Pricing Life Insurance Contracts under Financial Uncertainty

by

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Contents

Chapter 1

INTRODUCTION

1. Objective

The purpose of this dissertation is to derive valuation theories for life insurance contracts based on economic theory. Life insurance companies are exposed to two major sources of uncertainty: Mortality risk and financial risk. In this dissertation mortality risk is treated as in the classical actuarial models, i.e., from a risk neutral perspective. In traditional actuarial theory, see, e.g., Borch (1980) and Sverdrup (1969), financial uncertainty is not modeled explicitly. We introduce two sources of financial uncertainty, one related to the interest rate, the other one to the amount of benefit. In the traditional models the interest rate is assumed to be constant and the amount of benefit deterministic. Recently, however, new life insurance products have been introduced, where the amount of benefit is linked to a financial asset, whose market value fluctuates randomly. We present models where both these sources of financial uncertainty are taken into account.

2. Organization

This dissertation consists of four chapters, in addition to this introductory chapter. Each chapter is written as a self-contained paper. The first, entitled "Pricing of Unit-linked Life Insurance Policies", is accepted for publication in the Scandinavian Actuarial Journal with Knut Aase as co-author. The second, "Valuation of a Multistate Life Insurance Contract with Random Benefits", was presented at the first Nordic Symposium on Contingent Claims Analysis in Naantali, Finland, 8-9 May 1992 and published in a supplementary issue of Scandinavian Journal of Management, Vol. 9, 1993. The third paper is entitled "Interest Rate Risk in Life Insurance". The fourth paper, "Random Benefits and Stochastic Interest Rates in Life Insurance", was presented at the Second Nordic Symposium on Contingent Claims Analysis at Solstrand Fjord Hotel outside Bergen, Norway, 5-8 May 1994. As mentioned, the papers are intended to be self-contained, which, unfortunately, implies some duplications and to some extent varying notation between the different chapters.

3. Overview

The dissertation can naturally be categorized according to the two sources of financial uncertainty treated, i.e., the interest rate and the amount of benefit.

Figure 1. Structure of dissertation.

		Benefit	
		Deterministic	Random
Interest Rate	Deterministic	Traditional actuarial theory	Chapter 2 Chapter 3
	Random	Chapter 4	Chapter 5

In Chapter 2 a theory for pricing unit-linked contracts is presented. Unit-linked insurance is characterized by the fact that the benefit is linked to a mutual fund or another financial asset. In the unit-linked version of a term insurance contract the insured's heirs receive, say, the value of 10 units of a mutual fund upon death. These types of insurances may also include a guarantee, i.e., the heirs receive, say, the maximum of the value of 10 units in a mutual fund and 100 000 NOK. The model of the financial market in Chapter 2 is the same as the one used by Black and Scholes (1973) in their derivation of the option pricing formula. In this model the interest rate is constant and the price of the risky asset follows a geometric Brownian motion. In Chapter 2 we treat unit-linked versions of term insurance and pure endowment insurance contracts. These two contracts can be combined into endowment insurance which is a popular contract on a single life.

In Chapters 3, 4 and 5 we employ an extended model of the insurance contract. The insurance policy is at each point in time assumed to be in one of a finite number of states and moves between the states according to an inhomogenous Markov-process. This model of the insurance contract is quite general and somewhat standard in the actuarial sciences. The insurance contracts just mentioned are special cases of the Markov-model, and it can also be used to model contracts on several lives. It is natural to use an inhomogenous Markov-process in life insurance to reflect facts of life, such as that the probability of death or of becoming disabled generally increases with age.

In Chapter 3 we also generalize the model of the risky security to a geometric Gaussian process. The major advantage of this model compared to the geometric Brownian motion is that the volatility of the risky security is allowed to be a deterministic function of time as opposed to a constant when using the geometric Brownian motion. This added flexibility may in particular be useful when long-lived contracts such as life insurance policies are considered.

In classical actuarial theory the interest rate is assumed to be constant. In Chapter 4 we develop a pricing model where the interest rate is random and the benefits are deterministic. Whereas unit-linked products are relatively specialized life insurance products, the model in Chapter 4 is applicable for most traditional life insurance products. The valuation theory is based on models of the term structure known from financial economics.

In Chapter 5 we again allow for random benefits linked to risky assets where also the interest rate is random. In this model there is an arbitrary finite number of risky assets modeled by somewhat more general processes than in the previous chapters. One example of application of this model is the valuation of unit-linked contracts in the case of random interest rate.

4. The theory and model assumptions

We use models in continuous time. This approach is standard in the actuarial sciences as well as in the theories we apply from financial economics. From an actuarial perspective, the new component of the model is a financial market. This addition is natural when dealing with unit-linked insurance, but can also be applied when the benefit is deterministic. Our model of the financial market is highly idealized. There are no transaction costs or taxes, and short-sale and continuous trading are allowed and considered feasible.

The body of the financial theories we apply are known as arbitrage pricing theories. They are characterized by the fact that the processes for the market prices of the financial assets are taken as primitives. Furthermore, no arbitrage profit can be generated by trading with these securities – a necessary condition for an economic equilibrium. However, these theories are not general equilibrium theories where the market prices of the securities may be derived from more fundamental primitives such as the agents' preferences and technology factors. Arbitrage pricing theory is sometimes called preference free pricing, meaning that the resulting pricing formulas do not explicitly depend on the agents' preferences. Continuous time arbitrage theories are based on the seminal papers by Black and Scholes (1973) and Merton (1973) and developed further by Harrison and Kreps (1979) and Harrison and Pliska (1981), see, e.g., Duffie (1991) for a current overview of this theory. The purpose of this dissertation is not to create any new financial theory (nor do we apply the existing theories in their full generality), but rather to merge the central ideas of these theories with actuarial models, attempting to find valuation principles for life insurance contracts consistent with economic theory as well as with traditional actuarial valuation principles.

This work is based on two important assumptions.

Assumption 1.
The financial market is independent of the state of the insurance policy.

Assumption 2.
The insurer is risk neutral with respect to transition risk.

More precisely, the content of Assumption 1 is: All stochastic processes representing the market values of the financial assets are statistically independent of the stochastic process representing the state of the policy. We find this assumption rather plausible, though more or less realistic counter-examples may be constructed. One counter-example is the situation when a person gets a heart attack and dies because a dramatic decrease occurs at the stock market.

To explain the concept of transition risk we use term insurance as an example. In term insurance the insured is in one of two states, alive or dead. The insured may die immediately after the contract is signed, or at least much sooner than anticipated. This obviously represents a risk for the insurance company, called mortality risk (a term used earlier) for this particular contract. Transition risk is just the natural generalization when there are more than two possible states of the policy. In the actuarial literature transition risk is often referred to as mortality risk and even only risk. In our models also financial risk is present, so we prefer the term transition risk, a terminology that also fits well to the underlying Markov-model.

The following example is intended to explain the concept of risk neutrality with respect to transition risk. An insurer promises to pay the insured 100 000 NOK if he dies tomorrow (we impose this short time horizon to ignore any problems connected to the time value of money). The true probability for death tomorrow is known and equal $\frac{1}{10\,000}$. If the insurer is risk neutral with respect to transition risk, he charges 10 NOK for the policy, i.e., the insurer does not demand any premium in excess of the expected pay-out to offer the insurance.

The justification for the risk neutrality assumption with respect to transition risk is based on a pooling argument, i.e., the insurance company have a large number of independent and identical contracts. From the strong law of large numbers, the aggregate number of deaths (and other transitions causing the expiration of benefits) approaches the population's average as the number of policies gets large.

The pooling argument does not hold for financial risk, because all policies are generally affected by financial risk in the same direction. For example, every policy is exposed to the same interest rates, or at least to highly correlated interest rates. The amount of financial risk is therefore increased, rather than decreased, by increasing the number of identical policies.

5. Existing literature on the valuation problem

The pricing of unit-linked insurance is discussed in a number of papers. These can broadly be classified in two categories. The papers of the first category were published in the 1970's and culminated by the book of Brennan and Schwartz (1979a). The approach used was based on the Black and Scholes (1973) methodology for the financial part, and on a discrete time model of an endowment insurance for the actuarial part. The majority of the second category of papers were published in 1993 or 1994. The timing of these papers is probably connected to the recent introduction of unit-linked contracts in several European countries. Also here a discrete time model of an endowment insurance together with continuous time finance models were used, though these works apply the more modern martingale-based theory. Most of these papers are reviewed in Chapter 2 of this dissertation.

There exist a few papers in the actuarial literature dealing with stochastic interest rate in life insurance. We refer to Parker (1994) for a review. However, we are not aware of other works using the approach of Chapter 3, which is based on the existence of a financial market without arbitrage opportunities. In Chapter 3 we have included a comparison between our approach and a work of Norberg and Møller (1993) which is based on the traditional actuarial valuation principle, or "classical theory of risk", originating more than a century ago.

6. The results

As mentioned above, the classical actuarial models are based on a deterministic rate of return and deterministic amounts of benefits. The only remaining source of uncertainty is at what time the benefits expire. In classical actuarial theory the single premium may be found in one of two ways: Either as the expected present value of the future cashflows, or by solving a deterministic differential equation. The first approach even has its own name, the principle of equivalence. This principle was established by Jan de Witt in 1671 (see, e.g., Borch (1990)). The underlying idea is that an insurer's income and expenses should balance on average. The differential equation was first discovered by the Danish actuary Thorvald N. Thiele in 1875 and was derived by Hoem (1968) for the Markov-model we use.

It is striking that the two main methodologies of the modern arbitrage pricing theories are quite similar. The essence of the arbitrage theories is that the market price of a financial asset may be found either by solving a deterministic partial differential equation or as an expectation of the present value of the future cashflows, but where the expectation is calculated under a risk adjusted probability measure. The famous option pricing formula was originally derived by Black and Scholes (1973) by solving a partial differential equation. Another example is the term structure model of Brennan and Schwartz (1979b). In the financial literature these differential equations are often referred to as fundamental differential equations. The risk adjusted probability measure is called an equivalent martingale measure. Again we refer to Harrison and Kreps (1979), but this approach is currently a central topic of every advanced textbook in finance, see, e.g., Duffie (1992).

In this dissertation we construct probability measures so that the market prices of the insurance contracts may be found as expectations under this risk-adjusted measure. This principle is referred to as the principle of equivalence under Q, where Q denotes the risk-adjusted probability measure. In the case of no financial risk, the probability measure Q is identical to the original probability measure, so that our pricing principle coincides with the traditional principle of equivalence. Also in the case of no life insurance specific factors, our probability measure is identical to the probability measure from the financial theory. The principle of equivalence under Q differs from the traditional principle of equivalence, where the single premium of a policy is found as an expectation under the original probability measure. In our idealized model the insurance company's income and expenses will balance on average. This corresponds to the same fairness-idea underlying the traditional principle of equivalence. By using any other pricing principle, e.g., the traditional principle of equivalence, the insurance contracts will be systematically mis-priced. That is, the company will either go bankrupt, or other companies can offer the same policy at a lower cost.

In Chapters 2 and 4 examples are given where the premiums calculated by our principle are lower than if they were calculated by the traditional principle of equivalence. A possible explanation for this is the following: An investor buying financial assets generally demands higher returns than the riskfree rate of return, to be compensated for the financial risk. The return in excess of the riskfree rate is called a financial risk premium. By buying financial securities, the insurance company accepts financial risk, and consequently receives –on average– a financial risk premium. Then the insurance customers may benefit from this by lower prices on their insurance contracts. In our models the investments in the financial market do not expose the insurance company to financial risk. The purpose of the investments is to hedge the payoffs connected to the insurance benefits. Hence the financial investment reduces the company's exposure to financial risk. This explanation may be plausible for a mutual insurance company, i.e., a company owned by the policy-holders. For a an insurance company owned by shareholders, one would expect that the shareholders also want a part of the financial risk premium.

We also derive partial differential equations which can be considered as generalizations of both the classical Thiele equation and various fundamental differential equations known from the arbitrage pricing theory. These equations are derived using 3 different methodologies. In Chapter 3 we use arguments involving duplicating trading strategies. In Chapters 2 and 4 we use the martingale property of the financial assets under the risk adjusted probability measure. The approach in Chapter 5 involves a general stochastic differential equation for the premium reserve under the risk adjusted probability measure. The complexity of both these equations and the risk adjusted probability measure are connected to the complexity of the financial model. In the case of random interest rate the partial differential equations also depend on the choice of term structure model.

In the two following sections we present our results in some more detail.

7. The pricing principles

First we fix a time horizon T and a probability space $(\Omega,\mathcal{F},P)$. See Chapter 2 for a more detailed description of the notation. Let A_t denote the random accumulated payment stream in the period $[0,t]$ of an insurance contract. The random variable A_T represents the sum of all payments in the insurance period. See equation (12) of Chapter 3 for the expression for this quantity in the Markov-model. Let r_t denote the interest rate prevailing at time t. In Chapter 2 and 3 the interest rate is constant and will be referred to as r (without subscript). The money market account is defined by

$$\beta_t = \exp\left(\int_0^t r_s ds\right)$$

and can be interpreted as the value of an investment at time zero of one unit currency, accruing interest according to the short interest rate. We assume that the following expression for the random payment stream discounted by the money market account is well-defined,

$$V_0 = \int_0^T \frac{1}{\beta_s} dA_s.$$

Observe that V_0 is a random variable and represents the random present value of all cashflows related to a particular insurance contract. We denote the market price of the insurance policy by Π_0. From the principle of equivalence under Q it follows that

$$\Pi_0 = E^Q[V_0] = E\left[\frac{dQ}{dP} V_0\right], \tag{1}$$

where $E^Q[\,\cdot\,]$ denotes the expectation under Q and $E[\,\cdot\,]$ the expectation under the original probability measure P. Formally, $\frac{dQ}{dP}$ represents the Radon-Nikodym derivative of Q with respect to P, and is a random variable on $(\Omega, \mathcal{F}, P)$. The probability measures P and Q are equivalent provided $P(A) = 0$ if and only if $Q(A) = 0$ for all $A \in \mathcal{F}$. An equivalent probability measure Q is an equivalent martingale measure if $\frac{dQ}{dP}$ has finite variance and the price processes of the financial assets under Q, after a change of numeraire, are martingales. An equivalent martingale measure imposes the following conditions on the Radon-Nikodym derivative,

$$\text{i) } E\left[\frac{dQ}{dP}\right] = 1, \text{ ii) } \frac{dQ}{dP} > 0 \text{ P-a.s. and iii) } Var\left[\frac{dQ}{dP}\right] < \infty. \tag{2}$$

Both the economic interpretation and the economic purpose of the Radon-Nikodym derivative are important. It can be interpreted as the shadow price of risk per unit probability, is sometimes referred to as the pricing kernel. In our model $\frac{dQ}{dP}$ represents the shadow price of transition risk and financial risk per unit probability. From the independence assumption, Assumption 1, it follows (see Chapter 4) that $\frac{dQ}{dP}$ splits nicely into the product of two factors, i.e.,

$$\frac{dQ}{dP} = \xi_1 \cdot \xi_2, \tag{3}$$

where ξ_1 represents the shadow price of transition risk per unit probability and ξ_2 represents the shadow price of financial risk per unit probability. We have assumed that the insurers are risk neutral with respect to transition risk (Assumption 2), implying that $\xi_1 = 1$, and

$$\frac{dQ}{dP} = \xi_2.$$

That is, under the assumptions of independence between the financial market and the state of the policy and risk neutrality with respect to transition risk, the pricing kernel for financial risk following from economic theory should be used to price life insurance contracts. In the arbitrage pricing theories $\frac{dQ}{dP}$ is on the following form,

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \alpha(s)dW_s - \frac{1}{2}\int_0^T \alpha(s)^2 ds \right), \tag{4}$$

where W_t is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$ and $\alpha(t)$ depends on the model used. In Chapter 2 and 3 both W_t and $\alpha(t)$ denote one-dimensional processes. For the models in Chapter 4 and 5, W_t represents a d-dimensional vector of independent Brownian motions and $\alpha(t)$ represents a d-dimensional vector of processes (in which case $\alpha(t)^2$ should be interpreted as the dot product). We sometimes refer to the multi-dimensional Brownian motion as the d sources of uncertainty.

The quantity $\frac{dQ}{dP}$ serves a similar role as the marginal utility of the representative agent in general equilibrium models. If the arbitrage theory we apply is consistent with a more general equilibrium model possessing a representative agent with a utility function, the agent's normalized marginal utility at time zero for consumption at time T would be identical to expression (4).

The following table shows the expressions for $\alpha(t)$, together with the model of the risky securities, denoted by dS, used in the different chapters.

Table 1. Pricing principles.

Ch.	$\alpha(t)$	dS
2	$\frac{\eta - r}{\sigma}$	$dS = \eta S dt + \sigma S dW$
3	$\frac{\eta(S,t) - r}{\sigma(t)}$	$dS = \eta(S,t) S dt + \sigma(t) S dW$
4	$\begin{pmatrix} \lambda_t^1 \\ \vdots \\ \lambda_t^d \end{pmatrix}$	–
5a	$\begin{pmatrix} \psi_t^1 \\ \vdots \\ \psi_t^d \end{pmatrix}$	$dS = \eta(S,t) S dt + \sigma(S,t) S dW$
5b	$\begin{pmatrix} \psi_t^1 \\ \frac{1}{\sigma_2}[\eta - r_t - \sigma_1 \psi_t^1 \end{pmatrix}$	$dS = \eta S dt + \sigma_1 S dW^1 + \sigma_2 S dW^2$
5c	$\frac{1}{s_2\sigma_1 - s_1\sigma_2} \begin{pmatrix} s_2(\eta - r_t) - \sigma_2(\gamma - r_t) \\ \sigma_1(\gamma - r_t) - s_1(\eta - r_t) \end{pmatrix}$	$dS^1 = \eta S^1 dt + \sigma_1 S^1 dW^1 + \sigma_2 S^1 dW^2$ $dS^2 = \gamma S^2 dt + s_1 S^2 dW^1 + s_2 S^2 dW^2$

In Chapter 2 the drift and volatility processes of $\frac{dS}{S}$, η and σ, respectively, are constants. Hence $\alpha(t)$ is a constant and the conditions in (2) are satisfied. For the other models restrictions on the parameters of the price processes must be imposed to ensure that these conditions hold.

In Chapter 3, $\eta(S,t)$ is allowed to be a function of S and t, and $\sigma(t)$ is a general function of t.

In the model of Chapter 4 there are no risky securities. Here $\alpha(t)$ is a vector of market prices of risk related to each of the d sources of uncertainty. The theory does not provide any insight in the parametric form of these functions. In applications, a parametric form usually must be assumed before any estimation of parameters can take place, except for the cases where the data are rich enough to permit non-parametric estimation.

In Chapter 5 the parametric form of the market prices of risk depends on the number of risky assets. In the case of no risky assets, this model is the same as in Chapter 4, i.e., $\psi_t^i = \lambda_t^i$, for $i = 1,\ldots,d$. In the case of d or more risky assets, $\alpha(t)$ will be determined in terms of the parameters of the processes governing the risky assets. When there are between 1 and $(d-1)$ risky assets, the parametric form of some of the ψ_t^i's may be chosen arbitrarily, the remaining will depend on these in addition to the parameters of the risky assets. To illustrate, we have included examples in the table, labeled 5b and 5c, where there are two sources of uncertainty. In 5b there is one risky asset and we are not able to determine ψ_t^1, but ψ_t^2 can be determined in terms of ψ_t^1 and the parameters of the price process of the risky security. In 5c we introduce another risky asset and $\alpha(t)$ is now completely determined in terms of the parameters of the risky assets.

We would like to emphasize that in our life insurance model, knowledge of $\alpha(t)$ completely determines the pricing of the life insurance policies under our set of assumptions.

8. Partial differential equations for the market value of the insurance contracts

We also develop equations describing the evolution over time of the market value of the insurance contract. They are on the form

$$\frac{\partial \Pi_t^g}{\partial t} = r_t \Pi_t^g - a_g(t) - \sum_{h \neq g} \mu_{gh}(t)[a_{gh}(t) + \Pi_t^h - \Pi_t^g] - \kappa. \tag{5}$$

The term κ is not a constant, but consists of several partial derivatives with respect to various state variables and/or risky assets. These equations can be considered both as generalizations of various fundamental differential equations from the theories of financial economics and also possibly as a generalization of Thiele's equation from the actuarial sciences. Excluding the term κ, equation (5) is identical to the traditional Thiele equation. This equation can be interpreted in an intuitive and straight-forward manner (see below equation (26) of Chapter 3). The above equation also deals with economic risk, and as a consequence the new collection of terms κ appears. The κ terms depend on the financial model, and in the following table the κ terms for the models in the different chapters are listed.

Table 2. Differential equations of the market value of the insurance contract.

Ch.	κ
2	$rS\frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2}$
3	$rS\frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma(t)^2 S^2 \frac{\partial^2 \Pi}{\partial S^2}$
4	$\frac{\partial \Pi}{\partial Z}^{T}(\eta_Z - \sigma_Z \lambda) + \frac{1}{2}\mathrm{tr}\left[\sigma_Z \sigma_Z^T \frac{\partial^2 \Pi}{\partial Z^2}\right]$
5a	$\frac{\partial \Pi}{\partial Z}^{T}(\eta_Z - \sigma_Z \lambda) + \frac{\partial \Pi}{\partial S}^{T} S r_t + \frac{1}{2}\mathrm{tr}\left[\sigma_Z \sigma_Z^T \frac{\partial^2 \Pi}{\partial Z^2} + \sigma_Z \sigma_S^T \frac{\partial^2 \Pi}{\partial S \partial Z}\right] + \frac{1}{2}\mathrm{tr}\left[\sigma_S \sigma_S^T \frac{\partial^2 \Pi}{\partial S^2} + \sigma_S \sigma_Z^T \frac{\partial^2 \Pi}{\partial Z \partial S}\right]$
5b	$\frac{\partial \Pi}{\partial X}^{\xi}\left[r_t - \frac{1}{2}\sigma_a^T \sigma_a\right] + \frac{\partial \Pi}{\partial S}^{T} S r_t + \frac{1}{2}\sigma_a \sigma_a^T \frac{\partial^2 \Pi}{\partial X^2} + \sigma_a^T \sigma_S^T \frac{\partial^2 \Pi}{\partial S \partial X} + \frac{1}{2}\mathrm{tr}\left[\sigma_S \sigma_S^T \frac{\partial^2 \Pi}{\partial S^2}\right]$
5c	$\frac{\partial \Pi}{\partial B} B_t(T) r_t + \frac{\partial \Pi}{\partial S}^{T} S_t r_t + \frac{1}{2}\sigma_a \sigma_a^T B_t(T)^2 \frac{\partial^2 \Pi}{\partial B^2} + \sigma_a^T \sigma_S^T \frac{\partial^2 \Pi}{\partial S \partial B} B_t(T) + \frac{1}{2}\mathrm{tr}\left[\sigma_S \sigma_S^T \frac{\partial^2 \Pi}{\partial S^2}\right]$

These terms cannot be interpreted in a straight forward manner, as is the case for the other terms in equation (5). In Chapter 2 the full-fledged version of this equation is not developed, but the κ term labeled 2 can be considered as a special case of the model in Chapter 3 where the volatility process of the risky security is constant, i.e., $\sigma(t) = \sigma$. The κ terms for the equations in Chapter 2 and 3 involve first and second order partial derivatives with respect to the risky security.

In the model in Chapter 4 the economy is described by a vector of state variables Z and no risky assets. The κ term of this model involves the first and second derivatives with respect to Z and depends also on λ, the vector of market prices of risk.

In the model of Chapter 5 risky assets are introduced. The model labeled 5a) is also based on a vector of state variables Z and involves the first and second order partial derivatives with respect to the state variables. In addition, the first and second order partial derivatives with respect to the risky assets, and two terms representing the covariation between the state variables and the risky securities, are included. The models in 5b and 5c are based on the Heath, Jarrow and Morton (1992) model of the term structure. Here we suggest using either the market price of a bond, B, as a state variable, or minus the integral of the forward rates, X, as a state variable. In both cases the structure of the κ terms are similar to the model in 5a, containing terms involving the first and second order partial derivatives with respect to the state variable and the risky assets, and a term representing the covariation between the state variables and the risky assets. However, the two last expressions do not involve the market prices of risk.

9. Concluding remarks

In this dissertation we attempt to model the life insurance business as a part of an economic environment and investigate how pricing in the financial market affects the pricing of life insurance products. The model of the financial market is highly idealized and somewhat ad hoc, as it does not provide any explanation for the financial price processes, which are our primitives. However, this financial theory currently seems to be the industry standard both among academicians and practitioners working in the field of financial economics. This dissertation provides insights into how these theories may be applied in the actuarial sciences.

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Chapter 2

PRICING OF UNIT-LINKED LIFE INSURANCE POLICIES[1]

The key feature of unit-linked or equity linked life insurance policies is the uncertain value of the future insurance benefit. By issuing unit-linked insurances that guarantees the policy-holder a minimum benefit, the insurance company is exposed to financial risk.

The value of the insurance benefit is assumed to be a function of a particular stochastic process. We use the financial theory of arbitrage pricing and martingale theory to derive single premiums for different policies. We derive risk-minimizing trading strategies describing how the issuing company can reduce financial risk. We derive a partial differential equation for the market value of the premium reserve which we compare to Thiele's equation of the actuarial sciences. Our equation contains some new terms stemming from our economic model.

The interpretation of the principle of equivalence may be revisited in this framework; the principle still holds but under a new risk adjusted probability measure, equivalent to - but different from - the originally given probability measure.

Key words: Unit-linked Insurance, Equity-linked Insurance, Arbitrage Pricing Theory, Thiele's Differential Equation, Principle of Equivalence.

1. Introduction

1.1 Focus

A life insurance contract or policy is an agreement between a customer and an insurance company which specifies an *event* that must occur for the policy-holder to get a *benefit* from the insurance company, a specified time-period, the *insurance period*, in which the contract is valid and a *premium-plan* which specifies how the customer shall pay for the benefit.

A unit-linked or equity-linked insurance (called variable life insurance in the United States) is a certain kind of life insurance where the amount of insurance benefit is *linked* to the market value of some specified reference portfolio. This portfolio may consist of stocks, bonds and/or other

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financial assets. The typical example seems to be shares in a mutual fund. As opposed to traditional insurance the benefit is random. To reflect this fact we model the amount of benefit by a stochastic process. The principle of equivalence, which is the basis of pricing traditional life insurance products, does not include random benefits. We therefore use financial theory to value the benefit and then take mortality into account – assuming that the financial market is independent of the insured's health condition. We call the resulting pricing principle *the principle of equivalence under Q*. Like the traditional approach, it is also implicit in this procedure that the insurer is risk neutral with respect to mortality. It does not assume that the insurer is risk neutral with respect to financial risk.

Another feature of unit-linked insurance is that the components, i.e., benefits, premiums etc., may be measured in *units* of the reference portfolio. Furthermore, the insurance company is supposed to have several portfolios available so that the unit-linked customer can choose a (financial) risk-level of his insurance by choosing an appropriate portfolio. These factors will only to a limited extent be taken into consideration.

We restrict attention to endowment and term insurances, which are the building blocks for most of the interesting policies written on one life. The model is extended to more general life insurance contracts in Persson (1994b).

1.2 Existing literature

The first treatments of unit-linked contracts with guarantees by modern financial techniques that we are aware of, seem to have been conducted by Brennan and Schwartz (1976, 1979a, 1979b) and Boyle and Schwartz (1977). While this problem had been discussed in the actuarial literature for several years, no satisfactory theory had been developed (see, e.g., Corby, 1977). This last reference also demonstrates that the actuaries were reluctant to accept these results.

Boyle, Brennan and Schwartz recognized that the payoff from a unit-linked insurance at expiration is identical to the payoff from a European call option plus a certain amount (the guaranteed amount) or to the payoff from a European put option plus the value of the reference fund. Options are specialized financial instruments and will be described in Section 3. Finally, the option theory initiated by the results of Black and Scholes (1973) was utilized to value the unit-linked contract.

Delbaen (1990) and Bacinello and Ortu (1993) also analyzed unit-linked products by using the martingale-based theory credited to Harrison and Kreps (1979).

In practice most life insurance contracts are paid by periodic premiums. In the traditional life insurance policies this fact does not influence the amount of benefit. If a person arranges to pay, say, a term insurance with periodic premiums and dies the day after he signs the contract, his heirs will receive the full benefit. This is generally not the case for unit-linked contracts. The amount of the benefit will in general depend on, firstly, the time since issue and, secondly, the random value of the reference portfolio. This contract therefore involves two new properties compared to traditional life insurance contracts.

Fix a time horizon T. Let N(t) and S(t) be the prescribed number of shares of the reference portfolio included in the benefit and the market value of one share, respectively, at time t, $0 \leq t \leq T$. Boyle, Brennan, Delbaen and Schwartz considered a certain contract where N(t) is random and depends on the path of S(t). To obtain the market price of the policy Brennan, Boyle and Schwartz numerically evaluated a complex differential equation and Delbaen used Monte Carlo simulation.

In contrast, we assume that N(t) is non-random. In the single premium case our results coincide with the earlier results. However, by our assumption we are able to get analytical results in the case of periodic premiums and to treat contracts not previously addressed in the literature, which also may be of interest from an applied point of view.

The basic assumptions are essentially the same in our model as in the model used by Black and Scholes, but like Delbaen, Bacinello and Ortu, we use the theory that originated from the papers by Harrison and Kreps (1979) and Harrison and Pliska (1981) to value unit-linked insurance contracts. The present stage of this theory (see, e.g., Duffie, 1991) is rather general and can in principle be used for valuing any contingent claim. When comparing option pricing results with unit-linked results one has to take into account that options and life insurance contracts are different products with different characteristics. For example while options usually expire within one year, life insurance contracts have typically long contract periods (more than 40 years are not unusual). Except solely for the purpose of comparisons, we are therefore reluctant to state the prices of unit-linked products in terms of options prices which is commonly seen in the literature.

The earlier papers presented results for an endowment insurance which consisted of a pure endowment insurance and a term insurance–both with guarantee. In contrast, we consider an endowment insurance to be a combination of a pure endowment insurance and a term insurance and concentrate our effort on valuing those building-blocks separately. It is then a simple task to combine the building-blocks into various kinds of endowment insurances. Note that by our approach we have a traditional, a pure unit-linked and a guaranteed unit-linked version of both of the building-blocks. This means that we can make a total of 9 different endowment insurances by combining them in different ways (of which one is the traditional endowment insurance). Not all of these contracts may be offered by the insurance companies. Also Bacinello and Ortu (1993) apply the martingale based valuation approach to other types of contracts.

Also contrary to Brennan, Boyle, Delbaen and Schwartz, we use time-continuous death probabilities which is common in the actuarial literature. This leads to results that can be directly compared to the corresponding actuarial, as well as the pure financial, counterparts. As a consequence, we find a connection between the celebrated Black and Scholes partial differential equation (Black and Scholes, 1973) encountered in financial economics and the familiar Thiele differential equation from the theory of life insurance, the latter dating back to 1875. The principle of equivalence in life insurance still holds formally, but now under a risk adjusted probability measure, which means that the real interpretation of this principle is changed in our approach.

1.3 Pure contracts and guaranteed contracts

The intention of this paragraph is threefold. First to distinguish two classes of unit-linked contracts, then to provide examples of unit-linked contracts and finally to illustrate the equivalence principle under Q.

A unit-linked insurance can be equipped with a guarantee that assures the policy-holder a minimum amount even though the value of the reference portfolio at expiration is below this level. We denote such a contract a unit-linked contract *with guarantee* and a contract without a guarantee a *pure* unit-linked contract. The latter contract transfers all financial risk to the customer, so for the issuer there is even less financial risk connected to this contract than to the traditional products.

We will now demonstrate how the methodology suggested in this article may be applied to find the single premium of pure unit-linked contracts. This example also introduces notation and assumptions that will be maintained throughout the paper.

First we abstract from the insurance aspects of the policy and look at the valuation of different financial assets. These financial assets will be used to model insurance benefits when the insurance aspect is incorporated. Let C(t) and $\pi_0(t)$ represent the payoff of the financial asset payable at time t and the market value at time zero of C(t), respectively.

