Pricing of Rate of Return Guarantees on Multi-period Assets.

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Abstract

The basis for this paper is the pricing of multi-period rate of return guarantees. These guarantees can typically be found in life insurance and pension contracts. We derive closed form solutions, expressed by the cumulative multivariate normal probability distribution, for multi-period rate of return guarantees on both a money market account and a stock. The guarantees of Hipp (1996), Persson and Aase (1997), and Miltersen and Persson (1999) can be seen to be special cases of our results.

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1 Introduction.

Most financial investments are exposed to the risk of getting a low rate of return. By including a minimum rate of return guarantee in a financial contract, the risk of getting a low rate of return on the investment is eliminated. However, the rate of return is still risky.

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Minimum rate of return guarantees are typically embedded in life insurance contracts and index linked bonds. How the guarantees embedded in life insurance contracts are priced in practice seems to be somewhat insufficient. According to Donselaar (1999), as much as 75% of the Dutch life insurers offered minimum rate of return guarantees free of any charge.[1] One has seen several life insurance companies that have gone into bankruptcy because they were unable to fulfil the liabilities imposed by minimum rate of return guarantees, see e.g., Briys and de Varenne (1997). This demonstrates that the pricing of minimum rate of return guarantees is an important issue.

One of the earliest treatments of guarantees is due to Brennan and Schwartz (1976). They consider maturity guarantees, and they show, by using the framework and the results of Black and Scholes (1973), that a maturity guarantee is equivalent to holding a European put option and the underlying asset (or, alternatively, a risk free investment and a European call option). They also include mortality risk and extend the results to periodic premium payments. This is the same kind of guarantee that can be found in index linked bonds and has been thoroughly analysed in the literature.

Life insurance contracts with their embedded guarantees are often far more complicated than the maturity guarantee. Both legal requirements in different countries and different company policies will determine how returns are distributed between the insurer and the insured. These distribution mechanisms may be fairly involved, and life insurance contracts may therefore be embedded with several option and guarantee elements. Grosen and Jørgensen (2000), Hansen and Miltersen (2000), and Miltersen and Person (2000) analyse different mechanisms for distributing the return between the insurer and the insured. The complexity of the contracts forces, in all of the three papers mentioned above, the main results to be solved by numerical methods.

Hipp (1996) recognises that the guarantees included in many life insurance contracts are not maturity guarantees, but annual, or multi-period, guarantees. A multi-period guarantee secures a minimum rate of return in *each* period. This turns out to be a totally different guarantee than the maturity guarantee that only lasts for one period. Within the framework of Black and Scholes (1973), Hipp (1996) derives closed form solutions for the market value of a multi-period rate of return guarantee. For deterministic interest rates, the market value of an N-period guarantee is given by a fairly simple expression. Persson and Aase (1997) investigate a two-period guarantee when interest rates are stochastic. They find that the market value is given as a function of the cumulative bivariate normal probability distribution. This work is continued by Miltersen and Persson (1999) in a Heath, Jarrow, and Morton setting. They find the market value of a two-period rate of return guarantee on both the short term interest rate and the stock

[1] It seems unlikely that this is only a Dutch phenomena.

return.[2]

In this paper we derive closed form solutions for the market value of multi-period rate of return guarantees when interest rates are both deterministic and stochastic. The results of Hipp (1996), Persson and Aase (1997), and Miltersen and Persson (1999) can be seen to be special cases of our results. These contracts are stylised in the sense that mortality factors, periodical premiums, surrender option[3], annual distribution of surplus, and bonus mechanisms are not included, and the contracts may therefore not be directly found in the market. However, we believe that results in this line are important for the understanding of models which incorporates some of the factors just mentioned.

An outline of the paper goes as follows. In section 2 we give a description of the general framework we work within. In section 3 we calculate the market value of multi-period rate of return guarantees. In subsection 3.1 interest rates are assumed deterministic, while in subsection 3.2 interest rates are assumed stochastic. In section 4 we calculate market values of multi-period rate of return guarantees. In section 5 we end the paper with some concluding remarks.

2 The Economic Model.

We work within an extended Heath, Jarrow, and Morton (1992) model, also called an Amin and Jarrow (1992) model. A description of this model can be found in an advanced textbook in finance, see e.g., Musiela and Rutkowski (1997).

We assume that trading takes place on a continuous basis on the time interval $[0, \mathcal{T}]$, for some fixed horizon $\mathcal{T} > 0$. A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is fixed, where Ω is the state space, $\mathcal{F}$ is a σ-algebra, $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq \mathcal{T}\}$ is a filtration where $\mathcal{F}_\mathcal{T} = \mathcal{F}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$, where $\emptyset$ is the empty set, and P is a probability measure. The σ-algebra is generated by a $d \geq 1$ dimensional Brownian motion, B_t.[4]

We assume, under the equivalent martingale measure Q, that the continuously compounded forward rates from time t to s, $f(t, s)$, $0 \leq t \leq s \leq \mathcal{T}$,

[2] Reffs (1998) considers an instantaneous rate of return guarantee where the investment, at all times in the contract period, bears the maximum of the short term interest rate and the minimum guaranteed rate of return. This kind of guarantee can be seen as the limit of the multi-period rate of return guarantee when the length of each period approaches zero and the number of periods approaches infinite.

[3] A surrender option is the right a policy holder has to terminate the policy prior to maturity. This kind of problem can be analysed as an optimal stopping problem, or in financial terms, as an American option. Grosen and Jørgensen (1997) find that the market value of the surrender option can be quite significant.

[4] For the case with both stochastic interest rates and a stock, we require, in order to avoid perfect correlation between the stock and the interest rates, that $d > 1$.

are given by

$$f(t,s) = f(0,s) + \int_0^t \sigma_f(v,s)\int_v^s \sigma_f(v,u)dudv + \int_0^t \sigma_f(v,s)dB_v, \qquad (1)$$

where sufficient regularity conditions for $\sigma_f(t,s)$, $0 \le t \le s \le \mathcal{T}$ are given in Heath, Jarrow, and Morton (1992).

The short term interest rate $r_t = f(t,t)$. We will throughout assume that $\sigma_f(t,s)$, is a deterministic function, a fact which implies Gaussian interest rates. When considering deterministic interest rates we formally set $\sigma_f(t,s) = 0$. We also assume that there is a continuum of zero coupon bonds trading in the market.

We let the market value of the stock, S_t, be given under the equivalent martingale measure Q by the equation

$$S_t = S_0 + \int_0^t r_v S_v dv + \int_0^t \sigma_S(v) S_v dB_v, \qquad (2)$$

where $r_t S_t$ satisfies the integrability condition $\int_0^t r_v S_v dv < \infty$ almost surely for all $t \le \mathcal{T}$. $\sigma_S(t)$ is the instantaneous standard deviation of the return on the stock and satisfies the square integrability condition $\int_0^t (\sigma_S(v) S_v)^2 dv < \infty$ almost surely (for further details on integrability conditions, see e.g., Duffie (1996)).

