use of the compensated partial derivatives:

$$\varepsilon_{h1} = \left.\frac{\partial h}{\partial q_1}\right|_{V_1} \tag{24}$$

and

$$\varepsilon_{hy} = \left.\frac{\partial h}{\partial q}\right|_{V_1}. \tag{25}$$

For given s the individual is compensated in period 1. So for no change in s we can look at compensated effects. But in general s will also change so that in period 1 there is an income effect on labour supply in addition to the compensated price effects. Hence we find that

$$\frac{dh}{d\tau_1} = \varepsilon_{h1} q_1' + \varepsilon_{hy} q' - \frac{\partial h}{\partial I_1} \frac{ds}{d\tau_1} \Big/ (1-t). \tag{26}$$

From (22) and (23):

$$\varepsilon_{h1} q_1' + \varepsilon_{hy} q' = \frac{(1+\tau)(x_1+x_2)(y_1+y_2)}{(1-t)(c_1+c_2)} \left(\frac{\varepsilon_{h1}}{x_1+x_2} - \frac{\varepsilon_{hy}}{y_1+y_2} \right), \tag{27}$$

where the sign is determined by the sign of the expression in parentheses. This is exactly the criterion of Corlett and Hague (1953–54, formula (8)), allowing for the fact that each commodity is consumed in two periods. The effect of differentiating the commodity taxation depends on the substitution effects pointed out by Corlett and Hague. To use their own words: 'If there are three goods, X, Y and L[2], a consumer will always work harder as the result of the introduction of the indirect tax (total tax paid remaining constant) if it is levied on that good (X or Y) which is more complementary

[2] L denotes leisure.

with leisure.' If in the current analysis commodity 1 is more complementary with leisure than the composite commodity it follows that y is more complementary with work effort than x, which means that $\varepsilon_{h1}/(x_1+x_2)>\varepsilon_{hy}/(y_1+y_2)$. Hence we have a benchmark which allows us to extend the analysis of the effects of differentiating commodity taxation directly from the result of Corlett and Hague. We shall refer to (27) as the Corlett–Hague effect.

A modern and more general version of the Corlett–Hague result is theorem 6 of Dixit (1975): 'In a competitive economy with constant producer prices and an initial equilibrium with equal proportional distortions, a small change in tax rates holding commodity tax revenue constant will increase welfare if all commodities whose prices are lowered are better substitutes for the numeraire than all those whose prices are raised.' Although Dixit's analysis and modern exposition may be a more adequate reference than the original Corlett–Hague article,[3] we shall continue to associate the relevant effect with the names of the pioneers in this field, as done by Dixit in his article.

The existence of savings adds two kinds of effects to the Corlett–Hage effect. First, a change in savings affects the labour supply. Since normally the income effect is negative, a positive effect on savings reducing the income left for consumption in period 1 will bring about a greater supply of labour. Thus there is a connection between the distortion of savings and the distortion of labour supply. Second, the effect on savings is important as such since it can mitigate or aggravate the original distortion of savings.

Let us now analyse the effect on savings. We then recall formula (12), the first-order optimum condition for savings:

$$-\lambda_1/(1-t)+\lambda_2(1+i-it)/(1-t)=0.$$

We can now make use of the compensated price effects on the marginal utilities of income:[4]

$$\left.\frac{\partial\lambda_i}{\partial q_1}\right|_{V_i}=-\lambda_i\frac{\partial x_i}{\partial I_i},\quad i=1,2, \tag{28}$$

$$\left.\frac{\partial\lambda_i}{\partial q}\right|_{V_i}=-\lambda_i\frac{\partial y_i}{\partial I_i},\quad i=1,2. \tag{29}$$

[3]This was strongly pointed out by one of the referees.

[4]Let there in general be n commodities with quantities written as $x_1,\dots,x_n$ and prices as $q_1,\dots,q_n$, or simply q in vector notation. Let $V(q,I)$ and $e(q,V)$ be the indirect utility function and the expenditure function, respectively. Let subscripts, apart from the commodity index of x_i, denote derivatives with respect to the relevant arguments. Also let a superscript c indicate that compensation is provided. Then from the equilibrium condition $e(q,V(q,I))=I$ we find that the marginal utility of income, $\lambda(q,I)$, equals e_v^{-1}. Hence

$$\lambda_i^c=-e_v^{-2}e_{vi}=-e_v^{-2}e_{iv}=-e_v^{-2}x_{iI}e_v=-\lambda x_{iI}.$$

For further discussion of this result, see Christiansen (1983).

Differentiating the first-order optimum condition we then get:

$$V''\frac{\mathrm{d}s}{\mathrm{d}\tau_1}+\frac{1}{1-t}\lambda_1\left(\frac{\partial x_1}{\partial I_1}q_1'+\frac{\partial y_1}{\partial I_1}q'\right)$$

$$-\frac{1}{1-t}\lambda_2(1+i-it)\left(\frac{\partial x_2}{\partial I_2}q_1'+\frac{\partial y_2}{\partial I_2}q'\right)-\frac{\lambda_2}{1-t}it'=0. \tag{30}$$

Eliminating λ_2 by means of the first-order condition and solving for $\mathrm{d}s/\mathrm{d}\tau_1$, we get:

$$\frac{\mathrm{d}s}{\mathrm{d}\tau_1}=\frac{\lambda_1}{-V''(1-t)}\left[\left(q_1\frac{\partial x_1}{\partial I_1}-q_1\frac{\partial x_2}{\partial I_2}\right)\frac{q_1'}{q_1}-\left(q\frac{\partial y_2}{\partial I_2}-q\frac{\partial y_1}{\partial I_1}\right)\frac{q'}{q}\right.$$

$$\left.+\frac{i(1-t)}{(1+i-it)}\cdot\frac{(1+\tau)c_1c_2(x_1/c_1-x_2/c_2)}{s(c_1+c_2)}\right], \tag{31}$$

where (20) has been used. The right-hand side is an expression in square brackets with a positive coefficient. The expression in square brackets consists of three main terms. The first term is positive if $q_1\,\partial x_1/\partial I_1>q_1\,\partial x_2/\partial I_2$. This partial effect is easily interpreted. At optimum a marginal unit of expenditure must be equally valuable to the consumer regardless of whether it is consumed in the first period or in the second period. When a price rises, the real value (or purchasing power) of a marginal expenditure unit is reduced. By how much it is reduced depends on the propensity to spend marginal expenditure on the good which becomes more expensive. If this propensity is lower in period 2 than in period 1, the value of marginal spending is reduced less in period 2. Then it becomes advantageous to transfer some spending from period 1 to period 2, which simply means saving more. The interpretation is, of course, analogous for the opposite sign.

The marginal consumption propensities do, of course, in general depend on preferences as well as the parameters of the economy (prices, interest rate and tax rates). Even if preferences were in some sense identical over periods,[5] marginal consumption propensities might still differ.

The interpretation of the second term is quite similar. The term is positive if the marginal propensity to spend income on other market commodities (than commodity 1) is higher in period 2 than in period 1 (since $q'<0$). In the present model the marginal propensity to consume the composite commodity in period 2 is obviously higher the lower is the marginal propensity to consume commodity 1 in that period. This is because $q\,\partial y_2/\partial I_2+q_1\,\partial x_2/$

[5]Preferences for consumption bundles might be identical in the sense that $U_1(x_1,y_1,h)\equiv\varphi(x_1,y_1)+\psi(h)$ and $U_2(x_2,y_2)\equiv\varphi(x_2,y_2)$.

$\partial I_2 = 1$ from the budget constraint. In the first period there is not such a tight relationship because marginal income can in addition be spent on leisure. Yet there is a tendency towards one commodity having the higher propensity in one period and the other commodity having the higher propensity in the other period.

The third term of (31) is the effect of the change in the after-tax interest rate which arises if the income tax rate changes. The income tax rate will fall and savings be stimulated by a higher net interest rate if the expenditure share of commodity 1 is higher in period 1 than in period 2.

To sum up we have found that *savings will be encouraged if*:

(i) the marginal propensity to spend income on the more strongly taxed commodity is higher in the first period than in the second period;

(ii) the marginal propensity to spend income on other commodities is higher in the second period than in the first period;

(iii) the share of the more strongly taxed commodity in expenditure is higher in the first period than in the second period.

In a more compact way we may say that *savings will be more encouraged the greater the propensity, both at the margin and on the average, to spend income on commodity 1 in period 1 as compared to period 2.*

Let us now bring together the various partial effects by making use of (21), (26), (27) and (31) to get:

$$\frac{\mathrm{d}R}{\mathrm{d}\tau_1} = (1+i)\frac{(\tau+t)(x_1+x_2)(y_1+y_2)}{(1-t)(c_1+c_2)}\left(\frac{\varepsilon_{h1}}{x_1+x_2} - \frac{\varepsilon_{hy}}{y_1+y_2}\right)$$
$$-(1+i)\frac{(\tau+t)}{(1+\tau)(1-t)}\frac{\partial h}{\partial I_1}\frac{\mathrm{d}s}{\mathrm{d}\tau_1} + i\frac{t}{1+\tau}\frac{\mathrm{d}s}{\mathrm{d}\tau_1}, \tag{32}$$

where the savings effect is given by formula (31). The first term is the Corlett–Hague effect. The second and third term express the fact that stimulation of savings has an indirect positive effect on welfare through the labour supply response as well as a direct positive effect by reducing the initial distortions. Each effect has been discussed above.

We have considered a reform which introduces a marginal excise tax on one commodity and adjusts existing taxes to keep the utility level constant. (The particular choice of compensating adjustments is arbitrary because of the optimality of the original taxes.) Then, if tax revenue increases, a welfare improvement is obtainable by increasing government expenditure, or by taking the tax reform one step further, whereby taxes are lowered so as to make the after-tax price vector of the economy more favourable to the consumers without loss of initial government revenue. In this sense the tax reform may be welfare-improving. The formulae (31) and (32) establish a number of characteristics which can serve as guides in identifying such welfare-improving reforms whereby marginal excise taxes are imposed in addition to the existing income and purchase taxes. In addition to the Corlett–Hague characteristics these are such simple characteristics as marginal and average propensities to consume the various commodities.

Proposition. Starting from a situation with an optimum proportional income tax and optimum uniform commodity taxes, a marginal tax reform which introduces an excise tax on a particular commodity is efficient under the following sufficient conditions:

(a) *the particular commodity is more complementary with leisure than other commodities;*

(b) *the marginal consumption propensity of this commodity is higher in the first period than in the second period;*

(c) *the marginal consumption propensity of other commodities is higher in the second period than in the first period; and*

(d) *the expenditure share of the particular commodity is higher in the first period than in the second period.*

5. Concluding remarks

In the tax literature the choice of commodity taxes has mainly been discussed in relation to labour supply distortions in a timeless economy. The tax distortion of savings has mainly been attended to in analyses of the balance been the taxation of income and the taxation of total consumption expenditure (in the form of a general purchase tax or an expenditure tax). The present analysis has combined the roles of labour supply and savings distortions due to income and purchase taxes in the search for excise taxes to supplement an initially uniform purchase tax and a proportional income tax.

This has been done within the simplest possible model capturing only the most essential features of an economy with tax-distorted labour supply and savings. All taxes are proportional. Capital market imperfections and uncertainty are non-existent. Distributional considerations are ignored. Economic growth problems are not discussed. A tax reform approach has been adopted by analysing the effects of deviating slightly from uniform commodity taxation. The optimum choice of commodities for imposing small excise taxes has been shown to depend on the degree of complementarity with leisure, as demonstrated by Corlett and Hague, and the marginal and average propensities to consume the various commodities at different stages in life. If the marginal and average propensities to consume a commodity are falling over the life cycle, an excise tax on that commodity will stimulate savings. An intuitive explanation is that the consumer will be encouraged to postpone more of his spending as the future real value of marginal spending increases as compared to the present one. In addition the after-tax return to savings rises as the income tax is reduced.[6]

Appendix

We have defined a direction of marginal tax reform which implies that for fixed savings utility levels in both periods are kept constant. To keep utility levels constant the introduction of marginal excise tax $d\tau_1$ on commodity 1 has to be accompanied by compensating variations $\tau' d\tau_1$ and $t' d\tau_1$ in τ and t, respectively. Compensation implies that

$$\frac{d}{d\tau_1} V_1((1+\tau)(1+\tau_1)/(1-t), (1+\tau)/(1-t), -s/(1-t)) = 0 \tag{A.1}$$

and

$$\frac{d}{d\tau_1} V_2((1+\tau)(1+\tau_1)/(1-t), (1+\tau)/(1-t), (1+i-it)s/(1-t)) = 0. \tag{A.2}$$

When we recall that initially $\tau_1 = 0$, it follows from (A.1) that

$$\lambda_1\left(-\frac{1+\tau}{1-t}x_1 - \frac{\tau'}{1-t}c_1 - \frac{1+\tau}{(1-t)^2}t' c_1 - \frac{t'}{(1-t)^2}s\right) = 0. \tag{A.3}$$

Similarly from (A.2):

$$\lambda_2\left(-\frac{1+\tau}{1-t}x_2 - \frac{\tau'}{1-t}c_2 - \frac{1+\tau}{(1-t)^2}t' c_2 + \frac{t'}{(1-t)^2}s\right) = 0. \tag{A.4}$$

From the budget constraints (2) and (3) we have that

$$(1+\tau)c_1 + s = (1-t)h \tag{A.5}$$

and

$$(1+\tau)c_2 - s = (1-t) \text{ is,} \tag{A.6}$$

when initially $\tau_1 = 0$. Due to (A.5), eq. (A.3) is equivalent to

$$-c_1\tau' - ht' = (1+\tau)x_1. \tag{A.7}$$

[6]Since income effects of changing the balance between taxes cancel out, the ambiguous effects of a change in the interest rate on savings, demonstrated in every elementary textbook, is not relevant in the present context.

And, due to (A.6), eq. (A.4) is equivalent to

$$-c_2\tau' - ist' = (1+\tau)x_2. \tag{A.8}$$

We can then solve for τ' and t' to obtain:

$$\tau' = \frac{-(1-t)(1+\tau)x_1 is + (1+\tau)x_2 h(1-t)}{(1-t)c_1 is - c_2 h(1-t)} \tag{A.9}$$

and

$$t' = \frac{(1-t)(1+\tau)(c_2 x_1 - c_1 x_2)}{(1-t)c_1 is - c_2 h(1-t)}. \tag{A.10}$$

Eq. (A.5) and eq. (A.6) imply that

$$(1-t)is\,c_1 - c_2 h(1-t) = -s(c_1+c_2) < 0. \tag{A.11}$$

Inserting this into (A.9) and (A.10), we get:

$$\tau' = \frac{(1-t)(1+\tau)is\,h(x_1/h - x_2/is)}{s(c_1+c_2)} \tag{A.12}$$

and

$$t' = \frac{(1-t)(1+\tau)c_1 c_2(x_2/c_2 - x_1/c_1)}{s(c_1+c_2)}. \tag{A.13}$$

We are now prepared to derive the total effect on tax revenue, $\mathrm{d}R/\mathrm{d}\tau_1$, which takes into account the compensating variations in τ and t. Differentiating eq. (15) from the main text, we find that

$$\frac{\mathrm{d}R}{\mathrm{d}\tau_1} = (1+i)(c_1\tau' + \tau\,\mathrm{d}c_1/\mathrm{d}\tau_1 + (1+\tau)x_1 + t'h + t\,\mathrm{d}h/\mathrm{d}\tau_1)$$

$$+ c_2\tau' + \tau\,\mathrm{d}c_2/\mathrm{d}\tau_1 + (1+\tau)x_2 + t'\,is + ti\,\mathrm{d}s/\mathrm{d}\tau_1. \tag{A.14}$$

Differentiating the budget constraint (2), we get:

$$c_1\tau' + (1+\tau)\,\mathrm{d}c_1/\mathrm{d}\tau_1 + (1+\tau)x_1 + \mathrm{d}s/\mathrm{d}\tau_1 - (1-t)\,\mathrm{d}h/\mathrm{d}\tau_1 + ht' = 0. \tag{A.15}$$

Also employing (A.7), we get:

$$\frac{dc_1}{d\tau_1} = -\frac{1}{1+\tau}\frac{ds}{d\tau_1} + \frac{1-t}{1+\tau}\frac{dh}{d\tau_1}. \tag{A.16}$$

Also differentiating the budget constraint (3), we get:

$$\tau' c_2 + (1+\tau)\, dc_2/d\tau_1 + (1+\tau)x_2 - (1+i-it)\, ds/d\tau_1 + ist' = 0. \tag{A.17}$$

Then employing (A.8), we obtain:

$$\frac{dc_2}{d\tau_1} = \frac{1}{1+\tau}(1+i-it)\, ds/d\tau_1. \tag{A.18}$$

Substituting from (A.7), (A.8), (A.16), and (A.18) in (A.14), we obtain:

$$\frac{dR}{d\tau_1} = (1+i)\frac{\tau+t}{1+\tau}\frac{dh}{d\tau_1} + i\frac{t}{1+\tau}\frac{ds}{d\tau_1}. \tag{A.19}$$

Moreover, we find the effects on the prices defined by eq. (4) and eq. (5) in the main text:

$$\begin{aligned} q_1' &= \frac{d}{d\tau_1}\left(\frac{1+\tau}{1-t}(1+\tau_1)\right) \\ &= \frac{1+\tau}{1-t}\left(1 - \frac{x_1+x_2}{c_1+c_2}\right) \\ &= \frac{1+\tau}{1-t}\cdot\frac{y_1+y_2}{c_1+c_2} > 0, \end{aligned} \tag{A.20}$$

$$\begin{aligned} q' &= \frac{d}{d\tau_1}\left(\frac{1+\tau}{1-t}\right) = \frac{(1-t)\tau' + (1+\tau)t'}{(1-t)^2} \\ &= \frac{-(1-t)^2(1+\tau)x_1 is + (1-t)^2(1+\tau)x_2 h + (1-t)(1+\tau)^2 c_2 x_1 - (1-t)(1+\tau)^2 c_1 x_2}{-(1-t)^2 s(c_1+c_2)} \\ &= -\frac{1+\tau}{1-t}\cdot\frac{x_1+x_2}{c_1+c_2} < 0, \end{aligned} \tag{A.21}$$

where (A.5) and (A.6) have been used.

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OPTIMUM TAXATION OF MIXED ENDOGENOUS AND EXOGENOUS INCOME

by

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Abstract

In optimum tax theory all income is usually assumed to be endogenous. The present paper analyses the optimum uniform taxation of a mixture of exogenous income and endogenous labour earnings. The tax policy is a linear income tax. The population consists of individuals with different wage rates and exogenous income endowments. Special attention is focused on the role of the composition of income. Counter-intuitive results are derived.

THE OPTIMUM TAXATION OF MIXED ENDOGENOUS AND EXOGENOUS INCOME

By

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1. INTRODUCTION

In optimum taxation analysis income is usually treated as completely endogenous, e.g. as a result of labour supply decisions. The opposite polar case would of course be simple from an analytical point of view since any income tax scheme would then be equivalent to a system of lump sum taxation. There would be no loss of efficiency from redistribution or net taxation. The policy problem would be reduced to that of selecting the first best allocation with the desired income distribution.

For practical purposes the intermediate case with mixed endogenous and exogenous income may, however, be more relevant than the purely endogenous income case. My argument is that the <u>primary</u> sources of income are labour <u>and</u> inherited capital. The latter source of income can for many purposes be treated as exogenous. In particular, this will be true in a one generation model where the bequest behaviour of the preceding generation is a fact of history. Moreover, certain kinds of inherited capital, for instance land, is essentially of an exogenous nature.

I shall assume that endogenous and exogenous income cannot be or for some reason are in fact not taxed differently. There may be administrative and other practical difficulties in distinguishing between the two kinds of income. One kind of income may fairly easily be disguised as another type of income. In particular, income from inherited capital can hardly be identified from the return to savings of own labour income, which is not a <u>primary</u> source of income.

The purpose of this note is to analyse the uniform taxation of mixed endogenous and exogenous income. Apart from the distinction between types of income the model is firmly rooted in the tradition of optimum income tax analysis. A comparison with Sandmo and Dixit (1977) is of particular relevance.

2. THE MODEL

A timeless model is considered. Each individual is endowed with an exogenous income and faces an exogenous wage rate which may be taken to reflect his skill level. There is a number of individuals simultaneously distributed by these two characteristics. A linear income tax is considered. Each individual chooses his supply of labour giving rise to endogenous labour earnings. The following symbols are used:

w = wage rate
e = exogenous income
t = the marginal tax rate
h = labour supply
a = lump sum transfer payment
w^* = $(1 - t)w$ = net wage rate
I = $wh + e$ = gross income
C = $(1 - t)I + a$ = consumption
$V((1 - t)w, (1 - t)e + a)$ = indirect utility function
λ = $\partial V/\partial a$ = marginal utility of income
R = total net tax revenue

An additive welfare function is used as the objective function of the government. The cardinalisation of the indirect utility function is then chosen so as to reflect the distributional preferences of the government. The net tax revenue is required to be equal to a preset level R^o. Everybody is assumed to have some earned income. The size of the population is set equal to n. A subscript i is used to assign a variable to the ith individual.

3. THE TAX DESIGN

The tax revenue requirement may be expressed as

(1) $$R = \sum_{i=1}^{n} t\, I_i - na = R^o$$

or

(2) $$R = \Sigma t w_i h_i((1 - t)w_i,\ (1 - t)e_i + a) + \Sigma t e_i - na = R^o.$$

(2) implicitly defines a as a function of t, a(t). Let a' = da/dt. The optimum tax problem is to maximize

(3) $$W = \Sigma V((1 - t)w_i,\ (1 - t)e_i + a)$$

taking into account the relationship between t and a imposed by the tax revenue requirement. The first order condition becomes

(4) $$W' = \frac{dW}{dt} = -\Sigma\lambda_i I_i + \Sigma\lambda_i a' = 0.$$

The second order condition is

(5) $$W'' < 0.$$

It is convenient to rewrite (4) as

(6) $$W' = -\Sigma\lambda_i(I_i - \bar{I}) - \Sigma\lambda_i(\bar{I} - a') = 0$$

or

(7) $$-n\,\mathrm{cov}(\lambda, I) - \Sigma\lambda_i(\bar{I} - a') = 0$$

where $\bar{I} = \Sigma I_i/n$. The first term on the left hand side of (7) may be interpreted as the distributional effect of marginally increasing t while the second term captures the effect on efficiency. When the marginal utility of income is negatively correlated to income this is an argument in favour of redistributing more income to the relatively poor by increasing the marginal tax rate and the lump sum transfer.

But since economic incentives are adversely affected, the increase in the transfer payment will be less than the average gross burden which is imposed on the taxpayers, $a' < \bar{I}$. Society as a whole suffers a loss of income in the process of redistribution. So the two opposing effects must be balanced against each other.

Differentiating (2) we find that

$$\Sigma I_i - \Sigma t w_i I_i \frac{\partial h_i}{\partial a} - \Sigma t w_i^2 s_{hw} + (\Sigma t w_i \frac{\partial h_i}{\partial a} - n) a' = 0,$$

and hence

$$(8) \qquad a' = \frac{- \Sigma t w_i I_i \frac{\partial h_i}{\partial a} - \Sigma t w_i^2 s_{hw} + \Sigma I_i}{- \Sigma t w_i \frac{\partial h_i}{\partial a} + n},$$

where s_{hw} is the Slutsky derivative of h with respect to the net wage rate. (For simplicity the subscript is omitted.)

We then find that

$$\bar{I} - a' = \frac{\Sigma I_i}{n} - \frac{\Sigma I_i - \Sigma t w_i I_i \frac{\partial h_i}{\partial a} - \Sigma t w_i^2 s_{hw}}{n - \Sigma t w_i \frac{\partial h_i}{\partial a}}$$

$$= \frac{1}{n(n - \Sigma t w_i \frac{\partial h_i}{\partial a})} (n\Sigma I_i - (\Sigma I_i)(\Sigma t w_i \frac{\partial h_i}{\partial a}) - n\Sigma I_i + n\Sigma t w_i I_i \frac{\partial h_i}{\partial a} + n\Sigma t w_i^2 s_{hw})$$

$$(9) \qquad \bar{I} - a' = \frac{t}{n - \Sigma t w_i \frac{\partial h_i}{\partial a}} (\Sigma(w_i \frac{\partial h_i}{\partial a} (I_i - \bar{I})) + \Sigma w_i^2 s_{hw}).$$

We can then rewrite (7) as

$$(10) \qquad W' = -n\,\mathrm{cov}(\lambda, I) - t\bar{\lambda} \frac{n\Sigma w_i \frac{\partial h_i}{\partial a} (I_i - \bar{I})}{n - \Sigma t w_i \frac{\partial h_i}{\partial a}} - t\bar{\lambda} \frac{n\Sigma w_i^2 s_{hw}}{n - \Sigma t w_i \frac{\partial h_i}{\partial a}} = 0$$

The covariance still represents the distributional concern. The second term normally differs from zero if the marginal propensity to pay income tax differs across individuals. Then marginal redistribution will change the total tax payment and hence aggravate or alleviate the initial efficiency loss from second best taxation. The third term captures the marginal efficiency loss due to the private incentive to substitute socially less valuable leisure for socially more valuable income when the marginal tax rate increases. The social opportunity cost of leisure in terms of income foregone, w, exceeds the social and private willingness to pay for leisure, $(1-t)w$, when $t>0$.

Formally, the optimum tax condition is no different from the one obtained in the case of earned income only, but the contents of the various expressions are generally different. Hence a further exploration is called for.

4. THE RELATION BETWEEN INCOME AND MARGINAL UTILITY OF INCOME

The covariance between the marginal utility of income and total income plays a crucial role in the analysis above. Let us therefore explore this relationship in more detail. A change in total income is made up of a change in earned income and a change in exogenous income. Earned income is affected directly by the wage rate and indirectly through effects on work effort both by the wage rate and the exogenous income. Since

$$I = \frac{1}{1 - t} C((1 - t)w, (1 - t)e + a) - \frac{a}{1 - t} \tag{19}$$

where C is total consumption of an individual, we can express a change in total income due to changes in w and e as

$$dI = \frac{\partial C}{\partial w^*} dw + \frac{\partial C}{\partial a} de \; . \tag{20}$$

Letting s_{Cw} denote the Slutsky effect of w^* on C, writing C_a for $\partial C/\partial a$, and making use of the Slutsky equation, we can rewrite (20) as

$$dI = s_{Cw} dw + h\, C_a\, dw + C_a\, de \; . \tag{21}$$

For constant income, dI = 0, (20) defines a downward sloping curve in the (de,dw)-diagram below. We easily see that

$$\left.\frac{dw}{de}\right|_I = \frac{-\, C_a}{s_{Cw} + h\, C_a} \; . \tag{22}$$

The curve defined by dI = 0 divides the (de,dw)-diagram into two half-spaces showing the directions of change which increase total income and those which reduce total income. The shaded area shows the changes which increase total income.

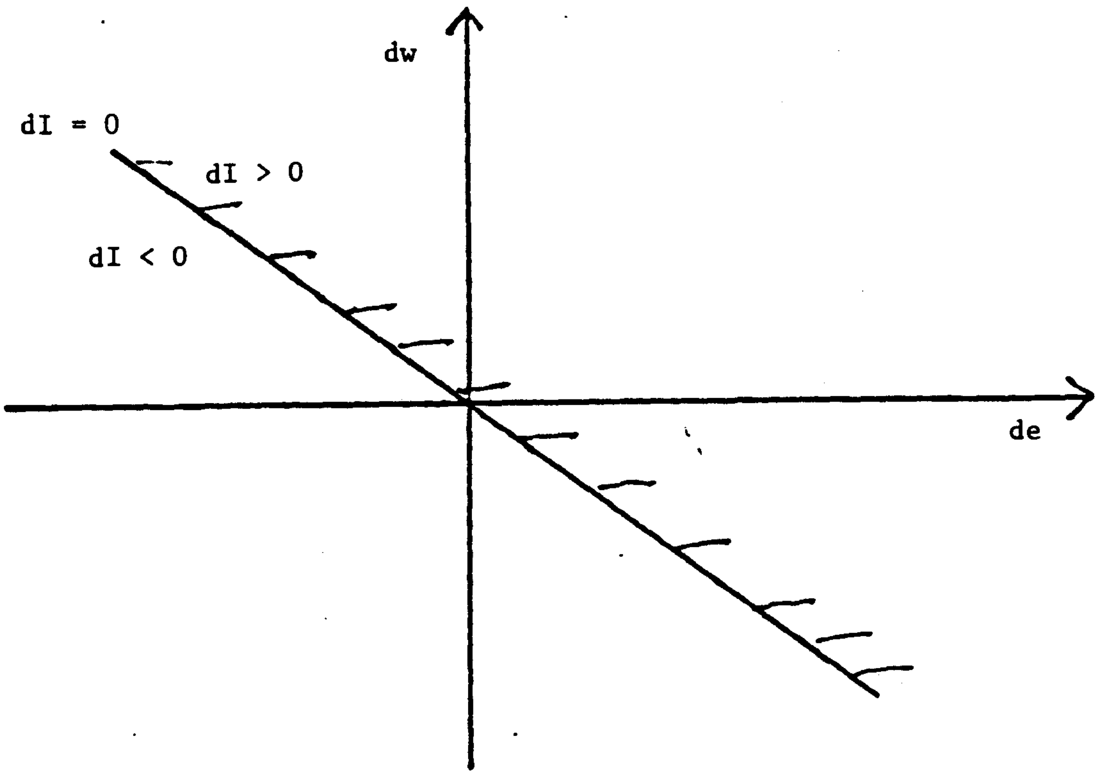

Figure 1

Changes in the wage rate and the exogenous income will affect the marginal utility of income in the following way:

$$(23) \qquad d\lambda = \frac{\partial\lambda}{\partial w^*}(1 - t)dw + \frac{\partial\lambda}{\partial a}(1 - t)de$$

$$= \lambda_w^C (1 - t)dw + \frac{\partial\lambda}{\partial a}(1 - t)h\ dw + \frac{\partial\lambda}{\partial a}(1 - t)de$$

where λ_w^C is the compensated (or purely marginal) effect of w^* on λ. [1)]
We further obtain

$$(24) \qquad d\lambda = \lambda\frac{\partial h}{\partial a}(1 - t)dw + \frac{\partial\lambda}{\partial a}(1 - t)h\ dw + \frac{\partial\lambda}{\partial a}(1 - t)de\ .$$

Setting $d\lambda = 0$ we get another curve in our diagram. The slope is given by

$$(25) \qquad \left.\frac{dw}{de}\right|_{\lambda} = \frac{-\lambda_a}{\lambda h_a + h\lambda_a},$$

where a partial derivative is indicated by subscript a.

