FULL COST AND OPTIMAL PRICE

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FULL COST AND OPTIMAL PRICE

A Study in the Dynamics of Multiple Production

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Preface

This book is a report on a research project in which computer simulation techniques were used to test the rationality of alternative methods of price calculation in industrial firms. Most of the work was done at the Norwegian School of Economics, with the aid of the Computation Centre of the University of Bergen. Drawing as it must from the various fields of economics, accounting, mathematics, statistics, and computer science, the book is partly a product of a rich interdisciplinary academic milieu, in which I am grateful for having had the opportunity to work. Its defects, however, may be attributed to me. The project was financed by grants from A/S Norsk Varekrigsforsikrings Fond and Norges Handelshøyskoles Forskningsfond.

O. L.

1. The Issues

1.1 The trouble with price theory

One of the minor conflicts caused by new economic thinking in the decade preceding the Second World War had to do with industrial pricing. In 1933 the two books on imperfect and monopolistic competition[1] restated neo-classical price theory in a way which seemed to imply a shifting of the basis of explanation from impersonal market forces to the deliberate maximizing behavior of individual price makers. If so, this was a theory which could be tested, and towards the end of the decade there appeared the first of a series of empirical studies of pricing behavior.[2] Generally unfamiliar with marginal concepts, industrial price makers were reported to adhere almost unanimously to a formula according to which price is determined by adding an estimated profit margin to average or "full" cost.

To some critics of the neo-classical system this was clear evidence against the marginal theory. In defense of the system, some "marginalists" sought to discredit the empirical evidence by casting doubt on the mental capacity of the full cost price makers and the analytical abilities of their professional interpreters. Needless to say, this exchange did little to clear the issue.[3]

Only gradually did it come to be realised that a confrontation of traditional theory with practice on the question of industrial price determination makes little sense. While no foundation can be found in theory for the insistence of practitioners on full cost coverage, this does not necessarily force us to reject either theory or practice. It may simply be an expression of the fact that theory so far has addressed itself to other tasks than that of guiding the practitioner.

[1] Cp. references [2] and [12].

[2] Cp. [5]. On main points its findings are confirmed by recent research, e.g. [1] and [3].

[3] High point of the debate was a series of articles in *The American Economic Review* 1946–8, including [4], [9], and [10]. A critical survey of pricing literature from the thirties is made by this author in [8].

The marginal theory of price was never intended to serve as a blueprint for entrepreneurial decision-making at all nor indeed to describe or explain in detail what takes place in the firm. It is of the nature of an explanatory device on a much higher level of abstraction, permitting only broadly generalised deductions about the aggregate effects of entrepreneurial behavior. Its merit as such was never a settled question. But obviously it takes more to disprove it than demonstrating that actual price makers do without marginal reasoning. The crucial question is whether the prices they reach in a different way for all that produce aggregate market effects which are predictable in the marginal system. This is not so easy to test. So the theory still stands, invulnerable to facts.

For some years now there has been a tendency to leave the full cost formula alone. This is understandable in view of the confused and bitter dialogues it provoked in the past. But in consequence an important question remains unanswered, besides that of the predictive value of abstract marginal theory. This is the question to which this book is devoted. It concerns the logic of full cost pricing from the point of view of the price maker himself.

While it is recognised that full cost practice does not immediately overthrow marginal theory, the reverse fact does not seem to have registered in the minds of many economists. It is still not unusual for those educated in the neo-classical tradition to adopt a slightly condescending attitude to the pricing procedures encountered in practice. The full cost formula is seen as a rough rule of thumb, capable, it is to be hoped, of producing results not too far inferior to those reached in theory and thus not invalidating theoretical prediction too much, but certainly due for replacement if only somebody could spare the necessary time and effort to educate business men on the finer points of theory.

This position is not tenable. It should be realised that a theoretically founded principle of pricing was never available to the industrial decision maker as an operationally meaningful alternative to what he actually does. From his point of view, motivated as he is by long-run objectives and saddled from time to time with the task of pricing new members in an ever-changing family of technically and economically interrelated products, the marginal constructions of neo-classical economic theory are too naïve to be taken seriously. But this is all that theory has managed to come up with in the line of pricing rules. So, on the basis of experience, practice has evolved its own rules. A priori they ought to command a certain respect.

Next they should, if possible, be subjected to scientific scrutiny.

1.2 Directions for new research

To evaluate the rationality of observed behavior in the field of pricing, models must be built in which more is preserved of the complicated environment of real life pricing than is the case with current theory. It seems to be clearly indicated by business men's responses to questions about pricing that some explanation of the full cost rule may be sought by extending the neo-classical model in three dimensions, taking explicit account of the empirical facts of multiple firms, multiple products, and multiple periods, each of which is treated in that model only by implication, if at all. If stated explicitly, the former fact completely erases the simple picture of a demand curve for an individual product, from which the marginal revenue curve is derived, while the latter two as effectively obliterate that pleasant piece of art the individual cost-output curve, which is the basis for deriving marginal cost. It is on the resultant tabula rasa that practice has made its own tentative drawings.

The fact that more than one product is produced causes the price maker to seek some allocation of common costs to products, thereby introducing the characteristic concept of the *burden rate*. The fact that his horizon extends over more than one period forces upon him some recognition of capital costs as an element in the burden. Finally the fact that he is usually not alone in the market reduces his consideration of demand relations to an experienced guess as to what *profit margin* he dare add to total costs. In the maze of inter-product, inter-period, inter-firm relations in which most industrial price makers are ensnared, they have thus found a way out which is certainly not the only feasible way and perhaps not the best way. But if the optimality of observed behavior is to be tested scientifically, it seems a good start to build a model of the firm in which some or all of these three complicating dimensions are present simultaneously.

It should be possible to construct in the terms of such a model a set of feasible pricing formulae in which those encountered in practice are recognised as a subset. By operating the model over a long sequence of periods, using alternative pricing rules and measuring the degree to which stated objectives are attained in each case, it is reasonable to hope that some conclusion may be drawn as to the optimality of the empirical rules and in what way they may perhaps be improved. Eventually both practical business management and theoretical explanation on various levels of abstraction might benefit from this line of research.

Let me emphasise that I do not think its primary purpose ought to be that of assessing the relative merits of marginalism and full costing. In fact it seems rather meaningless to state the problem in those terms. Marginalism in its broadest sense of an application of a maximum condition for some objective function involving cost and revenue elements is, of course, always valid. By this token any pricing formula or class of pricing formulae arrived at by a process of maximizing such a function may be termed a marginal formula or class of formulae, although the descriptive merit of this term may be more or less evident. As for marginalism in the narrow sense of an application of the particular maximum condition of the neo-classical model to which the term has referred in the discussion of pricing, this is a different matter entirely. The only chance for this kind of marginalism to approach relevance in a more realistic environment must be to redefine its cost and revenue functions to take implicit account of relations to other products, firms, and periods. This has sometimes been tried. There are some rather tricky problems involved in it. In any case it can be attempted in many different ways and so comes to involve virtually any feasible pricing rule for the more complex situation. Thus marginalism in that narrow sense disappears as a well-defined analytical alternative. To conclude, I cannot see that marginalism is a relevant issue at all once the question of optimal pricing has been released from its artificial tie-up with the static equilibrium conditions of economic theory.

Full cost coverage is a relevant issue, however. This is the key element in the pricing formulae evolved by practice. Its definition is very simple. Full cost pricing means that the burden rate or rates employed by the firm are such that as an average per period in the long run, total costs carried by products sold converge on total costs incurred. Moreover, the degree of cost coverage in this sense can readily be measured for any pricing rule formulated in a test model which describes a firm's production and marketing activities with any degree of realism. For a given set of pricing rules defined in such a model, if not entirely unrealistic, there is a subset of full cost rules. The performance of these rules would be a main object of study.

So it is in the study to be reported in this book. What I propose to do here is to make an attempt at the pricing problem along the general line of approach described above. However, it is a peculiar fact, which requires some additional introductory remarks, that although the problem thus attacked has been in the minds of able economists for more than a generation, the present study must generally break its own path.

1.3 The failure of deductive methods

Naturally the discrepancies between existing theory and empirical findings, as soon as they were realised, fostered requests for new theoretical studies of pricing in less restricted models. And of course this challenge has not gone entirely unanswered. Over the years all three of the complicating dimensions of multiplicity mentioned above have repeatedly been unfolded for theoretical observation. But nearly always this has had to be done partially and in severely simplified descriptions of the firm's activities. Hence little useful information has come out of this research, and in the end we do not seem to know very much more about optimal pricing procedure than was on record thirty years ago.

Admittedly this is to some extent due to a certain lack of interest among the majority of economists. Official price regulation during and after the war and stickiness of prices for other reasons have drawn some attention away from price to other market parameters. But this only takes us a short way towards explaining the sparseness of analytical achievement. The main reason is the inadequacy of the analytical tools so far available.

Economists relying on deductive mathematical analysis are traditionally resigned to study rather simple problems or, which may not always come to the same thing, to radically simplify the problems they want to study. By this yardstick the problem before us is one of almost prohibitive complexity. This is true even after the postwar introduction of the more powerful management science tools, which decisively broadened the scope of analysis.

The credo of management science used to emphasise the importance of analysing the firm's decisions as integral parts of a total system rather than isolated fragments. With the growing realisation of the immensity of this task the point has been played somewhat down lately. Nevertheless it would be possible to compile an impressive list of successfully completed mathematical analyses of comprehensive industrial decision systems by management scientists. The systems operated by price decisions would seem to be eminently eligible for such study. But in fact the representation in the list of achievement of management science of problems involving industrial pricing is conspicuously poor. This speaks with eloquence of the complexity of the problem and the futility of attacking it with analytical methods at the present time.

This may not always remain so. I think we may hope with confidence that the continuing rapid development of management science techniques will some day bring even such problems as this within the compass of mathematical deduction. But as things stand at present it seems as though we must follow some other route if we are to get any further for some time to come.

In related problem areas promising results are shown by digital computer simulation. It seemed worth while to try to bring these newly developed techniques to bear on the problem of optimal pricing. So this is what I have tried to do in this book.

1.4 A philosophy of simulation

But simulation raises problems of research strategy all its own. The revolutionary feature of simulation as a method of research in the social sciences is its vastly increased capacity for processing descriptive detail. The simulator is free to include in his model of analysis any type of quantitative relation between any number of variables and can have the computer work out any complex result of their interaction, exactly and speedily. In a field of enquiry checked so severely and for so long by the limitations of mathematical deduction, as is the case with economics, it would not be human to arrest the impulse to explore this capacity to the full. Looking back upon the first decade of simulation research such comprehensiveness is very much in evidence. It has been tempting to admit almost any detail that promises to lend more realism to the model. The result is often a realistic mess.

The drawbacks of simulation are the necessity of working with numerical prototypes and the limited possibility for tracing observed cause and effect relations through the system. When the model is very complicated, the combined impact of these phenomena can be very troublesome. The advantages of a controlled experiment may slip away, the research situation reverting to something not much different from that of empirical research in a complicated area. In a mass of confusing detail some results stand out which the researcher is at a loss to explain or the significance of which he is unable to assess. Empirical research in the field of industrial pricing is exactly an instance of this dilemma. Applied uncritically, simulation may offer little advantage over it.

However, the problems of interpretation of simulation results are now recognised by workers in the field as involving some peculiar aspects of prime importance, and the call is out for a uniform effort to stake out rational research strategies. It is natural that anybody who attempts simulation should take some interest in these questions, both for the benefit of his own research achievement and because his study also adds material for the general methodological discussion. Unlike the case in some stagnant fields, the researcher is also a methodologist. This has been the economist's lot for a long time. It happens again for the economist as simulator. And it is inevitable that this book should reflect some of its author's preoccupation with what may perhaps be called a philosophy of simulation.

It seems to me that past experience ought to teach the economic simulator a lesson of more restraint in model construction. Some successfully completed simulation studies of industrial decision systems have employed models of moderate complexity. And observations of the conditions of controlled experimentation in other fields can but confirm the wisdom of such restraint.

Time and effort gained by sacrificing some of the "realism" of a complex simulation model may perhaps be better spent on experimentation with alternative sets of numerical values to get a better grasp of how the system works. This is necessary if the simulator is to approach even remotely the knowledge of the deductive analyst, who can explore the relations of his simpler, general system at leisure.

Obviously the simulation model could be stripped too much of detail. After all, the whole idea of using simulation rather than traditional methods is to permit more complex descriptions. In transition something is necessarily lost in lucidity. It is a question here only of striking a reasonable balance.

Moreover, these statements refer to general research only. The consulting analyst will probably benefit much more from a comprehensive representation of the details of his problem. And in the second instance such overall studies of special cases may also prove important as bases for constructing a valid theory for general case. So there are no doubt relevant lines of research approach along which simulation may be taken other than the one advocated here.

I suggest that in many cases it may be wise to start by exploring the possible avenues of extension of existing theory as far as possible by deductive analysis and then to attempt a further advance by simulation in carefully measured steps. This will assure a modicum of continuity, which is essential. As familiarity with the new techniques increases and results accumulate, the process may be carried on to gradually more ambitious projects.

The present study is only a first step in such a process. Still, I do not want to leave the impression that the model employed is not complicated. Even when only relations of obvious importance are included and each is drawn in broad outline, the composite picture of multiple production and marketing over time is bound to be rather involved.

Some of the building of the model is done in each of the two following chapters. The single-product, static model of the theory of price is taken as a point of departure. In Chapter 2 this model is extended partially in different directions by deductive analysis. This serves to limit the search for optimal pricing rules to a particular class of formulae. In Chapter 3 the partial extensions are merged, some new elements are added, and the total system is operated by simulation to locate optimal rules for different sets of numerical values of important variables in the model.

2. Deductive Explorations

2.1 Theory and fact in pricing: Two points of divergence

The theory of price referred to in this book is a system of reasoning which has gone through a series of adaptations from classical monopoly assumptions to the conditions of present-day markets with very little formal change. In the following statement of the theory we rewrite one of its basic functions to prepare for an explicit analysis of intertemporal relations. Furthermore, we introduce two or three specific assumptions regarding the shapes of some functions. These are the first in a series of specifications through which we shall arrive at a complete statement of the properties of the simulation model to be analysed in Chapter 3.

Let $q = q(p)$[1] be the quantity demanded of a given product from a given firm in a given period and let $c = c(q)$ be the total costs of producing this quantity. In total costs there is usually an easily recognisable element of short-run, variable costs such as material, some types of labor, etc. In practice such costs are generally assumed to vary linearly with output. We shall accept this assumption and write the total cost function in the form

$$c = vq + \nu(q)$$

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where v is a constant while the function $\nu(q)$ comprises all other costs, present and future, which may be relevant to the pricing decision.

Let η be demand elasticity, defined as a positive variable by

$$\eta = -\frac{dq}{dp}\frac{p}{q},$$

[1] In the literature, quantity is commonly treated as the independent variable rather than price. The mathematics then tends to be simpler. For the purpose of the present analysis, however, it is better to state the problem in terms of price throughout.

assuming dq/dp to be negative. The following analysis is further limited to the normal case of $\eta > 1$. We also define

$$\alpha = \frac{1}{\eta - 1}.$$

The firm seeks to maximize profit

$$\pi = pq - c.$$

The (first order) condition of maximum, stated as a pricing rule, is then[1]

$$p = (1+\alpha)\,[v + v'(q)] \tag{1}$$

This is the form in which we shall compare the theory with empirical pricing procedures.

The rule used in practice is, slightly modified, of the form

$$p = (1+\beta)\,(v + xu) \tag{2}$$

where p and v are defined as above, while β is an estimated profit margin, x is the estimated cost of using the firm's capital production equipment per unit of its capacity (the burden rate), and u is the number of capacity units required for making one unit of the product.

The modification made in the formula is two-fold:

i) In some accounting systems elements of short-run variable costs are treated differently according as they are classified as direct or indirect costs. If this scheme is strictly adhered to, only the direct element should be included in v, while the indirect variable costs, which may sometimes amount to as much as ten or fifteen per cent of total costs, should be included in the burden.[2] The distinction is purely one of book-keeping expediency, however, and there seems to be now a growing recognition of the rationality of treating all short-run variable costs in a uniform manner. So in interpreting contemporary accounting practice we shall assume that there are no indirect variable costs in the traditional sense, all genuinely short-run variable costs being included in v. Then the burden consists only of certain overhead costs which bear no recognisable relation to short-run variations in output. In this book such costs are called capacity costs[3].

[1] Cp. Mathematical Appendix 1.

[2] Cp. [7].

[3] This is a general definition of capacity costs. However, the term may not be as appropriate when inter-product and inter-period relations in production and demand are more complex than assumed in the following. Then there may be overhead cost elements less directly related to capacity.

ii) In most industrial firms capacity costs are allocated to a number of different departments or processes and a burden rate is computed for each of these. In this study that number is reduced to a single burden rate. The reduction is motivated by a tremendous gain in analytical simplicity, while there does not seem to be any immediate reason to think that we have lost much in generality. But on this point we have not much more than intuition to guide us, and this is true all along the line of model specifications that we have now embarked upon. While sacrificing for the sake of manageability certain facets of the very complicated problem before us, we can only hope that we have succeeded in preserving its fundamental logical structure.

Comparing formulae (1) and (2) we find that they differ in two respects only. To the profit margin and to the burden applied per product unit in the full cost formula there correspond certain theoretical expressions which may or may not amount to the same things. Each point has been the subject of much discussion. In this book we address ourselves only to the latter point. The former is deliberately avoided by an assumption which reconciles the conflicting views.

2.2 Not to be analysed: The profit margin

Some critics have seen a serious defect in the application of the profit margin of the full cost formula. While α of (1) is a function of demand elasticity, β of (2), it is held, is a fixed mark-up on cost, applied entirely automatically with no attention to market forces.

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I am inclined to doubt the weight of this criticism. Schmalenbach, who may deserve to be called the founder of scientific cost accounting, stated very emphatically that "der zugeschlagene Gewinn ist . . . eine veränderliche Grösse, mit der der Kalkulator sich an den erziehlbaren Marktpreis heranfühlt"[1], and it is convincingly borne out by the empirical evidence that this search for an appropriate profit margin is indeed an operative fact.

It is true that the margin sometimes shows only slight alteration from one period to the next and from one product to the next. Theorists unfamiliar with the inside workings of industrial decision processes may easily be led astray by this peculiar rigidity. But it can be explained as a natural result, partly of market strategy, partly of insufficient information which tends, in practice as well as in theory, to favor the status quo.

[1] [13], p. 273.

The underlying relations are indeed very complicated. In this study no attempt is made to unravel them. A simple assumption is made which leaves out the entire question of the size of the profit margin. We assume that there exists for each product, in each period, a known demand function with *constant elasticity*. Then α is also a constant, given for each product and period, but not necessarily identical for all products in all periods The price maker applies this constant in each case, putting $\beta = \alpha$.

Two points should be commented upon concerning the implications of this assumption. One tends to weaken it somewhat, the other to strengthen it considerably.

