E C O N O M I C S   B U L L E T I N

Testing for Granger causality in the presence of measurement errors

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Abstract

In this paper a potential problem with tests for Granger−causality is investigated. If one of the two variables under study, but not the other, is measured with error the consequence is that tests of forecastablity of the variable without measurement error by the variable with measurement error will be rejected less often than it should. Since this is not the case for the test of forecastability of the variable with measurement error by the one without there is a danger of concluding that one variable leads the other while it is in fact a feed−back relationship. The problem is illustrated by an example.

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1 Introduction

Since the ground-breaking work by Granger (1969), tests for what is now called Granger-causality have been employed to evaluate forecasting ability of one time series variable by another. Even though sometimes mixed up with the everyday-use word "causality" it can, at least rule out that one variable is causing another by the reasonable idea that for an event to cause another event it must at least precede it. Therefore it is perhaps as close as we can get in using data analysis to evaluate the philosophical concept of causality. Some variables in macroeconomics and finance are arguably measured with error. Examples are inflation, economic growth and volatility in financial markets. The purpose of this paper is to investigate the impact of measurement errors on a test of Grange causality and this is done by means of a Monte Carlo study. In the next section, a brief review of Granger causality and how to test it is given. Section 3 investigates properties of the test when one of the variables is measured with error and Section 4 presents a simulation study to illustrate the consequences of this. Section 5 concludes.

2 Granger causality

A time series variable x is said to fail to *Granger-cause* another variable y if the mean squared error (MSE) of a forecast of y_{t+s} based on $\mathcal{F}_t^{xy} = \{x_t, x_{t-1}, ..., y_t, y_{t-1}, ...\}$ is equal to the MSE of a foreacast based on $\mathcal{F}_t^{y} = \{y_t, y_{t-1}, ...\}$, $s > 0$. Tests of Granger-causality can e.g. be based on a vector autoregressive model, a multivariate MA-representation or a regression of y_{t+s} on $\mathcal{F}_t^{xy}$. See Hamilton (1994) for a review of such tests. For the purpose of this paper, the last of these approches is particularly helpful and therefore chosen. The test I consider is simply performed by testing the hypothesis

$$\begin{cases} H_0 : \alpha_1 = ... = \alpha_p = 0 \\ H_1 : \text{At least one } \alpha_j \neq 0 \end{cases} \tag{1}$$

where the parameters are given by the model

$$y_t = \alpha_0 + \alpha_1 x_{t-1} + ... + \alpha_p x_{t-p} + \beta_1 y_{t-1} + ... + \beta_p y_{t-p} + a_{y,t} \tag{2}$$

where $a_{y,t}$ is a zero mean strict white noise. The choice of the lag length, p, is of great importance for this type of analysis but is not the object of this paper. Therefore, it is assumed to be known. The possibility that $a_{y,t}$ can be autocorrelated in practice is not considered either.

The model is estimated under the null hypothesis as well. We form

$$S_1 = \frac{T(RSS_0 - RSS_1)}{RSS_1} \tag{3}$$

where

$$RSS_i = \sum_{t=1}^{T} \hat{a}_{it}^2, \tag{4}$$

and $i = 0, 1$, are the residual sum of squares for the null and alternative hypothesis, respectively. Then, under the null hypothesis, S_1 is asymptotically $\chi^2(p)$-distributed.

3 Measurement error

It is common that tests of Granger-causality are used both to investigate whether x fails to Granger-cause y and vice versa. This can be made, e.g. in order to establish whether events connected with inflation are preceding events connected with consumer behaviour or whether the opposite is true.

Assume now that x is measured with error while y is not. Thus, x can be written

$$X_t = x_t + e_t \tag{5}$$

where X_t is the observed value of x_t and e_t is a measurement error which is assumed to be a strict white noise with zero mean and variance σ_e^2. We will use the notation σ_x^2, σ_y^2 and σ_{xy} for the variances of x_t and y_t and the covariance between them, respectively. As an example we consider the case where $p = 1$ and the intercepts are zero.

$$\begin{cases} x_t = \gamma_1 x_{t-1} + \delta_1 y_{t-1} + a_{x,t} \\ y_t = \alpha_1 x_{t-1} + \beta_1 y_{t-1} + a_{y,t} \end{cases}$$