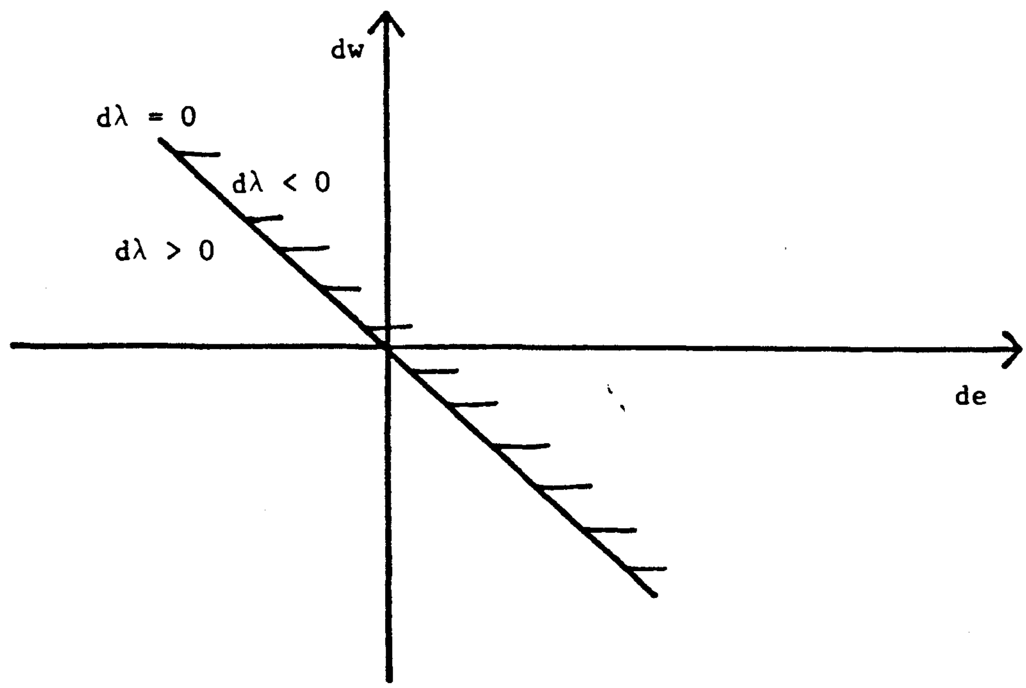

Figure 2

When consumption is normal and the welfare function exhibits inequality aversion, h_a and λ_a are both negative and the slope is obviously negative. The shaded area shows the changes which reduce the marginal utility of income. In general the curve defined by $dI = 0$ and the one defined by $d\lambda = 0$ do not collapse into one. They do only if

$$\frac{C_a}{s_{Cw} + h\, C_a} = \frac{\lambda_a}{\lambda\, h_a + h\, \lambda_a}, \tag{26}$$

which is equivalent to

$$\frac{C_a}{s_{Cw}} = \frac{\lambda_a}{\lambda\, h_a} = \frac{\lambda_a}{\lambda_w^C}. \tag{27}$$

If the curves do not collapse we have either the situation depicted in figure 3 or the one depicted in figure 4 below.

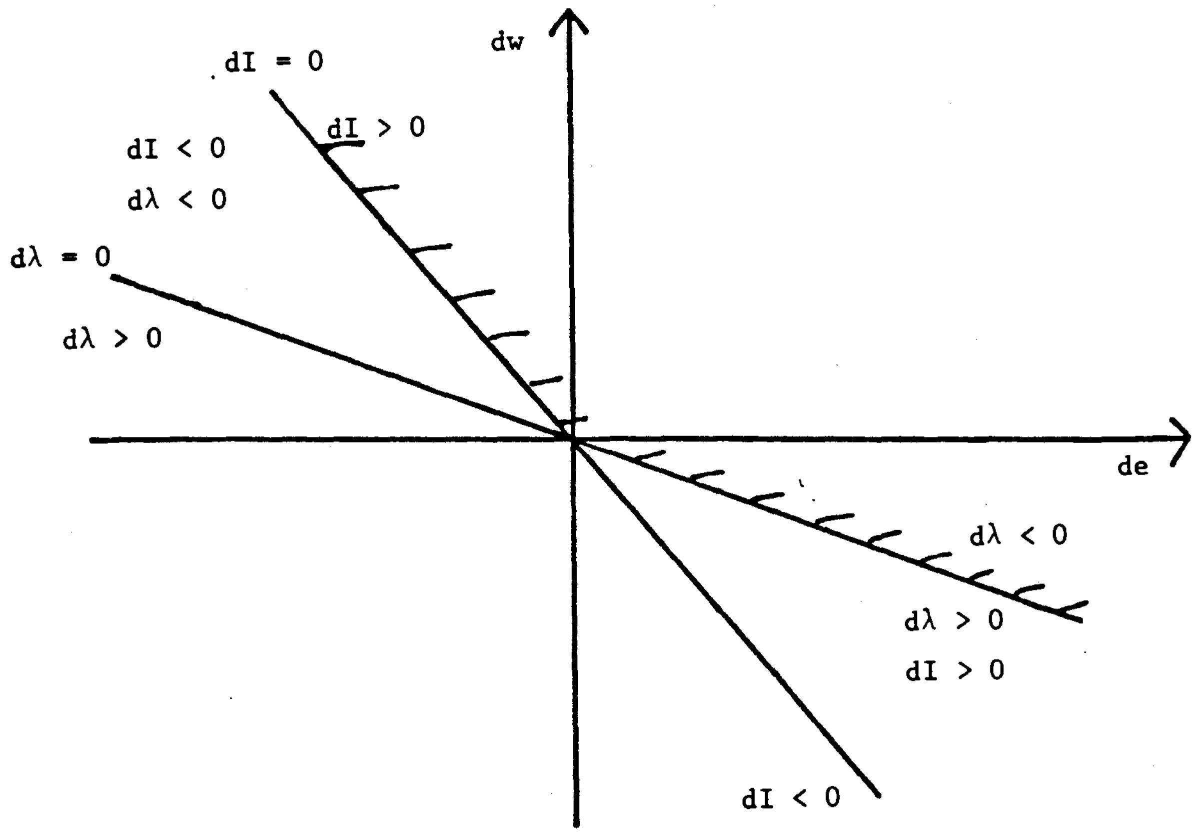

Figure 3

The shaded area shows the changes which imply that when total income increases the marginal utility of income goes down. We see that there are two cones of directions of change in which total income and marginal utility of income move in the same direction.

A similar picture is obtained in figure 4.

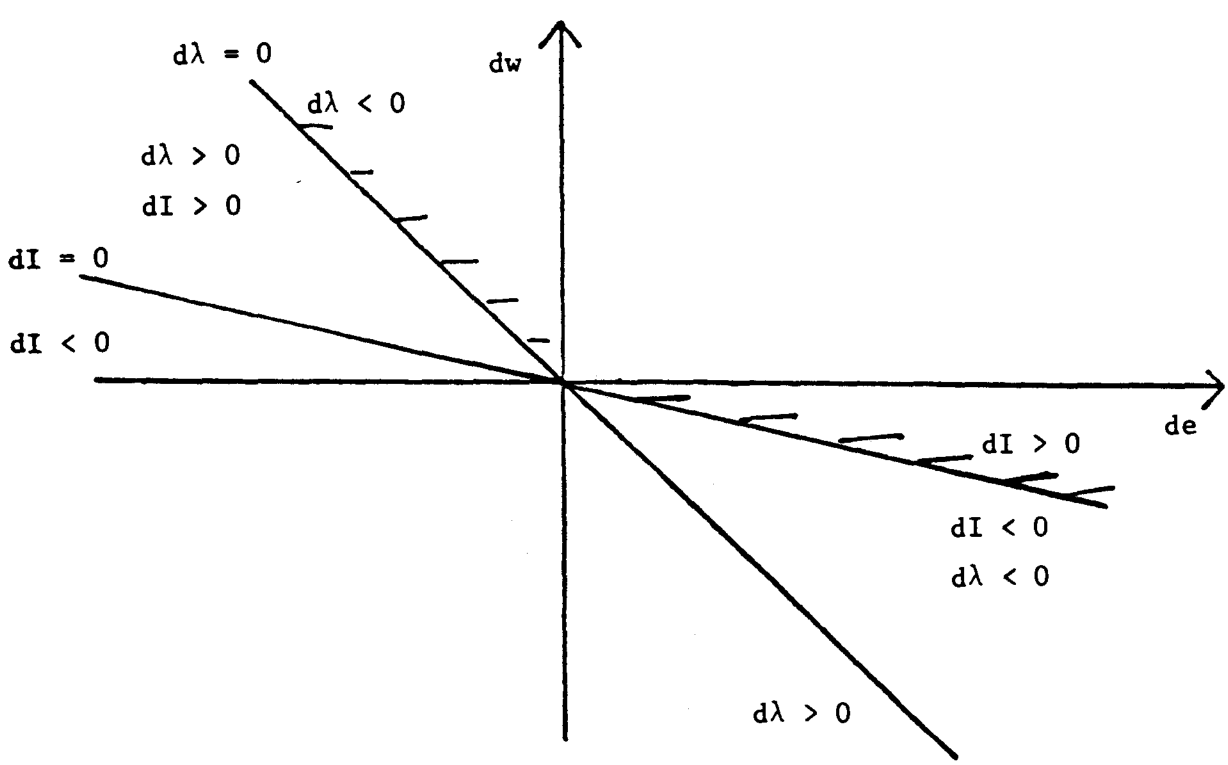

Figure 4

If the typical increase in total income is generated by directions of movement into the shaded area there will be a negative covariance between total income and marginal utility of income. Note that this will always be the case if w and e are both increasing.

The effect of a change in total income on the marginal utility of income depends on whether the change is due to a change in the wage rate or a change in the exogenous income. Let us examine the two alternatives. Let dI be a given change in total income.

$$dI = s_{Cw}\, dw + h\, C_a\, dw + C_a\, de$$

For de = 0,

$$(28)\qquad dw = \frac{dI}{s_{Cw} + h\, C_a}$$

For dw = 0,

$$(29)\qquad de = \frac{dI}{C_a}$$

Let $- d\lambda_w$ and $- d\lambda_e$ denote the reductions in λ corresponding to (28) and (29), respectively.

Combining (24) and (28) we see that

$$(30)\qquad - d\lambda_w = \frac{- \lambda\, h_a - h\, \lambda_a}{s_{Cw} + h\, C_a}\, (1 - t)dI = \frac{-\lambda_a - \lambda h_a/h}{C_a + s_{Cw}/h}(1-t)dI.$$

Combining (24) and (29) we get

$$(31)\qquad - d\lambda_e = \frac{- \lambda_a}{C_a}\, (1 - t)dI.$$

We see that

$$- d\lambda_w \gtrless - d\lambda_e \quad \text{according as}$$

$$(32)\qquad h_a\, \lambda/\lambda_a = \frac{\lambda_w^C}{\lambda_a} \gtrless s_{Cw}/C_a.$$

We may note that $C_a = (1 - t)wh_a + 1$. Some polar cases may be interesting. If the income effect on labour supply is very small an increase in total income will always have a stronger effect on the marginal utility of income if it is due to a change in exogenous income than if it is brought about by a change in the wage rate. If the cross substitution effect is very small, i.e. s_{Cw} is close to zero, the opposite will be true.

Let us now try to get more insight into the formal results derived above. When a person's income increases and he becomes better off, the marginal utility of income decreases. This is true for an increase in exogenous income as well as the pure real <u>income effect</u> of an increase in the wage rate. However, an increase in the wage rate will in addition reduce the marginal value of income because a marginal income unit buys less leisure than before as leisure becomes more expensive. This effect occurs when marginal income is used to buy some leisure, $h_a < 0$. To focus on one effect at a time let us for the moment assume away this effect. The magnitude of the real income effect of increasing the wage rate depends on how strongly the wage rate increases. Since a rise in the wage rate has a positive substitution effect on earned income, it must be smaller the stronger the substitution effect is, to bring about a preset change in observed income. If for a moment we also neglect the substitution effect, $s_{cw} = 0$, (30) and (31) coincide. A wage change is then equivalent to a change in exogenous income, both affecting equally full (potential) and observed income. The effects on the marginal utility of income are pure income effects which are the same in both cases. Then observed income and the marginal utility of income obviously move in opposite direction no matter why income changes, as is confirmed by formula (26).

When there is a positive substitution effect, the increase in the wage rate required to produce a given rise in earned income gets lower, and the real income effect of such an increase in the wage rate falls below that of an increase in exogenous income. Hence the effect on the marginal utility of income also becomes smaller when w changes than when e changes, as we also see from (32) since by assumption the left hand side is zero and the right hand side is positive.

However, if there is also a <u>marginal</u> real income effect of changing w, $h_a < 0$, the total effect of changing w on the marginal utility of income gets stronger. The individual gets better off <u>and</u> the real value of marginal income is reduced.[3)] So this will be an effect in the opposite direction. If $|h_a|$ is sufficiently large, this effect will dominate, and a change in earned income will have a stronger effect on the marginal utility of income than an equally large change in exogenous income.

It might be possible to have increasing exogenous and total income combined with a diminishing wage rate. Since w, unlike exogenous income, has a marginal real income effect on λ which is positive when w decreases, it might also be possible to have a positive covariance between I and λ as shown in figure 3 and figure 4. But I do not expect this to be the prevailing case in practice.

The above analysis is useful for discussing the effects of the composition of income on the marginal utility of income. These effects are important when considering the impact of the composition of income on the optimum tax design to which we shall turn in the next section.

5. EFFECTS OF THE COMPOSITION OF INCOME ON THE OPTIMUM TAX DESIGN

It would be interesting to know how changes in the composition of income would affect the optimum tax policy. For instance would a larger exogenous component work in favour of higher or lower progressivity? More precisely we may ask: If the population with the original characteristics were replaced by a population with the same observed distribution of actual income but with higher exogenous income and lower earned income, would the government then want to change the tax policy. This is obviously a complicated question. If we consider the first order condition for the optimum tax rate, there are a number of effects to take into account. In general little or nothing can be said about how substitution effects, income effects, etc. change with the wage rate, exogenous income, etc. It may, however, be of interest to consider simple cases.

Let us first make clear some implications of the experiment we conceive of. First, there is no change in I_i for $i = 1,\ldots,n$ at the original tax policy. It follows that C_i is also left unchanged for all i. Moreover, the tax revenue constraint remains fulfilled without changing any tax parameters. However, it may be desirable to change the tax policy.

Let us consider the case where the preferences of the individuals are represented by a Stone-Geary utility function:

(33) $$u = \alpha \ln(L - \bar{L}) + \ln(C - \bar{C})$$

where L denotes leisure, α is a positive parameter and $\bar{L}$ and $\bar{C}$ are parameters usually interpreted as minimum requirements.
Let us define

(34) $$x = L - \bar{L}$$

and

(35) $$y = C - \bar{C}\ .$$

The budget constraint can then be expressed as

(36) $$m = w^* + (1 - t)e + a - \bar{C} - w^*\bar{L} = w^*x + y$$

where the available amount of time (for labour and leisure) has been set equal to unity.

The Gossen condition becomes

$$\frac{u_L}{u_C} = \frac{\alpha y}{x} = w^*$$

which is equivalent to

$$w^*x = \alpha y$$

or

$$y = \frac{1}{\alpha} w^*x\ .$$

Also employing the budget constraint, we then get the demand functions:

(37) $$y = \frac{m}{1 + \alpha} = C - \bar{C},$$

(38) $$x = \frac{m\alpha}{(1 + \alpha)w^*} = L - \bar{L} = 1 - h - \bar{L}\ .$$

We see that

$$(39) \qquad w\,\frac{\partial h}{\partial a} = -\,\frac{\alpha}{(1 + \alpha)(1 - t)}$$

which is independent of w and e.

The Slutsky derivatives with respect to the net wage rate, s_{xw} and s_{yw}, are derived from the Gossen condition and the condition that utility is constant. Hence

$$x + w^* s_{xw} = \alpha\, s_{yw}$$

and

$$u_L\, s_{xw} + u_C\, s_{yw} = 0$$

which is equivalent to

$$\frac{\alpha}{x}\, s_{xw} + \frac{1}{y}\, s_{yw} = 0$$

and hence

$$s_{yw} = -\,\frac{\alpha y}{x}\, s_{xw} = -\, w^*\, s_{xw}\ .$$

We find that

$$(40) \qquad w^2\, s_{xw} = -\,\frac{w^* x}{(1 + \alpha)(1 - t)^2} = -\,\frac{\alpha\, y}{(1 + \alpha)(1 - t)^2} = -\, w^2\, s_{hw},$$

$$(41) \qquad s_{yw} = s_{Cw} = \frac{\alpha\, y}{(1 + \alpha) w^*}\ .$$

Since our experiment leaves C and y unchanged, $w^2\, s_{hw}$ also retains its initial value.

Inspecting (9) we now see that if preferences are of the Stone-Geary type, there will be no change in $(\bar{I} - a')$. The absolute value of $\bar{\lambda}$ can of course be manipulated by the conventional choice of units of welfare. It is convenient to keep $\bar{\lambda}$ fixed. Then there will be no change in the second and third term of (10), i.e. the efficiency terms.

The remaining question is how the first term, i.e. the equity term, is affected. When $\bar{\lambda}$ is fixed, the change in the covariance equals $\Sigma(d\lambda_i - 0)I_i/n = \mathrm{cov}(d\lambda_i, I_i)$. If the changes in the marginal welfare weights are negatively correlated with income, the marginal tax rate should be increased. Let us assume that initially marginal welfare weights are negatively correlated with income. Then the covariance will be reduced if marginal welfare weights are reduced <u>relatively</u> more at higher income levels than they are at lower income levels neglecting the requirement that the mean value should be preserved.[2)] Since the mean restoring adjustments are proportional changes, absolute mean preserving changes in the marginal welfare weights will then be negatively correlated with income. The relative change in some marginal welfare weight, λ, when e increases and w changes to keep income unaltered, is from (22) and (24):

$$\text{(42)} \qquad \frac{d\lambda}{\lambda} = \frac{\lambda_a/\lambda - h_a\, C_a/s_{Cw}}{1 + h\, C_a/s_{Cw}}\,(1 - t)de\,.$$

Unfortunately there is not much to say about how this expression changes with income in general. However, it may be interesting to consider the special case in which distributional preferences are adequately represented by the special cardinalisation of the utility function presented in formula (33). Let us first introduce the corresponding indirect utility function, denoted, by v, which is obtained by plugging (37) and (38) into (33):

$$\text{(43)} \qquad v = \alpha \ln\left(\frac{m\alpha}{(1+\alpha)w^*}\right) + \ln\frac{m}{1+\alpha}$$

$$= \ln\left[\left(\frac{m\alpha}{(1+\alpha)w^*}\right)^{\alpha}\frac{m}{1+\alpha}\right]$$

$$= \ln\left[\left(\frac{m}{1+\alpha}\right)^{1+\alpha}\left(\frac{\alpha}{w^*}\right)^{\alpha}\right] = \ln(m^{1+\alpha}\,\beta)$$

where β is defined implicitly.

We easily derive that

$$(44) \qquad \lambda = \frac{1+\alpha}{m} = \frac{1}{y} ,$$

$$(45) \qquad \lambda_a = - \frac{1+\alpha}{m^2} ,$$

and hence

$$(46) \qquad \frac{\lambda_a}{\lambda} = - \frac{1}{m} .$$

I would like to argue that this is not an arbitrary special case, but rather one which may have a special claim for interest. In applied welfare economics the welfare weight is frequently assumed to be some isoelastic function of total consumption expenditure

$$(47) \qquad \lambda = C^{-\sigma}.$$

(See for example Stern (1977)).

As a special case σ is often assumed to be unity, which may even have some empirical support. (See Christiansen and Jansen (1978, p. 233). If in addition the minimum consumption requirement, $\bar{C}$, is zero, (44) and (47) are equal. Even if $\bar{C} > 0$, one can hardly argue that (44) is a less plausible specification than (46).

We find from (39) and (41) that

$$\frac{h_a}{s_{Cw}} = \frac{-\dfrac{\alpha}{(1+\alpha)w^*}}{\dfrac{\alpha y}{(1+\alpha)w^*}} = - \frac{1}{y} ,$$

and from(39):

$$C_a = w^* h_a + 1 = 1 - \frac{\alpha}{1+\alpha} = \frac{1}{1+\alpha} .$$

Hence

$$C_a \frac{h_a}{s_{Cw}} = - \frac{1}{(1+\alpha)y} = - \frac{1}{m}$$

due to (37).

The numerator of the fraction in (42) then becomes

$$\lambda_{\ddot{a}}/\lambda - h_a \; C_a/s_{Cw} = - \frac{1}{m} + \frac{1}{m} = 0.^{4)}$$

The denomimator becomes

$$\frac{1-\bar{L}}{\alpha} \; \frac{w^*}{y} = \frac{(1-\bar{L})w^*(1+\alpha)}{\alpha m}.$$

We see that if the utility function (33) represents the individual preferences as well as the distributional preferences of the government, then the optimum degree of income tax progressivity is unaffected by the composition of income.

Intuitively it might be tempting to believe that the existence of exogenous income should lead to a higher marginal tax rate because efficiency effects might be believed to be less important and perhaps because high income people might be believed to have more unearned income. But in general this intuition does not hold. As we have seen the composition of income is not necessarily important. But in general we cannot tell whether studies of optimum taxation neglecting exogenous income tend to over - or underestimate the optimum marginal tax rate.

FOOTNOTES

1) Let $E(w^*, V)$ denote the expenditure function. From the condition $E(w^*, V(w^*, (1-t)e+a)) = (1-t)e+a$ we find that λ equals E_V^{-1}. Hence

$$\lambda^c_{w^*} = -E_V^{-2} E_{w^*V} = E_V^{-2} h_a E_V = \lambda h_a .$$

2) As an alternative formulation we could divide (10) by $\bar{\lambda}$ and consider the covariance $\Sigma(\lambda_i/\bar{\lambda} - 1)I_i/n$. The change would then be $\mathrm{cov}(d(\lambda_i/\bar{\lambda}), I_i)$.

3) These effects are analyzed in more detail in Christiansen (1983).

4) A more straightforward route is to observe that since $\lambda = u_C$, which only depends on C, λ does not change when C remains the same.

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VC/SH

INCOME TAXATION OF TWO-PERSON HOUSEHOLDS

By

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ABSTRACT

A model is constructed to analyze the tax treatment of secondary wage earners in two-person households. The households have different income opportunities, and potential secondary wage earners differ in their willingness to take a job. In a variety of numerical cases the optimum tax structure is computed allowing for income distribution and the tax distortions of the labour market participation of secondary wage earners. Special tax systems of the kinds actually in operation are analyzed and compared.

1. INTRODUCTION

Most normative tax theory has explicitly or tacitly been concerned with the taxation of single individuals. The special problems raised by the taxation of couples have received little attention. One exception is Blundell and Walker (1982). At the same time the taxation of families has been an issue in the tax debate in many countries. The choice of taxable unit has been a source of controversy. Different countries have adopted different schemes such as joint taxation (the United States), single filing (Sweden), optional single or joint filing (Norway) or income splitting.

There has, however, been considerable interest in the positive analysis of the labour supply of spouses, which is highly relevant to the tax design problem. A large number of empirical studies are reported in the literature. Although estimates vary, there is general agreement that the labour supply of married women is far more responsive to economic factors (wages, taxes, etc.) and non-economic circumstances (number of children, etc.) than men's labour supply. The latter is often found to be very inelastic. In the wake of these econometric studies there has been a growing interest in exploring the effects of alternative tax treatments of the family, see for example Feenberg and Rosen (1983).

The purpose of the present analysis is to study the taxation of two-person households from a welfare point of view. We consider an exogenous population of two-person households. The households may be married couples or people living together as unmarried couples. If both categories are included we assume that they are treated equally by the tax authorities. Our assumption implies that in the world of our model tax rules do not motivate the formation or breaking up of households. The couples may have children, but they are not treated explicitly in the model. Hence child allowances are neglected. Single persons are not included in the analysis.

Each person is endowed with an exogenous earnings capacity equal to the wage that the person will obtain if entering the labour market. There is a discrete number of wage levels. A pair of potential wage levels, w_i, w_j, will then characterize the earnings capacity of a household. By convention $w_i > w_j$ if and only if $i > j$. Let n_{ij} denote the number of households with the pair of potential wages w_i, w_j, where by convention $w_i \geq w_j$. These numbers then characterize the discrete distribution of potential wages.

In each household the person with the higher potential wage is always in the labour force. This person is called the primary wage earner. The other person chooses whether to take a job or not. This is the secondary wage earner. In parts of the analysis she or he is also assumed to have the option of taking a part-time job. These are the only choices made by households in the model. Thus we abstract from a large part of household behaviour in order to focus on participation in the labour force.

A household will incur a certain disutility from having a secondary wage earner. In other words the potential secondary wage earner has a certain reservation wage, or a certain willingness to take a job, which varies among households due to a number of circumstances. Persons may have different abilities to do a job. Hence efforts required to manage a job may vary. People with children or other family members in their care are less inclined to take a job than those without such duties. Preferences with respect to income and leisure may vary. Health, age, etc. may be important. To capture the effect of such factors a parameter z is included as argument in the utility funciton. The value of z is a characteristic of the household. We assume that there is a continuous distribution of z which is independent of potential incomes. In other words, if households are grouped by pairs of potential wages, the same distribution of z applies to all groups. This is mainly a reflection of the desire to keep the analysis fairly simple.

An income tax is imposed on each household. The tax liability of a household is a function of the actual earnings of the two persons.

Since potential earnings and the reluctance to take a job (the value of z) are unobservable in practice, the tax policy is obviously constrained to a second best system. Further constraints may be added by requiring taxation on an individual base or, at the other extreme, joint taxation of the two household members. The alternatives will be discussed below. Throughout the paper optimum taxation is used synonymous to optimum second best taxation with no further constraint on the choice of system. When there is a need, we shall make explicit reference to first best taxation or special constrained second best systems. The government has a fixed tax revenue requirement.

For the assessment of alternative tax policies an additive welfare function is formulated. The purpose of the analysis is to explore the optimum taxation of secondary income and to compare special tax systems such as taxation on an individual base, joint taxation etc. The analysis will be carried out by means of numerical examples using special functional forms and parameter values.

We should like to emphasize that our interest is in the broad structure of the taxation of secondary income. Our concern is with the main features and not the details of the tax schedule. Our ambition is not to prove that a tax rate should be 0.4 rather than 0.5. Our purpose is rather to explore if the marginal tax rate should increase, be constant or decrease, whether a special tax system is likely to be close to or far from the optimum, etc.

The paper is organized in the following way: The formal model is presented in a general form in Section 2. The special functional forms used in the computations are introduced and discussed in Section 3. Section 4 defines more precisely the scope and limits of the analysis. Optimum tax rules are derived in Section 5. The computational method is described very briefly in Section 6. Results are presented and discussed in Section 7. There is a short concluding section. A couple of computations considered in the main text are presented in more detail in an appendix.

2. THE FORMAL MODEL

Let t_{ij} denote the tax imposed on a household earning incomes w_i and w_j, where $w_0 = 0$. (For convenience we shall often write only t_i instead of t_{i0}.) Moreover, let w_{ij} denote the corresponding after-tax income:

$$(1) \qquad w_{ij} = w_i + w_j - t_{ij} \qquad\qquad \forall i, j$$

The preferences of a household are described by a utility function

$$(2) \qquad u(w_{ij}, \delta, z),$$

where $\delta = 0$ if there is no secondary wage earner, $\delta = 1$ if there is a secondary wage earner working full time, and $0<\delta<1$ if the secondary wage earner takes a part-time job.
Moreover,

$$(3) \qquad \partial u/\partial w_{ij} > 0,\ \partial u/\partial \delta<0,\ \partial u/\partial z<0.$$

Let us first consider the choice between a full-time job and no job. Then a household characterized by a special value of z will have a secondary wage earner if

$$u(w_{ij}, 1, z)>u(w_{i0}, 0, z).$$

It will have no secondary wage earner if the opposite inequality holds. Normally there will be a critical value of z denoted by z_{ij} such that households endowed with this value of z are indifferent between having one or two wage earners. Formally z_{ij} is implicitly defined by the equation

$$(4) \qquad u(w_{ij}, 1, z_{ij}) - u(w_{i0}, 0, z_{ij}) = 0.$$

There is a secondary wage earner if $z < z_{ij}$.

Let z be distributed by the density function f(z). The cumulative distribution is described by F(z), where f(z)=dF/dz. Let us assume that $z \geqq z_0$.

