Discussion paper

Income risk aversion with quantity constraints

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4/2-2010

Abstract: In this paper, I consider a consumer with a concave utility function over n commodities and trace out the consequences of quantity constraints on product markets for the consumer's aversion towards income risk. I show that the effect can be decomposed in a cardinal and ordinal term, that both terms may add up to a non-linear effect on the coefficient of relative risk aversion, and that a severely rationed consumer may even become less risk averse then when unconstrained.

Keywords: household demand**,** income risk aversion, quantity constraints.

JEL Classification: D11; D81.

1 Introduction

A consumer facing an uncertain income prospect will evaluate this prospect in terms of the opportunities for using this income. These are defined by the prices of the different commodities she cares about, and possibly other market restrictions, like quotas. If the consumer is certain w.r.t. the terms at which she can trade, and if these trading opportunities do not change, then for the purpose of analyzing her attitude towards a risky income, it suffices to work with the standard single argument Bernoulli utility function summarizing the optimal trading for any income level m.

The purpose of this paper is to analyze the change in the consumer's willingness to bear income risk when her trading opportunities get restricted because of quantity constraints. Understanding the impact of such constraints is useful for two reasons. On descriptive grounds, a consumer may not always be successful in realizing her notional trades. Prices can be regulated or sticky on other grounds, requiring that available supply is allocated according to a quantity rationing mechanism–an example is health care in countries with a National Health Service. Also, many household services are derived from durable household goods which are purchased in lumpy amounts. Marginal adjustments of these goods are very costly, implying that a household is committed to a service flow that may differ from the ideal amount. Understanding the consequences of such trading constraints on the willingness to bear income risks is required for correctly explaining and interpreting the variation in empirical measures of risk aversion (cf Barski *et al.* 1997).

Second, such an understanding is also useful to sharpen the normative arguments in favour of price rigidities. For example, it has been argued–see Drèze and Gollier (1993)–that downward wage rigidities can implement second-best Pareto efficient allocations when labour market contracts are incomplete. Compared with a situation with competitive spot markets for labour, these rigidities balance the gain in risk sharing efficiency with a loss in allocational efficiency. However, these arguments are made on the assumption that the employment status of a worker does not bear directly upon her willingness to accept risks. If it does, then the normative role of wage rigidities may need to be reexamined.

The subject of the present paper is related to a recent paper by Gollier (2009) who considers a general dynamic choice problem and asks whether an agent who can choose a lottery and take some action after observing the outcome of the lottery, has a larger willingness to bear risks than an

agent who has to commit to an action before observing the lottery outcome. Gollier derives a set of sufficient conditions for the flexible context to lead to a higher risk tolerance. He then examines how rigidities may induce a household to *more* risk-prone behaviour in portfolio allocation and/or savings decisions. While both papers address a similar question, their focus is very different. Gollier's focus is on decision taking under risk: does the ability to postpone an action until the uncertainty is resolved always lead to more risk taking? In the present paper, I examine the effect of one particular set of constraints–quantity constraints on purchased levels of goods and services– on the willingness to accept small income risks, and decompose it in terms of consumer preferences.

Section 2 gives a reminder of the consumer's decision problem, its properties, and the definition of her willingness to bear income risks. In section 3, I introduce quantity constraints and derive the consumer's aversion w.r.t income risks and its relation to her aversion when quantity constraints are absent. Section 4 looks at the 'second order' effects of quantity constraints and shows, by means of examples, that these effects may exceed the first order one. Concluding remarks are presented in Section 5.

2　Income risk aversion without quantity constraints

A consumer cares about n commodities whose quantities are given by the bundle $q \in R_+^n$. Let the price vector be certain and given by $p \in R_+^n$. The consumer's income $\widetilde{m}$, however, is random with expectation m and variance σ_m^2. Her preferences are represented by a cardinal Bernoulli utility function $u(\cdot)$ which is monotone and strongly concave.