The payoff of the first asset is C(t) = 1, i.e., one unit of currency paid at the fixed time t. The present value of C(t) is

$$\pi_0(t) = e^{-\delta t}, \tag{1}$$

where δ represents the constant riskless rate of return.

Now let C(t) = S(t), i.e., one unit of the reference portfolio is paid at the fixed time t. In any reasonable economic model the market value of the benefit at time zero must be the market value of one unit of the fund at time zero, otherwise either the buyer or the seller would benefit from not taking part in the deal. Hence

$$\pi_0(t) = S(0). \tag{2}$$

Observe that $\pi_0(t)$ is independent of t.

The asset described above is used to model the benefit of the pure unit-linked contracts. Even though the time of expiration is uncertain for an insurance, we would like to emphasize that the benefit basically is a financial asset that the customer equally well could have bought directly in the financial market. This observation implies that the insurance company can no longer calculate this present value by using certain tables or discounting techniques, but has to watch the financial markets to calculate correct prices of insurance contracts. We therefore denote the value of the benefit the *market value* instead of the usual present value.

We now turn to insurance aspects again and commence with the pure unit-linked contracts. Let $U^1_{x:\overline{T}|}$ denote the single premium, or market value, of a contract which gives the policy-holder (or his heirs), who is x years old when he buys the insurance, right to receive 1 unit of the reference portfolio upon death within T years. This contract is the pure unit-linked version of the traditional term insurance. Let the random variable T_x denote the remaining life time of an x-year old person. We assume that the probability density function for T_x exists and denote it f_x. The single premium for this policy is calculated as

$$U^1_{x:\overline{T|}} = E\left[\int_0^T \pi_0(t)d(1_{\{T_x \le t\}})\right], \tag{3}$$

where $\pi_0(t)$ is given by (2) for this contract and $1_{\{T_x \le t\}}$ denotes the indicator function which takes the value 1 if $\{T_x \le t\}$ and 0 otherwise. In the case of a traditional life insurance contract $\pi_0(t)$ is given by relation (1) and the above expression is simply the traditional equivalence principle.

We get from relation (2) and since S(0) is observable at time zero that

$$U^1_{x:\overline{T|}} = \int_0^T S(0)f_x(t)dt.$$

Let ${}_tp_x = P(T_x > t)$ denote the survival probability for an x-year old policy buyer. The force of mortality is defined by $\mu_{x+t} = \frac{f_x(t)}{{}_tp_x}$. Then it follows that $f_x(t) = {}_tp_x\mu_{x+t}$ and that

$$\frac{\partial}{\partial t}\, {}_tp_x = -\mu_{x+t}\, {}_tp_x.$$

Then we can write

$$U^1_{x:\overline{T|}} = \int_0^T S(0)\, {}_tp_x\mu_{x+t}dt,$$

which can be simplified to

$$U^1_{x:\overline{T|}} = P(T_x \le T)S(0) = (1 - {}_Tp_x)S(0). \tag{4}$$

Let ${}_TU_x$ denote the single premium for a contract which gives the policy-holder, who is x years old when he buys the insurance, right to receive 1 unit of the reference portfolio if he is alive after T years. This is the pure unit-linked version of the traditional pure endowment insurance. A similar approach on this contract gives

$$_TU_x = P(T_x > T)S(0) = {}_Tp_xS(0). \tag{5}$$

Note that no assumptions regarding the stochastic process governing the evolution of S(t) are necessary to obtain (4) and (5).

We have demonstrated how to find the market value of the pure unit-linked contracts, but problems arise when the contracts include guarantees. Buying shares are risky investments in that the investor may lose money, as well as make profits. In order to protect buyers of unit-linked life insurances from the general downside risks of the stock markets, a guarantee may be issued. This guarantee can be arranged in many different ways and in most countries it is required and regulated by law. Denoting the guarantee at time t by G(t), one example of a possible guarantee is $G(t) \equiv G$, for all t, so that this guarantee is constant through time. Another example is

$$G(t) = \int_0^t \bar{p}(s)ds,$$

where $\bar{p}(t)$ is the premium rate at time t. By this guarantee the customer is sure to get the nominal value of his money back. The same guarantee may be imposed with an interest rate, r, $0 < r < 1$,

$$G(t) = \int_0^t \bar{p}(s)e^{r(t-s)}ds.$$

The customer is in this case guaranteed $r \cdot 100\%$ return on his insurance. In this paper we will work with a general, non-random guarantee. A guarantee that is functionally dependent on the premium rate is called an endogenous guarantee by Bacinello and Ortu (1993).

We now consider a financial asset with payoff C(S,t), typically on the form $C(S,t) = S(t) \vee G(t)$, where $\vee$ is forming the maximum. This financial asset will be used to model the life insurance benefit in the case of guaranteed unit-linked contracts. Below we sketch the idea for this more complex case which is the topic of the remainder of the paper.

Now we assume that the value of the fund evolves according to a given stochastic process and a financial asset with payoff $C(S,t) = S(t) \vee G(t)$ for some deterministic function G(t). The essence of an important result from financial economics can in this setting be formulated as follows:

$$\pi_0(t) = E^Q[e^{-\delta t}C(S,t)], \tag{6}$$

where $E^Q[\ ]$ denotes the expectation under an equivalent probability measure. At this point we can consider $E^Q[\ ]$ as a market consistent pricing principle. We return to its description in Section 2.

To value a contract similar to the one described by equation (3) with benefit $C(S,t)$ described above, we still use the right hand side of relation (3), but where $\pi_0(t)$ is given by (6). This ought to explain what we mean by the term the principle of equivalence under Q, a topic we return to in Section 3.

In Section 4 we derive an equation for the market value of the premium reserve. This equation we compare with both the Thiele equation of the actuarial science and the Black and Scholes equation from the theory of financial economics, and we find that our equation is a generalization of both. We also discuss the concepts of economic risk premiums, mortality risk premiums and savings premiums. In Section 5 we list certain risk minimizing or replicating trading strategies which may be used by the issuing company to reduce the financial risk associated with issuing unit-linked products. Some concluding remarks are included in Section 6.

2. The economic model

2.1 Further references

In this section we reproduce some important results from the arbitrage pricing theory of financial economics. This presentation must necessarily be brief and we can only refer to the seminal works mentioned in the introduction or to textbooks in finance such as Dothan (1990), Duffie (1988, 1992) and Huang (1991). In Aase (1988) this theory is extended to include price processes including possible jumps for the underlying security. Cox and Huang (1989) give a comprehensive introduction and, as mentioned, Duffie (1991) presents an overview of the current status of this theory. All relations involving random variables are understood to hold almost surely, though the short hand notation a.s. is sometimes added for emphasis.

2.2 The model of uncertainty

We consider a finite time horizon [0,T] and a given probability space $(\Omega, \mathcal{F}, P)$. The set Ω consists of all the possible states of the world. Here $\mathcal{F}$ is a σ-algebra of subsets of Ω and P is a probability measure. Events are revealed over time according to a filtration, $\mathbf{F} = \{\mathcal{F}_t, t \in [0,T]\}$, a collection of increasing σ-algebras, i.e., $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $t \geq s$. In addition we assume that $\mathcal{F}_0$ contains all the sets of probability zero and that the filtration is right continuous. A filtration satisfying these conditions is said to satisfy the *usual conditions*. We also take $\mathcal{F} = \mathcal{F}_T$ and $\mathcal{F}_0$ to be almost trivial. This can roughly be interpreted as follows: At time zero no information is available, at time t the agents can determine whether the events in $\mathcal{F}_t$ have occurred or not, and at time T all uncertainty is resolved.

To model the market value of the reference portfolio we use a standard Brownian motion W(t) on $(\Omega, \mathcal{F}, P)$ which includes that the increment $\{W(t) - W(s)\}$ is normally distributed and independent of $\mathcal{F}_s$, with mean 0 and variance (t-s), and $W(0) = 0$. Let $\mathcal{G}_t$ be the σ-algebra generated by the Brownian motion and the sets of probability zero from time 0 to time t.

A random time U is a stopping time with respect to a filtration F if the event $\{U \leq t\}$ belongs to $\mathcal{F}_t$ for all $t \in [0,T]$.

We recall that T_x represents an x-year old person's remaining life time. This random variable generates a σ-algebra $\mathcal{H}_t = \sigma(\{T_x > s\}, 0 \leq s \leq t)$. We observe that T_x is a stopping time with respect to the filtration $\{\mathcal{H}_t, t \in [0,T]\}$.

We assume that the σ-algebras $\mathcal{G}_t$ and $\mathcal{H}_t$ are independent which basically says that the value of the reference portfolio is independent of the insured's health condition. We also assume that $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$, where $\mathcal{G}_t \vee \mathcal{H}_t$ is the σ-algebra generated by the union of $\mathcal{G}_t$ and $\mathcal{H}_t$. This can be interpreted as the total information available in the economy at time t is the information one can get by recording the value of the reference portfolio and the state of the insured from time 0 to time t. We observe that T_x is then a stopping time with respect to the filtration **F**.

A stochastic process X: $\Omega \times [0,T]$ is called measurable if it is product measurable with respect to the smallest σ-algebra on $\Omega \times [0,T]$ containing all sets of the form $A \times B$, where $A \in \mathcal{F}$ and B is a set in the Borel σ-algebra on [0,T]. A stochastic process X is adapted to the filtration **F** if X_t is measurable with respect to $\mathcal{F}_t$ for all $t \in [0,T]$.

The security market model consists of two securities. Let B(t) denote the value of a riskless bond and S(t) the value of the reference portfolio at time $t \in [0,T]$. These securities are traded in a frictionless market (no taxes, no transaction costs, short-sales allowed). We choose the following price system where the bond price at time t equals

$$B(t) = e^{\delta t}. \tag{7}$$

As before, δ may be interpreted as the constant riskless rate of return. The price process for the reference portfolio (the mutual fund) is

$$S(t) = S(0)e^{\left(\eta - \frac{1}{2}\sigma^2\right)t + \sigma W(t)} \tag{8}$$

The constants η and σ may be interpreted as the instantaneous expected rate of return of the fund and the instantaneous standard deviation of the rate of return of the fund, respectively. Also S(0) is assumed to be a constant, interpretable as the price of one unit of the reference portfolio at time zero. These interpretations may become clearer if we write (8) as (heuristic notation)

$$\frac{dS(t)}{S(t)} = \eta dt + \sigma dW(t), \text{ given an initial value } S(0).$$

It follows by Itô's lemma that (8) is the solution of this stochastic differential equation. Neither of the securities pay dividends during (0,T). Here we observe that B(t), which is not stochastic, and S(t), which is uniquely determined by W(t), are adapted processes.

We denote the insurance benefit payable at time U by C(U), where U is a stopping time. In the life insurance context U can be interpreted as the time of expiration of the benefit which is T for a pure endowment insurance and T_x, if $T_x \le T$, for a term insurance. For simplicity we only present results for the case where the benefit is payable at the fixed time T (the pure endowment case).

Let C(T) be a random variable with finite variance, representing the benefit payable at time T. In this paper C(T) will be a measurable function of S(T) and since $S = \{S(t), t \in [0,T]\}$ is adapted, its value can be determined based on $\mathcal{F}$.

The discounted price system, denoted by the *-symbol, is simply (7) and (8) divided by B(t), or

$$B^*(t) = \frac{B(t)}{B(t)} \equiv 1$$

and

$$S^*(t) = \frac{S(t)}{B(t)} = S(0)e^{\left(\eta - \delta - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}.$$

2.3 Results from the theory of financial economics
An outline of the arbitrage pricing theory now follows.

First we define

$$\xi_t = \exp\left(-\frac{1}{2}\left(\frac{\eta-\delta}{\sigma}\right)^2 t - \frac{\eta-\delta}{\sigma}W(t)\right),$$

for $t \in [0,T]$. It is easy to verify that $E[\xi_t] = 1$ and $Var[\xi_t] = \exp\left(\left(\frac{\eta-\delta}{\sigma}\right)^2 t\right) - 1 < \infty$ and that ξ_t is a strictly positive random variable (almost surely) for $t \in [0,T]$. We then define a probability measure by $Q(A) = E\left[1_A \frac{dQ}{dP}\right]$ for $A \in \mathcal{F}$, where $\frac{dQ}{dP} = \xi_T$ and 1_A denotes the indicator function that takes the value 1 if the event A occurs and 0 otherwise. Q thus defined is equivalent to P, meaning that $P(A) = 0 \Leftrightarrow Q(A) = 0$ for any $A \in \mathcal{F}$. From Girsanov's theorem it follows that

$$\hat{W}(t) = W(t) + \frac{\eta-\delta}{\sigma}t$$

is a standard Brownian motion under Q which is also adapted to F.

It follows that the discounted price process under Q is

$$S^*(t) = S(0)e^{-\frac{1}{2}\sigma^2 t + \sigma \hat{W}(t)}. \tag{9}$$

Here we notice that $E^Q[S^*(u) \mid \mathcal{F}_t] = S^*(t)$ for $0 \le t \le u \le T$, so $S^* = \{S^*(t), t \in [0,T]\}$ is a martingale with respect to F under Q. The probability measure Q thus satisfies: (1) P and Q are equivalent, (2) S^* is a martingale under Q and (3) $Var\left(\frac{dQ}{dP}\right) < \infty$ and we say that S^* admits an *equivalent martingale measure*. A proof of uniqueness of Q may for example be found in Huang (1991).

Now we turn to the trading strategies and the definition of arbitrage. A trading strategy is an adapted measurable process or a dynamic investment rule describing how many shares of the fund and bonds to hold at each point in time.

Let H be the set of admissible trading strategies in this model and let $\alpha(t)$ and $\beta(t)$ denote the numbers of shares in the fund and bonds held at time t, respectively. As a matter of notation we sometimes refer to the pair $\{\alpha(t), \beta(t), t \in \tau\}$ as (α,β), where the time period τ should be clear from the context.

A self-financing trading strategy is defined as a trading strategy which does not generate capital gains or require inflow of capital during the investment period and satisfies for $t \le T$:

$$\alpha(T)S(T) + \beta(T)B(T) = \alpha(t)S(t) + \beta(t)B(t) + \int_t^T \alpha(s)dS(s) + \int_t^T \beta(s)dB(s) =$$

$$\alpha(t)S(t) + \beta(t)B(t) + \int_t^T [\alpha(s)\eta S(s) + \beta(s)\delta B(s)]ds + \int_t^T \alpha(s)\sigma S(s)dW(s) \text{ a.s.}$$

The integrals involving dS and dW are well-defined only as stochastic integrals. The similar expression for the discounted price system under the equivalent martingale measure Q is

$$\alpha(T)S^*(T) + \beta(T) = \alpha(t)S^*(t) + \beta(t) + \int_t^T \alpha(s)\sigma S^*(s)d\hat{W}(s) \text{ a.s.} \tag{10}$$

To avoid technical difficulties let H consist of the self-financing trading strategies such that

$$E^Q\left[\int_0^T [\alpha(t)S^*(t)]^2 dt\right] < \infty.$$

This restriction limits the size and the speed of the trades that may take place and ensures that the stochastic integral in (10) is a martingale with respect to **F**.

An arbitrage opportunity is a trading strategy that, loosely speaking, generates something out of nothing or

$$\alpha(T)S(T) + \beta(T)B(T) \geq 0 \text{ and } \alpha(t)S(t) + \beta(t)B(t) < 0 \text{ or}$$

$$\alpha(T)S(T) + \beta(T)B(T) > 0 \text{ and } \alpha(t)S(t) + \beta(t)B(t) \leq 0, \text{ for } t \leq T.$$

In this setting, a *complete* economy means that any C(U) with finite variance can be obtained as the terminal value $\alpha(T)S(T) + \beta(T)B(T)$ of some $(\alpha,\beta) \in H$, meaning that $C(T) = \alpha(T)S(T) + \beta(T)B(T)$ a.s.

Lemma 1

If S^* admits an equivalent martingale measure, then there is no arbitrage.

Proof:

See, e.g., Duffie (1992), Chapter 6, paragraph F. ❑

Lemma 2

The economy given by $(\Omega, \mathcal{F}, P)$, **F**, $S = \{S(t), t \in [0,T]\}$, $B = \{B(t), t \in [0,T]\}$ and H is complete.

Proof:

The proof is based on the martingale representation theorem, see, e.g., Duffie (1992), Chapter 6, paragraph I or Cox and Huang (1989), Theorem 4. ❑

Lemma 3

In the economy given by $(\Omega, \mathcal{F}, P)$, **F**, S, B and H the unique market price of C(T) at time t is given by $\pi_t(T) = \alpha(t)S(t) + \beta(t)B(t)$, for some $(\alpha,\beta) \in H$.

Proof:

From Lemma 2 there exists some $(\alpha,\beta) \in H$ which duplicates C(T). From Lemma 1 there is no arbitrage opportunities in this economy, so by investing $\alpha(t)S(t) + \beta(t)B(t)$ at time t and employing the strategy (α,β) from t to T, C(T) will be obtained at no extra cost. If $(\hat{\alpha},\hat{\beta})$ is another strategy which duplicates C(T) and $[\alpha(t)S(t) + \beta(t)B(t)] \neq [\hat{\alpha}(t)S(t) + \hat{\beta}(t)B(t)]$, then there is an arbitrage opportunity, so uniqueness follows. ❑

Lemma 4

In the economy given by $(\Omega,\mathcal{F}, P)$, **F**, S, B and H, the market price at time t for a benefit payable at time T is given by

$$\pi_t(T) = E^Q[e^{-\delta(T-t)}C(T) \mid \mathcal{F}_t] \text{ a.s.} \tag{11}$$

Proof:

Let $C^*(T) = C(T)e^{-\delta T}$ and $\pi_t^*(T) = \pi_t(T)e^{-\delta t}$. Consider the following equalities:

$$E^Q[C^*(T) \mid \mathcal{F}_t] = E^Q[\alpha(T)S^*(T) + \beta(T) \mid \mathcal{F}_t] = \alpha(t)S^*(t) + \beta(t) = \pi_t^*(T) \text{ a.s.}$$

The first equality follows from Lemma 2. The second equality follows from (10) by observing that the stochastic integral is a Q-martingale. The third equality follows from Lemma 3. Relation (11) now follows immediately. ❑

Observe that $\pi_t^*(T)$, as a function of t, is a martingale under Q with respect to **F**.

The philosophy behind the traditional principle of equivalence is that the insurer's expenses and income will average in the long run. The results from the previous paragraph imply that if the insurer values the benefit different from the principle given in (11), he may systematically either make a positive profit or lose money. However, by using (11) he will, by using the corresponding trading strategies, neither lose nor win money. Observe that this principle forces the insurer to trade actively in the market. The principle given by (11) states that the market value of the benefit must equal its price. So, also the market values of premiums must be equal to the market value of the benefits under our principle.

Intuitively, in this model we use finance theory to value the benefit and incorporate the resulting market values into the standard actuarial models. There are two independent sources of uncertainty, one related to the financial market, the other related to mortality. Formally we may model each source on its own probability space so that we can consider $(\Omega,\mathcal{F}, P)$ as a product space. From the discussion above we can consider ξ_T as a pricing rule for financial risk. By risk neutrality with respect to mortality it follows that the corresponding pricing rule for mortality risk is identical to 1. From the assumed independence between financial risk and mortality risk it follows that the pricing rule on the product space is $1\xi_T = \xi_T$. Then the pricing principle presented below follows formally from Fubini's theorem. This idea is explained in Persson (1994c).

By considering the market value at time zero of a pure endowment policy and a term insurance, it follows from this principle that

$$V(0) = E^Q[e^{-\delta T}C(T)]_T p_x$$

and

$$V(0) = \int_0^T E^Q[e^{-\delta t}C(t)]f_x(t)dt$$

respectively, for the unit-linked pure endowment policy and the unit-linked term insurance.. In the next section we treat these contracts with one particular example of a benefit.

3. Single premiums of unit-linked contracts

3.1 Arbitrage pricing

The insurance company is exposed to financial risk by issuing unit-linked insurance contracts involving guarantees. This implies that, even though the probability distributions for the value of the reference portfolio and of the time of death are both supposed to be known, it is not correct to compute premiums based on risk neutrality with regard to the financial assets.

Given the price system in (7) and (8), and transforming this system to a discounted price system, and correspondingly transforming the probability measure, the discounted price system becomes a martingale under the new probability measure. This measure is often called a risk-adjusted probability measure. The Radon-Nikodym derivative ξ_T can be interpreted as an infinite dimensional vector of weights so that after multiplying a cashflow C by this weight ω by ω, the expectation of the product, i.e., $E[C\xi_T] = E^Q[C]$, represents the market price. As a consequence we can still calculate single premiums as expectations, but where we use this new probability measure Q in the computations instead of the originally given P, on which everyone is assumed to agree, and which represents the agents' beliefs regarding the occurrence of the future uncertain events. Thus Q has no meaning as a probability measure representing the beliefs of the agents. Instead we can say that Q really represents state prices and also happen to satisfy the formal requirements of a probability measure. The formal similarity between this approach using Q, and the principle of equivalence is, however, striking and somewhat elegant.

3.2 Single premium for a guaranteed unit-linked pure endowment insurance

Recall the definitions of C(t), N(t) and G(t), $t \in [0,T]$ from the previous sections. A person at age x buys a policy specifying that the benefit $C(T) = (N(T)S(T) \vee G(T))$ is to be paid at time T if he is alive then. Let ${}_TG_x$ denote the single premium. Then we can formulate the following:

Theorem 1.

Assume risk–neutrality with respect to mortality, let the financial model be given as in paragraph 2.2. and maintain our other idealized market assumptions. Then the net market premium ${}_TG_x$ of the insurance contract described above is given by

$$ {}_TG_x = {}_Tp_x[G(T)e^{-\delta T}\Phi[-d_2^0(T)] + S(0)N(T)\Phi[d_1^0(T)]], \tag{12}$$

where $\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$ is the standard normal distribution function and for $s \geq t$

$$d_1^t(s) = \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta + \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}} \tag{13}$$

and

$$d_2^t(s) = \frac{\ln\left[\frac{N(s)S(t)}{G(s)}\right] + \left(\delta - \frac{1}{2}\sigma^2\right)(s-t)}{\sigma\sqrt{s-t}}. \tag{14}$$

Proof.

From (11) it immediately follows that this market value at time zero of the benefit is given by $E^Q[C^*(T)]$. Let us simplify the notation by writing Z instead of $S^*(T)$. Now, combining insurance and finance using the policy specification and the independence between the value of the stock and the policy-holder's remaining life time as well as risk-neutrality with respect to mortality, we get that ${}_TG_x = {}_Tp_x E^Q[(N(T)Z \vee G^*(T))]$.

From (9) we have that $Z = S(0)e^{-\frac{1}{2}\sigma^2 T + \sigma\hat{W}(T)}$ where $\hat{W}(T)$ is normally distributed with mean 0 and variance T under Q. By taking the expected value as indicated above, we obtain the expression

$${}_TG_x = {}_Tp_x \int_{-\infty}^{\infty} [N(T)S(0)e^{-\frac{1}{2}\sigma^2 T + \sigma w} \vee G^*(T)] f(w)dw,$$

where $f(w) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T}w^2}$ is the normal density function corresponding to mean 0 and variance T. We find that when $w > \frac{1}{\sigma}\left[\ln\left(\frac{G^*(T)}{N(T)S(0)}\right) + \frac{1}{2}\sigma^2 T\right] = \bar{w}$, $N(T)Z > G^*(T)$. We can thereby split the expression into two integrals

$${}_TG_x = {}_Tp_x\left(G^*(T)\int_{-\infty}^{\bar{w}} f(w)dw + N(T)S(0)\int_{\bar{w}}^{\infty} e^{-\frac{1}{2}\sigma^2 T + \sigma w} f(w)dw\right).$$

It follows that $e^{-\frac{1}{2}\sigma^2 T + \sigma w} f(w) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(w - \sigma T)^2}$ which is the normal probability density function for a random variable with mean σT and variance T. The rest is a matter of routine calculations. We substitute to the standard normal density function using $u = \frac{w}{\sqrt{T}}$ in the first integral and $v = \frac{w - \sigma T}{\sqrt{T}}$ in the second integral. The integration limit in the first integral then becomes $\bar{u} = \frac{\bar{w}}{\sqrt{T}}$ and in the second integral $\bar{v} = \frac{\bar{w} - \sigma T}{\sqrt{T}}$. By using the definition of the cumulative normal distribution function of the standard normal variable, denoted by $\Phi(\cdot)$, $G^*(T) = G(T)e^{-\delta T}$ and standard properties of the normal density, we obtain (12). ❑

Readers familiar with the theory of options will immediately recognize the similarities between the payoff at expiration of this unit-linked contract and an option. A European call option entitles the owner the amount $[S(T) - G(T)] \vee 0$ at expiration, while the owner of a European put option is entitled to the amount $[G(T) - S(T)] \vee 0$ at time T. The insurance contract of Theorem 1 entitles the owner, if he is alive, the amount $G(T) + [N(T)S(T) - G(T)] \vee 0 = N(T)S(T) + [G(T) - N(T)S(T)] \vee 0$ at time T.

The single premium could alternatively be expressed as

$${}_TG_x = {}_Tp_x[G(T)e^{-\delta T}(1 - \Phi[d_2^0(T)]) + N(T)S(0)\Phi[d_1^0(T)]].$$

The Black and Scholes European call option formula values the claim $[S(T) - G(T)] \vee 0$ at time zero. The two last terms with $N(T) \equiv 1$ in the square bracket are identical to this formula. The first term is simply the net present value of G(T) which must be included and has the given form since G(T) is assumed non-random. The benefit will only be paid if the policyholder is alive at time T, the whole expression is therefore multiplied with the survival probability ${}_Tp_x$, where we have used the independence assumption regarding S and T_x.

The formula (12) may also be written as

$${}_TG_x = {}_Tp_x[N(T)S(0) + G(T)e^{-\delta T}\Phi[-d_2^0(T)] - N(T)S(0)\Phi[-d_1^0(T)]],$$

where the last two terms with $N(T) = 1$ inside the brackets are identical to the European put option formula and the first term is the market value at time zero of N(T)S(T).

3.3 Single premium for a unit-linked term insurance with guarantee

A person at age x buys a policy specifying that the benefit $C(t) = (N(t)S(t) \vee G(t))$ is payable upon death at time t before T. Let $G^1_{x:\overline{T}|}$ denote the single premium. We then have

Theorem 2.

Consider the above contract with payoff C(t) if death occurs before time T, zero otherwise. Then the market value, or the single premium, of this contract is given by the expression

$$G^1_{x:\overline{T}|} = \int_0^T [G(t)e^{-\delta t}\Phi[-d_2^0(t)] + N(t)S(0)\Phi[d_1^0(t)]]_t p_x \mu_{x+t} dt. \tag{15}$$

Proof.
The main difference from the policy of Theorem 1 is that the point of expiration for the term insurances is random. However, this time point is distributed according to the probability distribution $f_x(t) = {}_tp_x\mu_{x+t}$ for $t \leq T$, since for term policies the time of expiration coincides with the time of death if death occurs before T.

We have that $G^1_{x:\overline{T}|} = E^Q[C^*(\tilde{T})]$. In this expression $\tilde{T} = \{T_x \wedge T\}$, where $\wedge$ is forming the minimum. Since nothing is paid when $T_x > T$, we get by conditioning on T_x that $G^1_{x:\overline{T}|} = \int_0^T E^Q[N(t)Z \vee G^*(t)]f_x(t)dt$, where $Z = S^*(t)$. We now proceed as in the proof of Theorem 1 to complete the integrand above, and this yields the desired result. ❑

As illustrated in the following example, Theorems 1 and 2 provide the building-blocks from which the market values of many interesting unit-linked contracts can be found.

Example: Endowment insurance
Let ${}_TE_x = {}_Tp_x e^{-\delta T}K$ be the single premium of a traditional pure endowment insurance, where K is a constant, and $\overline{A}^1_{x:\overline{T}|} = K[1 - \delta\overline{a}_{x:\overline{T}|} - {}_Tp_x e^{-\delta T}]$ be the single premium of a traditional term insurance, where $\overline{a}_{x:\overline{T}|} = \int_0^T {}_tp_x e^{-\delta t}dt$. By combining traditional, pure unit-linked and guaranteed unit-linked pure endowment and term insurances, we get the following single premiums for the possible endowment insurances:

$${}_TE_x + \overline{A}^1_{x:\overline{T}|},\ {}_TE_x + U^1_{x:\overline{T}|},\ {}_TE_x + G^1_{x:\overline{T}|},\ {}_TG_x + G^1_{x:\overline{T}|},\ {}_TG_x + U^1_{x:\overline{T}|},$$

$${}_TG_x + \overline{A}^1_{x:\overline{T}|},\ {}_TU_x + U^1_{x:\overline{T}|},\ {}_TU_x + \overline{A}^1_{x:\overline{T}|},\ {}_TU_x + G^1_{x:\overline{T}|},$$

of which the first is the traditional endowment insurance.

3.4 Economic risk premiums

Abstracting from administrative expenses, a general insurance premium is often modeled as (see, e.g., Borch, 1990)

$$\pi_0 = E[C] + RP. \tag{16}$$

where π_0 is the single premium of a given policy, E[C] is the expected payoff of the benefit and RP is an economic risk premium. The nature of the economic risk premium varies from one type of insurance to another and may be interpreted as the insurer's compensation for bearing risk.

Life insurance contracts usually have longer insurance periods than non-life insurance contracts which often are renewed yearly. Therefore it is important to take the time perspective into account in the expectation term in (16). We denote the expected present value E[PV(C)] so that

$$\pi_0 = E[PV(C)] + RP \tag{17}$$

for life insurance. For traditional life insurance, the economic risk premium term RP is equal to zero and (17) is thus simply a restatement of the traditional principle of equivalence.

In unit-linked insurance the value of the benefit is supposed to behave according to the value of the reference portfolio. The benefit may be considered as an asset which the customer equally well could have bought in the financial market. An investor buying financial assets is also generally demanding a risk premium to carry financial risk. We denote by $\pi_0^\delta(t)$ the market price at time zero of the benefit $C(t) = [S(t) \vee G(t)]$ payable at time t, as a function of the interest rate δ. From the results in paragraphs 3.2 and 3.3 it follows that

$$\pi_0^\delta(t) = G(t)e^{-\delta t}\Phi[-d_2^t(t)] + N(t)S(0)\Phi[d_1^t(t)]$$

The expected payoff discounted by the risk-free rate δ for this benefit equals $e^{(\eta-\delta)t}\pi_0^\eta(t)$. Consider the pure endowment unit-linked contract. From (17) we get that RP $= {}_TG_x -$ E[PV(C)].

By inserting the expressions for the single premium (12) and E[PV(C)] (where the expectation also includes mortality), we obtain

$$RP = -{}_Tp_x[e^{(\eta-\delta)T}\pi_0^\eta(T) - \pi_0^\delta(T)]. \tag{18}$$

It is easily shown that $\frac{\partial}{\partial r}\pi_0^r(t) > 0$, so it follows that RP < 0 if and only if $\eta - \delta > 0$. Since the portfolio is risky, normally we would have the situation that $\eta - \delta > 0$.

Also note that the risk premium of a European call option calculated this way is RP = $e^{(\eta-\delta)T}\pi_0^{\eta}(T) - \pi_0^{\delta}(T)$. For the unit-linked term insurance we get from (17) that RP = $G^1_{x:\overline{T}|}$ − E[PV(C)]. By inserting the expressions for the single premium (15) and E[PV(C)], we get

$$RP = -\int_0^T \{e^{(\eta-\delta)t}\pi_0^{\eta}(t) - \pi_0^{\delta}(t)\}\,{}_tp_x\mu_{x+t}dt. \tag{19}$$

Also for this policy the economic risk premium is negative if $\eta > \delta$.

By issuing pure unit-linked contracts the issuing company acts merely as a broker between the customer and the financial market. The products with a guarantee naturally involves risk bearing for the company. The customers are generally risk averse and will accept to pay a positive risk premium for the insurance product, whereas they expect a positive profit on the pure financial investments. Comparing the unit-linked products to the traditional ones, we see that the customer can expect a higher payoff for unit-linked products when $\eta > \delta$. But naturally unit-linked products are more risky than traditional life insurance. This also means that the companies can not treat these products the way they treat ordinary insurance risks. The reinsurance market may be employed to relieve the company of some of the risk, but the aggregate risk remains in this market. On the other hand, we will demonstrate section 5 how the insurance company may completely diversify the financial risk by reversing the risk minimizing trading strategies in the financial market. Thus the insurance companies must actively "play the financial market" in this line of insurance in order to avoid losing money in the long run.