We can then write the welfare function as

$$(5)\quad W=\sum_i \sum_{j\leqq i} \left(\int_{z_0}^{z_{ij}} u(w_{ij}, 1,z)n_{ij}\, f(z)dz + \int_{z_{ij}}^{\infty} u(w_{i0}, 0,z)n_{ij} f(z)dz\right).$$

Total tax revenue, R, becomes

$$(6)\quad R= \sum_{i,j\leqq i}(t_{ij}n_{ij}F_{ij}+ t_i\, n_{ij}(1-F_{ij})),$$

where $F_{ij}=F(z_{ij})$.

The structure of income taxation is then determined by the tax parameters t_{ij} and t_i for all i and j.

3. SPECIAL FUNCTIONAL FORMS

In this section we shall introduce special functional forms for the utility function and the distribution of z to be used in the numerical calculations. But before we do so a general discussion of principles may be of interest.

We would like the form of the utility function to satisfy a number of requirements:

(i) It should comply with reasonable behavioural assumptions.

(ii) It should, with a proper choice of cardinalisation, reflect plausible distributional preferences.

(iii) It should allow us to work with simple analytical expressions and carry out fairly simple calculations (avoiding numerical integration, etc.)

The final choice is, of course, likely to be some compromise between conflicting concerns.

There may be many reasons why households with the same income opportunities choose to supply different amounts of labour. Number of children, age, health condition, differences in pure preferences and a number of other characteristics are obviously important. It would not be feasible to include explicitly in the model all characteristics of this kind. But it might be desirable to model explicitly some of the more important factors. In our present model there is, however, only one variable (z) to capture the various factors. It is then important to keep in mind the underlying factors when the role of z is considered.

A particularly difficult question is how households with different characteristics should be treated in the distributional preferences. Should a household with a high reservation wage for a secondary wage earner be given a higher or a lower distributional weight than a household with a lower reservation wage? One problem is that the factors behind the reservation wage are a mixture of very different factors. It may be very useful to consider various kinds of factors.

One category consists of properties of pure preferences. This concept is not easily defined in a rigorous way. We have in mind that different households choose to behave differently although there are no observable differences in the circumstances under which they make their choices. For some reason of their minds their indifference curves slope differently. This phenomenon is phrased in various ways. Some people are said to be lazy or to cultivate the non-materialistic values of life. Others are characterized as industrious or as greedy. There seems to be little or nothing to go by in deciding how differences in pure preferences should be allowed for in distributional preferences.

A second category of factors are such that are commonly considered to reduce welfare <u>if</u> a person is working. A person may be physically or mentally less able to work than some other person. And it may somehow take more effort to do a job if one is less able or less qualified, or opportunities are unfavourable. It may seem intuitively reasonable that those who are less favourably endowed should be given a higher distributional weight when the persons in question are working. But on second thoughts the matter is not so simple.

To see this let us consider the following illustration. Let r be some observable measure of labour supply, say number of working hours. Then let γ be some parameter which converts working hours into units of effort, where γ is a characteristic of the individual. Now suppose that work effort is represented in the utility function by $\gamma r = z$. Let us also assume that r is fixed. We then consider a utility function $u(w,z)$ where w denotes disposable income.[1)]

If u denotes the utility level, $u(w,z)=u$, which implicitly defines the disposal income required to obtain a particular utility level for a given work effort: $w(u,z)$. Let subscripts denote partial derivatives. Then the social marginal utility of income can be expressed as

$$(7) \qquad \lambda = u_w(w(u,z),z) \equiv \Lambda(u,z).$$

This formulation allows us to distinguish between the effects on λ of a change in z along a given indifference curve (the compensated effect) and the effect via the change in utility level (the real income effect). Analytically the former effect is

$$\Lambda_z = u_{ww} w_z + u_{wz} = - \frac{u_{ww} u_z}{u_w} + u_{wz}.$$

The latter effect is

$$\Lambda_u u_z \equiv u_{ww} w_u u_z = u_{ww} \frac{u_z}{u_w}.$$

Then we can write

$$(8) \qquad \lambda_z = \Lambda_u u_z + \Lambda_z,$$

and we can easily establish that

$$(9) \qquad \frac{\lambda_z}{\lambda} = \frac{-u_{ww}}{u_w} \left(- \frac{u_z}{u_w}\right) - \frac{\partial}{\partial w} \left(\frac{-u_z}{u_w}\right).$$

1) Even though z varies in our model because households have different characteristics (γ varies), the utility function is also valid if we consider a change in working hours (r) for a fixed characteristic (γ).

The first term is positive when inequality aversion prevails ($u_{ww}<0$), and implies that cet.par. a lower utility level tends to increase the marginal utility of income. The second term is negative under normal assumptions (work effort is inferior) and implies that the relative marginal valuation of income is reduced when income increases. There is in general no reason why any one of these effects should dominate the other.

On the other hand there may be other persons who are particularly well qualified for do-it-yourself activities such as cooking, growing vegetables, painting, doing repair work on cars, houses, etc. They may be reluctant to work even though income prospects may be good because their opportunity cost in terms of self-made goods foregone may be too high. It may seem reasonable that a relatively lower distributional weight should be assigned to such people if they are not working. [1)]

At last there are factors which tend to reduce welfare in general and also the ability to work such as a poor health. The low welfare may tell in favour of relatively high distributional weight. It may, however, be the case that a person with a lower level of welfare may also be considered as less able to benefit from a marginal unit of income.

This discussion leaves considerable scope for making alternative assumptions. But we are not entirely free to choose. A simple assumption would be that the social marginal utility of income is independent of work effort, so that

(10) $$u_w = a(w)$$

and hence

(11) $$u = b(w) + d(z)\ .$$

But let us now consider the case in which there is no income effect on labour supply:

(12) $$\frac{u_w}{u_z} = g(z).$$

Empirical studies for Norway by Strøm et al. suggest that this may be close to reality. See Strøm and Ljones (1985, p. 24).

1) This point may, however, be of minor relevance in the present model since household production activities are not explicitly dealt with.

(11) and (12) together obviously imply that

(13) $u = cw + d(z)$,

where c is a constant. But this means that there is no inequality aversion. The distributional weight is the same for everybody. This is not anattractive property. In order to retain the possiblity that there is no income effect we have to discard assumption (10), and thus accept that

(14) $u_{wz} \neq 0$.

The class of functions consistent with no income effect so that (12) is valid is found by solving this partial differential equation. We obtain the class of functions

(15) $u = \phi(w+G(z))$,

where $u_w/u_z = 1/G'(z) = g(z)$.

Including δ as an explicit argument, we can write $u = \phi(w+G(z,\delta))$ or, since $\phi' > 0$, we can use the transformation $u = w + G(z,\delta)$. There is obviously no income effect on the choice of δ.

We shall consider two alternative specifications of the utility function. For the sake of reference we shall label the first alternative as the logarithmic specification and the second alternative as the exponential specification.

The utility function - logarithmic specification.

The first alternative is the utility function:

(16) $u(w_{ij}, \delta, z) = k \ln w_{ij} - \delta \ln z$,
where k is a positive constant.
It is convenient to introduce the notation:

(17) $h = \ln z$.

We can then rewrite the utility function as

$u(w_{ij}, \delta, h) = k \ln w_{ij} - \delta h$.

We have retained the function symbol u, since no confusion is likely to arise. We easily see that

$u = k \ln w_{ij} - h$ if both persons are working, and,

$u = k \ln w_{i0}$ if only one person is working.

The critical value of h separating the two cases is then given by

$$(18) \quad h_{ij} = k \ln \frac{w_{ij}}{w_{i0}} = k \ln \frac{w_i + w_j - t_{ij}}{w_i - t_i} .$$

The utility function plays a double role. It is assumed to motivate household behaviour. The chosen cardinalisation also appears in the welfare function of the government. Let us first discuss the utility function from the point of view of household behaviour which in the current analysis is synonymous to labour supply behaviour.

We see that the critical value h_{ij} increases as the secondary income, w_j increases. More households will then find their h-values to be below the critical level, and more households will have a secondary wage earner. Under the reasonable assumption that a secondary income will raise total disposable income, an increase in primary income will lower h_{ij}. Thus there is a negative income effect on the participation rate of secondary wage earners.
We see from (18) that the labour supply of a category of households depends only on the ratio between disposable income if both persons are working and the disposable income if only one person is working. Leaving taxes aside, this means that the negative income effect of say a 10 per cent increase in the income of the primary wage earner will be exactly offset by the positive effect on labour supply from a 10 per cent increase in the income of the (potential) secondary wage earner. This is of course a special assumption implied by the choice of functional form.

By varying the value of the parameter k the volume of labour supply from secondary wage earners can be adjusted to a reasonable level.

The welfare function implied by the choice of cardinalisation of the utility function has two important properties which may be subject to discussion. First, the marginal social utility of income (or the distributional weight assigned to a household) is independent of h. It is independent of the number of wage earners in the household and the disutility incurred by having a secondary wage earner. Assessing this assumption raises difficult questions as discussed in some detail above.

The utility function - exponential specification.

Let the utility function be

$$(19) \qquad u = -e^{-aw^b} z^{m\delta} z^p$$

where a, b, m and p are parameters; $a > 0$, $0 < b \leqq 1$, $m > 0$, $p \geqq 0$, and $m + p < 1$. As we shall see below there is no income effect on labour supply if $b=1$ and a negative effect if $b<1$. The social marginal utility of income then becomes

$$(20) \qquad u_w = abe^{-aw^b} w^{b-1} z^{m\delta} z^p.$$

We see that

$$u_{w\delta} > 0.$$

For a given income the social marginal utility of income is higher when there is a secondary wage earner.
The elasticity of u_w with respect to w is

$$(21) \qquad \hat{u}_{ww} = \frac{u_{ww} w}{u_w} = b-1-abw^b$$

which is obviously negative and decreasing as income rises.

Let us consider two households, labelled 1 and 2, respectively, both with the same value of z. The relative distributional weight is then

$$(22)\qquad \frac{u_w^1}{u_w^2} = e^{a(w_2^b - w_1^b)}\left(\frac{w_2}{w_1}\right)^{1-b}$$

If there is no income effect, b=1, the relative distributional weight depends only on the absolute difference between disposable incomes. Unfortunately this is hardly an attractive property. If there is an income effect, b<1, relative income matters too.

The elasticity of u_w with respect to z is

$$(23)\qquad \hat{u}_{wz} = m\,\delta + p,$$

which is higher when there is a secondary wage earner (δ=1)than otherwise. The parameters m and p then determine how strongly the social marginal utility income is affected by the level of z. We shall assume that u_w gets higher as z increases.

A useful transformation of u is

$$(24)\qquad u^* = \frac{1}{m}\ln(-u)^{-1}$$

$$= \frac{a}{m}w^b - (\delta + \frac{p}{m})\ln z$$

$$= \frac{a}{m}w^b - (\delta + \frac{p}{m})h.$$

If disposable income is w_{ij} when there are two wage earners in the household, and disposable income is w_{i0} if there is only one wage earner in the household, then the critical value of h is given by

$$\frac{a}{m}w_{ij}^b - h_{ij} = \frac{aw_{i0}^b}{m},$$

which implies that

$$(25)\qquad h_{ij} = \frac{a}{m}w_{ij}^{\;b} - \frac{a}{m}w_{i0}^b.$$

If b = 1, (25) reduces to

$$h_{ij} = \frac{a}{m}(w_j - t_{ij} + t_i)$$

and the income effect vanishes.

In general

(26) $$\frac{\partial h_{ij}}{\partial w_i} = \frac{ab}{m} \left(w_{ij}^{b-1} - w_{i0}^{b-1}\right)$$

which is negative if b<1.

(27) $$\frac{\partial h_{ij}}{\partial w_j} = \frac{ab}{m} \; w_{ij}^{b-1}$$

The participation rate is an increasing function of the extra income obtained by a secondary wage earner.

The distribution of h and z.

We now restrict h to be non-negative ($z \geq 1$) and choose a Gamma distribution with density :

(28) $$g(h) = he^{-h} \; ; \; h \geq 0,$$

and the cumulative distribution:

(29) $$G(h) = 1-(1+h)e^{-h} .$$

Then z is distributed by the density function

(30) $$f(z) = \ln z/z^2,$$

and the cumulative distribution is

(31) $$F(z) = 1- \frac{1}{z} - \frac{\ln z}{z} .$$

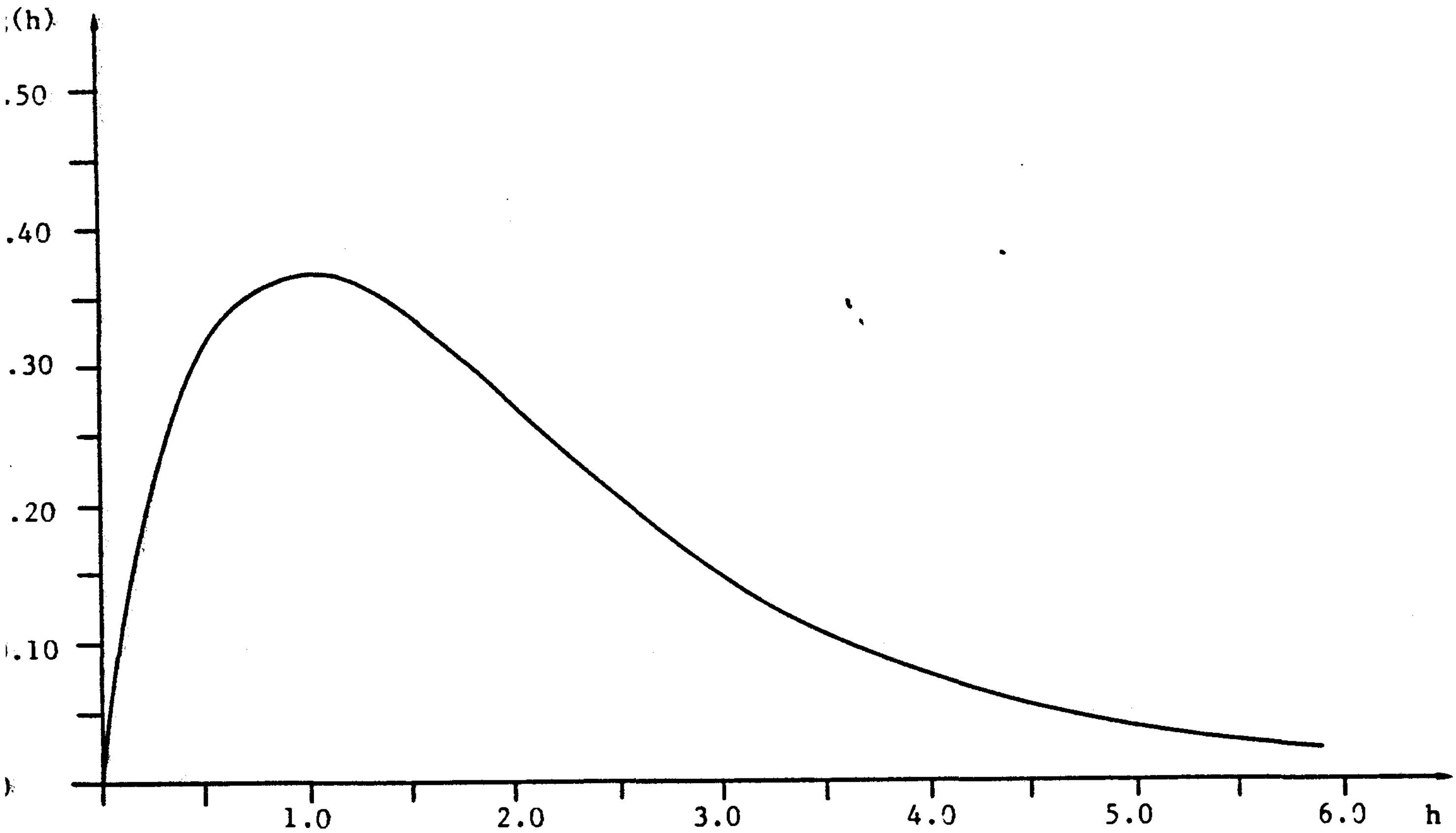

Figure 1. The Gamma distribution with density $g(h) = he^{-h}$.

This distribution implies that no potential secondary wage-earner would be willing to take a job without being paid. Some, but few, would be willing to accept a very low pay. Some would demand an extremely high pay in order to take a job. Most people are in between. The distribution is singlepeaked. These very general features are hardly very controversial. Apart from this we shall make no attempt to justify this particular distribution in its own right. We consider the essential test to be whether the labour supply function derived from this distribution <u>and</u> the utility function is judged to be reasonable.

A further argument is that the chosen distribution in combination with either utility function allows us to compute the welfare level by simple methods since the relevant integrals are expressed by analytical functions. Being able to compute the welfare level (at a reasonable computational cost) is a great advantage since different tax vectors can then be compared directly from a welfare point of view. This can be used in the search for an optimum. We are not restricted to rely on first order conditions alone, but can actually check numerically that deviations from the assumed optimum do reduce welfare, that constrained optima are actually less favourable than unconstrained optima, etc. And different tax systems can be compared rather easily.

Labour supply functions.

For each group of households defined by pair of potential incomes there is a labour supply function expressing the participation rate of potential secondary wage earners as a function of wage rates and taxes. For wages w_i and w_j the participation rate is $F_{ij} = F(z_{ij}) = G(h_{ij})$. Applying the logarithmic specification of the utility function, the critical h-level is given by formula (18) and the supply function becomes

$$(32) \qquad F_{ij} = 1 - (1+k \ln \frac{w_{ij}}{w_{i0}})(\frac{w_{i0}}{w_{ij}})^k$$

From (32) and the definition of disposable income we can derive a number of elasticities in order to measure labour supply responses to changes in economic data. One measure which may be of interest is the elasticity of F_{ij} with respect to the net income of the secondary wage earner defined as $w_j - t_{ij} + t_i$, when w_i is the primary income and w_j is the secondary income. We may note that $t_{ij}-t_i$ is the net increase in tax payment due to the secondary wage income. This elasticity is by definition:

$$(33) \qquad E_{ijj} = \frac{w_j - t_{ij} + t_i}{F_{ij}} \cdot \frac{\partial F_{ij}}{\partial w_j} \quad .$$

It may also be interesting to consider the elasticity of F_{ij} with respect to the net primary income $w_i - t_i$:

$$(34) \qquad M_{iji} = \frac{w_i - t_i}{F_{ij}} \frac{\partial F_{ij}}{\partial w_i}$$

It turns out that a property of our labour supply function is that

$$(35) \qquad M_{iji} = -E_{ijj}.$$

Applying the exponential specification of the utility function the participation rate becomes

$$(36) \qquad F_{ij} = 1-(1+\frac{a}{m}(w_{ij}^{b} - w_{i0}^{b}))\, e^{-\frac{a}{m}(w_{ij}^{b} - w_{i0}^{b})}.$$

Part-time work.

We shall also consider the case where a secondary wage earner has the option of taking a part-time job. We shall then apply the exponential specification of the utility function. To allow for part-time work δ can now assume three different values: 0 if there is no secondary wage earner, x if there is a secondary wage earner taking a part-time job, and 1 if there is a secondary wage earner working full time. Let w_{pj} be the part-time wage rate corresponding to the full-time wage rate w_j. Employing the transformation of the utility function

$$u^* = \frac{a}{m} w^b - (\delta + \frac{p}{m})h,$$

we now obtain two critical h-levels. Households with h close to zero are almost solely concerned with income and choose to have secondary wage earners in full-time jobs. For a value of h equal to h_{ij} (potential full time wages being w_i and w_j), the secondary wage earner is just indifferent between taking a full-time job and taking a part-time job:

$$\frac{a}{m} w_{ij}^{b} - h_{ij} = \frac{a}{m} w_{ipj}^{b} - x\, h_{ij}$$

where w_{ipj} is the disposable income of a household with gross incomes w_i and w_{pj}. For some larger value of h, h_{ipj}, the potential secondary wage earner is just indifferent between taking a part-time job and not taking a job at all:

$$\frac{a}{m} w_{ipj}^{b} - x\, h_{ij} = \frac{a}{m} w_{i0}^{b}$$

So the choice of the potential secondary wage earner becomes:

full-time job if $h < h_{ij}$,

part-time job if $h_{ij} \leqq h \leqq h_{ipj}$,

no job if $h > h_{ipj}$.

The participation rate becomes $F(h_{ipj})$. The share of potential secondary wage earners in this group working full time is $F(h_{ij})$ and the share of part-timers is $F(h_{ipj}) - F(h_{ij})$. If the full-time second wage increases, more people will work full time instead of part time. If the part-time wage increases the part time group will attract more people both from the group of non-workers and the group of full-time workers.

In the case of part-time work the welfare function takes the following form:

$$(37)\; W = \sum_i \sum_{j \leqq i} \int_{z_0}^{z_{ij}} u(w_{ij}, 1, z)\, n_{ij}\, f(z)dz + \sum_i \sum_{j \leqq i} \int_{z_{ij}}^{z_{ipj}} (w_{ipj}, x, z) n_{ij} f(z)dz$$

$$+ \sum_i \sum_{j \leqq i} \int_{z_{ipj}}^{\infty} u(w_{i0}, 0, z) n_{ij} f(z)dz$$

The corresponding tax revenue becomes:

$$(38)\quad R=\sum_i\sum_{j\leqq i}\int_{z_0}^{z_{ij}} t_{ij}\, n_{ij} f(z)dz+\sum_i\sum_{j\leqq i}\int_{z_{ij}}^{z_{ipj}} t_{ipj} n_{ij} f(z)dz$$

$$+\sum_i\sum_{j\leqq i}\int_{z_{ipj}}^{\infty} t_i\, n_{ij}\, f(z)dz$$

4. SCOPE AND LIMITS OF THE ANALYSIS

We are interested in the optimum tax structure. Our model has, however, strong limitations which imply that it is suitable for analysing only part of the tax structure. In particular, actual wages of primary wage earners are exogenous. It follows that the marginal tax rate on primary income for a fixed secondary income has by itself no distortionary effect. It could therefore become quite high. It could even exceed 100 per cent. To see the mechanism involved let us consider on the one hand households with low primary income and on the other hand households with high primary income. Assuming that inequality aversion prevails, one would like to tax high income households at a high rate. From a pure distributional point of view one would like to impose particularly high taxes on those households who add a secondary income to a high primary income. However, by taxing the additional income at a high marginal rate considerable distortions may arise. To combine the concerns with equity and efficiency, already those households who earn only a high primary income should be faced with a very high tax rate compared to those who earn a low primary income. Hence a very high marginal tax on primary income may be implied.

Due to various incentive problems this is not realistic. A person may be able to earn a lower income than his potential income and would prefer to do so if the marginal tax rate exceeds 100 per cent. In practice also the primary wage earner has some scope for varying his work effort. Also the concern with tax evasion does not allow extremely high marginal tax rates. As a compromise between elaborately modelling and completely neglecting these factors we shall simply restrict marginal tax rates from "exploding" by imposing upper limits. As long as marginal tax rates are restricted to a reasonable level we consider the completely exogenous behaviour of primary wage earners to be an acceptable approximation which allows us to focus on the behaviour of secondary wage earners. This is in line with the common view and empirical evidence that the most important variations are observed in the female labour supply.

By our approach the policy analysis is confined to the taxation of secondary incomes. The advantage of the model is that it allows us to analyze this issue within a relatively simple framework. Refinements in other respects would add a good deal of complexity.

Let us now consider more closely the role of an upper limit on marginal tax rates related to primary incomes. Let us consider two income levels, w_2 and w_1. Restrict the marginal tax rate to be no greater than c. Then

$$(39) \qquad \frac{t_2 - t_1}{w_2 - w_1} \leqq c ,$$

$$(40) \qquad \frac{t_{21} - t_{11}}{w_2 - w_1} \leqq c .$$

Let us then consider the trivial relation:

$$t_{21}-t_2 = t_{21}-t_{11} -(t_2-t_1)+t_{11}-t_1,$$

which is easily transformed to:

$$\frac{t_{21}-t_2}{w_1} = \frac{t_{21}-t_{11}}{w_2-w_1} \frac{w_2-w_1}{w_1} - \frac{t_2-t_1}{w_2-w_1} \frac{w_2-w_1}{w_1} + \frac{t_{11}-t_1}{w_1} .$$

We immediately see that if the constraints (39) and (40) are both binding then

$$(41) \qquad \frac{t_{21}-t_2}{w_1} = \frac{t_{11}-t_1}{w_1} ,$$

which means that the marginal tax rate on a low secondary income is the same for a high and low primary income. So constraints on marginal tax rates associated with primary income may have immediate implications for marginal tax rates on secondary incomes. If (41) holds, the remaining issue to be analyzed is how the marginal tax rate on a low secondary income compares with that on a higher secondary income.

In practice family taxation usually takes the form of joint taxation, taxation on an individual base or a mixture (as presently in Norway). Joint taxation by definition implies that $t_{rs}=t_{ij}$ if $w_r+w_s=w_i+w_j$. The tax liability depends only on the total income of the household. Also by definition taxation on an individual base implies that $t_{ij}=t_{i0}+t_{j0}$. Each person of a household is taxed as a single person. The total tax imposed on two incomes is the same if they are earned by members of the same household as when they are earned by persons belonging to different households.

5. OPTIMUM TAXATION

The optimum set of taxes is the one that maximizes the welfare function (5) subject to the tax revenue requirement (6) and the constraints on marginal tax rates. These constraints take the following form:

(42) $$t_{i+1,j}-t_{i\,j}\leq c(w_{i+1}-w_i)$$ for i = 1,2,.., n-1, j=0,.., n-1, and $j \leq i$, where j=0 implies that there is no secondary income.

The number of strictly positive wage levels has been denoted by n. The parameter c is the upper limit imposed on the marginal tax rate. To avoid an arbitrary asymmetri, the same upper limit is in principle valid for marginal taxes on secondary income.
But since this constraint is less likely to be effective, we disregard it at this stage.

The Lagrange function of our optimization problem is then

(43) $$L = W + \mu(R-R^{o}) - \sum_i \sum_{j\leq i} \mu_{ij}(t_{i+1,j} - t_{ij} - c(w_{i+1} - w_i)).$$

Necessary first order conditions become:

(44) $$L_{kj} = \frac{\partial L}{\partial t_{kj}} = \frac{\partial W}{\partial t_{kj}} + \mu \frac{\partial R}{\partial t_{kj}} - \mu_{k-1,j} + \mu_{kj} = 0,\ k=1,2,\ldots,n\ j=0,\ldots,n,$$

and $j \leqq k$,

$\mu_{ij} > 0$ and $t_{i+1,j} - t_{ij} = c(w_{i+1} - w_i)$

or

(45) $\mu_{ij} = 0$ and $t_{i+1,j} - t_{ij} \leqq c(w_{i+1} - w_i)$ for $i=1,2\ldots,n-1$

and $j \leqq i$,

$\mu_{ij} = 0$ for all other indices, and $R = R^o$.

Similar conditions are derived in the case of opportunities for part-time work.

So far the number of income levels has been arbitrary. To deal with system-constrained optimization it is convenient to consider a small number of income levels as we shall do in the subsequent computations. Let us introduce three income levels: $w_1=1$, $w_2=2$, and $w_3=3$.

The conditions of (44) are then equivalent to

(46) $$L_{11} = L_{10} = L_{22} = L_{21} = L_{20} = L_{33} = L_{32} = L_{31} = L_{30} = 0.$$

Joint taxation implies that

$t_{11} = t_{20}$ since $2w_1 = w_2$,

$t_{21} = t_{30}$ since $w_2 + w_1 = w_3$,

$t_{22} = t_{31}$ since $2w_2 = w_3 + w_1$.

In place of the conditions (46) we now get:

(47) $L_{11}+L_{20}=L_{21}+L_{30}=L_{22}+L_{31}=L_{10}=L_{33}=L_{32}=0.$

Taxation on an individual base implies that

$t_{11}=2t_{10},$

$t_{21}=t_{20}+t_{10},$

$t_{22}=2t_{20},$

$t_{31}=t_{30}+t_{10},$

$t_{32}=t_{30}+t_{20},$

$t_{33}=2t_{30}.$

The conditions (46) are now replaced by the conditions:

(48) $L_{10}+2L_{11}+L_{21}+L_{31}=L_{20}+L_{21}+2L_{22}+L_{32}=L_{30}+L_{31}+L_{32}+2L_{33}=0$

6. COMPUTATIONAL METHOD [1)]

To solve the respective optimization problems we used a programme for solving a system of non-linear equations. The special programme used is the TK-solver programme.[2)] The programme is not capable of handling inequality constraints. We therefore assumed various subsets of constraints to be binding neglecting the remaining ones. The budget constraint was always assumed to be binding. We then solved the equation system consisting of these effective constraints and the first order conditions (46) [or (47) or (48)] with respect to the tax parameters and the shadow prices (Lagrange multipliers) assigned to the constraints. If neglected constraints were then violated this candidate for a solution

1) The computations are partly joint work with Gunnar Bramness.

2) For a description see Konopasek and Jayaraman (1984).

of the original problem was discarded. So it was too if shadow prices turned out to get a wrong sign, thus violating the Kuhn-Tucker conditions (45). By this procedure we reached solutions of all the first order Kuhn-Tucker conditions of the original optimization problem. We also made use of the objective function to check directly that imposing a constraint did reduce its value. Also systematic searches in the neighbourhood of some of the solutions were carried out to verify that they do imply maxima. Also different starting-points were tried. And direct welfare comparisons with quite different tax vectors were made. Thus we did not rely solely on first order conditions.

7. NUMERICAL RESULTS

In this section we report the numerical results obtained by applying the respective specifications of the utility function.

Logarithimic specification of the utility function

We first consider two cases in which there are two income levels. In the first case income levels are 1 and 2. In the second case income differences are made larger by choosing income levels 1 and 3. The tax revenue requirement is chosen so as to make the resulting tax level equal to one third of the national income. Other exogenous characteristics of the economy are the same in both cases. Optimum taxes and other endogenous variables characterizing the economy at optimum taxation have been calculated. The data and results are reported in table 1.