Suppose that the consumer is informed about the income draw before she makes her consumption decision. Suppose as well that the income draw coincides with the expected income m.[1] Her problem is then to solve

$$\max_q u(q) \text{ s.t. } p'q = m \ \ (\lambda).$$

Let the unique solution be given by the bundle $q(p,m)$ satisfying the first order conditions[2]

$$u_q(q(p,m)) = \lambda(p,m) \ p, \tag{1}$$

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[1]This is for notational convenience, since I will later evaluate the risk aversion measures at $m = \mathrm{E}\widetilde{m}$.

[2]Subscripts with u (and with λ and v below) denote derivatives.

where $\lambda(p, m)$ is the equilibrium value of the Lagrange multiplier.

The local properties of $q(p, m)$ are well known but repeated here for future reference:

$$\text{(i) } p'q_m = 1, \text{ (ii) } \frac{\partial q}{\partial p'} = K - q_m q', \text{ (iii) } K = K', \tag{2}$$

$$\text{(iv) } Kp = 0, \text{ and (v) } y'Ky < 0 \text{ for } y \neq \alpha p \text{ (α real scalar)},$$

where q_m stands for the vector of income effects $\frac{\partial q}{\partial m}$. Expression (2-ii) is the Slutsky decomposition. A similar decomposition of the price effect on the marginal utility of income, λ, is

$$\frac{\partial \lambda}{\partial p} = -\lambda_m q - \lambda q_m. \tag{3}$$

The first *rhs* term is a real income effect that can be neutralized by an appropriate change in income. The second *rhs* term is a substitution effect: the change in the marginal utility of income when the consumer is compensated so as to remain at the same utility level (Silberberg, 1978, pp 260-1).

The indirect utility function is defined as $v(p, m) \overset{\text{def}}{=} u(q(p, m))$ and satisfies $v_m = \lambda(p, m)$. By assumption, the Hessian of $u(\cdot)$, u_{qq}, has full rank. Then it can be shown (see, e.g., Barten, 1977) that

$$K = \lambda u_{qq}^{-1} - \frac{\lambda}{\lambda_m} q_m q_m'. \tag{4}$$

Using the adding-up and homogeneity conditions (2-(i) and (iv)), we get $v_{mm} = \lambda_m = q_m' u_{qq} q_m$. Hence, the Arrow-Pratt coefficient of absolute risk aversion, measuring twice the risk premium the consumer is willing to pay (per unit of variance) to get rid of the income risk, is given by

$$A(p, m) \overset{\text{def}}{=} -\frac{v_{mm}}{v_m} = -\frac{q_m' u_{qq} q_m}{\lambda}. \tag{5}$$

Since $\lambda = q_m' u_q$, expression $-\frac{q_m' u_{qq} q_m}{q_m' u_q}$ may be added to Hanoch's (1977, Theorem 1) list of alternative representations of relative risk aversion.

3 (Weakly binding) quantity constraints

Suppose now that $q' = (x', z')$, $p' = (p_x', p_z')$ and that the consumer can no longer choose the sub-bundle z which is fixed at $\overline{z}$. Her problem then turns into

$$\max_x u(x, \overline{z}) \text{ s.t. } p_x' x + p_z' \overline{z} = m \quad (\lambda^r).$$

Let the solution be given by $x^r(p, m, \bar{z})$, satisfying the first order condition $u_x = \lambda^r p_x$. The indirect utility function is now $v^r(p, m, \bar{z}) \overset{\text{def}}{=} u(x^r(p, m, \bar{z}), \bar{z})$. Repeating the procedure of section 2, the coefficient of absolute risk aversion for income risk is given by

$$A(p, m|\bar{z}) \overset{\text{def}}{=} -\frac{v_{mm}^r}{v_m^r} = -\frac{x_m^{r\prime} u_{xx} x_m^r}{\lambda^r}. \tag{6}$$