On the one hand, once we assume the existence of a known demand function, the assumption of a constant elasticity is not as strong as it may seem. If it only serves to determine β for pricing purposes, it does not necessarily amount to assuming $q = Qp^{-\eta}$ with Q and η constant for all positive p. It is sufficient to assume the function known with such constants in the *relevant price range*. This is a much weaker assumption, for the relevant price range is usually quite narrow, and for most shapes of demand functions it is possible to find constants Q, η which give a very good fit within limited ranges.

On the other hand, the very assumption of a given demand function for each product is much stronger than may perhaps appear at first sight. In the pricing theorem of the monopolistic and imperfect competition theories the demand function has gradually come to be interpreted as a subjective entity. In this way its meaning has also been preserved in cases of oligopoly, where actual demand, measurable ex post, is a function both of the firm's price and of competitors' reactions to that price. By the nature of things these reactions cannot be foreseen. So it would seem more appropriate, as has been attempted lately, to describe the pricing process explicitly as a game of strategy with incomplete information on the parts of all players. Defenders of the traditional theory argued, however, that all that is needed to explain the behavior of each individual firm is its ex ante, subjectively estimated demand function. In this function the firm may take account implicitly of all intermediate effects on demand caused by its own price decision, thus also of estimated competitors' reactions.

The theory of games has not done much to explain industrial pricing so far and there is much to be said for the simpler approach by subjective demand curves. But it should be clearly realised that it is limited to a static theory. If we are to analyse a series of successive price decisions, it is impossible to leave out actual, "objective" demand relations. In each new period, as prices are to be chosen, the effects of those chosen in previous periods come back to the decision maker in the form of actual sales figures.

Hence if we assume in a dynamic model constant demand elasticities in the sense that the optimal profit margin can be determined solely by the parameter η of the demand function, it can only mean that we have left out all oligopolistic uncertainty. We assume, in effect, a monopolistic market (if $\alpha > 0$). This is in reality the sacrifice which is made in the present study to avoid all discussion concerning the profit margin.

To put it in terms of the dimensions discussed in Chapter 1: We retain the single-firm limitation of traditional price theory. By doing this we shall be able to extend the analysis to multiple products and to multiple periods in a model which is still tolerably lucid.[1]

2.3 To be analysed: The burden rate

We are left with two formulae which are now completely reconciled except for the terms $v'(q)$ of (1) and xu of (2).[2] We have certainly chopped off large hunks of the problem involved in a final theoretical evaluation of the significance of empirical pricing procedures. But I believe that the question which remains is the essential one: How well does the burden applied in full cost calculation express the costs relevant to optimal pricing other than short-run variable expense?

Much of the confusion about this question is due to the deceptive formal simplicity of the theory which gave rise to the debate. If its formal character of a single-product, single-period theory is taken literally to mean that the firms considered actually produce only one product and seek maximum profit within a single-period horizon, then the term $v'(q)$ disappears and any burden included in the pricing formula is a clear-cut deflection from rational procedure. This conclusion is evidently susceptible of misinterpretation. The reason must be that the real assumptions of the theory were not always stated sufficiently clearly. This mistake may

[1] There is one further dimension in which we may even be said to have reduced the multiplicity of the traditional theory. There is no mention in this book of other market parameters than price, such as quality and selling costs, which played a celebrated part in certain versions of the theory. However, the analyses of the different parameters were essentially partial. In the standard treatment of price the optimality condition, as evidenced by (1), includes no reference to other parameters.

[2] The possibility of reconciling full costing with marginal theory in the case of linear costs and constant demand elasticity is well known. Cp. for instance, [11] and [14]. The present study adopts these views, but places a greater emphasis on the long-run cost elements.

seem naïve, but it is nevertheless potentially very harmful, for instance when expressed in those modern cost-accounting devices which ignore long-run costs. So we must devote a few words to explain it.

The trouble is rooted in historical tradition. The price theory of the early thirties is best understood when seen as an attempt to extend the Marshallian industry analysis to the economics of the individual firm. This immediately explains the formal limitation to a single product, since the product is the very basis for defining the industry. When the importance of multiple production for the analysis of pricing in the firm is realised, the theorist will of course try to take this into account, but in a theory which employs a formally single-product model he can do so only implicitly, for instance by redefining marginal cost of a given product to include the opportunity loss on rival products. The circumstances are almost identical in the case of the time horizon. The theory writhes in the cruel grip of Marshall's static model, where intertemporal relations affecting present decisions, however clearly acknowledged, can only be conceived of as somehow projected into the shapes of the short-run curves.

On both points all recent responsible expositions of the theory are quite unmistakable. It is clearly meant thus implicitly to take account of both inter-product and inter-period relations to the extent that these do exist and influence decisions. Hence if we are to approach the problem of the burden rate in full cost pricing in terms of a comparison with the marginal cost curve of the theory of price, we must envisage a general case where $v'(q)$ does exist. As for the shape of $v'(q)$, however, the theory in its implicit form can tell us nothing. To get within reach of this problem we need a model which is explicitly multi-product and dynamic.

We now proceed to build such a model. The way we shall go about this is to take the slightly modified theoretical model in which (1) was deduced and carefully loosen its two remaining singularity assumptions. In the process we shall have to specify a whole host of new inter-period and inter-product relations. By keeping these very simple we shall just be able to extend the mathematical analysis partially to multiple products in a single period and to a single product in a sequence of periods. We do this in the following sections of this chapter preparatory to a simultaneous analysis by simulation in Chapter 3.

2.4 Multiple production in the static case

Analysis of multiple production has a long tradition. The classical expositions of the theory of monopoly, which in a sense was merged with the main body of value theory to form the theory of price we have discussed above, usually included a section on the pricing of "joint" products. But this treatment was limited to some extreme cases of substitution and complementarity in production and demand. In modern industry there is an important intermediate class of assorted production, characterised by zero or negligible cross-elasticities of demand and by more or less constant rates of substitution within most of the capacity range of production. This class has only recently found its way into the theoretical literature. Following the general trend towards a linear theory of the firm, attention has then focused on the simplest case of assortment, where independence between products is absolute except for competing claims on a fixed capacity. When extending the traditional single-product theory to multiple products, we shall stay with this simple case.

Consider a firm which produces n products in a given period with a given capacity M.[1] For product no. i $(i = 1, \ldots, n)$ let v_i be unit variable cost and u_i the number of capacity units required to produce one unit of the product (unit capacity requirement), v_i and u_i both being positive constants.

Further let

$$q_i = Q_i p_i^{-\eta_i}$$

be demand at price p_i. Q_i and η_i are constants, $Q_i > 0$ and $\eta_i > 1$. Finally put

$$\alpha_i = \frac{1}{\eta_i - 1}.$$

The firm seeks a set of prices which maximize period profit

$$\Pi = \sum_{i=1}^{n} q_i(p_i - v_i)$$

under the constraint

$$\sum_{i=1}^{n} q_i u_i \leqq M$$

The (first order) maximum condition, stated as a pricing rule, is[2]

$$p_i = (1 + \alpha_i)(v_i + \lambda u_i) \quad (i = 1, \ldots, n) \tag{3}$$

= [illegible] multiplicator

[1] Measured in some unspecified capacity unit, e.g. one machine hour, one square foot of factory space, or the like.

[2] Cp. Mathematical Appendix 2.

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where $\lambda = 0$ if the capacity constraint is not effective in optimum, λ is determined (along with the prices) by (3) and

$$\sum_{i=1}^{n} q_i u_i = M$$

if the constraint is effective in optimum.

λ is the cost of using scarce capacity on the margin of production. Thus (3) is certainly a marginal pricing rule and the term is highly meaningful in this case. At the same time the formula shows a further approximation to the full cost rule (2), the general term $v'(q)$ of (1) having been reduced to λu of (3), which differs from the xu of (2) only in the interpretation of unit capacity costs.

But this result is circumstantial. It follows from the linear production technology assumed in both cases. The difference is still a real one. While λ is an opportunity cost, the burden rate x is an actual or normal average expense computed on the basis of accounting data. So there is no assurance that λ will approximate x and thus, when applied repeatedly, exactly cover capacity costs. However, this question is of no great importance. A meaningful comparison requires an extended time horizon.

What we have achieved by the assumption of linearity is to further narrow the field of comparison to a single, multiplicative factor. By retaining the linearity assumption in the dynamic case where the actually incurred capacity costs must enter the theoretical model also, we shall be able to discuss the significance of the full cost burden rate in equally simple, but empirically more relevant terms.

2.5 Intertemporal relations

The crux of dynamic planning is present binding of future behavior without full knowledge of the consequences. In a dynamic version of the pricing model of the preceding sections, capacity change by investment in capital equipment must be treated as a variable along with price. The binding of future behavior involves both types of variable and is caused by the fact that both products and capital have lives of more than one period. We shall assume some simple intertemporal relations whereby uncertainty is limited to product life.

The completely deterministic investment process is described by

$$M_t = \sum_{j=t-z_1+1}^{t} m_j$$

where *total capacity* M_t is the number of capacity units available in period no. t, *part capacity* m_j is the number of units added by investment in period no. j, and z_1 is the life (in number of periods) of capital equipment. Investment may be made in any period. We may reasonably assume investment costs to be proportional to volume and life of new capital. To add one unit of capacity then costs ψz_1 in the period of investment, no matter how many units are added. The investment entails no further cost in this or subsequent periods, regardless of whether the unit in question is used or not. We may say that the constant ψ expresses investment costs per period per capacity unit or, less clumsily but using a more dangerous term, the costs of depreciation per capacity unit. Capital equipment is useless after z_1 periods and has no scrap value.

In production future behavior is bound by price rigidity. Each product is assumed to be priced once and for all in its period of introduction. If we were to permit all types of intertemporal relations between different products we should lose no generality by excluding the possibility of price changes, since we might immediately define a new product, appropriately related to previous products, when a new price is set. But since we assume very simple inter-product relations, over time as well as in each period, there is a certain loss of generality involved in the rigid price assumption. To some extent it is justified empirically. Reluctance to change prices prevails in many industries.

Except for a chance element, the price once set for a product uniquely determines future demand and profit. There is no relation to other products except through the competing claims on present and future capacity. In the precise shape of the life cycle of demand we cannot hope even to approach full realism. The chief merit of the relations assumed here is that they lend themselves readily to numerical analysis.

Let q_t[1] be demand for a given product in period no. t. Let $t = \tau$ be the period of introduction of the product and p its price chosen in that period. Retaining the previous assumption as to the shape of the demand function, we have

$$q_\tau = Qp^{-\eta}$$

with Q and η constant for each product. We now assume these parameters to be constant over time also, so that demand, once determined by the price chosen, is stable throughout the life of the product. Product life is

[1] Since we now proceed to an analysis of single production in the dynamic case, we may omit the indices identifying individual products.

determined by a random process. For any $t > \tau$ we assume

$$P(q_t = 0) = \frac{1}{z_2}$$

$$P(q_t = q_{t-1}) = 1 - \frac{1}{z_2}$$

where P is probability. Hence product life has a geometric probability distribution with expectation z_2.

In analysing the simultaneous process of production and investment over time, our prime concern is still with the pricing decision. So we deliberately reduce investment to a secondary variable, dependent on price. In any period where scrapping of old equipment or introduction of new products or both require an addition to available capacity, the necessary investment is made automatically. Thus while the consideration of capital costs will play a dominant part in the dynamic problem, the problem is still expressed solely in terms of a set of prices.

In conclusion of this section a few words must also be said about pricing objectives. In the static analyses performed above we tacitly adopted the traditional assumption of profit maximization. We now want to retain the limitation to profit as a measure of preference, but in the dynamic case, under risk, the precise statement of the objective function needs an amendment. We state that the firm's objective shall be understood to be that of maximizing *expected average period profit* within its horizon.

There is no snag in this when used in our main simulation analysis of the multi-product case. There we include an extended, randomly determined sequence of products over a very large number of periods so that chance influence on the average is reduced to a minimum. But before approaching this main problem we are also to consider a dynamic model involving a single product. Here such seriability is not at work. Strictly speaking this means that the objective function just formulated expresses a "zero risk preference" or, in a more recent parlance, a "linear utility function" of money profits. But this is accidental. The case in question is merely a limiting one, included by way of introduction to the main analysis of multiple production. When maximization of expected profit is extended to this preliminary case, it is only because corresponding objectives are necessary for the purpose of comparing the results.

Finally it should be mentioned that "time preference" of all kinds is omitted from the dynamic analysis. Since the model employed only describes the way profits accrue in a given productive activity without relating it to alternative activities in which these profits may be employed or indeed touching on the financial aspects of investment at all, it seems more reasonable to let all period profits weigh equally in the average than to discount future profits by some arbitrary rate of interest.

2.6 Single production in the dynamic case. Optimal price

We now approach the problem of pricing an isolated product in the period of its introduction, assuming that the firm throughout the life of the product will make sure that there is capacity available to satisfy demand at the chosen price. The firm's horizon within which it desires to maximize expected average period profit on this product we assume to coincide with the termination of the life of the product. This life is unknown at the date of the pricing decision, but since product life is also independent of the price chosen, the optimal price may be determined by maximizing expected total profit defined as

$$\bar{\Pi} = z_2 q(p-v) - I$$

where I is expected total capacity costs. The problem turns on the nature and exact definition of I.

There may be a certain capacity already available and paid for before the period of introduction of the given product and thus to be used for its production without further expense. This is not included in I. We include only costs of investments made during the life of the given product and for the purpose of its production, i.e. initial investment if available capacity is insufficient and in addition possible reinvestments made necessary in subsequent periods. When $z_1 > 1$ (which is the general case), a certain capacity may remain for a number of periods after demand suddenly drops off. Although this capacity is not used for the production of the given product in these remaining periods of the life of the capital equipment in question (and may indeed be used for the production of some new product), the capital costs are to be included in I in extenso, provided the investment is made during the life of the given product.

If previous investment is staggered over time, the different units making up initial capacity may not be all of the same age. Referring to the description of the investment process in Section 2.5. we may simplify the analysis of I if we rewrite the part capacities in terms of remaining life. In the period $t = \tau$ of investment of the given product, before any possible new investment is made in that period, the available capacity consists of a number of part capacities

$$\bar{m}_j = m_{\tau-z_1+j} \quad (j = 1, \ldots, z_1-1)$$

with remaining life j periods (including the present period). Before investment there is no part capacity with remaining life z_1 periods, so we put

$$\bar{m}_{z_1} = 0 .$$

The part capacity with remaining life zero periods was just scrapped (if there ever existed any such part capacity). Its actual size, therefore, will not influence the subsequent investment process. For analytical purposes we assume some large $\bar{m}_0$ so that

$$\sum_{j=0}^{z_1} \bar{m}_j > uq$$

for any q considered.
Then for any q there exists a number κ so that

$$\sum_{j=\kappa}^{z_1} \bar{m}_j \geq uq > \sum_{j=\kappa+1}^{z_1} \bar{m}_j$$

and it can be shown that[1]

$$I = z_2\varrho(\kappa)\psi[uq-\sigma(\kappa)]$$

where

$$\varrho(\kappa) = \frac{z_1}{z_2} \frac{\left(1-\frac{1}{z_2}\right)^{\kappa}}{1-\left(1-\frac{1}{z_2}\right)^{z_1}}$$

and

$$\sigma(\kappa) = \sum_{j=\kappa+1}^{z_1} \bar{m}_j \left[1-\left(1-\frac{1}{z_2}\right)^{j-\kappa}\right] .$$

Hence

$$\bar{\Pi} = z_2q[p-v-\varrho(\kappa)\psi u] + z_2\psi\varrho(\kappa)\sigma(\kappa) .$$

We now seek maximum of this function to determine optimal price and to see whether this price corresponds to an exact coverage of capacity costs.

[1] Cp. Mathematical Appendix 3.

Since the first derivative of $\bar{\Pi}$ is in general not continuous, we cannot rely entirely on differentiation to find optimal price. It may be that it is optimal to employ a number of the existing part capacities exactly, no more, no less. It may then further turn out that if more had been available of the oldest (or of course of a younger) of these employed part capacities, it would have paid to extend production somewhat, but in fact it does not pay to do so because this means using still older capacity units involving higher expected reinvestment costs. The probability that optimum shall be thus located depends on the ages of the initially available part capacities and of their number relative to the optimal requirement. However, it may be shown[1] that the important conclusion as to cost coverage in this limiting case does not differ materially from that of the general case where optimum corresponds to a point in the interior of one of the available part capacities. So we are content to pass over the problem of how to determine when a general case exists and assume that this is in fact the case.

The location of optimal price is then very simple. Mathematically our assumption is that there exists a κ so that the demand quantity corresponding to optimal price falls in the interval

$$\sum_{j=\kappa}^{z_1} \bar{m}_j > uq > \sum_{j=\kappa+1}^{z_1} \bar{m}_j$$

In this interval $\varrho(\kappa)$ and $\sigma(\kappa)$ are constant. The problem is thereby reduced to one of simple, linear costs. Differentiating $\bar{\Pi}$ we find the following (first order) maximum condition, stated as a pricing rule,[2]

$$p = (1+\alpha)\,[v+\varrho(\kappa)\psi u] \tag{4}$$

where α is defined as before.

2.7 Single production in the dynamic case. Cost coverage

We have thus found that in the single-product, dynamic case the pricing formula also conforms closely to the one used in practice. To x of (2) there corresponds a term $\varrho(\kappa)\psi$ of (4) for which we shall henceforth adopt the name of its empirical counterpart and call it a burden rate. The question which still remains, however, is whether this theoretical burden rate also ensures exact coverage of capacity costs as the empirical one is usually intended to do.

[1] Cp. Mathematical Appendix 6.

[2] Cp. Mathematical Appendix 4.

The answer is that in the single-product case, in general, it does not. But there are some important exceptions. Moreover, a closer inspection of the problem reveals some features which point to a rather different hypothesis for the multi-product, dynamic case, which is to be analysed in the next chapter.

Consider first the near-trivial case where capital life is a single period. When $z_1 = 1$, we must have $\kappa = 0$, and it follows that $\varrho(\kappa) = 1$, $\sigma(\kappa) = 0$ for all z_2. Capacity costs are reduced to variable costs. Each unit of the product produced carries a burden of ψu, i.e. the costs of investment in a capacity unit multiplied by the number of capacity units required to produce a product unit. There is of course full cost coverage.

When $z_1 > 1$, risk is introduced. Investment in productive equipment entails a fixed cost, the exact coverage of which cannot be guaranteed. However, in view of the proposed extension of the analysis to the multi-product case where seriability works to average out individual product risks, it is relevant to restate the problem of cost coverage in terms of expected values. Is the burden rate $\varrho(\kappa)\psi$ such that the mathematically expected capacity costs incurred are exactly covered by the mathematically expected burden, i.e. is

$$I = R$$

where

$$R = z_2 \varrho(\kappa)\psi u q$$

is the total burden expected to be carried by the product during its life?