Table 1. Two income levels. Data and numerical results.

	wage levels		distribution of earning capacities				preferences	tax revenue requirement	marginal tax rates
exogenous parameters	w_1	w_2	n_{11}	n_{21}	n_{22}	Σ	k	R	
case 1	1	2	0.2	0.6	0.2	1	4	0.884	≤ 0.75
case 2	"	3	"	"	"	"	"	1.157	≤ 0.75

	optimum taxes					marginal tax rates				
endogenous variables	t_1	t_2	t_{11}	t_{21}	t_{22}	$\frac{t_2-t_1}{w_2-w_1}$	$\frac{t_{21}-t_{11}}{w_2-w_1}$	$\frac{t_{11}-t_1}{w_1}$	$\frac{t_{21}-t_2}{w_1}$	$\frac{t_{22}-t_{21}}{w_2-w_1}$
case 1	0.09	0.84	0.23	0.98	1.61	0.75	0.75	0.14	0.14	0.63
case 2	-0.28	1.22	-0.18	1.32	2.50	0.75	0.75	0.10	0.10	0.59

	participation rates			labour supply elasticities			observed income distribution			national income	tax level
endogenous variables	F_{11}	F_{21}	F_{22}	E_{111}	E_{211}	E_{222}	s_0	s_1	s_2	Y	R/Y
case 1	0.74	0.65	0.78	0.49	0.63	0.42	0.15	0.37	0.48	2.65	1/3
case 2	0.63	0.49	0.75	0.66	0.88	0.47	0.21	-0.31	0.48	3.47	1/3

F_{ij} = the labour market participation rate of secondary wage earners in households with potential incomes w_i and w_j.

s_i = the relative number of persons with actual income i; i = 0, w_1, w_2.

E_{ijj} = the elasticity of F_{ij} with respect to the net wage of the secondary wage earner, $w_j-(t_{ij}-t_i)$.

We see that the marginal tax rate on primary income is effectively constrained to 0.75 as might be excepted. The most striking feature of the results is that the marginal tax rate on a low secondary income is very low, while the marginal tax rate on high secondary income is high.

We then consider an economy with three wage levels. Four different tax systems are then considered. First, optimum taxes are calculated. Second, optimum taxes ~~are~~ calculated on the condition that people should be taxed on an individual basis. Third, joint taxation of household members is imposed and optimum taxes are calculated under this restriction. Fourth, a presumably far from optimal tax system is considered. The system is constructed by reversing main characteristics of the optimum system making marginal taxes on secondary income high at intervals where they are low in the optimum system and vice versa. More precisely, marginal taxes are exogenously set equal to 0.75 on primary and low secondary income and equal to 0.25 on medium and high secondary income. Then there is only one degree of freedom left, and t_1 is determined so as to satisfy the revenue requirement. The system is referred to as "the reversed system". A survey of the data and results is presented in table 2.

Also in this model we obtain a pattern with a second best marginal tax rate which is very low on a low secondary income, but steeply increasing as the secondary income increases.

The welfare weights or social marginal utilities of income which materialize at the second best optimum are easily computed. As a normalisation the weight assigned to a household earning two low incomes is set equal to unity. The following results are obtained:

gross incomes	welfare weight
1,0	1.9
1,1	1.0
2,0	1.5
2,1	0.9
2,2	0.8
3,0	1.3
3,1	0.8
3,2	0.7
3,3	0.6

Since a marginal primary income is taxed at a higher rate than a marginal secondary income for efficiency reasons, a household earning only one income is left with a lower disposable income than a household earning

Table 2. Three income levels. Data and numerical results.

	wage levels			number of households of each type						preferences	tax revenue requirement	marginal tax rates
exogenous parameters	w_1	w_2	w_3	n_{11}	n_{21}	n_{22}	n_{31}	n_{32}	n_{33}	k	R	
	1	2	3	0.1	0.15	0.3	0.15	0.25	0.05	4	1.18	≤ 0.75

	tax	marginal tax rates						
endogenous variables	t_1	$\frac{t_2-t_1}{w_2-w_1}$	$\frac{t_3-t_2}{w_3-w_2}$	$\frac{t_{21}-t_{11}}{w_2-w_1}$	$\frac{t_{31}-t_{21}}{w_3-w_2}$	$\frac{t_{32}-t_{22}}{w_3-w_2}$	$\frac{t_{11}-t_1}{w_1}$	$\frac{t_{21}-t_2}{w_1}$
optimum taxation	-0.12	0.75	0.75	0.75	0.75	0.75	0.01	0.01
individual base	-0.11	0.75	0.75	0.75	0.75	0.75	-0.11	-0.11
joint taxation	0.01	0.53	0.56	0.56	0.08	0.56	0.53	0.56
"reversed" system	-0.24	0.75	0.75	0.75	0.75	0.75	0.75	0.75

	marginal tax rates				national income	tax level	welfare
endogenous variables	$\frac{t_{31}-t_3}{w_1}$	$\frac{t_{22}-t_{21}}{w_2-w_1}$	$\frac{t_{32}-t_{31}}{w_2-w_1}$	$\frac{t_{33}-t_{32}}{w_3-w_2}$	Y	R/Y	W
optimum taxation	0.09	0.57	0.57	0.69	3.54	1/3	2.36
individual base	-0.11	0.75	0.75	0.75	3.55		2.35
joint taxation	0.08	0.08	0.56	0.65	3.41		2.25
"reversed" system	0.75	0.25	0.25	0.25	3.15		2.14

the same gross income by having two wage earners. Hence the one-income household also gets a higher welfare weight at the optimum. This fact reflects that the optimum welfare weights are affected both by the distributional concern and the efficiency effects present in our model.

As a benchmark which may be of some interest, we have also considered the first best tax system. In this system the tax levied on a household is determined by its potential incomes and reluctance to supply labour, which are exogenous characteristics. Taxes vary so as to equalize social marginal utilities of income. Hence disposable incomes are equalized. For each pair of potential incomes there is a critical first best value of z (and h) drawing a line between those households which should have a secondary wage earner and those which should not. Households with a lower value of z are faced with a higher tax and choose to have a secondary wage earner, while those with a higher value of z are faced with a lower tax and choose not to have a secondary wage earner. Since the first best system is a purely hypothetical one we shall not elaborate any further on details.

It may be of some interest to compare the labour market participation rates of potential secondary wage earners under the various systems. Let us recall that F_{ij} is the participation rate in the group of households with potential incomes w_i and w_j. The following results are obtained:

Tax system:

	first best	optimum taxation	individual base	joint taxation	reversed system
F_{11}	0.49	0.72	0.76	0.50	0.17
F_{21}	0.49	0.64	0.69	0.30	0.13
F_{22}	0.84	0.78	0.76	0.76	0.61
F_{31}	0.49	0.57	0.62	0.49	0.10
F_{32}	0.84	0.72	0.70	0.65	0.54
F_{33}	0.95	0.79	0.76	0.74	0.77

We observe that joint taxation reduces the participation rate for low secondary income significantly compared to the optimum second best system. With the reversed system this effect is dramatic.

To compare the welfare levels achieved under the various systems we have computed the increase in tax revenue under optimum (second best) taxation which would reduce welfare to the level achieved under the alternative system. The extra tax revenue is assumed to be collected, but used for no purpose - neither redistribution nor public spending. This would be a waste of resources equivalent to the welfare cost of an inefficient tax system. We then find that the tax revenue would have to be raised by only 0.24 per cent to reduce welfare to the level achieved under taxation on an individual base. It would have to be raised by 5.5 per cent to bring down the welfare level to that of the joint taxation system. An increase of 11 per cent would give a welfare level equal to that of the reversed system. The first best tax system would yield a welfare level of 4.66 which is clearly of a different order of magnitude from the figures in table 2. This suggests that the comparisons between the welfare achievements of a first best and a second best system, sometimes presented in the literature, may tell us very little about the welfare effects of practically feasible tax reforms.

Exponential specification of the utility function

We consider the income levels 1 and 2. Exogenous parameters are varied to produce four cases. The results are reported in table 3.

The general pattern is the same as it was with the previous specification. The marginal tax rate on a low secondary income is low, while the marginal tax rate on a high secondary income is high. The optimum progressivity is, however, sensitive to the degree of inequality aversion. This is seen by comparing case 1 and case 2. In case 2 the parameters a and m have been increased. This is tantamount to an increase in inequality aversion as we can see from the formulas (20) - (23). We see from (36) that there is no shift in the labour supply function. The switch to this case produces a negative income tax and there is a substantial rise in the marginal tax rate on low secondary income. The marginal tax rate on a high secondary income was already close to the upper limit.

Table 3. Exponential specification. Two income levels.
Data and numerical results.

$w_1 = 1$, $w_2 = 2$, $n_{11} = 0.2$, $n_{21} = 0.6$, $n_{22} = 0.2$

	Case 1	Case 2	Case 3	Case 4
a	1	2	1	1.3
b	1	1	1	0.75
m	0.3	0.6	0.3	0.3
p	0.01	0.01	0.01	0.01
c	0.75	0.75	0.75	0.75
R	0.839	0.802	1.119	0.813
		Numerical results		
t_1	0.02	-0.13	0.30	0.001
$(t_2-t_1)/(w_2-w_1)$	0.75	0.75	0.75	0.75
$(t_{21}-t_{11})/(w_2-w_1)$	0.75	0.75	0.75	0.75
$(t_{11}-t_1)/w_1$	0.10	0.29	0.10	0.13
$(t_{21}-t_2)/w_1$	0.10	0.29	0.10	0.13
$(t_{22}-t_{21})/(w_2-w_1)$	0.749	0.75	0.749	0.69
F_{11}	0.80	0.68	0.80	0.73
F_{21}	0.80	0.68	0.80	0.71
F_{22}	0.90	0.83	0.90	0.84
E_{111}	0.56	0.77	0.56	0.64
E_{211}	0.56	0.77	0.56	0.68
E_{222}	0.36	0.50	0.36	0.44
M_{111}	-0.61	-1.22	-0.44	-0.86
M_{212}	-0.77	-1.49	-0.59	-1.11
M_{222}	-0.38	-0.72	-0.29	-0.55
R/Y	0.3	0.3	0.4	0.3

Looking at case 3 and case 1 we can trace the effects of raising the total tax level. The remarkable result is that this change does only produce an upward shift in the whole tax schedule, while marginal tax rates are left unchanged.

In case 1 to case 3 there is no income effect on labour supply since b=1. In case 4 there is an income effect, $b < 1$. At the same time distributional preferences are obviously affected by changing a and b. The resulting tax rates lie between those of case 1 and those of case 2 apart from a small reduction in the top interval marginal tax rate.

The distributive weight or social marginal utility of income assigned to a household is a function of the characteristics of the household. To get some feeling about the weights which are embodied in the optimum tax situation, we have computed the average weights obtained by the various income groups in case 1 and case 2, respectively. In each case the weight given on average to households earning two low incomes has been set equal to unity. The results are presented in table 4 below.

Table 4. Average marginal welfare of income.

Household characteristics		Average distributive weights	
Potential incomes	Actual incomes	Case 1	Case 2
1,1	1,0	1.60	2.87
1,1	1,1	1.00	1.00
2,1	2,0	1.25	1.76
2,1	2,1	0.78	0.61
2,2	2,0	1.24	1.74
2,2	2,2	0.66	0.42

The greater differences in case 2 obviously reflect that a higher inequality aversion has been assumed. We observe that the weights given to households earning one high income differ slightly between households with a low potential secondary income and those with a high potential secondary income. This is due to the fact that the marginal welfare of income is also a function of the reluctance to take a job (the value of z).

In section 4 we discussed the need to impose constraints on marginal tax rates because of the exogenous primary incomes. To give an idea of what the optimum would look like in the absence of such constraints we have computed one free optimum. The data used were $n_{11} = 0.2$, $n_{21} = 0.6$, $n_{22} = 0.2$, $w_1 = 1$, $w_2 = 2$, $a = 1$, $b = 1$, $p = 0.01$. The following tax rates were obtained: $t_1 = -0.25$, $t_2 - t_1 = 1.09$, $t_{21} - t_{11} = 1.03$, $t_{11} - t_1 = 0.15$, $t_{21} - t_2 = 0.10$, $t_{22} - t_{21} = 0.72$. The tax level is one third of the total income. We note that marginal tax rates on primary income now exceed unity, while the taxation of secondary income remains strongly progressive.

Exponential specification and part-time work

Optimum taxes have been computed for alternative values of the strategic parameters in the presence of opportunities for part-time work. The full-time wages are set equal to $w_1=1$ and $w_2=2$. A secondary wage-earner capable of earning a full-time wage w_j has the option of taking a part-time job at the wage $0.5\ w_j$. This will be a half-time job if total pay is proportional to working hours, but there is in principle no need to make this assumption. The relevant part-time wages in our case are $w_{p1}=0.5$ and $w_{p2}=1$. A survey of parameters and results are provided in table 4 and table 5.

The following notation is used: The index .5 refers to the part-time income 0.5. For instance the tax levied on a household with a primary income w_i and a secondary part-time income 0.5 is denoted by $t_{i.5}$. The upper limit imposed on marginal tax rates is denoted by c. As before F_{ij} denotes the share of potential secondary wage earners from the group of households with potential incomes w_i, w_j in full-time jobs. Let P_{ij} denote the similar share of potential wage earners in part-time jobs, and let the total participation rate be denoted by $R_{ij} = F_{ij} + P_{ij}$. The following elasticities have been computed:

E_{ijj} = the elasticity of the full-time participation rate F_{ij} with respect to the net wage increment from working full time instead of part time,

p_{ijk}= the elasticity of the part-time participation rate P_{ij} with respect to the net wage from working part time, $w_k-(t_{ik}-t_i)$, where w_k=0.5 w_j,

e_{ijk}= the elasticity of the total participation rate R_{ij} with respect to the net wage from working part time, $w_k-(t_{ik}-t_i)$.

The data are varied to produce four different cases. But the general tax pattern is uniform. The marginal tax rate on secondary income is rising. It is considerably higher on a high secondary income than on a low secondary income. The introduction of part-time work has, however, had a strong dampening effect on the marginal tax rates on high income. This is not surprising since one would expect the part-time option to make labour supply on the whole more elastic and the concern with efficiency more prominent.

We may observe that the effect of raising the total tax level or lowering the upper limit on marginal tax rates is mainly to push up the base tax rate, t_1.

Table 5. Exponential specification and part-time work.
Data and numerical tax results.

	Case 1	Case 2	Case 3	Case 4
	Data			
a	1	1.2	1.2	1
b	1	0.9	0.9	1
p	0.01	0.01	0.01	0.01
m	0.3	0.3	0.3	0.3
x	0.45	0.45	0.45	0.45
c	0.75	0.75	0.75	0.65
R	0.969	1.111	0.971	0.968
	Optimum taxes			
t_1	0.15	0.27	0.13	0.23
$_{2}-t_{1})/(w_{2}-w_{1})$	0.75	0.75	0.75	0.65
$_{2.5}-t_{1.5})/(w_{2}-w_{1})$	0.75	0.75	0.75	0.65
$_{21}-t_{11})/(w_{2}-w_{1})$	0.75	0.75	0.75	0.65
$_{1.5}-t_{1})/\ w_{.5}$	0.16	0.18	0.18	0.16
$t_{2.5}-t_{2})/\ w_{.5}$	0.16	0.18	0.18	0.16
$t_{11}-t_{1.5})/(w_{1}-w_{.5})$	0.20	0.22	0.22	0.20
$t_{21}-t_{2.5})/(w_{1}-w_{.5})$	0.20	0.22	0.22	0.20
$t_{22}-t_{21})/(w_{2}-w_{1})$	0.47	0.48	0.48	0.47
R/Y	0.35	0.40	0.35	0.35

Table 6. Exponential specification and part-time work. Data and computed participation rates and elasticities.

Data

$n_{11}=0.2$, $n_{21}=0.6$, $n_{22}=0.2$, $w_1=1$, $w_2=2$,

part-time wages = 0.5 · full-time wages

	Case 1	Case 2	Case 3	Case 4
a	1	1.2	1.2	1
b	1	0.9	0.9	1
p	0.01	0.01	0.01	0.01
m	0.3	0.3	0.3	0.3
x	0.45	0.45	0.45	0.45
c	0.75	0.75	0.75	0.65
	Participation rates and elasticities			
F_{11}	0.70	0.71	0.71	0.70
P_{11}	0.12	0.13	0.13	0.11
R_{11}	0.82	0.84	0.84	0.81
F_{21}	0.70	0.70	0.70	0.70
P_{21}	0.12	0.13	0.13	0.11
R_{21}	0.82	0.83	0.83	0.81
F_{22}	0.83	0.83	0.83	0.83
P_{22}	0.15	0.16	0.16	0.15
R_{22}	0.98	0.99	0.99	0.98
E_{111}	0.75	0.71	0.72	0.75
$P_{11.5}$	8.28	7.23	7.33	8.26
$e_{11.5}$	0.53	0.47	0.48	0.53
E_{211}	0.75	0.73	0.74	0.75
$P_{21.5}$	8.28	7.41	7.48	8.26
$e_{21.5}$	0.53	0.49	0.50	0.53
E_{222}	0.50	0.50	0.51	0.50
P_{221}	4.76	4.58	4.58	4.73
e_{221}	0.09	0.08	0.08	0.09

8. CONCLUSION

A model framework for analyzing the taxation of secondary income in two-person households has been presented. The numerical cases that have been analyzed provide strong evidence that a low secondary income should be taxed at a low rate. The marginal tax rate should be steeply increasing to reach a high value at a high secondary income. This outcome differs from the result frequently obtained in traditional optimal tax models that the optimal income tax schedule is close to a linear one. Options for part-time work do, however, have a strong dampening effect on the marginal tax rate on high secondary income. Our results suggest that choices among different family tax systems produce welfare differences that would justify the concern of tax designers, even though it is not a question of changing the order of magnitude of our standard of living. In our contest between special systems taxation on an individual base comes out as the superior one.

The present research can be extended along various lines. One could, of course, always try different assumptions to a reasonable extent. But rather than extensively pursuing the implications of other data and specifications, it would be desirable to adapt these more closely to the empirical knowledge about the actual economy. In particular this would be the appropriate direction for further research if the ambition extends beyond that of achieving a rough outline of the optimal tax system. A widening of the scope of the analysis would be to consider simultaneously the taxation of couples and single persons, and to deal more explicitly with the presence of children and child allowances, which clearly affect incomes and labour supply behaviour. It would also be desirable to allow for explicit labour supply reactions of primary wage earners. In the present analysis potential incomes are taken to be exogenous. A modelling of wage responses to tax induced changes in labour supply behaviour would bring the model closer to a general equilibrium analysis. Such responses would affect both distribution and efficiency, and it is not clear a priori what the net effect on the tax design would be.

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APPENDIX

A detailed description of all the computations would fill many pages and will not be presented. We shall, however, display two important computations. The other computations are very similar. Let us first present the equation system which yields the optimum second best solution for the logarithmic specification of the utility function. The welfare function is expressed as

$$W = \sum_{i,j} \int_0^{h_{ij}} (k \ln w_{ij} - h) n_{ij} \; he^{-h} dh + \sum_{i,j} \int_{h_{ij}}^{\infty} k \ln w_{i0} n_{ij} he^{-h} dh,$$

for $i=1,2,3$ and $j \leq i$, and the equation system becomes:

```
g11/(2*w1-t11)+u1*H11+u5-u6=0
(k*n11-g11)/(w1-t1)+u1*(n11-H11)-u2-u5=0
g21/(w2+w1-t21)+u1*H21 +u3-u4+u6-u8=0
(k*n2-g21-g22)/(w2-t2)+u1*(n2-H21-H22)+u2-u3-u7=0
g22/(2*w2-t22)+u1*H22+u4-u9=0
g31/(w3+w1-t31)+u1*H31+u8+u10-u11=0
(k*n3-g31-g32-g33)/(w3-t3)+u1*(n3-H31-H32-H33)+u7-u10=0
g32/(w3+w2-t32)+u1*H32+u9+u11-u12=0
g33/(2*w3-t33)+u1*H33+u12=0
h11=k*ln((2*w1-t11)/(w1-t1))
g11=k*(H11-n11*h11*exp(-h11)*(t11-t1)*u1)
H11=n11*(1-(1+h11)*exp(-h11))
h21=k*ln((w2+w1-t21)/(w2-t2))
g21=k*(H21-n21*h21*exp(-h21)*(t21-t2)*u1)
H21=n21*(1-(1+h21)*exp(-h21))
h22=k*ln((2*w2-t22)/(w2-t2))
g22=k*(H22-n22*h22*exp(-h22)*(t22-t2)*u1)
H22=n22*(1-(1+h22)*exp(-h22))
h31=k*ln((w3+w1-t31)/(w3-t3))
g31=k*(H31-n31*h31*exp(-h31)*(t31-t3)*u1)
H31=n31*(1-(1+h31)*exp(-h31))
h32=k*ln((w3+w2-t32)/(w3-t3))
g32=k*(H32-n32*h32*exp(-h32)*(t32-t3)*u1)
H32=n32*(1-(1+h32)*exp(-h32))
h33=k*ln((2*w3-t33)/(w3-t3))
g33=k*(H33-n33*h33*exp(-h33)*(t33-t3)*u1)
H33=n33*(1-(1+h33)*exp(-h33))
```

```
u1*(t11*H11+t1*(n11-H11)+t21*H21+t2*(n2-H21-H22)+t22*H22)=tax
tax+u1*(t31*H31+t32*H32+t33*H33+t3*(n3-H31-H32-H33)-R)=0
t2-t1-w2+w1+c21=0
w1-t11+d21-w2+t21=0
w2-t2+t3-w3+c32=0
t31-t21-w3+w2+d32=0
t32-t22-w3+w2+e32=0
Y=(n11+H11+H21+H31)*w1+(n2+H22+H32)*w2+(n3+H33)*w3
3*R=Y

F11=1-(1+h11)*exp(-h11)
W11=n11*(h11*F11+k*ln(w1-t1)-2+2*(1+h11+.5*h11*h11)*exp(-h11))
F21=1-(1+h21)*exp(-h21)
W21=n21*(h21*F21+k*ln(w2-t2)-2+2*(1+h21+.5*h21*h21)*exp(-h21))
F22=1-(1+h22)*exp(-h22)
W22=n22*(h22*F22+k*ln(w2-t2)-2+2*(1+h22+.5*h22*h22)*exp(-h22))
F31=1-(1+h31)*exp(-h31)
W31=n31*(h31*F31+k*ln(w3-t3)-2+2*(1+h31+.5*h31*h31)*exp(-h31))
F32=1-(1+h32)*exp(-h32)
W32=n32*(h32*F32+k*ln(w3-t3)-2+2*(1+h32+.5*h32*h32)*exp(-h32))
F33=1-(1+h33)*exp(-h33)
W33=n33*(h33*F33+k*ln(w3-t3)-2+2*(1+h33+.5*h33*h33)*exp(-h33))
W=W11+W21+W22+W31+W32+W33
```

Variables are explained below. The first nine relations are first order conditions for the welfare maximum which are obtained by deriving partial derivatives of the Lagrangian with respect to t_{11}, t_1, t_{21}, t_2, t_{22}, t_{21}, t_3, t_{32} and t_{33}. The next eighteen relations define the critical values of h for the various groups of households and a number of auxiliary variables used to shorten the equations. The first seven relations on this page reflect the binding constraints. Then total income, Y, is defined and the relative tax level is determined. The remaining equations determine the participation rates and the welfare level. In the welfare equations it is useful to recall that $h_{ij} = k \ln(w_{ij}/w_{io})$.

The output and input table of the computation is

================= VARIABLE SHEET ==

Input	Name	Output	Unit	Comment
	t1	-.1193935		
	t2	.63060650		
	t3	1.3806065		The nine
	t11	-.1108518		tax rates
	t21	.63914816		
	t22	1.2056852		
	t31	1.3891482		
	t32	1.9556852		
	t33	2.6432426		
	u1	-1.706201		Shadow price of revenue requirement
	u2	.05118791		
0	u3			
0	u4			The twelve
0	u5			shadow prices
	u6	.01415030		associated with
	u7	.05468336		marginal tax constraints
	u8	.01392124		
	u9	.00523418		
0	u10			
0	u11			
0	u12			
	g11	.28927184		
	g21	.38626885		Auxiliary variables
	g22	1.1270739		defined in the model
	g31	.34400897		
	g32	.91613278		
	g33	.22562740		
	h11	2.5372183		
	h21	2.1786183		
	h22	2.8528760		The critical values of h
	h31	1.9104994		
	h32	2.5248967		
	h33	2.9156950		
	H11	.07202552		
	H21	.09602810		
	H22	.23333191		Auxiliary variables
	H31	.08538410		Hij = nij*Fij
	H32	.17944339		

	H33	.03939505	
.1	n11		
.15	n21		
.3	n22		Number of households
.15	n31		of each type,
.25	n32		ni is the sum over j of nij
.05	n33		
.45	n2		
.45	n3		
1	w1		
2	w2		Wage levels
3	w3		
	R	1.1823912	Total tax revenue
4	k		Preference parameter
.25	c21		
.25	d21		Parameters constraining
.25	c32		marginal tax rates
.25	d32		
.25	e32		
	Y	3.5471735	Total income
	F11	.72025523	
	F21	.64018730	
	F22	.77777305	The six
	F31	.56922730	participation rates
	F32	.71777356	
	F33	.78790101	
	W11	.13471972	
	W21	.28636508	Six auxiliary welfare
	W22	.71707587	variables associated
	W31	.36262280	with the various groups
	W32	.70384912	of households
	W33	.15550836	
	W	2.3601410	Total welfare
	tax		Auxiliary variable

Applying the exponential specification of the utility function, the welfare function is expressed as

$$W = -\sum_{i,j} e^{-a\, w_{ij}^{b}}\, n_{ij} \int_{1}^{z_{ij}} z^{m\delta+p}(\ln z/z^2)dz$$

$$-\sum_{i,j} e^{-a\, w_{i0}^{b}}\, n_{ij} \int_{z_{ij}}^{\infty} z^{p}\,(\ln z/z^2)dz, \qquad i,j = 1,2 \text{ and } j \leqq i.$$

Solving the integration problem we find that

$$\int z^{m\delta + p-2} \ln z\, dz = \frac{z^{m\delta + p-1}}{m\delta + p-1}(\ln z - \frac{1}{m\delta + p-1}) + C = I(z)$$

We find that

$$I(1) = -\frac{1}{(m\delta + p-1)^2} + C$$

and

$$\lim_{z\to\infty} I(z) = C, \text{ when } m\delta + p < 1.$$

Hence

$$\int_{1}^{z_{ij}} z^{m\delta + p-2} \ln z\, dz = \frac{z_{ij}^{q}}{q}(\ln z_{ij} - \frac{1}{q}) + \frac{1}{q^2},$$

where $q = m\delta + p-1$, and

$$\int_{z_{ij}}^{\infty} z^{p-2} \ln z\, dz = \frac{z_{ij}^{p-1}}{p-1}(\frac{1}{p-1} - \ln z_{ij}).$$

The equation system which yields the optimum second best solution is

```
n11*v11*(z11^q*(ln(z11)-1/q)/q+1/q^2)+u1*n11*(F11+F1111*(t11-t1))-u5+u6=0
n11*v1*z11^P*(1/P-ln(z11))/P+u1*n11*(1-F11+F111*(t11-t1))+u2+u5=0
n21*v21*(z21^q*(ln(z21)-1/q)/q+1/q^2)+u1*n21*(F21+F2121*(t21-t2))-u3-u6=0
n22*v22*(z22^q*(ln(z22)-1/q)/q+1/q^2)+u1*n22*(F22+F2222*(t22-t2))=0
21=-n21*v2*z21^P*(1/P-ln(z21))/P+u1*n21*(1-F21+F212*(t21-t2))-u2+u3
22=-n22*v2*z22^P*(1/P-ln(z22))/P+u1*n22*(1-F22+F222*(t22-t2))
21+r22=0
10=w1-t1
11=2*w1-t11
20=w2-t2
21=w2+w1-t21
22=2*w2-t22
11=exp((w11^b-w10^b)*a/m)
21=exp((w21^b-w20^b)*a/m)
22=exp((w22^b-w20^b)*a/m)
1=a*b*exp(-a*w10^b)*w10^(b-1)
11=a*b*exp(-a*w11^b)*w11^(b-1)
21=a*b*exp(-a*w21^b)*w21^(b-1)
22=a*b*exp(-a*w22^b)*w22^(b-1)

2=a*b*exp(-a*w20^b)*w20^(b-1)
1*n11+(t11-t1)*n11*F11+t2*(n21+n22)+(t21-t2)*n21*F21+(t22-t2)*n22*F22=R
11=1-(1+ln(z11))/z11
21=1-(1+ln(z21))/z21
22=1-(1+ln(z22))/z22
1111=-ln(z11)*w11^(b-1)*a*b/(m*z11)
2121=-ln(z21)*w21^(b-1)*a*b/(m*z21)
2222=-ln(z22)*w22^(b-1)*a*b/(m*z22)
111=ln(z11)*w10^(b-1)*a*b/(m*z11)
212=ln(z21)*w20^(b-1)*a*b/(m*z21)
222=ln(z22)*w20^(b-1)*a*b/(m*z22)
w10+c21-w20)=0
w20+c10-w21)*u3=0
w10+c10-w11)*u5=0
w11+c21-w21)=0
=w1*n11*(1+F11)+w2*(n21+n22)+w1*n21*F21+w2*n22*F22
*Y=R
```

The variables are defined below. The first four equations and the seventh are first order conditions for the welfare maximum which emerge when deriving the partial derivatives of the Lagrangian with respect to t_{11}, t_1, t_{21}, t_{22} and t_2. The next five equations define disposable incomes. Then the three critical values of z are determined. Equation twentyone is the total tax revenue constraint. The participation rates and a number of derivatives are defined by the following nine equations. There are four equations reflecting the constraints imposed on marginal tax rates. The last two equations define Y and the relative tax level, respectively.