Rather than comparing $x_m^{r\prime} u_{xx} x_m^r$ with $q_m' u_{qq} q_m$ in order to relate $A(p, m|z)$ to $A(p, m)$, I will use a 'virtual price' approach (cf Neary and Roberts, 1980). This consists in defining a virtual price vector π_z for the sub-bundle z, and adjusting the consumer's income to $m + (\pi_z - p_z)'\bar{z}$ such that the consumer's notional demand for that bundle coincides with the imposed quantities, yielding the following identities:

$$\bar{z} \equiv z(p_x, \pi_z, m + (\pi_z - p_z)'\bar{z}), \tag{7}$$
$$x^r(p_x, p_z, m, z) \equiv x(p_x, \pi_z, m + (\pi_z - p_z)'\bar{z}), \tag{8}$$
$$v^r(p_x, p_z, m, z) \equiv v(p_x, \pi_z, m + (\pi_z - p_z)'\bar{z}). \tag{9}$$

Implicitly differentiating (7) and using the Slutsky equation (2-ii) shows that

$$\frac{\partial \pi_z}{\partial m} = -K_{zz}^{-1} z_m. \tag{10}$$

Intuitively, the consumer would like to respond to a marginal income increase by $\mathrm{d}z = z_m \mathrm{d}m$. However, the quantity constraint prevents her from doing so, and therefore the virtual price of that bundle has to go up with $-K_{zz}^{-1}\mathrm{d}z$.

The marginal utility of income is then

$$v_m^r \equiv (v_{\pi_z}' + v_m \bar{z}')\frac{\partial \pi_z}{\partial m} + v_m = v_m,$$

where the equality sign follows from Roy's identity. Differentiating one more time w.r.t. m yields

$$\begin{aligned} v_{mm}^r &\equiv v_{m\pi_z}'\frac{\partial \pi_z}{\partial m} + v_{mm}(1 + \bar{z}'\frac{\partial \pi_z}{\partial m}) \\ &= -v_{mm}\bar{z}'\frac{\partial \pi_z}{\partial m} - v_m z_m'\frac{\partial \pi_z}{\partial m} + v_{mm}(1 + \bar{z}'\frac{\partial \pi_z}{\partial m}) \\ &= v_{mm} - v_m z_m'\frac{\partial \pi_z}{\partial m}, \end{aligned}$$

where the second equality follows upon using (3). Since $\frac{\partial \pi_z}{\partial m} = -K_{zz}^{-1} z_m$, the coefficient of absolute risk aversion with quantity constraints is

$$-\frac{v_{mm}^r(p,m)}{v_m^r(p,m)} = -\frac{v_{mm}(p_x, \pi_z, m + (\pi_z - p_z)'\overline{z})}{v_m(p_x, \pi_z, m + (\pi_z - p_z)'\overline{z})} - z_m' K_{zz}^{-1} z_m. \qquad (11)$$

Assume first that the quantity constraints $\overline{z}$ are weak, i.e., that they exactly coincide with $z(p,m)$, the levels the consumer would have chosen if her income takes the expected value. Then $\pi_z = p_z$ and the first *rhs* term reduces to $-\frac{v_{mm}(p,m)}{v_m(p,m)}$. The following proposition immediately follows:

Proposition 1 *If the quantity constraints are just binding,*

$$A(p, m | \overline{z}) = A(p, m) - z_m' K_{zz}^{-1} z_m.$$

Since K_{zz} is a negative definite matrix, so is its inverse. Therefore the quadratic form $z_m' K_{zz}^{-1} z_m$ is strictly negative (and entirely ordinal).

This result can be explained as follows. Ideally, the consumer would like to respond to a small deviation in income, dm, from its expected value, by increasing the demand for z commodities with $dz = z_m dm$. Since this is not feasible, the virtual price vector of z-goods increases with $d\pi_z = -K_{zz}^{-1} z_m dm$. This price increase has a double effect on the marginal utility of income: $d\lambda = -\lambda_m z' d\pi_z - \lambda z_m' d\pi_z$. The first effect is the change in marginal utility because real income falls, while the second effect is the compensated price effect on marginal utility. The first effect is eliminated, however, because the consumer's virtual income, $m + (\pi_z - p_z)'\overline{z}$, is by definition adjusted with exactly $z d\pi_z$. Hence, the change in marginal utility due to the virtual price change is $\lambda z_m' K_{zz}^{-1} z_m dm$, and the relative change in marginal utility is $z_m' K_{zz}^{-1} z_m dm$.