Two cases should be distinguished. One important case is defined by the assumption that $\bar{m}_j = 0$ for all $j = \kappa+1, \ldots, z_1$, while $\bar{m}_\kappa$ is large enough to support any relevant production volume. In general this means that all capacity units which are employed for the production of the given product in any given period of its life are of a uniform age and will be replaced simultaneously. Two special cases may be mentioned. $\kappa = z_1 - 1$ means that no part capacity is zero except $\bar{m}_{z_1}$, which is zero by definition, but a part capacity with remaining life $z_1 - 1$ periods is found to be sufficient. $\kappa = 0$ means that no capacity is available at all before investment is to be made in the period of introduction of the product.

In the other main case, to which we shall return presently, different part capacities employed may be of different ages. Mathematically, this is of course the more general case. But I would like to stress that the assumption of a uniform capacity age is by no means a far-fetched one in the single-product case. If the production of a given product is seen in isolation from other uses of capacity, it is quite natural to imagine that capacity is either bought outright for the occasion or otherwise that it is available in a uniform bulk sufficiently large for the product in question. Staggering of investment over time is primarily an effect of multiple production where changes in product family causes sudden shifts in capacity requirement.

This is emphasised because if capacity is uniformly old, the optimal pricing rule is, in terms of expected values, a full cost rule. It follows from $\bar{m}_j = 0 \quad (j = \kappa+1, \ldots, z_1)$ that $\sigma(\kappa) = 0$, hence

$$I = z_2\varrho(\kappa)\psi uq = R \, .$$

We shall find in the next chapter by simulating the multi-product case that the size of the burden rate necessary to cover full cost exactly varies considerably with the lives of products and capital. There is a correspondence between these results and those that may be deduced in the present case of single production with capacity of a uniform age. Since this latter lends itself readily to analysis, brief attention should now be given to the function $\varrho(\kappa)$. To state verbally what this function measures is not possible in any simple terms. It is the burden rate per unit of investment costs per period (or of depreciation costs) of capacity employed in production. The significance of the function may become clearer when we describe it numerically. As it is larger than, equal to, or smaller than unity, the product has to carry, in order to give expected full cost coverage, a burden which is larger than, equal to, or smaller than the streightforward costs of depreciation of the capacity used to produce the product.

Although for economy of notation we have included only κ in the argument of the function, the burden rate depends on z_1 and z_2 as well. As already noted, $\varrho = 1$ for all z_2 when $z_1 = 1$ (i.e. $\kappa = 0$). New investment in single-period equipment is then made for each new period. So all capacity units are already employed, and each unit pays evenly for its investment costs. The other case which entails no risk is $z_2 = 1, \kappa \geq 1$, for which $\varrho = 0$. When product life is a single period while there is available sufficient capital equipment with at least this time to go, there are no investment costs at all and hence no burden. The same does not apply when $\kappa \geq z_2$ for some $z_2 > 1$, since in this case product life is a stochastic variable, and it may happen to exceed the remaining life of existing equipment so that some reinvestment costs must be incurred. Then there is also some risk that this new equipment will lay idle for one period or more.

In all cases save the two stated above, the value of ϱ is determined by the risk of non-use of capacity invested in and paid for, weighed against the profitable use of existing, free capacity. These risks and consequent losses and the gains counterweighing them are in turn determined by κ, z_1, and z_2 so that, depending on these variables, each capacity unit employed should, in order to give expected full cost coverage, sometimes carry more, sometimes less burden than its straightforward depreciation costs. As might be expected, the fraction κ/z_1 is crucial in this respect. While it is impossible to describe in simple terms the detailed shape of the function ϱ, it can be shown[1] that when the remaining life of existing capacity is at least half that of new capacity, the rate is never above unity. Otherwise it is sometimes above, sometimes below, depending on z_1 and z_2.

When the pricing rule for the case of uniform capacity age was described as a full cost rule, this may have struck the critical reader as being something of a subreption. The justification for full cost coverage in this case is clearly the fact that it coincides with a readily evident marginal principle. When all productive equipment is of the same age, the expected cost of capacity per product unit on the margin of production is equal to average expected or full cost of capacity.

When we allow for different ages of parts of total capacity, this simple equivalence no longer prevails. Then the expected capacity cost per product unit on the margin of production is higher than the average because the marginal unit employs capacity with a shorter remaining life. Hence optimum price, determined by considering costs on the margin, includes a burden which is expected to cover more than full cost, i.e.

$$R > I .$$

Mathematically, this follows from the fact that for finite z_2 and at least one $\bar{m}_j > 0 \quad (j = \kappa+1, \ldots, z_1)$ we have

$$\sigma(\kappa) > 0 .$$

It can be shown[2] that this result obtains also when optimum does not, as assumed here, correspond to a point in the interior of one of the available part capacities, but consists in employing a number of the existing part capacities exactly.

However, the more we generalise the description of the capacity situation, the less satisfactory is a limitation to a partial analysis of a single product. The capacity vacated by one product when its demand drops off may give room for one or more new products. So the former product

[1] Cp. Mathematical Appendix 5.
[2] Cp. Mathematical Appendix 6.

should perhaps not carry all the investment costs. By the same token, idle capacity is no longer necessarily free to the new product. The full cost hypothesis in this case would be that the possibility of reallocating capacity after all tends in the long run to favor a pricing rule which aims at covering only the costs which are actually incurred.

There is nothing inherently unreasonable about that. It may perhaps even be said that the mathematical analyses in this chapter have indicated that it cannot be far off the mark. But a full-fledged dynamic analysis of multiple production is required to substantiate the hypothesis.

2.8 A class of pricing formulae

Before this task, mathematical deduction breaks down. The main obstacle is the description of the life and death process of the family of products which becomes too complicated even if stripped down to a minimum of realistic detail. So in the next chapter we approach the problem by means of computer simulation.

What this chapter has given us is a class of pricing rules to simulate. We have analysed multiple products under single-period conditions and a single product under multi-period conditions and have found in both cases that the optimal pricing formula is equivalent to the empirical formula (2) save for the burden rate x, which may or may not ensure exact coverage of (expected) capacity costs. In the simulation experiment the class of pricing rules to investigate cannot be found by deduction but must be part of the assumptions of the model. I propose to exclude from consideration all pricing formulae except those of the general form (2).

This amount to taking it as sufficiently established by the analyses in this chapter that in any situation in which the firm may find itself there is a burden rate so that formula (2) applied with this burden rate is the optimal pricing formula for all products to be priced in that period. It remains, of course, to determine these burden rates for different periods. The optimal rate must be presumed to be a function of certain descriptive properties of the pricing environment. Some important properties are therefore specified and the functional relationships are estimated by simulation.

With this evidence on hand we may then again turn to the full cost hypothesis and see whether the application of the burden rates thus located will or will not in the long run tend to exactly cover investment costs.

3. The Simulation Experiment

3.1 Multiple production in the dynamic case

In the dynamic case the analysis of multiple production centers on the notion of an ever-changing product family. To get to the core of the problem we must conceive of this change as a stochastic phenomenon.

Risk was introduced in the dynamic, single-product case by chance termination of product lives. Now risk is enhanced by chance introduction of new products. We shall assume that the number of new products to be priced in an arbitrary period is a Poisson variable with expectation z_3. Different products might well be assumed to have different life expectations. But in this study I want to isolate expected product life as a main determining variable and to define it as simply and clearly as possible. For this reason we retain the assumption of constant demand through a product life terminated or extended from one period to the next by a random draw, and we assume the chances of termination to be the same for all products and independent of each other. Then all products have independent geometric life distributions with expectation z_2.

It follows from these assumptions[1] that the number of products in the family in any period (after a number of periods sufficient for the system to attain the equilibrium level) is also Poisson distributed. The expected size of the product family is $y = z_2 z_3$, in other words the expectation is independent of the individual values of z_2 and z_3. This will enable us, by varying one of these parameters inversely to the other, to study the effect of changes in product life on optimal pricing under constant conditions as to fluctuability and average size of the product family.

Though the products are assumed to have equal life expectations, they may differ in the values of the parameters Q_i and η_i of the demand function and of the cost parameters v_i and u_i. These parameters will also be treated as stochastic. The simulation experiment requires specification of the probability distributions. For each product we assume the four par-

[1] Cp. Mathematical Appendix 7.

ameters to be independent of each other. We further assume for each paramenter the same probability distribution to apply to all products. The expectations we denote by $\bar{Q}$, $\bar{\eta}$, $\bar{v}$, and $\bar{u}$. As for the shapes of the distributions themselves, the computations required for the stochastic generation are reduced to a minimum if we assume them to be rectangular with a range equal to the expectation in each case.

The actual numerical values of the expectations were chosen so as to give the model a businesslike appearance and to simplify computations. An average elasticity of $\bar{\eta} = 4$ seems reasonably high. $\bar{Q}$ is purely a scale adjuster. $\bar{Q} = 1000000$ gives a suitable number of digits in aggregate output figures. The appropriate relation between $\bar{v}$ and $\bar{u}$ depends somewhat on the value of ψ. When investment costs are assumed to be linear both in life and volume of capital, we may put $\psi = 1$ without loss of generality by an imagined scale adjustment. Then empirically we should expect $\bar{v}$ and $\bar{u}$ to be of somewhat the same order of magnitude. The assumption $\bar{v} = \bar{u} = 1$ was chosen for convenience.[1]

Pooling the partial descriptions from several sections of Chapter 2 and adding the life and death process of the product family, we now have the complete outline of our multi-product, multi-period model. It remains only to specify some secondary relations for the purpose of the actual execution of the simulation experiment. A sample of such specifications was just included. They grow increasingly technical as we proceed. Let us pause now to assemble from the somewhat scattered assumptions above the main features of the model.

* A firm's production and marketing activities are examined over a sequence of discrete time periods.

* In each period a number of new products are introduced, the number being a Poisson variate.

* Parameters of the cost and demand functions of the products are also generated by a random process.

* Each product is priced in the period of its introduction and the price is held constant during the life of the product.

[1] Note that when $\psi = 1$, $\bar{u} = 1$, several measures of capital costs coincide, simplifying notation. When $\psi = 1$ the burden rates per capacity unit and per unit of periodic depreciation costs are both expressed by x. On the average this also coincides with the burden rate per product unit when $\bar{u} = 1$ and exactly so for any product no. i for which $u_i = 1$.

* The products are independent in demand.
* Demand is constant through product life.
* For each product, life is terminated or extended after each period according to a probability which is the same for all products and constant over time.
* Demand functions have constant elasticities. hyperbolic
* Variable costs are linear.
* All products compete for a single, common capacity.
* Capacity requirements are linear.
* Capacity is extended when required to support the production of old or new products at chosen prices.
* Investment expence is proportional to volume and life of capital.
* Capital life is fixed and known.
* There is no capital maintenance cost.
* There is no cost of removing obsolete capital and no scrap value.
* The objective of the firm is to maximize average period profit, disregarding time preference.
* The means to attain this objective is the pricing rule.

3.2 Pricing in terms of burden rate limits

As stated in the conclusion to Chapter 2, we assume that the firm will always price its new products according to the formula (2), but with the burden rate x adjusted to the different situations in which it may find itself. Once those different situations are specified, i.e. once a class of pricing rules is defined, this problem is reduced to that of locating an optimal value of x for each situation considered. But there are of course infinitely many different classes of pricing rules, depending on what properties of the pricing environment are chosen as pertinent bases for the adjustment. In attempting to locate an optimal pricing rule by simulation we cannot consider all classes of rules but have to limit the analysis to a single, more or less broad class. This is the problem now before us.

Again we are inevitably influenced by the merits of simplicity. The danger involved in this is obvious. We must believe that the closer we examine environmental properties and the more narrowly we specify different situations according to them, the better in terms of period profit, although the more complicated to locate and to apply, is the pricing rule we are able to construct. So it is imperative that some obviously superior class of rules is not permitted to escape our attention only because it is slightly more complicated than the one focused on.

If we are properly aware of this danger, however, there is no demerit in cutting through a mass of descriptive detail and holding on to some few really important determinants. The savings in computational costs by this is tremendous in simulation, as will be readily appreciated when the experimental procedure is explained. So it is not only legitimate, and rational, but absolutely necessary, to omit some detail. In principle this problem is no different from that which is encountered by scientific generalization anywhere, and there is no means of solving it but a familiar compromise.

In our case the purpose of the analysis is to test a specific hypothesis concerning the degree of cost coverage as we approach maximum in a set of alternative pricing rules. After examining carefully the pricing environment we construct a set of alternatives and carry out the experiment. We find no reason to believe that the conclusions reached are much damaged by the fact that the alternatives are not described in more detail. But there is no way of proving this except by actually restating the problem on a broader scale and carrying out a new experiment. This grows rapidly more cumbersome and the process has no natural end.

There may be some solace in an analogy to real life pricing. Any business consultant could testify to the need for sacrificing some operational advantages in administrative routines for reasons of analytical and operational costs. So the details that we omit in the simulation model might very well have had to be omitted in real life too if pricing routines were to be worked out for a firm in this kind of environment.

But in the final evaluation we must rely on our own scrutiny of the environment. And then there is one aspect of it which seems to stand out as all-important, namely the capacity condition. Other things being equal, the firm will not charge a higher price and thereby get a lower volume of production and sales of a new product in a period with sufficient capacity than it would in a period when new investment would have to be made to produce it. However, it is not obvious that the burden rate would be zero even in periods of idle capacity. This was seen not to be the case generally in the single-product, dynamic model of Chapter 2.

Hence if we want to extend the findings of that chapter to a simple yet pertinent formulation of a class of pricing rules to try out in the simulation experiment, it seems that we may pose the problem of pricing in terms of *burden rate limits:* there is an upper limit, x_1, which is used in periods of investment. If investment is not necessary at this rate, the rate is lowered until either all existing capacity is employed or until a lower limit, x_2, is reached.

If we consider the combined activities of pricing and investment, we may distinguish between five different types of periods, in three of which prices are to be made. They are periods with

1. new investment to support existing production, no new products,
2. no new investment, no new products,
3. new investment, new products priced with burden rate x_1,
4. no new investment, new products priced with burden rate x_2,
5. no new investment, new products priced with some intermediate burden rate just sufficient to use up all available capacity.

In the simulation experiment this intermediate rate is located in each case by a sequence of approximations. The pricing routine can then be illustrated by a flow chart as in Figure 1.

In the general case where the two limits do not coincide, the number of different situations considered by such a pricing rule is infinite (or in practice finite, but very large), since there is an infinite number (or actually a very large finite number determined by the number of decimals allowed) of intermediate rates. But given the routine of Figure 1, it is only necessary to state the two limits x_1, x_2 in order to define a pricing rule completely. Mathematically the set of alternative rules to be examined is a point set in the (x_1, x_2)-plane.

The range of adaptation to changing conditions needed to price optimally is measured by the interval between the two burden rate limits. If all costs are covered in optimum, the pricing rule is a full cost rule in the broadest sense, but it may be a little out of tune with empirical procedure. It is true that students disagree about the extent to which industrial price makers are apt to vary the burden rate to meet changing conditions, but I think the consensus may be said to be that such variation is rather limited. In a narrower sense often encountered in practice, full cost pricing means covering all capacity costs by means of a uniform burden rate. In the simulation experiment we shall keep an eye on this limiting case also, defined by $x_1 = x_2$, in addition to the general case.

Obviously our description of the pricing situations could have been elaborated. It is reasonable to think that we might increase period profit above the level attained by optimum in our two-dimensional model if we could find an optimal rule which discriminated between different age distributions of existing capital, for instance, or between different uses of that capital as to size and volume distribution of the product family. But any such extension would require a vast increase in the number of dimensions required to describe each situation. In the first place this probably means that it would simply be impossible to find the desired optimum by simulation, at least for more than a single set or a few sets of values of important structural variables in the problem. In the second place it is highly dubious whether such an analysis would really tell us much more than the simpler one, even if it could be completed. So we stay with the two-dimensional formulation in terms of the limits imposed on the burden rate according to the capacity condition, and use the gain in time and cost of computation for a more thorough examination of the structural relations in the model.

3.3 Strategy of the experiment

In the simulation model there are still three important parameters which have not been specified numerically. They are

z_1 = capital life
z_2 = average product life
z_3 = average number of new products per period.

In the experiment these are treated as structural variables in the sense that optimal pricing rules are located for a number of combinations of values of the variables to trace the basic fundamental dependence upon them.

A standard combination $z_1 = 16, z_2 = 8, z_3 = 2$ (i.e. $y = 16$) was chosen for intensive study. For this combination, the whole (x_1, x_2)-plane was scanned and period profit (as a fraction of profit in optimum), capacity employment, and cost coverage were recorded. For each of a number of supplementary combinations of values of main structural variables, analysis was limited to locating and examining the conditions in optimum. In addition to the absolute optimum in terms of an upper and a lower burden rate limit, the constrained optimum along the line $x_1 = x_2$ was located in each case. First, keeping y constant at the standard value $y = 16$ and varying z_3 inversely to z_2, optimal pricing rules were located for all combinations $z_1 = 2^i \quad (i = 0, \ldots, 8)$ and $z_2 = 2^j \quad (j = 0, \ldots, 6)$. Next z_1 and z_2 were kept constant at their standard values and optima were located for all $y = 2^k \quad (k = 0, \ldots, 8)$,[1] z_3 varying accordingly. Finally some scattered combinations were examined to test inferences drawn from the main study.

For each combination of values of structural variables optimal x_1 and x_2 were determined to the nearest percentage point. Closer approximation would have been very costly even with the speed of the large computer available, and two decimals were deemed adequate to trace the fundamental relations and test our hypothesis. Less than fifty points in the (x_1, x_2)-plane sufficed in most cases to locate the optimal point. The constrained optima were of course much easier to find.

In addition to numerous computations for each period, the project was rendered time-consuming because of a heavy burden of serial correlation between periods, which necessitated extensive simulation runs. Much effort had to be devoted to reducing the run lengths to the minimum requirement in each case.

For each combination of values of structural variables and burden rate limits examined, average period profit and other variables were computed in a "production run" following a "start run" estimated to be sufficient to bring the system from zero product family up to its equilibrium level and from zero capital to the equilibrium age distribution. As for the production run, the aim was for all combinations of main structural variables to reduce random fluctuation to a point where the average period profit function was monotonously increasing in both x_1 and x_2 (or nearly so, to the nearest percentage point) up to the absolute maximum. This simplified the location operation and probably did not increase total computation time at all, since it reduced the need for investigating points surrounding a located maximum area to make sure that the maximum was not a local one.

After a long series of test runs, a system was chosen according to which the lengths of both start run and production run were determined so as to satisfy certain minimum conditions in terms of the structural variables The start run was set at four times the maximum length of capital life

[1] As for the exponential expansion in the values examined for the structural variables, this was based on a guess, partly confirmed by test runs, that some of the basic relations might be approximately linear in the logarithms.

and average product life. The production run should be either 1250 periods or else the maximum of $10000/z_3$ and $10000z_1/2^7$ if this maximum should exceed 1250. As a result the computation time for the combined run came to vary considerably with the combination of structural variables, from less than half a minute to almost an hour in one extreme case. The standard combination required less than two minutes per run. Of course computation time was determined not only by the length of the run but also by the varying number of individual computations required in each period for different combinations.