The output and input table of the computation is presented below.

================ VARIABLE SHEET ================================

Input	Name	Output	Unit	Comment
	t1	.02222239		
	t2	.77222239		
	t11	.12364093		The five tax rates
	t21	.87364093		
	t22	1.6229591		
	R	.83946051		Total tax revenue
	w10	.97777761		
	w20	1.2277776		
	w11	1.8763591		Disposable incomes
	w21	2.1263591		
	w22	2.3770409		
	Y	2.7982017		Total income
	F11	.80014436		
	F21	.80014436		Participation rates
	F22	.89521550		
	v1	.37614611		
	v2	.29294289		
	v11	.15314669		Five auxiliary variables
	v21	.11927076		
	v22	.09282485		
	z11	19.990787		
	z21	19.990787		
	z22	46.102980		Critical values of z
	u1	.21899781		Shadow price of public revenue
	u2	.00471603		
	u3	1.497E-12		Shadow prices associated
0	u4			with marginal tax constraints
	u5	1.497E-12		
	u6	.00652925		
	F1111	-.4994420		
	F2121	-.4994420		
	F2222	-.2769798		Tax derivatives of Fij
	F111	.49944198		
	F212	.49944198		
	F222	.27697975		
	r21	-.0084535		Auxiliary
	r22	.00845348		variables
.2	n11			
.6	n21			Number of households
.2	n22			of each type
1	a			
1	b			Six preference parameters
.01	p			
-.99	P			P=p-1
.3	m			
-.69	q			q=m+p-1
1	w1			Wage
2	w2			levels
.25	c10			Parameters constrainig
.25	c21			marginal tax rates
.3	s			Relative tax level

CHOICE OF OCCUPATION, TAX INCIDENCE AND PIECEMEAL TAX REVISION*

by

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Abstract

A model is presented in which workers move between two different occupations in response to economic incentives which are distorted by an income tax. Prices and wages assume equilibrium values which are affected by the tax parameters. Incidence and welfare effects of small tax revisions are analysed within different variants of the basic model and with particular attention paid to the role of taxinduced wage and price changes. It is demonstrated that within the economic setting of the model one may neglect such wage and price effects in assessments of piecemeal tax revisions. It is also shown that provided that there is an optimum income tax, there is no need to employ commodity taxation.

1. INTRODUCTION

In a general equilibrium setting wages and other prices and pretax incomes will in general be endogenous variables which are affected by the tax policy. Taxes which in the first place are imposed on the income or consumption of one group, may be passed on to some other group when the market responds to tax-induced shifts in supply and demand. But in most optimum tax literature assumptions are chosen which rule out changes in pretax prices and wages and thereby simplify the incidence problems considerably.[1)] A notable exception is Feldstein (1973). His model distinguishes between two classes of workers whose wages are determined by the market. Both kinds of labour are used in a production process displaying constant returns to scale so that no pure profit arises. A linear income tax is imposed. Each class of workers then choose how much labour to supply and leisure to enjoy in accordance with the standard textbook theory of labour supply. There is no mobility between classes. Wages then adjust to clear the markets for the two kinds of labour, and equilibrium wages are obviously sensitive to the choice of tax parameters. Employing numerical examples Feldstein found that the optimum tax rates with endogenous wage rates changed little from those obtained with wages held constant (at no tax equilibrium values).

Allen (1982) argues that these results are mainly due to the use of a Cobb-Douglas production function. Using an analytical approach to the same basic model he argues that the endogeneity of wages may make a considerable difference to optimal linear tax rates.

The purpose of the present paper is to analyse a tax policy which affect pretax wages and other prices within a somewhat different economic model, which may have claim for interest in its own right. The basic assumption of the model is that workers move between two different occupations in response to economic incentives which are distorted by taxes. Work effort may be different in the two occupations, but within each occupation there is no choice of work effort. A person has to choose a fixed bundle of work characteristics when he chooses occupation. This seems at least as realistic as the assumption of perfectly free choice of working hours within a given occupation. Labour from the two occupations are used in production which exhibits constant returns to scale. From the production sector(s) there is a demand for labour belonging to each occupation, and wages are then determined as equilibrium wages in the labour market. Simultaneously prices adjust to clear the market for consumer goods.

In the simplest version of the model the two kinds of labour are inputs in a macro production function with constant returns to scale. This model differs from that of Feldstein in that changes in the supply of the two kinds of labour are due to mobility between the two categories, and not to changes in the hours worked by each person. In other respects the models are similar.

With the assumption of constant returns to scale a major simplification from most optimum tax theory is retained. This simplification may be a great help in the search for clear-cut benchmark results. Otherwise one would also expect results to be sensitive to how the claims for profit are assumed to be distributed.

The typical outcome of optimum tax analysis is to characterize the trade-off between efficiency and distributional considerations. The model of the current paper obviously requires that such a trade-off is considered. A more progressive taxation will shift a greater tax burden on to the better-off tax payers, which is presumably desirable from egalitarian distributional preferences. On the other hand a higher marginal tax will more strongly distort the wage difference between the two occupations and lead to a more inefficient allocation of labour between occupations. The priority of the paper is, however, to focus on other aspects than the familiar trade-off problem.

As in the articles by Feldstein and Allen the important question is how tax-induced price changes must be allowed for in the process of tax design. Although we shall later distinguish between price and wage changes, we now use price changes as a common term. Since price effects are awkward to cope with in tax analysis, it is interesting to see to what extent they may be neglected. Neglecting tax shifting through price changes can mean different things depending on how taxes are assumed to be designed. One interpretation is that prices are assumed to remain fixed as we move from a situation without taxation to the tax optimum. But this is hardly an interesting situation in practice, because taxes are already there when we start thinking about their optimal structure. An alternative interpretation is that prices are assumed to retain their initial values when the tax structure is taken from the initial one to the optimal one. It might then be analysed if and when such an extreme neglect of price adjustments would be permissible.

The present analysis will adopt a different approach. The basic idea is that taxes are changed gradually through piecemeal revisions.[2)] A full optimization will then be carried out by a step-wise procedure which stops when a marginal tax adjustment has got no further effect on welfare. One way to neglect price changes is then to assume at each step that prices are fixed at the values *actually* resulting from the previous step. We can say that *price changes are neglected at the margin*. It is this case that will be analysed in the current paper.

To provide a formal treatment, let us take as our starting point the basic structure of the government's tax design problem. The government is to choose the value of a tax parameter, t, taking into account the effect on social welfare. (Without changing anything t may be thought of as a vector). The welfare level is ultimately a function of the tax parameter (and other exogenous parameters suppressed here). It is affected by the tax parameter through various channels. We may distinguish between effects via induced changes in prices and direct effects which are independent of price reactions.[3)] Let p denote the vector of prices. The welfare level can then be written as a function of t directly and of p which in turn depends on t:

$$(1.1) \qquad W(t,p) = W(t,p(t)).$$

The direct effect of t on welfare is equal to the partial effect of the first argument which is expressed by the partial derivative $\partial W/\partial t$. The total effect is expressed by the total derivative

$$(1.2) \qquad dW/dt = \partial W/\partial t + (\partial W/\partial p)(dp/dt),$$

where $\partial W/\partial p$ and dp/dt are the proper vectors of derivatives. The second term is the effect via changes in prices. Let us introduce some more notation. Let t^o be some arbitrary value of t and let $p^o = p(t^o)$ be the corresponding value of p. To neglect price changes at the margin implies that the welfare effect of a marginal tax revision at (t^o,p^o) is judged by considering at that point $\partial W(t,p^o)/\partial t$ instead of $dW(t,p(t))/dt$. This is clearly permissible only if the partial and total effects have the same sign.

Within the framework outlined above we shall analyse tax incidence and in particular the opportunities for neglecting tax-induced price changes when contemplating piecemeal tax revisions. Before taking up the very tax analysis, the modelling of the choice of occupation is described in detail in Section 2. Tax revisions are then explored within three different models sharing the basic features which have been set out in the introduction. Section 3 deals with a model of one production sector with two occupations (labelled model I). Section 4 presents a model of two production sectors with one occupation in each (labelled model II), while Section 5 presents a model of two production sectors with two occupations in each (labelled model III). Some concluding remarks are presented in Section 6.

2. *THE CHOICE OF OCCUPATION*

A basic model of choice of occupation is an important ingredient of the analyses to be carried out in this paper. There are two occupations, labelled 1 and 2. Let N_1 and N_2 denote the number of workers in the respective occupations. The total number of workers is exogenous and equal to the number of consumers, while the distribution of workers between occupations is endogenous. The size of the work force is normalized at unity so that $N_1 + N_2 = 1$.

The workers differ with respect to some sort of ability measure, a, which is taken to be non-negative. There is a continuous distribution of workers with respect to a described by the density function f(a). In this respect the set-up is inspired by the Mirrlees model where people have different productive skills (see Mirrlees (1971)). But whereas productive efforts of people with unequal abilities are perfect substitutes in the Mirrlees model, implying that relative wages reflect relative exogenous abilities, quite different implications of the ability structure are assumed in the present analysis.

We shall assume that occupation 1 is the more demanding occupation, and that it is too demanding to the workers of lowest ability. Workers of somewhat higher ability may be able to be in occupation 1 with strong enough efforts, but will prefer the less demanding occupation 2 unless they are compensated by a sufficiently higher wage rate in occupation 1. A person will find occupation 1 less demanding the higher ability he has got, and may even find it the more attractive alternative at a sufficiently high level of ability.

It is important to note that there may well be people who are skilful enough to enjoy a very demanding job as well as people who are unfit for the demanding job. The crucial assumption is that there is a marginal group of workers who must have an economic compensation to be attracted to the more demanding job.

The choice problem of an individual is to choose occupation and consumption bundle at given prices, wages, ability and tax policy. Let I denote after-tax income in occupation j and p the vector of commodity prices. The utility of an individual in a given occupation as expressed by the indirect utility function is

$$(2.1) \qquad V(p,I_j,a,j), \qquad j = 1,2.$$

Occupation is chosen by picking j so as to maximise V.

Let x_{ij} denote the amount of commodity i purchased by a person choosing occupation j. The demand functions are written as

$$(2.2) \qquad x_{ij} = x_{ij}(p,I_j,a), \qquad i,j = 1,2.$$

We know from duality theory that

$$(2.3) \qquad x_{ij} = \frac{-1}{\lambda_j} \partial V(p,I_j,a,j)/\partial p_i,$$

where $\lambda_j = \partial V(p,I_j,a,j)/\partial I_j$.

We shall assume that the market situation is such that a higher wage (w) is paid in occupation 1 in order to attract workers to the more demanding job: $w_1 > w_2$ and $I_1 > I_2$. Then at some ability level $\bar{a}$ workers are indifferent between the two occupations:

$$(2.4) \qquad V(p,I_1,\bar{a},1) = V(p,I_2,\bar{a},2).$$

Those with higher ability choose occupation 1, while those with lower ability choose occupation 2. The amount of labour in occupation 1 equals

(2.5) $$N_1 = \int_{\bar{a}}^{\infty} f(a)da.$$

The amount of labour in occupation 2 equals

(2.6) $$N_2 = \int_{0}^{\bar{a}} f(a)da.$$

(2.4)- (2.6) thus describe the supply side of the labour market. If $\bar{a}$ is eliminated from this relation system we get

(2.7) $$N_1 = N_1(p,I_1,I_2)$$

and

(2.8) $$N_2 = N_2(p,I_1,I_2).$$

We can then analyse the responses of labour supply to price and income changes. Since $N_2 = 1-N_1$, it will be sufficient to study the effects on N_1. Let us define

(2.9) $$D(a) = V(p,I_1,a,1) - V(p,I_2,a,2).$$

From the assumptions made above

$$D(a) < 0 \text{ for } a < \bar{a},$$

$$D(\bar{a}) = 0,$$

and

$$D(a) > 0 \text{ for } a > \bar{a}.$$

Hence

(2.10) $$D'(\bar{a}) > 0.$$

Differentiating

(2.11) $$D(\bar{a}) = V(p,I_1,\bar{a},1) - V(p,I_2,\bar{a},2) = 0$$

we find that

$$(2.12) \qquad \frac{\partial\bar{a}}{\partial I_1} = - \frac{\lambda_1^o}{D'} ,$$

$$(2.13) \qquad \frac{\partial\bar{a}}{\partial I_2} = \frac{\lambda_2^o}{D'} ,$$

$$(2.14) \qquad \frac{\partial\bar{a}}{\partial p_i} = \frac{\lambda_1^o x_{i1}^o - \lambda_2^o x_{i2}^o}{D'} ,$$

where $\lambda_1^o = \partial V(p,I_1,\bar{a},1)/\partial I_1$, $\lambda_2^o = \partial V(p,I_2,\bar{a},2)/\partial I_2$, $x_{i1}^o = x_{i1}(p,I_1,\bar{a})$ and $x_{i2}^o = x_{i2}(p,I_2,\bar{a})$.

We see that

$$(2.15) \qquad \frac{\partial\bar{a}}{\partial p_i} = - x_{i1}^o \frac{\partial\bar{a}}{\partial I_1} - x_{i2}^o \frac{\partial\bar{a}}{\partial I_2} .$$

Making use of (2.5) and (2.12)-(2.14) we see that

$$(2.16) \qquad \frac{\partial N_1}{\partial I_1} = \frac{\lambda_1^o}{D'} f(\bar{a})$$

$$(2.17) \qquad \frac{\partial N_2}{\partial I_2} = \frac{\lambda_2^o}{D'} f(\bar{a})$$

$$(2.18) \qquad \frac{\partial N_1}{\partial p_i} = - x_{i1}^o \frac{\partial N_1}{\partial I_1} - x_{i2}^o \frac{\partial N_1}{\partial I_2}$$

We can notice the rather obvious result that a higher income in one occupation makes that occupation more attractive and the alternative less attractive. A higher price on a commodity has a far more uncertain effect on the choice of occupation. It tends to reduce the real income in both states, and by how much depends on the consumption in both states of the commodity which becomes more expensive. If the consumption depends only on income and not on the consumer's occupation as such, and the commodity is non-inferior, then $x_{i1}^o > x_{i2}^o$. In that case the loss of real income is larger for a person in occupation 1 than in occupation 2. On the other hand the effect of a one unit income loss in occupation 2 may well have a larger effect than a one unit income loss in occupation 2. To see this, assume that people become more inclined to choose the less demanding job if the income level rises without changing income differentials. This will happen if $\partial N_1/\partial I_1 + \partial N_1/\partial I_2 < 0$ which implies that $-\partial N_1/\partial I_2 > \partial N_1/\partial I_1$. Hence the ultimate price effect is likely to be ambiguous.

Before the model is completed we have to describe the tax policy. The government is assumed to impose a linear tax function

$$T(w) = tw - b \tag{2.19}$$

where $0 < t < 1$ is the marginal tax rate and $-b$ is a headtax which is negative if $b > 0$. It follows straight away that

$$I_1 = (1-t)w_1 + b \tag{2.20}$$

and

$$I_2 = (1-t)w_2 + b. \tag{2.21}$$

We can then write

$$N_1 = N_1(p,(1-t)w_1 + b,\ (1-t)w_2 + b), \tag{2.22}$$

and we see straight away that

$$\frac{\partial N_1}{\partial w_1} = (1-t)\,\frac{\partial N_1}{\partial I_1}\ , \tag{2.23}$$

$$\frac{\partial N_1}{\partial w_2} = (1-t)\,\frac{\partial N_1}{\partial I_2}\ , \tag{2.24}$$

$$\frac{\partial N_1}{\partial t} = -\,w_1\,\frac{\partial N_1}{\partial I_1} - w_2\,\frac{\partial N_1}{\partial I_2}\ , \tag{2.25}$$

$$\frac{\partial N_1}{\partial b} = \frac{\partial N_1}{\partial I_1} + \frac{\partial N_1}{\partial I_2}\ . \tag{2.26}$$

In order to add some practical flavour to the theoretical choice of occupation model, we may briefly review some relevant labour market observations. A number of jobs are obviously tough, maybe risky, require hard work, offer bad working conditions, imply long spells of separation from families etc., but are in return well paid. Oil drilling in the North Sea and several kinds of construction work may provide good examples. Managers of big firms are extremely well paid to take heavy responsibility and become nearly full-time servants of their firms. Liberally interpreted we may also think of necessary efforts in taking education and acquiring special competence as the features making certain jobs more than normally demanding and more than normally well paid. These are all examples of characteristics captured by the theoretical model.

On the other hand there are occupations which do not fit well into the model. An obvious objection is that there are jobs which are both unattractive (for instance tedious or physically strenuous) and offering a rather low pay. This phenomenon may be explained by lack of workers in these occupations who are able to compete for more attractive jobs. Moreover, there are people who are highly paid for other reasons than the need to attract marginal workers to the occupation in question. There are occupations with barriers to entry and other departures from competitive conditions. In contrast to the assumptions of the present model there are occupations where people differ in their performances, and differences in quality or efficiency are being screened and reflected in wages. Conspicuous examples are people with talents as artists, professional sportsmen, etc.[4)]

3. *MODEL I: ONE PRODUCTION SECTOR WITH TWO OCCUPATIONS*

3.1 *The model*

We shall assume that only one (aggregate) commodity is produced. The output, x, is related to the amount of labour in two occupations by the macro production function

$$(3.1) \qquad x = F(N_1, N_2),$$

which is assumed to be homogeneous of degree one. As assumed already

$N_1 + N_2 = 1$. The output is used as numeraire. The wage per unit of N_1 is w_1, and the wage per unit of N_2 is w_2.

The condition that production takes place on the expansion path is, of course, that

$$(3.2) \qquad \frac{F_1}{F_2} = \frac{w_1}{w_2},$$

where $F_i = \partial F/\partial N_i$, $i = 1,2$. If we write the unit cost function as $c(w_1,w_2)$, we also know that in equilibrium

$$(3.3) \qquad c(w_1,w_2) = 1.$$

It is a well-known property of the cost function that

$$(3.4) \qquad \partial c/\partial w_i = N_i/x, \qquad i = 1,2.$$

Hence from (3.3)

$$(3.5) \qquad \frac{dw_2}{dw_1} = -\frac{N_1}{N_2}.$$

Equations (3.2) and (3.3) are equivalent to the conditions

$$(3.6) \qquad w_1 = F_1(N_1,1-N_1),$$

$$(3.7) \qquad w_2 = F_2(N_1,1-N_1).$$

Let us then turn to the supply side of the labour market. As special cases of the relations (2.1), (2.7) and (2.8) of the previous section, we now get

$$(3.8) \qquad V(I_i,a,i),$$

$$(3.9) \qquad N_1 = N_1(I_1,I_2),$$

$$(3.10) \qquad N_2 = N_2(I_1,I_2).$$

The budget constraint of the government is

$$(3.11) \qquad w_1N_1t + w_2N_2t = b + x_g,$$

where x_g is exogenous government expenditure. When t is determined by the government, the equilibrium values of w_1,w_2,N_1,N_2,x and b are determined by the relations

$$(3.12) \qquad x = F(N_1,N_2),$$

$$(3.13) \qquad \frac{F_1}{F_2} = \frac{w_1}{w_2},$$

$$(3.14) \qquad c(w_1,w_2) = 1,$$

$$(3.15) \qquad N_1 = N_1((1-t)w_1 + b,\ (1-t)w_2 + b),$$

$$(3.16) \qquad N_2 = 1-N_1,$$

$$(3.17) \qquad b = w_1N_1t + w_2N_2t - x_g,$$

which form the complete equation system of the model. All equilibrium values are then functions of t.

In the previous section we discussed at a rather abstract level the distinction between the direct effects of a change in tax policy and the effects through induced price changes. Considering the equation system above, we see that by using eq. (3.12) and the equations (3.15)-(3.17) we can express all quantities as functions of t,w_1 and w_2. Also eliminating w_2 by means of equation (3.14), we are left with only one wage variable. We can then distinguish between the partial effects of changes in t and changes in w_1, respectively, as described in the introductory section. We now adopt this approach and treat b,N_1,N_2 and x as functions of t and w_1, keeping in mind that w_1 is ultimately a function of t at the general equilibrium which is determined when the expansion path condition (3.13) is also taken into account.[5)]

It is useful to derive the partial effects of t and w_1 on b. Let the partial derivatives be denoted by b_t and b_w, respectively. From (3.15) -

(3.17) we obtain

$$b_t = \frac{1}{n}\left[w_1N_1 + w_2N_2 + t(w_1-w_2)\frac{\partial N_1}{\partial t}\right], \tag{3.18}$$

where

$$n = 1 - t(w_1-w_2)\frac{\partial N_1}{\partial b} = 1 - t(w_1-w_2)\left(\frac{\partial N_1}{\partial I_1} + \frac{\partial N_1}{\partial I_2}\right). \tag{3.19}$$

We assume that $n > 0$ which rules out the possibility that a one unit lump sum transfer from the government generates a more than one unit increase in tax revenue.

Similarly we find that

$$b_w = \frac{1}{n}\left[t(w_1-w_2)\left(\frac{\partial N_1}{\partial w_1} - \frac{N_1}{N_2}\frac{\partial N_1}{\partial w_2}\right)\right]. \tag{3.20}$$

Making use of (2.23) - (2.25) we find that

$$b_t = \frac{1}{n}\left[w_1N_1 + w_2N_2 - t(w_1-w_2)w_1\frac{\partial N_1}{\partial I_1} - t(w_1-w_2)w_2\frac{\partial N_1}{\partial I_2}\right] \tag{3.21}$$

and

$$b_w = \frac{1}{n}\left[t(w_1-w_2)(1-t)\frac{\partial N_1}{\partial I_1} - t(w_1-w_2)(1-t)\frac{N_1}{N_2}\frac{\partial N_1}{\partial I_2}\right]. \tag{3.22}$$

3.2 Incidence effects of more progressive taxation

Let us assume that from some arbitrary starting point where $0 < t < 1$ the tax policy is made slightly more progressive by raising t a little and adjusting b accordingly. We shall consider the gain in disposable income, g_i, obtained by the workers in each occupation; $i = 1,2$. The gain g_i is made up of the direct effect of the change in t, denoted by g_{it}, and the consequence of the general equilibrium effects on wages, denoted by g_{iw}.

Hence, recalling (2.20) and (2.21),

$$(3.23) \qquad g_1 = g_{1t} + g_{1w} = (-w_1+b_t) + (1-t)\, w_1' + b_w w_1'$$

where $w_1' = dw_1/dt$.

$$(3.24) \qquad g_2 = g_{2t} + g_{2w} = (-w_2+b_t) + (-(1-t)\, \frac{N_1}{N_2}\, w_1' + b_w w_1').$$

We make use of (3.16), (3.19) and (3.21) and find that

$$(3.25) \qquad g_{1t} = \frac{1}{n}\Big[-w_1 + w_1 t(w_1-w_2)\, \frac{\partial N_1}{\partial I_1} + w_1 t(w_1-w_2)\, \frac{\partial N_1}{\partial I_2}$$

$$+ w_1 N_1 + w_2 N_2 - t(w_1-w_2) w_1\, \frac{\partial N_1}{\partial I_1} - t(w_1-w_2) w_2\, \frac{\partial N_1}{\partial I_2}\Big]$$

$$= \frac{1}{n}(w_1-w_2)\left(-N_2 + t(w_1-w_2)\, \frac{\partial N_1}{\partial I_2}\right).$$

The sign is obviously negative. Hence we have got:

Conclusion 1: A higher marginal tax rate has a negative direct effect on the disposable income of the high-wage group.

We make use of (3.16), (3.19) and (3.22) and find that

$$(3.26) \qquad g_{1w} = \frac{1}{n}\Big[1-t-t(w_1-w_2)\, \frac{\partial N_1}{\partial I_1}\, (1-t)$$

$$-t(w_1-w_2)\, \frac{\partial N_1}{\partial I_2}\, (1-t) + t(w_1-w_2)(1-t)\, \frac{\partial N_1}{\partial I_1}$$

$$-t(w_1-w_2)(1-t)\, \frac{N_1}{N_2}\, \frac{\partial N_1}{\partial I_2}\Big] w_1'$$

$$= -\frac{1-t}{nN_2}\left(-N_2+t(w_1-w_2)\, \frac{\partial N_1}{\partial I_2}\right) w_1' .$$

In a quite analogous manner we find that

$$(3.27) \qquad g_{2t} = \frac{1}{n}(w_1-w_2)\left(N_1-t(w_1-w_2)\, \frac{\partial N_1}{\partial I_1}\right)$$

and

$$g_{2w} = -\frac{1-t}{nN_2}\left(N_1 - t(w_1-w_2)\frac{\partial N_1}{\partial I_1}\right) w_1'. \tag{3.28}$$

Taking a closer look at g_1, we see that

$$\begin{aligned} g_1 &= \frac{1}{n}\left(-N_2 + t(w_1-w_2)\frac{\partial N_1}{\partial I_2}\right)\left(w_1 - w_2 - (1-t)w_1' - (1-t)\frac{N_1}{N_2}w_1'\right) \\ &= \frac{1}{n}\left(-N_2 + t(w_1-w_2)\frac{\partial N_1}{\partial I_2}\right)(g_2 - g_1). \end{aligned} \tag{3.29}$$

From our general knowledge of tax distortions we expect a more progressive tax to cause a loss of allocative efficiency, and we do not expect that g_1 and g_2 both can be positive. Another question is whether a higher marginal tax rate may create general equilibrium effects which outweigh the direct effect on the disposable income of the high-wage group. Let us now look into these matters. We let a prime indicate total derivatives with respect to t. Since total production equals private disposable income plus the government's share we can write

$$x = ((1-t)w_1+b)N_1 + ((1-t)w_2+b)N_2 + x_g. \tag{3.30}$$

Differentiating we get

$$\begin{aligned} x' = F_1N_1' + F_2N_2' &= ((1-t)w_1+b)N_1' + ((1-t)w_2+b)N_2' \\ &+ N_1(1-t)w_1' + N_2(1-t)w_2' - w_1N_1 - w_2N_2 + b'N_1 + b'N_2, \end{aligned}$$

where the production function has been used. Since $F_1=w_1$, $F_2=w_2$ and $N_2' = -N_1'$, we further get

$$x' = (w_1-w_2)N_1' = (1-t)(w_1-w_2)N_1' + g_1N_1 + g_2N_2. \tag{3.31}$$

One more manipulation gives us

$$g_1N_1 + g_2N_2 = t(w_1-w_2)N_1'. \tag{3.32}$$

We also observe that

$$g_1 - g_2 = -w_1 + (1-t)w_1' + w_2 - (1-t)w_2',$$

and since $w_2' = -w_1' N_1/N_2$,

$$(3.33) \qquad g_1 - g_2 = -(w_1 - w_2) + (1-t)w_1'/N_2.$$

Let us hypothetically assume that g_1 were positive. Then it would follow from (3.29) that $g_1 - g_2 > 0$, and from (3.33) that w_1 would have to increase since $w_1 > w_2$. But then it would follow from the production side that N_1 would be reduced. According to (3.32) this could only happen if $g_2 < 0$. But if g_1 were positive and g_2 negative, more labour would definitely move into occupation 1, while on the production side labour belonging to occupation 1 would be substituted by labour from occupation 2. So adjustments on the supply side and the demand side of the labour market would not match, and these effects would not be compatible with the maintenance of a market equilibrium. Thus we can rule of the possibility that $g_1 > 0$, and we can state:

Conclusion 2: A higher marginal tax rate has a negative total effect on the disposable income of the high-wage group.

We see that the direct effect and the total effect have the same sign and general equilibrium effects on wages do not outweigh the direct effect.

We know that when $g_1 < 0$ and $g_2 > 0$, labour will move from occupation 1 to occupation 2 so that $N_1' < 0$. We also see from (3.32) that this would be true also if g_1 and g_2 were both negative. Hence we can state:

Conclusion 3: A higher marginal tax rate will always cause a movement of labour from the high-wage occupation to the low-wage occupation.

Such a movement will only be consistent with necessary adjustments on the production side if the high wage increases. So we can state:

Conclusion 4: A higher marginal tax rate will lead to an increase in the high wage and a lowering of the low wage. So induced general equilibrium effects on wages will to some extent, but not entirely, offset the direct effect on the disposable income of the high-wage group.

Since more workers move from the high-wage occupation to the low-wage occupation, it follows from (3.31) that the total income will fall. This confirms our expectation that a higher tax progressivity entails a loss of allocative efficiency.

Conclusion 5: A higher marginal tax rate tends to lower the total income of society.