The intuition for Proposition 1 comes about most clearly in the case where the utility function is quasi linear in one good, e.g., in leisure. Because *preferences* are quasi-linear, all exogenous income risk is then absorbed by leisure. Since also the *utility function* is linear in leisure, the consumer is risk neutral w.r.t. this income risk. But if she faces a binding quantity constraint on her labour supply, the exogenous income risk is absorbed by the consumption of other goods, whose marginal utility is strictly falling. Hence, the quantity constraint turns the consumer into a strictly risk averse person w.r.t. income risk.

Proposition 1 is a generalization of a result by Drèze and Modigliani (1972). They considered a consumer deciding about the amount to save while

facing an uncertain future income. They compared the attitudes towards income risk under two settings: (i) a timeless income risk where the consumer is informed about her income draw before making her savings decision, and (ii) a temporal income risk where the savings decision is made before the income draw is known. Drèze and Modigliani (1972, eq 2.9) showed that the risk aversion for temporal income risks exceeds that for timeless income risks by an ordinal term positively related to the (squared) income effect on current consumption and reciprocally related to the degree of substitution between current and future consumption.

A similar relation between the coefficients of relative risk aversion obtains by using the Rotterdam parameterization for the income and substitution effects (Theil, 1976). Using a ˆ above a vector to denote the diagonal matrix with the vector as its main diagonal elements, we can write

$$b_z \stackrel{\text{def}}{=} \widehat{p}_z z_m, \text{ and } S_{zz} \stackrel{\text{def}}{=} \frac{1}{m} \widehat{p}_z K_{zz} \widehat{p}_z.$$

Defining the Arrow-Pratt coefficient of relative risk aversion without and with quantity constraints as $R(p, m) \stackrel{\text{def}}{=} A(p, m) \cdot m$ and $R(p, m | \overline{z}) \stackrel{\text{def}}{=} A(p, m | \overline{z}) \cdot m$, respectively, the next result immediately follows:

Corollary 1 *The coefficient of relative risk aversion under quantity constraints is given by*

$$R(p, m | \overline{z}) = R(p, m) - b_z' S_{zz}^{-1} b_z. \tag{12}$$

The next result (proven in appendix) shows that these measures of risk aversion w.r.t. income are monotone in the number of commodities subject to a quantity constraint.

Corollary 2 *Suppose that the consumption bundle q is partitioned as (x, y, z). Then*
$$A(p, m | \overline{y}, \overline{z}) \geq A(p, m | \overline{z}).$$

Up til now, I have assumed that the quantity constraints are just binding, so that $\pi_z = p_z$. In the next section, I relax this assumption.

4 Strictly binding quantity constraints

If quantity constraints are strictly binding, then $\pi_i > (<)p_i$ depending on whether the consumer's notional demand for commodity i exceeds (falls short

of) the quantity constraint. The first *rhs* term in (11), which may be written as $A(p_x, \pi_z, m + (\pi_z - p_z)'\bar{z})$, then no longer coincides with $A(p, m)$, and we need to account for the influence of $\bar{z}$ on this measure of absolute risk aversion. For simplicity, I focus in the remainder on the case where z is a scalar. Now (12) can be written as

$$
\begin{aligned}
R(p, m|\bar{z}) &= A(p_x, \pi_z, m^v) \cdot m^v \cdot \frac{m}{m^v} - \frac{(b_z^v)^2}{s_{zz}^v} \cdot \frac{m}{m^v} \\
&= R(p_x, \pi_z, m^v) \cdot \frac{m}{m^v} - \frac{(b_z^v)^2}{s_{zz}^v} \cdot \frac{m}{m^v}.
\end{aligned}
$$