We will test the hypotheses

$$H_{0xy} : x \text{ fails to Granger-cause } y$$

and

$$H_{0yx} : y \text{ fails to Granger-cause } x$$

which can be translated as

$$H_{0xy} : \alpha_1 = 0$$

and

$$H_{0yx} : \delta_1 = 0$$

The OLS-estimator of α_1 under the alternative to H_{0xy}, H_{1xy}, has the property

$$\hat{\alpha}_1^{H_1} \xrightarrow{p} \alpha_1 \frac{\sigma_{xy}^2 - \sigma_x^2\sigma_y^2}{\sigma_{xy}^2 - (\sigma_x^2 + \sigma_e^2)\sigma_y^2} \equiv \alpha_1^{H_1} \tag{6}$$

From (6) it can be seen that $|\alpha_1^{H_1}| < |\alpha_1|$. However, this does not necessarily mean that the test statistic is performing badly. The entire distribution of the test statistic, or an approximation of it, under measurement errors would be the quantity to consider if one should theoretically evaluate size and power properties of the test under measurement errors. However, evaluating the probability limit of it, which is arguably easier, could at least hint what the measurement errors are doing asymptotically to the test statistic. As for the size and power of the test, I refer to the Monte Carlo study in Section 4.

In order to do this I consider the components of (3) and their probability limits, or rather of normalized versions of them. To do this, in turn, we will need the probability limits of the parameter estimators of β_1, γ_1 and δ_1 as well as the one for α_1. In order to study (3) we actually need these limits both under the null and alternative hypotheses.

In the same way as we obtained (6) we obtain the probability limit of $\hat{\beta}_1^{H_1}$, the estimate of β_1 under the alternative hypothesis.

$$\hat{\beta}_1^{H_1} \xrightarrow{p} \beta_1 + \alpha_1 \frac{\sigma_e^2\sigma_{xy}}{(\sigma_x^2 + \sigma_e^2)\sigma_y^2 - \sigma_{xy}^2} \equiv \beta_1^{H_1} \tag{7}$$

We see that the bias of $\hat{\beta}_1^{H_1}$ can be both positive and negative, depending on the signs of α_1 and σ_{xy}. From (6) and (7) we can deduce the probabilty limit of $\frac{1}{T}RSS_1$ under measurement error.

$$\begin{aligned} \frac{1}{T}RSS_1^{xy} \xrightarrow{p} \sigma_y^2 &- 2\alpha_1^{H_1}(\alpha_1\sigma_x^2 + \beta_1\sigma_{xy}) - 2\beta_1^{H_1}(\alpha_1\sigma_{xy} + \beta_1\sigma_y^2) \\ &+2\alpha_1^{H_1}\beta_1^{H_1}\sigma_{xy} + \left(\beta_1^{H_1}\right)^2\sigma_y^2 + \left(\alpha_1^{H_1}\right)^2(\sigma_x^2 + \sigma_e^2) \end{aligned} \tag{8}$$

Under the null hypothesis, the variable measured with error is absent and thereby $\hat{\beta}^{H_0} \xrightarrow{p} \beta_1$. This implies

$$\frac{1}{T}RSS_0^{xy} \xrightarrow{p} \sigma_y^2(1 - \beta_1^2) \tag{9}$$

If we instead test if y is useful in forecasting x, the measurement error ends up both in the dependent and independent variables. The regression

$$x_t = \gamma_1 x_{t-1} + \delta_1 y_{t-1} + a_{xt} \tag{10}$$

is estimated. The probability limits of the OLS estimators of γ_1 and δ_1 are then

$$\hat{\gamma}_1^{H_1} \xrightarrow{p} \gamma_1 \frac{\sigma_x^2 \sigma_y^2 - \sigma_{xy}^2}{(\sigma_x^2 + \sigma_e^2)\sigma_y^2 - \sigma_{xy}^2} \equiv \gamma_1^{H_1} \tag{11}$$

and

$$\hat{\delta}_1^{H_1} \xrightarrow{p} \delta_1 + \gamma_1 \frac{\sigma_e^2 \sigma_{xy}}{(\sigma_x^2 + \sigma_e^2)\sigma_y^2 - \sigma_{xy}^2} \equiv \delta_1^{H_1} \tag{12}$$

respectively. Again, there is no clearcut inequality as it is in (6). Whether $\hat{\delta}_1$ converges to a quantity larger or smaller than δ_1 depends on the size and sign of σ_{xy}, the correlation between x and y, and the parameter γ_1. Say, as an example, that $\sigma_{xy} > 0$ and $\gamma_1 > 0$. Then, the inequality $\delta_1^{H_1} > \delta_1$ occur.