In practice companies use a loading on mortality, which gives them the right safety margins on the death elements of these contracts. This loading may be determined by the market (see, e.g., Aase, 1992). We also note that if the single premium of a unit-linked contract, incorrectly, was calculated by the traditional principle of equivalence, the single premium would have been too large.

Example. The benefits of the pure unit-linked contracts.
Assume that the price models (7) and (8) hold. The expected net present value of the benefit S(t) then becomes $E[e^{-\delta t}S(t)] = E\left[S(0)\exp\left\{\left(\eta - \delta - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right]\right. = S(0)e^{(\eta-\delta)t}$. From the introduction we have that $\pi_0(t) = S(0)$, so the risk premiums $RP = {}_Tp_xS(0)[1 - e^{(\eta-\delta)t}]$ and $RP = {}_T(1 - p_x)S(0)[1 - e^{(\eta-\delta)t}]$ for the pure endowment unit-linked policy and the pure unit-linked term insurance, respectively. Both are seen to be less than zero if $\eta > \delta$.

4. A generalization of Thiele's differential equation

4.1 Periodic premiums

We now consider the situation where the premiums are paid continuously over the term of the contract. In the first paragraphs the periodic premiums are represented by a non-random rate $\bar{p}(t)$. Therefore the principle of equivalence under Q is the same as the traditional principle, so that $\bar{p}(t)$ is determined from the equation

$$_TG_x = \int_0^T \bar{p}(t)e^{-\delta t}\, {}_tp_x dt$$

for the guaranteed unit-linked pure endowment contract and from

$$G^1_{x:\overline{T}|} = \int_0^T \bar{p}(t)e^{-\delta t}\, {}_tp_x dt$$

for the guaranteed unit-linked term insurance contract.

4.2 Prospective premium reserves

The prospective reserve is defined in traditional life insurance theory as the conditional expected present value of future benefits less premiums on the policy, given its present state. For the unit-linked contracts, we are interested in the market value of the of future benefits less the market values of the future premiums on the policy given the current state of the policy. These are found using the methodology outlined above. For the guaranteed unit-linked pure endowment contract we get

$$V(t) = {}_{T-t}p_{x+t}\pi_t(T) - \int_t^T \bar{p}(u)e^{-\delta(u-t)}\, {}_{u-t}p_{x+t} dt, \tag{20}$$

and for the guaranteed unit-linked insurance

$$V(t) = \int_t^T (\pi_t(u)\, f_{x+t}(u-t) - \bar{p}(u)e^{-\delta(u-t)}\, {}_{u-t}p_{x+t}) du, \tag{21}$$

where $\pi_t(s) = G(s)e^{-\delta(s-t)}\Phi[-d_2^t(s)] + N(s)S(t)\Phi[d_1^t(s)]$.

It is worth noticing that these reserves depend on S(t), the market value of the reference portfolio at time t and are valid given that the policy-holder is alive at time t.

4.3 Derivation of an equation describing the evolution of the market value

We now present the differential equations describing the evolution of the market value of the premium reserve through time. The corresponding equations for the traditional life products are called Thiele's differential equations in the actuarial sciences.

Theorem 3.

The market value of the premium reserve for the guaranteed unit-linked pure endowment contract paid with periodic premium rate $\bar{p}(t)$ satisfies the following partial differential equation:

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta)V(t) - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} - \delta S(t)\frac{\partial V}{\partial S}. \tag{22}$$

Proof. Recall the definition of $\pi_t^*(T)$ from Section 2. From (20) it follows that

$$V(t) = {}_{T-t}p_{x+t}\pi_t^*(T)\cdot e^{\delta t} - \int_t^T \bar{p}(u)e^{-\delta(u-t)}\,{}_{u-t}p_{x+t}dt.$$

We first solve this equation for $\pi_t^*(T)$, and get

$$\pi_t^*(T) = \psi(t)\left(V(t) + \int_t^T \bar{p}(u)e^{-\delta(u-t)}\,{}_{u-t}p_{x+t}dt\right),$$

where $\psi(t) = \frac{e^{-\delta t}}{{}_{T-t}p_{x+t}}$.

We now want to express the dynamics of $\pi_t^*(T)$ by using the partial derivatives of V. By noting that $\frac{\partial}{\partial t}\,{}_{u-t}p_{x+t} = \mu_{x+t}\,{}_{u-t}p_{x+t}$ and $\frac{\partial}{\partial t}\psi(t) = -(\mu_{x+t} + \delta)\psi(t)$ and also by considering $\pi_t^*(T)$ as a function of S and t, we get the partial derivatives

$$\frac{\partial \pi_t^*}{\partial S} = \psi(t)\frac{\partial V}{\partial S},$$

$$\frac{\partial^2 \pi_t^*}{\partial S^2} = \psi(t)\frac{\partial^2 V}{\partial S^2}$$

and

$$\frac{\partial \pi_t^*}{\partial t} = \psi(t)\left(\frac{\partial V}{\partial t} - (\mu_{x+t} + \delta)V(t) - \bar{p}(t)\right).$$

The undiscounted price process for S under Q is $dS(t) = \delta S(t)dt + \sigma S(t)d\hat{W}(t)$, so by Itô's lemma we get for $s \geq t$

$$\pi_s^*(T) = \pi_t^*(T) + \int_t^s \psi(u)\frac{\partial V}{\partial S}\sigma S d\hat{W}(u) +$$

$$\int_t^s \psi(u)\left(\frac{\partial V}{\partial S}\delta S + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - (\mu_{x+u} + \delta)V(u) + \frac{\partial V}{\partial u} - \bar{p}(u)\right)du. \tag{23}$$

By the fact that $\pi_t^*(T)$ is a martingale under Q (See Lemma 2), the drift term must be zero and (22) follows since $\psi(u) \neq 0$ for all $u \in (t,s)$. ❑

Theorem 4.

The market value of the premium reserve for the guaranteed unit-linked term insurance contract paid with periodic premium rate $\bar{p}(t)$ satisfies the following partial differential equation:

$$\frac{\partial V}{\partial t} = \bar{p}(t) + (\mu_{x+t} + \delta)V(t) - C(t)\mu_{x+t} - \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V}{\partial S^2} - \delta S(t)\frac{\partial V}{\partial S}. \tag{24}$$

Proof: The structure of this policy is more complicated since the benefit may expire at any time between t and T. Apart from this fact the proof is very similar to the previous proof.

We first define

$$V^u(t) = \pi_t^u\, f_{x+t}(u-t) - \bar{p}(u)e^{-\delta(u-t)}\ {}_{u-t}p_{x+t}.$$

From (21) we have that

$$V(t) = \int_t^T V^u(t)du.$$

Also note that

$$\frac{\partial V}{\partial S} = \int_t^T \frac{\partial V^u}{\partial S}du,$$

$$\frac{\partial^2 V}{\partial S^2} = \int_t^T \frac{\partial^2 V^u}{\partial S^2}du$$

and

$$\frac{\partial V}{\partial t} = \int_t^T \frac{\partial V^u}{\partial t}du - V^t(t). \tag{25}$$

For this insurance $V^t(t) = C(t)\mu_{x+t} - \bar{p}(t)$, so

$$\int_t^T \frac{\partial V^u}{\partial t} du = \frac{\partial V}{\partial t} + C(t)\mu_{x+t} - \bar{p}(t). \tag{26}$$

Again we express $\pi_t^*(u)$ as a function of $V^u(t)$,

$$\pi_t^*(u) = \chi(t)[V^u(t) + p(u)e^{-\delta(u-t)}\ {}_{u-t}p_{x+t}],$$

where $\chi(t) = \frac{e^{-\delta t}}{f_{x+t}(u-t)}$.

By the same arguments as in the previous proof it follows that

$$\frac{\partial V^u}{\partial t} = (\mu_{x+t} + \delta)V^u(t) - \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V^u}{\partial S^2} - \delta S(t)\frac{\partial V^u}{\partial S}$$

for $u \in [t,T]$. By integrating this expression on both sides with respect to u from t to T, using (25) and (26), (24) follows. ❑

Note that the solutions of (22) subject to $V(T) = C(T)$ and of (24) subject to $V(T) = 0$ are given by (20) and (21), respectively.

Also note that (22) follows by letting $C(t) = 0$ in (24), so expression (24) may be termed the full-fledged Thiele's equation in our set-up. The three first terms on the right hand side may be interpreted as: The value of the premium reserve in the time interval $(t,t + dt)$ increases with interest δV and premiums paid $\bar{p}(t)$ and decreases with expected net payments from death $\mu_{x+t}(C(t) - V(t))$. The two last terms are changes in the market value of the premium reserve caused by changes in value of the reference portfolio and are not present in the corresponding Thiele equation of traditional life insurance, which really does not deal with economic risk. By letting the insurance specific factors equal zero, i.e., $\mu_{x+t} = 0$ and $\bar{p}(t) = 0$, we indeed get the Black and Scholes partial differential equation, which is the original starting point for the derivation of the celebrated option pricing formula. From financial theory we know that in this economic model every contingent claim on a non-dividend paying security has to satisfy this equation. Expression (24) therefore shows explicitly how insurance factors such as periodic premium and mortality affect this equation. We are tempted to state that the equation resulting from the Cauchy problem, or the "heat equation" of Black and Scholes equals the fundamental Thiele partial differential equation in the special case of zero mortality. However, here we hasten to add that the traditional Thiele equation of ordinary life insurance does not contain the terms containing V_{SS} and V_S since in traditional insurance the benefit is not random. We remark here that the traditional Thiele equation obviously does not deal with stock market risk and treats mortality risk from the perspective of risk-neutrality, whereas the Black and Scholes equation deals with economic risk. The above equation thus incorporates economic risk also in life insurance.

4.4 Mortality risk premium and savings premium

It is common in life insurance to split the periodic premium rate into a mortality risk premium rate and a savings premium rate. The mortality risk premium is usually called only the risk premium. As mentioned earlier, unit-linked contracts with guarantee expose the issuer (and also the buyer) to financial risk, so we should expect to find economic risk premiums as well as mortality risk premiums. As showed in the previous section the economic risk premiums for unit-linked products are negative, as compared to zero for traditional products. Thus buyers may expect better return and must accept a higher risk than for ordinary products.

The quantity $[C(t) - V(t)]$ is called the uncovered amount, and the mortality risk premium in the context of life insurance is defined as $\mu_{x+t}(C(t) - V(t))$, i.e., the conditional expected benefit payments in excess of the reserves.

Let $\bar{p}_t^m$ and $\bar{p}_t^s$ denote the mortality risk premium rate and savings premium rate, respectively.

For unit-linked pure endowment insurance the mortality risk premium rate is

$$\bar{p}_t^m = -V(t)\mu_{x+t}. \tag{27}$$

For a unit-linked term insurance, on the other hand, we have that

$$\bar{p}_t^m = [C(t) - V(t)]\mu_{x+t}. \tag{28}$$

The savings premium is the part of the premium rate which is due to external inflows or outflows of funds and it is convenient to use the concept of self-financing strategies to derive expressions for these quantities.

In Section 5 below we will calculate the trading strategies in the fund and in the bond (α^+, β^+) such that

$$V(t) = \alpha^+ S(t) + \beta^+ B(t). \tag{29}$$

If we denote the change in value of the reserve due to capital gains by ΔV, we have from the definition of self-financing in Section 2 that this change is

$$\Delta V = \beta^+ dB(t) + \alpha^+ dS(t). \tag{30}$$

By considering V as a function of S in addition to t, we get from Itô's lemma that

$$dV = \left[\eta S(t)\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right]dt + \sigma S(t)\frac{\partial V}{\partial S}dW(t). \tag{31}$$

From the above definition of $\bar{p}_t^s$ we have that $\bar{p}_t^s dt = dV - \Delta V$, where dV is given by (31). We then get

$$\bar{p}_t^s dt = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \delta\beta^+ B(t)\right)dt + \left(\frac{\partial V}{\partial S} - \alpha^+\right)dS(t). \tag{32}$$

By choosing $\alpha^+ = \frac{\partial V}{\partial S}$ the stochastic part, dS, cancels out. Since $\beta^+ B(t) = V(t) - \alpha^+ S(t)$, we then obtain

$$p_t^s = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \delta V(t) + \delta \frac{\partial V}{\partial S} S. \tag{33}$$

For both insurances we have that $\bar{p}_t^m + \bar{p}_t^s = \bar{p}(t)$. By adding (27) and (33) we obtain (22), which is just our version of Thiele's differential equation for a unit-linked pure endowment insurance. By adding (28) and (33) we obtain (24), which is the same fundamental equation governing a unit-linked term insurance.

We may also note that the terms involving $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$ are connected to the savings premium and not to the mortality risk premium.

4.5 Premium rate as a constant fraction of the value of the reference portfolio

One of the ideas of unit-linked insurances is that the components of a contract should be measured in *units* of the mutual fund. In our models the benefit is specified to N(t) units. It is also possible to let the periodic premium be a given number of units of the reference portfolio. A consequence of this strategy is that the amount the customer is supposed to pay must vary with the value of the reference portfolio. As this may be considered somewhat unusual to life insurance customers, many potential customers may prefer the familiar constant premium rate.

Let λ denote a constant fraction of the reference portfolio. Then $\bar{p}(t)$ may be written as

$$\bar{p}(t) = \lambda S(t).$$

In this model the market value at time zero of the benefits equal the market value at time zero of the premiums, so that

$${}_TG_x = E^Q\left[\int_0^T \lambda S(s)\,{}_sp_x e^{-\delta s} ds\right] = \int_0^T \lambda S(0)\,{}_sp_x ds$$

for the guaranteed pure endowment insurance and

$$G^1_{x:\overline{T}|} = E^Q\left[\int_0^T \lambda S(s)\,{}_sp_x e^{-\delta s} ds\right] = \int_0^T \lambda S(0)\,{}_sp_x ds$$

for the guaranteed term insurance.

The fraction λ then may be found as

$$\lambda = \frac{{}_TG_x}{S(0)\int_0^T {}_sp_x ds}$$

and

$$\lambda = \frac{G^1_{x:\overline{T}|}}{S(0)\int_0^T {}_sp_x ds},$$

respectively, for the two policies.

5. Trading Strategies

5.1 The market value of the premium reserve and self-financing strategies

The following theorem identifies the only case when the market value of the premium reserve may be duplicated by a self-financing strategy.

Theorem 5.

The periodic premium rate equals the mortality risk premium rate if and only if the value of the premium reserve can be duplicated by a self-financing trading strategy in the fund and the bond.

Proof: We will first show that (α^+,β^+) self-financing implies that $\bar{p}(t) = \bar{p}_t^m$. Assume that (α^+,β^+) is self-financing. By definition it follows that $dV = \beta^+ dB + \alpha^+ dS$, hence $dV = \Delta V$ and $\bar{p}_t^s \equiv 0$. The other direction of the proof follows by reversing the arguments. ❑

From (28) it follows that when $\bar{p}(t) = \bar{p}_t^m$, Thiele's differential equation is reduced to Black and Scholes' differential equation. The reason for this is that the two insurance specific factors, the mortality rate and the periodic premiums, exactly offset each other.

It is also known from actuarial theory that the situation when $\bar{p}(t) = \bar{p}_t^m$ is the lower limit of the periodic premium rate to avoid the market value of the premium reserve to take negative values.

We also observe that in the single premium case, where $\bar{p}(t) \equiv 0$, it is never possible to duplicate the market value of the premium reserve by a self-financing strategy.

5.2 The duplicating strategies of the benefit

In most of this paper we have worked with the benefit $C(u) = [N(u)S(u) \vee G(u)]$. From the previous analysis we know that the market value at time t of this benefit is

$$\pi_t(u) = G(u)e^{-\delta(u-t)}\Phi[-d_2^t(u)] + N(u)S(t)\Phi[d_1^t(u)]. \tag{34}$$

Now we want to derive the duplicating strategy for this benefit.

From (10) and Lemma 2 we have for u > t that

$$C^*(u) = \pi_t^*(u) + \int_t^u \alpha(s)\sigma S^* dW. \tag{35}$$

By substituting for $\frac{\partial V}{\partial S}$ in relation (20) and by noticing that $\pi_u^*(u) = C^*(u)$, it follows that

$$C^*(u) = \pi_t^*(u) + \int_t^u \frac{\partial \pi_t}{\partial S}\sigma S^* dW. \tag{36}$$

By comparing (35) and (36) we see that $\alpha(t) = \frac{\partial \pi_t}{\partial S}$ duplicates the benefit. So from (34) we get

$$\left.\begin{aligned} \alpha(t) &= N(u)\Phi[d_1^t(u)] \\ \beta(t) &= G(u)e^{-\delta u}\Phi[-d_2^t(u)] \end{aligned}\right\}. \tag{37}$$

The latter expression follows since $\beta(t) = \pi_t^*(u) - \alpha(t)S^*(t)$.

5.3 Single premium contracts

We now use the strategies (37) to compute the duplicating trading strategies for the insurance contracts we have treated. We will find trading strategies (α^+, β^+) satisfying (29).

The strategies can now be interpreted as the "risk minimizing" trading strategies that in our model eliminate the financial risk associated with issuing unit-linked insurance contracts. By continuously adjusting the portfolio of stocks and bonds, which in this model is costless, the issuing company duplicates the payoff of the unit-linked insurance. By reversing those strategies in the financial markets, the financial risk inherent in these policies is eliminated.

These strategies may or may not be followed by the insurance company. The idea is that the potential use of these strategies gives a hedge against financial risks. In option pricing theory such strategies play a fundamental role, both in theory and in practical use in the market. Here we may mention portfolio insurance, which is totally based on the existence of such duplicating strategies. For issuers of unit-linked products these strategies may clearly be of practical importance in reducing the financial risk.

The market value of the premium reserve for the guaranteed unit-linked pure endowment paid with single premium may be written

$$V(t) = {}_{T-t}p_{x+t}\pi_t(T).$$

The duplicating strategy for this contract is

$$\left.\begin{aligned} \alpha^+ &= {}_{T-t}p_{x+t}N(T)\Phi[d_1^t(T)] \\ \beta^+ &= {}_{T-t}p_{x+t}G(T)e^{-\delta T}\Phi[-d_2^t(T)] \end{aligned}\right\}. \tag{38}$$

For guaranteed unit-linked term insurance we can write the market value of the premium reserve as

$$V(t) = \int_t^T \pi_t(u)\, f_{x+t}(u)du.$$

Here we get the duplicating strategy

$$\left.\begin{aligned} \alpha^+ &= \int_t^T \Phi(d_1^t(u))f_{x+t}(u)du \\ \beta^+ &= \int_t^T G(u)e^{-\delta u}\Phi[-d_2^t(u)]f_{x+t}(u)du \end{aligned}\right\}. \tag{39}$$

5.4 Periodic premiums

Case 1: p(t) deterministic

The market value of the premium reserve for the guaranteed unit-linked pure endowment insurance paid with a periodic premium rate $\bar{p}(t)$ is given by (20), and we get duplicating strategy

$$\left.\begin{aligned} \alpha^+ &= {}_{T-t}p_{x+t}N(T)\Phi[d_1^t(T)] \\ \beta^+ &= {}_{T-t}p_{x+t}G(T)e^{-\delta T}\Phi[-d_2^t(T)] - \int_t^T \bar{p}(u)e^{-\delta u}\, {}_{u-t}p_{x+t}du \end{aligned}\right\}. \tag{40}$$

For guaranteed unit-linked term insurance paid with a periodic premium rate $\bar{p}(t)$ the market value of the premium reserve is given by (21) and the corresponding strategy is

$$\left.\begin{aligned}\alpha^{+} &= \int_t^T \Phi(d_1^t(u)) f_{x+t}(u)du \\ \beta^{+} &= \int_t^T [G(u)e^{-\delta u}\Phi[-d_2^t(u)]f_{x+t}(u) - \bar{p}(u)e^{-\delta u}\, {}_{u-t}p_{x+t}]du\end{aligned}\right\}. \tag{41}$$

Case 2: P(t) is a fraction of the value of one share of the fund

Note first that in this case the market value of the remaining benefits is

$$E^Q\left[\int_t^T \lambda S(u)\, {}_{u-t}p_{x+t}du\right] = \lambda S(t)\int_t^T {}_{u-t}p_{x+t}du,$$

so that the market value of the premium reserve for the guaranteed unit-linked pure endowment contract is

$$V(t) = {}_{T-t}p_{x+t}\pi_t(T) - \lambda S(t)\int_t^T {}_{u-t}p_{x+t}du.$$

Then it follows that

$$\left.\begin{aligned}\alpha^{+} &= {}_{T-t}p_{x+t}N(T)\Phi[d_1^t(T)] - \lambda\int_t^T {}_{u-t}p_{x+t}du \\ \beta^{+} &= {}_{T-t}p_{x+t}G(T)e^{-\delta T}\Phi[-d_2^t(T)]\end{aligned}\right\}. \tag{42}$$

For the guaranteed unit-linked term insurance the expression for the market value of the premium reserve is

$$V(t) = \int_t^T [\pi_t(u)f_{x+t}(u) - \lambda S(t)\, {}_{u-t}p_{x+t}]du.$$

The duplicating strategy is accordingly given as

$$\left.\begin{aligned}\alpha^{+} &= \int_t^T [\Phi(d_1^t(u))f_{x+t}(u) - \lambda\, {}_{u-t}p_{x+t}]du \\ \beta^{+} &= \int_t^T G(u)e^{-\delta u}\Phi[-d_2^t(u)]f_{x+t}(u)du\end{aligned}\right\}. \tag{43}$$

When the periodic premium rate is a deterministic function of time, the expected present value of the future premiums is known at each point in time. Since we also model the value of a bond as a deterministic function of time, in this set-up the value of the bonds held can be reduced by exactly the expected present value of future premiums.

When the periodic premium is a constant fraction of the value of the reference portfolio, the expected number of units of the reference portfolio to be received in the future is known. The company can therefore at every point in time reduce the number of units held by just this number.

We see from (40) and (41) that the trading strategy in bonds is adjusted while the strategy in shares is unchanged, and from (42) and (43) that the trading strategy in stocks is adjusted while the trading strategy in bonds is unchanged – both compared to the single premium case.

5.5 A stochastic version of the Thiele differential equation

We now present a stochastic differential equation for the market value of the premium reserve. Here we deduce the stochastic Thiele equation corresponding to the non-stochastic version given in equation (24). First we consider $V(t)$ in equation (21) as a function of $\pi_t(u)$ and t. By differentiation we get $dV=\frac{\partial V}{\partial t}dt + \int \frac{\partial V}{\partial \pi_t(u)} d\pi_t(u)du$. From Lemma 2 we can write $d\pi_t(u) = \alpha(t)dS(t) + \beta(t)dB(t)$, where $\alpha(t)$ and $\beta(t)$ are given in (37). By using (41) and (29), we now get

$$dV = [\bar{p}(t) + (\mu_{x+t} + \delta)V(t) - C(t)\mu_{x+t} + \alpha^+ S(t)(\eta - \delta)]dt + \alpha^+ \sigma S(t)dW(t). \quad (44)$$

This is a stochastic differential equation which also governs the evolution of the market value of the premium reserve through time. The terms $(\mu_{x+t} + \delta)V(t)$, $\bar{p}(t)$ and $\mu_{x+t}C(t)$ have the same interpretation as before. The term $(\eta - \delta)\alpha^+ S(t)$ represents the additional expected capital gain (which is negative if $\eta < \delta$) of the part of the premium reserve invested in the reference portfolio. The term $dW(t)$ of (44) represents the stochastic change of the value of the reference fund. This term has expectation zero. If $\eta > \delta$, which is the normal case, and $\alpha^+ > 0$ is not chosen so that all financial risk is eliminated, V will have a higher expected drift rate than is the case for traditional life insurance contracts.

6. Concluding remarks

Unit-linked insurance is characterized by a random amount of benefit which is linked to some financial asset. Therefore new valuation techniques are required to value these products. We have shown how to use arbitrage pricing theory to derive expressions for the single premium for a unit-linked pure endowment insurance contract and a unit-linked term insurance contract in a continuous time model. Compared to the classical version, the equations for the market value of the premium reserves for unit-linked products now contain some new terms. These terms are not interpretable in the same way as the terms in the traditional Thiele equation, and we are tempted to state that by economic theory we have developed this equation one important step further.

This analysis is extended to a more general life insurance contract in Persson (1994b). While unit-linked products are highly specialized life insurance products, the same kind of results may be expected for traditional life insurance products in presence of a stochastic interest rate. The latter problem is the topic of Persson (1994c).

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Chapter 3

VALUATION OF A MULTISTATE INSURANCE CONTRACT WITH RANDOM BENEFITS[1]

We present a model where the value of the life insurance benefit is random. The policy is at each point in time assumed to be in one of a finite number of states and the evolution of the policy through time is modeled by a time-continuous, non-homogeneous Markov chain.

The insurance period of a life insurance contract is long compared to the contract period of a typical financial contingent claim. The value of the insurance benefit is assumed to follow a geometric Gaussian process which has certain appealing properties when dealing with such long contract periods. We use the martingale-based arbitrage pricing theory to derive the market value of a quite general life insurance policy and deduce a corresponding generalization of Thiele's differential equation.

Key words: Unit-linked Insurance, Equity-linked Insurance, Thiele's Differential Equation, Arbitrage Pricing Theory, Continuous Time Markov Chains.

1. Introduction

Life insurance companies are exposed to mortality risk and financial risk. Financial risks have, traditionally been avoided by guaranteeing a low rate of return on life insurance contracts. If the companies then realized a higher rate of return, which they usually did, the excess return was credited the insurance customer by various means such as bonuses, reduced premiums, higher benefits etc. During the last two decades a new kind of life insurance contracts have been introduced in several countries, most recently also in the Nordic countries. The insurance contracts have different names in different countries such as unit-linked, equity linked, variable life, universal life, universal variable life; this list is probably not even complete. In this paper we will focus on the property these policies have in common[2], namely that the value of the insurance benefits depends on some economic factor which cannot be controlled by the

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[2] However, depending on the design of the policy, this is not the case for all versions of the universal life insurance. See, e.g., Adelman and Dorfman (1992).

insurance company. An example of a benefit may be that the insured receives the maximum of the value of a given number of shares in a mutual fund and a given amount upon expiration. The benefit could alternatively be linked to money market instruments, stock-indexes, etc. Crucial for our model, however, is that this factor somehow must be traded in a competitive market.

The focus of this article is to demonstrate how results from the arbitrage pricing theory from financial economics may be connected to the application of continuous-time Markov processes in life insurance. The traditional theory of life insurance is based upon risk neutrality, meaning, among other things, that net premiums can be calculated as expectations. This method of calculating premiums even has its own name in the actuarial science, the equivalence principle. In this model the value of the insurance benefit is uncertain. By guaranteeing a minimum benefit the insurer is exposed to financial risk, and by assuming that the insurer is risk-averse with respect to financial risk, the traditional equivalence principle can no longer be applied to premium calculations.

The Markov set-up provides a common framework to model features which are usually included in life insurance policies such as accidental death benefits, premium waivers and family term coverage. An important result from the arbitrage pricing theory is that exogenously given price models having a certain martingale representation property yield a complete market where every contingent claim has a unique price. See Harrison and Kreps (1979), Harrison and Pliska (1981) and Aase (1988). This result is utilized in the Markov set-up for life insurance where the policy holder is assumed to be in one of a finite number of states at each point in time. Markov-chains in life insurance are discussed by Hoem (1968), (1969), (1988) and Norberg (1991), among others.

Brennan and Schwartz (1976), (1979), Boyle and Schwartz (1977) and Delbaen (1990) have presented some results for equity linked insurances. The present analysis is basically an extension of the two-state model of Persson (1994a).

Motivating example.
To be more specific we specialize the uncontrollable economic factor to be a mutual fund. In addition, a riskless bond exists, and these securities are traded in a frictionless market. The values of the bond and the fund at time t are denoted B_t and S_t. Their price processes are exogenously given by

$$dB_t = \delta B_t dt, \tag{1}$$

and

$$dS_t = \eta S_t dt + \sigma S_t dW_t.$$

Here dW_t is the increment of a standard Brownian motion, δ, η and σ are constants to be interpreted as the risk-free rate of return, the instantaneous drift rate of the stock and the instantaneous standard deviation rate of the stock, respectively. Neither of the securities pay dividends.

Imagine a life insurance contract which entitles the insured (or, to be precise, the insured's inheritors) the value $C(t) = Max[S_t,G]$, where G is the non-random guaranteed amount, upon death at time t within a given time horizon T. This contract is called a unit-linked term insurance. The single premium of this contract is

$$G^{1}_{x:\overline{T|}} = \int_0^T \{Ge^{-\delta t} + h(S_0,t)\}\frac{l_{x+t}}{l_x}\mu_{x+t}dt, \tag{2}$$

where $h(S_0,t)$ is a known function, which can be interpreted as the option pricing formula for a European call option (Black and Scholes, 1973) with expiration date t and exercise price G. In addition to t and G it depends on S_0, δ and σ. The function $f_x(t) = \frac{l_{x+t}}{l_x}\mu_{x+t}$ is the probability density function of the insured's remaining life-time which is customarily assumed to only depend on x, the insured's age at the point of issue.

If this policy is paid by periodic premium rates $\bar{p}(t)$, the following equation describes the evolution of the value V of the policy through time

$$\frac{\partial V}{\partial t} = \delta V(t) + \bar{p}(t) - \mu_{x+t}(C(t) - V(t)) - \left\{\delta S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right\}, \tag{3}$$

where μ_{x+t} is the force of mortality. By letting $\mu_{x+t} = 0$ and $\bar{p}(t) = 0$, this equation is reduced to the differential equation which originally was the starting point for the option pricing formula. It may also be interpreted as the similar equation for an option where the owner of the option receives a dividend. In that case the terms involving $\bar{p}(t)$ and μ_{x+t} describe the dividend. In the actuarial science these types of equations are called Thiele's differential equations after the Danish actuary Thorvald N. Thiele.

These results will appear as special cases in the model presented in this paper.

Many financial, contingent claims have short contract periods, e.g., options usually expire within a few months of the date of issue. Life insurance contracts on the other hand, typically have long contract periods. Contracts lasting 40 or more years are not unusual.

In the previous example and in many applications of contingent claim theory, the price of the risky security is assumed to follow a geometric Brownian motion. We propose a more general price-model, a geometric Gaussian process, for describing the value of the risky security where the geometric Brownian motion is a special case. By other specifications of the parameters of the geometric Gaussian process, we obtain price models having more appealing features compared to those of the geometric Brownian motion, especially when contingent claims with long contract periods are concerned.

In the next section we present some elements of the economic model. In Section 3 we demonstrate how to value a quite general life insurance contract. In Section 4 we deduce expressions for the prospective premium reserve and the corresponding generalization of Thiele´s differential equation. In Section 5 we give examples of the geometric Gaussian process and apply the results to different insurance contracts.

2. Economic model

In this section we will concentrate on the parts of the present model which differ from the general theory. For complete treatments of the arbitrage pricing theory or Markov-chains in life insurance we refer to the works mentioned in the introduction.

The geometric Brownian motion seems to have been accepted as a reasonable model of stock prices in the financial literature. It is easy to estimate the two parameters included from observations. Other price models are described in the literature, see, e.g., Merton (1971) or Aase (1988). Here we use the geometric Gaussian process for describing the value of the risky security. The mathematical complexity is quite similar to the geometric Brownian motion, but as opposed to this process, it includes three functions whose mathematical structure must be determined before any estimation of parameters can take place.

The following price system is exogeneously given:

$$B_t = \exp\{\delta t\}, \tag{4}$$

$$S_t = S_0 \exp\{R(t)\}, \tag{5}$$

where $R(t) = \int_{(0,t]} \gamma(s)ds + g(t) \int_{(0,t]} j(s)dW_s$.

Here $\gamma(t)$ is a integrable, adapted process, $g(t)$ and $j(t)$ are two non-random square integrable functions where $g(t)\cdot j(t) > 0 \;\forall\, t$ and $R(0) \equiv 0$.