where the superscript v denotes evaluation at the virtual price and income level, and $m^v \stackrel{\text{def}}{=} m + (\pi_z - p_z)\bar{z}$. The influence of $\bar{z}$ on the *rhs* factors is intricate. In order to proceed, I assume that the utility function is homogenous of degree $1 - \gamma$. Then $R(p_x, \pi_z, m^v) = \gamma$ and the marginal budget share b_z^v coincides with the average budget share $w_z^v \stackrel{\text{def}}{=} \frac{\pi_z \bar{z}}{m^v}$. Moreover, the compensated own price elasticity of good z, given by $s_{zz}^v w_z^v$, can be written as $-\sigma^v w_z^v (1 - w_z^v)$, where σ^v denotes the elasticity of substitution between good z and the other goods (evaluated at the virtual prices and income). Then we get:

$$
R(p, m|\bar{z}) = \gamma \cdot \frac{m}{m^v} + \frac{1}{\sigma^v} \frac{w_z^v}{1 - w_z^v} \cdot \frac{m}{m^v}. \tag{13}
$$

Thus, there are two factors that regulate the relationship between $R(p, m|\bar{z})$ and γ. One is the degree of substitutability between the z-good and the other commodities. The lower this degree, the higher is the second, ordinal, term. The other is the relationship between nominal income m and virtual income m^v. The latter is given by $m + (\pi_z - p_z)\bar{z}$. If the consumer is forced to consume more than her ideal demand, then $\pi_z < p_z$ and $m^v < m$. In this case, $R(p, m|\bar{z})$ will exceed γ both because of a low degree of substitutability and because of forced consumption. On the other hand, if the consumer is rationed in the sense that her notional demand exceeds $\bar{z}$, then $\pi_z > p_z$ and $m^v > m$. The coefficient $R(p, m|\bar{z})$ may now fall below γ.

The size of the scaling factor $\frac{m}{m^v}$ depends on the quantity constraint in the following way (see appendix):

$$
\frac{\mathrm{d} \log \frac{m}{m^v}}{\mathrm{d} \log \bar{z}} \gtrless 0 \iff \frac{\pi_z - p_z}{\pi_z} \lessgtr -\frac{1}{\frac{\mathrm{d} \log z}{\mathrm{d} \log p_z}}. \tag{14}
$$

The *lhs* is the relative increase in the willingness to pay for the z-good when this good falls from the notional demand level to the rationed level.

The *rhs* is the inverse of the uncompensated price elasticity of the z-good. If z is a normal good, the *rhs* is always positive. The *lhs* is only positive when the quantity constraint lies below the notional demand for the z-good. Thus, with forced consumption, $\frac{m}{m^v} > 1$ and increasing. Forced consumption will then always make a consumer more risk averse. With rationing, $\frac{m}{m^v} < 1$ and possibly decreasing for sufficiently strong rationing. Strong rationing may turn the consumer into a more risk tolerant person then when unconstrained.

The following two examples show the behaviour of $R(p, m|\overline{z})$ and its two components when preferences are given by a symmetric CES function over two goods (see the appendix for the derivations). In both examples, $m = 10$ and $p_x = p_z = 1$, so that the notional demand for each good is 5 units. In the first example, illustrated in figure 1, $\sigma = 2$ and $\gamma = 2$. If the quantity constraint on z is less than 2.95 units, the consumer turns less risk averse than without facing any constraint at all. The figure also shows that $R(p, m|\overline{z})$ need not be monotone in $\overline{z}$, and here the decreasing part is due to $\frac{m}{m^v}$ falling for low levels of $\overline{z}$.