In some cases an automatic search program was employed to locate optimum. In other cases the search was made by intutition, since after a while it became rather easy to guess the location of the optimal burden rate limits for new combinations, and it was not deemed worth while to make this information the basis for any formal procedure.[1]

3.4 Execution

The bulk of the computational work was done on the IBM 360/50 installation of the University of Bergen, Norway, and took some 150 hours on the central processing unit of that computer. Much exploratory work before the definite choice of a model and some of the running in and testing of parameters in this model was done on the smaller (and much slower) IBM 1130 computer operated in the author's own Business Research Institute at the Norwegian School of Economics in Bergen. Later some supporting computations were undertaken at the Northern Europe University Computing Center at Lyngby, Denmark, on an IBM 7090 computer.[2] Converting total time spent on each machine to '360'-hours on the basis of a rough estimate of relative speeds, another fifty hours must be added for a total of more than 200 hours for the entire project.

Technically the simulation was mostly a straightforward application of standard computational procedures. Since the study retains the periodicity of accounting and of the economic theory to which it refers, the features which sometimes require a special simulation language were absent. The program was written in FORTRAN.

[1] This decision partly reflects the fact that the author had ready and often continuous access to the computer for long stretches of the day (or rather night).

[2] On this computer (though of course not on the 1130) the pseudo-random number generators (cp. below) adopted for the IBM 360 could be exactly reproduced because of the longer computer word.

The random determination of the number of new products in each period and of the values of their cost and demand parameters, as well as the random termination or extension of the lives of existing products, were simulated by means of pseudo-random number generators in the machine. This generation had to be handled with some care, since the sequences generated were quite long, running well into six digits in some cases. However, the word length of the IBM 360 is 32 bits, and this is still easily sufficient to construct a simple multiplicative congruential generation formula with the required period length.

The standard formula of the IBM 360 Scientific Subroutine Package was therefore used without any advance testing, but with different initial values and different multipliers for several generators running in parallel. A program for generating Poisson variates from a random fraction series was developed and tested, but although this technique was not found in the literature, it is probably quite well known.

These and other features of the FORTRAN program are best explained by the appended program listing itself. A macro flow chart of the program is found in Figure 2.

3.5 Output

According to the program listing, a total of 20 output figures are to be listed after each simulation run. They include some data which serve to identify the run, such as values of structural variables and burden rate limits examined and lengths of start run and production run. The data which describe the actual results are average period profit and, in addition to a number of intermediate sums and averages, the following crucial variables:[1]

ξ = relative unused capacity in production run
φ = relative coverage of capacity costs in production run.

The number and selection of output figures varied somewhat in introductory and supporting computations and even in the main study. The numbers of periods of different types were sometimes computed, as seen in the program listing. In the following discussion these numbers are presented as fractions:

$$\mu_i = \text{relative frequency of period type } i \quad (i = 1, \ldots, 5)\,.$$

[1] Greek letters are used here to replace, for simplicity, the longer FORTRAN variable names.

For the standard combination of values of the main structural variables, output for non-optimal pricing alternatives also included

ζ = period profit in production run relative to observed maximum.

The findings are presented in Figures 3–12 and in Tables 1–11. Their contents and economic significance will be discussed in the following sections. Here some notes af a more technical nature are required.

Figures 3–5 deal with the standard combination of values of structural variables. In the first two figures loci are drawn in the (x_1, x_2)-plane for some select values of ζ and φ (Figure 3) and ξ (Figure 4). Negative burden rates are not considered. Since by definition $x_1 \geq x_2$, the area above the line $x_1 = x_2$ is also left out.[1] In the three sections of Figure 5 curves are drawn for the values of ζ, φ, and ξ along this particular line.

Special mention should be made of the peculiarly distorted shapes of the curves in Figure 3. They are not due to poor draftsmanship or to a scarcity of observations, but came out like this as a result of the stochastic element in the model. Prolonged simulation runs would presumably have revealed more regular elliptic shapes for the ζ-curves and perhaps straight lines for the φ-curves. With the limited runs slight deviations from these norms occur, and there is a certain regularity in these deviations, because exactly the same sequences of random numbers are generated for each of the points observed.

In Tables 1–3 and in the companion Figures 6–8 the standard combination of values of structural variables is taken as a point of departure and results are shown for three partial variations. In Table 1 and Figure 6 we vary z_1 (with z_2 and z_3 and hence y constant). In Table 2 and Figure 7 we vary z_2 (with z_1 and y constant, varying z_3 inversely to z_2). And in Table 3 and Figure 8 we vary y (and z_3, with z_1 and z_2 constant). The Tables record the optimal values of x_1 and x_2 for these combinations and the values of ξ, φ and μ_i $(i = 1, \ldots, 5)$ in the optimal points. In the Figures the optimal values of x_1 and x_2 and the corresponding values of ξ are plotted. Logarithmic scales are used for the structural variables.

Tables 4–6 show some of the same results for simultaneous variations in z_1 and z_2 (with y constant, varying z_3 inversely to z_2). Optimal values of x_1 and x_2 are found in Table 4. Table 5 shows the corresponding values

[1] Pricing rules corresponding to points outside the boundaries of Figures 3 and 4 are technically possible, however, and a sample of such rules was tested. The general shapes of the curves indicated in the Figures continue unbroken by the boundaries.

of ξ and Table 6 the values of φ, with row and column averages. Figures 9–11 are companion Figures to Tables 4 and 5. In the (z_1, z_2)-plane some fixed level loci are shown for optimal x_1 and x_2 and for ξ in optimum, based on interpolations between the 63 observations for $y = 16$ recorded in the Tables and a few supplementary observations. Continuous curves are drawn to simplify exposition, although z_1 (unlike z_2) is actually a discrete variable in number of periods. Logarithmic scales are used for both structural variables.

Tables 7–11 are devoted to the constrained optima, i.e. the case of a uniform burden rate. In addition to the optimal rate $x = x_1 = x_2$ itself, ζ, ξ, and φ are shown for each combination of values of structural variables. ζ is again defined as period profit in the given point, i.e. in the constrained maximum point, as a fraction of period profit in the absolute optimum of burden rate limits. Table 7 records these data for variations in y and Tables 8–11 for variations in z_1 and z_2 from the standard combination. Figure 12 is a companion to Table 9, showing some fixed level loci for ζ. Technical notes to Figures 9–11 also apply to Figure 12.

3.6 Findings: capacity and burden rates

Before we launch on a discussion of the general findings of the study, it is best to explain some peculiar results observed in the limiting cases $z_1 = 1$ and $z_2 = 1$. This may also serve to familiarise the reader with the workings of the model.

When $z_1 = 1$, i.e. when capital life is a single period, there is never any capacity left over at the beginning of a new period. Capacity needed to produce a new product is always acquired by investment, hence the upper burden rate limit is always the one used in pricing. One instance of this is recorded in Table 1, where it is seen that $\mu_4 = \mu_5 = 0$ for $z_1 = 1$. Hence for this and all other combinations corresponding to an element in the first row in Table 4, average profit is independent of the value of x_2 used in the experiment.[1] Optimal values of x_2 do not exist and are omitted in the first row of Table 4 (and of Table 1).

What happens in the case of $z_2 = 1$ is rather more strange. In this case the probability distribution of the number of new products in a period is identical to that of the size of the product family. When the average is as high as $y = z_3 = 16$, as is the case for $z_2 = 1$ in Table 4, the deviation

[1] Different values were tried and confirmed this, providing in fact a test of the logic of the model itself.

relative to the average is so small that investment volume is fairly stable and there is never any large discrepancy between capacity required and capacity left over from the preceding period. Hence it is not optimal to set a lower limit to the burden rate used in pricing. Available capacity is never so large that it should not be entirely used up.

Ideally, this should mean that any value of x_2 sufficiently low never to be actually used, serves equally well as a lower limit. However, because of chance variations coupled with the approximative computational procedure used to locate the appropriate intermediate rates in the frequently occurring type 5 periods, this proved in the experiment to be true only in an average sense. When x_2 was gradually increased, x_1 being held constant at a near optimal level (optimal x_1 varied slightly with x_2), average period profit fluctuated around a horizontal level until falling off abruptly when x_2 had become so large that it was actually used in some periods. These chance fluctuations are irrelevant to our study. Hence for $z_2 = 1$ the optimal value of x_1 and other relevant parameter values were located for the minimum value $x_2 = 0$, this value appearing in parentheses in each element in the first column of Table 4 (and in the first row of Table 2).[1,2]

The choice of $x_2 = 0$ to represent the optima for $z_2 = 1$ was not arbitrary. A product existing for a single period does not bind the firm's future behavior. Hence if there is vacant capacity to fill, it does not seem rational to stop short of zero in reducing the burden rate. This is in accordance with the results obtained in section 2.7. for an isolated product. It means that if relative fluctuations in capacity available and required were considerably larger than in the cases depicted in Table 4, so that much lower burden rates were sometimes needed to use up all capacity, then one would expect a minimum level to be set at $x_2 = 0$. When the average product family is smaller, relative fluctuations increase. Therefore the combination $z_1 = 16$, $z_2 = 1$, $y = 4$ (instead of $y = 16$, as in Table 4) was analysed. Optimal burden rate limits were then located at $x_1 = 1.57$, $x_2 = 0$, while small positive values were found both for the

[1] The explanation given above is not documented in the appended Tables for other than the optima thus located, which in a sense begs the question. It is seen in Table 2 that $\mu_4 = 0$, $\mu_5 = .754$ for $z_2 = 1$ and in Table 5 that $\xi = 0$ for all $z_2 = 1$.

[2] It follows from the discussion of the case $z_1 = 1$ that all capacity costs are always covered. This is also seen in Table 6, where $\varphi = 1$ for all elements in the first row. The values of φ appearing in the first column of the Table are not exactly comparable to the rest because of the special condition under which the optima are located for $z_2 = 1$. Hence row and column averages in Table 6 are computed exclusive of the elements in the first row and column.

relative frequency of type 4 periods (when the lower limit is used) and for relative unused capacity in optimum, $\mu_4 = .017$, $\xi = .019$, which confirms the hypothesis.

For all combinations of values of structural variables where $z_1 > 1$, $z_2 > 1$, optimal values exist for the lower as well as for the upper limit of the burden rate. These optima do not in the general case consist in full employment of all available capacity in all periods where this is possible. On the contrary there exists, in all but the limiting cases, a definite, positive level below which the burden rate should not be lowered, even if this means letting some capacity occasionally lie idle.

As seen in Tables 3 and 5, the relative size of this unused capacity varies with the structural variables according to no simple function. In general part of it remains, even when x_2 is much lower than in optimum. Figure 4 demonstrates this for the standard combination. The amount of unused capacity depends very much on the frequency of type 2 periods, when there are no new products to fill idle capacity at any price. A striking example is offered by the first row of Table 3. Here the very small product family means great fluctuation in available capacity relative to the requirement. We find that no less than 83 per cent of all periods in the production run are of type 2,[1] with a consequent unused capacity of 38.5 per cent, which is the highest recorded for any optimal pricing rule. Table 3 shows that when the average size of the product family increases, relative unused capacity falls off rapidly. With short capital life and long average product life, a product family of a moderate size also gives negligible unused capacity, as seen in Table 5.

Like capacity employment in optimum, the optimal burden rate limits themselves vary considerably with the structural variables, and in rather complex fashions. This is particularly true of the upper limit. Table 3 shows that it decreases with increasing size of the product family. For the standard size, recorded in Table 4, it reaches a maximum for moderate average product lives. The locus of this maximum varies only little with capital life, but the maximum level itself does so considerably. The highest value of x_1 recorded in Table 4 is 2.36 for $z_1 = 256$, $z_2 = 4$. See also the companion Figures 8 and 9.

The variation in the lower burden rate limit in optimum is smaller and depends mostly on average product life, as best seen in Figure 10. Most surprising may be the generally high level on which x_2 is to be set. Except

[1] In optimum. The frequency of type 2 relative to type 1 periods may vary a little with the pricing rule.

for very short product lives, there is no cutting down to variable costs in periods of idle capacity to get as much "contribution" as possible. In the model such a policy does not pay because it binds capacity which may be better employed in future periods.

The study indicates two general conclusion about the optimal levels of the burden rate limits which accord well with the findings in Chapter 2 for a single product.[1] It seems clearly established that the upper limit should never be below unity while the lower limit, though this is less clear, should not be above unity. In fact there is a single observation in Table 4, for $z_1 = 256$, $z_2 = 32$, where $x_2 = 1.02$. But this is most probably due to stochastic variations. I think we are safe in concluding that x_1 and x_2 should be above and below unity respectively for finite values of the structural variables but approach this level asymptotically in limiting cases.

To describe the conditions in the optima of a uniform burden rate, a few words suffice. Except in the irregular case of $z_2 = 1$, it is seen in Tables 7-8 compared to Tables 3–4 that a uniform rate should always be set between the limits found to be optimal in the unconstrained case. Comparing Tables 7 and 10 to Tables 3 and 5, we also find that relative unused capacity is always at least as high and usually much higher in the constrained than in the unconstrained optima. This was to be expected, since the lower limit is valuable exactly as a means to sop up idle capacity.

We can sum up this section by stating a principle which may now seem obvious but which is nevertheless often overlooked in discussions about pricing. When students are told to ignore capacity costs in price decisions because they are sunk costs, the reciprocity of the relation between pricing and investment is forgotten. Past investment is given, but present pricing determines present *and future* investment. In our model the extended product lives at fixed prices claim future capacity directly. In real life there are numerous ways in which the future investment activity of a firm is influenced, directly or indirectly, by its price policy. In this sense capacity costs are also variable costs and a burden rate should be included in the pricing formula to take account of these costs on the margin. Only when there is definitely no competition for present capacity and a product is definitely a single-period product with no future ties at all, should the rate be zero. In all other cases it should be positive. It is naturally higher when there is present investment to pay for, but it is positive even when there is not.

[1] Cp. Section 2.7 and Mathematical Appendix 5.

3.7 The optimality of full cost pricing

We proceed then to examine our main hypothesis about the long-run cost coverage achieved when these rates are chosen optimally. The answer we get is beyond doubt. The study shows that in the model employed the optimal pricing rule is indeed a full cost rule in the broad sense that all capacity costs ought to be exactly covered by the burden applied to products sold. Consider first the standard combination of values of the main structural variables. In the section of the (x_1, x_2)-plane shown in Figure 3 total burden applied varies from zero to more than twice the amount of total investment costs. The locus of all pricing rules where $\varphi = 1$ is an approximately linear curve cutting steeply through the central part of the diagram. The point of maximum period profit lies almost exactly on this curve. Located to the nearest percentage point for both variables, the maximum is $x_1 = 1.23$, $x_2 = .85$, and relative cost coverage in this point is $\varphi = .996$.

This result obtains with very small variations for all other combinations examined. Though, as we have seen, both the optimal burden rate limits and the corresponding unused capacity vary considerably with the structural variables, in each case the limits should be so chosen that actual production exactly carries the cost of this unused capacity. In most of the cases recorded in Tables 3 and 6, φ lies between .99 and 1 in optima located to the nearest percentage point. The average is significantly below unity.[1] The reason for this deviation is not quite evident. It is probably due at least to some extent to approximative computational methods. The deviation is minute, however, and does not refute the very strong general conclusion as to the optimality of full cost pricing in the broad sense used here.

Now it should again be emphasised that this full cost coverage is achieved in most cases by means of a varying burden rate. The interval between the lower and the upper burden rate limits can be considerable, depending on the structural variables. Thus the practice of covering full cost by means of a uniform rate is clearly not optimal in our experiment.

However, if the interval between the limits is reasonably small, which after all it is in many cases, full cost pricing in this narrow sense may be

[1] The output recorded in Table 6 was actually given with five decimals. Omitting the first row and column, the arithmetic mean of the remaining 48 figures in the Table is .9948 and the standard deviation is .0053, so that the standard deviation of the mean (the standard error) can be estimated at about .0008. Hence we find (10000–9948)/8=6.5, which is highly significant.

almost optimal. As seen in Tables 7 and 9 and the companion Figure 12, it is possible to achive 90 per cent or more of absolute maximum by a uniform rate in about two-thirds of the cases recorded. Only when product life is very short and capital life very long is the percentage considerably lower. Moreover, there are certain features in the relations involved which make the application of this single-rate pricing rule more than ordinarily simple.

This is demonstrated in Figures 3–5 for the standard combination of values of the structural variables, which is a typical case of a moderate interval between the burden rate limits. It is seen in Figure 3 that the locus of exact coverage of capacity costs (the curve $\varphi = 1$) passes through the point of maximum average period profit. But in addition it passes, if not through, then at least very close to, the constrained maximum on the line $x_1 = x_2$. This is in fact true for all combinations tested. φ is generally lower than in the unconstrained maximum points, but in no case recorded in Tables 7 and 11 is it below .92. For the standard combination the degree of approximation is best judged from Figure 5. Located to the nearest percentage point, the optimum is $x = x_1 = x_2 = 1.12$, where $\varphi = .982$, $\zeta = .963$. The latter figure means that the constrained maximum is only about four per cent short of absolute maximum.

Note also in Figures 4–5 that relative unused capacity is nearly constant along the line $x_1 = x_2$. This means that all the price maker has to do to come within a few percentage points of absolute maximum is to estimate normal capacity in the sense of maximum capacity less average unused capacity and then to allocate all capacity costs to products on the basis of normal capacity. This is precisely what full-costers are usually reported to do.

3.8 Summary and conclusion

By extending the traditional model of the theory of price to multiple products in multiple periods and trying out alternative pricing rules by simulation, we have found in this study that it pays to use a rule which somewhat resembles the full cost pricing rule observed by empirical investigators. This result is remarkable only because full cost pricing has been treated with scorn by many theorists.

I anticipate the objection that the result is due to some special quirk in the simulation model, absent in real life. This is of course very possible. It is the sorry lot of the simulator to be unable to state precisely what interaction of factors causes the results that he observes. The most that can be said is that nothing has been put in deliberately to invite this result.

The model is a much simplified replica of the industrial pricing situation. Simplicity is partly a virtue of necessity in this case. Partly it expresses a research policy. I wanted to see what happens when we move a first, carefully measured distance away from the barren assumptions of traditional price theory. We have found, perhaps surprisingly, that even at this short distance the rules observed in the much more complicated real life pricing situations have already replaced those of the theory left behind.

This may not suffice to remove the stigma from full cost pricing. But if the study reported here could incite some new interest in a dormant field, bolder ventures in the same direction might give us more and better evidence for evaluating empirical behavior. This is the long-run objective of the book. In itself, to use the words of one of its early critics, it is less a guide for entrepreneurial action than a comment on economic theory.