The equation (6) also indicates that for the null hypothesis H_{0xy} we are dealing with, mainly, a small sample problem. The explanation to this is that, if $\alpha_1 \neq 0$, the estimator $\hat{\alpha}_1$ will converge, in probability, to a quantity not equal to zero at the same time as the standard error converges to zero and thereby cause a rejection of the null hypothesis that x is Granger causing y. The null hypothesis will thus, asymptotically, be rejected regardless of the measurement error. However, for the null hypothesis H_{0yx}, the problem is not only a small sample problem. In this case, the result depends, as can be seen by equation (12), among other things, on the value of γ_1.

As a consequence of (11) and (12)

$$\begin{aligned}\frac{1}{T} RSS_1^{yx} \xrightarrow{p} \sigma_x^2 + \sigma_e^2 - 2\gamma_1^{H_1}(\gamma_1 \sigma_x^2 + \delta_1 \sigma_{xy}) \\ -2\delta_1^{H_1}(\gamma_1 \sigma_{xy} + \delta_1 \sigma_y^2) + 2\gamma^{H_1}\delta_1^{H_1}\sigma_{xy} \\ +\delta_1^{H_1}\sigma_y^2 + (\gamma_1^{H_1})^2(\sigma_x^2 + \sigma_e^2)\end{aligned}$$

Under the null hypothesis, the estimator of γ_1 is asymptotically biased, according to a standard result on simple regression with measurement error in the explanatory variable

$$\hat{\gamma}_1^{H_0} \xrightarrow{p} \gamma_1 \frac{\sigma_x^2}{\sigma_x^2 + \sigma_e^2}$$

From this, it follows that

$$\frac{1}{T} RSS_0^{yx} \xrightarrow{p} \sigma_x^2 + \sigma_e^2 - 2\gamma_1^{H_0}\gamma_1\sigma_x^2 + (\gamma_1^{H_0})^2(\sigma_x^2 + \sigma_e^2) \tag{13}$$

To summarize the results we now consider how the test statistic (3) will be affected under the null and alternative hypotheses H_{0xy} and H_{0yx}, respectively. To this end, it will be useful to rewrite it as

$$S_1 = T\left(\frac{RSS_0}{RSS_1} - 1\right) \tag{14}$$

and just consider the probability limit of the ratio RSS_0/RSS_1. Under H_{0xy}, according to (8), RSS_1 is affected by the measurement error and is thus a possible source for size distortions. Under the alternative to H_{0xy}, H_{1xy}, RSS_1 is dependent on the sign and size of α_1 and σ_{xy} since these are determining the bias of the parameter estimators. As for the hypothesis if y is Granger-causing x, both RSS_0 and RSS_1 are affected under both the null and the alternative hypotheses. In general, this implies size distortions. Because of these size distortions, not necessarily hitting both tests to the same extent, we will tend to reject one of the hypotheses H_{0yx} and H_{0xy} more often than the other.

In the next section this problem is illustrated by means of a simulation study.

4 Simulation study

The calculations in Section 3 was instructive in order to see that the asymptotic consequence of measurement error in x was different for the test of forecasting power in x on y than for the test of forecasting power in y on x. However, the power properties of such a test was not studied explicitly. The question now is: Given that there is a feedback between x and y, does a measurement error in x cause more rejections in one of the tests than in the other?

In order to study this in the finite sample case data from a bivariate VAR(1)-models is generated. The two null hypotheses

$$H_{0xy} : x \text{ fails to Granger-cause } y$$

and

$$H_{0yx} : y \text{ fails to Granger-cause } x$$

are investigated by simulation from the process

$$\begin{cases} x_t = 0.5x_{t-1} + a_{x,t} \\ y_t = 0.5y_{t-1} + a_{y,t} \end{cases} \tag{15}$$

where $(a_{x,t}, a_{y,t})'$ is a normally distributed bivariate white noise with covariance matrix Σ_a and we observe $X_t = x_t + e_t$ instead of x_t itself. In the simulation $\{e_t\}$ is assumed to be a normally distributed white noise with mean zero and variance σ_e^2. For the case of a signal-to-noise ratio of one the rejection rates under a nominal 5% level are given in Figure 1. In Figure 2 the corresponding results are given for the case when the signal-to-noise ratio is four. As shown in Section 3, size distortions are only present under H_{0yx}.