The money market account is an asset where interest accrues according to the short term interest rate. Under the equivalent martingale measure the market value, M_t, is given by

$$M_t = M_0 + \int_0^t r_v M_v dv, \quad M_0 = 1, \qquad (3)$$

where $r_t M_t$ satisfies the integrability condition $\int_0^t r_v M_v dv < \infty$ almost surely for all $t \le \mathcal{T}$.

From (2) and (3) we can see that the money market account, under the equivalent martingale measure Q, is just a special case of the stock since the money market account has no diffusion term.

In the rest of this paper we divide the time into periods. Period n will be the time interval between time t_{n-1} and t_n. The initial investment is normalised to one.

3 Pricing Multi-Period Rate of Return Guarantees.

Before we turn to the pricing of multi-period minimum rate of return guarantees, we review some useful relationships and results from Miltersen and

Persson (1999). They find that the return on the money market account in period n is given by

$$\begin{aligned}\beta_n &= \int_{t_{n-1}}^{t_n} r_v dv = -\ln F(0, t_{n-1}, t_n) + \frac{1}{2}\sigma_{\beta_n}^2 + \sum_{k=1}^{n} c_{(k-1),n} \\ &+ \int_0^{t_{n-1}} \int_{t_{n-1}}^{t_n} \sigma_f(v,u) du dB_v + \int_{t_{n-1}}^{t_n} \int_v^{t_n} \sigma_f(v,u) du dB_v,\end{aligned}$$

where

$$F(0, t_{n-1}, t_n) = \frac{P(0, t_n)}{P(0, t_{n-1})}$$

is the forward price at time zero for delivery at time t_{n-1} of a zero coupon bond maturing at time t_n ($P(0, t_n)$). Also

$$\sigma_{\beta_n}^2 = \int_0^{t_{n-1}} \Big(\int_{t_{n-1}}^{t_n} \sigma_f(v,u) du\Big)^2 dv + \int_{t_{n-1}}^{t_n} \Big(\int_v^{t_n} \sigma_f(v,u) du\Big)^2 dv$$

is the variance of the return on the money market account in period n, and, finally,

$$\begin{aligned}c_{m,n} &= \int_0^{t_{m-1}} \Big(\int_{t_{m-1}}^{t_m} \sigma_f(v,u) du\Big)\Big(\int_{t_{n-1}}^{t_n} \sigma_f(v,u) du\Big) dv \\ &+ \int_{t_{m-1}}^{t_m} \Big(\int_v^{t_m} \sigma_f(v,u) du\Big)\Big(\int_{t_{n-1}}^{t_n} \sigma_f(v,u) du\Big) dv\end{aligned}$$

is the covariance between the return on the money market account in period m and n, $1 \le m < n$. The market value of the money market account at time t_n can be written as

$$M_{t_n} = M_{t_{n-1}} e^{\beta_n}.$$

The return on the stock in period n can be written as

$$\delta_n = \int_{t_{n-1}}^{t_n} \Big(r_v - \frac{1}{2}\sigma_S(v)^2\Big) dv + \int_{t_{n-1}}^{t_n} \sigma_S(v) dB_v, \qquad (4)$$

so that the time t_n market value of the stock may be expressed as

$$S_{t_n} = S_{t_{n-1}} e^{\delta_n}.$$

Since $c_{m,n}$ is the covariance between the rate of return on the money market account in period m and n, we can from the above clearly see that the rate of return on both the money market account and on the stock in one period (under the equivalent martingale measure Q) is dependent on the return in the previous periods. Since we use an interest rate with contineous path (trajectory), this seems intuitive. A high interest rate at the end of one period will typically be followed by a high interest rate in the beginning of the next period.

In addition, we also need to calculate the covariance ($\bar{c}_{m,n}$) between the return on the stock (in period m) and the money market account (in period n) and between the return on the stock in different periods ($\bar{\bar{c}}_{m,n}$). Using the Itô isometry, we get

$$\bar{c}_{m,n} = c_{m,n} + \int_{t_{m-1}}^{t_m} \sigma_S(v) \int_{t_{n-1}}^{t_n} \sigma_f(v,u) du dv$$

for $n > m$.

$$\bar{c}_{n,n} = \sigma_{\beta_n}^2 + \int_{t_{n-1}}^{t_n} \sigma_S(v) \int_{v}^{t_n} \sigma_f(v,u) du dv,$$

for $m = n$, and

$$\bar{c}_{m,n} = c_{n,m},$$

for $n < m$.

$$\bar{\bar{c}}_{m,n} = c_{n,m} + \int_{t_{m-1}}^{t_m} \sigma_S(v) \int_{t_{n-1}}^{t_n} \sigma_f(v,u) du dv.$$

The variance of the return on the stock in period n is given by

$$\sigma_{\delta_n}^2 = \sigma_{\beta_n}^2 + 2 \int_{t_{n-1}}^{t_n} \sigma_S(v) \int_{v}^{t_n} \sigma_f(v,u) du dv + \int_{t_{n-1}}^{t_n} \sigma_S(v)^2.$$

Maturity guarantees are important building blocks for multi-period rate of return guarantees. We therefore write down the pricing formula for the maturity guarantee. Let $X_t \in \{M_t, S_t\}$ and $\sigma_X(t) \in \{\sigma_\beta(t), \sigma_\delta(t)\}$. The market value at time $t < 1$ of a maturity guarantee is given by (see e.g. Miltersen and Persson (1999))

$$\pi_t = \frac{X_t}{X_0} \Phi(d_{1,t}) + P(t,1) e^g \Phi(-d_{2,t}),$$

where $\Phi(\cdot)$ is the cumulative normal distribution function, g is the minimum guaranteed rate of return, and

$$d_{1,t} = \frac{\ln(X_t/X_0) - g - \ln P(t,1)}{\sigma_X(t)} + \frac{1}{2}\sigma_X(t),$$

and

$$d_{2,t} = d_{1,t} - \sigma_X(t).$$

Let us now concentrate on the case with deterministic interest rates.

3.1 Deterministic Interest Rates.

In this subsection we find the market value of a multi-period rate of return guarantee with the return on a stock as the underlying asset (this is indicated by the superscript d).

The terminal payoff at time t_N for an N-period guarantee is given by

$$\theta_{t_N}^{d,N} = \prod_{i=1}^{N} \max\left(\frac{S_{t_i}}{S_{t_{i-1}}}, e^{g_i}\right). \tag{5}$$

Hipp (1996) and Miltersen and Persson (1999) show that the time t_0 market value of the claim in (5) is given by

$$\theta_{t_0}^{d,N} = \prod_{i=1}^{N} \Big(\Phi(d_{1,0}^i) + e^{g_i} F(t_0, t_{i-1}, t_i)\Phi(-d_{2,0}^i)\Big),$$

where

$$\begin{aligned} d_{1,t}^i &= \frac{\ln(S_t/S_t) - g_i - \ln(F(t, t_{i-1}, t_i))}{\sigma_{d_{i,t}}} + \frac{1}{2}\sigma_{d_{i,t}} \\ d_{2,t}^i &= d_{1,t}^i - \sigma_{d_{i,t}}, \end{aligned}$$

and $\sigma_{d_{i,t}}^2 = \int_t^{t_i} \sigma_S^2(v)dv$ for $t \in [t_{i-1}, t_i)$.