3.3 *Welfare effects of tax policy*

The social welfare function is written as

$$(3.34)\qquad W = \int_{\bar{a}}^{\infty} V((1-t)w_1 + b,a,1)f(a)da + \int_{0}^{\bar{a}} V((1-t)w_2 + b,a,2)f(a)da.$$

As usual when adopting an additive welfare function the cardinalisation of V is assumed to reflect the distributional values of the government. Differentiating with respect to t we find that

$$(3.35)\qquad W' = \int_{\bar{a}}^{\infty} \lambda_1 g_1 f(a)da + \int_{0}^{\bar{a}} \lambda_2 g_2 f(a)da + (V(I_2,\bar{a},2) - V(I_1,\bar{a},1))\bar{a}'f(\bar{a}).$$

Due to (2.4) the last term vanishes. Let us define

$$(3.36)\qquad \bar{\lambda}_1 = \frac{1}{N_1} \int_{\bar{a}}^{\infty} \lambda_1 f(a)da,$$

$$(3.37)\qquad \bar{\lambda}_2 = \frac{1}{N_2} \int_{0}^{\bar{a}} \lambda_2 f(a)da$$

which are average marginal utilities of income in the two categories of consumers. We can then write

$$(3.38)\qquad W' = \bar{\lambda}_1 N_1 g_1 + \bar{\lambda}_2 N_2 g_2.$$

Substituting from (3.25) - (3.28) we get

$$(3.39)\qquad W' = \bar{\lambda}_1 N_1 \frac{1}{n}(w_1-w_2)(-N_2+t(w_1-w_2)\frac{\partial N_1}{\partial I_2})$$

$$+ \bar{\lambda}_1 N_1 \frac{-(1-t)}{nN_2}(-N_2+t(w_1-w_2)\frac{\partial N_1}{\partial I_2})\, w_1'$$

$$+ \bar{\lambda}_2 N_2 \frac{1}{n}(w_1-w_2)(N_1-t(w_1-w_2)\frac{\partial N_1}{\partial I_1})$$

$$+ \bar{\lambda}_2 N_2 \frac{-(1-t)}{nN_2}(N_1-t(w_1-w_2)\frac{\partial N_1}{\partial I_1})\, w_1'$$

$$= \frac{1}{n}\left[\bar{\lambda}_1 N_1(-N_2+t(w_1-w_2)\frac{\partial N_1}{\partial I_2}) + \bar{\lambda}_2 N_2(N_1-t(w_1-w_2)\frac{\partial N_1}{\partial I_1})\right](w_1-w_2-(1-t)w_1'/N$$

$$(3.40)\qquad W' = k(w_1-w_2-(1-t)w_1'/N_2) = k(g_2-g_1)$$

where k is the expression in square brackets divided by n.

Suppose that in some arbitrary tax situation a small adjustment of the tax parameters is considered, and we want to know whether it will increase social welfare or not. An important question is then whether general equilibrium effects on wages can be neglected. Since w_1' does not appear in the expression for k, the sign of W' can obviously be determined without knowing w_1' if

$$(3.41)\qquad \mathrm{sgn}(w_1-w_2) = \mathrm{sgn}(g_2-g_1).$$

We know from the previous sub-section that $w_1 > w_2$ implies that $g_1 < 0$, and from (3.29) we then see that $g_2 > g_1$, and (3.41) is satisfied. We can therefore state:

Proposition 3.1: Induced wage changes can be neglected when piecemeal tax revisions are considered.

Let us assume that the tax parameters are determined by a step-wise procedure which stops when a marginal tax adjustment has got no further effect on welfare when possible wage effects of that marginal adjustment are ignored. Hence at each step wages are assumed to be fixed at the values *actually* resulting from the former step. Since in practice most changes in tax policy are small adjustments, and also rather infrequent major revisions are followed up by small adjustments, this seems to be a relevant and interesting case. Now it is interesting to notice that if the tax policy has been worked out by this procedure, the discovery that tax incidence through wage changes have been neglected makes no difference in our model. The optimum attained when neglecting wage changes at the margin is also the true optimum.

Proposition 3.2: A welfare maximum when wage changes are neglected at the margin, is also a true welfare maximum.

3.4 Further interpretations

Define

(3.42) $$k_1 = (-N_2 + t(w_1-w_2)\frac{\partial N_1}{\partial I_2})$$

and

(3.43) $$k_2 = (N_1 - t(w_1-w_2)\frac{\partial N_1}{\partial I_1}).$$

Our findings in the formulae (3.25) - (3.28) imply that

(3.44) $$g_{1t} = \frac{1}{n}(w_1-w_2)k_1,$$

(3.45) $$g_{1w} = \frac{-1}{n}\frac{1-t}{N_2}k_1w_1',$$

(3.46) $$g_{2t} = \frac{1}{n}(w_1-w_2)k_2,$$

(3.47) $$g_{2w} = \frac{-1}{n}\frac{1-t}{N_2}k_2w_1'.$$

Hence

(3.48) $$\frac{g_{1w}}{g_{1t}} = \frac{g_{2w}}{g_{2t}}$$

and .

$$(3.49) \qquad \frac{g_{1t}}{g_{2t}} = \frac{g_{1w}}{g_{2w}} = \frac{g_{1t}+g_{1w}}{g_{2t}+g_{2w}} = \frac{g_1}{g_2}$$

The optimum condition

$$W' = \bar{\lambda}_1 N_1 g_1 + \bar{\lambda}_2 N_2 g_2 = 0$$

is equivalent to

$$(3.50) \qquad \frac{N_1 g_1}{N_2 g_2} = -\frac{\bar{\lambda}_2}{\bar{\lambda}_1}$$

But since $g_1/g_2 = g_{1t}/g_{2t}$, we may as well neglect the marginal wage changes. The fact is that the marginal change in wage income will change the net gains of both categories of workers by the same percentage. Hence it is immaterial when choosing the optimum trade-off between income taken from the one category and income given to the other category.

To give an example, let us assume that, neglecting marginal changes in wage-rates, an optimum has been established where the social marginal utility of income assigned to an average person in occupation 2 is twice that assigned to an average person in occupation 1. It is then implied that in order to transfer one income unit to persons in occupation 2, persons in occupation 1 will have to give up 2 units of income. Since a weight equal to 2 is assigned to the one unit received, and a weight 1 is assigned to the two units foregone, the welfare effects just cancel out as they must do at the optimum.

Suppose that we are now told that wage rates do in fact change, and, as a consequence, both income effects are in fact only half of what they were assumed to be. We then realize that it will still be true that one income unit for the benefit persons in occupation 2 can only be raised if persons in occupation 1 are deprived of two income units. Hence the same trade-off is retained, and there is no reason to choose a different policy.

If at some point we have found that

$$\bar{\lambda}_1 N_1 g_{1t} + \bar{\lambda}_2 N_2 g_{2t} > 0 \quad (\text{or} < 0),$$

the information that induced wage changes do occur, implies that g_{1t} and g_{2t} should be multiplied by the same positive figure, which makes no difference to the inequality. If, say, nine income units to one group is more highly valued than three income units to the other group, six income units to the former are also more highly valued than two income units to the latter.

Arguments related to tax shifting are also used in practical policy discussion. One argument sometimes heard is that the presumably favourable distributional effects of a marginal increase in income tax progressivity are likely to be small because the extra burden first imposed on those who are better off, are to a large extent shifted on to the people who are less well off through the adjustment process of the market. Under this assumption it is sometimes argued that the negative effects on efficiency of an increase in progressivity tend to dominate the distributional impact. But this may be a too hasty conclusion. The reason is that the effect on efficiency also tends to be mitigated by the tax shifting. In the present model a higher marginal tax tends to deter people from seeking the more demanding job and thus entails a loss of efficiency. But this effect is clearly modified by the induced rise in the pretax wage received by workers doing the more demanding job. In fact both effects are modified by the same relative amount. Hence the partially dominant effect is never overthrown.

Distinctions between the effects which allow for induced wage changes and the effects which are generated in the absence of general equilibrium effects, must be interpreted with caution. Since, in general equilibrium, wages are in fact endogenous, the case of no induced wage changes represents a partial view, and can only be defined by cancelling some relation of the general equilibrium system. In the current analysis the relation to be suppressed is the expansion path condition. This has allowed us to carry out a partial study of local labour supply side reactions for fixed wages to be compared with the complete general equilibrium effects.

4. MODEL II: TWO PRODUCTION SECTORS WITH ONE OCCUPATION IN EACH

4.1 The model

In model I only one commodity was produced which implied that we barred ourselves from discussing effects on relative commodity prices. In model II we assume that one commodity is produced in a sector using labour from occupation 1 and a different commodity is produced in another sector using labour from occupation 2. In order to keep down the number of complications a very simple production structure will be assumed. Labour is the only factor of production. Let N_1 and N_2 denote the number of workers in sector 1 and sector 2 respectively, and let x_1 and x_2 be the corresponding output levels. Each worker in a sector is assumed to do some specified job which is exogenously given. The production volume in each sector is assumed to be proportional to the number of workers, and the units of measurement are chosen so as to make the factor of proportionality equal to unity. The production functions of the two sectors are then

$$x_1 = N_1 \tag{4.1}$$

$$x_2 = N_2 \tag{4.2}$$

Let p_1, p_2 denote the product prices and w_1, w_2 the wage rates. It follows immediately from (4.1) and (4.2) that in a competitive market equilibrium

$$w_1 = p_1 \tag{4.3}$$

$$w_2 = p_2 \tag{4.4}$$

We choose commodity 2 as the numeraire so that

$$w_2 = p_2 = 1$$

The amount of each commodity purchased for government purposes is assumed to be exogenously given equal to x_1^g and x_2^g respectively.

At market equilibrium the following relations must also hold

$$x_1 = x_1^g + \int_{\bar{a}}^{\infty} x_{11}(p_1, I_1, a, 1) f(a)da + \int_0^{\bar{a}} x_{12}(p_1, I_2, a, 2) f(a)da \tag{4.5}$$

$$(4.6)\qquad x_2 = x_2^g + \int_{\bar{a}}^{\infty} x_{21}(p_1, I_1, a, 1) f(a) da + \int_0^{\bar{a}} x_{22}(p_1, I_2, a, 2) f(a) da$$

$$(4.7)\qquad tw_1N_1 + tw_2N_2 - b = p_1x_1^g + p_2x_2^g$$

$$(4.8)\qquad x_1 = N_1(p_1, I_1, I_2)$$

$$(4.9)\qquad x_2 = N_2(p_1, I_1, I_2)$$

$$(4.10)\qquad V(p_1, I_1, \bar{a}, 1) = V(p_1, I_2, \bar{a}, 2)$$

$$(4.11)\qquad I_1 = (1-t)w_1 + b$$

$$(4.12)\qquad I_2 = (1-t)w_2 + b$$

One relation can be eliminated by Walras' law. We choose to eliminate (4.6). There are 10 endogenous variables: $p_1, w_1, b, I_1, I_2, \bar{a}, N_1, N_2, x_1$ and x_2. To determine the equilibrium values of these variables we have got the 10 relations (4.1), (4.2), (4.3), (4.5) and (4.7)-(4.12). $w_2 = p_2 = 1$ by (4.4) and the choice of numeraire, and x_1^g, x_2^g and t are determined by the government.

4.2 *Income tax analysis*

Let us now examine the welfare effects of changing the marginal tax rate t. We shall stick to an additive welfare function which is written as

$$(4.13)\qquad W = \int_{\bar{a}}^{\infty} V(p_1, (1-t)w_1 + b, a, 1) f(a)\, da + \int_0^{\bar{a}} V(p_1, (1-t)w_2 + b, a, 2) f(a) da$$

Let us define

$$w_1' = \frac{dw_1}{dt} = \frac{dp_1}{dt}, \quad b' = \frac{db}{dt} \quad \text{and} \quad \bar{a}' = \frac{d\bar{a}}{dt}$$

We can then derive

$$(4.14) \qquad \frac{dW}{dt} = \int_{\bar{a}}^{\infty} -\lambda_1 x_{11} w_1' f(a)da - \int_0^{\bar{a}} \lambda_2 x_{12} w_1' f(a)da$$

$$- \int_{\bar{a}}^{\infty} \lambda_1 w_1 f(a)da - \int_0^{\bar{a}} \lambda_2 w_2 f(a)da + \int_{\bar{a}}^{\infty} \lambda_1 (1-t) w_1' f(a)da$$

$$+ \int_{\bar{a}}^{\infty} \lambda_1 b' f(a)da + \int_0^{\bar{a}} \lambda_2 b' f(a)da - D(\bar{a})\bar{a}'$$

$D(\bar{a}) = 0$ and can be eliminated. Let us now make a further simplification by assuming that the quantities consumed only depend on income and prices and not on the consumer's occupation. Then define

$$(4.15) \qquad \bar{\lambda}_1 = \frac{1}{N_1} \int_{\bar{a}}^{\infty} \lambda_1 f(a)da$$

$$(4.16) \qquad \bar{\lambda}_1 = \frac{1}{N_2} \int_0^{\bar{a}} \lambda_2 f(a)da$$

which are average marginal utilities of income in the two categories of consumers. (4.14) can then be rewritten as

$$(4.17) \qquad \frac{dW}{dt} = \bar{\lambda}_1((1-t) - x_{11})w_1'N_1 - \bar{\lambda}_2 x_{12} w_1' N_2$$

$$- \bar{\lambda}_1 N_1 w_1 - \bar{\lambda}_2 N_2 w_2 + (N_1\bar{\lambda}_1 + N_2\bar{\lambda}_2)(b_t + b_w)$$

Through (4.1) - (4.4) and (4.7) - (4.12) b, I_1, I_2, $\bar{a}$, N_1, N_2, x_1 and x_2 are implicitly defined as functions of t and w_1. We use these functions to distinguish between direct effects of t and effects through prices. The equilibrium value of w_1 can ultimately be found by also invoking (4.5). As before the gain in real disposable income obtained by a worker in occupation i, g_i, is made up of a direct effect of t, g_{it}, and an effect from price changes, g_{iw}. Using (4.17) we can identify the effects as:

$$g_{1t} = -w_1 + b_t \tag{4.18}$$

$$g_{1w} = \left((1-t) - x_{11} + b_w\right)w_1' \tag{4.19}$$

$$g_{2t} = -w_2 + b_t \tag{4.20}$$

$$g_{2w} = (-x_{12} + b_w)w_1' \tag{4.21}$$

A change in w_1 and p_1 will affect real disposable income by changing the nominal income after tax, by changing the real value of income, and by changing the transfer payment. Making use of (4.7), (4.3) and the fact that $N_1+N_2 = 1$, we find that

$$b_t = \frac{1}{n}\left(w_1N_1 + w_2N_2 + t(w_1-w_2)\partial N_1/\partial t\right) \tag{4.22}$$

and

$$b_w = \frac{1}{n}\left(tN_1 + t(w_1-w_2)\partial N_1/\partial w_1 - x_1^g\right) \tag{4.23}$$

where

$$n = 1+t(w_1-w_2)\partial N_2/\partial b = 1-t(w_1-w_2)\left(\frac{\partial N_1}{\partial I_1} + \frac{\partial N_1}{\partial I_2}\right) \tag{4.24}$$

We assume that $n > 0$ which rules out the possibility that a one unit lump sum transfer from the government generates a more than one unit increase in tax revenue. From (4.5) and (4.8) we get

$$x_1^g = N_1 - N_1x_{11} - N_2x_{12} \tag{4.25}$$

Since $p_1 = w_1$ the effect of a change in w_1 on the labour supply is found by combining the partial effects established in (2.18) and (2.23). Hence

$$\frac{\partial N_1}{\partial w_1} = (1-t-x_{11})\,\frac{\partial N_1}{\partial I_1} - x_{12}\,\frac{\partial N_1}{\partial I_2}\,. \tag{4.26}$$

We know from (2.25) that

$$\frac{\partial N_1}{\partial t} = - w_1 \frac{\partial N_1}{\partial I_1} - w_2 \frac{\partial N_1}{\partial I_2}$$

Inserting these equations, we get from (4.18) and (4.22) that

$$(4.27) \qquad g_{1t} = \frac{-1}{n} (w_1 - w_2)\left(N_2 - t(w_1 - w_2) \frac{\partial N_1}{\partial I_2}\right)$$

and from (4.19) and (4.23) we obtain

$$(4.28) \qquad g_{1w} = \frac{1}{n} (1-t-x_{11}+x_{12})\left(N_2 - t(w_1 - w_2) \frac{\partial N_1}{\partial I_2}\right) w_1'$$

In a similar way we find that

$$(4.29) \qquad g_{2t} = \frac{-1}{n} (w_1 - w_2)\left(- N_1 + t(w_1 - w_2) \frac{\partial N_1}{\partial I_1}\right)$$

and

$$(4.30) \qquad g_{2w} = \frac{1}{n} (1-t-x_{11}+x_{12})\left(- N_1 + t(w_1 - w_2) \frac{\partial N_1}{\partial I_1}\right) w_1'.$$

Using these results we can write the total effect on welfare as

$$(4.31) \qquad W' = \frac{1}{n} \left[\bar{\lambda}_1 N_1 \left(N_2 - t(w_1 - w_2) \frac{\partial N_1}{\partial I_2}\right)\right.$$

$$\left. + \bar{\lambda}_2 N_2 \left(- N_1 + t(w_1 - w_2) \frac{\partial N_1}{\partial I_1}\right)\right] \left(- w_1 + w_2 + (1-t-x_{11}+x_{12}) w_1'\right)$$

$$= k\left(- w_1 + w_2 + (1-t-x_{11}+x_{12}) w_1'\right)$$

where k is the expression in square brackets divided by n. Since $g_1 = - w_1 + (1-t-x_{11}) w_1' + b'$ and $g_2 = - w_2 - x_{12} w_1' + b'$ we easily see that

(4.32) $$W' = k(g_1 - g_2)$$

Using (4.27) and (4.28) we find that

(4.33) $$g_1 = g_{1t} + g_{1w} =$$

$$\frac{1}{n}\left(N_2 - t(w_1 - w_2)\,\frac{\partial N_1}{\partial I_2}\right)\left(-\,w_1 + w_2 + (1-t-x_{11}+x_{12})w_1'\right)$$

$$= \frac{1}{n}\left(N_2 - t(w_1 - w_2)\,\frac{\partial N_1}{\partial I_2}\right)(g_1 - g_2)$$

from which it follows that

(4.34) $$\operatorname{sgn} g_1 = \operatorname{sgn}\,(g_1 - g_2)$$

From (4.31), (4.32) and (4.34) we see that price effects can be neglected at the margin if and only if $\operatorname{sgn}\,(w_2 - w_1) = \operatorname{sgn} g_1$. We have assumed that $w_2 - w_1 < 0$. The crucial question is then whether g_1 may become positive. From (4.27) and (4.28) follows that

(4.35) $$g_1 = \frac{-1}{w_1 - w_2}\left(-\,w_1 + w_2 + (1-t-x_{11}-x_{12})w_1'\right)g_{1t} = \alpha g_{1t}$$

where α is defined by the last equation. From (4.29) and (4.30) follows that

(4.36) $$g_2 = \alpha g_{2t}$$

We see from (4.27) that $g_{1t} < 0$. g_1 will then be negative if α is positive and positive if α is negative. Let us now recall the equilibrium condition of the market for commodity 1, (4.5), which we can rewrite as

(4.37) $$N_1 = x_1^g + N_1 x_{11}(p_1, I_1) + N_2 x_{12}(p_1, I_2)$$

We shall now assume that there is a stable equilibrium in the sense that the excess supply of commodity 1 is an increasing function of p_1 when all repercussions are taken into account. Let s_{11} denote the direct compensated price derivative of x_{11}, and s_{12} denote the direct compensated price derivative of x_{12}. With the ordinary assumptions these derivatives are negative. The assumption of increasing excess supply implies that

$$(4.38) \qquad \frac{\partial N_1}{\partial I_1}(1-x_{11}+x_{12})\frac{g_{1w}}{w_1'} + \frac{\partial N_1}{\partial I_2}(1-x_{11}+x_{12})\frac{g_{2w}}{w_1'}$$

$$- N_1\frac{\partial x_{11}}{\partial I_1}\ \frac{g_{1w}}{w_1'} - N_2\frac{\partial x_{12}}{\partial I_2}\ \frac{g_{2w}}{w_1'} - N_1 s_{11} - N_2 s_{12} > 0.$$

Let us then differentiate both sides of the equilibrium condition (4.37) with respect to t. We get

$$(4.39) \qquad \left(\frac{\partial N_1}{\partial I_1}(1-x_{11}+x_{12}) - N_1\frac{\partial x_{11}}{\partial I_1}\right)g_{1t}$$

$$+ \left(\frac{\partial N_1}{\partial I_2}(1-x_{11}+x_{12}) - N_2\frac{\partial x_{12}}{\partial I_2}\right)g_{2t}$$

$$+ \left(\frac{\partial N_1}{\partial I_1}(1-x_{11}+x_{12}) - N_1\frac{\partial x_{11}}{\partial I_1}\right)g_{1w}$$

$$+ \left(\frac{\partial N_1}{\partial I_2}(1-x_{11}+x_{12}) - N_2\frac{\partial x_{12}}{\partial I_2}\right)g_{2w}$$

$$+ (-N_1 s_{11} - N_2 s_{12})w_1' = 0$$

Let us define

$$(4.40) \qquad \beta_1 = \frac{\partial N_1}{\partial I_1}(1-x_{11}+x_{12}) - N_1\frac{\partial x_{11}}{\partial I_1}$$

(4.41) $$\beta_2 = \frac{\partial N_1}{\partial I_2}(1-x_{11}+x_{12}) - N_2 \frac{\partial x_{12}}{\partial I_2}$$

We can then write (4.39) as

(4.42) $$\beta_1 g_{1t} + \beta_2 g_{2t} + \beta_1 g_{1w} + \beta_2 g_{2w}$$

$$- (N_1 s_{11} + N_2 s_{12}) w_1' = 0$$

Combining (4.38) and (4.42) we see that

(4.43) $$\begin{cases} \beta_1 g_{1t} + \beta_2 g_{2t} < 0 \text{ if } w_1' > 0 \\ \beta_1 g_{1t} + \beta_2 g_{2t} > 0 \text{ if } w_1' < 0 \end{cases}$$

We can also write (4.39) as

(4.44) $$\beta_1 g_1 + \beta_2 g_2 - (N_1 s_{11} + N_2 s_{12}) w_1' = 0$$

If we invoke (4.35) and (4.36) we can rewrite the equation once more as

(4.45) $$\alpha(\beta_1 g_{1t} + \beta_2 g_{2t}) = (N_1 s_{11} + N_2 s_{12}) w_1'$$

Since the direct Slutsky derivatives are negative we can use (4.43) to conclude that α is always positive. Hence g_1 will always be negative, and we can conclude that price effects can be neglected at the margin.

Proposition 4.1: Induced price changes can be neglected at the margin when piecemeal tax revisions are considered.

From (4.34) we know that since g_1 is negative, g_1-g_2 is negative too.

(4.32) then tells us that the first order optimum tax condition is

$$W' = k = \bar{\lambda}_1 N_1\left(N_2 - t(w_1-w_2)\,\frac{\partial N_1}{\partial I_2}\right.$$

$$\left. + \bar{\lambda}_2 N_2 (-N_1 + t(w_1-w_2)\,\frac{\partial N_1}{\partial I_1}\right) = 0$$

4.3 *Should a commodity tax be imposed?*

Let us now examine how the introduction of a small commodity tax in addition to an optimum income tax would affect the welfare level. Let s be the tax per unit of x_1, and let P_1 denote the consumer price per unit of x_1. Obviously

$$(4.46) \qquad P_1 = p_1 + s$$

N_1 is then determined by the supply function

$$(4.47) \qquad N_1(w_1+s, (1-t)w_1+b, (1-t)w_2 + b).$$

The budget constraint of the government now becomes

$$(4.48) \qquad b = tw_1N_1 + tw_2N_2 + sx_1 - (p_1+s)x_1^g - p_2x_2^g$$

We keep in mind that $p_1=w_1$ and $p_2=1$. We differentiate at the point where s=0 to obtain

$$(4.49) \qquad b_s = t(w_1-w_2)\,\frac{\partial N_1}{\partial P_1} + t(w_1-w_2)\,\frac{\partial N_1}{\partial b}\,b_s$$

$$+t(w_1-w_2)\,\frac{\partial N_1}{\partial w_1}\,w_{1s} + tN_1w_{1s} + x_1 - x_1^g - x_1^g w_{1s}$$

where subscript s indicates a derivative with respect to s. Hence

$$(4.50) \qquad b_s = \frac{1}{n}[t(w_1-w_2)(-x_{11}\frac{\partial N_1}{\partial I_1} - x_{12}\frac{\partial N_1}{\partial I_2}) + x_1 - x_1^g]$$

$$+ \frac{1}{n}[t(w_1-w_2)(1-t-x_{11})\frac{\partial N_1}{\partial I_1} - t(w_1-w_2)x_{12}\frac{\partial N_1}{\partial I_2}$$

$$+ tN_1 - x_1^g]w_{1s}$$

Substituting from (4.5) and (4.8) we further obtain

$$(4.51) \qquad b_s = \frac{1}{n}[t(w_1-w_2)(-x_{11}\frac{\partial N_1}{\partial I_1} - x_{12}\frac{\partial N_1}{\partial I_2}) + N_1x_{11} + N_2x_{12}]$$

$$+ \frac{1}{n}[t(w_1-w_2)(1-t-x_{11})\frac{\partial N_1}{\partial I_1} - t(w_1-w_2)x_{12}\frac{\partial N_1}{\partial I_2}$$

$$-(1-t-x_{11})N_1 + N_2x_{12}]w_{1s}.$$

Invoking the Envelope Theorem we find the effect on welfare as

$$(4.52) \qquad W_s\Big|_{s=0} = \frac{1}{n}[-\bar{\lambda}_1N_1x_{11} - \bar{\lambda}_2N_2x_{12} + \bar{\lambda}_1N_1t(w_1-w_2)\frac{\partial N_1}{\partial I_1}x_{11}$$

$$+ \bar{\lambda}_1N_1x_{11}t(w_1-w_2)\frac{\partial N_1}{\partial I_2} + \bar{\lambda}_2N_2x_{12}t(w_1-w_2)\frac{\partial N_1}{\partial I_1}$$

$$+ \bar{\lambda}_2N_2x_{12}t(w_1-w_2)\frac{\partial N_1}{\partial I_2} - t(w_1-w_2)x_{11}\bar{\lambda}_1N_1\frac{\partial N_1}{\partial I_1}$$

$$- \bar{\lambda}_1N_1t(w_1-w_2)x_{12}\frac{\partial N_1}{\partial I_2} + \bar{\lambda}_1N_1x_{11}N_1 + \bar{\lambda}_1N_1x_{12}N_2$$

$$- \bar{\lambda}_2N_2t(w_1-w_2)x_{11}\frac{\partial N_1}{\partial I_1} - \bar{\lambda}_2N_2t(w_1-w_2)x_{12}\frac{\partial N_1}{\partial I_2}$$

$$+ \bar{\lambda}_2N_2N_1x_{11} + \bar{\lambda}_2N_2x_{12}N_2 + (1-t-x_{11})kw_{1s} + x_{12}kw_{1s}]$$

$$= \frac{1}{n}[-x_{11}(N_1\bar{\lambda}_1(N_2-t(w_1-w_2)\frac{\partial N_1}{\partial I_2}) + N_2\bar{\lambda}_2(-N_1+t(w_1-w_2)\frac{\partial N_1}{\partial I_1}))$$

$$+ x_{12}(N_1\bar{\lambda}_1(N_2-t(w_1-w_2)\frac{\partial N_1}{\partial I_2}) + N_2\bar{\lambda}_2(-N_1+t(w_1-w_2)\frac{\partial N_1}{\partial I_1}))$$

$$+ (1-t-x_{11})kw_{1s} + x_{12}kw_{1s}]$$

$$= \frac{1}{n}[-x_{11}k + x_{12}k + (1-t-x_{11})kw_{1s} + x_{12}kw_{1s}].$$

Since t is already at its optimum, we know that $k = 0$ and hence

$$(4.53) \qquad W_s \Big|_{s=0} = 0$$

Proposition 4.2: Provided that there is an optimum income tax, no commodity tax should be applied.

5. MODEL III: TWO PRODUCTION SECTORS WITH TWO OCCAPATIONS IN EACH.

5.1. The model.

There are two production sectors in the economy, each producing one commodity. The sectors are labelled 1 and 2 respectively, and x_1 and x_2 denote the respective production levels. Two kinds of labour are used in both sectors. Each type belongs to an occupation which from the workers' point of view is the same in both sectors. The two occupations are labelled 1 and 2. We let w_1 and w_2 denote the respective wage-rates of the two occupations. As before let N_1 be the number of workers in occupation 1 and N_2 be the number of workers in occupation 2, and the total workforce is set equal to unity. Let n_{ij} denote the number of workers in occupation j in sector i. Each sector has a production function which is assumed to be homogeneous of degree one. We let p_i denote the price per unit of x_i, and c_i denote the unit cost function of sector i; i = 1,2. The unit cost function is, of course, homogeneous of degree one in w_1 and w_2.

The production side of the economy is then described by the following relations in equilibrium:

(5.1) $$N_1 + N_2 = 1$$

(5.2) $$N_1 = n_{11} + n_{21}$$

(5.3) $$N_2 = n_{12} + n_{22}$$

(5.4) $$x_1 = x_1(n_{11},n_{12})$$

(5.5) $$x_2 = x_2(n_{21},n_{22})$$

(5.6) $$p_1 = c_1(w_1,w_2)$$

(5.7) $$p_2 = c_2(w_1,w_2)$$

(5.8) $$n_{11}/x_1 = c_{11}(w_1,w_2)$$

(5.9) $$n_{21}/x_2 = c_{21}(w_1,w_2)$$

where $c_{ij} = \partial c_i/\partial w_j$; $i,j = 1,2$. (5.1)-(5.3) are true by definition. (5.4) and (5.5) are the production functions. (5.6) and (5.7) are equilibrium conditions, and (5.8) and (5.9) are optimum conditions for factor intensities.

The workers choose occupation as previously described. The number of workers in occupation 1 then becomes a function of prices and after tax income levels in the two occupations .

(5.10) $$N_1 = N_1(p_1,p_2,(1-t)w_1 + b,(1-t)w_2 + b).$$

Commodity demand is assumed to depend on prices and disposable income only. Let x_{kh} denote the demand for commodity k from a person in occupation h; $k,h = 1,2$. x_k^g is the government demand for commodity k, $k = 1,2$. Market equilibrium are then ensured by the relations:

(5.11) $$x_1 = x_1^g + N_1x_{11}(p_1,p_2,(1-t)w_1 + b) + N_2x_{12}(p_1,p_2,(1-t)w_2 + b)$$

(5.12) $$x_2 = x_2^g + N_1x_{21}(p_1,p_2,(1-t)w_1 + b) + N_2x_{22}(p_1,p_2,(1-t)w_2 + b)$$

(5.13) $$b + p_1x_1^g + p_2x_2^g = tN_1w_1 + tN_2w_2.$$

We choose commodity 2 as numeraire so that

(5.14) $$p_2 = 1.$$

One of the relations (5.11) - (5.13) is redundant by Walras' law. The model then provides 12 independent relations to determine the equilibrium values of the 12 variables $N_1,N_2,n_{11},n_{12},n_{21},n_{22},x_1,x_2w_1,w_2,p_1$ and b when t is determined directly by the government.