Here we list some properties of R for $0 < \tau < t$:

$$E[R(t)|\mathcal{F}_0] = E\left[\int_{(0,t]} \gamma(s)ds\right], \tag{6}$$

$$Var[R(t)|\mathcal{F}_0] = g(t)^2 \int_{(0,t]} j(s)^2 ds,$$

$$E[R(t)|\mathcal{F}_\tau] = \frac{g(t)}{g(\tau)}R(\tau) - \left(1 - \frac{g(t)}{g(\tau)}\right)\int_{(0,\tau]}\gamma(s)ds + E\left[\int_{(\tau,t]}\gamma(s)ds)|\mathcal{F}_\tau\right], \tag{7}$$

and

$$Var[R(t)|\mathcal{F}_\tau] = g(t)^2\int_{(\tau,t]} j(s)^2ds.$$

From Itô´s lemma we also get the following representation of this price system,

$$dB_t = \delta B_t dt, \tag{8}$$

$$dS_t = \kappa(S_t,t)S_t dt + \upsilon(t)S_t dW_t, \tag{9}$$

where

$$\kappa(S_t,t) = \gamma(t) + \frac{g'(t)}{g(t)}\left\{\ln\frac{S_t}{S_0} - \int_{(0,t]}\gamma(s)ds\right\} + \frac{1}{2}\{g(t)j(t)\}^2,$$

and

$$\upsilon(t) = g(t)j(t).$$

For example, by letting $g(t) \equiv \sigma$, $j(t) \equiv 1$ and $\gamma(t) \equiv \eta - \frac{1}{2}\sigma^2$, Equation (9) is reduced to the geometric Brownian motion given in Equation (1).

By applying this price system we have the possibility to choose the functions g(t), j(t) and γ(t) and thereby obtain a flexibility not inherent in the simple geometric Brownian motion. It allows us to work with price models where the limiting distributions have finite expectations and variances and to model non-homogeneous variance.

This price system also has the required martingale representation property, yielding a complete market so that every claim, contingent on the traded uncontrollable factor denoted S_t, may be duplicated by a dynamic trading strategy of the available securities. The stochastic process S_t is defined on a given probability space $(\Omega, \mathcal{F}, P)$, where the sample space Ω is the set of all possible outcomes, P is a probability measure on which everyone agrees. Here $\mathcal{F}$ is a σ-algebra of measurable subsets of Ω, generated by the involved stochastic processes.

The filtration $\mathcal{F}_t^S$ is defined by $\mathcal{F}_t^S = \sigma\{S_s : s \in [0,t]\}$. At time T all the uncertainty is resolved. $\mathcal{F}_0^S$ is almost trivial, $\mathcal{F}^S = \mathcal{F}_T^S$ and $\{\mathcal{F}_t^S : 0 \leq t \leq T\}$ satisfy "the usual conditions" (increasing, right-continuous and augmented).

There is an insurance policy and a set $\mathcal{J} = \{0, \ldots, J\}$ of possible states of the policy. The policy is issued at time 0 and expires within a given time horizon T. At any time $t \in [0,T]$ it is in one unique state in $\mathcal{J}$, commencing in state 0. In some situations more than one payment may occur at the same time. In such situations we consider the net cashflow. The benefits can be of two kinds: Net transitions benefits $\mathring{a}_{gh}(S_t,t)$ which are payable upon transitions from state g to state h where $g, h \in \mathcal{J}$. Net annuities $\mathring{a}_g(S_t,t)$ are paid to the insured as long as the policy is staying in state $g \in \mathcal{J}$. Premiums from the customer to the insurance company may be considered as negative annuities. For transitions or sojourns which do not entitle the policyholder to benefits $\mathring{a}(S_t,t) \equiv 0$. Note that the benefits are in general functions of S_t.

In addition, the benefits of the insurance contract may include a guarantee.

Some states are absorbing or external, meaning that once entered they are never left.

Endowment insurances are not treated explicitly. The benefit of an endowment insurance expires if the policy is in a given state or set of states at a given point in time. Therefore, an endowment insurance may be considered as a special case of an annuity.

The evolvement of the policy through time is modeled by a continuous time non-homogeneous Markov chain $X(t)$, $t \geq 0$ on the state space $\mathcal{J}$, defined on the same probability space. $X(t)$ generates a filtration $\mathcal{F}_t^X = \sigma(X(s) : s \in [0,t])$, the σ-algebra of all information provided by the process $X(t)$. Let $\mathcal{F}^X = \mathcal{F}_T^X$ and $\mathcal{F} = \mathcal{F}^X \vee \mathcal{F}^S$. The Markov property means that for a fixed present state of the process $X(t)$ its future and past are conditionally independent.

We assume that the processes S_t and $X(t)$ are statistically independent under P (this assumption may seem reasonable, but it is violated, for example, in situations where the insured´s death is caused by, say heart attach, due to rapid changes at the stock market). The generic element ω of Ω contains at least two pieces of information; one describing an outcome of $X(T)$, another describing an outcome of S_T.

Furthermore, the insurer is assumed risk-neutral with respect to a policy´s transitions between states during the insurance period. This is an extension of the classical assumption saying that the insurer is risk-neutral with respect to mortality risk, usually being justified by arguing that the company is holding a great number of identical contracts and referring to the law of large numbers. Risk-neutralism with respect to mortality is discussed in Aase (1993). However, by increasing the number of states or by other means specializing the contracts according to the customer's particular needs, the argument for using the strong law of large numbers is somewhat weakened. Since we are concerned with valuation, we must be explicit on this point.

We assume that the insurers are risk neutral with respect to financial risk. We would like to point this out because the existing financial risk explains why we can not immediately use the traditional principle of equivalence to calculate premiums. If we, incorrectly, applied the equivalence principle, the prices the insurer had to pay for the policy would be too high (see discussion in Persson, 1994a).

Let $p_{ij}(s,t)$ denote the transition probability

$$p_{ij}(s,t) = P\{X(t) = j \mid X(s) = i\}.$$

We impose the following regularity assumption

$$\lim_{t \downarrow s} p_{ij}(s,t) = I_{ij},$$

where I_{ij} is the Kronecker delta and equals 1 if $i = j$ and zero otherwise.

We assume that the following transition intensities exist. They are the basic entities in the system and are easily interpretable. They are functions of only one variable, in addition to certain characteristics of the person or persons being insured at the point of issue, for example age, sex, health condition etc.

The transition intensity is defined by

$$\mu_{ij}(t) = \lim_{u \downarrow t} \frac{p_{ij}(t,u)}{u-t}.$$

We also have the total transition intensity from state j

$$\mu_j(t) = \sum_{k \neq j} \mu_{jk}(t).$$

These intensities depend on t, the time elapsed since issue. They do not depend on other factors which may be relevant for practical considerations, e.g., how long or how many times a certain state has been visited. By carefully constructing the state space $\mathcal{J}$, some factors of that kind may be included, see Hoem (1968) for details.

If $j \in \mathcal{J}$ is external, $\mu_{jk}(t)$ for all t and $k \in \mathcal{J}$ equals zero.

From the Chapman-Kolmogorov equations

$$p_{ij}(s,t) = \sum_{g \in J} p_{ig}(s,\tau)p_{gj}(\tau,t),$$

for $s \leq \tau \leq t$, we may deduce Kolmogorov´s forward differential equations

$$\frac{\partial p_{ij}(s,t)}{\partial t} = \sum_{g \neq j} p_{ig}(s,t)\mu_{gj}(t) - p_{ij}(s,t)\mu_{j}(t) \tag{10}$$

and Kolmogorov´s backward differential equations

$$\frac{\partial p_{jk}(t,u)}{\partial t} = p_{jk}(t,u)\mu_{j}(t) - \sum_{g \neq j} \mu_{jg}(t)p_{gk}(t,u). \tag{11}$$

3. Valuation

The valuation process consists of two steps. First we use results from the contingent claim analysis to value each benefit. This valuation is consistent with the risk aversion present in the market. Then we utilize the Markov-chain set-up (and the risk-neutrality assumption with respect to transitions) to value the final insurance treaty.

Let N_{gh} be the function counting transitions from state g to h, that is $N_{gh}(t) = \#\{\tau \in [0,t]: X(\tau^-) = g, X(\tau) = h\}$. The random stream of net payments A may be written:

$$A(t) = \int_{(0,t]} \sum_{j \in J} \left\{ 1_{\{X(\tau)=j\}} a_j^{\circ}(S_\tau,\tau)d\tau + \sum_{k \neq j} a_{jk}^{\circ}(S_\tau,\tau)dN_{jk}(\tau) \right\}. \tag{12}$$

Here $1_{\{X(t)=j\}}$ is the indicator function which takes the value 1 if $X(t) = j$ and zero otherwise and $A(t)$ is simply the undiscounted sum of the net payments from time zero to time t.

The existence of a complete market implies the existence of a probability measure Q equivalent to P so that the market prices, denoted by π's, may be found as (dropping the subscripts on π and a° which only describe under what conditions the benefits are being paid)

$$\pi^t(S_0) = e^{-\delta t} E^Q[a^{\circ}(S_t,t)], \tag{13}$$

where $E^Q[\ ]$ is the expectation with respect to Q.

Here $\pi^t(S_0)$ denotes the market price at time zero of the benefit $a^{\circ}(S_t,t)$ payable at time t, which is called the expiration date of the benefit. If a guarantee is included, the benefit itself can be considered as a contingent claim with respect to S_t. This process of finding market values takes both the uncertain value of $a^{\circ}(S_t,t)$ and the time dimension into account as well as the attitude towards financial risk in the market. From Equation (13) we see that the discounting is carried out by the risk free rate δ, and that the original probability measure P is replaced by Q, the risk adjusted probability measure.

Equation (13) may equivalently be stated as

$$\pi^t(S_0) = e^{-\delta t} E[a^{\circ}(S_t,t)\xi_t].$$

Here the expectation is taken with respect to the original probability measure P and

$$\xi_t = \exp\left\{ - \int_{(0,t]} \frac{\kappa(S_s,s) - \delta}{\upsilon(s)} dW - \frac{1}{2} \int_{(0,t]} \left(\frac{\kappa(S_s,s) - \delta}{\upsilon(s)} \right)^2 ds \right\}$$

and $\frac{dQ}{dP} = \xi_T$. We may interpret $\frac{dQ}{dP}$ as the shadow price of risk per unit P probability.

The market value at time zero of the benefits included in the insurance contract are determined by Equation (13). To obtain the market value of the complete insurance treaty the probabilities for the different benefits to expire should also be taken into account. By the market's risk neutrality with respect to the policy's transition between the states, the independence between X(T) and S_T and the given initial state 0, we obtain the following general valuation formula

$$\Pi(S_0) = \int_{(0,T]} \sum_{j \in J} p_{0j}(0,\tau) \left\{ \pi_j^\tau (S_0) + \sum_{k \neq j} \mu_{jk}(\tau) \pi_{jk}^\tau (S_0) \right\} d\tau. \tag{14}$$

In accordance with actuarial terminology we may say that Equation (14) is derived from an application of the equivalence principle, but under a risk-adjusted probability measure. Here Π is the market value of the policy at time zero and represents the amount the insured has to pay to the insurer at time zero. It is tempting to interpret Π as the single premium of the contract, but this term is reserved for the case when the policy is paid fully at time zero. In general, $\mathring{a}_g$, $g \in \mathcal{K}$ for some $\mathcal{K} \subseteq \mathcal{J}$, with corresponding market price π_g, may be the periodic premium rate.

The probabilities p_{0j} in Equation (14) must in general be deduced from the Kolmogorov differential equations and can only in special cases be replaced by explicit formulas.

In the example given in the introduction based on the price system (1), the policy's state space consists of two states; State 0 {policyholder alive} and State 1 {policyholder dead}. Furthermore, all possible benefits equal zero except $\mathring{a}_{01}(t) = C(t)$, with a market value at time zero of $h(S_0,t) + e^{-\delta t}G$. We also have that $p_{00}(0,t) = \frac{l_{x+t}}{l_x}$ and $\mu_{01}(t) = \mu_{x+t}$ so that Equation (14) is reduced to Equation (2).

4. Premium reserves

The premium reserve or the cash value at time t represents the value of the policy at time t. For traditional life insurance products it can also be interpreted as the insurer's debt to the insured at time t or as the necessary funds the insurer should reserve at time t for future net claims. Another characteristic feature of life insurance policies compared to other contingent, financial claims is that several payments may take place between the issuer and the buyer before expiration.

The value at time $t \in [0,T]$ of a contingent claim is usually determined as the risk-adjusted net present value of the future cash flows. In life insurance this is called the prospective premium reserve. In addition, there is a retrospective premium reserve obtained, at time t, by considering the cash flows from time zero to t.

At any $t \in [0,T]$ a policy's complete payment stream A(T) given by (12) splits into payments after time t and payments up to and including time t,

$$A(T) = \{A(T) - A(t)\} - \{-A(t)\}. \tag{15}$$

At time t, the terms included in the first bracket on the right hand side may be interpreted as the future net expenditures for the insurer. The term in the second bracket may similarly be interpreted as the past net income (due to the minus sign).

4.1 Prospective premium reserves

The market value of the first bracket in (15) is identical with the prospective premium reserve at time t which is defined as the conditional expected present value of future benefits less premiums on the policy given its present state. It is derived by the same arguments as we used to arrive at Π.

We have the following expression, analogous to Equation (12), for the stream of payments from time t to time T:

$$A(T) - A(t) = \int_{(t,T]} \sum_{j \in J} \left\{ 1_{\{X(t)=j\}} \mathring{a}_j(S_\tau, \tau) d\tau + \sum_{k \neq j} \mathring{a}_{jk}(S_\tau, \tau) dN_{jk}(\tau) \right\}. \tag{16}$$

In order to determine the market values of the future benefits at time t, we have for $t \leq \tau \leq T$,

$$\pi^{\tau}(S_t,t) = e^{-\delta(\tau - t)}E^Q[a^{\circ}(S_{\tau},\tau) \mid \mathcal{F}_t]. \tag{17}$$

Notice that $\pi^t(S_t,t)$ equals $a(S_t,t)$.

Our next task is to obtain the market value of the stream (16) given the state of the policy at time t. Let $V_g^+(t)$ denote the prospective value given current state g at time t.

From Equation (16) and the valuation formula of Equation (17) it follows that

$$V_g^+(t) = \int_{(t,T]} \sum_{j \in J} p_{gj}(t,\tau)\left\{\pi_j^{\tau}(S_t,t) + \sum_{k \neq j} \mu_{jk}(\tau)\pi_{jk}^{\tau}(S_t,t)\right\}d\tau. \tag{18}$$

Here $V_g^+(t)$ is the market value at time t of the remaining benefits from the policy in the time interval (t,T] given current state g at time t.

4.2 A generalization of Thiele´s differential equation

We will now derive a differential equation describing the evolution of the value of the policy through time.

We consider V_g^+ given in Equation (18) as a function of the market prices of the benefits at time t, in addition to t, and denote it $V_g^+(\pi,t)$,

$$V_g^+(\pi,t) = \int_{(t,T]} \sum_{j \in J} p_{gj}(t,\tau)\left\{\pi_j^{\tau} + \sum_{k \neq j} \mu_{jk}(\tau)\pi_{jk}^{\tau}\right\}d\tau. \tag{19}$$

We then obtain the following equation for the differential of V_g^+,

$$dV_g^+ = \frac{\partial V(\pi,t)}{\partial t}dt + \int_{(t,T]} \sum_{j \in J} p_{gj}(t,\tau)\left\{d\pi_j^{\tau} + \sum_{k \neq j} \mu_{jk}(\tau)d\pi_{jk}^{\tau}\right\}d\tau. \tag{20}$$

The assumption about complete markets implies that the value of every benefit may be duplicated by a self-financing trading strategy. Furthermore, the market prices of the benefits at time t may also be duplicated by self-financing trading strategies, meaning that for all benefits included in the insurance contract, we have (again dropping the subscripts on $d\pi$, α and β)

$$d\pi = \alpha(t)dS_t + \beta(t)dB_t, \tag{21}$$

where $\alpha(t)$ and $\beta(t)$ are the number of stocks and bonds, respectively, held at time t in the strategy duplicating the market price at time t of the benefit.

We define

$$\alpha^+ = \int_{(t,T]} \left\{ \sum_{j \in J} p_{gj}(t,\tau) \left\{ \alpha_j^\tau(S_t,t) + \sum_{k \neq j} \mu_{jk}(\tau) \alpha_{jk}^\tau(S_t,t) \right\} \right\} d\tau \tag{22}$$

and

$$\beta^+ = \int_{(t,T]} \left\{ \sum_{j \in J} p_{gj}(t,\tau) \left\{ \beta_j^\tau(S_t,t) + \sum_{k \neq j} \mu_{jk}(\tau) \beta_{jk}^\tau(S_t,t) \right\} \right\} d\tau.$$

Note that $V_g^+(t) = \alpha^+ S_t + \beta^+ B_t$, so α^+ and β^+ may be interpreted as the trading strategies in the stock and the bond which duplicates the policy. As we should expect, α^+ and β^+ are functions of the conditional probabilities of future transitions given current state g, the duplicating strategies of the benefits and the remaining time of the insurance period at time t. The duplicating trading strategy is not necessarily self-financing due to inflows of premiums and outflows of benefits during the insurance period. The trading strategy is risk-minimizing in the following sense: Strategy (22) duplicates the benefit of the policy and by implementing the reverse strategy in the financial market, the insurer has eliminated the financial risk connected to the contract.

Now dV_g^+ may be expressed as

$$dV_g^+ = \frac{\partial V(\pi,t)}{\partial t} dt + \alpha^+ dS_t + \beta^+ dB_t. \tag{23}$$

It follows from Equation (19) that

$$\frac{\partial V_g^+(\pi,t)}{\partial t} = -\overset{\circ}{a}_g(S_t,t) - \sum_{h\neq g}\mu_{gh}(t)\{a_{gh}(t) + V_h^+(t) - V_g^+(t)\}. \tag{24}$$

Inserting Equation (24) into Equation (23) and by using the price system (8) and (9) and the identity $V_g^+(t) = \alpha^+S_t + \beta^+B_t$, we obtain:

$$dV_g^+ = \alpha^+\upsilon(t)S_tdW_t +$$

$$\left\{\delta V_g^+(t) - \overset{\circ}{a}_g(S_t,t) - \sum_{h\neq g}\mu_{gh}(t)[\overset{\circ}{a}_{gh}(S_t,t) + V_h^+(S_t,t) - V_g^+(S_t,t)] + [\kappa(S_t,t) - \delta]\alpha^+S_t\right\}dt. \tag{25}$$

By equating Equation (25) and the expression obtained by Itô's lemma applied to V_g^+ in Equation (18) considered as a function of S_t in addition to t, we get

$$\frac{\partial V_g^+}{\partial t} = \delta V_g^+(t) - \overset{\circ}{a}_g(S_t,t) -$$

$$\sum_{h\neq g}\mu_{gh}(t)\{\overset{\circ}{a}_{gh}(S_t,t) + V_h^+(t) - V_g^+(t)\} - \left\{\frac{\partial V}{\partial S}S_t\delta + \frac{1}{2}\upsilon(t)^2S_t^2\frac{\partial^2 V}{\partial S^2}\right\}. \tag{26}$$

This is the multistate generalization of Thiele's classical equation for this set-up and may be interpreted as is usual in the actuarial science.

The term $\delta V_g^+(t)$ represents the capital gain of the premium reserve in the time interval (t,t+dt). The term $\overset{\circ}{a}_g(S_t,t)$ represents the benefit to the insured in the time interval (t,t+dt).

The terms $\sum_{h\neq g}\mu_{gh}(t)\overset{\circ}{a}_{gh}(S_t,t)$ represent the expected benefit to be paid in the time interval (t,t + dt) upon transitions from state g. Payments connected to possible future transitions are included in the terms $\sum_{h\neq g}\mu_{gh}(t)\{V_h^+(t) - V_g^+(t)\}$ which represent the expected premium reserve in the state to which the policy arrives in the time interval (t, t+dt), in excess of what is covered by the premium reserve in state g.

The term $\left\{\frac{\partial V}{\partial S}S_t\delta + \frac{1}{2}\upsilon(t)^2S_t^2\frac{\partial^2 V}{\partial S^2}\right\}$ represents changes in the premium reserve caused by changes in the value of S_t in the time interval (t,t+dt) and is, contrary to the other terms, not present in Thiele's differential equation of traditional life insurance. It is not easy to give a meaningful interpretation "letter by letter" for this term as we can do with the other terms.

The terms $\sum_{h \neq g} \mu_{gh}(t)V_h^+(t)$ are added compared to the introductory example. Recall that the state space of that contract consists of only two states; State 0 {"policyholder is alive"} and State 1 ("policyholder is dead"), where State 1 is external so that $\mu_{10}(t) = 0 \ \forall \ t$. In addition no further payments occur if the policyholder is dead and the inheritors have received the benefit, so also $V_1(t) = 0$ and the term $\mu_{10}(t)V_1(t)$ equals zero. Furthermore, $V_0(t)$ is denoted $V(t)$, $\mathring{a}_{01}(t) = C(t)$, $\upsilon(t) = \sigma$ for the price system (1) and $\mathring{a}_0(t) = -\bar{p}(t)$. Equation (26) is then reduced to Equation (3).

In order to give a perhaps more intuitive interpretation of the change of the value of the premium reserve in the time interval (t,t+dt), we may consider the expression in Equation (25). Disregarding the terms $[\kappa(S_t,t) - \delta]\alpha^+S_t$ and $\left\{\frac{\partial V}{\partial S}S_t\delta + \frac{1}{2}\upsilon(t)^2S_t^2\frac{\partial^2 V}{\partial S^2}\right\}$ in Equation (26), all the terms in (26) are contained in the dt term of (25). The term $[\kappa(S_t,t) - \delta]\alpha^+S_t$ is readily interpretable as the additional (if $\kappa > \delta$) expected gain of the part of the premium reserve invested in the risky security and the dW_t term represents the stochastic variation of the same part. However, as opposed to the Equation (26), the expression in Equation (25) is stochastic.

While the duplicating trading strategy for an insurance contract is not necessarily self-financing, our approach demonstrates that the valuation formula of Equation (18) depends on the completeness of the economy. The duplicating trading strategies of the individual benefits, and consequently those of the market values of the benefits, included in the insurance treaty, must be self-financing.

4.3 A comment on retrospective premium reserves

In principle, the retrospective premium reserve is the market value of the second set of brackets of Equation (15), and expressions for this premium reserve are easily established for policies which expire immediately upon transition from the initial state. There seem to be several ways, although some appear incompatible, of defining expressions for more complex policies, see Hoem (1988) and Norberg (1991).

No matter what has happened in the past, it seems reasonable for the insurer to consider the future net claims when determining the appropriate level of the premium reserve. We do not make any attempt to deduce expressions for the retrospective premium reserve for our model.

5. Examples

In this section we show examples of possible price models, market values of different benefits and possible insurance contracts.

5.1 Examples of price processes

Consider the following Ornstein-Uhlenbeck process

$$dR_t = k\{\hat{\psi} - R_t\}dt + \sigma dW_t,$$

where the parameters $\hat{\psi}$, k and σ may be interpreted as the long range mean to which R_t tends to revert, the speed of adjustment and the volatility, respectively.

At time zero $E[R_t] = \hat{\psi}(1 - e^{-kt})$ and $Var[R_t] = \frac{\sigma^2}{2k}(1 - e^{-2kt})$ for $t > 0$. The variance increases with time, which may seem reasonable when modeling uncertain events. This is also the case for the Brownian motion, but contrary to this process, with probability one, $E[R_t] \to \hat{\psi}$ and $Var[R_t] \to \frac{\sigma^2}{2k}$, so R_t converges in distribution to a well-defined normally distributed random variable with expectation $\hat{\psi}$ and variance $\frac{\sigma^2}{2k}$. This property may be advantageous when long-lived securities are concerned.

From Equation (5) we get

$$dS_t = k\{\psi - \ln S_t\}S_t dt + \sigma S_t dW_t, \tag{27}$$

where $\psi = \hat{\psi} + \ln S_0 + \frac{1}{2k}\sigma^2$.

Here ψ, which is connected to the long range mean of the process, depends on the price of the security at time zero. The analyst may express his beliefs of the future price via the constant $\hat{\psi}$.

This price process follows immediately from Equation (7) by letting $\gamma(t) = k\hat{\psi}e^{-kt}$, $g(t) = \sigma e^{-kt}$ and $j(t) = e^{kt}$. We also see that $\upsilon(t) = g(t)j(t) = \sigma$. Notice that the volatility term in Equation (27) equals the volatility term in Equation (1), so formulas that only involve the volatility term will be the same for the geometric Brownian motion and the geometric Ornstein-Uhlenbeck process. The famous option pricing formula is one example of this.

Since the logarithm of S_t is a well-defined normally distributed random variable, it follows that

S_t is an equally well-behaved lognormally distributed random variable and has the same appealing features compared to the geometric Brownian motion as those mentioned above.

We obtain a price model having non-homogeneous variance by letting $g(t) \equiv \sigma$ and $j(t) = e^{\theta t}$, where θ is a constant. From Equation (9) we get

$$dS_t = \left\{\gamma(t) + \frac{1}{2}\sigma e^{\theta t}\right\} S_t dt + \sigma e^{\theta t} S_t dW_t. \tag{28}$$

The variation term exponentially increases with time. For this price model the limiting distribution does not have finite variance. The variance for the corresponding normally distributed variable from Equation (5) is $\mathrm{Var}\,[R_t] = \frac{\sigma^2}{2\theta}[e^{2\theta t} - 1]$, which does not converge. The limiting distribution may or may not have finite expectation depending on $\gamma(t)$. For example by letting $\gamma(t) = ae^{-bt}$, $E[R_t] = \frac{a}{b}[1 - e^{-bt}]$, which converges to $\frac{a}{b}$ as t gets large.

5.2 Examples of market values of benefits

Some examples of possible benefits and their corresponding market values given price system (27)/(1) and (28) are given in Table 1.

Table 1. Market value at time 0 for different benefits.

no	$a^{\circ}(S_t,t)$	$\pi(S_0)$ (dS by (27))	$\pi(S_0)$ (dS by (28))
1	K (a constant)	$e^{-\delta t}K$	$e^{-\delta t}K$
2	S_t	S_0	S_0
3	$\mathrm{Max}[S_t,G_t]$	$h(S_0,t) + e^{-\delta t}G_t$	$\hat{h}(S_0,t) + e^{-\delta t}G_t$

Here

$$\hat{h}(S_0,t) = S_0\Phi(d_1) - G_t e^{-\delta t}\Phi(d_2),$$

$$d_1 = \frac{\ln\frac{S_0}{G_t} + \delta t + \frac{1}{2}\rho_t^2}{\rho_t}, \qquad d_2 = \frac{\ln\frac{S_0}{G_t} + \delta t - \frac{1}{2}\rho_t^2}{\rho_t},$$

$\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, and

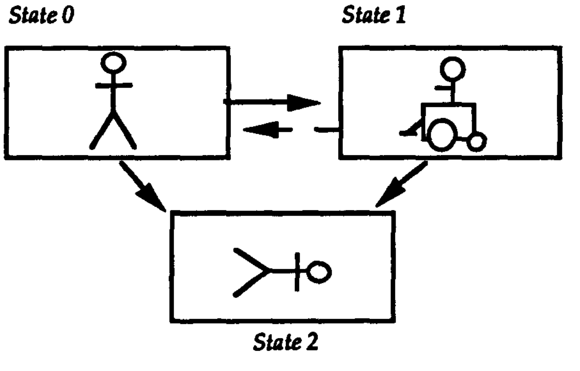
$$\rho_t = \int_{(0,t]} \upsilon(\tau)^2 d\tau = \frac{\sigma^2}{2\theta}(e^{2\theta t} - 1). \tag{29}$$

See for example Corollary 1 of Theorem 5 of Aase (1988). By letting $\upsilon(t) \equiv \sigma$, ρ_t equals $\sigma^2 t$ and $\hat{h}(S_0,t)$ equals $h(S_0,t)$, the standard European call option pricing formula.

5.3 Examples of insurance contracts

First we consider a term insurance paid by leveled premiums including a premium waiver if the policy-holder becomes disabled. The premium rate is $\bar{p}$ (a constant) when the customer is active and changes to $\lambda\bar{p}$, where $0 \leq \lambda < 1$, upon disability. The insurance benefit is $C(t) = \text{Max}[S_t,G_t]$. The policy's state space is described in Figure 1.

Figure 1. Premium waiver.



State 0. Policyholder active.
State 1. Policyholder disabled.
State 2. Policyholder dead.

State 2 is external. By disregarding the possibility of recovery, the state space becomes hierarchical, i.e., it is not possible to return to a state once it is left. We will here consider both cases.

Case 1. Recovery impossible.
Even though people recover in real life, this assumption is common in many actuarial models.

Here $\mathring{a}_0 = -\bar{p}$, $\mathring{a}_1 = -\lambda\bar{p}$, $\mathring{a}_3 = 0$ and $\mathring{a}_{01} = 0$, $\mathring{a}_{02} = \mathring{a}_{12} = C(t) = \max[S_t,G_t]$. The non-zero transition intensities are $\mu_{01}(t)$, $\mu_{02}(t)$ and $\mu_{12}(t)$ and are indicated by solid arrows in Figure 1.

From Equation (14) we obtain the following market value at time zero

$$\Pi(S_0) = \int_{(0,T]} \{\pi^{\tau}[p_{00}(0,\tau)\mu_{02}(\tau) + p_{01}(0,\tau)\mu_{12}(\tau)] - e^{-\delta\tau}\bar{p}[p_{00}(0,\tau) + \lambda p_{01}(0,\tau)]\}d\tau,$$

where $\pi^t = h(S_0,t) + e^{-\delta t}G_t$ for the price system (27) and $\pi^t = \hat{h}(S_0,t) + e^{-\delta t}G_t$ for the price system (28).

From the Kolmogorov forward equations it follows that

$$p_{00}(0,t) = \exp\left\{-\int_{(0,t]} \mu_0(s)ds\right\}$$

and

$$p_{01}(0,t) = \int_{(0,t]} \mu_{01}(s)\exp\left\{-\int_{(0,s]} \mu_0(s)ds\right\}\exp\left\{-\int_{(s,t]} \mu_1(s)ds\right\}ds.$$

When a policy has a hierarchical state space it is possible to get closed form solutions for the transition probabilities. This is not the case in general.

If this policy is being paid only by the periodic premium rates (no initial lump-sum payment), $\bar{p}$ is determined by equating $\Pi(S_0)$ to zero.

Here $V_2^+(t) = 0$ and Thiele's differential equations for State 0 and State 1 become

$$\frac{\partial V_0^+}{\partial t} = \{\delta + \mu_0(t)\}V_0^+(t) + \bar{p} - \mu_{01}(t)V_1^+(t) - \mu_{02}(t)C(t) - \left\{\frac{\partial V_0}{\partial S}S_t\delta + \frac{1}{2}\upsilon(t)^2S_t^2\frac{\partial^2 V_0}{\partial S^2}\right\}$$

and $$\frac{\partial V_1^+}{\partial t} = \{\delta + \mu_1(t)\}V_1^+(t) + \lambda\bar{p} - \mu_{12}(t)C(t) - \left\{\frac{\partial V_1}{\partial S}S_t\delta + \frac{1}{2}\upsilon(t)^2S_t^2\frac{\partial^2 V_1}{\partial S^2}\right\}.$$

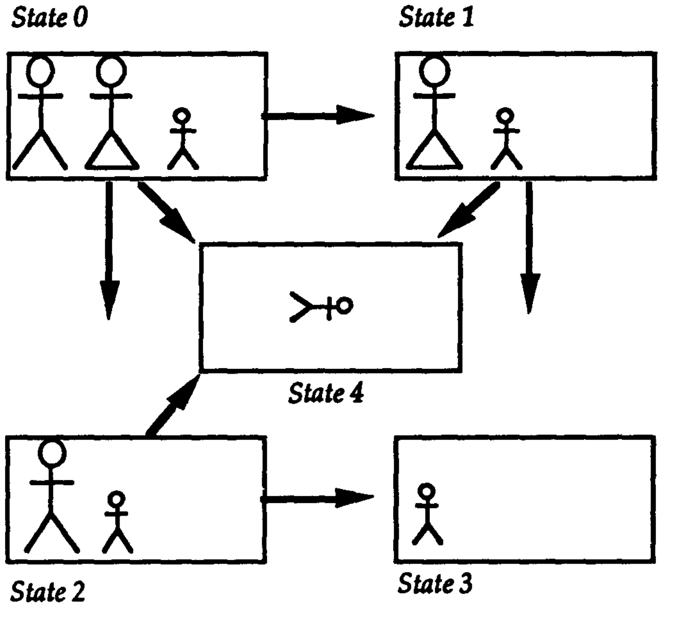
Case 2. Recovery possible.