To study the power of the tests the example

$$\begin{cases} x_t = 0.5x_{t-1} + 0.2y_{t-1} + a_{x,t} \\ y_t = 0.2x_{t-1} + 0.5y_{t-1} + a_{y,t} \end{cases} \tag{16}$$

is used. This example is a situation where there is a symmetry in the sense that y_t is Granger-caused by x_t "as much as" the opposite is true. However x_t is measured with error according to (5). Given the data generating process above, both these hypotheses should, optimally, be rejected as often as possible. The results are presented in Figure 3. The figure shows empirical rejection rates when the nominal significance level is 5%. In the case of a signal-to-noise ratio of one and a positive correlation between $a_{x,t}$ and $a_{y,t}$, as can be seen in Figure 3, yield a power for the test of H_{0yx} which is substantially larger than for the test of H_{0xy}. The implication of this is that it is more likely that the conclusion is that y is driving x is more likely than the opposite. In the cases with negative correlation between $a_{x,t}$ and $a_{y,t}$, the difference are not as remarked. In Figure 4 the case with a signal to noise ratio of 4 is considered. The tendency is the same while the relative decrease in measurement error is moving the power of the two tests closer.

5 Conclusion

The problem of measurement errors in one of the variables in tests of Granger-causality has been studied. When the correlation between the two variables

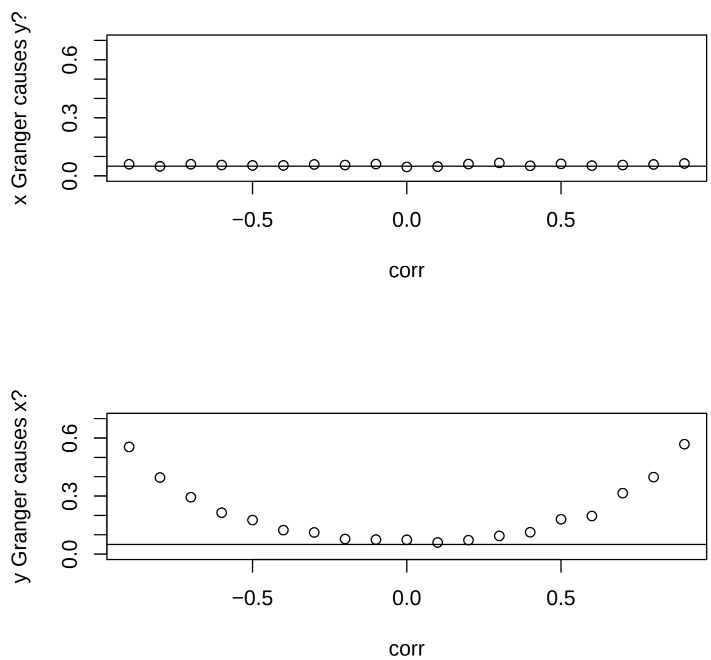

Figure 1: Monte Carlo sizes of the two tests H_{0xy} and H_{0yx} for different correlations between a_{xt} and a_{yt}. The nominal significance level is 5% which is indicated by the horizontal solid line, the signal to noise ratio is one, the sample size, T, is 100 and the number of replicates 1000.