Mathematical Appendixes

1 *Appendix to Section* 2.1

Single-product, static price optimum

We seek maximum with respect to p of the function

$$\pi = qp - qv - v(q)$$

$$\text{where } \eta = -\frac{dq}{dp}\frac{p}{q}, \quad \alpha = \frac{1}{\eta - 1}.$$

Differentiating, we find the first order, necessary maximum condition

$$\frac{d\pi}{dp} = p\frac{dq}{dp} + q - v\frac{dq}{dp} - v'(q)\frac{dq}{dp} = 0$$

$$p + \frac{dp}{dq}q = v + v'(q)$$

$$p\left(1 - \frac{1}{\eta}\right) = v + v'(q)$$

$$p = (1+\alpha)\left[v + v'(p)\right].$$

Whether this is in fact a maximum depends on the relative rates of increase in costs and revenue. For the particular functions to be assumed in this book the second order, sufficient condition of a maximum is satisfied. Cp. Mathematical Appendix 2. For an analysis of the second order condition in the general case, cp. [6].

2 *Appendix to Section* 2.4

Multi-product, static price optimum

We seek first the unconstrained maximum of

$$\Pi = \sum_{i=1}^{n} q_i(p_i - v_i)$$

where

$$q_i = Q_i p_i^{-\eta_i}, \quad \alpha_i = \frac{1}{\eta_i - 1}, \quad (i = 1, \ldots, n).$$

We find

$$\frac{\partial \Pi}{\partial p_i} = q_i + (p_i - v_i) \frac{dq_i}{dp_i}$$

$$\frac{\partial^2 \Pi}{\partial p_i \partial p_j} \begin{cases} = 0 & (i \neq j) \\ = 2 \dfrac{dq_i}{dp_i} + (p_i - v_i) \dfrac{d^2 q_i}{dp_i^2} & (i = j) \end{cases}$$

$$\frac{dq_i}{dp_i} = -\frac{\eta_i q_i}{p_i}$$

$$\frac{d^2 q_i}{dp_i^2} = \frac{\eta_i(\eta_i + 1) q_i}{p_i^2}$$

$$(i = 1, \ldots, n, \quad j = 1, \ldots, n).$$

The first order maximum condition is

$$\frac{\partial \Pi}{\partial p_i} = q_i - (p_i - v_i) \frac{\eta_i q_i}{p_i} = 0$$

i.e.

$$p_i = (1 + \alpha_i) v_i \qquad (i = 1, \ldots, n).$$

The second order condition is

$$A_1 < 0, \quad A_2 > 0, \quad A_3 < 0, \quad \ldots$$

where

$$A_m = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{3m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1m} & a_{2m} & a_{3m} & \cdots & a_{mm} \end{vmatrix} \qquad (m = 1, \ldots, n)$$

$$a_{ij} = \frac{\partial^2 \Pi}{\partial p_i \partial p_j} \qquad (i = 1, \ldots, n, \quad j = i, \ldots, n).$$

Since $a_{ij} = 0$ for all $i \neq j$, the condition is satisfied if all $a_{ii} < 0$. To prove that this is the case, we substitute in the formula for $\dfrac{\partial^2 \Pi}{\partial p_i^2}$ the expressions for $\dfrac{dq_i}{dp_i}$ and $\dfrac{d^2 q_i}{dp_i^2}$ and, according to the first order condition, put

$$p_i - v_i = \frac{p_i}{\eta_i}$$

to find

$$a_{ii} = \frac{q_i}{p_i}(1-\eta_i) < 0 \quad (i = 1, \ldots, n)$$

which completes the proof for the unconstrained case. It is of course valid also in the special case of a single product. Cp. Mathematical Appendix 1.

If in the maximum point now located, $\sum_{i=1}^{n} q_i u_i \leqq M$, we say that the capacity constraint is not effective. Prices are set at the unconstrained optima. However, if $\sum_{i=1}^{n} q_i u_i > M$, these unconstrained optima are not feasible. The capacity constraint is effective. We then seek maximum of

$$\Pi = \sum_{i=1}^{n} q_i(p_i - v_i)$$

subject to

$$\sum_{i=1}^{n} q_i u_i = M$$

and define

$$U = \Pi - \lambda \left(\sum_{i=1}^{n} q_i u_i - M\right).$$

The first order maximum condition is

$$\frac{\partial U_i}{\partial p_i} = q_i + (p_i - v_i)\frac{dq_i}{dp_i} - \lambda u_i \frac{dq_i}{dp_i} = 0 \qquad (i = 1, \ldots, n)$$

which may be reduced, analogously to the unconstrained case, to

$$p_i = (1+\alpha_i)(v_i + \lambda u_i) \qquad (i = 1, \ldots, n).$$

These n equations plus the capacity constraint suffice to determine λ and the n optimal prices.

The second order maximum condition is in this case

$$B_2 > 0, \quad B_3 < 0, \quad B_4 > 0, \quad \ldots$$

where

$$B_m = \begin{vmatrix} 0 & b_1 & b_2 & \ldots & b_m \\ b_1 & b_{11} & b_{12} & \ldots & b_{1m} \\ b_2 & b_{12} & b_{22} & \ldots & b_{2m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_m & b_{1m} & b_{2m} & \ldots & b_{mm} \end{vmatrix} \qquad (m = 2, \ldots, n)$$

$$b_i = \frac{\partial M}{\partial p_i} \qquad (i = 1, \ldots, n),$$

$$b_{ij} = \frac{\partial^2 \Pi}{\partial p_i \partial p_j} - \frac{\dfrac{\partial \Pi}{\partial p_i}}{\dfrac{\partial M}{\partial p_i}} \frac{\partial^2 M}{\partial p_i \partial p_j} \qquad (i = 1, \ldots, n, \quad j = i, \ldots, n).$$

This condition is clearly satisfied if the sign matrix of B_n is

$$\begin{bmatrix} 0 & - & - & - & \cdots & - \\ - & - & 0 & 0 & \cdots & 0 \\ - & 0 & - & 0 & \cdots & 0 \\ - & 0 & 0 & - & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ - & 0 & 0 & 0 & \cdots & - \end{bmatrix}$$

We shall prove that this is the case. Since $b_{ij} = 0$ for all $i \neq j$, it suffices to prove that

$$b_i < 0, \quad b_{ii} < 0, \quad (i = 1, \ldots, n).$$

Substituting

$$\frac{\partial M}{\partial p_i} = u_i \frac{dq_i}{dp_i}, \qquad \frac{\partial^2 M}{\partial p_i^2} = u_i \frac{d^2 q_i}{dp_i^2}$$

and the expressions for $\dfrac{\partial \Pi}{\partial p_i}$, $\dfrac{\partial^2 \Pi}{\partial p_i^2}$, $\dfrac{dq_i}{dp_i}$, and $\dfrac{d^2 q_i}{dp_i^2}$ derived for the unconstrained case, we find

$$b_i = u_i \frac{dq_i}{dp_i} = -\frac{\eta_i}{p_i} u_i q_i < 0 \qquad (i = 1, \ldots, n)$$

$$\begin{aligned} b_{ii} &= 2\frac{dq_i}{dp_i} + (p_i - v_i)\frac{d^2 q_i}{dp_i^2} - \frac{q_i + (p_i - v_i)\dfrac{dq_i}{dp_i}}{u_i \dfrac{dq_i}{dp_i}} u_i \frac{d^2 q_i}{dp_i^2} \\ &= -2\frac{\eta_i q_i}{p_i} + q_i \frac{p_i}{\eta_i q_i} \frac{\eta_i(\eta_i + 1) q_i}{p_i^2} \\ &= \frac{q_i}{p_i}(1 - \eta_i) < 0 \qquad (i = 1, \ldots, n) \end{aligned}$$

which completes the proof.

3 *Appendix to Section* 2.6

Capacity cost function

To produce the product uq capacity units are needed in each period of its life. It pays to wait as long as possible before reinvesting, hence at the start the firm employs the existing capacity units with the longest remaining life, i.e. the part capacities $\bar{m}_j \quad (j = \kappa+1, \ldots, z_1)$ in full and what is required of $\bar{m}_\kappa$, namely

$$\bar{\mu} = uq - \sum_{j=\kappa+1}^{z_1} \bar{m}_j \, .$$

Let $I_j \quad (j = \kappa+1, \ldots, z_1)$ be the expected costs of reinvestment of $\bar{m}_j$ and let I_κ be the expected costs of reinvestment of $\bar{\mu}$. Then

$$I = \sum_{j=\kappa}^{z_1} I_j \, .$$

Let k be the actual life of the product to be priced. We recall that k is a geometrically distributed stochastic variable with expectation z_2. This means that after each period there is a probability $\frac{1}{z_2}$ that the product will not remain in the next period, there is a probability $\left(1-\frac{1}{z_2}\right)^j$ that it will remain for at least j periods, a probability $1-\left(1-\frac{1}{z_2}\right)^j$ that it will remain for less than j periods, and hence a probability

$$\left(1-\frac{1}{z_2}\right)^j \left[1-\left(1-\frac{1}{z_2}\right)^i\right]$$

that it will remain for at least j periods but for less than $j+i$ periods.

Consider now first the reinvestment process for $\bar{\mu}$. The firm makes no investment if $k \leqq \kappa$. (It may be that $\kappa = 0$. This means that total capacity available in the initial period is insufficient. New investment must be made immediately. Mathematically this is expressed by the fact that $k \leqq \kappa = 0$ is impossible, since the product, once introduced, will remain for at least this first period. Minimum expected product life is $z_2 = 1$, which is a limiting case of the geometric distribution where the event $k = 1$ has probability 1.) The firm invests i times $(i > 0)$ if $\kappa + iz_1 \geqq k > \kappa+(i-1)z_1$, i.e. if the product lasts at least the remaining life of the existing capacity units $\bar{\mu}$ plus more than $i-1$ but not more than i full reinvestment lives. The probability of this event is

$$\left(1-\frac{1}{z_2}\right)^{\kappa+(i-1)z_1}\left[1-\left(1-\frac{1}{z_2}\right)^{z_1}\right]$$

Each reinvestment costs $\psi z_1\bar{\mu}$. Multiplying costs and probabilities and taking the sum over all positive i, we find expected reinvestment costs for this part capacity,

$$\begin{aligned} I_\kappa &= \psi z_1\bar{\mu}\left(1-\frac{1}{z_2}\right)^{\kappa}\left[1-\left(1-\frac{1}{z_2}\right)^{z_1}\right]\sum_{i=1}^{\infty} i\left(1-\frac{1}{z_2}\right)^{(i-1)z_1} \\ &= \frac{z_1}{1-\left(1-\frac{1}{z_2}\right)^{z_1}}\,\psi\bar{\mu}\left(1-\frac{1}{z_2}\right)^{\kappa} \\ &= \frac{z_1}{1-\left(1-\frac{1}{z_2}\right)^{z_1}}\,\psi\left(uq-\sum_{j=\kappa+1}^{z_1}\bar{m}_j\right)\left(1-\frac{1}{z_2}\right)^{\kappa}. \end{aligned}$$

To analyse the reinvestment process for any of the part capacities that are fully employed in the initial period, substitute j and $\bar{m}_j$ $(j=\kappa+1,\ldots,z_1)$ for κ and $\bar{\mu}$ and proceed as above. We find

$$I_j=\frac{z_1}{1-\left(1-\frac{1}{z_2}\right)^{z_1}}\,\psi\bar{m}_j\left(1-\frac{1}{z_2}\right)^{j}\quad (j=\kappa+1,\ \ldots,z_1)\,.$$

Hence

$$\begin{aligned} I &= \sum_{j=\kappa+1}^{z_1} I_j+I_\kappa \\ &= \frac{z_1}{1-\left(1-\frac{1}{z_2}\right)^{z_1}}\,\psi\left[\sum_{j=\kappa+1}^{z_1}\bar{m}_j\left(1-\frac{1}{z_2}\right)^{j}+\left(uq-\sum_{j=\kappa+1}^{z_1}\bar{m}_j\right)\left(1-\frac{1}{z_2}\right)^{\kappa}\right] \\ &= \frac{z_1\left(1-\frac{1}{z_2}\right)^{\kappa}}{1-\left(1-\frac{1}{z_2}\right)^{z_1}}\,\psi\left[uq-\sum_{j=\kappa+1}^{z_1}\bar{m}_j\left(1-\left(1-\frac{1}{z_2}\right)^{j-\kappa}\right)\right] \\ &= z_2\varrho(\kappa)\psi[uq-\sigma(\kappa)] \end{aligned}$$

with $\varrho(\kappa)$ and $\sigma(\kappa)$ defined as in the text.

4 *Appendix to Section* 2.6

Single-product, dynamic price optimum

We seek maximum of

$$\bar{\Pi} = z_2 q[p - v - \varrho(\kappa)\psi u] + z_2 \psi \varrho(\kappa)\sigma(\kappa) .$$

We have assumed that this maximum occurs in an interval where $\varrho(\kappa)$ and $\sigma(\kappa)$ are constant, hence it coincides with maximum of

$$\bar{\bar{\Pi}} = q(p - \bar{v})$$

where

$$\bar{v} = v + \varrho(\kappa)\psi u .$$

Putting $n = 1$ in Mathematical Appendix 2 (i.e. assuming a single product), dropping the indices identifying individual products, substituting $\bar{v}$ for v and proceeding as in the unconstrained case, we find the first order maximum condition

$$p = (1+\alpha)\bar{v}$$

which was proved also to satisfy the second order condition.

5 *Appendix to Section* 2.7

Analysis of burden rate function

We shall analyse the function

$$\varrho = \frac{z_1}{z_2} \frac{\left(1 - \frac{1}{z_2}\right)^{\kappa}}{1 - \left(1 - \frac{1}{z_2}\right)^{z_1}}$$

for finite z_2, finite integer z_1 and κ, $z_1 > 1$, $z_2 > 1$, $0 \leqq \kappa < z_1$. Defining $\upsilon = 1 - \frac{1}{z_2}$ $(0 \leqq \upsilon < 1)$, we write

$$\varrho = \frac{z_1(1-\upsilon)\upsilon^{\kappa}}{1-\upsilon^{z_1}} = \frac{z_1 \upsilon^{\kappa}}{1+\upsilon+\upsilon^2+\ldots+\upsilon^{z_1-1}} .$$

Consider first the case $\kappa = 0$. Then for all z_2, $\varrho = 1$ for $z_1 = 1$, $\varrho > 1$ for $z_1 > 1$.

Consider next the case $\kappa \geqq 1$, $z_2 = 1$ (i.e. $\upsilon = 0$). We also have $z_1 > 1$ by definition of κ. We find $\varrho = 0$ for all z_1.

Finally consider the general case $\kappa \geqq 1$ (hence $z_1 > 1$), $z_2 > 1$. Keeping z_1 and κ constant at finite values, we differentiate ϱ partially with respect to υ and find

$$\frac{\partial \varrho}{\partial \upsilon} = \frac{z_1 \upsilon^{\kappa-1} \Theta}{(1+\upsilon+\upsilon^2+\ldots+\upsilon^{z_1-1})^2}$$

where

$$\Theta = (1+\upsilon+\upsilon^2+\ldots+\upsilon^{z_1-1})\kappa-(\upsilon+2\upsilon^2+3\upsilon^3+\ldots+(z_1-1)\upsilon^{z_1-1})$$

so that

$$\lim_{\upsilon \Rightarrow 1} \Theta = \kappa z_1-(z_1-1)\frac{z_1}{2} = z_1\left(\kappa-\frac{z_1-1}{2}\right).$$

Consider first the case $\kappa \geqq \frac{z_1-1}{2}$. Then

$$\lim_{\upsilon \Rightarrow 1} \frac{\partial \varrho}{\partial \upsilon} = z_1 \lim_{\upsilon \Rightarrow 1} \Theta \geqq 0.$$

It is also easily verified that in this case $\frac{\partial \varrho}{\partial \upsilon} > 0$ for all $\upsilon < 1$. Hence maximum of ϱ with respect to υ is found at

$$\lim_{\upsilon \Rightarrow 1} \varrho = 1$$

which means that $\varrho < 1$ for all $\upsilon < 1$, i.e. for all finite z_2.

Then consider the case $\kappa < \frac{z_1-1}{2}$. Here

$$\lim_{\upsilon \Rightarrow 1} \frac{\partial \varrho}{\partial \upsilon} < 0.$$

Hence maximum of ϱ with respect to υ is found for $\upsilon < 1$ and is higher than unity. However, it is also easy to find examples of $\kappa < \frac{z_1-1}{2}$ where $\varrho < 1$ for some finite z_2.

6 *Appendix to Section* 2.7

Cost coverage

With reference to assumptions and definitions in the text (cp. also Section 2.6) a complete statement is given here of the conditions of cost coverage. For any optimal quantity q there exists a κ so that

$$I = z_2\varrho(\kappa)\psi[uq-\sigma(\kappa)] .$$

When optimal quantity falls in the interior of one of the existing part capacities, optimal price is found by differentiation, and total burden is

$$R = z_2\varrho(\kappa)\psi uq$$

so that

$$R-I = z_2\psi\varrho(\kappa)\sigma(\kappa) ,$$

where the three first factors to the right are always positive. When capacity is uniformly old, all $\bar{m}_j = 0 \quad (j = \kappa+1, \ldots, z_1)$. Hence $\sigma(\kappa) = 0$, i.e. $R = I$. When part capacities with different ages are employed, there must be at least one $\bar{m}_j > 0 \quad (j = \kappa+1, \ldots, z_1)$. Hence $\sigma(\kappa) > 0$, i.e. $R > I$ (except when $z_2 \to \infty$; then $\sigma(\kappa) \to 0$).
Now consider the case when optimal quantity exactly coincides with a number of the existing part capacities, i.e. $\bar{\mu} = \bar{m}_k$ and

$$uq = \sum_{j=\kappa}^{z_1} \bar{m}_j .$$

The problem is here that optimal price cannot be found by differentiation, since the marginal cost curve has a break in optimum. However, assuming that pricing rule (4) is used as in the general case, there is a marginal cost r such that $p = (1+\alpha)r$ where p is the price chosen. It is possible to reason about the size of r. Varying quantity from the optimal level, we find marginal cost

(i) $$r_1 = v+\varrho(\kappa)\psi u \quad \text{for} \quad uq < \sum_{j=\kappa}^{z_1} \bar{m}_j .$$

(ii) $$r_2 = v+\varrho(\kappa-1)\psi u \quad \text{for} \quad uq > \sum_{j=\kappa}^{z_1} \bar{m}_j .$$

((i) is impossible if all $\bar{m}_j = 0 \quad (j = \kappa, \ldots, z_1)$, but we may ignore this trivial case which means that the optimal quantity is zero. As for (ii), this means investing in new capacity if $\kappa = 1$. $\kappa = 0$ is impossible by definition of $\bar{m}_0$.)