In this case we allow for transitions between state 1 and 0, so $\mu_{10}(t) > 0$ (indicated by the dotted arrow in Figure 1). Therefore we do not have a hierarchical state space and do not get any closed form solution for $p_{00}(0,t)$ and $p_{01}(0,t)$ as we did in the previous case. Apart from that, the expression for $\Pi(S_0)$ is the same. There are no changes in Thiele's differential equation for State 0, but a new term is added for State 1,

$$\frac{\partial V_1^+}{\partial t} = \{\delta + \mu_1(t)\}V_1^+(t) + \lambda\bar{p} - \mu_{10}(t)V_0^+(t) - \left\{\frac{\partial V_1}{\partial S}S_t\delta + \frac{1}{2}\upsilon(t)^2 S_t^2 \frac{\partial^2 V_1}{\partial S^2}\right\}.$$

The next example is an orphan insurance where the child receives a benefit of Type 3 in Table 1 upon death of the last of the parents within a given time horizon T. The contract is paid by a premium rate $\bar{p}$ as long as at least one of the parents is alive. The premium rate is of the Type 1 from the table above. See Figure 2 for a description of this policy.

Figure 2. Orphan insurance.



State 0. Father, mother and child alive.
State 1. Mother and child alive.
State 2. Father and child alive.
State 3. Only child alive.
State 4. Child dead.

States 3 and 4 are external. The child becomes an orphan and the benefit expires upon transition into state 3. The policy moves into state 4 if the child dies, and no further premiums are to be paid. For this insurance $\dot{a}_0 = \dot{a}_1 = \dot{a}_2 = -\bar{p}$, $\dot{a}_3 = \dot{a}_4 = 0$, and $\dot{a}_{ij} = 0$ for all transitions except $\mathring{a}_{23} = \mathring{a}_{13} = C(t) = \text{Max}[S_t,G_t]$.

The non-zero transition intensities are $\mu_{01}(t)$, $\mu_{02}(t)$, $\mu_{04}(t)$, $\mu_{13}(t)$, $\mu_{14}(t)$, $\mu_{23}(t)$ and $\mu_{24}(t)$ and are given and indicated by arrows in the figure. Notice that for this insurance the transition probabilities and consequently the transition intensities depend on the state of several individuals.

From Equation (14) we obtain the following market value at time zero for this contract

$$\Pi(S_0) = \int_{(0,T]} \{\pi^{\tau}[p_{01}(0,\tau)\mu_{13}(\tau) + p_{02}(0,\tau)\mu_{23}(\tau)] - e^{-\delta\tau}\bar{p}[1 - p_{04}(0,\tau)]\}d\tau,$$

where $\pi^t = h(S_0,t) + e^{-\delta t}G_t$ for the price system (28) and $\pi^t = \hat{h}(S_0,t) + e^{-\delta t}G_t$ for the price system (29). Also for this policy the state space is hierarchical so the expressions for the probabilities will be on the same form as for Case 1 of the previous policy.

For this contract $V_3^+(t) = V_4^+(t) = 0$. Thiele's differential equations (23) for the other states become

$$\frac{\partial V_0^+}{\partial t} = \{\delta + \mu_0(t)\}V_0^+(t) + \bar{p} - \mu_{01}(t)V_1^+(t) - \mu_{02}(t)V_2^+(t) - \left\{\frac{\partial V_0}{\partial S}S_t\delta + \frac{1}{2}\upsilon(t)^2 S_t^2\frac{\partial^2 V_0}{\partial S^2}\right\}$$

and

$$\frac{\partial V_i^+}{\partial t} = \{\delta + \mu_i(t)\}V_i^+(t) + \bar{p} - \mu_{i3}(t)C(t) - \left\{\frac{\partial V_i}{\partial S}S_t\delta + \frac{1}{2}\upsilon(t)^2 S_t^2\frac{\partial^2 V_i}{\partial S^2}\right\} \quad, i = 1,2.$$

The benefit is not payable immediately upon transition from State 0, therefore C(t) is not included in the equation for this state. From State 1 or State 2 the policy can move only to external states. Thiele's equations for these states do not depend on the premium reserve in any other states. In contrast, for State 0 Thiele's equation depends on $V_1^+(t)$ and $V_2^+(t)$. All the expressions include a term on the form $\left\{\frac{\partial V}{\partial S}S_t\delta + \frac{1}{2}\upsilon(t)^2 S_t^2\frac{\partial^2 V}{\partial S^2}\right\}$.

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Chapter 4

INTEREST RATE RISK IN LIFE INSURANCE

We derive an economic valuation theory for life insurance contracts in a model with random interest rates. Here we deduce a partial differential equation for the market values of the assurances, which corresponds to the traditional Thiele equation of classical actuarial sciences, but contains some interesting new terms. By using various models of the term structure, we derive some new formulas for the market value of life insurance contracts.

The interpretation of the principle of equivalence may be revisited in this framework; the principle still holds but under a new risk adjusted probability measure, equivalent to - but different from - the originally given probability measure. This risk adjustment comes from the economics of uncertainty.

Key words: Financial Risk, Arbitrage Pricing Theory, Thiele's Differential Equation, Principle of Equivalence, Stochastic Interest Rate.

1. Introduction

1.1 Focus

In life insurance two major sources of uncertainty prevail, one related to the future development of the return on the financial investments, the other related to the future flow of payments, which again typically is connected to the development of the population of life insurance customers' health. We refer to the first source as financial risk and to the second source as transition risk. The term transition risk is more general and corresponds better to the model we use than the term mortality risk which is seen in other treatments.

In traditional actuarial models the rate of return on the financial investments is modeled by a constant or at most, a deterministic function of time. In this paper we allow for a stochastic development of the rate of return. We address the question whether the traditional principle of equivalence can be used as basis for valuation in this stochastic model. After introducing a simple model of a financial market, it turns out that the answer to this question is no. To obtain values for the insurance contracts which are consistent with the economic model we have to use a new pricing principle. Still the insurance premiums are found as expectations, but now under an equivalent probability measure which is constructed by the use of economic theory.

The equivalent martingale measure, also called a pricing measure, is the topic of Section 2. We take as primitives a theory for the pricing of certain financial assets and a theory for pricing mortality risk. We show how these theories can be combined to give consistent prices for life insurance contracts. To be more specific we assume there exist theories such that market prices are given as expectations under pricing measures. A pricing measure is formally a probability measure equivalent to the one given in the model, but does not necessarily represent probabilities of any future events in the model. Examples of such theories from financial economics are included. We assume that the financial market is independent of the state of the policy and in this situation we obtain a nice representation of the pricing measure, represented by the Radon-Nikodym derivative of the pricing measure with respect to the originally given probability measure. To simplify further, we assume risk neutrality with respect to transition risk. This assumption is implicit in the traditional principle of equivalence.

In Section 3 we derive expressions for the single premiums and premium reserves for some policies based on the independence between the state of the policy and the financial market. One notable difference between these formulas and the corresponding traditional ones is that the traditional discount function is replaced by an expression for the market value of a unit discount bond. The unit discount bond is a financial security which is traded in the financial market. The expression for its market value will generally depend on the term structure model being used.

A model of the term structure is described in Section 4.

In Section 5 we obtain the differential equations governing the evolution of the market value of the insurance contract. These equations also depend on the underlying term structure model and may be considered as a combination of the partial differential equations based on the no-arbitrage condition known from the theory of financial economics and Thiele's differential equations encountered in the actuarial sciences. Our equation differs from a corresponding equation derived by Norberg and Møller (1993) in a model without a financial market. An explanation for this is presented. We also give explicit examples of pricing formulas by specializing to term structure models known from the financial literature.

Section 6 contains some concluding remarks.

Differential equations similar to the one described above is also the topic of Persson (1994a) and Persson (1994b). These papers consider a special kind of life insurance called unit-linked or equity-linked contracts where the interest rate is assumed to be constant, i.e., deterministic.

2. Pricing in the presence of two independent sources of risk

2.1 Two sources of risk

Here we present one way to formalize the situation when two independent sources of risk are present. We have in mind a continuous time model with finite time horizon T. However, in this section the time dimension will not be given special attention. In the next section we introduce filtrations to incorporate the dynamic aspects of the model.

We assume there are two independent sources of uncertainty. In this paper one source represents the financial market and the other source the state of the insurance policy. One way to model this is by using two separate probability spaces and model each source of uncertainty on its own space. In this section we will focus on the use of a Radon-Nikodym derivative to construct a pricing measure, i.e., a probability measure which does not represent the agents' beliefs, but is constructed solely for the purpose of pricing and in addition happens to satisfy the formal requirements of a probability measure.

We take as primitives the probability spaces $(\Omega_1, \mathcal{G}, P_1)$ and $(\Omega_2, \mathcal{H}, P_2)$, the first used for modeling the financial market, the second for the state of the policy.

In several continuous time models from financial economics we have a result like this (Harrison and Kreps (1979)):

Subject to some technical conditions, no arbitrage opportunities implies the existence of an equivalent martingale measure Q_1 such that, after a change of numeraire, the price of a financial security may be found as an expectation under Q_1. We denote by ξ_1 the Radon-Nikodym derivative of Q_1 with respect to P_1.

The models we have chosen to work with will be presented later, but we now give two examples. Example 2.4 will be further explained and generalized later in the paper. First we define a quantity called *the money market account* or *the savings account* which represents the value at time t of one unit currency invested at time zero where interest is accrued according to the short term interest rate.

Definition 2.1

Let r_t denote the short term interest rate prevailing at time t, $t \in [0,T]$. Formally, r_t is a stochastic process defined on $(\Omega_1, \mathcal{G}, P_1)$. We define the money market account as

$$\beta_t = \exp\left(\int_0^t r_s ds\right). \tag{1}$$

This means that $d\beta_t = r_t \beta_t dt$, with the initial condition $\beta_0 = 1$.

Definition 2.2

A unit discount bond is a financial asset that entitles its owner to one unit currency at maturity without any intermediate coupon payments. There is a continuum of such bonds maturing at all times $s \in [0,T]$. We denote by $B_t(s)$ the market price at time t for a bond maturing at a fixed date $s \geq t$. From the definition $B_s(s) = 1$ (assuming no default risk).

Example 2.3 (Black and Scholes (1973))

This model we use for pricing derivative securities. The underlying risky security is a stock with price process

$$dS_t = \eta S_t dt + \sigma S_t dW_t,$$

where r, η and σ are positive constants and W_t a Brownian motion on $(\Omega_1, \mathcal{G}, P_1)$. In addition there is a constant short term interest rate so the money market account is given by $\beta_t = e^{rt}$.

The Radon-Nikodym derivative of Q_1 with respect to P_1 is

$$\xi_1 = \exp\left(-\frac{1}{2}\left(\frac{\eta - r}{\sigma}\right)^2 T - \frac{\eta - r}{\sigma} W_T\right).$$

Let C denote the payoff at time T of a derivative security with finite variance. We use β_T as numeraire and the market value at time zero of C is given as the expectation of $\frac{C}{\beta_T}$ under Q_1, i.e.,

$$\pi = E^{P_1}\left[\frac{C}{\beta_T}\xi_1\right] = E^{Q_1}\left[\frac{C}{\beta_T}\right].$$

Example 2.4 (Vacisek (1977))

A set of unit discount bonds maturing at all times $s \in [0,T]$ is given. The objective is to find expressions for the market values of the bonds. The short term interest rate is the only factor in addition to time which explains the price development of the bonds and is modeled by the stochastic differential equation

$$dr_t = q(m - r_t)dt + vdW_t,$$

where m, q and v are positive constants and can be interpreted as the long-range mean to which r_t tends to revert, the speed of adjustment and the volatility factor, respectively. Assuming no arbitrage possibilities, there exists a function called the market price of risk $\lambda(r_t,t)$. This function does not depend on time to maturity, its importance stems from the fact that the Radon-Nikodym derivative of Q_1 with respect to P_1 is expressed by $\lambda(r_t,t)$ as

$$\xi_1 = \exp\left(- \int_0^T \lambda(r_t,t)dW_t - \int_0^T \lambda(r_t,t)^2 dt \right),$$

assuming that $\lambda(r_t,t)$ is well-behaved so that the above expression exists and ξ_1 has unit expectation and finite variance. Also here we use the money market account as numeraire, where r_t is given by the above stochastic differential equation. Recall that $B_0(s)$ denotes the market value at time 0 of a unit discount bond expiring at time s. Then

$$B_0(s) = E^{Q_1}\left[\frac{1}{\beta_s} \right] = E^{Q_1}\left[\exp\left(- \int_0^s r_u du \right) \right]. \qquad \square$$

Assume there is some probability measure Q_2 such that the price of a policy may be found as the expectation under Q_2. Let ξ_2 denote the Radon-Nikodym derivative of Q_2 with respect to P_2.

Now we proceed to construct a product space based on the two described spaces. The objective is to characterize the Radon-Nikodym derivative of Q with respect to P, where P and Q are the product measures on the product space.

The cartesian product between Ω_1 and Ω_2 is defined as $\{(\omega_1,\omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$, i.e., the set of all ordered pairs from Ω_1 and Ω_2, and is denoted $\Omega_1 \times \Omega_2$. Any set on the form $A \times B = \{(\omega_1,\omega_2) : \omega_1 \in A \subset \Omega_1, \omega_2 \in B \subset \Omega_2\}$is called a rectangle. A rectangle $A \times B$ is measurable if $A \in \mathcal{G}$ and $B \in \mathcal{H}$. Define $\Omega = \Omega_1 \times \Omega_2$ and let $\mathcal{F}$ be the σ-algebra generated by the measurable rectangles. Now we consider the product space $(\Omega, \mathcal{F})$.

From, e.g., Theorem 18.2 in Billingsley (1986) it follows that the measure P on $\mathcal{F}$ defined by

$$P(E) = P_1(A)P_2(B), \tag{2}$$

for measurable $E = A \times B$, is well-defined and the unique probability measure with the property described in relation (2).

The same result also holds for the equivalent measures given by the previous lemmas such that Q defined by

$$Q(E) = Q_1(A)Q_2(B), \tag{3}$$

for a measurable rectangle $E = A \times B$, is well-defined and the unique probability measure with the property described in relation (3).

We now want to find an expression for the Radon-Nikodym derivative of Q with respect to P. We can then show the following proposition.

Proposition 2.5

The Radon-Nikodym derivative $\xi(\omega_1,\omega_2)$ such that

$$Q(E) = \int_E \xi(\omega_1,\omega_2)dP(\omega_1,\omega_2), \tag{4}$$

where $E = A \times B$ is a measurable rectangle,

is given by

$$\xi(\omega_1,\omega_2) = \xi_1(\omega_1)\xi_2(\omega_2),$$

for $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$.

Proof:

One way to calculate Q for a measurable set $E = A \times B$ is

$$Q(E) = \int_{\Omega_1} Q_2(B) 1_A(\omega_1) dQ_1(\omega_1),$$

for $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ (see, e.g., expression 18.1 in Billingsley (1986), see expression 18.2 for an alternative way). Observe that the relations (2) and (3) follow immediately from this expression.

By the definition of ξ_1 we can write this as

$$Q(E) = \int_{\Omega_1} Q_2(B) 1_A(\omega_1) \xi_1(\omega_1) dP_1(\omega_1),$$

for $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ and $E = A \times B$, a measurable rectangle.

Notice that $Q_2(B) = \int_{\Omega_2} \xi_2(\omega_2) 1_B(\omega_2) dP_2(\omega_2)$, so we can write

$$Q(E) = \int_{\Omega_1} \int_{\Omega_2} \xi_2(\omega_2) 1_B(\omega_2) dP_2(\omega_2) 1_A(\omega_1) \xi_1(\omega_1) dP_1(\omega_1).$$

By observing that $1_B(\omega_2) 1_A(\omega_1) = 1_E(\omega_1, \omega_2)$ and by the Fubini theorem, expression (4) follows. It is easy to verify that $\xi(\omega_1, \omega_2)$ is strictly positive P-almost surely and $\int_{\Omega} \xi(\omega_1, \omega_2) dP(\omega_1, \omega_2) = 1$. ❑

In the case of two independent sources of uncertainty the Radon-Nikodym derivative of the product space splits nicely into the product of the two Radon-Nikodym derivatives corresponding to the pricing rules from the given probability spaces.

2.2 The principle of equivalence under Q

In this paragraph we explain the principle we will use to price life insurance contracts in the remainder of the paper. First we impose an assumption:

Assumption 2.6
In this paper we assume risk neutrality with respect to transition risk. Then the measures Q_2 and P_2 are the same, hence $\xi_2 = 1$.

Risk neutrality with respect to the state of the policy is implicit in the traditional principle of equivalence in the actuarial sciences.

As a consequence of Proposition 2.5 and Assumption 2.6 it follows that the pricing measure Q is represented by

$$\xi(\omega_1,\omega_2) = \xi_1(\omega_1) 1_{\Omega_2}(\omega_2).$$

That is, the Radon-Nikodym derivative of Q with respect to P is given by the Radon-Nikodym derivative from the finance model times one. The market price of an insurance contract will accordingly be found as the expectation with respect to this measure. This pricing principle will be referred to as the principle of equivalence under Q.

3. The market value of a payment stream

3.1 The introduction of a financial market

In this paper we present an alternative approach to the use of the so called discount function encountered in actuarial works. We introduce a financial market and the discount function is replaced by market-based discounting using unit discount bonds. By using economic theory we derive expressions for the market value of the unit discount bonds. These expressions will depend on the chosen model of the financial market. There is a number of so called term structure models in the financial literature. In the next section we present an example which include many of the most popular models. In this section we demonstrate how our approach is based on the use of economic theory and hence differs from the classical discount function approach, whether the discount function is stochastic or not.

3.2 Insurance factors

To model the insurance contract we use the multi-state Markov model which seems to be standard in the actuarial sciences. See Hoem (1968), (1969) and (1988) and Norberg (1991) for details.

The state of the contract is assumed to evolve according to the right continuous stochastic process X_t defined on $(\Omega_2, \mathcal{H}, P_2)$ with left limits. Here X_t is a continuous time, inhomogenous Markov-chain with finite state space $\mathcal{J} = \{1,\ldots,J\}$. The transition probabilities are denoted by $p_{ij}(s,t) = P(X_t = j \mid X_s = i)$. The intensities $\mu_{ij}(s) = \lim_{t \downarrow s} \frac{p_{ij}(s,t)}{t-s}$, $i \neq j$, are assumed to exist for $i, j \in \mathcal{J}$.

To model the flow of information in the time period $[0,T]$ we use the filtration $\mathcal{H} = \{\mathcal{H}_t : t \in [0,T]\}$. Here we let $\mathcal{H}_t = \sigma(X_s: 0 \leq s \leq t)$ augmented by the sets of probability zero, so the process X_t is adapted to $\mathcal{H}$. Generally, a filtration is a right-continuous collection of increasing σ-algebras, i.e., $\mathcal{H}_s \subset \mathcal{H}_t \subset \mathcal{H}$ for $t \geq s$. In addition we let $\mathcal{H}_T = \mathcal{H}$.

We work with general deterministic insurance benefits. At any time $t < T$ the policy is in one of the states, commencing in state 0. There are two types of benefits, a general life insurance $a_{jk}(t)$ payable upon transition from state j to state k at time t and a general annuity rate $a_j(t)$ the insurer receives in state j at time t. Payments from the insured to the insurer, such as premiums paid during the term of the contract, are considered as negative benefits.

3.3 The financial market

The financial uncertainty is generated by a d-dimensional standard Brownian motion $W = \{W_t: t \in [0,T]\}$ on $(\Omega_1, \mathcal{G}, P_1)$. Let $\mathcal{G}_t$ be the filtration generated by W and the collection of P_1-null sets of Ω_1, i.e., an increasing and right-continuous filtration. We take $\mathcal{G}_T = \mathcal{G}$.

All trade is assumed to take place in a frictionless market (no transaction cost or taxes and short-sale allowed) with continuous trading opportunities.

The total information available is given by $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$, the smallest σ-algebra containing $\mathcal{G}_t$ and $\mathcal{H}_t$. By construction $\mathcal{G}_t$ and $\mathcal{H}_t$ are independent. We assume $\mathcal{F}_t$ is completed, i.e., contains all the sets of P-measure zero.

To fit the ideas presented here into the framework of the last section we assume that the market values of unit discount bonds can be represented in the following way.

Assumption 3.1

We assume that $B_t(s)$ for fixed $s > t$ is an Itô-process on $(\Omega_1, \mathcal{G}, P_1)$ and that there exists some probability measure Q_1 equivalent to P_1 so that

$$\frac{1}{\beta_t} B_t(s) = E_t^{Q_1}\left[\frac{1}{\beta_s}\right],$$

where β_t is given by (1), or

$$B_t(s) = E_t^{Q_1}\left[\exp\left(-\int_t^s r_u du\right)\right], \tag{5}$$

where $E_t^{Q_1}[\cdot]$ denotes the expectation conditional upon $\mathcal{G}_t$. This is a fairly general description, since most term structure models allow this representation. One example will be given in the following section. The above relations may be interpreted as follows: After both the market value at time t and the payoff at time s are divided by the numeraire, β_t and β_s, respectively, the market value at time t is equal to the conditional expectation under Q_1 of the payoff.

As is common in the financial literature, we refer to $B_t(s)$, as a function of s, as the term structure of interest rate at time t. In this section we will frame our results in terms of $B_t(s)$ and postpone a discussion of term structure models to the following section.

3.4 Pricing principles

The random payment stream in the period [0,T] of this general insurance policy can be described by

$$A_T = \int_0^T \sum_{j \in \mathcal{J}} \left\{ 1_{\{X_s = j\}} a_j(s)ds + \sum_{k \neq j} a_{jk}(s)dN_{jk}(s) \right\},$$

where $1_{\{X_t = j\}}$ is the indicator function taking the value 1 if $X_t = j$ and zero otherwise and $N_{jk}(t)$ counts the number of transitions from state j to state k by time t.

We denote by V_0 the present value of the payments in the period [0,T] after discounting by the money market account. Then

$$V_0 = \int_0^T \frac{1}{\beta_s} \sum_{j \in \mathcal{J}} \left\{ 1_{\{X_s = j\}} a_j(s)ds + \sum_{k \neq j} a_{jk}(s)dN_{jk}(s) \right\},$$

Let π_0 denote the market value at time zero of the payments in the period [0,T]. From the principle of equivalence under Q from the last section we get that $\pi_0 = E^Q[V_0]$. This is different from the traditional principle of equivalence which states that the price of the policy is equal to the present value under the original probability measure, i.e., $E^P[V_0]$. By using Proposition 2.5,

$$\pi_0 = E^P[\xi_1 V_0],$$

where ξ_1 denotes the Radon-Nikodym derivative of Q_1 with respect to P_1, implied by the representation in expression (5). By using the independence of the financial market and the state of the policy and the expression (5) we get that

$$\pi_0 = \int_0^T \sum_{j \in \mathcal{J}} p_{0j}(0,s)B_0(s) \left\{ a_j(s) + \sum_{k \neq j} \mu_{jk}(s)a_{jk}(s) \right\} ds. \tag{6}$$

The above expression represents our valuation principle for insurance contracts in our model. One notable difference between this expression and the corresponding classical one is that the discount function is replaced by an expression for the market value of a unit discount bond.

By the same arguments it follows that the premium reserve at time t, given that the policy is in state g is

$$\pi_t^g = \int_t^T \sum_{j \in \mathcal{J}} p_{gj}(t,s) B_t(s) \left\{ a_j(s) + \sum_{k \neq j} \mu_{jk}(s) a_{jk}(s) \right\} ds \quad \text{a.s.} \tag{7}$$

Example 3.2

A) Pure endowment insurance

Let ${}_TR_x$ denote the single premium of a contract with benefit C_T if the insured is alive at time T and 0 if not. This contract is an important building block, for example in pension plans.

By the above principle it follows that

$${}_TR_x = B_0(T) C_T \, {}_Tp_x,$$

where ${}_Tp_x$ represents the probability for an x-year old insured to be alive at time T. In the deterministic case $B_0(t) = e^{-rt}$ and the above expression is reduced to the familiar classical formula for the pure endowment insurance contract.

B) Term insurance

By this contract the insured receives the benefit C_t upon death before time T. Let $R^1_{x:\overline{T}|}$ denote the single premium. By the same arguments as above we get that

$$R^1_{x:\overline{T}|} = \int_0^T B_0(t) C_t \, {}_tp_x \mu_{x+t} dt,$$

where ${}_tp_x\mu_{x+t}$ is common actuarial notation for the probability density function of an x-year old insurance customer's remaining life time. By assuming deterministic interest rate and constant benefit, also this expression is reduced to the classical formula for term insurance.

4. A term structure model

4.1 Term structure models

The following model is a d-factor model of the term structure, i.e., the value of a unit discount bond depends on d factors in addition to time. Examples of these factors are the short term interest rate, inflation and various long term interest rates. In this section we describe a model which is related to the models by Vacisek (1977), Richard (1978), Brennan and Schwartz (1979), Hull and White (1990).

4.2 State variables

We assume that the economy is described by n state variables of which one is the short term interest rate. The short term interest rate is given by the stochastic differential equation

$$r_t = r_0 + \int_0^t \eta(r_s,s)ds + \int_0^t \sigma(r_s,s)dW_s \text{ a.s,} \tag{8}$$

where r_0 is a constant to be interpreted as the short interest rate prevailing at time zero. The continuous function η and the $(d\times 1)$ vector σ, are assumed to satisfy technical conditions so that equation (8) is a well-defined Itô-process and a solution exists.

In addition to r_t there is an $(n-1)$-dimensional Itô-process $\hat{Z}$ of state variables. The n-dimensional vector of state variables is given by $Z = \begin{pmatrix} r \\ \hat{Z} \end{pmatrix}$ or

$$Z_t = Z_0 + \int_0^t \eta_Z ds + \int_0^t \sigma_Z dW_s \text{ a.s,} \tag{9}$$

where $\eta_Z = \eta_Z(Z_t,t)$, an n-dimensional vector and $\sigma_Z = \sigma_Z(Z_t,t)$ an $(n\times d)$ dimensional matrix. Z_0 is an n-dimensional vector of constants interpretable as the initial values of the state variables.

4.3 The Securities

In this paper the security market consists only of unit discount bonds. We assume that $B_t(s)$ is a sufficiently smooth function of Z_t and t for fixed s. From Itô's lemma it follows that $B_t(s)$ can be represented as

$$B_t(s) = B_0(s) + \int_0^t \eta_B(u,s)B_u(s)du + \int_0^t B_u(s)\sigma_B^T(u,s)dW_u \quad \text{a.s.,} \tag{10}$$

where
$$\eta_B(t,s) = \frac{1}{B_t(s)}\left[\frac{\partial B}{\partial Z}^T \eta_Z + \frac{\partial B}{\partial t} + \frac{1}{2}\mathrm{tr}\left[\sigma_Z \sigma_Z^T \frac{\partial^2 B}{\partial Z^2}\right]\right]$$

and
$$\sigma_B(t,s) = \frac{1}{B_t(s)}\left[\sigma_Z^T \frac{\partial B}{\partial Z}\right],$$

where $\frac{\partial B}{\partial Z}$ represents the $(n \times 1)$-vector of first order derivatives of the bond price with respect to the state variables and tr[A] denotes the trace, i.e., the sum of the diagonal elements, of the square matrix A. We impose the boundary condition $B_s(s) = 1$.

The following lemma describes the local no-arbitrage condition from the theory of financial economics.

Lemma 4.1
Let $\lambda_i(Z_t,t)$, $i = 1,\ldots, d$, be a function of current time and the state variables and λ the $(d \times 1)$ vector of such functions. No arbitrage opportunities implies the existence of a vector λ such that

$$\eta_B(t,s) - r_t = \lambda^T \sigma_B(t,s). \tag{11}$$

Remarks
Each function $\lambda_i(Z_t,t)$ is independent of the expiration date s and can be interpreted as the market price of risk related to the i'th source of uncertainty at time t. Observe that the market prices of risk are related to the d sources of uncertainty, i.e., the d Brownian motions and not to the n factors characterizing the economy. The arbitrage pricing theory used here does not give any insight into the mathematical structure of $\lambda_i(Z_t,t)$. The quantity on the left hand side in relation (11) is sometimes called the instantaneous excess expected return of the bond. The product on the right hand side may be interpreted as the market price of risk (which may be negative) connected to source of uncertainty i multiplied by the amount of risk related to source of uncertainty i, added up for all sources, so relation (11) relates the instantaneous excess expected return to the market value of the risk associated with a bond with given maturity. The functions $\lambda_i(Z_t,t)$ sometimes occur with the opposite sign in other treatments.

Proof:

We refer to, e.g., Vasicek (1977) and Richard (1978) for an arbitrage setting and Cox, Ingersoll, Ross (1985) for a general equilibrium formulation for a proof. Here is a proof in our model:

We form a portfolio of (d + 1) bonds with distinct maturities. This portfolio can be written

$$dP = Px^T\eta dt + Px^T\sigma dW, \tag{12}$$

where x is a $(d+1) \times 1$ vector of portfolio weights, η is a $(d+1) \times 1$ vector of drift processes of the (d + 1) bonds and σ is a $(d+1) \times d$ matrix of diffusion processes representing d diffusion coefficients for the (d + 1) bonds. Now, find a portfolio $\hat{x}$ so that

$$\hat{x}^T\mathbf{1} = 1$$

and

$$\hat{x}^T\sigma = \mathbf{0},$$

where $\mathbf{1}$ is a $(d+1) \times 1$ vector of the real number 1 and $\mathbf{0}$ is a $1{\times}d$ vector of zeros.

We assume a solution to this problem exists, which imposes conditions on σ. The portfolio $\hat{x}$ has the property that the term involving dW in relation (12) vanishes so no risk is present. Hence the portfolio is (locally) risk-free and the instantaneous expected drift rate of this portfolio must equal the risk free rate, otherwise there is an arbitrage opportunity, i.e.,

$$\hat{x}^T\eta = r.$$

Consider now the following linear programming problem:

$$\min_{\hat{x}} \eta^T\cdot\hat{x}$$

subject to
$$\begin{pmatrix} \mathbf{1}^T \\ \sigma^T \end{pmatrix}\hat{x} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \tag{P}$$

and no sign restrictions on the elements of the vector $\hat{x}$. We recognize the constraints in this problem as the constraints described above. The object function is artificial in the sense that it does not have any economic interpretation, but is constructed solely for the purpose of studying the equivalent dual problem. First we observe that from the no-arbitrage condition, the value of the object function is r and a solution to the problem exists (by the previous assumption on σ). The dual problem is

$$\max_{\psi,\lambda} \psi$$

subject to $$\psi 1 + \sigma\lambda = \eta, \tag{D}$$

where ψ is the dual variable corresponding to the first constraint in (P) and λ is a $(d \times 1)$ vector of unconstrained dual variables corresponding to the d remainder constraints. By strong duality the value of the objective function of problem (P), which is identical to r, must take the same value as the objective function of problem (D), so $\psi = r$. By substituting for ψ in the constraints in the dual problem, expression (11) is obtained. ❑

Assumption 4.1
We assume that the vector λ does not depend on properties, such as the expiration dates, of the particular d+1 bonds in the portfolio constructed in the proof above.

In the case of a 1-dimensional Brownian motion this assumption will automatically be satisfied, but it is not apparent from the above proof that this also is the case for multi-dimensional Brownian motions.

Inserting the above expressions for η_B and σ_B into relation (11) leads to

$$\frac{\partial B}{\partial t} + \frac{\partial B}{\partial Z}^T(\eta_Z - \sigma_Z\lambda) + \frac{1}{2}\mathrm{tr}\left[\sigma_Z\sigma_Z^T\frac{\partial^2 B}{\partial Z^2}\right] - r_t B_t = 0. \tag{13}$$

This equation together with the boundary condition $B_s(s) = 1$, is a version of the Cauchy-problem.

Now we impose an assumption of technical nature.