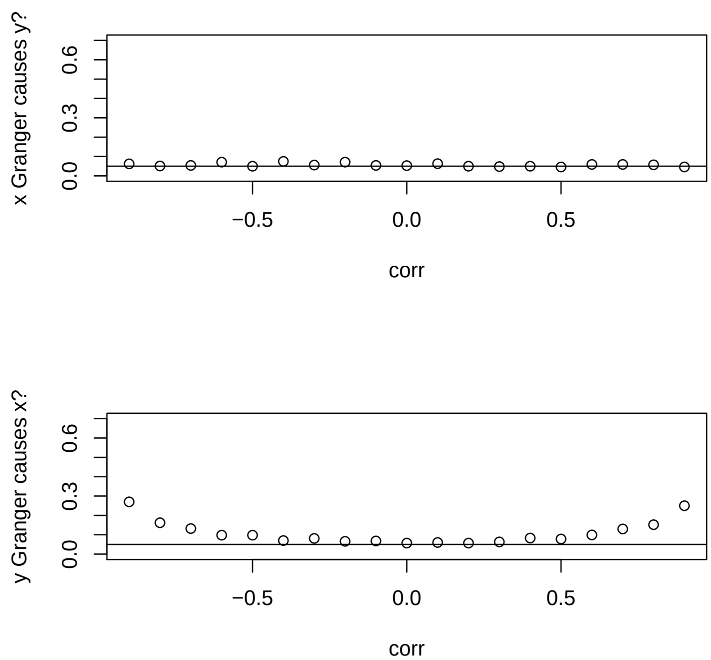

Figure 2: Monte Carlo sizes of the two tests H_{0xy} and H_{0yx} for different correlations between a_{xt} and a_{yt}. The nominal significance level is 5% which is indicated by the horizontal solid line, the signal to noise ratio is four, the sample size, T, is 100 and the number of replicates 1000.

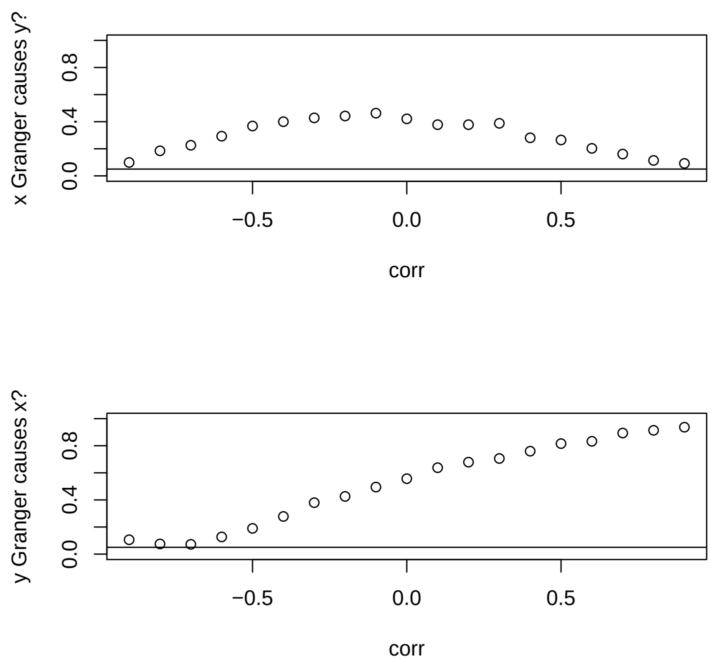

Figure 3: Monte Carlo rejection rates (power) of the two tests H_{0xy} and H_{0yx} for different correlations between a_{xt} and a_{yt}. The nominal significance level is 5% which is indicated by the horizontal solid line, the signal to noise ratio is one, the sample size, T, is 100 and the number of replicates 1000.

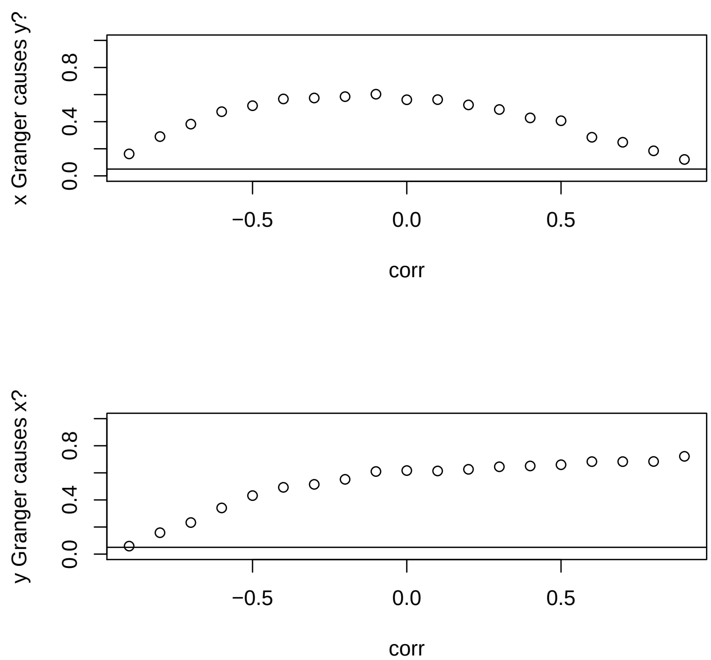

Figure 4: Monte Carlo rejection rates (power) of the two tests H_{0xy} and H_{0yx} for different correlations between a_{xt} and a_{yt}. The nominal significance level is 5% which is indicated by the horizontal solid line, the signal to noise ratio is four, the sample size, T, is 100 and the number of replicates 1000.

is positive the problem occurs in that the variable measured with error is often mistakenly concluded to fail to Granger-cause the other variable while the Granger causality in the other direction is more often detected. This causes a tendency to conclude that one variable is driving the other while there is indeed a feedback relationship present.