Now, assume that we want to find the market value of the guarantee at time t in period $\tau \in \{2, 3, \ldots, N\}$. For $\tau \geq 2$, the realized return in the $n = \tau - 1$ previous periods is given by

$${}^{n}R^d = \prod_{i=1}^{n} \max\left(\frac{S_{t_i}}{S_{t_{i-1}}}, e^{g_i}\right).$$

Proposition 1. *The time $t \in [t_{\tau-1}, t_\tau)$ market value in period τ of an N-period rate of return guarantee on the stock return is given by*[5]

$$\pi_{N,t}^{d,\tau} = {}^{\tau-1}R^d \cdot \pi_t^d \cdot \theta_{t_\tau}^{d,(N-\tau)}.$$

Proof. Let $E_Q[\cdot]$ be the expectation under the equivalent martingale measure

[5] $\pi_{N,t}^{d,\tau}$ should be read as $\pi_{\text{total number of periods, point in time}}^{\text{type of underlying asset, current period}}$.

Q. We have that

$$
\begin{aligned}
\pi_{N,t}^{d,\tau} &= E_Q\Big[e^{-\int_t^{t_N} r_v dv} \prod_{i=1}^{N} \max\Big(\frac{S_{t_i}}{S_{t_{i-1}}}, e^{g_i}\Big)\Big|\mathcal{F}_t\Big] \\
&= \prod_{i=1}^{\tau-1} \max\Big(\frac{S_{t_i}}{S_{t_{i-1}}}, e^{g_i}\Big) E_Q\Big[e^{-\int_t^{t_N} r_v dv} \prod_{i=1}^{N} \max\Big(\frac{S_{t_i}}{S_{t_{i-1}}}, e^{g_i}\Big)\Big|\mathcal{F}_t\Big] \\
&= {}^{(\tau-1)}R^d E_Q\Big[\prod_{i=\tau}^{N} e^{-\int_t^{t_i} r_v dv} \max\Big(\frac{S_{t_i}}{S_{t_{i-1}}}, e^{g_i}\Big)\Big|\mathcal{F}_t\Big] \\
&= {}^{(\tau-1)}R^d E_Q\Big[e^{-\int_t^{t_\tau} r_v dv} \max\Big(\frac{S_{t_i}}{S_{t_{i-1}}}, e^{g_i}\Big)\Big|\mathcal{F}_t\Big] \cdot \\
&\quad \prod_{i=\tau+1}^{N} E_Q\Big[e^{-\int_{t_{i-1}}^{t_i} r_v dv} \max\Big(\frac{S_{t_i}}{S_{t_{i-1}}}, e^{g_i}\Big)\Big|\mathcal{F}_t\Big] \\
&= {}^{\tau-1}R^d \cdot \pi_t^d \cdot \theta^{d,(N-\tau)}.
\end{aligned}
$$

The second equality follows since the return in the first $\tau-1$ periods is $\mathcal{F}_{t_{\tau-1}}$-measurable. The third and fourth follows since the return on the stock in period i and j are independent for all i, j, $i \neq j$. □

From the pricing formula in Proposition 1, we see that the market value consists of the product between three parts; the realised return in the $\tau - 1$ previous periods, the market value of the guarantee in the current period, τ, and, finally, the market value of the guarantees in the remaining $N - \tau$ periods. For the special case when $\tau = 1$, the same pricing formula follows with ${}^0R^d = 1$.

3.2 Stochastic Interest Rates.

As already mentioned, considering a stochastic interest rate environment, the rate of return in one period is dependent on the rate of return in earlier periods. This makes it necessary to involve the multivariate probability distribution when pricing multi-period rate of return guarantees.

Highly inspired by the results of Persson and Aase (1997) and Miltersen and Persson (1999), we follow their approach rather closely when deriving the pricing formulas. We find closed form solutions for the initial market value of guarantees on both the money market account and on the stock return for an N-period guarantee, $N \geq 2$. The solutions are expressed by the N-dimensional multivariate normal probability distribution. Setting $N = 2$, we obtain the results of Persson and Aase (1997) and Miltersen and Persson (1999) as special cases. We start by considering the money market account.

3.2.1 Pricing the Guarantee on the Money Market Account.

Let N be the total number of periods. To find the market value of the guarantee we have to find the expected deflated cash flow at time $t_N \leq \mathcal{T}$ under the equivalent martingale measure. This is given by the expectation

$$\pi^{\beta}_{N,t_N} = E_Q\Big[e^{(g_1-\beta_1)\vee 0} \cdot e^{(g_2-\beta_2)\vee 0} \cdot \ldots \cdot e^{(g_N-\beta_N)\vee 0}\Big]. \tag{6}$$

An N-period guarantee has two different possibilities in each period; (0) the guarantee is not binding, and (1) the guarantee is binding. For an N-period guarantee, this yields the possibility of in total 2^N different "states" of the world.

To evaluate the expectation in (6), we first define some vectors and matrices. Let $\mathbf{c}_j$, $j \in \{1, 2, \ldots, 2^N\}$ be an $N \times 1$ dimensional vector giving the "state" of the world. The i'th element of $\mathbf{c}_j$, $i \in \{1, 2, \ldots, N\}$, takes the value 1 when the guarantee is binding in the i'th period and 0 otherwise. This, of course, yields 2^N unique $\mathbf{c}_j$'s, each having a unique combination of 0's and 1's.[6]

$\hat{\mathbf{c}}_j$, $j \in \{1, 2, \ldots, 2^N\}$, is an $N \times N$ dimensional symmetric matrix with only non-zero elements on the diagonal. The diagonal of $\hat{\mathbf{c}}_j$ is given by $2\mathbf{c}_j - \mathbf{1}$, where $\mathbf{1}$ is a vector only containing ones, i.e., the i'th diagonal element of $\hat{\mathbf{c}}_j$ takes the value 1 when the guarantee is binding in the i'th period and minus one otherwise. The minimum guaranteed rate of return in each period is given by the column vector $\mathbf{g} = (g_1, g_2, \ldots, g_N)'$. The expected return on the money market account under the equivalent martingale measure Q is given by $\mathbf{\Lambda}$, an $N \times 1$ dimensional vector with i'th element $\Lambda_i = -F(0, t_{i-1}, t_i) + \frac{1}{2}\sigma^2_{\beta_i} + \sum_{k=1}^{i} c_{(k-1),i}$. $\mathbf{\Sigma}$ is the variance-covariance matrix of the multivariate normal distributed random variables $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_N)'$. $\underline{\mathbf{\Sigma}}$ is the standardized version of $\mathbf{\Sigma}$. $\hat{\boldsymbol{\alpha}}_j$ is an $N \times 1$ dimensional vector, whose rational follows from the proof of Proposition 2. The i'th element of $\hat{\boldsymbol{\alpha}}_j$ is given by

$$\hat{\alpha}_{j,i} = \frac{g_i - \Lambda_i + (\mathbf{\Sigma}\mathbf{c}_j)_i}{\sigma_{\beta_i}}, \tag{7}$$

where $(\mathbf{\Sigma}\mathbf{c}_j)_i$ is the i'th element of the vector $\mathbf{\Sigma}\mathbf{c}_j$, and is due to a property for the multivariate normal probability distribution that is given in Lemma 1.