Since $\varrho(j)$ is a non-increasing function of j, it follows that $r_1 \leqq r_2$. Hence r should be chosen anywhere in the interval $r_1 \leqq r \leqq r_2$ and total burden

$\bar{R}$ applied in this case should be

$$\bar{R} = z_2 q(r-v) \geqq z_2 q(r_1-v) = z_2 q \varrho(\kappa)\psi u = R\,.$$

But we have just found $R > I$, hence $\bar{R} > I$ (except if all $\bar{m}_j = 0$ $(j = \kappa+1, \ldots, z_1)$ and $r = r_1$, in which case $\bar{R} = I$).

7 *Appendix to Section* 3.1

Product family distribution

The proposition stated in the text holds even if the product life distribution is not geometric. But then the process is not a life and death process in the proper sense. There is no simple way to describe the termination of product lives in a given period. When product lives are geometrically distributed, the number of products dropping off in any one period has a binomial distribution, which makes the proof very simple. We shall consider only this case.

Assume then that m is the number of products produced by the firm in an arbitrary period. Let

$$\gamma_1(m,j) \qquad (m = 0, \ldots, \infty\,, \quad j = 0, \ldots, m)$$

be the probability that exactly j of these products remain in the next period and let

$$\gamma_2(k) \qquad (k = 0, \ldots, \infty)$$

be the probability that exactly k new products are added in an arbitrary period to those remaining from the previous one. Hence the conditional probability that n products will be produced in a period immediately succeeding one in which m products were produced, is

$$\gamma_3(m,n) \begin{cases} = \sum_{j=0}^{m} \gamma_1(m,j)\gamma_2(n-j) & (m = 0, \ldots, n-1) \\ = \sum_{j=0}^{n} \gamma_1(m,j)\gamma_2(n-j) & (m = n, \ldots, \infty) \end{cases}$$
$$(n = 0, \ldots, \infty)\,.$$

The life and death process of products is then completely described by the transition matrix

$$[\gamma_3(m, n)] \qquad (m = 0, \ldots, \infty\,, \quad n = 0, \ldots, \infty)$$

with a given initial number of products.

When the products have independent geometric life distributions, as assumed in the text, we have

$$\gamma_1(m,j) = \binom{m}{j}\left(\frac{1}{z_2}\right)^{m-j}\left(1-\frac{1}{z_2}\right)^{j}.$$

We have further assumed

$$\gamma_2(k) = \frac{z_3^k\, \mathrm{e}^{-z_3}}{k!}.$$

We then want to prove that the life and death process, starting with zero products, converges to a stationary probability distribution

$$\omega(\mathrm{n}) = \frac{(z_2 z_3)^n\, \mathrm{e}^{-z_2 z_3}}{n!} \qquad (n = 0, \ldots, \infty)$$

for the number n of products produced in an arbitrary period.

A complete proof by iteration is possible. However, since all $\gamma_3(m, n)$ are positive, the process is an irreductible Marcow chain, which is ergodic if there exists a stationary distribution. Hence it suffices to prove that the postulated distribution is stationary, i.e. that

$$\omega(n) = \sum_{m=0}^{\infty} \omega(m)\gamma_3(m, n) \qquad (n = 0, \ldots, \infty).$$

This proof is now given.

Since for any matrix $[a_{m,j})$,

$$\sum_{m=0}^{n-1}\sum_{j=0}^{m} a_{m,j} + \sum_{m=n}^{\infty}\sum_{j=0}^{n} a_{m,j} = \sum_{j=0}^{n}\sum_{m=j}^{\infty} a_{m,j}$$

and since

$$\sum_{m=j}^{\infty} \frac{z_3^{m-j}}{(m-j)!} = \mathrm{e}^{z_3}$$

and

$$\sum_{j=0}^{n} \binom{n}{j}\left(\frac{1}{z_2}\right)^{n-j}\left(1-\frac{1}{z_2}\right)^{j} = 1,$$

we have

$$\sum_{m=0}^{\infty} \omega(m)\gamma_3(m,n) = \sum_{m=0}^{n=1}\sum_{j=0}^{m} \omega(m)\gamma_1(m,j)\gamma_2(n-j)$$
$$+ \sum_{m=n}^{\infty}\sum_{j=0}^{n} \omega(m)\gamma_1(m,j)\gamma_2(n-j)$$

$$= \sum_{j=0}^{n} \sum_{m=j}^{\infty} \omega(m)\gamma_1(m,j)\gamma_2(n-j)$$

$$= \sum_{j=0}^{n} \sum_{m=j}^{\infty} \frac{(z_2 z_3)^m \, \mathrm{e}^{-z_2 z_3}}{m!} \binom{m}{j} \left(\frac{1}{z_2}\right)^{m-j} \left(1-\frac{1}{z_2}\right)^{j} \frac{z_3^{n-j} \, \mathrm{e}^{-z_3}}{(n-j)!}$$

$$= \frac{(z_2 z_3)^n \, \mathrm{e}^{-z_2 z_3}}{n!} \, \mathrm{e}^{-z_3} \sum_{j=0}^{n} \left[\binom{n}{j} \left(\frac{1}{z_2}\right)^{n-j} \left(1-\frac{1}{z_2}\right)^{j} \sum_{m=j}^{\infty} \frac{z_3^{m-j}}{(m-j)!}\right]$$

$$= \frac{(z_2 z_3)^n \, \mathrm{e}^{-z_2 z_3}}{n!} .$$

$$= \omega(n)$$

which completes the proof.

Fortran Program

```
C     START

C     PRICING SIMULATION. MODEL A. APRIL 1968.
C     OPTIMAL BURDEN RATES IN PRICE CALCULATION.

C     MULTI-PRODUCT FIRM. INDEPENDENT PRODUCTS USING A COMMON CAPACITY.
C     LINEAR COST RELATIONS. CONSTANT ELASTICITIES OF DEMAND.
C     PRODUCTS PRICED AT INTRODUCTION. DEMAND CONSTANT THROUGH PRODUCT LIVES.
C     RANDOM GENERATION OF NEW PRODUCTS. RANDOM TERMINATION OF PRODUCT LIVES.
C     CAPACITY EXTENDED WHEN REQUIRED. LIFE OF CAPITAL EQUIPMENT FIXED.
C     BURDEN RATE VARYING BETWEEN X1 AND X2 ACCORDING TO CAPACITY SITUATION.

C     NAMES OF VARIABLES, VECTORS AND CONSTANTS.

C     X1    UPPER LIMIT OF BURDEN RATE. (DECISION VARIABLE.)
C     X2    LOWER LIMIT OF BURDEN RATE. (DECISION VARIABLE.)
C     Z1    LIFE OF CAPITAL EQUIPMENT. (MAIN STRUCTURAL VARIABLE.)
C     Z2    AVERAGE LIFE OF PRODUCTS. (MAIN STRUCTURAL VARIABLE.)
C     Z3    AVERAGE NUMBER OF NEW PRODUCTS PER PERIOD. (MAIN STRUCTURAL VAR.)
C     IS1   INTEGER LINEAR TRANSFORM OF X1.
C     IS2   INTEGER LINEAR TRANSFORM OF X2.
C           (IS1,IS2 USED IN INPUT TO PERMIT READING POS. INT. FOR NEG. X1,X2.)
C     IZ1   INTEGER FORM OF Z1.
C     IZ2   INTEGER FORM OF Z2.
C     Y     AVERAGE SIZE OF PRODUCT FAMILY. (NUMBER OF PRODUCTS.)
C     IY    INTEGER FORM OF Y.

C     MTS   LENGTH OF START RUN.
C     AMTS  REAL FORM OF MTS.
C     MTP   LENGTH OF PRODUCTION RUN.
C     AMTP  REAL FORM OF MTP.
C     MT    TOTAL LENGTH OF RUN.
C     NT    PERIOD NUMBER IN RUN.
C     ANT   REAL FORM OF NT.
C     NP    NUMBER OF PRODUCTS IN PERIOD.
C     NOP   NUMBER OF OLD PRODUCTS.
C     NOP1  ONE MORE THAN NUMBER OF OLD PRODUCTS.
C     NNP   NUMBER OF NEW PRODUCTS.
C     KPOIS RANDOM NUMBER GENERATED PERIODICALLY TO COMPUTE POISSON VARIATE NNP.
C     POIS  REAL FORM OF KPOIS.
C     APOIS AUXILIARY VARIABLE USED IN THIS COMPUTATION.
C     IPOIS DITTO.
C     MPOIS MULTIPLIER USED IN GENERATION OF KPOIS.
C     E(I)  ELASTICITY OF DEMAND FOR PRODUCT I.
C     Q(I)  SCALE DETERMINANT OF DEMAND FOR PRODUCT I.
C     V(I)  VARIABLE COSTS PER UNIT OF PRODUCT I.
C     U(I)  CAPACITY REQUIRED PER UNIT OF PRODUCT I.
C     K(I)  RAND. FRACTION GEN. PERIODICALLY TO LOCATE END OF LIFE OF PRODUCT I.
C     AKI   AUXILIARY VARIABLE USED WITH K(I) TO LOCATE ENDS OF PRODUCT LIVES.
C     KREC1 RANDOM NUMBER GENERATED FIVE TIMES FOR EACH NEW PRODUCT TO DETERMINE
C           THE RECTANGULARLY DISTRIBUTED E(I),Q(I),V(I),U(I) AND INITIAL K(I).
C     S(J)  AUXILIARY VECTOR USED WITH KREC1 TO DETERMINE THESE VARIABLES.
C     MREC1 MULTIPLIER USED IN GENERATION OF KREC1.
C     MREC2 MULTIPLIER USED IN GENERATION OF K(I) IN SUBSEQUENT PERIODS.
C     X(I)  BURDEN RATE APPLIED TO PRODUCT I.
C     XH    AUXILIARY RATE EMPLOYED IN SOME PERIODS TO LOCATE X(I).
C     XM,XL DITTO.
C     P(I)  PRICE OF PRODUCT I.
C     D(I)  DEMAND FOR PRODUCT I.
C     R(I)  CAPACITY REQUIRED FOR PRODUCT I.
```

```
C     C(I)  CONTRIBUTION OF PRODUCT I.

C     A(J)  CAPACITY AVAILABLE OF AGE J.
C     AN    CAPACITY AVAILABLE FOR NEW PRODUCTS.
C     AA    TOTAL CAPACITY AVAILABLE IN PERIOD.
C     RO    CAPACITY REQUIRED FOR OLD PRODUCTS.
C     RN    CAPACITY REQUIRED FOR NEW PRODUCTS.
C     RR    TOTAL CAPACITY REQUIRED IN PERIOD.
C     BB    TOTAL BURDEN APPLIED IN PERIOD.
C     PROF  PERIOD PROFIT.

C     N1-N5 PERIOD CLASSICATION COUNTERS, DENOTING FREQUENCY OF PERIODS WITH
C           (N1) NEW INVESTMENT, NO NEW PRODUCTS.
C           (N2) NO NEW INVESTMENT, NO NEW PRODUCTS.
C           (N3) NEW INVESTMENT, BURDEN RATE X1 APPLIED TO NEW PRODUCTS.
C           (N4) NO NEW INVESTMENT, BURDEN RATE X2 APPLIED TO NEW PRODUCTS.
C           (N5) NO NEW INVESTMENT, INTERMEDIATE RATE APPLIED TO NEW PRODUCTS.
C     TA    TOTAL CAPACITY AVAILABLE IN PRODUCTION RUN.
C     TR    TOTAL CAPACITY REQUIRED IN PRODUCTION RUN.
C     TB    TOTAL BURDEN APPLIED IN PRODUCTION RUN.
C     TPROF TOTAL PROFIT IN PRODUCTION RUN.
C     ARATE WEIGHTED AVERAGE OF BURDEN RATES APPLIED IN PRODUCTION RUN.
C     RELUN RELATIVE UNUSED CAPACITY IN PRODUCTION RUN.
C     SRATE BURDEN RATE WHICH EXACTLY COVERS ALL COSTS IN PRODUCTION RUN.
C     RELCO RELATIVE COVERAGE OF CAPACITY COSTS IN PRODUCTION RUN.
C     APROF AVERAGE PROFIT PER PERIOD IN PRODUCTION RUN. (OBJECTIVE VARIABLE.)

C     DIMENSION AND FORMAT STATEMENTS.

      DIMENSION E(900), Q(900), V(900), U(900), K(900)
      DIMENSION X(900), P(900), R(900), C(900), D(900)
      DIMENSION S(5), A(256)
C
  700 FORMAT (3I3,2I4)
  701 FORMAT (1H1)
  702 FORMAT (1H0,2F9.0,F10.2,2F8.3,7I9)
  703 FORMAT (1H0,3F16.0,4F11.5,F15.2)

C     READ VALUES OF DECISION VARIABLES AND MAIN STRUCTURAL VARIABLES.

      READ(1,700) IZ1,IZ2,IY,IS1,IS2
      Y=IY
      Z1=IZ1
      Z2=IZ2
      Z3=Y/Z2
      X1=FLOAT(IS1)/100.-10.
      X2=FLOAT(IS2)/100.-10.

C     COMPUTE LENGTHS OF START RUN AND PRODUCTION RUN.

      MTS=4*IZ1
      IF(IZ1-IZ2) 33,34,34
   33 MTS=4*IZ2
   34 AMTS=MTS
      AMTP=10000./Z3
      IF(AMTP-Z1*10000./256.) 36,37,37
   36 AMTP=Z1*10000./256.
   37 IF (AMTP-1250.) 38,39,39
   38 AMTP=1250.
   39 MTP=AMTP
      MT=MTS+MTP

C     SET INITIAL PARAMETER VALUES.

      NT=0
      NP=0
      KPOIS=588782217
      KREC1=418519179
      MPOIS=65539
      MREC1=65533
      MREC2=65531
```

```
      DO 80 I=1,IZ1
   80 A(I)=0.
      TA=0.
      TB=0.
      TR=0.
      TPROF=0.
      N1=0
      N2=0
      N3=0
      N4=0
      N5=0
C     START PERIODIC LOOP.
      DO 20 NT=1,MT
      ANT=NT
C     IF START RUN IS TERMINATED,
C     SET NEW INITIAL VALUES OF RANDOM NUMBER GENERATORS,
C     ROTATE MULTIPLIERS,
C     ZERO PERIOD CLASSIFICATION COUNTERS.
      IF(NT-MTS-1) 42,41,42
   41 KPOIS=915394631
      KREC1=386056621
      MPOIS=65533
      MREC1=65531
      MREC2=65539
      N1=0
      N2=0
      N3=0
      N4=0
      N5=0
   42 CONTINUE
C     COMPUTE NUMBER OF NEW PRODUCTS.
      IPOIS=0
      APOIS=0
    1 IPOIS=IPOIS+1
      KPOIS=KPOIS*MPOIS
      IF(KPOIS) 505,506,506
  505 KPOIS=KPOIS+2147483647+1
  506 POIS=KPOIS
      APOIS=APOIS-ALOG(POIS*.4656613E-9)
      IF(APOIS-Z3) 1,1,2
    2 NNP=IPOIS-1
C     INSPECT OLD PRODUCT LINE.
      NOP=NP
      IF(NOP) 6,6,3
C     PREPARE NEW NUMBERING OF RETAINED PRODUCTS.
    3 M=0
      DO 5 I=1,NOP
C     DELETE TERMINATED PRODUCTS.
      K(I)=K(I)*MREC2
      IF(K(I)) 515,516,516
  515 K(I)=K(I)+ 2147483647+1
  516 AKI=K(I)
      AKI=AKI*.4656613E-9
      IF(AKI-1./22) 5,4,4
C     CHANGE NUMBERING OF RETAINED PRODUCTS IN OLD LINE.
    4 M=M+1
      K(M)=K(I)
```

```
      X(M)=X(I)
      R(M)=R(I)
      C(M)=C(I)
    5 CONTINUE
      NOP=M

C     COMPUTE PARAMETER VALUES OF NEW PRODUCTS.

    6 NP=NNP+NOP
      IF(NNP) 10,10,7
    7 NOP1=NOP+1
      DO 9 I=NOP1,NP
      DO 8 J=1,5
      KREC1=KREC1*MREC1
      IF(KREC1) 525,526,526
  525 KREC1=KREC1+2147483647+1
  526 S(J)=KREC1
    8 S(J)=S(J)*.4656613E-9+0.5
      E(I)=4.*S(1)
      Q(I)=1000000.*S(2)
      V(I)=S(3)
      U(I)=S(4)
    9 K(I)=KREC1

C     INSPECT NEW PRODUCT LINE.

C     COMPUTE CAPACITY REQUIRED FOR OLD PRODUCTS.

   10 RO=0.
      IF(NOP) 72,72,601
  601 DO 602 I=1,NOP
  602 RO=RO+R(I)

C     COMPUTE CAPACITY AVAILABLE FOR NEW PRODUCTS.

   72 AA=0
      IF(IZ1-1) 14,14,11
   11 DO 12 I=2,IZ1
      J=IZ1+2-I
      A(J)=A(J-1)
   12 AA=AA+A(J)
   14 AN=AA-RO

C     COMPUTE PRICES OF NEW PRODUCTS.
C     COMPUTE CAPACITY REQUIRED FOR NEW PRODUCTS.
C     DECIDE ON NEW INVESTMENT.
C     CLASSIFY PERIOD.

      IF(NNP) 603,603,606
  603 IF(AN) 604,605,605
  604 A(1)=-AN
      N1=N1+1
      GO TO 15
  605 A(1)=0.
      N2=N2+1
      GO TO 15
  606 RN=0.
      DO 607 I=NOP1,NP
      X(I)=X1
      P(I)=(E(I)/(E(I)-1.))*(V(I)+U(I)*X(I))
      D(I)=Q(I)/P(I)**E(I)
      R(I)=D(I)*U(I)
  607 RN=RN+R(I)
      IF(AN-RN) 608,608,609
  608 A(1)=RN-AN
      N3=N3+1
      GO TO 617
  609 RN=0.
      DO 610 I=NOP1,NP
      X(I)=X2
      P(I)=(E(I)/(E(I)-1.))*(V(I)+U(I)*X(I))
```

```
      D(I)=Q(I)/P(I)**E(I)
      R(I)=D(I)*U(I)
  610 RN=RN+R(I)
      IF(RN-AN) 611,611,612
  611 A(1)=0.
      N4=N4+1
      GO TO 617
  612 A(1)=0.
      N5=N5+1
      XH=X1
      XL=X2
      DO 616 IJ=1,10
      XM=(XL+XH)/2.
      RN=0.
      DO 613 I=NOP1,NP
      X(I)=XM
      P(I)=(E(I)/(E(I)-1.))*(V(I)+U(I)*X(I))
      D(I)=Q(I)/P(I)**E(I)
      R(I)=D(I)*U(I)
  613 RN=RN+R(I)
      IF(RN-AN) 614,614,615
  614 XH=XM
      GO TO 616
  615 XL=XM
  616 CONTINUE

C     COMPUTE TOTAL CAPACITY AVAILABLE IN PERIOD.
C     COMPUTE TOTAL BURDEN APPLIED IN PERIOD.
C     COMPUTE TOTAL CAPACITY REQUIRED IN PERIOD.
C     COMPUTE PERIOD PROFIT.

  617 DO 618 I=NOP1,NP
  618 C(I)=D(I)*(P(I)-V(I))
   15 PROF=-Z1*A(1)
      AA=AA+A(1)
      BB=0.
      RR=0.
      IF(NP) 74,74,73
   73 DO 16 I=1,NP
      PROF=PROF+C(I)
      BB=BB+R(I)*X(I)
   16 RR=RR+R(I)

C     END PERIODIC LOOP.

C     COMPUTE AGGREGATE VALUES.
C     SKIP IF STILL IN START RUN.

   74 IF(NT-MTS) 20,20,17
   17 TA=TA+AA
      TB=TB+BB
      TR=TR+RR
      TPROF=TPROF+PROF

C     IF PRODUCTION RUN IS TERMINATED.
C     COMPUTE RELATIVE VALUES.
C     PRINT RESULTS.

      IF(NT-MT) 20,18,18
   18 ARATE=TB/TR
      RELUN=(TA-TR)/TA
      SRATE=TA/TR
      RELCO=TB/TA
      APROF=TPROF/(ANT-AMTS)
      WRITE(3,701)
      WRITE(3,702) Z1,Z2,Z3,X1,X2,MTS,MTP,N1,N2,N3,N4,N5
      WRITE(3,703) TA,TB,TR,ARATE,RELUN,SRATE,RELCO,APROF
   20 CONTINUE

      CALL EXIT
      END
```