Assumption 4.2
We assume that $E^{P_1}\left[\exp\left(\frac{1}{2}\int_0^T \lambda^T\lambda\right)\right] < \infty$. This condition is known as Novikov's condition.

Lemma 4.3

The solution of the problem described above has the following probabilistic representation

$$B_t(s) = E_t\left[\exp\left(-\int_t^s r_u du\right)\exp\left(-\int_t^s \lambda^T dW_u - \frac{1}{2}\int_t^s \lambda^T\lambda du\right)\right] \text{ a.s.,}$$

where $E_t[\cdot]$ denotes the conditional P_1-expectation with respect to $\mathcal{G}_t$.

Proof:

We define $A(u) = -\int_t^u \left(r_v + \frac{1}{2}\lambda^T\lambda\right)dv - \int_t^u \lambda^T dW_v$ and $Y(u) = B_u(s)e^{A(u)}$. A simple calculation shows that $Y(u)$, $u \in [t,s]$ is a martingale, so $E_t[Y(s)] = Y(t)$. The result follows by observing that $Y(t) = B_t(s)$ and that $Y(s) = \exp\left(-\int_t^s r_u du\right)\exp\left(-\int_t^s \lambda dW_u - \frac{1}{2}\int_t^s \lambda^T\lambda du\right)$.

❑

We now define $\xi_t = \exp\left(-\int_0^t \lambda^T dW_u - \frac{1}{2}\int_0^t \lambda^T\lambda du\right)$. Here ξ_t is a strictly positive random variable on $(\Omega_1, \mathcal{G}, P_1)$. Novikov's condition is sufficient to ensure that ξ_t has unit expectation. We then define the probability measure Q_1 by $Q_1(D) = E[1_D\xi_T]$ for $D \in \mathcal{G}$.

Lemma 4.4

The market price at time t for a bond maturing at time s, $0 \le t \le s \le T$, is given by

$$B_t(s) = E_t^{Q_1}\left[\exp\left(-\int_t^s r_u du\right)\right]. \tag{14}$$

Proof:

First let $\xi_{t,s} = \exp\left(-\int_t^s \lambda^T dW_u - \frac{1}{2}\int_t^s \lambda^T\lambda du\right)$ and observe that $E_t^{P_1}[\xi_{t,T}] = 1$ for all $t \le T$. The following result is standard:

$$E_t^{Q^1}\left[\exp\left(-\int_t^s r_u du\right)\right] = E_t^{P^1}\left[\exp\left(-\int_t^s r_u du\right)\xi_{0,T}\right]\frac{1}{E_t^{P^1}[\xi_{0,T}]}.$$

Now,

$$E_t^{P^1}[\xi_{0,T}] = \xi_{0,t}E_t^{P^1}[\xi_{t,T}] = \xi_{0,t},$$

and by the law of iterated expectations

$$E_t^{P^1}\left[\exp\left(-\int_t^s r_u du\right)\xi_{0,T}\right] = E_t^{P^1}\left[\exp\left(-\int_t^s r_u du\right)\xi_{0,s}E_s^{P^1}[\xi_{s,T}]\right] = \xi_{0,t}\left[\exp\left(-\int_t^s r_u du\right)\xi_{t,s}\right],$$

so the result now follows from Lemma 4.3. ❑

From Girsanov's theorem it follows that under Q

$$Z_t = Z_0 + \int_0^t [\eta_Z - \sigma_Z\lambda]du + \int_0^t \sigma_Z^T d\hat{W}_u \quad \text{a.s.} \tag{15}$$

and

$$B_t(s) = B_0(s) + \int_0^t r_u B_u(s)du + \int_0^t B_u(s)\sigma_B^T d\hat{W}_u \quad \text{a.s.}, \tag{16}$$

where $\hat{W}_s$ is a standard Brownian motion under Q. We note that under Q the drift process of the bond is $r_t B_t(s)$. The variation processes under Q are the same as under P. Observe that $\frac{B_t(s)}{\beta_t}$, where β_t is defined in (1) is a martingale under Q.

5. A partial differential equation for the market value of the insurance contract

5.1 Thiele's equation

The equations describing the evolution of the premium reserve through time are called Thiele's differential equations in the actuarial sciences. These kind of equations date back to 1875 and are also the topic of current work, such as Linneman (1993), Møller (1993), Norberg (1992), Norberg and Møller (1993) or Ramlau-Hansen (1991).

In this section we use the term structure model of the previous paragraph, we would expect that different term structure models would give rise to other differential equations.

We derive a partial differential equation describing the evolution of the market value of the policy. The idea is as follows: We know one representation of $B_t(s)$ under Q from (16). Then we derive another expression for $B_t(s)$ from the corresponding expression for the premium reserves given by the relation (7). By equating the expressions for $B_t(s)$ we derive one differential equation for the market value of the contract.

We now obtain the partial differential equation for the market value of the insurance contract as follows:

From (7) we may write $\pi_t^g = \int_t^T \pi^g(u)du$, where $\pi^g(u) = B_t(u)P_t^g$ and

$$P_t^g = \sum_j p_{gj}(t,u)\left\{a_j(u) + \sum_{k \neq j} \mu_{jk}(u)a_{jk}(u)\right\}.$$

We notice that

$$\int_t^T \frac{\partial \pi^g(u)}{\partial t}du = \frac{\partial \pi_t^g}{\partial t} + a_g(t) + \sum_{h \neq g} \mu_{gh}(t)a_{gh}(t).$$

For fixed u we calculate $B_t(u) = \frac{1}{P_t^g}\pi^g(u)$ and want to find an expression for dB_t under the probability measure Q. By using Kolmogorov's backward differential equation,

$$\frac{\partial p_{gj}(t,u)}{\partial t} = \sum_{h \neq g} \mu_{gh}(t)[p_{gj}(t,u) - p_{hj}(t,u)],$$

we get that

$$\frac{\partial B_t(u)}{\partial t} = \frac{1}{P_t^g}\left(\sum_{h \neq g} \mu_{gh}(t)[\pi^h(u) - \pi^g(u)] + \frac{\partial \pi^g(u)}{\partial t}\right).$$

Recall that $B_t(u)$ for fixed u is a function of Z, the vector of state variables, and t. From Itô's lemma it follows that the drift process for dB_t under Q is

$$\frac{1}{P_t^g}\left(\frac{\partial \pi^g(u)}{\partial Z}^T(\eta_Z - \sigma_Z\lambda) + \frac{1}{2}\mathrm{tr}\left[\sigma_Z\sigma_Z^T\frac{\partial^2 \pi^g(u)}{\partial Z^2}\right] + \sum_{h \neq g} \mu_{gh}(t)[\pi^h(u) - \pi^g(u)] + \frac{\partial \pi^g(u)}{\partial t}\right).$$

From expression (16) and since $B_t(u) = \frac{1}{P_t^g}\pi^g(u)$ we may also write this drift process as

$$\frac{1}{P_t^g}\pi^g(u)r_t.$$

By equating the above two expressions we get that

$$\frac{\partial \pi^g(u)}{\partial Z}^T(\eta_Z - \sigma_Z\lambda) + \frac{1}{2}\mathrm{tr}\left[\sigma_Z\sigma_Z^T\frac{\partial^2 \pi^g(u)}{\partial Z^2}\right] + \sum_{h \neq g} \mu_{gh}(t)[\pi^h(u) - \pi^g(u)] + \frac{\partial \pi^g(u)}{\partial t} - \pi^g(u)r_t = 0.$$

By integrating this expression with respect to u from t to T we get the following differential equation:

$$\frac{\partial \pi_t^g}{\partial t} = r_t\pi_t^g - a_g(t) - \sum_{h \neq g} \mu_{gh}(t)[a_{gh}(t) + \pi_t^h - \pi_t^g] - \left[\frac{\partial \pi_t^g}{\partial Z}^T(\eta_Z - \sigma_Z\lambda) + \frac{1}{2}\mathrm{tr}\left[\sigma_Z\sigma_Z^T\frac{\partial^2 \pi_t^g}{\partial Z^2}\right]\right]. \quad (17)$$

We may interpret equation (17) as the counterpart to Thiele's differential equation in the actuarial sciences. However, our equation (17) also deals with economic risk. This is not the case for the standard Thiele's equation. All the terms have reasonable and intuitive interpretations(see, e.g., Hoem, 1969) except the terms in the brackets on the right hand side. These terms stem from the economic theory of uncertainty. The above equation therefore incorporates financial risk also in the context of life insurance.

Møller and Norberg (1993) have derived an equation similar to our equation (17) in a model with stochastic interest rate. Their model did not include an economic model of a bond market and their equation did consequently not contain the term involving the market price of risk $\left(\frac{\partial \pi_t^g}{\partial Z}^T \sigma_Z \lambda\right)$. The major difference between the two models is the valuation principle being used. Møller and Norberg use the classical principle of equivalence, i.e., the single premium is calculated as expectations under the originally given probability measure P which implies risk neutrality with respect to financial risk. We argue that in the context of a financial market this principle must be replaced by the principle of equivalence under an equivalent probability measure Q which is constructed by imposing the no-arbitrage condition on the financial market. This means in particular that the principle of equivalence under Q involves equating market values of premiums to market values of benefits. Below we compare the two pricing principles in a single state variable model.

It is common in life insurance to split the periodic premium rate into a mortality risk premium rate and savings premium rate. For the multi-state contract we denote the savings premium rate $a_g^s(t)$ and the transition risk premium rate $a_g^r(t)$ given that the policy is in state g at time t. For this policy we get

$$a_g^r(t) = \sum_{h \neq g} \mu_{gh}(t)[a_{gh}(t) + \pi_t^h - \pi_t^g]$$

and

$$a_g^s(t) = \frac{\partial \pi_t^g}{\partial t} - r_t \pi_t^g + \left[\frac{\partial \pi_t^g}{\partial Z}^T (\eta_Z - \sigma_Z \lambda) + \frac{1}{2} \mathrm{tr}\left[\sigma_Z \sigma_Z^T \frac{\partial^2 \pi_t^g}{\partial Z^2}\right]\right].$$

From (17) it follows that $-a_g(t) = a_g^r(t) + a_g^s(t)$, i.e., net income equals the sum of the savings and the transition risk premium. The new terms are related to the savings premiums and can be traced back to the stochastic financial environment. Both financial risk and risk connected to the state of the policy are present in this model. We therefore find it natural to refer to what is usually called only risk premium as transition risk premium.

5.2 Examples

A) Vasicek (1977)-model

In this case n = d = 1 and the only state variable is the short-term interest rate.

First we specialize (8) by choosing $\eta(r_t,t) = q(m - r_t)$ and $\sigma(r_t,t) = v$, where q, m and v are positive constants.

This Ornstein-Uhlenbeck process is described in Example 2.4 and the following short-hand notation is common

$$dr_t = q(m - r_t)dt + vdW_t. \tag{18}$$

This model is stated in nominal terms so one disadvantage by using this process is that negative values of r_t are possible, a fact which implies arbitrage opportunities in the bond market.

Now we assume that the market price of risk $\lambda(r_t,t) = \lambda$, a constant.

It follows that

$$B_0(t) = E_0^Q\left[\exp\left(-\int_0^t r_s ds\right)\right] = G_t\, e^{-H_t \cdot r_0}, \tag{19}$$

where $H_t = \frac{1-e^{-qt}}{q}$ and $G_t = \exp\left(\left(m - \frac{\lambda v}{q} - \frac{1}{2}\left(\frac{v}{q}\right)^2\right)(H_t - t) - \frac{1}{q}\left(\frac{vH_t}{2}\right)^2\right)$.

This result is from Vasicek (1977) (in his paper the market price of risk is defined with the opposite sign). Observe that the formula depends on the market price of risk.

The fundamental differential equation of the market value of a term insurance now becomes:

$$\frac{\partial\pi}{\partial t} = (\mu_{x+t} + r_t)\pi_t + \bar{p}_t - C_t\mu_{x+t} - \left[\frac{\partial\pi}{\partial r}(q(m-r) - v\lambda) + \frac{1}{2}v^2\frac{\partial^2\pi}{\partial r^2}\right].$$

The single premiums of a pure endowment insurance and a term insurance follows from expression (19) and Example 3.2:

$${}_TR_x = {}_Tp_x \cdot G_T\, e^{-H_T \cdot r_0}$$

and

$$R^{1}_{x:\overline{T}|} = \int_0^T G_s e^{-H_s r_0} {}_s p_x \mu_{x+s} ds. \qquad (20)$$

B) A comparison of our model with Norberg and Møller's model in the Vacisek-setting

In Norberg and Møller's model the short interest rate is the only state variable and is given by (18). They also assume that the financial market is independent of the state of the policy, but calculate the premium as expectations under the originally given probability measure P. In particular this implies that the valuation equation corresponding to our equation (6) will be

$$\pi_0 = \int_0^T \sum_{j \in \mathcal{J}} p_{0j}(0,s) Z_0(s) \left\{ a_j(s) + \sum_{k \neq j} \mu_{jk}(s) a_{jk}(s) \right\} ds,$$

where

$$Z_0(t) = G_t^o e^{-H_t \cdot r_0},$$

$G_t^o = \exp\left(\left(m - \frac{1}{2}\left(\frac{v}{q}\right)^2\right)(H_t - t) - \frac{1}{q}\left(\frac{vH_t}{2}\right)^2\right)$, and H_t is as previously defined.

We observe that $Z_0(t)$ has the similar role in Norberg and Møller's model as $B_0(t)$ in our and would now like to compare the two pricing principles. From expression (19) we can write

$$B_0(t) = Z_0(t)\exp\left(-\frac{\lambda v}{q}(H_t - t)\right).$$

Since $1 - x \leq \exp(-x)$ for $x \geq 0$, it follows that $(H_t - t) \leq 0$ for $q \geq 0$ and $t \geq 0$.

By the arguments in Hull (1989) the market price of risk, when the underlying state variable is an interest rate, is negative, so $\lambda < 0$. Then $\exp\left(-\frac{\lambda v}{q}(H_t - t)\right) \leq 1$, hence $B_0(t) \leq Z_0(t)$.

With this result in mind and by comparing our valuation principle given in expression (6) with the one above, it should be clear that our principle implies lower prices for insurance than the principle of equivalence. In a world where all our idealized assumptions of frictionless markets were satisfied all the customers would buy insurance from our company and still our company would not go bankrupt. The company of Norberg and Møller would then go out of business.

C) CIR-model

This model of the term structure was developed in an equilibrium setting by Cox, Ingersoll and Ross (1985) and in an arbitrage free economy by Richard (1978). The short-term interest rate is the only state variable and is given by

$$dr_t = q(d - r_t)dt + v\sqrt{r_t}\,d\hat{W}_t$$

under Q. Compared to the previous model this one has the advantage that the interest rate can not take negative values. The price at time zero of a unit discount bond maturing at time t is

$$B_0(t) = G_t^{CIR} e^{-H_t^{CIR} \cdot r_0},$$

where

$$G_t^{CIR} = \left[\frac{2\gamma e^{(q+\gamma)t/2}}{(\gamma+q)(e^{\gamma t} - 1) + 2\gamma} \right]^{\frac{2dq}{v^2}},$$

$$H_t^{CIR} = \frac{2(e^{\gamma t} - 1)}{(\gamma+q)(e^{\gamma t} - 1) + 2\gamma},$$

and

$$\gamma = (q^2 + 2v^2)^{\frac{1}{2}}.$$

The market prices for the two single life contracts will take the same form as in the Vasicek-model:

$$_T R_x = {}_T p_x \cdot G_T^{CIR} e^{-H_T^{CIR} \cdot r_0}$$

and

$$R^1_{x:\overline{T}|} = \int_0^T G_s^{CIR} e^{-H_s^{CIR} r_0} {}_s p_x \mu_{x+s} ds.$$

6. Concluding remarks

We have derived pricing formulas for a general life insurance contract in a model with random interest rates based on the assumption that the state of the policy is independent of the financial markets and that no arbitrage opportunities exists in the financial market. The market price was found as an expectation under a different probability measure following from economic theory. Furthermore, our differential equation of the market value of the insurance contract in this model formally resemble the traditional Thiele's equation, but contains some new interesting terms dealing with economic risk. By specializing to term structure models we have also presented some new formulas for the market value of various life insurance contracts.

In a companion paper Persson (1994d) these results are generalized to a situation where also the amount of benefit is random and linked to the value of a financial asset. This situation is relevant for unit-linked insurance.

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Chapter 5

RANDOM BENEFITS AND STOCHASTIC INTEREST RATES IN LIFE INSURANCE

Unit-linked or equity-linked contracts are examples of life insurance contracts where the amount of benefit is contingent on the market value of some financial asset. Using contingent claims theory and traditional actuarial theory, we derive an economic valuation theory for such contracts in a model with random interest rates. We derive partial differential equations for the market values of the assurances, which may be considered as generalizations of both the traditional Thiele equation of classical actuarial sciences and well known differential equations from the theory of financial economics based on no arbitrage opportunities. Compared to the classical Thiele equation our equations contain some interesting new terms which depend on the choice of model of the term structure. Here we generalize the similar equation from Persson (1994c) to the case of a random benefit. We deduce similar equations based on the Heath, Jarrow and Morton (1992) term structure model both for deterministic and random benefits.

Key words: Life Insurance, Contingent Claims Analysis, Arbitrage Pricing Theory, Thiele's Differential Equation, Principle of Equivalence, Term Structure Models.

1. Introduction

1.1 Focus

In this paper we are concerned with pricing of life insurance contracts in the presence of interest rate risk. In particular we consider insurance policies where the amount of benefit is random and linked to a financial asset. The principle of equivalence, which traditionally has been the basis of the pricing of life insurance policies, neither deals with stochastic interest rates nor stochastic amounts of benefits.

In an economic model where risky investment opportunities are present and also the return on so-called riskfree investments is random, care must be taken regarding the valuation issue and in our model the traditional principle of equivalence cannot be applied. The philosophy behind this principle is that, abstracting from administrative expenses, a company's income (premiums), and expenses (paid benefits) should balance in the long run. Traditionally the discount factor used for the valuation purpose is interpreted as the company's return on its investments. In the described financial environment this return will depend on the chosen investment strategy, which again depends on the company's attitude towards financial risk. Here we adopt a conservative point of view, i.e., we assume that the company does not want to accept more financial risk than it is forced to. This corresponds to the common opinion that the insurance companies should not "play with other people's money", in most countries manifested by legislation restricting the insurance industry's investments possibilities. We do not address the important question whether an insurance company should accept more financial risk and, if yes, how much more. However, we should expect that companies which choose more risky strategies on average will get a higher yield on their investments and hence could offer cheaper insurance than the conservative companies. At the same time risky investment strategies increase the probability of bankruptcy of the insurance company.

Equity-linked or unit-linked insurance contracts link the amount of benefit to a financial asset, usually a mutual fund. For the insurance industry, one of the reasons for introducing these products is to take advantage of the higher yields in the financial markets and therefore offer products more competitive compared to alternative ways of saving. These products also offer the customers some flexibility in that they may choose more or less freely to which mutual fund to link the benefit.

We assume that the policy at each point in time is in one of a finite number of states and moves between the states according to a time inhomogenous Markov process. An important assumption we maintain throughout the paper is that the insurance company is risk neutral with respect to transition risk. The term transition risk is more general and corresponds better to the Markov-model than the term mortality risk which is seen in other treatments. This assumption is also implicit in the traditional principle of equivalence. Another important, though maybe more reasonable assumption is that the financial market is independent of the state of the policy.

In addition to the pricing issue of unit-linked contracts we develop a partial differential equation for the market value of the premium reserve which can be viewed as a generalization of the Thiele equation of the actuarial sciences. The traditional Thiele equation was first discovered by the Danish actuary Thorvald N. Thiele in 1875 and has been generalized and developed further in more recent works such as Hoem (1968), Hoem (1969), Møller (1993), Norberg (1992) and Norberg and Møller (1993) and Persson (1994a).

The current paper is basically an extension of a companion paper Persson (1994c) in two ways. First, we develop partial differential equations for the market value of the premium reserve based on the Heath, Jarrow and Morton (1992)-model (henceforth referred to as the HJM-model) of the term structure. We compare this equation with the corresponding equation from Persson (1994c) which is based on another term structure model. Second, we explain a pricing principle and develop the similar equations when the amounts of benefits are allowed to be random.

In Section 2 we describe two term structure models and investigate the consequences for a generalization of the traditional Thiele equation in Section 3. To focus on this difference we first assume only one dimensional uncertainty (i.e., the model is driven by a one-dimensional Brownian motion) and a deterministic benefit.

In the first bond pricing models (Vacisek (1977), Richard (1978), Brennan and Schwartz (1979c)) in financial economics, the bond price is assumed to depend on certain state variables of which one is the short interest rate. The HJM-model assumes that the development of the forward rates and the initial term structure are given. By assuming that no arbitrage opportunities exist, the drift term of the forward rate is restricted in a way so that pricing can be done from knowledge of only the volatility processes of the forward rate and the initial term structure.

In Section 4 we introduce risky assets, we have in mind the modern life insurance products mentioned above where the benefit is linked to the market value of some financial assets. Here it is natural to include multi-dimensional uncertainty modeled by a multi-dimensional Brownian motion. We develop pricing principles consistent with economic theory and, in Section 5, differential equations describing the evolution of the market value of the policies - similar to the ones described above. These equations are more complex than the ones developed in Section 2 because of the multi-dimensional uncertainty and the random benefits.

Examples are included in Section 6 which compare the pricing of a typical unit-linked benefit in the two term structure models. Here we also present some formulas for the unit-linked versions of some insurance contracts. Some concluding remarks are given in Section 7.

The mathematical presentation is not completely rigorous in that conditions of technical nature often are described very briefly, if mentioned at all. Technical conditions are required, e.g., on the drift and volatility processes of the Itô-processes (such as measurability, adaptedness, and integrability-conditions) to ensure that these are well-defined. Smoothness conditions are required to apply Itô's lemma, for example growth conditions are required to ensure that stochastic differential equations have solutions. However, we have tried to mention any condition of economic nature and to cite references where technical conditions can be looked up. Also all relations involving random variables are understood to hold almost surely.

2. Two term structure models

2.1 The financial market

This section presents a brief description of two different term structure models. In this section we limit the discussion to one factor models meaning that only one source of uncertainty is present. The generalization to several sources of uncertainty, which we will refer to as multi-factor models, is straight-forward and will be shown in Section 5.

A time horizon T is fixed and the financial uncertainty is generated by a 1-dimensional standard Brownian motion W defined on a probability space $(\Omega, \mathcal{F}, P)$. All trade is assumed to take place in a frictionless market (no transaction cost or taxes and short-sale allowed) with continuous trading opportunities.

Definition 2.1

A unit discount bond is a financial asset that entitles its owner to one unit currency at maturity without any intermediate coupon payments. There is a continuum of such bonds maturing at all times $s \in [0,T]$. We denote by $B_t(s)$ the market price at time t for a bond maturing at a fixed date $s \geq t$. From the definition $B_s(s) = 1$ (assuming no default risk).

2.2 The state variable model

Now we describe what we call the state variable model. In this model the bond price is assumed to only depend on one state variable, the interest rate, in addition to time.

The short-term interest rate is given by the Itô-process

$$r_t = r_0 + \int_0^t \eta_r(r_s,s)ds + \int_0^t \sigma_r(r_s,s)dW_s.$$

We refer to the functions η_r and σ_r as the as the drift process and the volatility process of the interest rate, respectively. These processes satisfy technical conditions so that there exists a solution to the stochastic differential equation above. The parametric form of these processes together with the constant r_0 are our primitives, so the dynamics governing the evolution of the underlying state variable (for the moment only one) is known.

For fixed $s \leq T$, we assume that $B_t(s)$ is a function of r_t and t. The functional relationship between $B_t(s)$ and r_t and t is at this step not known. By assuming that $B_t(s)$ is sufficiently smooth we obtain from Itô's lemma,

$$B_t(s) = B_0(s) + \int_0^t \eta_B(u,s)B_u(s)du + \int_0^t \sigma_B(u,s)B_u(s)dW_u. \tag{1}$$

Here $\eta_B(t,s) = \frac{1}{B_t(s)}\left[\frac{\partial B}{\partial r}\eta_r + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma_r^2 + \frac{\partial B}{\partial t}\right]$ and $\sigma_B(t,s) = \frac{1}{B_t(s)}\frac{\partial B}{\partial r}\sigma_r$. The value of the unit discount bond is 1 at maturity, so it is reasonable to expect that the volatility of the bond should decrease as time approaches maturity. This is not generally the case for the above price process.

From the theory of financial economics (see, e.g., Vacisek (1977)) we have that no arbitrage opportunities imply that there exists a function $\lambda(t)$ such that

$$\eta_B - r_t = \lambda(t)\sigma_B. \tag{2}$$

In the case of a single state variable the function $\lambda(t)$ is called the market price of risk of the interest rate, though it really is related to the underlying Brownian motion. A generalization of this result is presented in Section 4. The above relation requires that the bond price is an Itô-process on the form in expression (1), but does not require that the drift and the volatility processes are on the particular forms implied by the state variable model given in expression (1). However, by assuming the state variable model and substituting the expressions for η_B and σ_B in relation (2), we obtain the so-called fundamental partial differential equation governing the value of the bond.

Assuming that $\lambda(t)$ satisfies some technical conditions, we define

$$\xi_t = \exp\left(-\int_0^t \lambda(u)dW_u - \frac{1}{2}\int_0^t \lambda(u)^2 du\right) \tag{3}$$

and a probability measure Q by

$$Q(A) = \int_\Omega 1_A(\omega)\xi_T dP(\omega).$$

It can be shown that the market value at time t of a unit discount bond is

$$B_t(s) = E_t^Q[e^{-\int_t^s r_u du}]. \tag{4}$$

We see that the quantity $-\int_t^s r_u du$ serves a special role. If for example, this is normally distributed under Q, $\exp\left(-\int_t^s r_u du\right)$ is lognormally distributed and $B_t(s)$ follows from known properties of these distributions. This is the case, e.g., for the model by Vacisek (1977), where $B_t(s)$ can easily be calculated from the above formula (see Section 6) and where the resulting formula depends on the parameters of the drift and the volatility processes of the state variable, the expiration date (s) and the market price of risk ($\lambda(t)$). It also follows that

$$B_t(s) = B_0(s) + \int_0^t r_u B_u(s) du + \int_0^t \sigma_B(u,s) B_u(s) d\hat{W}_u \tag{5}$$

and

$$r_t = r_0 + \int_0^t [\eta_r(r_s,s) - \lambda(s)\sigma_r(r_s,s)] ds + \int_0^t \sigma_r(r_s,s) d\hat{W}_s \tag{6}$$

under Q, where $\hat{W}$ is a standard Brownian motion under Q. We observe that the drift process of r_t under Q depends on the market price of interest rate risk. To calculate the bond prices from formula (4) we need knowledge of $\lambda(t)$ to be able to use expression (6).

2.3 The HJM model

The starting point for the term structure model of Heath, Jarrow and Morton (1992) is the relationship between the instantaneous forward rates and the bond price,

$$B_t(s) = \exp\left(-\int_t^s f_t(u) du\right). \tag{7}$$

Here $f_t(u)$ is the instantaneous forward rate for time u prevailing at time t. The forward rates are modeled by Itô-processes on the form,

$$f_t(u) = f_0(u) + \int_0^t \alpha_v(u)dv + \int_0^t \sigma_v(u)dW_v, \qquad (8)$$

$u \in [0,T]$. Here $f_0(u)$ is given for all $u \leq T$. The function α_v is called the drift process and the function σ_v is called the volatility process of the instantaneous forward rate, respectively. Both these processes are assumed to satisfy technical conditions so that expression (8) is a well-defined Itô-process. From (7) we see that also $B_0(s)$ for $s \in [0,T]$ is given, so the whole initial term structure is taken as a primitive. We will refer to the collection $\{\sigma_v(u):u \in [0,T]\}$ as the volatility structure. In particular, the short interest is given by

$$r_t = f_t(t).$$

From expression (7) it follows that

$$B_t(s) = e^{X_t(s)},$$

where

$$X_t(s) = \ln(B_0(s)) + \int_0^t [r_v + \upsilon(v,s)]dv + \int_0^t a(v,s)dW_v, \qquad (9)$$

$$\upsilon(v,s) = -\int_v^s \alpha_v(u)du,$$

and

$$a(v,s) = -\int_v^s \sigma_v(u)du.$$

The process $X_t(s)$, represents minus the integrals of the forward rates from time t to time s. We assume that this process is a well-defined Itô-process which imposes some additional constraints on the drift and the volatility processes of the forward rates (see, e.g., Heath, Jarrow and Morton (1992)).

It follows from Itô's lemma that

$$B_t(s) = B_0(s) + \int_0^t [r_v + b(v,s)]B_v(s)dv + \int_0^t a(v,s)B_v(s)dW_v, \qquad (10)$$

where $$b(v,s) = \upsilon(t,s) + \frac{1}{2}[a(t,s)]^2.$$

This stochastic differential equation does not involve any first or second order partial derivatives of the (unknown) function B as the corresponding equation (1) does. Instead the dynamics of B are given in terms of the primitives, i.e., the drift and the volatility processes of the forward rates. This model of the bond price also has the theoretical advantage that the volatility process tends to zero as time approaches maturity.

The no-arbitrage condition corresponding to the condition (2) above is usually written as (substitute $\eta_B = r_t + b(t,s)$, $\sigma_B = a(t,s)$ and differentiate with respect to s)

$$\alpha_t(s) = \sigma_t(s)[\lambda(t) - a(t,s)]. \tag{11}$$

This condition is called the forward rate restriction. We now define the measure Q exactly as in expression (3). By using the no-arbitrage condition (11) it follows that the forward rate process under Q becomes

$$f_t(u) = f_0(u) + \int_0^t \sigma_v(u) \int_v^u \sigma_v(s) ds dv + \int_0^t \sigma_v(u) d\hat{W}_v \tag{12}$$

and the short interest rate under Q becomes

$$r_t = f_0(t) + \int_0^t \sigma_v(t) \int_v^t \sigma_v(s) ds dv + \int_0^t \sigma_v(t) d\hat{W}_v$$

We see that the instantaneous forward rates processes as well as the short term interest rate under Q are completely determined by the volatility structure and the initial term structure. As opposed to the previous model, we do not need knowledge of the function $\lambda(t)$ to value bonds by equation (4).

By using the no-arbitrage condition (2) the process in expression (9) becomes

$$X_t(s) = \ln(B_0(s)) + \int_0^t \left[r_v - \tfrac{1}{2} a(v,s)^2 \right] dv + \int_0^t a(v,s) d\hat{W}_v, \tag{13}$$

and the expression for the bond price may be written

$$B_t(s) = B_0(s) + \int_0^t r_v B_v(s)dv + \int_0^t a(v,s)B_v(s)d\hat{W}_v, \tag{14}$$

i.e., the same form as the corresponding expression (5) in the state variable model.

The state variable model takes the short-rate interest process and the market price of risk as primitives, while the HJM-model takes the initial term structure, volatility structure and the market price of risk as primitives. The bond prices depend on the parameters of the drift and volatility factors and the market price of risk in the state variable model whereas they depend on the initial term structure and the volatility structure in the HJM-model. The HJM-model has the advantage that it does not require direct knowledge of the market price of risk. Still $\lambda(t)$ must satisfy some technical conditions, an issue we have not addressed here, to ensure that the measure Q is well-defined (that $\lambda(t)$ is uniformly bounded is a sufficient condition). For a more extensive comparison of these models we refer to Jamshidian (1991).

3. Partial differential equations for the market value of the insurance contract based on the one-factor models

3.1 The premium reserve

The premium reserve of a policy at time t can be considered as the insurer's debt to the insured at time t. In this section we only deal with deterministic benefits and the one-factor models. We derive a stochastic differential equation under the equivalent martingale measure Q for the market value of the premium reserve. This equation is based on the assumption that the instantaneous expected return of the bond price equals the short interest rate under Q, which is the case for both the state variable model and the HJM-model. The economy in the state variable model is characterized by a finite number of state variables and considering the market price of the premium reserve as a function of the state variables naturally leads to another stochastic differential equation under Q also describing the market value of the premium reserve. By equating the drift terms of these two processes we obtain a deterministic differential equation that must be satisfied by the market value of the premium reserve. In the HJM-model the forward rate processes serve the same purpose as the state variables. However, there is not a finite number of these, but we suggest two ways to go around this problem and present the resulting deterministic differential equations.