[6] To construct all 2^N $\mathbf{c}_j$'s, consider an $N \times 2^N$ dimensional matrix with 2^N different columns equal to $\mathbf{c}_j$. In the first row, let the first 2^{N-1} elements equal 1, and the remaining 2^{N-1} elements equal 0. In row two, let the first 2^{N-2} elements equal 1, the next 2^{N-2} elements equal 0, the next 2^{N-2} elements equal 1, and finally the last 2^{N-2} elements equal 0. Let this continue, so that the elements in row N are equal to 1, 0, 1,..., 1, and 0. The first column then corresponds to the state where the guarantee is binding in each period, and column 2^N the state where the guarantee is never binding.

Lemma 1. *For multivariate normal distributed random variables* $\mathbf{X}$ *with expectation* $\boldsymbol{\mu}$*, variance-covariance matrix* $\mathbf{V}$*, and probability density function* $\phi(\mathbf{X}; \boldsymbol{\mu}, \mathbf{V})$*, we have that*

$$\phi(\mathbf{X}; \boldsymbol{\mu}, \mathbf{V}) \exp(-\mathbf{m}'\mathbf{X}) = \phi(\mathbf{X}; \boldsymbol{\mu} - \mathbf{V}\mathbf{m}, \mathbf{V}) \exp(-\mathbf{m}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{m}'\mathbf{V}\mathbf{m}),$$

where $\mathbf{m}$ *can be any column vector with the same dimension as* $\mathbf{X}$*.*

Proof. For the k-dimensional multivariate distributed random variables $\mathbf{X}$, we have that

$$\begin{aligned} & \phi(\mathbf{X}; \boldsymbol{\mu}, \mathbf{V}) e^{-\mathbf{m}'\mathbf{X}} \\ = \; & (2\pi)^{-1/2k} |\mathbf{V}|^{-1/2} e^{-1/2(\mathbf{X}-\boldsymbol{\mu})'\mathbf{V}^{-1}(\mathbf{X}-\boldsymbol{\mu})-\mathbf{m}'\mathbf{X}}. \end{aligned} \tag{8}$$

Using the symmetry properties of $\mathbf{V}$, it follows by straight forward calculations that (8) can be rewritten as

$$\begin{aligned} & (2\pi)^{-1/2k} |\mathbf{V}|^{-1/2} e^{-1/2(\mathbf{X}-\boldsymbol{\mu}+\mathbf{V}\mathbf{m})'\mathbf{V}^{-1}(\mathbf{X}-\boldsymbol{\mu}+\mathbf{V}\mathbf{m})-\mathbf{m}'\boldsymbol{\mu}+1/2\mathbf{m}'\mathbf{V}\mathbf{m}} \\ = \; & \phi(\mathbf{X}; \boldsymbol{\mu} - \mathbf{V}\mathbf{m}, \mathbf{V}) \exp(-\mathbf{m}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{m}'\mathbf{V}\mathbf{m}). \end{aligned}$$

□

Finally, $\boldsymbol{\alpha}_j = \hat{\mathbf{c}}_j \hat{\boldsymbol{\alpha}}_j$ is an $N \times 1$ dimensional vector.

The solution of the expectation in (6) is given in Proposition 2.

Proposition 2. *The initial market value of an* N*-period guarantee on the money market account is given by*

$$\pi_N^{\beta} = \sum_{j=1}^{2^N} e^{\mathbf{c}_j'\mathbf{g} - \mathbf{c}_j'\boldsymbol{\Lambda} + \frac{1}{2}\mathbf{c}_j'\boldsymbol{\Sigma}\mathbf{c}_j} \Phi(\boldsymbol{\alpha}_j, \hat{\mathbf{c}}_j \underline{\boldsymbol{\Sigma}} \hat{\mathbf{c}}_j),$$

where

> $\Phi(\mathbf{a}, \mathbf{V})$ *is the cumulative multivariate normal distribution evaluated at the points determined by the vector* $\mathbf{a}$ *and with variance-covariance matrix* $\mathbf{V}$*.*

Proof. See Appendix A. □

3.2.2 Pricing the Guarantee on the Stock Return.

To find the initial market value of the guarantee on the stock return, we have to, as for the guarantee on the money market account, take the expectation of the deflated payoff at time t_N under the equivalent martingale measure. This yields the expectation

$$\pi^{\delta}_{N,t_N} = E_Q\Big[e^{(g_1-\beta_1)\vee(\delta_1-\beta_1)} \cdot e^{(g_2-\beta_2)\vee(\delta_2-\beta_2)} \cdot \ldots \cdot e^{(g_N-\beta_N)\vee(\delta_N-\beta_N)}\Big]. \quad (9)$$

We now introduce some new vectors and matrices. $\bar{\mathbf{c}}_j$, $j \in \{1, 2, \ldots, 2^N\}$, is a $2N \times 1$ dimensional vector only containing -1's, 0's, and 1's. The first N elements are equal to 1 and the remaining N elements are equal to $\mathbf{c}_j - \mathbf{1}$. As in the previous subsection, the i'th element of $\mathbf{c}_j$ is equal to 1 when the guarantee is binding in the i'th period and 0 otherwise. It then follows that the $N+i$'th element of $\bar{\mathbf{c}}_j$, $i \in \{1, 2, \ldots, N\}$, is equal to 0 when the guarantee is binding in the i'th period and -1 otherwise. It is possible to construct 2^N unique $\bar{\mathbf{c}}_j$'s, each corresponding to a state of the world.