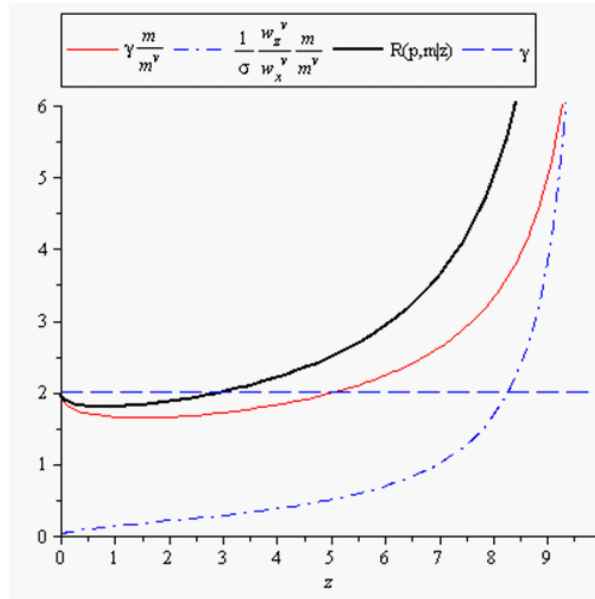

Figure 1

The next example, shown in figure 2, is for $\sigma = \frac{1}{4}$ and $\gamma = \frac{1}{2}$. Again, the ideal amount of the z-good is 5 units. Now, the non-monotone behaviour of $R(p, m|\overline{z})$ is due to the ordinal term whose relative importance shrinks. If the constraint is just binding, $R(p, m|\overline{z} = 5) = 4.5$, but it drops to almost 2.5 if there is forced consumption of two additional units.
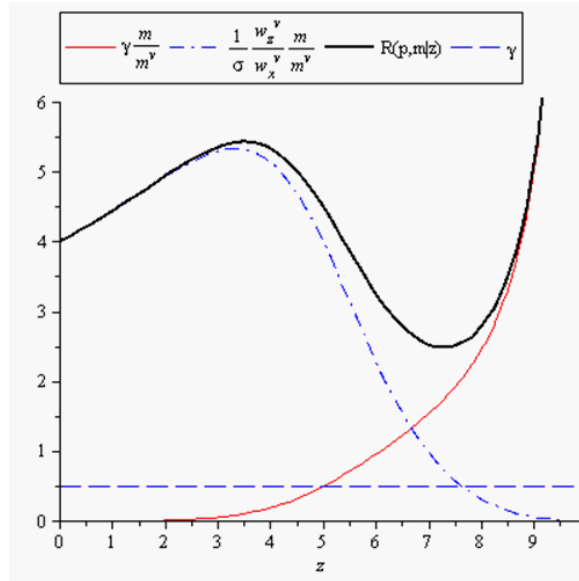
9

Figure 2

These examples illustrate that even with very 'regular' preferences (constant degree of relative risk aversion, homotheticity, a constant elasticity of substitution), quantity constraints have complicated effects, except in the neighbourhood of the notional demand for the constrained good(s). Stated differently, (income) insurance and the z-good can be both complements, as well as substitutes, depending on the level of the constraint.

5 Conclusion

In this paper, I have shown how quantity constraints on one or more goods or services have an effect on the consumer's willingness to accept income risk. Using the virtual price approach, I have decomposed the effect into an ordinal term that depends on the own price elasticities of the constrained goods, and a cardinal term that depends on the unconstrained degree of risk aversion. Numerical examples show that a rationed consumer may be less risk averse than without facing a quantity constraint, and that the relationship between the degree of relative risk aversion, and the quantity constraint can easily become very non-linear.

At a more general level, I believe these findings show that employment status of a worker/consumer, the imperfect malleability of durables, and transaction costs more generally, all may contribute to a person's willingness to bear risk, and not necessarily in a uniform manner. This suggests, e.g., the

use of a flexible form for employment status when explaining the empirical variation in risk aversion measures.

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Appendix

Proof of corollary 2.