3.2 The insurance contract

To model the insurance contract we use the multi-state Markov model, by now standard in the actuarial sciences. See Hoem (1968), (1969) and (1988) and Norberg (1991) for details.

The state of the contract is assumed to evolve according to a right continuous stochastic process X_t with left limits defined on a given probability space $(\Omega, \mathcal{F}, P)$. Here X_t is a continuous time, inhomogenous Markov-chain with finite state space $\mathcal{J} = \{1,\ldots,J\}$. The transition probabilities are denoted by $p_{ij}(s,t) = P(X_t = j \mid X_s = i)$. The intensities $\mu_{ij}(s) = \lim_{t \downarrow s} \frac{p_{ij}(s,t)}{t-s}$, $i \neq j$, are assumed to exist for $i, j \in \mathcal{J}$. Furthermore, we assume that X_t is independent of all the processes describing the financial assets, which we refer to as the state of the policy is independent of the financial market.

In this section we work with general deterministic insurance benefits. At any time $t \leq T$ the policy is in one of the states, commencing in state 0. There are two types of benefits, a general life insurance $a_{jk}(t)$ payable upon transition from state j to state k at time t and a general annuity rate $a_j(t)$ the insurer receives in state j at time t. Payments from the insured to the insurer, such as premiums paid during the term of the contract, are considered as negative benefits.

3.3 Partial differential equations for the market value of the insurance contracts

The Thiele equation of the actuarial sciences can be viewed as a description of how the value of an –in a certain sense– average insurance contract develops over time. The study of this equation is therefore important both from theoretical and practical points of view. By introducing models including financial uncertainty the Thiele equation based on the Markov-chain model developed by Hoem (1968) has been extended by Norberg and Møller (1993), Persson (1994b) and Persson (1994c). In Hoem's equation all terms have reasonable and intuitive interpretations. The corresponding equations in the papers mentioned all contain additional terms stemming from the financial risk. These terms can not be interpreted in a similar straight-forward and intuitive manner as the terms in Hoem's original equation.

The following lemma presents a formula for the market value of the premium reserve of an insurance contract and is a special case of the results in Section 4 of this chapter.

Lemma 2.1

Consider a life insurance contract as described in Section 3.2 with deterministic benefits. Assume that the financial market is independent of the state of the policy and that the insurer is risk neutral with respect to transition risk. The market value of the prospective premium reserve for the insurance contract at time t given that the policy is in state g, is given by

$$\Pi_t^g = \int_t^T \sum_{j \in \mathcal{J}} p_{gj}(t,u) B_t(u) \left\{ a_j(u) + \sum_{k \neq j} \mu_{jk}(u) a_{jk}(u) \right\} du, \tag{15}$$

where $B_t(u)$ represents the market value at time t of a default-free unit-discount bond expiring at time u.

Proof:

See Section 4 of this chapter. ❑

Observe that Π_t^g depends on the market values of the bonds.

It is convenient to rewrite expression (15) in a different notation. Let

$$F_t^g(u) = B_t(u)P_t^g, \tag{16}$$

where

$$P_t^g = \sum_j p_{gj}(t,u)\left\{a_j(u) + \sum_{k\neq j}\mu_{jk}(u)a_{jk}(u)\right\}.$$

By using Kolmogorov's backward differential equation,

$$\frac{\partial p_{gj}(t,u)}{\partial t} = \sum_{h\neq g}\mu_{gh}(t)[p_{gj}(t,u) - p_{hj}(t,u)],$$

we get that

$$\frac{\partial P_t^g}{\partial t} = \sum_{h\neq g}\mu_{gh}(t)[P_t^g - P_t^h].$$

From expression (15) the prospective premium reserve may be written as

$$\Pi_t^g = \int_t^T F_t^g(u)du.$$

We now focus on the quantity $F_t^g(u)$, but first observe that

$$F_t^g(t) = a_g(t) + \sum_{h\neq g}\mu_{gh}(t)a_{gh}(t).$$

Here $a_g(t)dt$ represents the benefit the insured is eligible to in the interval $(t,t+dt)$ in state g at time t, $\mu_{gh}(t)dt$ represents the conditional probability that the policy will jump to state h in the interval $(t,t+dt)$ given that it is in state g at time t and $a_{gh}(t)$ is the amount the insured receives upon a transition from g to h. In this sense $F_t^g(t)dt$ represents the benefits the insurer on average has to pay in the time interval $(t,t+dt)$.

Now we use property that the dynamics of the bond under Q is $dB_t(u) = r_tB_t(u)dt + \sigma_B(t,u)B_t(u)d\hat{W}_t$ (in the HJM-model $\sigma_B(t,u)$ was labeled $a(t,u)$, but this is just a notational issue). The critical property is that the instantaneous expected return of the bond equals r_t under Q.

From expression (16) and Itô's lemma follow that

$$F_t^g(u) = F_0^g(u) + \int_0^t \left(r_s F_s^g(u) + \sum_{h \neq g} \mu_{gh}(s)[F_s^g(u) - F_s^h(u)] \right) ds + \int_0^t F_s^g(u) \sigma_B(s,u) d\hat{W}_s.$$

We integrate this expression with respect to u from t to T, interchange the order of integration and add and subtract some terms to obtain

$$\int_t^T F_t^g(u)du =$$

$$\int_0^T F_0^g(u)du + \int_0^t \left(r_s \int_s^T F_s^g(u)du + \sum_{h \neq g} \mu_{gh}(s) \int_s^T [F_s^g(u) - F_s^h(u)]du \right) ds + \int_0^t \int_s^T F_s^g(u)\sigma_B(s,u)dud\hat{W}_s -$$

$$\left[\int_0^t \left[F_0^g(u)du + r_s \int_s^t F_s^g(u)du + \sum_{h \neq g} \mu_{gh}(s) \int_s^t [F_s^g(u) - F_s^h(u)]du \right] ds + \int_0^t \int_s^t F_s^g(u)\sigma_B(s,u)dud\hat{W}_s \right].$$

From the definition of Π_t^g and by interchanging the order of integration in the second line, we get

$$\Pi_t^g = \Pi_0^g + \int_0^t \left(r_s \Pi_s^g + \sum_{h \neq g} \mu_{gh}(s)[\Pi_s^g - \Pi_s^h] \right) ds + \int_0^t \int_s^T F_s^g(u)\sigma_B(s,u)dud\hat{W}_s -$$

$$\int_0^t \left[F_0^g(u) + \int_0^u \left(r_s F_s^g(u)du + \sum_{h \neq g} \mu_{gh}(s)[F_s^g(u) - F_s^h(u)] \right) ds + \int_0^u F_s^g(u)\sigma_B(s,u)d\hat{W}_s \right] du$$

We recognize the term in the square brackets inside the last integral above as $F_u^g(u)$ and write

$$\Pi_t^g = \Pi_0^g + \int_0^t \left(r_s \Pi_s^g - a_g(s) - \sum_{h \neq g} \mu_{gh}(s)[a_{gh}(s) + \Pi_s^h - \Pi_s^g] \right) ds + \int_0^t \int_s^T F_s^g(u)\sigma_B(s,u)dud\hat{W}_s. \quad (17)$$

This is our first stochastic differential equation describing the dynamics of the market value of the premium reserve under Q. The derivation is only based on the assumption that the expected instantaneous return of the bond price is r_t under Q, hence the equation is valid for both the state variable model and the HJM-model. The volatility process is not expressed in terms of Π_t^g, but as soon will become apparent this fact does not matter. Observe that the drift process of this equation contains exactly the same terms as the traditional Thiele equation.

For the state variable model another stochastic differential equation can easily be derived. We derive this equation under Q by noticing from (15) that Π_t^g is a function of $B_t(s)$, $s \in [t,T]$, which in the case of a single state is a function of r_t and t for $s \in [t,T]$. By considering Π_t^g as a function of r_t and t it follows immediately from Itô's lemma and the dynamics of r_t under Q given by equation (6) that

$$\Pi_t^g = \Pi_0^g + \int_0^t \left(\frac{\partial \Pi_s^g}{\partial r}(\eta_r - \sigma_r \lambda(s)) + \frac{1}{2}\sigma_r^2 \frac{\partial^2 \Pi_s^g}{\partial r^2} + \frac{\partial \Pi_s^g}{\partial s} \right) ds + \int_0^t \frac{\partial \Pi_s^g}{\partial r} \sigma_r d\hat{W}_s. \quad (18)$$

The stochastic differential equations in (17) and (18) represent the same quantity, hence their drift and volatility terms must be the equal. By equating the drift terms we obtain

$$\frac{\partial \Pi_t^g}{\partial t} = r_t \Pi_t^g - a_g(t) - \sum_{h \neq g} \mu_{gh}(t)[a_{gh}(t) + \Pi_t^h - \Pi_t^g] - \left[\frac{\partial \Pi_t^g}{\partial r}(\eta_r - \sigma_r \lambda(t)) + \frac{1}{2}\sigma_r^2 \frac{\partial^2 \Pi_t^g}{\partial r^2} \right]. \quad (19)$$

Here we observe that expression (19) depends on λ the market price of risk. From two stochastic differential equations under Q representing the same quantity we have derived one partial differential equation for the market value of the policy. The measure Q serves as a technical tool in the derivation of a deterministic partial equation which the market value of the insurance contract must satisfy. The above equation is a special case of an equation derived in Persson (1994c) using a somewhat different approach. We may interpret equation (17) as the counterpart to Thiele's differential equation of the actuarial sciences. However, our equation also deals with financial risk, which is not the case for the traditional Thiele equation.

As seen in Section 2, equations involving first and second order derivatives occur naturally in the state variable model. A similar equation for the HJM-model would necessarily involve the first and second order derivatives of Π_t^g with respect to a quantity corresponding to the state variables. The collection $\{f_t(u): u\in[t,T]\}$ are the primitives at time t of the HJM-model and, in many respects, serves the same purpose as the state variables in the state variable model. However, the number of these variables are not finite. One way to proceed is to represent this continuum of processes through one state variable. We suggest to consider

$$X_t = -\int_t^T f_t(u)du.$$

This quantity represents minus the integral from t to T over the instantaneous forward rates prevailing at time t. The only motivation for the use of the minus sign is the relation $X_t = X_t(T)$, where $X_t(T)$ is defined in expression (9) and where we already know the dynamics for $X_t(T)$ under Q from expression (13). Another possibility is to use $B_t(T)$ which essentially contains the same information as X_t (this is obvious from the definition of $X_t(T)$). We also know the dynamics of $B_t(T)$ under Q from expression (14). Maybe one advantage of using $B_t(T)$ instead of X_t is that $B_t(T)$ represents a well-defined economic quantity, the market price of a bond, whereas the economic interpretation of X_t is somewhat lose. The deterministic partial differential equations corresponding to equation (19) for these two cases are

$$\frac{\partial\Pi_t^g}{\partial t} = r_t\Pi_t^g - a_g(t) - \sum_{h\neq g}\mu_{gh}(t)[a_{gh}(t) + \Pi_t^h - \Pi_t^g] - \left\{\frac{\partial\Pi_t^g}{\partial X}r_t + \frac{1}{2}a(t,T)^2\left[\frac{\partial^2\Pi_t^g}{\partial X^2} - \frac{\partial\Pi_t^g}{\partial X}\right]\right\} \quad (20)$$

and

$$\frac{\partial\Pi_t^g}{\partial t} = r_t\Pi_t^g - a_g(t) - \sum_{h\neq g}\mu_{gh}(t)[a_{gh}(t) + \Pi_t^h - \Pi_t^g] - \left\{\frac{\partial\Pi_t^g}{\partial B}r_tB_t(T) + \frac{1}{2}[a(t,T)B_t(T)]^2\frac{\partial^2\Pi_t^g}{\partial B^2}\right\}. \quad (21)$$

These equations are derived exactly as equation (19), by equation the drift terms of two stochastic differential equations under Q. They may be considered as generalizations of the Thiele equation in the case of the HJM term structure model and deterministic benefits. The terms involving the first and second order derivatives are different from the similar terms from the state variable model. The equations involve the volatility structure of the forward rates through a(t,T) but do not depend on the market price of risk as the corresponding equation (19) of the state variable model do. Setting all the partial derivatives of equations (19), (20) and (21) equal to zero leads to the traditional Thiele equation which does not deal with financial risk. Setting the insurance specific factors, represented by a_g, a_{gh} (the benefits) and μ_{gh} (transition intensities), in equation (19) equal to zero leads to a well-known so-called fundamental, differential equation from financial economics governing the value of, e.g., bonds in the state variable model. We are not aware of any similar "fundamental" differential equations for the HJM-model.

4. The market value of an insurance contract with random benefits

4.1 Unit-linked insurance

The set-up in this section is applicable for unit-linked or equity-linked insurance (called variable life insurance in USA). The distinguishing property of unit-linked insurance is that the value of the benefit is linked to the value of a certain number of units in a mutual fund or another financial asset. Earlier work on this type of insurance may be found in Boyle and Schwartz (1977), Brennan and Schwartz (1976), (1979a), (1979b), Delbaen (1980), Baccinello and Ortu (1993), Nielsen (1993), Persson (1994a) and Persson (1994b). Some of these works are reviewed in Persson (1994a). The current model generalizes the models of Persson (1994a) and Persson (1994b) by introducing multiple sources of uncertainty including a stochastic interest rate.

First we describe the extended financial market and explain the pricing principles, which are common for both the term structure models. In Section 5 we look at the particular properties of the two term structure models. There are now several sources of uncertainty modeled by a multi-dimensional Brownian motion. The model of the financial market is from Amin and Jarrow (1992).

4.2 The financial market

The time horizon T is fixed and the financial uncertainty is now generated by a (d + e) dimensional standard Brownian motion $W = \{W_t: t \in [0,T]\}$ on a complete probability space $(\Omega, \mathcal{F}, P)$ together with a filtration $\{\mathcal{F}_t: t \in [0,T]\}$, representing the flow of information. All the martingales encountered throughout will be martingales with respect to this filtration.

All trade is assumed to take place in a frictionless market (no transaction costs or taxes and short-sale allowed) with continuous trading opportunities.

Definition 4.1

Let r_t denote the short term interest rate prevailing at time t, $t \in [0,T]$. Formally, r_t is a stochastic process defined on $(\Omega, \mathcal{F}, P)$ and will be described later. We define the money market account as

$$\beta_t = \exp\left(\int_0^t r_s ds\right), \tag{22}$$

i.e., the value at time t of one unit currency invested at time zero accruing interest according to the short term interest rate.

As before there is a continuum of unit discount bond maturing at all times $s \in [0,T]$. We assume that the market value of the unit discount bonds can be represented as Itô-processes on the following form,

$$B_t(s) = B_0(s) + \int_0^t \eta_B(u,s)B_u(s)du + \sum_{i=1}^{d}\int_0^t B_u(s)\sigma_B^i(u,s)dW_u^i\ , \tag{23}$$

As indicated in Section 2, the drift process η_B and the volatility processes σ_B^i will depend on the model of the term structure.

In addition to the bonds there are m risky securities. Neither of them pay dividends, which is not an unreasonable assumption in the current insurance setting. They are modeled by Itô-processes on the form

$$S_t^j = S_0^j + \int_0^t \eta_{S_j}(u)S_u^j du + \sum_{i=1}^{d+e}\int_0^t \sigma_{S_j}^i(u)S_u^j dW_u^i,\quad j = 1,2,\ldots,m. \tag{24}$$

We assume that $m \geq e$ and will sometimes refer to the $(m\times 1)$ vector of security prices at time t as S_t. The last e Brownian motions are reserved for modeling uncertainty related to the risky assets.

Our next task is to describe the concept of the market prices of risk corresponding to the d+e sources of uncertainty.

First fix d bonds with expiration dates $(T_1, \ldots, T_d)$, where $0 < T_1 < \ldots < T_d \leq T$ and fix e risky assets referred to by $(S_1,\ldots, S_e)$. The inclusion of the bonds is not strictly required (as long as at least d + e risky assets exist), but allow us to relate this analysis to the restricted term structure economy (i.e., no risky assets) used in Heath, Jarrow and Morton (1992) (see also Persson (1994c) for a similar insurance setting). We define

$$A_1 = \begin{bmatrix} \eta_B(t,T_1) - r_t \\ \vdots \\ \eta_B(t,T_d) - r_t \\ \eta_{S_1}(t) - r_t \\ \vdots \\ \eta_{S_e}(t) - r_t \end{bmatrix}, \quad A_2 = \begin{bmatrix} \sigma_B^1(t,T_1) & \cdots & \sigma_B^d(t,T_1) & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \sigma_B^1(t,T_d) & \cdots & \sigma_B^d(t,T_d) & 0 & \cdots & 0 \\ \sigma_{S_1}^1(t) & \cdots & \sigma_{S_1}^d(t) & \sigma_{S_1}^{d+1}(t) & \cdots & \sigma_{S_1}^{d+e}(t) \\ \vdots & & \vdots & \vdots & & \vdots \\ \sigma_{S_e}^1(t) & \cdots & \sigma_{S_e}^d(t) & \sigma_{S_e}^{d+1}(t) & \cdots & \sigma_{S_e}^{d+e}(t) \end{bmatrix}$$

Here A_1 is an (d+e)-vector representing the expected instantaneous return in excess of the short-term interest rate on the d bonds and the e risky securities. The rows in the $(d+e)\times(d+e)$ matrix A_2 represent the (d+e) volatility processes of the d bonds and the e risky assets.

We seek a (d+e)-vector λ so that

$$A_1 = A_2\lambda. \tag{25}$$

The following condition imposes restrictions on the volatility processes of the bonds and the risky assets and is sufficient to ensure that the inverse of A_2 is well-defined and hence that a solution to equation (25) exists.

Assumption 4.2
A_2 is non-singular (almost surely and almost everywhere).

Each λ_t^i, $i = 1,\ldots,d+e$, can be interpreted as the market price of risk of related to the i'th source of uncertainty at time t. In addition to t, the vector λ will depend on $(T_1, \ldots, T_d)$ and the e risky securities $(S_1,\ldots,S_e)$, so we may write $\lambda(t,T_1, \ldots, T_d,S_1,\ldots,S_e)$. Each row in the matrix on the right hand side of equation (25) may be interpreted as the market price of risk (which may be negative) connected to uncertainty source i multiplied by the amount of risk of source i, added up for all sources, so this expression relates the instantaneous excess expected return to the market value of the risk associated with a specific asset. The λ_t^i's sometimes occur with the opposite sign in other treatments.

Amin and Jarrow (1992) show that, assuming that the λ_t^i's are uniformly bounded, there exists a unique equivalent martingale measure for the d bonds and the e risky securities so that the price processes of these asset discounted by the money market account are martingales. However, we would like to find an equivalent measure so that the discounted price process of any bond and any risky asset are martingales. A sufficient condition for this now follows.

Assumption 4.3

We assume that $\lambda(t,T_1, \ldots, T_d, S_1, \ldots, S_e) = \lambda(t)$ for all choices of $(T_1, \ldots, T_d)$ and $(S_1, \ldots, S_e)$.

That is, we assume that the λ-vector in equation (25) does not depend on the expiration dates of the bonds or the particular e risky assets involved.

In particular, for a bond with expiration date s we can write

$$\eta_B(t,s) - r_t = \sum_{i=1}^{d} \sigma_B^i(t,s)\lambda_t^i. \qquad (26)$$

We notice that the expected instantaneous excess return of the bonds are not related to λ_t^i for $i = d+1, \ldots, d+e$. Similarly we can write for any risky asset j

$$\eta_{S_j}(t) - r_t = \sum_{i=1}^{d+e} \sigma_{S_j}^i(t)\lambda_t^i. \qquad (27)$$

These λ_t^i 's will be the basis of the construction of the equivalent martingale measure Q. To ensure that Q is well-defined we impose some conditions of technical nature.

Assumption 4.4

We assume that $E\left[\exp\left(\frac{1}{2}\sum_{i=1}^{d+e}\int_0^T (\lambda_t^i)^2 ds\right)\right] < \infty$. This condition is known as Novikov's condition. We also assume that $Var\left(\exp\left(-\sum_{i=1}^{d+e}\int_0^T \lambda_u^i dW_u^i - \frac{1}{2}\sum_{i=1}^{d+e}\int_0^T (\lambda_u^i)^2 du\right)\right) < \infty$.

A price system that satisfying Assumption 4.2 and 4.4 is said to be L^2-reducible by Duffie

(1992). A sufficient condition is that the λ_t^i 's are uniformly bounded (an economic interpretation of this assumption is that the agents are not allowed to be infinitely risk averse with respect to any of the d+e sources of uncertainty).

We define an equivalent probability measure by

$$\frac{dQ}{dP} = \exp\left(-\sum_{i=1}^{d+e} \int_0^T \lambda_u^i dW_u^i - \frac{1}{2} \sum_{i=1}^{d+e} \int_0^T (\lambda_u^i)^2 du \right) \tag{28}$$

and show the following lemma.

Lemma 4.5

The discounted price processes $\frac{S_t^j}{\beta_t}$ and $\frac{B_t(s)}{\beta_t}$, for j = 1,...,m and s ∈ [0,T], are martingales with respect to $\mathcal{F}_t$ under Q.

Proof:

From Girsanov's theorem

$$\hat{W}_t^i = W_t^i + \int_0^t \lambda_t^i ds, \quad i = 1,...d + e$$

are independent Brownian motions under Q. Substituting for dW_t^i in (23) and (24) and using (26) and (27) shows that the instantaneous expected rates of returns of the assets equal the short-term rate under Q,

$$B_t(s) = B_0(s) + \int_0^t r_u B_u(s) du + \sum_{i=1}^{d} \int_0^t B_u(s) \sigma_B^i(u,s) d\hat{W}_u^i, \tag{29}$$

$$S_t^j = S_0^j + \int_0^t r_u S_u^j du + \sum_{i=1}^{d+e} \int_0^t \sigma_{S_j}^i(u) S_u^j d\hat{W}_u^i, \quad j = 1,2,...,m.$$

The result follows by dividing the price processes by the money market account using Itô's lemma. Then the drift terms vanish. ❑

We have constructed an equivalent martingale measure in our economy, so no arbitrage opportunities are present (see, e.g., Duffie (1992)). Furthermore, it can be shown (see Amin and Jarrow (1992)) that Q is the unique measure with the property described in Lemma 4.5.

4.3 The insurance benefits

We allow the benefit to be linked to the market value of the risky securities. This means that the benefit is either a traded asset or can be duplicated by a portfolio of a finite number of traded assets. Note that also the payoff of deterministic benefits can be duplicated by portfolios of unit-discount bonds. As before there are two types of benefits, now denoted by $a_{jk}(S_t)$ for the general life insurance contract and by $a_j(S_t)$ for the general annuity rate. The benefits are allowed to be general (measurable) functions of the m risky securities, which in particular means that they may include guarantees.

Thus the insurance benefits can be viewed as contingent claims, in the usual sense known from financial economics. The contingent claim theory based on the works by Harrison and Kreps (1979) and Harrison and Pliska (1981) can then be applied for valuation. The uniqueness of the measure Q implies that the market is complete, i.e., there exists a market price for every contingent claim with finite variance. Let $\pi_t(u)$ denote the market value of the benefit $a(S_u)$, for the moment dropping the subscript of $a(S_u)$ (and the corresponding superscript of $\pi_t(u)$) describing upon which event the benefit expires. In the financial model where all uncertainty is modeled by Itô-processes, the market prices will also be Itô-processes on the form

$$\pi_t(u) = \pi_0(u) + \int_0^t \eta_\pi(s,u)\pi_s(u)ds + \sum_{i=1}^{d+e}\int_0^t \sigma_\pi^i(s,u)\pi_s(u)dW_s^i, \tag{30}$$

for some processes η_π and σ_π^i. To determine this market price we again use a result of Harrison and Kreps (1979) stating that the market price of a contingent claim can be found as the conditional expectation of the payoff under the equivalent martingale measure Q after a change of numeraire, or

$$\frac{\pi_t(u)}{\beta_t} = E_t^Q\left[\frac{a(S_u)}{\beta_u}\right], \tag{31}$$

using the money market account as numeraire. Observe that for $t \leq s \leq u$,

$$E_t^Q\left[\frac{\pi_s(u)}{\beta_s}\right] = E_t^Q\left[E_s^Q\left[\frac{a(S_u)}{\beta_u}\right]\right] = E_t^Q\left[\frac{a(S_u)}{\beta_u}\right] = \frac{\pi_t(u)}{\beta_t},$$

hence also the discounted market prices of the benefits are martingales under Q. The economic intuition for this is as follows: The price processes of the bonds and the risky assets are martingales under Q and the benefit can be considered as a portfolio consisting of a combination of these assets. If the price processes of the contingent claim is not a martingale, arbitrage opportunities would be present and riskless profit could be made by taking opposite positions in the duplicating portfolio and the contingent claim.

By the arguments used in the proof of Lemma 4.5 it is easy to show that the drift processes of $\pi_t(u)$ also satisfies

$$\eta_\pi - r_t = \sum_{i=1}^{d+e} \sigma_\pi^i(t)\lambda_t^i \tag{32}$$

and that the process for any π (before discounting) under Q is

$$\pi_t(u) = \pi_0(u) + \int_0^t r_t\pi_s(u)ds + \sum_{i=1}^{d+e}\int_0^t \sigma_\pi^i(s,u)\pi_s(u)d\hat{W}_s^i. \tag{33}$$

4.4 Pricing principles

The random payment stream in the period [0,T] of this general insurance policy can be described by

$$A_T = \int_0^T \sum_{j\in\mathcal{J}} \left\{ 1_{\{X_u=j\}} a_j(S_u)du + \sum_{k\neq j} a_{jk}(S_u)dN_{jk}(u) \right\},$$

where $1_{\{X_t=j\}}$ is the indicator function taking the value 1 if $X_t = j$ and zero otherwise and $N_{jk}(t)$ counts the number of transitions from state j to state k by time t.

We denote by V_0 the present value of the payments in the period [0,T] after discounting by the money market account. Then

$$V_0 = \int_0^T \frac{1}{\beta_u} \sum_{j\in\mathcal{J}} \left\{ 1_{\{X_u=j\}} a_j(S_u)du + \sum_{k\neq j} a_{jk}(S_u)dN_{jk}(u) \right\}.$$

Let Π_0 denote the market value at time zero of the payments in the period [0,T]. In this model there are two independent sources of uncertainty, one related to the financial market, the other related to the transition between states. Formally we may model each source on its own probability space and consider $(\Omega, \mathcal{F}, P)$ as a product space. The Radon-Nikodym derivative $\frac{dQ}{dP}$ in expression (28) can be considered as a pricing rule on the probability space describing financial risk. Risk neutrality with respect to transition risk implies that the corresponding pricing rule for that risk is identical to one on the other probability space. The independence between the two sources of risk then implies that the pricing rule on the product space is equal to $\frac{dQ}{dP}$ times one on $(\Omega, \mathcal{F}, P)$. This idea is explained in Persson (1994c). The market value of the policy is found as $\Pi_0 = E^Q[V_0]$, where Q is defined by $\frac{dQ}{dP}$ in expression (28). This is different from the traditional principle of equivalence which states that the price of the policy is equal to the present value under the original probability measure, i.e., $E^P[V_0]$. Therefore,

$$\Pi_0 = E^P\left[\frac{dQ}{dP} V_0\right].$$

By using expression (31) we obtain

$$\Pi_0 = \int_0^T \sum_{j \in \mathcal{J}} p_{0j}(0,u)\left\{\pi_0^j(u) + \sum_{k \neq j} \mu_{jk}(u)\pi_0^{jk}(u)\right\}du. \tag{34}$$

The above expression represents our valuation principle for insurance contracts in our model. By the same arguments it follows that the premium reserve at time t, given that the policy is in state g is

$$\Pi_t^g = \int_t^T \sum_{j \in \mathcal{J}} p_{gj}(t,u)\left\{\pi_t^j(u) + \sum_{k \neq j} \mu_{jk}(u)\pi_t^{jk}(u)\right\}du \ . \tag{35}$$

In the special case of a deterministic benefit equation (35) is reduced to equation (15). This can be seen from expression (31), $\pi_t(u) = a_t(u)\beta_t E_t^Q\left[\frac{1}{\beta_u}\right] = a_t(u)B_t(u)$ for a deterministic benefit.

5. Partial differential equations for the market value of the insurance contracts

5.1 The one-factor models generalized

In this section we first generalize the one-factor models described in Section 2 to multi-factor models. We assume that the uncertainty related to the term structure is generated by a d-dimensional Brownian motion. After a short description of the general state variable model and HJM-model, we develop partial differential equations for the market value of the insurance contract. These equations are similar to the ones derived in section 3, but generalize these in two respects. First we use multi-factor models of the term structure and, second, the benefits are allowed to be random.

5.2 The general state variable model

Now we present an extension of the model with one state variable described in Section 2. In the general state variable model the economy is described by n state variables of which one is the short-term interest rate. The short term interest rate is given by the stochastic differential equation

$$r_t = r_0 + \int_0^t \eta(r_s,s)ds + \int_0^t \sum_{i=1}^{d} \sigma^i(r_s,s)dW_s^i \,, \tag{36}$$

where r_0 is a constant to be interpreted as the short interest rate prevailing at time zero.

In addition to r_t, we assume there are $(n-1)$ Itô-process $\hat{Z}_j$ of state variables. We refer to the n-dimensional vector of state variables by $Z = \begin{pmatrix} r \\ \hat{Z} \end{pmatrix}$. For notational convenience we do not distinguish between the short interest rate and the other state variables and refer to the processes describing the state variables as

$$Z_t^j = Z_0^j + \int_0^t \eta_Z^j ds + \int_0^t \sum_{i=1}^{d} \sigma_{Z_j}^i dW_s^i \;, j = 1,\ldots,n \tag{37}$$

where Z_0^j can be interpreted as the initial values of the state variable j. Most popular term structure models in the current financial literature use one or two state variables.

For this model it is important to emphasize that the market prices of risk are related to the d sources of uncertainty, i.e., the Brownian motions, and not to the n state variables.

We assume that $B_t(s)$ is a sufficiently smooth function of the state variables in addition to t. We can now determine $\eta_B(t,s)$ and $\sigma_B(t,s)$ from equation (23) for the state variable model, but first we introduce a more compact notation.

We define

$$\eta_Z = \begin{pmatrix} \eta_Z^1 \\ \vdots \\ \eta_Z^n \end{pmatrix}, \sigma_Z = \begin{pmatrix} \sigma_{Z_1}^1 & \cdots & \sigma_{Z_1}^d \\ \vdots & & \vdots \\ \sigma_{Z_n}^1 & \cdots & \sigma_{Z_n}^d \end{pmatrix}, \frac{\partial B}{\partial Z} = \begin{pmatrix} \frac{\partial B}{\partial Z_1} \\ \vdots \\ \frac{\partial B}{\partial Z_n} \end{pmatrix} \text{and } \frac{\partial^2 B}{\partial Z^2} = \begin{pmatrix} \frac{\partial^2 B}{\partial Z_1^2} & \cdots & \frac{\partial^2 B}{\partial Z_1 \partial Z_n} \\ \vdots & & \vdots \\ \frac{\partial^2 B}{\partial Z_n \partial Z_1} & \cdots & \frac{\partial^2 B}{\partial Z_n^2} \end{pmatrix}.$$

Immediately from Itô's lemma it follows that

$$\eta_B(t,s) = \frac{1}{B_t(s)}\left[\frac{\partial B}{\partial Z}^T \eta_Z + \frac{\partial B}{\partial t} + \frac{1}{2}\mathrm{tr}\left[\sigma_Z \sigma_Z^T \frac{\partial^2 B}{\partial Z^2}\right]\right]$$

and

$$\sigma_B^i(t,s) = \frac{1}{B_t(s)} \sum_{j=1}^{n} \sigma_{Z_j}^i \frac{\partial B}{\partial Z_j}, \quad i = 1,\ldots,d,$$

where tr[A] denotes the trace, i.e., the sum of the diagonal elements of the square matrix A and A^T denotes the transposed of the matrix A.

Now we want to determine $\eta_\pi(t,s)$ and $\sigma_\pi^i(t,s)$ from expression (30) in a similar way. We assume that the market price of a benefit is a function of Z, S and t for fixed expiration date u.