The minimum guaranteed rate of return is given by the $N \times 1$ dimensional vector $\bar{\mathbf{g}} = (\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_N)'$. $\bar{\mathbf{\Lambda}}$ is a $2N \times 1$ dimensional vector giving the expectation of $\bar{\boldsymbol{\beta}} = (\beta_1, \beta_2, \ldots, \beta_N, \delta_1, \delta_2, \ldots, \delta_N)'$. The expectation of the i'th δ is given by $\bar{\Lambda}_{N+i} = \Lambda_i - \frac{1}{2}\sigma^2_{d_i}$. $\bar{\mathbf{\Sigma}}$ is the variance-covariance matrix of the multivariate normal distributed random variables $\bar{\boldsymbol{\beta}}$ and $\underline{\bar{\mathbf{\Sigma}}}$ is the standardized version of $\bar{\mathbf{\Sigma}}$. $\underline{\bar{\mathbf{\Sigma}}}_\delta$ is the standardized version of the variance-covariance matrix of the multivariate normal distributed random variables $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_N)'$. $\bar{\boldsymbol{\alpha}}_j$ is a $2N \times 1$ dimensional vector that gives the points to evaluate the cumulative multivariate normal probability distribution at, and it is given by $\bar{\mathbf{c}}_j \bar{\hat{\boldsymbol{\alpha}}}_j$, where the i'th element of $\bar{\hat{\boldsymbol{\alpha}}}_j$ is given by

$$\bar{\hat{\alpha}}_{j,i} = \infty,$$

for $i \in \{1, 2, \ldots, N\}$ and

$$\bar{\hat{\alpha}}_{j,i} = \frac{g_i - \bar{\Lambda}_i + (\bar{\mathbf{\Sigma}}\bar{\mathbf{c}}_j)_i}{\sigma_{\delta_i}},$$

for $i \in \{N+1, N+2, \ldots, 2N\}$.

$\bar{\boldsymbol{\alpha}}^{\delta}_j$ is an $N \times 1$ dimensional vector with i'th element, $i \in \{1, 2, \ldots, N\}$, equal to the $N+i$'th element of $\bar{\boldsymbol{\alpha}}_j$.

The solution of the expectation in (9) is given in Proposition 3.

Proposition 3. *The initial market value of an N-period guarantee on the stock return is given by*

$$\pi^{\delta}_N = \sum_{j=1}^{2^N} e^{\mathbf{c}'_j\bar{\mathbf{g}} - \bar{\mathbf{c}}'_j\bar{\mathbf{\Lambda}} + \frac{1}{2}\bar{\mathbf{c}}'_j\bar{\mathbf{\Sigma}}\bar{\mathbf{c}}_j}\, \Phi(\bar{\boldsymbol{\alpha}}^{\delta}_j, \hat{\mathbf{c}}_j \underline{\bar{\mathbf{\Sigma}}}_\delta \hat{\mathbf{c}}_j).$$

Proof. See Appendix B. ☐

Example ($N = 2$). Let us consider the same guarantee as in Miltersen and Persson (1999), i.e., $N = 2$. The first guarantee lasts from time 0 to 1, and the second from time 1 to 2. We then have that $\mathbf{c}_1 = (1 \quad 1)'$, $\mathbf{c}_2 = (1 \quad 0)'$, $\mathbf{c}_3 = (0 \quad 1)'$, and $\mathbf{c}_4 = (0 \quad 0)'$. We further have that

$$\hat{\mathbf{c}}_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\hat{\mathbf{c}}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\hat{\mathbf{c}}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\hat{\mathbf{c}}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The vector $\bar{\mathbf{\Lambda}}$ under Q is given by

$$\bar{\mathbf{\Lambda}} = \begin{pmatrix} -\ln P(0,1) + \frac{1}{2}\sigma_{\beta_1^2} \\ -\ln F_2 + \frac{1}{2}\sigma_{\beta_2^2} + c_{1,2} \\ -\ln P(0,1) + \frac{1}{2}\sigma_{\beta_1^2} - \frac{1}{2}\sigma^2_{d_1} \\ -\ln F_2 + \frac{1}{2}\sigma_{\beta_2^2} + c_{1,2} - \frac{1}{2}\sigma^2_{d_2} \end{pmatrix},$$

and $\bar{\mathbf{\Sigma}}$ is given by

$$\bar{\mathbf{\Sigma}} = \begin{pmatrix} \sigma^2_{\beta_1} & c_{1,2} & \sigma^2_{\beta_1} + k_1 & c_{1,2} \\ c_{1,2} & \sigma^2_{\beta_2} & c_{1,2} + k_{1,2} & \sigma^2_{\beta_2} + k_3 \\ \sigma^2_{\beta_1} + k_1 & c_{1,2} + k_{1,2} & \sigma^2_{\delta_1} & c_{1,2} + k_{1,2} \\ c_{1,2} & \sigma^2_{\beta_2} + k_3 & c_{1,2} + k_{1,2} & \sigma^2_{\delta_2} \end{pmatrix},$$

where

$$k_1 = \int_0^1 \sigma_S(v) \int_v^1 \sigma_f(v,u)\,du\,dv,$$

$$k_{1,2} = \int_0^1 \sigma_S(v) \int_1^2 \sigma_f(v,u)\,du\,dv,$$

and

$$k_3 = \int_1^2 \sigma_S(v) \int_v^2 \sigma_f(v,u)\,du\,dv.$$

$$\bar{\mathbf{\Sigma}} = \begin{pmatrix} 1 & \bar{\rho} \\ \bar{\rho} & 1 \end{pmatrix},$$

where $\bar{\rho} = \frac{c_{1,2}+k_{1,2}}{\sigma_{\delta_1}\sigma_{\delta_2}}$.

The exponent, $\mathbf{c}_j'\bar{\mathbf{g}} - \bar{\mathbf{c}}_j'\bar{\mathbf{\Lambda}} + \frac{1}{2}\bar{\mathbf{c}}_j'\bar{\mathbf{\Sigma}}\bar{\mathbf{c}}_j$, becomes

$$\mathbf{c}_j'\bar{\mathbf{g}} - \bar{\mathbf{c}}_j'\bar{\mathbf{\Lambda}} + \frac{1}{2}\bar{\mathbf{c}}_j'\bar{\mathbf{\Sigma}}\bar{\mathbf{c}}_j = \begin{cases} 0 & \text{for} \quad j=1, \\ g_2 + \ln F_2 - \sigma_{\delta_1}\sigma_{\delta_2}\bar{\rho} & \text{for} \quad j=2, \\ g_1 + \ln P(0,1) & \text{for} \quad j=3, \\ g_1 + g_2 + \ln P(0,1) + \ln F_2 & \\ \quad = g_1 + g_2 + \ln P(0,2) & \text{for} \quad j=4. \end{cases}$$