Figures

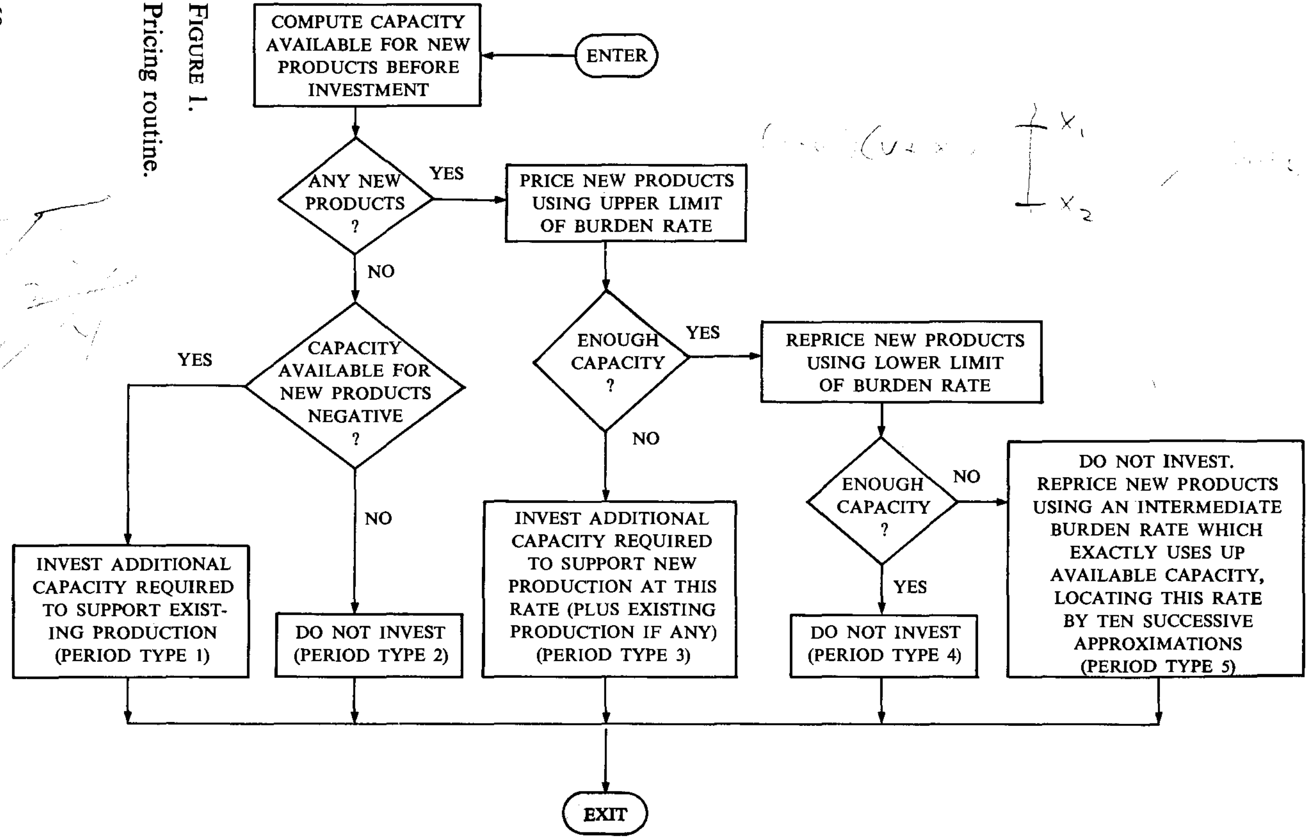


FIGURE 1.
Pricing routine.

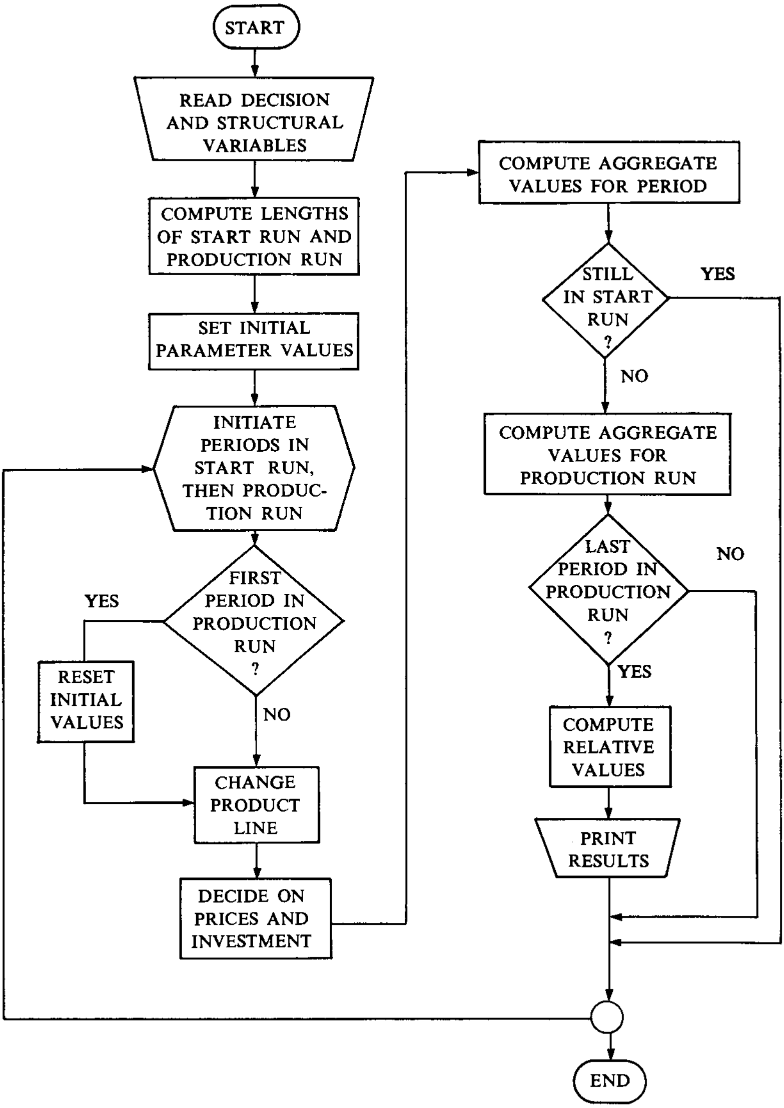
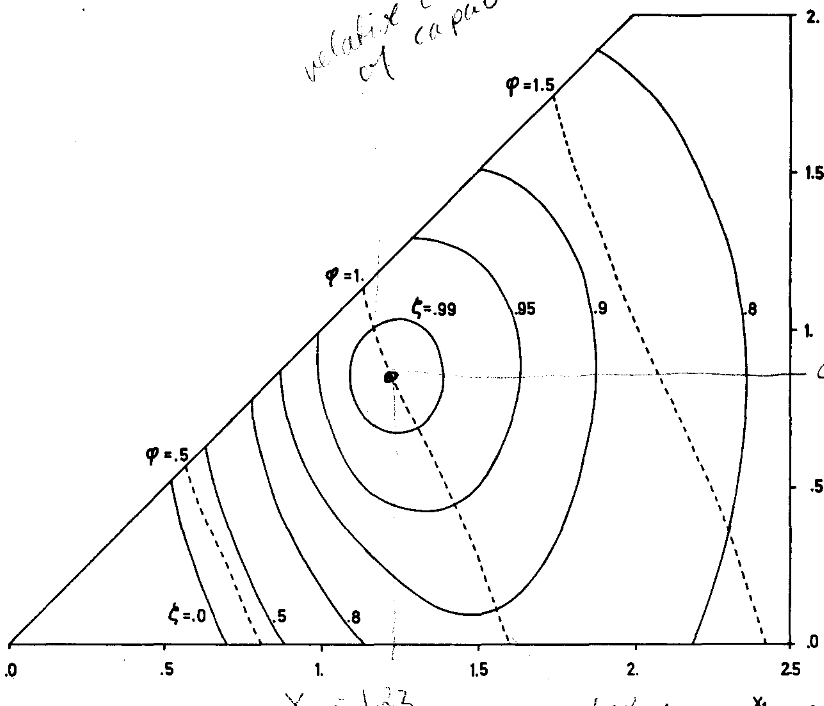


FIGURE 2.

General flow chart of simulation model.

relative coverage of capacity costs.



$0{,}85\ x_2$

$x_1 = 1{,}23$

Deler inn alt
$\Rightarrow$ går gjennom optimum.

FIGURE 3

Relative profit (ζ) and relative coverage of capacity costs (φ) as functions of upper (x_1) and lower (x_2) limits of burden rate, with constant capital life ($z_1 = 16$), average product life ($z_2 = 8$), and average size of product family ($y = 16$).

Forsøke finne maksimum langs skrålinja.
Profitten på skrålinja er der "kotene" skjærer linjen.

$\varphi = 1.0$ er selvkost regelen.

Ved å 6

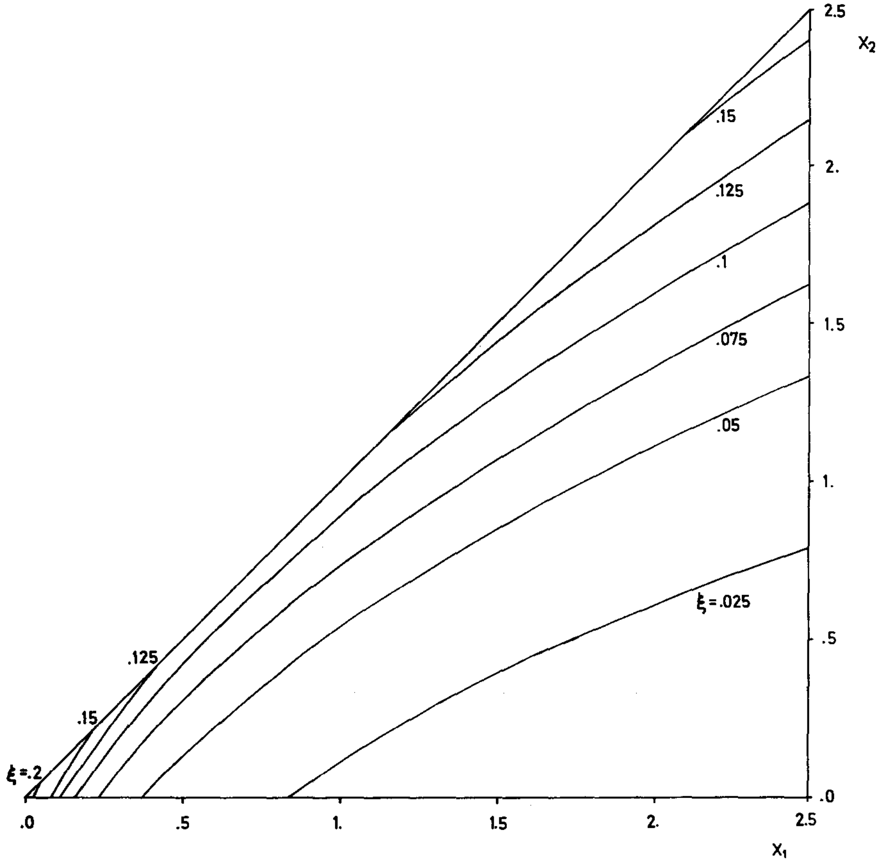


FIGURE 4

Relative unused capacity (ξ) as a function of upper (x_1) and lower (x_2) limits of burden rate, with constant capital life ($z_1 = 16$), average product life ($z_2 = 8$), and average size of product family ($y = 16$).

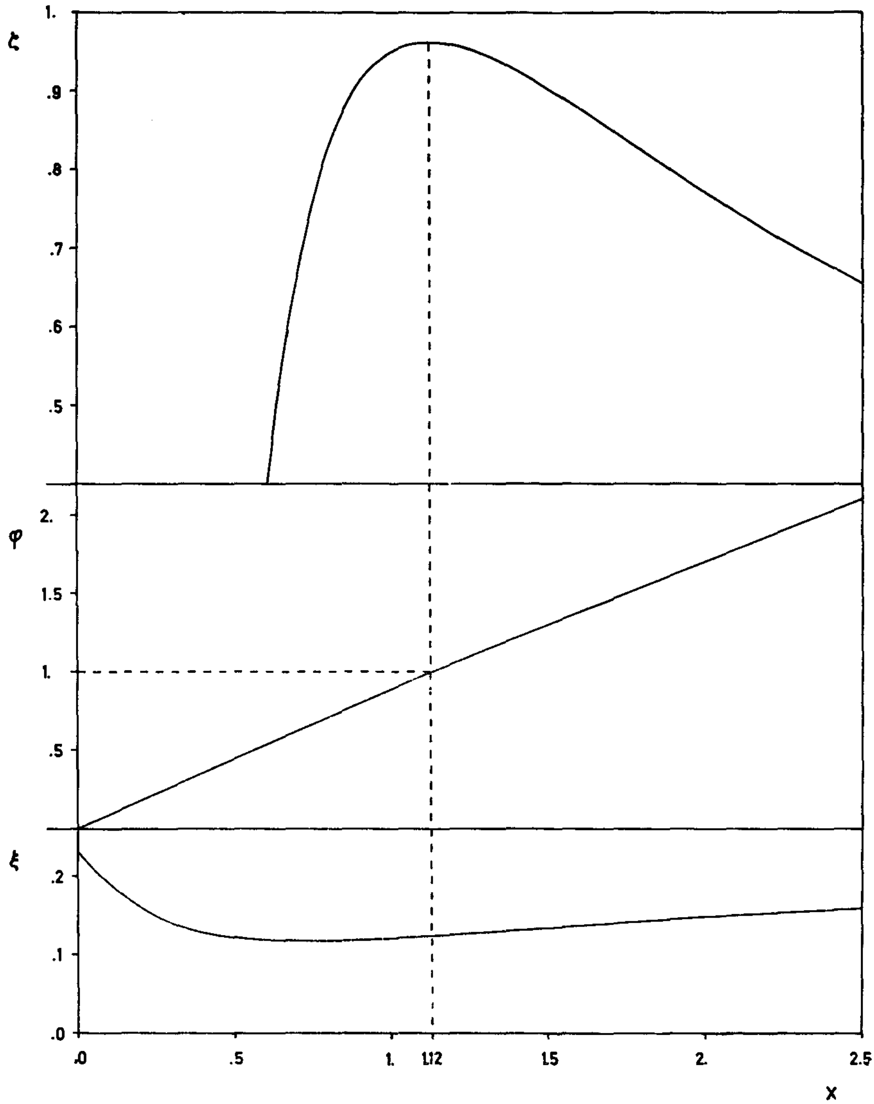


FIGURE 5

Relative profit (ζ), relative coverage of capacity costs (φ), and relative unused capacity (ξ) as functions of a uniform burden rate ($x = x_1 = x_2$), with constant capital life ($z_1 = 16$), average product life ($z_2 = 8$), and average size of product family ($y = 16$).

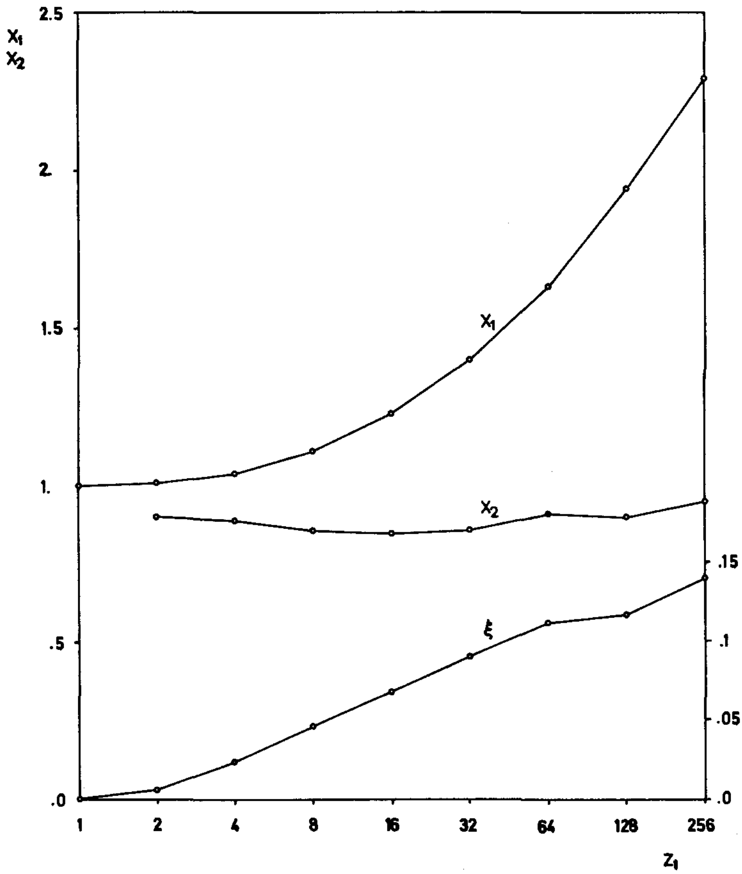


FIGURE 6

Optimal upper (x_1) and lower (x_2) limits of burden rate and relative unused capacity (ξ) as functions of capital life (z_1), with constant average product life ($z_2 = 8$) and average size of product family ($y = 16$).