Instead of (5.4) and (5.5) we could use

(5.15) $$n_{12}/x_1 = c_{12}(w_1,w_2)$$

(5.16) $$n_{22}/x_2 = c_{22}(w_1,w_2)$$

which can be derived by means of (5.8) and (5.9) and the fact that $w_1n_{11}/x_1 + w_2n_{12}/x_1 = w_1c_{11} + w_2c_{12}$ and $w_1n_{21}/x_2 + w_2n_{22}/x_2 = w_1c_{21} + w_2c_{22}$.

The prices of the model are related by the relations (5.6) and (5.7):

$$p_1 = c_1(w_1,w_2)$$

and

$$1 = c_2(w_1,w_2)$$

where p_2 has been set equal to unity. The two equations implicitly define w_1 and w_2 as functions of p_1. We can then analyse how a change in p_1 affects the wage-rates. Differentiating the system we find that

$$(5.17) \quad 1 = c_{11}\frac{dw_1}{dp_1} + c_{12}\frac{dw_2}{dp_1}$$

$$(5.18) \quad 0 = c_{21}\frac{dw_1}{dp_1} + c_{22}\frac{dw_2}{dp_1}$$

Let us define the determinant

$$(5.19) \quad A = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = c_{11}c_{22} - c_{12}c_{21}$$

$$= \frac{n_{11}}{x_1}\frac{n_{22}}{x_2} - \frac{n_{12}n_{21}}{x_1x_2} = \frac{n_{12}n_{22}}{x_1x_2}\left(\frac{n_{11}}{n_{12}} - \frac{n_{21}}{n_{22}}\right)$$

where (5.8), (5.9), (5.15) and (5.16) have been used. We choose the more intensive user of labour belonging to occupation 1 as sector 1. It then follows that $A > 0$. We easily find that

$$(5.20) \quad \frac{dw_1}{dp_1} = \frac{n_{22}}{Ax_2}$$

$$(5.21) \quad \frac{dw_2}{dp_1} = \frac{-n_{21}}{Ax_2}.$$

We may note that the production side of the model corresponds to the constant returns to scale, two-commodity, two-factor model which has received close attention in economic theory and in international trade theory in particular. The results in (5.20) and (5.21) are in fact the Stolper-Samuelson theorem.

5.2. Income tax analysis.

By means of the additive, indirect welfare function

$$(5.22) \qquad W = \int_{\bar{a}}^{\infty} V(p_1,(1-t)w_1 + b,a,1)f(a)da + \int_{0}^{\bar{a}} V(p_1,(1-t)w_2 + b,a,2)f(a)da$$

we can investigate the welfare effect of changing the marginal tax rate t. We let a prime indicate derivatives with respect to t. As before we can distinguish between the partial direct effects of t and the effects channelled through induced changes in equilibrium prices. More precisely, the relations (5.1) - (5.10) and (5.13) define all the other variables as functions of p_1 and t, and p_1 is in turn a function of t due to (5.11). The partial direct effects of t are expressed by the partial derivatives with respect to t. Since all other prices (wages) are functions of p_1 only, price changes can be represented by the change in p_1. Effects of price changes can therefore be expressed by partial derivatives with respect to p_1. A subscript t indicates direct effects, and a subscript p indicates effects of price changes. For instance $b_t = \partial b/\partial t$ and $b_p = \partial b/\partial p_1$.

The gain in terms of an increase in real disposable income obtained by a worker in occupation i, g_i, is made up of the direct effect of t, g_{it}, and the effect from price changes, g_{ip}. Differentiating the welfare function and making use of the average marginal utilities of income $\bar{\lambda} = \int_{\bar{a}}^{\infty} \lambda_1 f(a)da/N_1$ and $\bar{\lambda}_2 = \int_0^{\bar{a}} \lambda_2 f(a)da/N_2$, we can express the marginal effect of t on welfare by

$$(5.23) \qquad W' = dW/dt$$

$$= \bar{\lambda}_1 N_1(-x_{11}p_1' + (1-t)w_1' - w_1 + b_t + b_p p_1')$$

$$+ \bar{\lambda}_2 N_2(-x_{12}p_1' + (1-t)w_2' - w_2 + b_t + b_p p_1') - D(\bar{a})\ \bar{a}'$$

where the last term obviously vanishes due to (2.11). We obviously have that

(5.24) $$g_{1t} = - w_1 + b_t,$$

(5.25) $$g_{1p} = (-x_{11} + b_p)p_1' + (1-t)w_1',$$

(5.26) $$g_{2t} = - w_2 + b_t,$$

and

(5.27) $$g_{2p} =(-x_{12} + b_p)p_1' + (1-t)w_2'.$$

Real income effects here include changes directly in income as well as real income effects of price changes.

Let us define

(5.28) $$\Delta_1 = (1 - t) \frac{dw_1}{dp_1} - x_{11} = (1 - t) \frac{n_{22}}{Ax_2} - x_{11}$$

(5.29) $$\Delta_2 = (1 - t) \frac{dw_2}{dp_1} - x_{12} = (1 - t) \frac{-n_{21}}{Ax_2} - x_{12}$$

which are the real income effects from a one unit increase in p_1 when the effects on wages as well as the real income effects of a higher commodity price are taken into account. Let us then find the partial effect of p_1 on b. From (5.13)

$$b + p_1x_1^g + x_2^g = tN_1w_1 + tN_2w_2$$

we find that

(5.30) $$b_p + x_1^g = tN_1\frac{dw_1}{dp_1} + tN_2\frac{dw_2}{dp_1} + tw_1 \ \partial N_1/\partial p_1$$

$$+ tw_1(\partial N_1/\partial I_1)(1 - t) \frac{dw_1}{dp_1} + tw_1(\partial N_1/\partial b)b_p + tw_1(\partial N_1/\partial I_2)(1 - t) \frac{dw_2}{dp_1}$$

$$+ tw_2(-\partial N_1/\partial p_1) + tw_2(\partial N_2/\partial b)b_p + tw_2(-\partial N_1/\partial I_1)(1 - t) \frac{dw_1}{dp_1}$$

$$+ tw_2(-\partial N_1/\partial I_2)(1 - t)\frac{dw_2}{dp_1}.$$

Let

$$n = 1 + t(w_1 - w_2)\partial N_2/\partial b = 1 - t(w_1 - w_2)\left(\frac{\partial N_1}{\partial I_1} + \frac{\partial N_1}{\partial I_2}\right).$$

We assume that $n > 0$ which rules out the possibility that a one unit lump sum transfer from the government may generate a more than one unit increase in tax revenue.

Hence

$$\begin{aligned} b_p n &= tN_1\frac{dw_1}{dp_1} + tN_2\frac{dw_2}{dp_1} - x_1^g + t(w_1 - w_2)\partial N_1/\partial p_1 \\ &\quad + t(w_1 - w_2)(1 - t)(\partial N_1/\partial I_1)\frac{dw_1}{dp_1} + t(w_1 - w_2)(1 - t)(\partial N_1/\partial I_2)\frac{dw_2}{dp_1} \\ &= \left(tN_1\frac{n_{22}}{Ax_2} + tN_2\frac{-n_{21}}{Ax_2} - x_1^g\right) + t(w_1 - w_2)(-x_{11}\partial N_1/\partial I_1 - x_{12}\partial N_1/\partial I_2) \\ &\quad + t(w_1 - w_2)(1 - t)\frac{\partial N_1}{\partial I_1}\frac{n_{22}}{Ax_2} + t(w_1 - w_2)(1 - t)\frac{\partial N_1}{\partial I_2}\frac{-n_{21}}{Ax_2}. \end{aligned}$$

where (2.18), (5.20) and (5.21) have been invoked. Moreover,

$$\begin{aligned} b_p n &= t(w_1 - w_2)\Delta_1\partial N_1/\partial I_1 + t(w_1 - w_2)\Delta_2\partial N_1/\partial I_2 \\ &\quad + tN_1\frac{n_{22}}{Ax_2} + t\frac{-n_{21}}{Ax_2}N_2 - x_1^g. \end{aligned} \tag{5.31}$$

Making use of (5.11) we see that

$$\begin{aligned} & tN_1\frac{n_{22}}{Ax_2} + t\frac{-n_{21}}{Ax_2}N_2 - x_1^g = \\ & tN_1\frac{n_{22}}{Ax_2} + t\frac{-n_{21}}{Ax_2}N_2 - x_1 + N_1x_{11} + N_2x_{12} \\ & - N_1\frac{n_{22}}{Ax_2} + N_1\frac{n_{22}}{Ax_2} - N_2\frac{-n_{21}}{Ax_2} + N_2\frac{-n_{21}}{Ax_2} = \\ & - N_1\Delta_1 - N_2\Delta_2 - x_1 + N_1\frac{n_{22}}{Ax_2} + N_2\frac{-n_{21}}{Ax_2} \end{aligned} \tag{5.32}$$

$$= -N_1\Delta_1 - N_2\Delta_2 - x_1 + (n_{11} + n_{21})\frac{n_{22}}{Ax_2} + (n_{12} + n_{22})\frac{-n_{21}}{Ax_2}$$

$$= -N_1\Delta_1 - N_2\Delta_2 - x_1 + \frac{-n_{11}n_{22} - n_{12}n_{21}}{Ax_2}$$

$$= -N_1\Delta_1 - N_2\Delta_2 - x_1 + \frac{Ax_1x_2}{Ax_2}$$

$$= -N_1\Delta_1 - N_2\Delta_2,$$

where (5.19) has been invoked. Inserting this result into (5.31) we get

(5.33) $$b_p = \frac{1}{n}((-N_1 + t(w_1 - w_2)\partial N_1/\partial I_1)\Delta_1 + (-N_2 + t(w_1 - w_2)\partial N_1/\partial I_2)\Delta_2).$$

Rewriting (5.25) as

(5.34) $$g_{1p} = (-x_{11} + b_p + (1 - t)dw_1/dp_1)p_1'$$

and substituting from (5.33), we get

(5.35) $$g_{1p} = (\Delta_1 - \Delta_2)(N_2 - t(w_1 - w_2)\partial N_1/\partial I_2)p_1'/n.$$

In an analogous way we find that

(5.36) $$g_{2p} = (\Delta_1 - \Delta_2)(-N_1 + t(w_1 - w_2)\partial N_1/\partial I_1)p_1'/n.$$

Let us then find the *partial* effect of t on b. From (5.13),

$$b + p_1x_1^g + x_2^g = tN_1w_1 + tN_2w_2,$$

we find that

(5.37) $$b_tn = w_1N_1 + w_2(1 - N_1) + t(w_1 - w_2)\frac{\partial N_1}{\partial I_1}(-w_1)$$

$$+t(w_1 - w_2)\frac{\partial N_1}{\partial I_2}(-w_2)$$

where we have recalled that $N_1 + N_2 = 1$. Substituting for b_t in (5.24) we obtain

(5.38) $$g_{1t} = -(w_1 - w_2)(N_2 - t(w_1 - w_2)\ \partial N_1/\partial I_2)/n.$$

In an analogous manner we obtain

(5.39) $$g_{2t} = -(w_1 - w_2)(-N_1 + t(w_1 - w_2)\partial N_1/\partial I_1)/n.$$

We can now express the total effect on welfare by

(5.40) $$W' = \frac{1}{n}[\bar{\lambda}_1 N_1(N_2 - t(w_1 - w_2)\partial N_1/\partial I_2)$$

$$+ \bar{\lambda}_2 N_2(-N_1 + t(w_1 - w_2)\partial N_1/\partial I_1)]\ ((\Delta_1 - \Delta_2)p_1' - (w_1 - w_2))$$

or shorter

(5.41) $$W' = k((\Delta_1 - \Delta_2)p_1' - (w_1 - w_2))$$

where k is the expression in square brackets in (5.40) divided by n.

If we are only interested in the sign of W' as we are when we want to know whether a piecemeal tax revision is welfareaugmenting or not, we see that the price effects which are generated by the tax reform can be neglected if and only if

(5.42) $$\text{sgn}\ ((\Delta_1 - \Delta_2)p_1' - (w_1 - w_2)) = \text{sgn}\ (w_2 - w_1).$$

Since we obviously have that

(5.43) $$\Delta_1 p_1' - w_1 + w_2 - \Delta_2 p_1' = g_1 - g_2,$$

(5.42) is equivalent to

(5.44) $$\text{sgn}\ (g_1 - g_2) = \text{sgn}\ (w_2 - w_1).$$

From (5.35) and (5.38) we find that

$$(5.45)\qquad \dot{g}_1 = (\Delta_1 p_1' - \Delta_2 p_1' - w_1 + w_2)[N_2 - t(w_1 - w_2)\partial N_1/\partial I_2]/n$$

$$= (g_1 - g_2)[N_2 - t(w_1 - w_2)\partial N_1/\partial I_2]/n$$

due to (5.43). The expression in square brackets is obviously positive and g_1 and $g_1 - g_2$ have the same sign. Condition (5.44) is then equivalent to

$$(5.46)\qquad \operatorname{sgn} g_1 = \operatorname{sgn} (w_2 - w_1).$$

We have assumed that market conditions are such that $w_1 > w_2$. We shall now explore whether $g_1 > 0$ can be ruled out. From (5.35),(5.36),(5.38) and (5.39) we find that

$$(5.47)\qquad g_{1t} = \frac{w_2 - w_1}{\Delta_1 p_1' - \Delta_2 p_1' - w_1 + w_2} g_1 = \gamma g_1,$$

where γ is defined by the last equation, and

$$(5.48)\qquad g_{2t} = \gamma g_2.$$

We easily see from (5.38) that $g_{1t} < 0$. Hence if $\gamma > 0$ a positive g_1 can be ruled out. First we shall make the important assumption that the market for commodity 1 is stable in the sense that the excess supply of the commodity is an increasing function of the commodity price when all effects of that price on equilibrium prices (wages) in other markets are allowed for. In other words we take into account that the excess supply function is a function of p_1 also through the relations between w_2 and w_1 and p_1.

We write the demand for commodity 1 as x_1^d which is expressed by the right hand side of (5.11) where N_1 is given by (5.10). The supply side can be described by the relations (5.2)-(5.9). Taking commodity prices (p_1 and p_2) and factor endowments (N_1 and N_2) as given these are eight relations which determine x_1 (and x_2, n_{12}, n_{21}, n_{11}, n_{22}, w_1 and w_2). We denote the supply of x_1 by x_1^s. This model is well known from international trade literature. We can write the supply of x_1 as

$$(5.49)\qquad x_1^s\,(p_1, N_1, N_2)\,.$$

Obviously

$$\frac{\partial x_1^s}{\partial p_1} > 0 \tag{5.50}$$

The effects of changes in N_1 (and N_2) are easily found by appealing to the Rybczynski - theorem. Since sector 1 is the more intensive user of labour belonging to occupation 1, it follows that

$$\frac{\partial x_1^s}{\partial N_1} > 0 , \quad \frac{\partial x_1^s}{\partial N_2} < 0 \tag{5.51}$$

Recalling that $N_2 = 1 - N_1$ and that N_1 is related to prices through (5.10), the supply of commodity 1 has been related to p_1. Let us use the notation $(\partial x/\partial p)|u$ to denote a compensated price derivative (or Slutsky derivative) where u indicates that the utility level is fixed. Differentiating the excess supply with respect to p_1 we get

$$\begin{aligned}
\frac{\partial}{\partial p_1}(x_1^s - x_1^d) &= \frac{\partial x_1^s}{\partial N_1}\left(\frac{\partial N_1}{\partial I_1}\frac{g_{1p}}{p_1'} + \frac{\partial N_1}{\partial I_2}\frac{g_{2p}}{p_1'}\right) \\
&+ \frac{\partial x_1^s}{\partial N_2}\left(\frac{\partial N_2}{\partial I_1}\frac{g_{1p}}{p_1'} + \frac{\partial N_2}{\partial I_2}\frac{g_{2p}}{p_1'}\right) + \frac{\partial x_1^s}{\partial p_1} \\
&- N_1\left.\frac{\partial x_{11}}{\partial p_1}\right|_u - N_1\frac{\partial x_{11}}{\partial I_1}\frac{g_{1p}}{p_1'} - N_2\left.\frac{\partial x_{12}}{\partial p_1}\right|_u \\
&- N_2\frac{\partial x_{12}}{\partial I_2}\frac{g_{2p}}{p_1'} - x_{11}\frac{\partial N_1}{\partial I_1}\frac{g_{1p}}{p_1'} - x_{11}\frac{\partial N_1}{\partial I_2}\frac{g_{2p}}{p_1'} \\
&- x_{12}\frac{\partial N_2}{\partial I_1}\frac{g_{1p}}{p_1'} - x_{12}\frac{\partial N_2}{\partial I_2}\frac{g_{2p}}{p_1'} \\
&= \left(\xi_1 g_{1p} + \xi_2 g_{2p}\right)\Big/ p_1' - \left(N_1\left.\frac{\partial x_{11}}{\partial p_1}\right|_u + N_2\left.\frac{\partial x_{12}}{\partial p_1}\right|_u - \frac{\partial x_1^s}{\partial p_1}\right) > 0
\end{aligned} \tag{5.52}$$

where the inequality reflects the increasing excess supply assumption and

$$(5.53) \qquad \xi_1 = \frac{\partial x_1^s}{\partial N_1}\frac{\partial N_1}{\partial I_1} + \frac{\partial x_1^s}{\partial N_2}\frac{\partial N_2}{\partial I_1} - N_1\frac{\partial x_{11}}{\partial I_1} - x_{11}\frac{\partial N_1}{\partial I_1} - x_{12}\frac{\partial N_2}{\partial I_1}$$

$$(5.54) \qquad \xi_2 = \frac{\partial x_1^s}{\partial N_1}\frac{\partial N_1}{\partial I_2} + \frac{\partial x_1^s}{\partial N_2}\frac{\partial N_2}{\partial I_2} - N_2\frac{\partial x_{12}}{\partial I_2} - x_{11}\frac{\partial N_1}{\partial I_2} - x_{12}\frac{\partial N_2}{\partial I_2}$$

Also differentiating the equilibrium condition $x_1^s - x_1^d = 0$ with respect to t we get

$$(5.55) \qquad \frac{d}{dt}(x_1^s - x_1^d) = \xi_1 g_1 + \xi_2 g_2 - \left(N_1 \left.\frac{\partial x_{11}}{\partial p_1}\right|_u + N_2 \left.\frac{\partial x_{12}}{\partial p_1}\right|_u - \frac{\partial x_1^s}{\partial p_1}\right)p_1' = 0$$

which is equivalent to

$$(5.56) \qquad \left(\xi_1 g_{1t} + \xi_2 g_{2t}\right)/p_1' + \left(\xi_1 g_{1p} + \xi_2 g_{2p}\right)/p_1'$$

$$- \left(N_1 \left.\frac{\partial x_{11}}{\partial p_1}\right|_u + N_2 \left.\frac{\partial x_{12}}{\partial p_1}\right|_u - \frac{\partial x_1^s}{\partial p_1}\right) = 0 .$$

Combining (5.52) and (5.56) we can conclude that

$$(5.57) \qquad \begin{cases} \xi_1 g_{1t} + \xi_2 g_{2t} < 0 & \text{if } p_1' > 0 \\ \xi_1 g_{1t} + \xi_2 g_{2t} > 0 & \text{if } p_1' < 0 \end{cases}$$

Using that $g_{1t} = \gamma g_1$ and $g_{2t} = \gamma g_2$ (from (5.47) and (5.48))we can write instead that

$$(5.58) \qquad \begin{cases} \gamma(\xi_1 g_1 + \xi_2 g_2) < 0 & \text{if } p_1' > 0 \\ \gamma(\xi_1 g_1 + \xi_2 g_2) > 0 & \text{if } p_1' < 0 \end{cases}$$

Let us then recall (5.55). Since the direct Slutsky derivatives are negative and $\partial x_1^s/\partial p_1 > 0$, we see that

$$(5.59)\qquad \begin{cases} \xi_1 g_1 + \xi_2 g_2 < 0 & \text{if } p_1' > 0 \\ \xi_1 g_1 + \xi_2 g_2 > 0 & \text{if } p_1' < 0 \end{cases}$$

From (5.58) and (5.59) we can conclude that $\gamma > 0$, and (5.46) is satisfied.

Proposition 5.1: Under stability assumptions induced price changes can be neglected at the margin when tax revisions are considered.

5.3 *The effect of a commodity tax*

Let us consider the introduction of an excise tax, s, on commodity 1 under the assumption that the income tax has already been optimized. What are the effects of such a tax? We first observe that the relations (5.6) and (5.7) are unaffected so that the same relations hold between w_1 and p_1 and w_2 and p_1 where p_1 is now the producer price of commodity 1. The effects on the wage-rate will then occur only through the effect on p_1. In (5.11) - (5.13) and (5.22) p_1 is now replaced by p_1 + s which is the consumer price of commodity 1. In general p_1 now becomes a function of s and t. It is useful to express b as a function of p_1, t and s: $b(p_1,t,s)$. We can then derive the welfare effect of changing s marginally away from zero. The welfare function (formerly (6.22)) is now expressed as

$$(5.60)\qquad W = \int_{\bar{a}}^{\infty} V\,(p_1 + s, (1-t)w_1 + b(p_1,t,s), a, 1) f(a) da.$$
$$+ \int_{0}^{\bar{a}} V\,(p_1 + s, (1-t)w_2 + b(p_1,t,s), a, 2) f(a)\, da.$$

Since t is at its optimum initially, we can apply the Envelope Theorem to derive

$$(5.61)\qquad \frac{dW}{ds} = \frac{\partial W}{\partial (p_1 + s)} + \frac{\partial W}{\partial p_1}\frac{dp_1}{ds} + \frac{\partial W}{\partial b}\frac{\partial b}{\partial s}.$$

But we know already that at the initial tax optimum a change in p_1 has no effect on welfare. So the second term vanishes. Hence

$$(5.62)\qquad \frac{dW}{ds} = \frac{\partial W}{\partial(p_1+s)} + \frac{\partial W}{\partial b}\frac{\partial b}{\partial s}.$$

We easily see that

$$(5.63)\qquad \frac{\partial W}{\partial(p_1+s)} = -\bar{\lambda}_1 N_1 x_{11} - \bar{\lambda}_2 N_2 x_{12}.$$

The budget constraint of the goverment now becomes

$$(5.64)\qquad b + (p_1+s)x_1^g = tw_1N_1 + tw_2N_2 + sx_1.$$

Let $b_s = \partial b/\partial s$. Differentiating at the point where s=0 we find that

$$b_s + x_1^g = tw_1\partial N_1/\partial s + tw_2\partial N_2/\partial s + x_1$$

and further

$$(5.65)\qquad b_s = t(w_1 - w_2)\partial N_1/\partial s + N_1x_{11} + N_2x_{12}$$

where (5.1) and (5.11) have been employed.

Using

$$N_1 = N_1(p_1 + s,\ (1-t)w_1 + b,\ (1-t)w_2 + b),$$

we find that

$$(5.66)\qquad \frac{\partial N_1}{\partial s} = -x_{11}\frac{\partial N_1}{\partial I_1} - x_{12}\frac{\partial N_1}{\partial I_2} + \frac{\partial N_1}{\partial I_1}b_s + \frac{\partial N_1}{\partial I_2}b_s.$$

Inserting (5.66) into (5.65), we get

$$(5.67)\qquad b_s = \left(1-t(w_1-w_2)\left(\frac{\partial N_1}{\partial I_1} + \frac{\partial N_1}{\partial I_2}\right)\right)^{-1}\left(t(w_1-w_2)\left(-x_{11}\frac{\partial N_1}{\partial I_1} - x_{12}\frac{\partial N_1}{\partial I_2}\right) + N_1x_{11} + N_2x_{12}\right)$$

where the inverse expression has previously been denoted by $1/n$. Making use of (5.63) and (5.67) we find that

$$(5.68)\qquad \frac{dW}{ds} = \bar{\lambda}_1 N_1\left(-x_{11} + \frac{1}{n}\left(t(w_1-w_2)\left(-x_{11}\frac{\partial N_1}{\partial I_1} - x_{12}\frac{\partial N_1}{\partial I_2}\right) + N_1x_{11} + N_2x_{12}\right)\right)$$

$$+ \bar{\lambda}_2 N_2\left(-x_{12} + \frac{1}{n}\left(t(w_1-w_2)\left(-x_{11}\frac{\partial N_1}{\partial I_1} - x_{12}\frac{\partial N_1}{\partial I_2}\right) + N_1x_{11} + N_2x_{12}\right)\right)$$

It follows that

$$(5.69)\qquad n\frac{dW}{ds} = \bar{\lambda}_1 N_1[-x_{11}(N_1+N_2) + x_{11}t(w_1-w_2)\frac{\partial N_1}{\partial I_1}$$

$$+ x_{11}t(w_1-w_2)\frac{\partial N_1}{\partial I_2} - t(w_1-w_2)x_{11}\frac{\partial N_1}{\partial I_1} - t(w_1-w_2)x_{12}\frac{\partial N_1}{\partial I_2}$$

$$+ N_1x_{11} + N_2x_{12}] + \bar{\lambda}_2 N_2[-x_{12}(N_1+N_2) + t(w_1-w_2)x_{12}\frac{\partial N_1}{\partial I_1}$$

$$+ t(w_1-w_2)x_{12}\frac{\partial N_1}{\partial I_2} - t(w_1-w_2)x_{11}\frac{\partial N}{\partial I_1} - t(w_1-w_2)x_{12}\frac{\partial N_1}{\partial I_2}$$

$$+ N_1x_{11} + N_2x_{12}]$$

$$= \bar{\lambda}_1 N_1\left(-N_2 + t(w_1-w_2)\frac{\partial N_1}{\partial I_2}\right)(x_{11}-x_{12})$$

$$+ \bar{\lambda}_2 N_2\left(N_1 - t(w_1-w_2)\frac{\partial N_1}{\partial I_1}\right)(x_{11}-x_{12}) .$$

Then taking (5.40) into account and recalling that $dW/dt = 0$, we obtain

$$(5.70)\qquad \left.\frac{dW}{ds}\right|_{s=0} = 0.$$

Hence we can state:

Proposition 5.2: Provided that there is an optimum income tax, there is no need to employ commodity taxation.

8. CONCLUDING REMARKS

This paper has been concerned with the role of endogenous wages and prices in tax analysis. The subject has been studied within a model which has focused on the choice of occupation aspect of labour supply rather than the traditional free choice of working hours. In other respects the set-up has been similar to that used in earlier studies of the endogeneity of wages. (Feldstein (1973) and Allen (1982).) There are constant returns to scale, two market determined wage rates and a linear income tax. Both a one-sector version and a two-sector version of the basic model have been analysed. The latter version is closely related to the famous two by two model of international trade theory.

The paper has analysed the incidence effects of increasing the progressivity of the income tax. This would lead to an unambiguous redistribution of income which, however, would be modified by the induced general equilibrium effects on wages.

Central features of the model are the close relationship between the distributional effects and the effects on effiency, and the similarity between direct tax effects and the effects of induced wage (and price) effects. The behaviour of the workers is essentially governed by the income distribution between occupations. A reduction of income discrepancies through taxation directly reduces effiency via the labour responses to the very change in income distribution. The induced wage (and price) changes affect the economy in exactly the opposite direction, but to a lesser extent.

The very important implication is that if the direct redistributional effect is more (less) highly valued from a welfare point of view, than the direct effect on efficiency of a slightly more progressive taxation, so is the ultimate general equilibrium effect on distribution as compared to that on efficiency. Hence the main finding of the analysis is that, within the economic setting under survey, induced wage and price changes may be neglected in assessments of piecemeal tax revisions. This result differs from the conclusion of Allen (1982) that within a model of free choice of working hours allowing for general equilibrium effects on wages may be vital. The different results highlights the roles of different assumptions about the labour market.

The analysis is limited in a number of respects. Only two occupations have been considered. There is only a very simple tax structure. Ability has been treated as one-dimensional. Only one aspect of labour supply has been modelled. In reality the labour supply side is very complex and involve a number of decisions with respect to the number of household members taking a job, working hours, education, willingness to take responsibility, choice of occupation, etc. All these aspects seem to have a claim for interest although the scope for the various choices may differ between different economies. But at the present stage of research partial analyses of the various aspects in connection with tax theory seem likely to give more insight into the relevant mechanisms than a simultaneous treatment of a mixture of aspects, although a synthesis may well be the proper aim in the long run.

Until now general equilibrium effects on wages and prices have received little attention in normative tax theory. Yet, it may not be fair to blame the research in public economics for neglect in this respect. On the contrary, it may well have got priorities right by first developing optimum and tax revision theories under simpler assumptions. It seems important, however, to become increasingly aware of the incidence and efficiency aspects of tax induced general equilibrium effects on wages and prices.

Footnotes

*) A first draft of this paper was presented at the Summer 1982 meeting of the Yrjö Jahnsson Foundation Study Group on Public Economics. A second version was presented at the 1983 meeting of Nordic economists at Hankø. Comments by Avinash Dixit, Agnar Sandmo, Matti Tuomala and seminar participants are gratefully acknowledged.

1) For an introduction to optimum tax analysis, see for example Atkinson and Stiglitz (1980).

2) According to Feldstein (1976, p. 77) "Optimal tax reform must take as its starting point the existing tax system and the fact that actual changes are slow and piecemeal."