$\bar{\bar{\alpha}}_{j,i}$ for $i = 3, 4$ becomes

$$\bar{\bar{\boldsymbol{\alpha}}}_1 = \begin{pmatrix} \frac{g_1 - \bar{\Lambda}_3 + (\mathbf{\Sigma}\mathbf{c}_1)_3}{\sigma_{\delta_1}} \\ \frac{g_2 - \bar{\Lambda}_4 + (\mathbf{\Sigma}\mathbf{c}_1)_4}{\sigma_{\delta_2}} \end{pmatrix} = \begin{pmatrix} \frac{g_1 + \ln P(0,1) - \frac{1}{2}\sigma_{\delta_1}^2}{\sigma_{\delta_1}} \\ \frac{g_2 + \ln F_2 - \frac{1}{2}\sigma_{\delta_2}^2}{\sigma_{\delta_2}} - \sigma_{\delta_1}\bar{\rho} \end{pmatrix},$$

$$\bar{\bar{\boldsymbol{\alpha}}}_2 = \begin{pmatrix} \frac{g_1 - \bar{\Lambda}_3 + (\mathbf{\Sigma}\mathbf{c}_2)_3}{\sigma_{\delta_1}} \\ \frac{g_2 - \bar{\Lambda}_4 + (\mathbf{\Sigma}\mathbf{c}_2)_4}{\sigma_{\delta_2}} \end{pmatrix} = \begin{pmatrix} \frac{g_1 + \ln P(0,1) - \frac{1}{2}\sigma_{\delta_1}^2}{\sigma_{\delta_1}} + \sigma_{\delta_2}\bar{\rho} \\ \frac{g_2 + \ln F_2 + \frac{1}{2}\sigma_{\delta_2}^2}{\sigma_{\delta_2}} - \sigma_{\delta_1}\bar{\rho} \end{pmatrix},$$

$$\bar{\bar{\boldsymbol{\alpha}}}_3 = \begin{pmatrix} \frac{g_1 - \bar{\Lambda}_3 + (\mathbf{\Sigma}\mathbf{c}_3)_3}{\sigma_{\delta_1}} \\ \frac{g_2 - \bar{\Lambda}_4 + (\mathbf{\Sigma}\mathbf{c}_3)_4}{\sigma_{\delta_2}} \end{pmatrix} = \begin{pmatrix} \frac{g_1 + \ln P(0,1) + \frac{1}{2}\sigma_{\delta_1}^2}{\sigma_{\delta_1}} \\ \frac{g_2 + \ln F_2 - \frac{1}{2}\sigma_{\delta_2}^2}{\sigma_{\delta_2}} \end{pmatrix},$$

$$\bar{\bar{\boldsymbol{\alpha}}}_4 = \begin{pmatrix} \frac{g_1 - \bar{\Lambda}_3 + (\mathbf{\Sigma}\mathbf{c}_4)_3}{\sigma_{\delta_1}} \\ \frac{g_2 - \bar{\Lambda}_4 + (\mathbf{\Sigma}\mathbf{c}_4)_4}{\sigma_{\delta_2}} \end{pmatrix} = \begin{pmatrix} \frac{g_1 + \ln P(0,1) + \frac{1}{2}\sigma_{\delta_1}^2}{\sigma_{\delta_1}} + \sigma_{\delta_2}\bar{\rho} \\ \frac{g_2 + \ln F_2 + \frac{1}{2}\sigma_{\delta_2}^2}{\sigma_{\delta_2}} \end{pmatrix}.$$

Inserting these expressions into the formula in Proposition 3, the formula in Proposition 5.4 in Miltersen and Persson (1999) is obtained.

4 Implementation of the Pricing Formula.

The expression for the market value of the guarantee in Proposition 1 is easily implemented since it only involves the cumulative univariate normal probability distribution, and the time of maturity of the contract is therefore of no importance in regards to computer time. This is unfortunately not the case for the expressions in Proposition 2 and 3. These expressions involves the cumulative multivariate normal probability distribution, which has to be approximated by numerical methods, e.g., Monte Carlo-integration. Genz (1992) proposes a way of calculating multivariate normal probabilities.[7]

The guarantees considered here are typical long lasting, and the duration of the majority of the contracts are perhaps in the range from 20 to 40 years.

[7] A FORTRAN 77 code for this problem can be found at http://www.sci.wsu.edu/math/faculty/genz/homepage.

For a guarantee lasting for 30 years there are more than one billion (2^{30}) 30-tuppel integrals to be evaluated. This calculation is likely to be very time consuming.

By specifying the volatility in the Heath, Jarrow, and Morton model as (see e.g., Miltersen and Persson (1999))

$$\sigma_f(v,t) = e^{-\int_v^t \kappa_u du}\sigma_v, \tag{10}$$

the model of Vasicek (1977) is obtained. We will assume that $\sigma_v = \sigma$ and $\kappa_u = \kappa$ are constants. More precisely, when analysing the money market account we use the specification in (10), and when analysing the guarantee on the stock return (with stochastic interest rates), we let

$$\sigma_S(t) = \sigma_S \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\sigma_f(v,u) = \sigma e^{-\kappa(u-v)} \begin{pmatrix} \varphi \\ \sqrt{1-\varphi^2} \end{pmatrix},$$

where φ is a constant.

Using these specifications and inserting into the earlier expressions for the variances and covariances, the following equations follow (note that time $t_n = n$, $t_{n-1} = n-1$, $t_m = m$, and $t_{m-1} = m-1$)

$$\sigma^2_{\beta_n} = \frac{\sigma^2}{2\kappa^3}(2e^{-\kappa} - 2 - e^{-2\kappa n} + 2e^{\kappa(1-2n)} - e^{2\kappa(1-n)} + 2\kappa),$$

$$\begin{aligned} c_{m,n} &= \frac{\sigma^2}{2\kappa^3}(-2e^{\kappa(m-n)} - e^{\kappa(-m-n+2)} + 2e^{\kappa(-m-n+1)} \\ &\quad -e^{\kappa(-m-n)} + e^{\kappa(m-n-1)} + e^{\kappa(m-n+1)}), \end{aligned}$$

$$\sigma^2_{\delta_n} = \sigma^2_{\beta_n} + \frac{2\sigma\sigma_S\varphi}{\kappa^2}(\kappa - 1 + e^{-\kappa}) + \sigma^2_S.$$

$$\bar{c}_{m,n} = c_{m,n} + \frac{\sigma\sigma_S\varphi}{\kappa^2}(e^{-\kappa(n-m-1)} - 2e^{-\kappa(n-m)} + e^{-\kappa(n-m+1)}),$$

for $n > m$,

$$\bar{c}_{n,n} = \sigma^2_{\beta_n} + \frac{\sigma\sigma_S\varphi}{\kappa^2}(\kappa - 1 + e^{-\kappa}),$$

for $m = n$, and

$$\bar{c}_{m,n} = c_{n,m},$$

for $n < m$.

We will now use the results in Proposition 1 - 3 to calculate the market values of rate of return guarantees lasting from 2 - 5 periods. For the case with deterministic interest rates, we will assume the following parameter values (we assume an initial flat term structure of interest rates);

$$S_0 = 1, \quad g = \ln(1.04), \quad \sigma_S = 0.20, \quad r = 0.05.$$

For the case with stochastic interest rates, the following additional parameters are assumed;

$$\sigma = 0.03, \quad \kappa = 0.10, \quad \varphi = -0.5.$$

The market values are reported in Table 1. As we can see, introducing stochastic interest rates does not change the market value of the guarantee on the stock return much. The market value of the guarantee on the return on the money market account is lower than for the guarantee on the stock return. This is a consequence of the low volatility on the return on the money market account.

Table 1: Market value of multi-period rate of return guarantees.