Let $\tilde{z}' = (y', z')$. Then $R(p, m|\tilde{z}) = R(p, m) - b'_{\tilde{z}} S^{-1}_{\tilde{z}\tilde{z}} b_{\tilde{z}}$. Then using the rules for partitioned matrix inversion, it can be shown that $b'_{\tilde{z}} S^{-1}_{\tilde{z}\tilde{z}} b_{\tilde{z}} = b'_z S^{-1}_{zz} b - a'a$ where $a \overset{\text{def}}{=} -(S_{yy} - S_{yz} S^{-1}_{zz} S_{zy})^{-\frac{1}{2}} [b_y - S_{yz} S^{-1}_{zz} b_z]$. Since $R(p, m|z) = R(p, m) - b'_z S^{-1}_{zz} b$ and $a'a \geq 0$, the result follows. ■

Derivation of (14)

Since $m^v \overset{\text{def}}{=} m + (\pi_z - p_z)\overline{z}$, we have that

$$-\frac{\mathrm{d}\log m^v}{\mathrm{d}\log \overline{z}} = -\frac{\pi_z - p_z}{\pi_z} w^v_z - \frac{\mathrm{d}\log \pi_z}{\mathrm{d}\log \overline{z}} w^v_z \gtreqless 0$$

$$\Updownarrow$$

$$-\frac{\pi_z - p_z}{\pi_z} \gtreqless \frac{\mathrm{d}\log \pi_z}{\mathrm{d}\log \overline{z}}. \tag{15}$$

Totally differentiate (7) w.r.t. $\overline{z}$ to get

$$\frac{\mathrm{d}\pi_z}{\mathrm{d}\overline{z}} = k^{-1}_{zz} \left[1 - z_m(\pi_z - p_z)\right],$$

so that

$$\frac{\mathrm{d}\log \pi_z}{\mathrm{d}\log \overline{z}} = \frac{1 - \frac{\partial \log z}{\partial \log m} w^v_z \frac{\pi_z - p_z}{\pi_z}}{\frac{\partial \log z}{\partial \log p_z}\big|_{\mathrm{d}u = 0}}. \tag{16}$$

Using (16) in (15), and making use of the Slutsky identity then results in (14).

The numerical example.

Solving

$$\max_{x,z} u(x, z) = \frac{1}{1 - \gamma} [\alpha x^\rho + (1 - \alpha) z^\rho]^{\frac{1-\gamma}{\rho}}$$
$$\text{s.t.} \quad p_x x + p_z z = m$$

yields the notional demands

$$x(p_x, p_z, m) = \left(\frac{\alpha}{p_x}\right)^\sigma \left[\alpha^\sigma p^{1-\sigma}_x + (1 - \alpha)^\sigma p^{1-\sigma}_z\right]^{-1} m,$$

$$z(p_x, p_z, m) = \left(\frac{1 - \alpha}{p_z}\right)^\sigma \left[\alpha^\sigma p^{1-\sigma}_x + (1 - \alpha)^\sigma p^{1-\sigma}_z\right]^{-1} m.$$

The compensated price elasticity for good z is then

$$\frac{\partial \log z}{\partial \log p_z}\Big|_{du=0} = -\sigma(1 - w_z),$$

where the budget share w_z is given by

$$w_z = \frac{p_z z(p_x, p_z, m)}{m} = (1 - \alpha)^\sigma p_z^{1-\sigma} \left[\alpha^\sigma p_x^{1-\sigma} + (1 - \alpha)^\sigma p_z^{1-\sigma}\right]^{-1}.$$

Solving $\bar{z} = z(p_x, \pi_z, m + (\pi_z - p_z)\bar{z})$ for π_z gives

$$\pi_z = \frac{1 - \alpha}{\alpha} p_x^{1-\frac{1}{\sigma}} \left(\frac{m - p_z\bar{z}}{\bar{z}}\right)^{\frac{1}{\sigma}}.$$

This gives a virtual income

$$m^v = m + \left[\frac{1 - \alpha}{\alpha} p_x^{1-\frac{1}{\sigma}} \left(\frac{m - p_z\bar{z}}{\bar{z}}\right)^{\frac{1}{\sigma}} - p_z\right] \bar{z}.$$