3) Allen (1982) distinguished between direct effects via the fiscal system and general equilibrium effects due to changes in pre-tax wages.

4) But even in such instances there may be an element of economic compensation to the marginal individual. For instance a sportsman's career is usually short and risky (in terms of success or failure and physical injuries) and may require a number of sacrifices in other respects.

5) Feldstein (1973) obtained the necessary degree of freedom to keep wages fixed at the no-tax equilibrium values by giving up the expansion path condition.

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EVALUATION OF PUBLIC PROJECTS UNDER OPTIMAL TAXATION

by

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Evaluation of Public Projects under Optimal Taxation

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1. INTRODUCTION

The seventies have seen a revolt against the conventional cost-benefit criterion which simply sums up and compares the costs and benefits of a project without caring who benefit and who bear the costs. As an alternative the use of weighting systems has been proposed by which different weights are given to benefits obtained and costs incurred by persons who differ with respect to income level and possibly other characteristics which are assumed to bear strongly on welfare. There have, however, been conflicting views on the matter. One argument has been that the attainment of distributional ends should be the responsibility of tax policy and should not be allowed to interfere with the assessment of government projects.

Hylland and Zeckhauser (1979) have recently given a valuable contribution to the theoretical clarification of this issue. They showed that under certain assumptions (in their own words): distributional objectives should affect taxes but not programme choice or design. The purpose of this paper is to analyse the same question within a more general economic setting and using a different analytical approach. The more precise purpose is to establish conditions under which the conventional cost-benefit criterion (that sums up unweighted net benefits and neglects tax distortions[1]) or a simple modification of it is valid in the presence of distributional objectives and second best taxation. Hylland and Zeckhauser showed that certain assumptions are sufficient to permit the use of the conventional cost-benefit criterion in the presence of optimal income taxation. This paper derives conditions which are necessary and sufficient for this criterion to be valid under the same tax system. In general it extends the analysis to the case of mixed income and commodity taxation. It also tries to give more attention to the economic content of the formal results.

It should be made clear that the project considered in this analysis is of the public good type. A related strand of analysis is concerned with the social evaluation of publicly provided goods which are sold in the market at tax-distorted prices or market goods used in the public sector. The literature started by Diamond and Mirrlees (1971) has established that when optimal commodity taxes are in operation, private sector producer prices are appropriate public sector shadow prices for such goods even when distribution matters. This is an important result which could do a lot to simplify cost-benefit analyses in the public sector. The Diamond and Mirrlees paper also presented an optimality rule for the provision of public consumption in the presence of taxation (Diamond and Mirrlees, 1971, Section IX). But no attempt was made to derive conditions which would allow the use of simple cost-benefit criteria in this case. So this is the problem to which we address ourselves. We shall return to the relationship between the evaluation of public goods and that of market goods at the end of the paper.

Section 2 describes the economic setting within which the cost-benefit analysis is to be analysed. Section 3 presents the cost-benefit problem. The main analysis and results appear in Section 4, while a more special case is dealt with in Section 5.

2. THE ECONOMIC SETTING

We consider a population consisting of a continuum of individuals who differ in their earning ability, a. a is then a continuous variable which is taken to be nonnegative. The distribution of individuals by ability is given by the exogenous density function $f(a)$. For convenience the size of the population is normalized at unity.

There are four commodities, two private goods, work effort (i.e. hours of work or a more complicated measure), and a public good. All individuals have the same utility function $u(y, x, g, h)$ where y, x are the amounts of the two private goods consumed by the individual, g is the amount provided of the public good, and h is his work effort.

The wage rate faced by an individual is equal to his exogenous earning ability. Let I denote his gross income. Only earned income is considered, so that

$$I = ah. \tag{1}$$

$T(I)$ is the income tax imposed on a person earning I, and $T' = dT/dI$ is the marginal taxrate. Producer prices are assumed fixed. Commodity y is untaxed and serves as numeraire with its price set at unity, while a unit tax t is imposed on x. q denotes the consumer price, and p the producer price of x. By definition:

$$q = p + t. \tag{2}$$

It follows that the budget constraint of an individual becomes:

$$ah - T(ah) = y + qx. \tag{3}$$

Using (3) to substitute for y, the utility function can be written as

$$u(ah - T(ah) - qx, x, g, h) = v(h, x). \tag{4}$$

Each person is assumed to maximize his utility function with respect to h and x taking his ability, the shape of T, the price q, and the provision of the public good as given. This leads to the familiar first order conditions:

$$v_h = u_y a(1 - T') + u_h = 0 \tag{5}$$

$$v_x = -qu_y + u_x = 0 \tag{6}$$

where subscripts denote partial derivatives with respect to the appropriate arguments. Second order conditions are

$$v_{hh} < 0, \tag{7}$$

$$|A| > 0 \quad \text{where } A = \begin{bmatrix} v_{hh} & v_{hx} \\ v_{xh} & v_{xx} \end{bmatrix}. \tag{8}$$

The following demand functions and labour supply function are implied:

$$y(a, q, g) \tag{9}$$

$$x(a, q, g) \tag{10}$$

$$h(a, q, g). \tag{11}$$

The functional forms obviously depend on the shape of T.

The government's objective function is taken to be an additively separable social welfare function:

$$W = \int_{a=0}^{a=\infty} u(ah - T(ah) - qx, x, g, h) f(a)\, da. \tag{12}$$

u is interpreted as a particular cardinalization of the individual's utility function which is selected by the government so as to reflect its distributional preferences. As I have argued elsewhere (Christiansen and Jansen (1978, p. 221)) that I do not see the additive form as a serious restriction.

The limits of integration are the same throughout the analysis. To simplify the mathematical expressions they will be omitted in what follows. And so will arguments of functions when convenient and where no confusion is likely to arise.

The optimal tax policy for a given public project is obtained by maximizing (12) with respect to the income tax schedule and the tax rate t taking (9)–(11) and a revenue constraint into account. The government's revenue requirement is

$$\int T(ah)f da + t\int x(a)f\,da = g + k \tag{13}$$

where we assume that g is measured in terms of the cost of the public project. k is a constant which is zero if there is no public expenditure apart from g to be financed by tax revenue.

For our purpose there is no need to derive detailed optimality conditions. In order to characterize the optimal income tax we introduce a shift parameter S in the income tax function. The effect of S on T may be of any kind. A change in S can therefore be used to describe an arbitrary shift in the tax schedule. An arbitrary marginal shift is denoted by $T_S(I, S)$. $h_S(a)$ and $x_S(a)$ denote the corresponding changes in labour supply and demand for commodity x respectively of a person of ability a. We form the Lagrangean

$$\begin{aligned} L = \int u(ah - T(ah, S) - qx, x, g, h) f\,da \\ + \mu\Big(\int T(ah, S) f\,da + t\int x(a) f\,da - g - k\Big) \end{aligned} \tag{14}$$

where μ is the shadow price associated with (13).

A necessary condition for optimality of the income tax (for a given commodity tax t) is that

$$\partial L/\partial S = -\int u_y T_S f\,da + \mu\int T_S f\,da + \mu\int T' a h_S f\,da + \mu t\int x_S f\,da = 0 \tag{15}$$

where the individual optimization has been taken account of. The shadow price $\mu = -\partial W/\partial k$ evaluated at the optimum. Since there is a social cost to a marginal need for government revenue μ is positive. The interpretation of (15) is that an arbitrary marginal shift in the tax schedule should lead to no change in social welfare allowing for the resulting change in tax revenue evaluated at its shadow price. Otherwise a marginal shift in the tax schedule could always raise the welfare level, and then the initial schedule could not be optimal. A necessary condition for the commodity tax to be optimal as well is that in addition to (15):

$$\partial L/\partial t = -\int u_y x f\,da + \mu\int x f\,da + \mu\int T' a h_t f\,da + \mu t\int x_t f\,da = 0 \tag{16}$$

where $h_t = \partial h/\partial t$ and $x_t = \partial x/\partial t$.

We are now endowed with the necessary tools for coping with the cost-benefit problem.

3. THE COST-BENEFIT PROBLEM

Let us consider a government programme which implies a marginal change in g. How should this change be evaluated in a proper cost-benefit analysis? The effects are listed fairly easily. g has obviously a direct beneficial effect because it appears in the utility function. y, x and h will generally change in response to a change in g. Since y, x and h initially are set optimally by each individual, the only welfare effect of the induced changes in these variables is the effect of the ensuing change in tax revenue. g is financed by taxes, and the cost of a marginal unit of g equals one marginal tax unit. These effects can be brought out by applying the Envelope Theorem[2] to (14), i.e. by differentiating the Lagrange expression with respect to g at the appropriate tax optimum. Then we find that

$$dW/dg = \int u_g f\,da + \mu \int T'ah_g f\,da + \mu t \int x_g f\,da - \mu. \tag{17}$$

The marginal valuation of g by an individual in terms of the numeraire is

$$m = u_g/u_y \tag{18}$$

m is obviously a function of a (for given g and tax policy). We can then rewrite (17) as

$$dW/dg = \int mu_y f\,da + \mu \int T'ah_g f\,da + \mu t \int x_g f\,da - \mu. \tag{19}$$

u_y plays the role of a distributive weight which varies across individuals. When discussing distribution it is useful to establish a reference distribution as a benchmark for comparison. We choose as reference distribution one which is such that the sum of relative shares weighted by the appropriate distributive weights equals μ. This distribution is, of course, equivalent to giving the entire benefit to a person with distributive weight μ. We therefore rewrite (19) as

$$dW/dg = \mu \int mf\,da + \int [u_y - \mu]mf\,da + \mu \int T'ah_g f\,da + \mu t \int x_g f\,da - \mu \tag{20}$$

which we can easily interpret. The first term is the welfare effect which would obtain if the whole benefit were distributed by the reference distribution. The second term is the redistribution effect obtained because the actual distribution deviates from the reference distribution assumed in the first term. The remaining terms are the same terms as explained earlier.

We assume that gross income increases with the ability, i.e. I is an increasing function of a. This function can be inverted. It follows that we can write $h(a(I))$ and $m(a(I))$. Under these conditions a redistribution equal to that in the second term of (20) might have been implemented (or reversed) by means of the tax policy. Since the tax policy is optimal and all potential gains are exploited, the gain from such a marginal redistribution must be just offset by the cost of achieving it. To see this, consider the shift in the tax schedule which for every person is equivalent to the change in g:

$$T_s = -m = -u_g/u_y. \tag{21}$$

In contrast to S which was used to denote an arbitrary shift parameter, a lower case s is used to indicate this particular shift. Applying (15), which is true for any marginal shift in the tax schedule from the optimal one, to the shift expressed by (21), we get exactly the equality between the redistributional gain and the cost of achieving it:

$$\int [u_y - \mu]mf\,da = -\mu \int T'ah_s f\,da - \mu t \int x_s f\,da. \tag{22}$$

Let B be the total direct benefits from a one unit increase in g:

$$B = \int mf\,da = -\int T_s f\,da \tag{23}$$

where the last equation is due to (21). Similarly it may be convenient to let C denote the direct total cost which actually has been set equal to unity. Introducing B and C, and also inserting the result in (22) into (20), we obtain:

$$\frac{dW}{dg}\Big/\mu = [B-C]+\left[\int T'a(h_g-h_s)\,f\,da + t\int (x_g-x_s)f\,da\right]. \tag{24}$$

The welfare impact of the public programme is divided into two effects, each in square brackets. The former effect is the difference between total benefit and cost in the traditional sense, the sign of which is the conventional criterion for accepting or rejecting a programme in cost-benefit analysis. On its own B − C has two essential limitations. It gives the same weight to the net benefits of everybody, and it does not allow for the distortion effects of second best taxation. To make up for these shortcomings the second term is added. This is the difference between the effects on total tax revenue of the labour supply and demand reactions induced by the programme and those of the supply and demand responses to the equivalent tax revision. The former are efficiency effects of the programme, and, as we recall from (22) the latter reflect the distribution effects. If and only if these effects are equal, is the conventional cost-benefit criterion valid. Such an equality might obtain by chance because of the special values of the parameters of the economy, or because the utility function implies that the effects cancel out for each individual. We shall in the following section investigate under what conditions this situation will materialize.

4. WHEN IS THE CONVENTIONAL COST-BENEFIT CRITERION VALID?

Since the optimal income tax will depend on the values of all the parameters of the economy, and the marginal income tax will be positive, we cannot expect the effects on the tax payments of each individual to cancel out unless $h_g - h_s = t(x_g - x_s) = 0$ for all a. So far in our analysis we have only assumed explicitly that the income tax is optimal. No assumption has been made about the commodity tax. If t simply assumes an arbitrary value, the condition obviously becomes $h_g - h_s = x_g - x_s = 0$. We therefore need to analyse the responses of the labour supply and the demand for x. Keeping in mind that h_s and x_s are the derivatives of h and x with respect to the shift parameter s, we get from (5) and (6):

$$A\begin{bmatrix} h_s \\ x_s \end{bmatrix} = \begin{bmatrix} u_y a T'_s + a(1-T')u_{yy}T_s + u_{hy}T_s \\ -qu_{yy}T_s + u_{xy}T_s \end{bmatrix} \tag{25}$$

and

$$A\begin{bmatrix} h_g \\ x_g \end{bmatrix} = \begin{bmatrix} -a(1-T')u_{yg} - u_{hg} \\ qu_{yg} - u_{xg} \end{bmatrix}. \tag{26}$$

A sufficient condition to ensure that $h_g - h_s = x_g - x_s = 0$ is obviously that each element of the vector on the right hand side of (25) equals the corresponding element of the vector on the right hand side of (26). Making use of (5), (6) and (21) these equalities become

$$u_{yg}\frac{u_h}{u_y} - u_{hg} - u_y a T'_s - u_{yy}\frac{u_h}{u_y}\frac{u_g}{u_y} + u_{hy}\frac{u_g}{u_y} = 0 \tag{27}$$

and

$$u_{yg}\frac{u_x}{u_y} - u_{xg} - u_{yy}\frac{u_x}{u_y}\frac{u_g}{u_y} + u_{xy}\frac{u_g}{u_y} = 0$$

which is equivalent to

$$u_x\frac{\partial}{\partial y}\left(\frac{u_g}{u_y}\right) - u_y\frac{\partial}{\partial x}\left(\frac{u_g}{u_y}\right) = 0 \tag{28}$$

and further

$$\frac{d}{dy}\left(\frac{u_g}{u_y}\right)_{u,g,h}=0 \tag{28'}$$

which means that for a given utility level and constant g and h, a movement in y-direction does not change the marginal rate of substitution between g and y. In order also to reformulate (27), we first find T'_s by making use of (21):

$$\frac{\partial T'}{\partial s}=\frac{\partial}{\partial I}\frac{\partial T}{\partial s}=\left[\frac{\partial(u_g/u_y)}{\partial y}\frac{u_x}{u_y}-\frac{\partial(u_g/u_y)}{\partial x}\right]x_I-\frac{1}{u_y^2}(u_y u_{gy}-u_g u_{yy})(1-T')-\frac{\partial(u_g/u_y)}{\partial h}h_I \tag{29}$$

where $x_I=dx(a(I))/dI$ and $h_I=dh(a(I))/dI$. The term in square brackets vanishes because of (28). Then substituting for T'_s in (27) and performing some simple manipulations we find that (27) holds if and only if

$$u_y\frac{\partial(u_g/u_y)}{\partial h}(1-ah_I)=0. \tag{30}$$

We have assumed that a and I are positively related. By definition $I=ah$ which implies that

$$1-ah_I=hda/dI>0. \tag{31}$$

Then (30) holds if and only if

$$\frac{\partial(u_g/u_y)}{\partial h}=0. \tag{32}$$

Since u_g/u_y is independent of h, it follows immediately from (28) that so too is u_x/u_y. So the private goods and the public good are weakly separable from h. Hence u belongs to the class of utility functions:

$$u(y,x,g,h)\equiv\eta(\xi(y,x,g),h). \tag{33}$$

Thus the combination of (28') and (33) is sufficient to ensure that (24) is reduced to

$$\frac{dW}{dg}\Big/\mu=B-C \tag{34}$$

and the traditional cost-benefit analysis is valid.

Proposition 1. *If the utility function exhibits weak separability between h and combinations y, x, g, and the marginal valuation of g in terms of the numeraire is independent of y and x as long as g, h and the level of utility is constant, then it is always permissible to use the conventional cost-benefit criterion $B>C$ ($B<C$) for accepting (rejecting) a public project even if there is an arbitrary commodity tax, and even when distribution matters, but is taken care of by optimal income taxation.*

If consumers were identical or distribution simply did not matter, a uniform head tax would be the optimal tax, there would be no distortions, and the conventional criterion would, of course, always be adequate.

Let us now briefly consider the case where the commodity tax is zero. Then x vanishes from the government's budget constraint. Also when prices are given, the private goods may be aggregated in the utility function to be represented by the disposable income used to purchase these goods. Let us denote disposable or after-tax income by $Y=I-T(I)=y+qx$, and write the utility function as $u(Y,g,h)$. (For convenience the old function symbol u has been retained.) We can now perform exactly the same kind of analysis as above. The only difference will be that we are no longer concerned with what happens to x. What we do may be conceived of as leaving out x and substituting Y for y. So whether the conventional cost-benefit criterion is valid or not only depends on whether or not the term $(h_g - h_s)$ vanishes. It does if and only if (27) holds (with subscript y now referring to Y), which is tantamount to (32) being satisfied. Hence it can be stated:

Proposition 2. *It is always permissible to use the conventional cost-benefit criterion when distribution matters and is taken care of by optimal income taxation, and there is no commodity taxation, if and only if the utility function exhibits weak separability between h and combinations of Y and g.*

Now if the utility function is weakly separable between private goods and the public good taken together and work effort, it has very interesting implications. It has been shown by Atkinson and Stiglitz (1976) that this implies that an optimal tax system will include no commodity taxation (possibly apart from a uniform tax rate which is equivalent to an income tax), so that $t = 0$ is optimal. Obviously it also implies that the separability condition of Proposition 2 is satisfied. Once again (24) reduces to (34). Hence the conventional cost-benefit analysis holds if the utility function is weakly separable in the sense defined above and the overall tax policy is optimal.

Apparently distribution effects are neglected in (34), but this is only apparently. What happens is that in the presence of (33) and optimal income taxation distribution effects are just offset by the (in)efficiency effects of labour (dis)incentives. Both distribution effects and (dis)incentive effects are properly allowed for, but they happen to cancel out under the present conditions. Thus (33) and optimal income taxation permit us to use the conventional cost-benefit criterion without being guilty of neglecting income distribution effects.

The nature of the distributional objectives, expressed for instance by the degree of inequality aversion, will not affect the *form* of the decision rule set out above. But this is not to say that the choice of projects is unaffected by the distributional objectives since the valuation of benefits by the consumers (expressed by B) in general depends on the distribution of income. This would of course also be true if lump sum taxes were available.

5. A MODIFIED COST-BENEFIT CRITERION

So far we have considered an arbitrary commodity tax and the case where $t = 0$ is optimal. By assuming in general optimal design of both commodity and income taxation one might hope to find ways of deriving meaningful characteristics of wider classes of utility functions which allow the use of conventional cost-benefit analysis. However, since both kinds of taxes in general depend on all parameters of the economy, this turns out to become very difficult, and no further results to that effect can be reported.

It turns out, however, that under certain conditions the general second best cost-benefit criterion can be simplified in a way which leads to a modified version of the conventional criterion. As was briefly discussed in the introductory section a simple cost-benefit criterion for private goods is obtained by using producer prices as publi.. sector shadow prices in the presence of optimal commodity taxes. We shall now consider how an analogous modification of the conventional cost-benefit criterion is valid under certain conditions in the case of a public good.

To provide a link between the private good case and the public good case it is convenient to give the case to be dealt with a particular interpretation in terms of production of basic goods. As shown by Sandmo (1973) a public good can often be considered as an input which is used together with private goods to produce some kind of basic good. A standard example is road travelling being produced by means of road services and privately purchased commodities such as cars, petrol, oil, tyres etc. We shall consider the case where each consumer enjoys two basic goods in amounts y and z. y is itself a private good. z is produced by means of a private commodity in quantity x and a public good in quantity g. The production function is given by

$$z = z(x, g) \tag{35}$$

where $\partial z/\partial x = z_x > 0$ and $\partial z/\partial g = z_g > 0$. In other respects the assumptions are still those made earlier in the paper. Formally the only difference from the previous analysis is that the utility function formerly expressed by (4) is now written as

$$u(ah - T(ah) - qx, z(x, g), h) = v(h, x) \tag{36}$$

The optimality conditions of an individual are essentially those derived in Section 2. In the present formulation they become

$$v_h = u_y a(1 - T') + u_h = 0 \tag{37}$$

$$v_x = -qu_y + u_z z_x = 0. \tag{38}$$

It is useful to note that at this optimum

$$\frac{u_g}{u_y} = \frac{u_z z_g}{u_y} = q\frac{z_g}{z_x}. \tag{39}$$

In the previous section we considered the shift in the income tax schedule which was equivalent to a change in g. In this section we shall instead make use of an equivalent change in t. Let us therefore define Δt as the change in t which is equivalent to a one unit increase in g in the sense that it confers the same direct benefit on everybody. At this stage we obviously encounter a problem. t and Δt are required to be the same for everybody. But in general such a uniform Δt may not exist. We therefore have to impose such conditions on $z(x, g)$ as will allow the utility effects of a marginal rise in g for all consumers to be equalled by the utility effects of a suitable change in the unit tax t which is the same for everybody. By definition

$$u_z z_g = -u_y x \Delta t. \tag{40}$$

The left hand side is the direct benefit of increasing g, and the right hand side is the direct utility effect of a commodity tax change. Combining (39) and (40) we find that

$$\Delta t = -qz_g/z_x x. \tag{41}$$

Since Δt must be the same for everybody, it can only depend on factors which all consumers have in common. q and g are such factors, but x will vary between consumers with different income. Hence we see from (41) that we must impose the condition that $z_g/z_x x$ only depends on g (and not on x):

$$z_g/z_x x = \varphi(g). \tag{42}$$

This is a partial differential equation which can be solved to obtain

$$z = \psi(\Phi(g) + \ln x) \quad \text{or equivalently} \quad z = \Omega(\Lambda(g)x). \tag{43}$$

(43) defines the class of permissible z-functions. The main characteristic of this class, which is imposed by (42), is that the marginal rate of substitution between x and g

$$-\left(\frac{dx}{dg}\right)_{z\ \text{constant}} = \frac{z_g}{z_x} = \Phi'(g)x = \varphi(g)x \tag{44}$$

is multiplicatively separable in the two arguments and proportional to x for given g. We see immediately that the Cobb–Douglas function belongs to the class of functions defined by (43) which is much wider than the class of Cobb–Douglas functions. Production functions from this class, e.g. of the Cobb–Douglas type, are widely used in empirical analyses, and therefore interesting.

The total tax policy is assumed to be optimal so that both (15) and (16) hold. In the analysis that follows we shall make active use of (16) which it may therefore be convenient to repeat:

$$\int u_y x f\, da = \mu \int x f\, da + \mu \int T' a h_t f\, da + \mu t \int x_t f\, da. \tag{16}$$

The effect of g on W is found in exactly the same way as before. We have that

$$dW/dg = \int u_z z_g f\, da + \mu \int T' a h_g f\, da + \mu t \int x_g f da - \mu \tag{45}$$

which is equal to (17) except for the fact that g is now assumed to work through z. Our purpose is now to show how this cost-benefit expression can be simplified under the assumptions that have been made. First substituting from (40) into (45) and then invoking (16) and (41), we obtain

$$dW/dg = \mu q \int \frac{z_g}{z_x} f\, da - \mu + \mu t \int (x_g - x_t \Delta t) f\, da + \mu \int T' a (h_g - h_t \Delta t) f\, da. \tag{46}$$

$(x_g - x_t \Delta t)$ is the effect on x of combined changes in g and t where the change in t is the one which exactly offsets the direct beneficial effect of the change in g^3 $(h_g - h_t \Delta t)$ is the corresponding effect on h. For short we use the notations x' and h' for the respective effects. z' and y' are defined equivalently. To find these effects we have to use (37) and (38) which determine the optimum choice of an individual. We will show that the effects which satisfy (37) and (38) are:

$$y' = 0, \qquad h' = 0, \qquad x' = -\frac{z_g}{z_x} \text{ and hence } \quad z' = 0. \tag{47}$$

(47) implies that no argument in the utility function $u(y, z, h)$ is changed, and we immediately see that nothing is changed in (37) which thus remains satisfied. (38) is equivalent to $q/z_x = u_z/u_y$. Obviously nothing is changed on the right hand side. The only thing which remains is then to examine what happens to the left hand side. Differentiating q/z_x and taking into account that $dq = -\Delta t$, we find that

$$d\left(\frac{q}{z_x}\right) = \frac{z_x(-\Delta t) - q(z_{xx} x' + z_{xg})}{z_x^2} dg = \frac{q(z_g/x) + q z_{xx}(z_g/z_x) - q z_{xg}}{z_x^2} dg \tag{48}$$

which is obtained by making use of (41) and (47). From (43) we find

$$z_g = \psi' \varphi, \qquad z_x = \psi' \frac{1}{x}, \qquad z_{xx} = \frac{1}{x^2}(\psi'' - \psi'), \qquad z_{xg} = \psi'' \varphi \frac{1}{x}.$$

By inserting these results into (48) we obtain

$$d\left(\frac{q}{z_x}\right) = \frac{q\varphi}{x z_x^2}(\psi' + \psi'' - \psi' - \psi'')dg = 0.$$

Thus the left hand side is also left unchanged, and we have shown that (47) satisfies (37) and (38). Combining (46) and (47) we now obtain

$$\begin{aligned} dW/dg &= \mu q \int \frac{z_g}{z_x} f\, da - \mu t \int \frac{z_g}{z_x} f\, da - \mu \\ &= \mu (q - t) \int \frac{z_g}{z_x} f\, da - \mu \end{aligned}$$

or slightly reformulated:

$$\frac{dW}{dg}\Big/\mu = \int p\frac{z_g}{z_x} f\,da - 1. \tag{49}$$

This is the total welfare effect of a marginal increase in g measured in terms of government revenue. The expression is easily interpreted. -1 is simply the subtraction of the direct cost of one unit of g. z_g/z_x is the marginal value of g to a consumer in terms of x. It is the change in x which is equivalent to a one unit change in g. In the conventional cost-benefit criterion this marginal equivalent would be evaluated at the consumer price of x for every consumer, and the total value would then be calculated by taking the integral over all consumers. In Samuelson's famous formulation (Samuelson (1954)) the comparison is between the marginal cost of the project and the sum of marginal rates of substitution: $\int (u_g/u_y) f\,da$, which in this case equals $\int q(z_g/z_x) f\,da$. We see that (49) equals the conventional cost-benefit criterion except that the producer price of x is used instead of the consumer price. Thus (49) is simply a modified version of the conventional cost-benefit criterion.

Proposition 3. *With the utility function $u(y, \Omega(\Lambda(g)x), h)$, optimal income tax and an optimal unit tax on x, a cost-benefit assessment of increasing g can be carried out by means of the conventional cost-benefit criterion except for the modification that x is evaluated at its producer price.*

This means that if the petrol tax were imposed only for fiscal and distributional reasons, and (43) described the technology of road travelling, the benefit arising if a road improvement lowers the use of petrol is the amount of petrol which is saved evaluated at the petrol price net of tax. The fact that the kinds of technology very often assumed in applied work belong to the class defined by (43) seems to add some flavour to our result, although it must be admitted that the basis for making such assumptions is sometimes rather weak.

Our result implies that the conventional cost-benefit criterion evaluating x at its consumer price overestimates the social net benefit of a project when t is positive, and underestimates the social net benefit when t is negative, i.e. there is a subsidy.

The result would hold if there were also n other commodities on which unit taxes were imposed and which entered the utility function in addition to y, z and h. It is important, however, to notice that in the analysis t is only imposed for fiscal and distributional reasons. If t were imposed to charge consumers for social or external costs, our result would not hold. This is presumably the case with the petrol tax, which is partly levied to charge the drivers for the cost of providing road services.

We have seen that under optimal taxation and the existence of a commodity tax change which is equivalent to an extra unit of a public good, the social benefit from an extra unit of the public good can be found by evaluating the equivalent change in the consumption of a market good at its producer price. It is interesting to observe that this is a result which is indeed closely related to the result that market goods should be evaluated at producer prices when commodity taxes are set optimally. Thus the two strands of analysis, concerned with market goods and public goods respectively, are brought together under certain conditions.

6. CONCLUDING REMARKS

The present article has established conditions which permit the use of the conventional cost-benefit criterion or a simple modification of it even when the cost-benefit analysis is carried out in a second best economy where distribution matters. The conditions are separability properties of the utility function or properties of the technology available for producing basic goods, and optimality properties of the tax policy.

The analysis has a number of limitations. The individuals of the society differ only in one dimension, and constant returns to scale prevail so that producer prices are constant. (These are limitations which the present analysis has in common with the bulk of literature on optimal taxation so far.) Intertemporal problems of taxation have not been considered. Nor have the special problems raised by the combination of a national tax policy and the provision of local public goods.

The results of the analysis show that there are interesting cases in which it may be possible to simplify the cost-benefit analysis considerably even though the second best cost-benefit criterion in general is quite complicated. The limitations of the analysis show the need for further extension and generalisation of this work.

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NOTES

1. Public projects will generally induce changes in the use of taxed commodities which tend to increase or reduce the amount of inefficiency due to tax distortions. It is these effects which are neglected by the conventional cost-benefit criterion.
2. About the Envelope Theorem see for instance Dixit (1976, pp. 24–30).
3. Note that this change in t is $-\Delta t$, where Δt is determined by (41) so that $dq = -\Delta t$.

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