	Proposition 1 π_N^d	Proposition 2 π_N^β	Proposition 3 π_N^δ
N=2	1.1534	1.0105	1.1493
N=3	1.2388	1.0216	1.2341
N=4	1.3304	1.0511	1.3286
N=5	1.4288	1.0643	1.4268

The market values in Table 1 for the claim in Proposition 1 are easily found from the closed form solution. For $N = 2$ the market values follow directly from the closed form solutions in Miltersen and Persson (1999). For $N = 2, 3$, and 4, the multivariate normal probabilities used to calculate the market value of the claims in Proposition 2 and 3 are found by using a Fortran 77 code written by Genz and is based on an algorithm proposed by Genz (1992).

5 Conclusions.

We have in this paper derived closed form solutions for the market value of multi-period rate of return guarantees. First we extended the model of Hipp (1996) so that the market value can be calculated at any time within the contract period. We then went on to finding the market value of guarantees when interest rates are stochastic. We found closed form expressions for both multi-period guarantees on the short term interest rate and on the stock return. The two-period guarantees analysed by Persson and Aase (1997) and Miltersen and Persson (1999) were seen to be special cases of our formulas. Finally we gave some remarks on implementation of the pricing formulas.

A Appendix.

The proof follows the same lines as the proof in the appendix of Persson and Aase (1997). We let the vector $\mathbf{c}_j$, $j \in \{1, 2, \dots, 2^N\}$, represent a unique state. We let $1_{\mathbf{c}_j}$ be the indicator function for the state $\mathbf{c}_j$, returning the value 1 when $\mathbf{c}_j$ is true and 0 otherwise. An expectation is a linear operator, and we can therefore split the expectation in (6) into the expected deflated payoff in each state, i.e.,

$$\pi_N^\beta = \sum_{j=1}^{2^N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} 1_{\mathbf{c}_j} \phi(\boldsymbol{\beta}, \boldsymbol{\Lambda}, \boldsymbol{\Sigma}) \exp(\mathbf{c}_j'(\mathbf{g} - \boldsymbol{\beta})) d\beta_N \cdots d\beta_2 d\beta_1.$$

For ease of exposition, we will rewrite this as follows

$$\begin{aligned}
\pi_N^\beta &= \int_{-\infty}^{g_1} \int_{-\infty}^{g_2} \cdots \int_{-\infty}^{g_N} \phi(\boldsymbol{\beta}, \boldsymbol{\Lambda}, \boldsymbol{\Sigma}) \exp(\mathbf{c}_1'(\mathbf{g} - \boldsymbol{\beta})) d\beta_N \cdots d\beta_2 d\beta_1 \\
&\quad + \int_{-\infty}^{g_1} \int_{-\infty}^{g_2} \cdots \int_{g_N}^{\infty} \phi(\boldsymbol{\beta}, \boldsymbol{\Lambda}, \boldsymbol{\Sigma}) \exp(\mathbf{c}_2'(\mathbf{g} - \boldsymbol{\beta})) d\beta_N \cdots d\beta_2 d\beta_1 \\
&\quad + \ldots + \int_{-\infty}^{g_1} \int_{g_2}^{\infty} \cdots \int_{g_N}^{\infty} \phi(\boldsymbol{\beta}, \boldsymbol{\Lambda}, \boldsymbol{\Sigma}) \exp(\mathbf{c}_{(N-1)}'(\mathbf{g} - \boldsymbol{\beta})) d\beta_N \cdots d\beta_2 d\beta_1 \\
&\quad + \int_{g_1}^{\infty} \int_{g_2}^{\infty} \cdots \int_{g_N}^{\infty} \phi(\boldsymbol{\beta}, \boldsymbol{\Lambda}, \boldsymbol{\Sigma}) \exp(\mathbf{c}_N'(\mathbf{g} - \boldsymbol{\beta})) d\beta_N \cdots d\beta_2 d\beta_1.
\end{aligned}$$

Using the property in Lemma 1, this can be rewritten as

$$\begin{aligned}
\pi_N^\beta &= \int_{-\infty}^{g_1} \int_{-\infty}^{g_2} \cdots \int_{-\infty}^{g_N} \phi(\boldsymbol{\beta}, \boldsymbol{\Lambda} - \boldsymbol{\Sigma}\mathbf{c}_1, \boldsymbol{\Sigma}) \cdot \\
&\qquad \exp(-\mathbf{c}_1'\boldsymbol{\Lambda} + \mathbf{c}_1'(\mathbf{g} + \frac{1}{2}\boldsymbol{\Sigma}\mathbf{c}_1)) d\beta_N \cdots d\beta_2 d\beta_1 \\
&\quad + \int_{-\infty}^{g_1} \int_{-\infty}^{g_2} \cdots \int_{g_N}^{\infty} \phi(\boldsymbol{\beta}, \boldsymbol{\Lambda} - \boldsymbol{\Sigma}\mathbf{c}_2, \boldsymbol{\Sigma}) \cdot \\
&\qquad \exp(-\mathbf{c}_2'\boldsymbol{\Lambda} + \mathbf{c}_2'(\mathbf{g} + \frac{1}{2}\boldsymbol{\Sigma}\mathbf{c}_2)) d\beta_N \cdots d\beta_2 d\beta_1
\end{aligned}$$

$$+\ldots+\int_{-\infty}^{g_1}\int_{g_2}^{\infty}\cdots\int_{g_N}^{\infty}\phi(\boldsymbol{\beta},\boldsymbol{\Lambda}-\boldsymbol{\Sigma}\mathbf{c}_{(N-1)},\boldsymbol{\Sigma})\cdot$$
$$\exp(-\mathbf{c}'_{(N-1)}\boldsymbol{\Lambda}+\mathbf{c}'_{(N-1)}(\mathbf{g}+\frac{1}{2}\boldsymbol{\Sigma}\mathbf{c}_{(N-1)}))d\beta_N\cdots d\beta_2 d\beta_1$$
$$+\int_{g_1}^{\infty}\int_{g_2}^{\infty}\cdots\int_{g_N}^{\infty}\phi(\boldsymbol{\beta},\boldsymbol{\Lambda}-\boldsymbol{\Sigma}\mathbf{c}_8,\boldsymbol{\Sigma})\cdot$$
$$\exp(-\mathbf{c}'_N\boldsymbol{\Lambda}+\mathbf{c}'_N(\mathbf{g}+\frac{1}{2}\boldsymbol{\Sigma}\mathbf{c}_N))d\beta_N\cdots d\beta_2 d\beta_1.$$

Next, converting to standard multivariate random variables by using the relation in (7), it follows that the limits of the integrals given by $\mathbf{g}$, are changed to $\hat{\boldsymbol{\alpha}}_{\boldsymbol{j}}$. Finally, by using standard symmetry properties for the multivariate normal distribution, we find that the cumulative multivariate normal probability distribution must be evaluated at the points $\boldsymbol{\alpha}_j=\hat{\mathbf{c}}_j\hat{\boldsymbol{\alpha}}_j$ with variance-covariance matrix $\hat{\mathbf{c}}_j\underline{\boldsymbol{\Sigma}}\hat{\mathbf{c}}_j$. The desired pricing formula then follows.

B Appendix.