We let

$$\eta_S = \begin{pmatrix} \eta_{S_1} S_t^1 \\ \vdots \\ \eta_{S_m} S_t^m \end{pmatrix}, \sigma_S = \begin{pmatrix} \sigma_{S_1}^1 S_t^1 & \cdots & \sigma_{S_1}^{d+e} S_t^1 \\ \vdots & & \vdots \\ \sigma_{S_m} S_t^m & \cdots & \sigma_{S_m}^{d+e} S_t^m \end{pmatrix}, \frac{\partial \pi}{\partial Z} = \begin{pmatrix} \frac{\partial \pi}{\partial Z_1} \\ \vdots \\ \frac{\partial \pi}{\partial Z_n} \end{pmatrix}, \frac{\partial^2 \pi}{\partial Z^2} = \begin{pmatrix} \frac{\partial^2 \pi}{\partial Z_1^2} & \cdots & \frac{\partial^2 \pi}{\partial Z_1 \partial Z_n} \\ \vdots & & \vdots \\ \frac{\partial^2 \pi}{\partial Z_n \partial Z_1} & \cdots & \frac{\partial^2 \pi}{\partial Z_n^2} \end{pmatrix},$$

$$\frac{\partial \pi}{\partial S} = \begin{pmatrix} \frac{\partial \pi}{\partial S_1} \\ \vdots \\ \frac{\partial \pi}{\partial S_m} \end{pmatrix}, \frac{\partial^2 \pi}{\partial S^2} = \begin{pmatrix} \frac{\partial^2 \pi}{\partial S_1^2} & \cdots & \frac{\partial^2 \pi}{\partial S_1 \partial S_m} \\ \vdots & & \vdots \\ \frac{\partial^2 \pi}{\partial S_m \partial S_1} & \cdots & \frac{\partial^2 \pi}{\partial S_m^2} \end{pmatrix}, \frac{\partial^2 \pi}{\partial Z \partial S} = \begin{pmatrix} \frac{\partial^2 \pi}{\partial Z_1 \partial S_1} & \cdots & \frac{\partial^2 \pi}{\partial Z_1 \partial S_m} \\ \vdots & & \vdots \\ \frac{\partial^2 \pi}{\partial Z_n \partial S_1} & \cdots & \frac{\partial^2 \pi}{\partial Z_n \partial S_m} \end{pmatrix} \text{and}$$

$$\frac{\partial^2 \pi}{\partial S \partial Z} = \begin{pmatrix} \frac{\partial^2 \pi}{\partial S_1 \partial Z_1} & \cdots & \frac{\partial^2 \pi}{\partial S_1 \partial Z_n} \\ \vdots & & \vdots \\ \frac{\partial^2 \pi}{\partial S_m \partial Z_1} & \cdots & \frac{\partial^2 \pi}{\partial S_m \partial Z_n} \end{pmatrix}.$$

Directly from Itô's lemma it now follows that

$$\eta_\pi(t,u) =$$

$$\frac{1}{\pi_t(u)}\left[\frac{\partial \pi}{\partial Z}^T \eta_Z + \frac{\partial \pi}{\partial S}^T \eta_S + \frac{\partial \pi}{\partial t} + \frac{1}{2}\mathrm{tr}\left[\sigma_Z \sigma_Z^T \frac{\partial \pi}{\partial Z^2} + \sigma_Z \sigma_S^T \frac{\partial^2 \pi}{\partial S \partial Z}\right] + \frac{1}{2}\mathrm{tr}\left[\sigma_S \sigma_S^T \frac{\partial^2 \pi}{\partial S^2} + \sigma_S \sigma_Z^T \frac{\partial^2 \pi}{\partial Z \partial S}\right]\right]$$

and $$\sigma_\pi^i(t,u) = \frac{1}{\pi_t(\upsilon)}\left[\sum_{j=1}^{n} \frac{\partial \pi}{\partial Z_j} \sigma_{Z_j}^i + \sum_{j=1}^{m} \frac{\partial \pi}{\partial S_j} \sigma_{S_j}^i S_t^j\right], \text{ for } i = 1,\ldots,d,$$

$$\sigma_\pi^i(t,u) = \frac{1}{\pi_t(\upsilon)}\left[\sum_{j=1}^{m} \frac{\partial \pi}{\partial S_j} \sigma_{S_j}^i S_t^j\right], \text{ for } i = d+1,\ldots,d+e.$$

Our next task is to derive another fundamental differential equation. The starting point for this equation is expression (32) and the above expressions for η_π and σ_π^i. By also using expression (27), we obtain

$$\frac{\partial \pi}{\partial Z}^T(\eta_Z - \sigma_Z\lambda) + \frac{\partial \pi}{\partial S}^T S_t r_t + \frac{\partial \pi}{\partial t} +$$

$$\frac{1}{2}\mathrm{tr}\left[\sigma_Z\sigma_Z^T\frac{\partial \pi}{\partial Z^2} + \sigma_Z\sigma_S^T\frac{\partial^2 \pi}{\partial S\partial Z}\right] + \frac{1}{2}\mathrm{tr}\left[\sigma_S\sigma_S^T\frac{\partial^2 \pi}{\partial S^2} + \sigma_S\sigma_Z^T\frac{\partial^2 \pi}{\partial Z\partial S}\right] - r_t\pi_t(u) = 0. \qquad (38)$$

This equation is the pure financial counterpart to the equation we deduce in the next section.

From Girsanov's theorem follows that the processes of the state variables under Q are given by

$$Z_t^j = Z_0^j + \int_0^t\left[\eta_Z^j - \sum_{i=1}^{d}\sigma_{Z_j}^i\lambda_s^i\right]ds + \int_0^t\sum_{i=1}^{d}\sigma_{Z_j}^i d\tilde{W}_s^i \ , j = 1,\ldots,n. \qquad (39)$$

These equations now depend on the market prices of risk and will be used in the derivation of the similar partial differential equation of the insurance contract.

5.3 The HJM-model

The following family of Itô-processes describing the instantaneous forward rate is given,

$$f_t(u) = f_0(u) + \int_0^t \alpha_v(u)dv + \sum_{i=1}^{d}\int_0^t \sigma_v^i(u)dW_v^i.$$

The processes $\alpha_v(u)$ and $\sigma_v^i(u)$ are adapted and measurable, $\alpha_v(u)$ integrable and $\sigma_v^i(u)$ square integrable. The only economic restrictions imposed are that the forward rates have continuous sample paths and depend on a finite number of Brownian motions.

Additional technical assumptions (see Heath, Jarrow and Morton (1992)) are required to assure that the money market account and the value of the bonds are finite.

We determine η_B and σ_B^i in expression (23) as

$$\eta_B(t,u) = r_t + b(t,u)$$

and

$$\sigma_B^i(t,u) = a_i(t,u),$$

where

$$a_i(t,u) = -\int_t^u \sigma_t^i(s)ds,$$

$$b(t,u) = \upsilon(t,u) + \frac{1}{2}\sum_{i=1}^{d} a_i(t,u)^2$$

and $\upsilon(t,u)$ represents minus the integral of the drift rate process of the forward rate as defined in Section 2.

We have that $B_t(s) = \exp(X_t(s))$, where

$$X_t(s) = \ln(B_0(s)) + \int_0^t [r_v + \upsilon(v,s)]dv + \sum_{i=1}^{d}\int_0^t a_i(v,s)dW_v^i,$$

i.e., the only difference from Section 2 is the multi-dimensional Brownian motion. The multi-dimensional version of the forward rate restriction given in (11) is

$$\alpha_t(s) = \sum_{i=1}^{d} \sigma_t^i(s)[\lambda_t^i - a(t,s)]$$

By using this expression under the equivalent martingale measure Q we obtain that

$$f_t(u) = f_0(u) + \sum_{i=1}^{d}\int_0^t \sigma_v^i(u)\int_v^u \sigma_v^i(s)dsdv + \sum_{i=1}^{d}\int_0^t \sigma_v^i(u)d\hat{W}_v^i,$$

$$X_t(s) = \ln(B_0(s)) + \int_0^t \left[r_v - \frac{1}{2}\sum_{i=1}^{d} a_i(v,s)^2 \right] dv + \sum_{i=1}^{d}\int_0^t a_i(v,s)d\hat{W}_v^i$$

and

$$\beta_t = \frac{1}{B_0(t)}\exp\left(\sum_{i=1}^{d}\frac{1}{2}\int_0^t a_i(s,t)^2 ds - \sum_{i=1}^{d}\int_0^t a_i(s,t)d\hat{W}_s^i \right).$$

The above processes represent the forward rates, minus the integral of the forward rates and the money market account, respectively, under Q. Observe that these processes are determined by the initial term structure ($f_0(t)$, $B_0(t)$) and the volatility structure ($(\sigma_t^i(s), a_i(t,s))$).

5.4 Partial differential equations of the market value of the insurance contract

We now derive the generalized versions of the differential equations developed in the one-factor models for deterministic benefits in Section 3 using basically the same approach.

From expression (35) we may write $\Pi_t^g = \int_t^T \sum_j F_t^{gj}(u)du$, where

$$F_t^{gj}(u) = p_{gj}(t,u)\left\{\pi_t^j(u) + \sum_{k \neq j} \mu_{jk}(u)\pi_t^{jk}(u)\right\}. \tag{40}$$

We make the following observations,

$$\frac{\partial F_t^{gj}(u)}{\partial t} = \sum_{h \neq g} \mu_{gh}(t)[F_t^{gj}(u) - F_t^{hj}(u)]$$

and

$$F_t^{gj}(t) = a_g(t) + \sum_{h \neq g} \mu_{gh}(t) a_{gh}(t).$$

From (33) we know that the expected instantaneous return of any π under Q is r_t, so by using expression (40) and Itô's lemma we obtain

$$F_t^{gj}(u) = F_0^{gj}(u) + \int_0^t \left(r_s F_s^{gj}(u) + \sum_{h \neq g} \mu_{gh}(s)[F_s^{gj}(u) - F_s^{hj}(u)] \right) ds + \int_0^t \sigma_F^{gj}(s,u) d\hat{W}_s,$$

where

$$\sigma_F^{gj}(t,u) = p_{gj}(t,u)\left\{ \sigma_{\pi_j} \pi_t^j(u) + \sum_{k \neq j} \mu_{jk}(u) \sigma_{\pi_{jk}} \pi_t^{jk}(u) \right\}.$$

By adding together this equation for all j and repeating the steps leading to expression (17) in Section 3, we obtain

$$\Pi_t^g = \Pi_0^g + \int_0^t \left(r_s \Pi_s^g - a_g(s) - \sum_{h \neq g} \mu_{gh}(s)[a_{gh}(s) + \Pi_s^h - \Pi_s^g] \right) ds + \int_0^t \int_s^T \sum_j \sigma_F^{gj}(s,u)\, du\, d\hat{W}_s. \quad (41)$$

This equation corresponds to equation (17) from Section 3 and describes the dynamics of the market value of the premium reserve under Q. Also here the drift process contains exactly the same terms as the traditional Thiele equation.

Now we proceed as in Section 3, obtaining deterministic differential equations by equating two drift terms of stochastic differential equations under Q representing the same quantity. The volatility processes are not important in this exercise and will not be shown. For the state variable model we consider the market value of the premium reserve to be a function of Z, S and t. From Itô's lemma, expression (39) and the dynamics of S given in the previous section we obtain the drift process as

$$\frac{\partial \pi}{\partial Z}^T (\eta_Z - \sigma_Z \lambda) + \frac{\partial \pi}{\partial S}^T S_t r_t + \frac{\partial \pi}{\partial t} +$$

$$\frac{1}{2} \mathrm{tr}\left[\sigma_Z \sigma_Z^T \frac{\partial^2 \pi}{\partial Z^2} + \sigma_Z \sigma_S^T \frac{\partial^2 \pi}{\partial S \partial Z} \right] + \frac{1}{2} \mathrm{tr}\left[\sigma_S \sigma_S^T \frac{\partial^2 \pi}{\partial S^2} + \sigma_S \sigma_Z^T \frac{\partial^2 \pi}{\partial Z \partial S} \right]. \quad (42)$$

For the HJM-model we again use the state variables X_t and $B_t(T)$, in addition to the risky assets. Define

$$\sigma_a = \begin{pmatrix} a_1(t,T) \\ \vdots \\ a_d(t,T) \end{pmatrix}.$$

The drift process under Q by considering Π_t^g as a function of X_t, S_t and t is

$$\frac{\partial \Pi_t^g}{\partial X}\left[r_t - \frac{1}{2}\sigma_a^T \sigma_a \right] + \frac{\partial \Pi_t^g}{\partial t} + \frac{\partial \Pi_t^g}{\partial S}^T S_t r_t + \frac{1}{2}\sigma_a \sigma_a^T \frac{\partial^2 \Pi_t^g}{\partial X^2} + \sigma_a^T \sigma_S^T \frac{\partial^2 \Pi_t^g}{\partial S \partial X} + \frac{1}{2}\mathrm{tr}\left[\sigma_S \sigma_S^T \frac{\partial^2 \Pi_t^g}{\partial S^2} \right]$$

and the drift process under Q by considering Π_t^g as a function of $B_t(T)$, S_t and t is

$$\frac{\partial \Pi_t^g}{\partial B} B_t(T) r_t + \frac{\partial \Pi_t^g}{\partial t} + \frac{\partial \Pi_t^g}{\partial S}^T S_t r_t + \frac{1}{2}\sigma_a \sigma_a^T B_t(T)^2 \frac{\partial^2 \Pi_t^g}{\partial B^2} + \sigma_a^T \sigma_S^T \frac{\partial^2 \Pi_t^g}{\partial S \partial B} B_t(T) + \frac{1}{2}\mathrm{tr}\left[\sigma_S \sigma_S^T \frac{\partial^2 \Pi_t^g}{\partial S^2} \right].$$

We now derive our differential equation for the state variable model by equating the drift process of equation (41) by the drift process in expression (42),

$$\frac{\partial \Pi_t^g}{\partial t} = r_t \Pi_t^g - a_g(t) - \sum_{h \neq g} \mu_{gh}(t)[a_{gh}(t) + \Pi_t^h - \Pi_t^g] - \left\{ \frac{\partial \Pi_t^g}{\partial Z}^T (\eta_Z - \sigma_Z \lambda) + \right.$$

$$\left. \frac{\partial \Pi_t^g}{\partial S}^T S_t r_t + \frac{1}{2} \mathrm{tr}\left[\sigma_Z \sigma_Z^T \frac{\partial^2 \Pi_t^g}{\partial Z^2} + \sigma_Z \sigma_S^T \frac{\partial^2 \Pi_t^g}{\partial S \partial Z} \right] + \frac{1}{2} \mathrm{tr}\left[\sigma_S \sigma_S^T \frac{\partial^2 \Pi_t^g}{\partial S^2} + \sigma_S \sigma_Z^T \frac{\partial^2 \Pi_t^g}{\partial Z \partial S} \right] \right\}. \qquad (43)$$

In this section there are two sources of uncertainty, first the state variables, then the risky securities. Compared to equation (19) which also is based on a factor model, we now have two more (dot-products of) terms stemming from the risky securities and additional two representing the covariation between the state variables and the risky securities. By letting the insurance specific factors, represented by a_g, a_{gh} and μ_{gh}, equal zero this equation is reduced to equation (37) which is the fundamental partial differential equation from the theory of financial economics. In the case of a deterministic interest rate (and no other state variables) and only one risky security this equation is reduced to the similar expression found in Persson (1994b).

The similar equations of the to variations of the HJM-model are

$$\frac{\partial \Pi_t^g}{\partial t} = r_t \Pi_t^g - a_g(t) - \sum_{h \neq g} \mu_{gh}(t)[a_{gh}(t) + \Pi_t^h - \Pi_t^g] -$$

$$\left\{ \frac{\partial \Pi_t^g}{\partial X}\left[r_t - \frac{1}{2}\sigma_a^T \sigma_a \right] + \frac{\partial \Pi_t^g}{\partial S}^T S_t r_t + \frac{1}{2}\sigma_a \sigma_a^T \frac{\partial^2 \Pi_t^g}{\partial X^2} + \sigma_a^T \sigma_S^T \frac{\partial^2 \Pi_t^g}{\partial S \partial X} + \frac{1}{2}\mathrm{tr}\left[\sigma_S \sigma_S^T \frac{\partial^2 \Pi_t^g}{\partial S^2} \right] \right\} \qquad (44)$$

and

$$\frac{\partial \Pi_t^g}{\partial t} = r_t \Pi_t^g - a_g(t) - \sum_{h \neq g} \mu_{gh}(t)[a_{gh}(t) + \Pi_t^h - \Pi_t^g] -$$
$$\left\{ \frac{\partial \Pi_t^g}{\partial B} B_t(T) r_t + \frac{\partial \Pi_t^g}{\partial S}^T S_t r_t + \frac{1}{2}\sigma_a \sigma_a^T B_t(T)^2 \frac{\partial^2 \Pi_t^g}{\partial B^2} + \sigma_a^T \sigma_S^T \frac{\partial^2 \Pi_t^g}{\partial S \partial B} B_t(T) + \frac{1}{2}\mathrm{tr}\left[\sigma_S \sigma_S^T \frac{\partial^2 \Pi_t^g}{\partial S^2} \right] \right\}, \qquad (45)$$

respectively. These equations are our counterpart to the Thiele equations of the actuarial sciences, but contrary to the classical equation these deal with financial risk. We see that the terms stemming from the financial risk are on different forms for the two different term structure models and also differ with respect to how we choose the state variable in the HJM-model. All three equations (43), (44) and (45) involve the terms from the classical Thiele equation and, in addition, the terms $\frac{\partial \Pi_t^{f\,T}}{\partial S} S_t r_t$ and $\frac{1}{2}\mathrm{tr}\left[\sigma_S \sigma_S^T \frac{\partial^2 \Pi_t^f}{\partial S^2}\right]$ which are related to the risky assets. The terms involving the first and second order derivatives with respect to the state variables are of different form in the three equations. All equations involve terms representing the covariation between the risky assets and the state variables.

These equations may be of importance in the construction of certain complex insurance products, see, e.g., Ramlau-Hansen (1990). In cases where closed form solutions of these equations are not available, numerical solutions may be obtained from the Feyman-Kac formula.

6. Examples

6.1 The state variable model

First we describe a special case of the model presented in Section 4. Then we give an example of the pricing in the financial market before we also incorporate mortality factors. In this example there is one factor, the short-term interest rate, and one risky security, so the bond price is a function of the interest rate and time.

The short interest rate is given by an Ornstein-Uhlenbeck process

$$r_t = r_0 + \int_0^t q(m - r_u)du + \int_0^t v dW_u^1,$$

where r_0, q ,v and m are constants. The price process for the risky security is assumed to be a geometric Brownian motion:

$$S_t = S_0 + \int_0^t \eta S_u du + \int_0^t S_u \sigma_1 dW_u^1 + \int_0^t S_u \sigma_2 dW_u^2, \tag{46}$$

where S_0, η, σ_1 and σ_2 are constants and W^1 and W^2 are independent Brownian motions under P. Observe that the value of the risky asset depends on one more source of uncertainty than the bond.

Now we turn to the construction of the equivalent martingale measure. We assume that the market price of risk related to the first Brownian motion is a constant, i.e., $\lambda_t^1 = \lambda^1$, and let

$$\lambda_t^2 = \frac{1}{\sigma_2}[\eta - r_t - \sigma_1 \lambda^1].$$

Here λ_t^2 can be interpreted as the market price of risk at time t corresponding to the second Brownian motion W_t^2.

Define Q by

$$\frac{dQ}{dP} = \exp\left(- \int_0^T \lambda_u^1 dW_u^1 - \int_0^T \lambda_u^2 dW_u^2 - \frac{1}{2}\int_0^T [(\lambda_u^1)^2 + (\lambda_u^2)^2]\, du \right). \tag{47}$$

One can verify that $E\left[\frac{dQ}{dP}\right] = 1$ and $Var\left[\frac{dQ}{dP}\right] < \infty$ and that $\frac{dQ}{dP}$ is strictly positive almost surely so that Q is well-defined. The processes for the interest rate, for the risky security and for the bonds under Q are

$$r_t = r_0 + \int_0^t q(d - r_u)du + \int_0^t v d\hat{W}_u^1 \ ,$$

where $d = m - \frac{1}{q} v\lambda^1$,

$$S_t = S_0 + \int_0^t r_u S_u du + \int_0^t S_u \sigma_1 d\hat{W}_u^1 + \int_0^t S_u \sigma_2 d\hat{W}_u^2 \ ,$$

where $\hat{W}^1$ and $\hat{W}^2$ are independent Brownian motions under Q and

$$B_t(s) = B_0 + \int_0^t r_u B_u(s)du + \int_0^t v \frac{\partial B}{\partial r} d\hat{W}_u^1 \quad \text{for } t < s < T,$$

respectively. We observe that the instantaneous expected return of the risky asset and the bonds equal the short rate r_t. By taking S_t, r_t and λ^1 as primitives we have constructed an equivalent martingale measure Q.

The solutions of the stochastic differential equations for r_t and S_t are

$$r_t = d + (r_0 - d)e^{-qt} + \int_0^t ve^{-q(t-s)} d\hat{W}_s^1,$$

and

$$S_t = S_0 \exp\left(\int_0^t r_u du - \tfrac{1}{2}(\sigma_1^2 + \sigma_2^2)t + \int_0^t \sigma_1 d\hat{W}_u^1 + \int_0^t \sigma_2 d\hat{W}_u^2\right).$$

This can easily be verified using Itô's lemma. Now we want to derive some properties of the money market account. First we introduce a more compact notation by letting

$$R_t = \int_0^t r_u du, \quad \Lambda_t = d\cdot t + \tfrac{1}{q}(r_0 - d)(1 - e^{-qt}), \ H_t = \frac{1 - e^{-qt}}{q} \text{ and } \Gamma_t = \frac{v^2}{q^2}\left(t - H_t - \tfrac{1}{2q}H_t^2\right).$$

Observe that $\beta_t = e^{R_t}$, so the quantity R_t contains the same information as the money market account. It follows that

$$R_t = \Lambda_t + \int_0^t \frac{v}{q}(1 - e^{-q(t-s)})d\hat{W}_s^1 , \tag{48}$$

so $R_t \sim N(\Lambda_t, \Gamma_t)$, where $Z \sim N(\mu, \sigma^2)$ denotes a normally distributed random variable Z with expectation μ and variance σ^2.

Now we turn to the pricing of unit discount bonds. The price at time zero of a discount bond with maturity at t is

$$B_0(t) = E_0^Q[e^{-R_t}] = e^{-\Lambda_t + \frac{1}{2}\Gamma_t}. \tag{49}$$

The above formula is from Vacisek (1977) in a slightly different notation. We notice that this formula depends on the market price of risk of the first factor λ^1 through Λ_t which depends on d.

We consider the benefit payable at time t given by $a(S_t) = S_t \vee G_t$, where G_t is a deterministic function of t. This may be a realistic contract of a unit-linked insurance. Also the benefit is a function of a traded asset so contingent claims theory applies. We let $\dot{\pi}_t(T)$ denote the market value at time t of the benefit $a(S_T) = S_T \vee G_T$ payable at time T. From the pricing formula (31) and the definition of R_t we obtain the market price at time zero for a benefit expiring at time t,

$$\dot{\pi}_0(t) = E_0^Q[e^{-R_t}(G_t \vee S_t)]. \tag{50}$$

By noticing that $S_t e^{-R_t}$ and $G_t e^{-R_t}$ are lognormally distributed the above expression can be written as

$$\dot{\pi}_0(t) = E_0^Q[G_t e^{-\Lambda_t + x} \vee S_0 e^{-\frac{1}{2}(\sigma_1^2 + \sigma_2^2)t + y}],$$

where x and y are bivariate normally distributed with covariance matrix

$$\Sigma_{x,y} = \begin{pmatrix} \frac{v^2}{q^2}\left(t - H_t - \frac{1}{2q}H_t^2\right) & \frac{v\sigma_1}{q}(H_t - t) \\ \frac{v\sigma_1}{q}(H_t - t) & (\sigma_1^2 + \sigma_2^2)t \end{pmatrix} \equiv \begin{pmatrix} \Gamma_t & \Psi_t \\ \Psi_t & \Delta_t \end{pmatrix}. \tag{51}$$

In terms of the latter definition the problem can be restated as follows

$$\dot{\pi}_0(t) = E_0^Q[G_t e^{-\Lambda_t + x} \vee S_0 e^{-\frac{1}{2}\Delta_t + y}]. \tag{52}$$

The rest is simply a matter of algebra and properties of the bivariate normal distribution and we present the result below in Lemma 6.1.

6.2 The HJM-model

The instantaneous forward rates are given by a family of Itô-processes on the following form

$$f_t(s) = f_0(s) + \int_0^t \alpha_u(s)du + \int_0^t \sigma_u(s)dW_u^1,$$

for $u \in [0,T]$, where $\sigma_t(s)$ is a deterministic and bounded function. The price processes for the bonds are

$$B_t(u) = B_0(u) + \int_0^t [r_s + b(s,u)]B_s(u)ds + \int_0^t a(s,u)B_s(u)dW_u^1,$$

where b(t,u) and a(t,u) are defined in Section 2.

The price process of the risky security is also here given in expression (46).

We assume that the market price of risk related to the first source of uncertainty λ_t^1 is an arbitrary bounded process. In practical applications λ_t^1 can be estimated from the forward drift rate restriction in expression (11) since both $\alpha_t(u)$ and $\sigma_t(u)$ for $u \geq t$ are observable. It follows that the market price of risk related to the second source of uncertainty is given by

$$\lambda_t^2 = \frac{1}{\sigma_2}[\eta - r_t - \sigma_1\lambda_t^1].$$

Then we define the measure Q by an expression similar to expression (47).

Our next task is to derive the dynamics for the money market account. Also here we do that through the quantity R_t, defined previously. It follows that

$$R_t = \ln[B_0(t)] + \frac{1}{2}\int_0^t a(s,t)^2 ds - \int_0^t a(s,t) d\hat{W}_s^1 .$$

This quantity is also for this model normally distributed.

We consider the same benefit payable at time t given by $a(S_t) = S_t \vee G_t$, where G_t is a deterministic function of t. By the same arguments as for the state variable model it follows that

$$\dot{\pi}_0(t) = E_0^Q[G_t B_0(t) e^{\frac{1}{2}\int_0^t a(s,t)^2 ds + x} \vee S_0 e^{-\frac{1}{2}(\sigma_1^2+\sigma_2^2)t + y}],$$

where x and y are bivariate normally distributed with covariance matrix

$$\Sigma_{x,y} = \begin{pmatrix} \int_0^t a(s,t)^2 ds & \sigma_1 \int_0^t a(s,t) ds \\ \sigma_1 \int_0^t a(s,t) ds & (\sigma_1^2+\sigma_2^2)t \end{pmatrix} \equiv \begin{pmatrix} \Gamma_t & \Psi_t \\ \Psi_t & \Delta_t \end{pmatrix}. \qquad (53)$$

In terms of the latter definition the problem can be restated as follows

$$\dot{\pi}_0(t) = E_0^Q[G_t e^{-\frac{1}{2}\Gamma_t + x} \vee S_0 e^{-\frac{1}{2}\Delta_t + y}] . \qquad (54)$$

6.3 A formula for an asset expiring at time t

We present the solution of the described valuation problem in the following lemma.

Lemma 6.1

Consider the problem (52) and the described state variable model and the problem (54) in the described HJM-model. The market price at time zero of the claim $a(S_t) = S_t \vee G_t$ is for both models given by the expression

$$\dot{\pi}_0(t) = S_0\Phi(d_t^1) + G_t B_0(t)\Phi(-d_t^2), \qquad (55)$$

where

$$d_t^1 = \frac{1}{\Theta_t}\left(\frac{1}{2}\Theta_t^2 + \ln\left(\frac{S_0}{B_0(t)G_t}\right)\right),$$

$$d_t^2 = d_t^1 - \Theta_t,$$

$$\Theta_t = \sqrt{\Gamma_t + \Delta_t - 2\Psi_t},$$

where $B_0(t)$ is given in (49) for the state variable model and follows from the initial term structure in the HJM-model and Φ denotes the standard normal distribution function. Furthermore, Θ_t is given by (53) and (53) for the two models, respectively.

Proof:
Formula (55) follows by straight-forward calculations involving properties of bivariate normal random variables. ❑

The resulting formula depends on 10 parameters for the state variable model and 9 for the HJM-model. These are: 4 parameters of the interest rate process (v, r_0,q,m), 3 parameters of the stock price process (S_0, σ_1, σ_2), the market price of risk (λ^1), the guarantee (G_t) and time to expiration (t). The parameter λ^1 does not enter in the HJM-model. We also note that it does not depend on η, the instantaneous expected return of the risky security. In particular, it is worth noticing that it depends on λ_1 through $B_0(t)$ in the state variable model. After the bond prices are determined, a process which requires knowledge of λ_1, the contingent claim pricing is done in terms of the bond prices and consequently introduces dependence between λ_1 and the price of the contingent claim.

Rabinovitch (1989) values the contingent claim $(S_t - G_t) \vee 0$ in a similar setting. His formula is $S_0\Phi(d_t^1) - G_tB_0(t)\Phi(d_t^2)$. This corresponds to the relationship between the value of $(S_t \vee G_t)$ and $[(S_t - G) \vee 0]$ when the interest rate is deterministic given by the Black and Scholes formula (1973) and the results in Persson (1994a). We refer to the last reference for a discussion.

6.4 Market prices of insurance contracts

Let ${}_TC_x$ denote the single premium of a contract with benefit $a(S_T) = S_T \vee G_T$ if the insured is alive at time T and 0 if not. This contract is a unit-linked version of the traditional pure endowment insurance.

By the independence between the financial market and the state of the policy, risk neutrality with respect to transition risk and the expression (55) it follows that

$$ {}_TC_x = {}_Tp_x S_0 \Phi(d_T^1) + {}_Tp_x G_T B_0(T) \Phi(-d_T^2). \tag{56} $$

where ${}_Tp_x$ represents the probability for an x-year old insured to be alive at time T.

We now consider a unit-linked version of a term insurance which entitles the insured to the benefit $a(S_t) = S_t \vee G_t$ upon death before time T. Let $C^1_{x:\overline{T}|}$ denote the single premium. By the same arguments as above we get that

$$ C^1_{x:\overline{T}|} = \int_0^T [S_0 \Phi(d_t^1) + G_t B_0(t) \Phi(-d_t^2)]\ {}_tp_x \mu_{x+t} dt, \tag{57} $$

where ${}_tp_x \mu_{x+t}$ is common actuarial notation for the probability density function of an x-year old insurance customer's remaining life time.

Simple calculations show that formula (56) and (57) in the case where the interest rate is constant are identical to Theorem 1 and 2 from Aase and Persson (1994).

6.5 Introduction of another risky asset

The model described depends on the market price of risk λ_t^1, which some places in the literature is called a utility dependent parameter meaning that it will depend on the agents' attitude towards financial risk. It is interesting to note that by introducing another risky asset without increasing the number of sources of uncertainty, we would be able to also determine λ_t^1 in terms of the parameters of the model. For example if we let the second risky asset also be given by a geometric Brownian motion,

$$N_t = N_0 + \int_0^t \gamma N_u du + \int_0^t N_u s_1 dW_u^1 + \int_0^t N_u s_2 dW_u^2,$$

where γ, N_0, s_1 and s_2 are constants, the market prices of risk are

$$\lambda_t^1 = \frac{s_2(\eta - r_t) - \sigma_2(\gamma - r_t)}{s_2\sigma_1 - s_1\sigma_2}$$

and

$$\lambda_t^2 = \frac{\sigma_1(\gamma - r_t) - s_1(\eta - r_t)}{s_2\sigma_1 - s_1\sigma_2},$$

for the two sources of risk, respectively. This topic is discussed in Brennan and Schwartz (1979c).

7. Concluding remarks

This paper deals with pricing of life insurance contracts in a model with stochastic interest rates and generalizes the results of Persson (1994c) in two ways. First, generalizations of the Thiele equation based on the HJM-model are presented and then, a pricing principle for insurance contracts and versions of the Thiele equation are presented for random benefits. The derivations are based on the assumption that the state of the policy is independent of the financial market and that no arbitrage opportunities exist in the financial market. The market price was found as an expectation under a probability measure, derived by the use of economic theory, and different from the originally given probability measure.

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