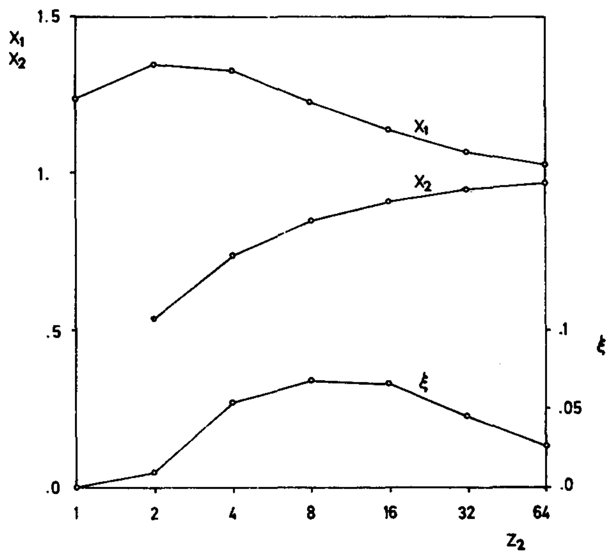


FIGURE 7

Optimal upper (x_1) and lower (x_2) limits of burden rate and relative unused capacity (ξ) as functions of average product life (z_2), with constant capital life ($z_1 = 16$) and average size of product family ($y = 16$).

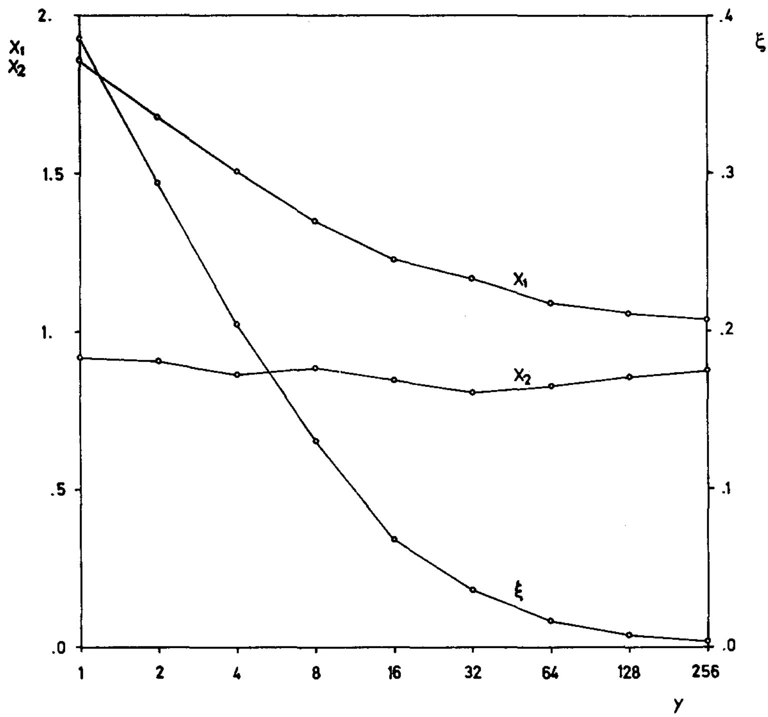


FIGURE 8

Optimal upper (x_1) and lower (x_2) limits of burden rate and relative unused capacity (ξ) as functions of average size of product family (y), with constant capital life ($z_1 = 16$) and average product life ($z_2 = 8$).

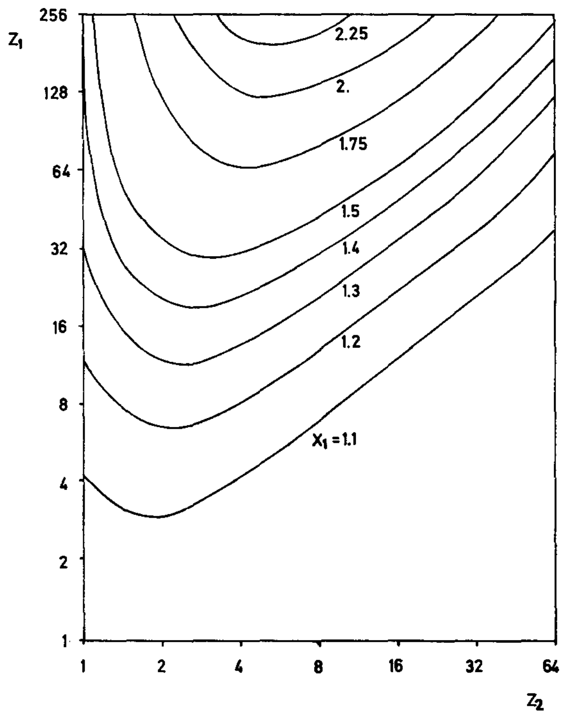


FIGURE 9

Optimal upper limit of burden rate (x_1) as a function of capital life (z_1) and average product life (z_2), with constant average size of product family ($y = 16$).

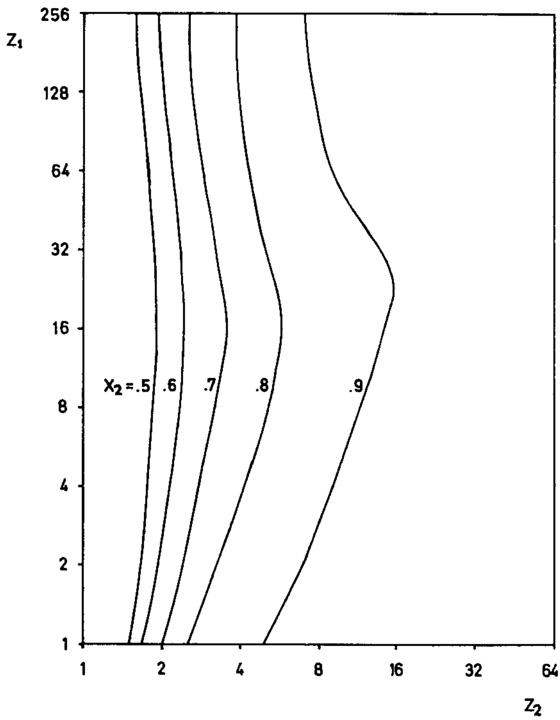


FIGURE 10

Optimal lower limit of burden rate (x_2) as a function of capital life (z_1) and average product life (z_2), with constant average size of product family ($y = 16$).

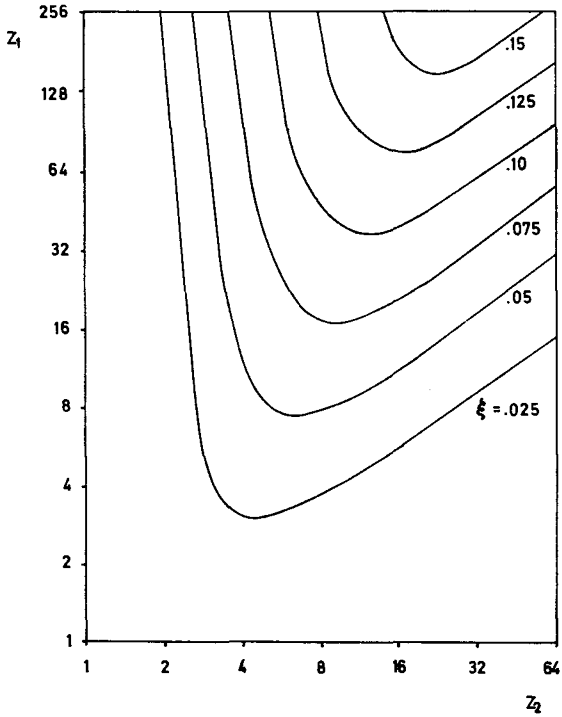


FIGURE 11

Relative unused capacity (ξ) in optimum of burden rate limits, as a function of capital life (z_1) and average product life (z_2), with constant average size of product family ($y = 16$).

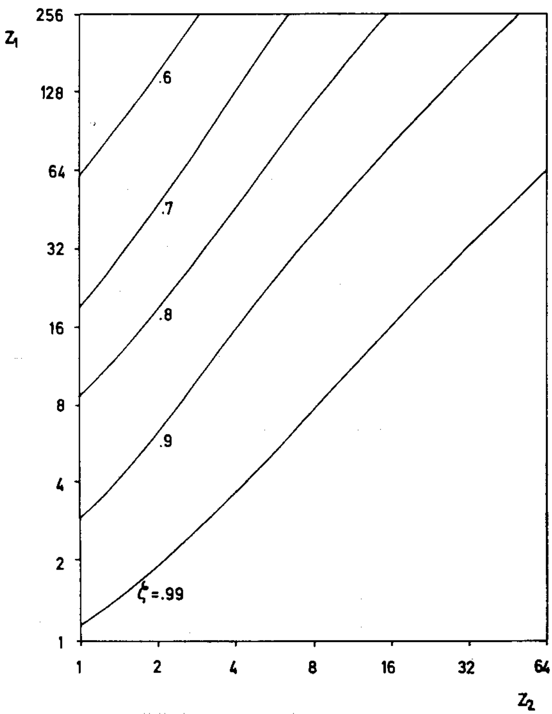


FIGURE 12

Relative profit (ζ) in optimum of a uniform burden rate, as a function of capital life (z_1) and average product life (z_2), with constant average size of product family ($y = 16$).

Tables

Table 1

Optimal upper (x_1) and lower (x_2) limits of burden rate, relative unused capacity (ξ), relative coverage of capacity costs (φ), and relative frequencies ($\mu_1, \ldots, \mu_5$) of period types 1–5, for different capital lives (z_1), with constant average product life ($z_2 = 8$) and average size of product family ($y = 16$)

z_1	x_1	x_2	ξ	φ	μ_1	μ_2	μ_3	μ_4	μ_5
1	1.00	–	.0	1.000	.131	.0	.869	.0	.0
2	1.01	.90	.006	1.001	.110	.021	.825	.038	.006
4	1.04	.89	.024	.997	.076	.055	.697	.139	.033
8	1.11	.86	.047	.993	.045	.086	.545	.247	.077
16	1.23	.85	.069	.996	.025	.107	.417	.317	.134
32	1.40	.86	.091	.989	.014	.117	.297	.390	.182
64	1.63	.91	.112	.993	.006	.125	.204	.458	.207
128	1.94	.90	.118	.982	.004	.127	.145	.471	.253
256	2.29	.95	.141	.981	.003	.128	.100	.516	.253

$z_1 = 16$

TABLE 2

Optimal upper (x_1) and lower (x_2) limits of burden rate, relative unused capacity (ξ), relative coverage of capacity costs (φ), and relative frequencies ($\mu_1, \ldots, \mu_5$) of period types 1–5, for different average product lives (z_2), with constant capital life ($z_1 = 16$) and average size of product family ($y = 16$)

period

z_2	x_1	x_2	ξ	φ	μ_1	μ_2	μ_3	μ_4	μ_5
1	1.24	(.00)	.0	1.010	.0	.0	.246	.0	.754
2	1.35	.54	.010	.996	.0	.0	.322	.056	.622
4	1.33	.74	.054	.989	.003	.012	.397	.272	.316
8	1.23	.85	.069	.996	.025	.107	.417	.317	.134
16	1.14	.92	.066	.996	.093	.275	.359	.228	.045
32	1.07	.95	.046	.996	.212	.396	.275	.106	.011
64	1.03	.97	.027	.996	.314	.466	.178	.040	.002

TABLE 3

Optimal upper (x_1) and lower (x_2) limits of burden rate, relative unused capacity (ξ), relative coverage of capacity costs (φ), and relative frequencies ($\mu_1, \ldots, \mu_5$) of period types 1–5, for different average sizes of product family (y), with constant capital life ($z_1 = 16$) and average product life ($z_2 = 8$)

y	x_1	x_2	ξ	φ	μ_1	μ_2	μ_3	μ_4	μ_5
1	1.86	.92	.385	.985	.055	.830	.077	.017	.021
2	1.68	.91	.294	.988	.083	.697	.126	.051	.043
4	1.51	.87	.205	.985	.092	.516	.200	.120	.072
8	1.35	.89	.131	.999	.063	.305	.300	.239	.093
16	1.23	.85	.069	.996	.025	.107	.417	.317	.134
32	1.17	.81	.037	.992	.004	.011	.496	.304	.185
64	1.09	.83	.016	.988	.0	.0	.560	.244	.196
128	1.06	.86	.007	1.002	.0	.0	.637	.174	.189
256	1.04	.88	.004	1.002	.0	.0	.696	.124	.180

TABLE 4

Optimal upper and lower limits of burden rate for different capital lives (z_1) and different average product lives (z_2), with constant average size of product family ($y = 16$)

z_1 \ z_2	1	2	4	8	16	32	64
1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	–	–	–	–	–	–	–
2	1.04	1.05	1.02	1.01	1.00	1.00	1.00
	(.0)	.60	.85	.90	.93	.94	1.00
4	1.10	1.13	1.08	1.04	1.02	1.01	1.00
	(.0)	.59	.80	.89	.95	.97	.99
8	1.15	1.23	1.20	1.11	1.05	1.02	1.01
	(.0)	.54	.76	.86	.93	.96	.97
16	1.24	1.35	1.33	1.23	1.14	1.07	1.03
	(.0)	.54	.74	.85	.92	.95	.97
32	1.34	1.48	1.50	1.40	1.29	1.17	1.08
	(.0)	.54	.77	.86	.90	.92	.96
64	1.35	1.63	1.77	1.63	1.48	1.31	1.17
	(.0)	.56	.78	.91	.93	.94	.96
128	1.38	1.74	1.99	1.94	1.81	1.54	1.32
	(.0)	.62	.82	.90	.96	.96	.97
256	1.43	1.95	2.36	2.29	2.19	1.85	1.53
	(.0)	.62	.80	.95	.97	1.02	.98

TABLE 5

Relative unused capacity in optium of burden rate limits, for different capital lives (z_1) and different average product lives (z_2), with constant average size of product family ($y = 16$)

z_1 \ z_2	1	2	4	8	16	32	64
1	.000	.000	.000	.000	.000	.000	.000
2	.000	.007	.011	.006	.004	.002	.001
4	.000	.010	.030	.024	.015	.008	.004
8	.000	.010	.047	.047	.034	.021	.012
16	.000	.010	.054	.069	.066	.046	.027
32	.000	.011	.065	.091	.093	.075	.051
64	.000	.015	.069	.112	.118	.109	.084
128	.000	.022	.085	.118	.139	.138	.113
256	.000	.023	.075	.141	.149	.165	.145

NB

TABLE 6

Relative coverage of capacity costs in optimum of burden rate limits, for different capital lives (z_1) and different average product lives (z_2), with constant average size of product family ($y = 16$). Row and column averages ($\bar{\varphi}$). (Elements in rows with $z_1 = 1$ and in columns with $z_2 = 1$ are not included in the averages.)

z_1 \ z_2	1	2	4	8	16	32	64	$\bar{\varphi}$
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	
2	.996	.992	.997	1.001	.995	.998	.999	.997
4	.999	1.001	.995	.997	1.000	1.001	.996	.998
8	.996	.994	.998	.993	.995	.993	.996	.995
16	1.010	.996	.989	.996	.996	.996	.996	.995
32	1.030	.995	.997	.989	.991	.993	.999	.994
64	1.003	.995	1.004	.993	.994	.992	.999	.996
128	.996	.991	.991	.982	.996	.984	1.002	.991
256	.999	.999	1.006	.981	.986	.999	.985	.993
$\bar{\varphi}$		.996	.997	.991	.994	.994	.996	.995

Full cost = optimal price
Stemmer for alle mulige komb.
av strukturvariablene!

TABLE 7

Optimal uniform burden rate (x), relative profit (ζ), relative unused capacity (ξ), and relative coverage of capacity costs (φ), for different average sizes of product family (y), with constant capital life ($z_1 = 16$) and constant average product life ($z_2 = 8$)

y	x	ζ	ξ	φ
1	1.76	.941	.450	.970
2	1.54	.928	.376	.962
4	1.36	.936	.290	.966
8	1.22	.951	.197	.980
16	1.12	.963	.123	.982
32	1.08	.968	.086	.987
64	1.04	.981	.044	.995
128	1.04	.986	.027	1.012
256	1.01	.993	.014	.996

TABLE 8

Optimal uniform burden rate, for different capital lives (z_1) and different average product lives (z_2), with constant average size of product family ($y = 16$)

z_1 \ z_2	1	2	4	8	16	32	64
1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	1.06	1.03	1.01	1.01	1.00	1.00	1.00
4	1.14	1.09	1.05	1.02	1.02	1.01	1.00
8	1.24	1.17	1.13	1.06	1.03	1.02	1.01
16	1.39	1.26	1.21	1.12	1.08	1.05	1.03
32	1.53	1.40	1.30	1.22	1.16	1.10	1.05
64	1.58	1.56	1.41	1.33	1.26	1.17	1.10
128	1.69	1.63	1.60	1.40	1.40	1.28	1.17
256	1.79	1.76	1.69	1.58	1.55	1.40	1.29

TABLE 9

Relative profit in optimum of a uniform burden rate, for different capital lives (z_1) and different average product lives (z_2), with constant average size of product family ($y = 16$)

z_1 \ z_2	1	2	4	8	16	32	64
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	.944	.985	.999	1.000	1.000	1.000	1.000
4	.872	.940	.989	.998	1.000	1.000	1.000
8	.807	.885	.949	.987	.998	1.000	1.000
16	.724	.822	.897	.963	.990	.998	1.000
32	.644	.743	.835	.916	.960	.987	.998
64	.596	.674	.768	.856	.913	.960	.991
128	.562	.611	.673	.805	.858	.914	.968
256	.519	.556	.644	.737	.787	.862	.922

TABLE 10

Relative unused capacity in optimum of a uniform burden rate, for different capital lives (z_1) and different average product lives (z_2), with constant average size of product family ($y = 16$)

z_1 \ z_2	1	2	4	8	16	32	64
1	.000	.000	.000	.000	.000	.000	.000
2	.066	.034	.014	.007	.004	.002	.001
4	.144	.094	.050	.028	.015	.008	.004
8	.210	.166	.119	.068	.037	.022	.012
16	.293	.237	.191	.123	.083	.050	.027
32	.368	.313	.263	.199	.150	.097	.055
64	.409	.374	.333	.274	.225	.163	.098
128	.442	.427	.418	.336	.303	.242	.157
256	.480	.475	.447	.404	.378	.313	.237

TABLE 11

Relative coverage of capacity costs in optimum of a uniform burden rate, for different capital lives (z_1) and different average product lives (z_2), with constant average size of product family ($y = 16$)

z_1 \ z_2	1	2	4	8	16	32	64
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	.990	.995	.996	1.003	.996	.998	1.000
4	.976	.988	.997	.992	1.004	1.002	.996
8	.979	.976	.995	.988	.992	.998	.998
16	.983	.962	.979	.982	.980	.998	1.002
32	.967	.962	.959	.977	.986	.993	.992
64	.934	.977	.940	.965	.977	.979	.992
128	.943	.934	.931	.929	.976	.970	.987
256	.931	.924	.935	.941	.964	.961	.984

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