References

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Appendix

Here, an indication of how the probability limits for the different quantities are derived, is given. Consider the model

$$y_t = \alpha_1 x_{t-1} + \beta_1 y_{t-1} + a_{yt} \tag{17}$$

and

$$X_t = x_t + e_t. \tag{18}$$

where e_t and a_{yt} are independent, e_t is orthogonal to x_t and y_t and $E(e_t) = 0$. The OLS-estimator of α_1 can be written

$$\hat{\alpha}_1 = \frac{\sum_{t=1}^{T-1} y_t^2 \sum_{t=1}^{T-1} X_t y_{t+1} - \sum_{t=1}^{T-1} X_t y_t \sum_{t=1}^{T-1} y_t y_{t+1}}{\sum_{t=1}^{T-1} X_t^2 \sum_{t=1}^{T-1} y_t^2 - (\sum_{t=1}^{T-1} X_t y_t)^2} \tag{19}$$

Multiplying both the numerator and denumenator by $1/T^2$ and taking each of the terms in probability limit we obtain

$$\hat{\alpha}_1 \xrightarrow{p} \frac{\sigma_y^2(\alpha_1\sigma_x^2 + \beta_1\sigma_{xy}) - \sigma_{xy}(\alpha_1\sigma_{xy} + \beta_1\sigma_y^2)}{(\sigma_x^2 + \sigma_e^2)\sigma_y^2 - \sigma_{xy}^2} \tag{20}$$

which can be rewritten as (6).

Now keeping the variables X_{t-1} and y_{t-1} on the right-hand side of (17) but replacing the left-hand side with X_t we can write the OLS-estimator of δ_1 in the regression

$$x_t = \gamma_1 x_{t-1} + \delta_1 y_{t-1} + a_{xt} \tag{21}$$

as

$$\hat{\delta}_1 = \frac{\sum_{t=1}^{T-1} X_t^2 \sum_{t=1}^{T-1} y_t X_{t+1} - \sum_{t=1}^{T-1} X_t y_t \sum_{t=1}^{T-1} X_t X_{t+1}}{\sum_{t=1}^{T-1} X_t^2 \sum_{t=1}^{T-1} y_t^2 - (\sum_{t=1}^{T-1} X_t y_t)^2} \tag{22}$$

Again, multiplying both the numerator and denumenator by $1/T^2$ we obtain

$$\hat{\delta}_1 \xrightarrow{p} \frac{(\sigma_x^2 + \sigma_e^2)(\gamma_1 \sigma_{xy} + \delta_1 \sigma_y^2) - \sigma_{xy}(\gamma_1 \sigma_x^2 + \delta_1 \sigma_{xy})}{(\sigma_x^2 + \sigma_e^2)\sigma_y^2 - \sigma_{xy}^2} \tag{23}$$

which can be rewritten as (12).

The probability limits in e.g. (8) can be obtain by using that e.g.

$$\begin{aligned} \hat{a}_{y,t}^2 &= (y_t - \hat{\alpha}_1 X_{t-1} - \hat{\beta}_1 y_{t-1})^2 \\ &= y_t^2 - 2\hat{\alpha}_1 y_t X_{t-1} - 2\hat{\beta}_1 y_t y_{t-1} \\ +2\hat{\alpha}_1 \hat{\beta}_1 X_{t-1} y_{t-1} &+ \hat{\beta}_1^2 y_{t-1}^2 + \hat{\alpha}_1^2 X_{t-1}^2 \end{aligned} \tag{24}$$

and then considering the probability limits of each of the terms in

$$\frac{1}{T} RSS_1 = \frac{1}{T} \sum_{t=1}^{T} \hat{a}_{y,t}^2. \tag{25}$$