The proof partially follows from the proof for the guarantee on the money market account.

We let $1_{\bar{\mathbf{c}}_j}$ be an indicator function returning the value 1 when $\bar{\mathbf{c}}_j$ is true and 0 otherwise. Again, using the linearity of the expectation operator, the expectation in (9) can be written as

$$\pi_N^{\delta}=\sum_{j=1}^{2^N}E_Q\Big[e^{\mathbf{c}'_j\bar{\mathbf{g}}-\bar{\mathbf{c}}'_j\bar{\boldsymbol{\beta}}}1_{\bar{\mathbf{c}}_j}\Big].$$

Let $E_Q[1_{\bar{\mathbf{c}}_j}]=Q(\bar{\mathbf{c}}_j)$ be the probability under the equivalent martingale measure Q for the state $\bar{\mathbf{c}}_j$. It follows directly from Lemma 1 that we can, for each j, construct a probability measure $Q_{\bar{\mathbf{c}}_j}$ equivalent to Q. $Q_{\bar{\mathbf{c}}_j}$ is defined by

$$\frac{dQ_{\bar{\mathbf{c}}_j}}{dQ}=\frac{e^{-\bar{\mathbf{c}}'_j\bar{\boldsymbol{\Lambda}}+\frac{1}{2}\bar{\mathbf{c}}'_j\bar{\boldsymbol{\Sigma}}\bar{\mathbf{c}}_j}}{e^{-\bar{\mathbf{c}}'_j\bar{\boldsymbol{\beta}}}}.$$

It then follows that

$$\pi_N^{\delta}=\sum_{j=1}^{2^N}e^{\mathbf{c}'_j\bar{\mathbf{g}}-\bar{\mathbf{c}}'_j\bar{\boldsymbol{\Lambda}}+\frac{1}{2}\bar{\mathbf{c}}'_j\bar{\boldsymbol{\Sigma}}\bar{\mathbf{c}}_j}Q_{\bar{\mathbf{c}}_j}(\bar{\mathbf{c}}_j),$$

where the expectation of $\bar{\boldsymbol{\beta}}$ under $Q_{\bar{\mathbf{c}}_j}$ is from Lemma 1 seen to be given by

$$\bar{\boldsymbol{\Lambda}}_{Q_{\bar{\mathbf{c}}_j}}=\bar{\boldsymbol{\beta}}-\bar{\boldsymbol{\Sigma}}\bar{\mathbf{c}}_j.$$

$Q_{\bar{\mathbf{c}}_j}(\bar{\mathbf{c}}_j)$ is determined by the cumulative multivariate normal probability distribution evaluated at the points determined by the vector $\bar{\boldsymbol{\alpha}}_j^{\delta}$. The i'th element of $\bar{\boldsymbol{\alpha}}_j^{\delta}$, $i \in \{N+1, N+2, \ldots, 2N\}$, follows after changing to standard multivariate normal random variables under the probability measures $Q_{\bar{\mathbf{c}}_j}$ and exploiting symmetry properties for the cumulative multivariate normal probability distribution. We have that

$$\bar{\bar{\alpha}}_{j,i} = \frac{g_i - (\bar{\boldsymbol{\Lambda}}_{Q_{\bar{\mathbf{c}}_j}})_i}{\sigma_{\delta_i}},$$

where $(\bar{\boldsymbol{\Lambda}}_{Q_{\bar{\mathbf{c}}_j}})_i$ is the $N+i$'th element of the vector $\bar{\boldsymbol{\Lambda}}_{Q_{\bar{\mathbf{c}}_j}}$. Since $\boldsymbol{\beta}$ has no upper or lower limit as the vector $\mathbf{g}$ for $\boldsymbol{\delta}$, it is easily seen that the $2N$-dimensional multivariate normal cumulative probability distribution is reduced to an cumulative N-dimensional multivariate normal probability distribution. Finally, using symmetry properties for the cumulative multivariate normal probability distribution, we can proceed as for the money market account, and it then follows that the distribution must be evaluated at the points determined by the vector $\bar{\boldsymbol{\alpha}}_j^{\delta}$. The desired pricing formula then follows.

References

Amin, K. I. and Jarrow, R. (1992). "Pricing Options on Risky Assets in a Stochastic Interest Rate Economy", *Mathematical Finance*, *2*(4), 217–237.

Black, F. and Scholes, M. (1973). "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, *81*(3), 637–654.

Brennan, M. J. and Schwartz, E. S. (1976). "The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee", *Journal of Financial Economics*, *3*(2), 195–213.

Briys, E. and de Varenne, F. (1997). "On the Risk of Insurance Liabilities: Debunking Some Common Pitfalls", *Journal of Risk and Insurance*, *64*(4), 673–694.

Donselaar, J. (1999). "Guaranteed returns: Risks assured?", *Proceedings AFIR 1999*, 195–203.

Duffie, D. (1996). *Dynamic Asset Pricing Theory.* Princeton University Press, Princeton, New Jersey.

Genz, A. (1992). "Numerical Computation of Multivariate Normal Probabilities", *Journal of Computational Graph Statistics*, pp. 141–149.

Grosen, A. and Jørgensen, P. L. (1997). "Valuation of Early Exercisable Interest Rate Guarantees", *Journal of Risk and Insurance*, *64*(3), 481–503.

Grosen, A. and Jørgensen, P. L. (2000). "Fair Valuation of Life Insurance Liabilities: The Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies", *Insurance: Mathematics and Economics*, *26*(1), 37–57.

Hansen, M. and Miltersen, K. (2000). "Minimum Rate of Return Guarantees: The Danish Case", Working Paper, Department of Management, School of Business and Economics, Odense University.

Heath, D., Jarrow, R., and Morton, A. (1992). "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation", *Econometrica*, *25*(1), 77–106.

Hipp, C. (1996). "Options for Guaranteed Index-Linked Life Insurance", *Proceedings AFIR 1996*, pp. 1463–1483.

Miltersen, K. and Persson, S.-A. (1999). "Pricing Rate of Return Guarantees in a Heath-Jarrow-Morton Framework", *Insurance: Mathematics and Economics*, *25*(3), 307–326.

Miltersen, K. R. and Person, S.-A. (2000). "Guaranteed Investment Contracts: Distributed and Undistributed Excess Return", Discussion Paper no. 1/00, Department of Finance and Management Science, Norwegian School of Economics and Business Administration.

Musiela, M. and Rutkowski, M. (1997). *Martingale Methods in Financial Modeling*. Springer Verlag, Berlin Heidelberg.

Persson, S.-A. and Aase, K. (1997). "Valuation of the Minimum Guaranteed Return embedded in Life Insurance Contracts", *Journal of Risk and Insurance*, pp. 599–617.

Reffs, C. (1998). *Rentegarantier på kredit- og forsikringsprodukter analyseret i et Markovmiljø*. Københavns Universitet, Forsikringsmatematisk Laboratorium.

Vasicek, O. A. (1977). "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics*, *5*, 177–188.