Representation of preferences á la Savage with a general probability measure.

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Abstract

This paper addresses two weaknesses of the subjective expected utility representation of Savage: The first is that the resulting subjective probability measure P is atomless only, the second is that P is only finitely additive. We give conditions under which a numerical representation of preferences is an expected utility á la Savage, but with respect to an arbitrary, countably additive probability measure.

Savage has seven axioms in his theory, some of which are rather hard to interpret in an economic setting. One advantage with the theorem of this paper is that, essentially, one has to relate to only four axioms for the general representation to hold, all of which easy to interpret in economic terms.

KEYWORDS: subjective expected utility, numerical representation of preferences, tightness of probability measures, relative compactness

1 Introduction

The two classical numerical representations of preferences in presence of uncertainty is the von Neumann-Morgenstern (1944) theory using objective probabilities and the Savage (1954) theory based on subjective probabilities. These theories appear in different settings, formally speaking, and also have somewhat different situations of choice in mind. The most widely applicable one to economists is that of Leonard Savage, especially for the purpose of modelling general equilibrium. In his theory a probability distribution is, for each agent, also part of the representation of preferences, not only a utility function as in the von Neumann-Morgenstern theory. In between, Anscombe and Aumann (1963) have derived a characterization for a two-stage decision problem with subjective probabilities in the first stage and objective probabilities in the second stage. Since in the general equilibrium model there are hardly any objective probabilities, at most Savage's approach seems applicable.

In a choice between subjective and objective probabilities, the recent unfortunate events on September 11, 2001 in New York and Washington is illustrative: Looking at the Twin Towers in isolation, they were owned by the Port Authority of New York, so the buildings had most likely not property insurance, but shortly before the incident took place, the towers had been leased to a private businessman, who bought insurance for the buildings in the amount of USD 3.5 Billion.

Another example is the award offered in 1971 by the whisky producer Cutty Sark of one million pounds for the capture of the monster assumed to exist in Loch Ness. Apparently somebody in the higher echelons of Cutty Sark got cold feet after the offer had been made, and approached Lloyd's in London. As usual Lloyd's was obliging, and agreed to cover the risk for a premium of 2500 pounds. The case was appropriately handled by the marine underwriters (see Borch (1976)).

If the requirement is that it must be possible to estimate the probabilities of the relevant events from past observations alone, in order for a risk to be insurable, insurance contracts would not have existed for the two situations described above, and in numerous other cases treated by the insurance industry every day.

On rare occasions real life presents the realization of events which are as outlandish as anything which can be dreamed up at a philosopher's desk. In the September 11. event it seems doubtful that the observed events were even included in the state space. These problems deserve particular attention, because they actually happened.

Savage originally developed his theory under a set of axioms, where one of them leads to the conclusion that this probability measure P is atomless, i.e., $P(\omega) = 0$ for all $\omega \in \Omega$. On the other hand, this theory is usually illustrated in microeconomics textbooks and scientific papers when there is only a finite number of outcomes in the state space Ω, so that the utility representation is claimed to be given by a sum

$$V(h) = \sum_{i=1}^{n} p_i u(h(\omega_i)). \tag{1}$$

Here Z is a set of consequences, or prizes, and H is the set of all functions from Ω to Z. The elements $h \in H$ are called acts, or horse race lotteries, and it is on the set H that the preference relation $\succsim$ is defined. Thus the decision maker will choose among the elements $h \in H$. In contrast, in the von Neumann-Morgenstern theory the decision maker will choose between the set of probability measures on $(Z, \mathcal{B})$, where $\mathcal{B}$ signify the Borel sets in Z. The utility function u is then defined from Z to R, the latter being the real numbers. The set Z is a separable, complete metric space, which in many cases can be thought of as the set of real numbers R, or the Euclidean m-space R^m.

Strictly speaking the representation given in (1) is not valid according to Savage's seven axioms. Stigum (1972), Hens (1992) and Lensberg (1999) all provide different sets of axioms ensuring that the representation above is a numerical representation of $\succsim$ on H, i.e., $h \succsim g$ if and only if $V(h) \geq V(g)$. By this we mean that provided these axioms are valid for the preference relation $\succsim$ on H, there exists a probability measure P given by a set of discrete probabilities $(p_1, p_2, \ldots, p_n)$ on $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$, where $P(\{\omega_i\}) = p_i$, and a Bernoulli utility function $u : Z \rightarrow R$ such that V on H given by (1) numerically represents $\succsim$ on H in the above described manner. This is thus a characterization of subjective probabilities in a model with a continuum of consequences and a finite number of states.

This paper will try to establish a Savage type representation in the general case, that is for a probability distribution that can be continuous, discrete, a combination thereof, or otherwise. In this connection we notice that Savage's probability P is only finitely additive. This has to do with the fact that it is defined on the set of *all* the subsets of Ω. In contrast, we will obtain a countably additive probability measure P in our representation. By taking as the starting point a representation in the finite case, we will give a condition on the set of probability measures generated on the space Z induced by the acts $h \in H$, as the number of states in Ω grows, that will provide the required representation of preferences, valid also for an arbitrary probability measure P. Since this probability also is a part of the representation of preferences, it is a subjective probability measure defined on a σ-field $\mathcal{F}$ in the set of states Ω.

We will think of the acts h as measurable mappings from Ω to Z, to wit, $h^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}$. The resulting probability measure P will then be defined on $\mathcal{F}$ only, but thus in turn be countably additive. More importantly, it will be a general probability measure, that need not be atomless, nor need it be purely discrete.

In short, to obtain this representation on an arbitrary state space $(\Omega, \mathcal{F})$, we start out with a finite state space, establish the representation (1) on this set, and then we refine the state space, and move to the induced state space $(Z, \mathcal{B})$, where we have the required mathematical structure.

As mentioned above, the first part, that of obtaining a representation on a finite state space, has been treated in Hens (1992). Lensberg established a corresponding representation at about the same time, published in 1999 (in Norwegian). Furthermore, Stigum (1972) has a an expected utility representation for a finite number of states of the world. His representation is based on preferences that can be represented by an absolutely continuous utility function that has finite positive partial derivatives almost everywhere on $[0, \infty]$. Hens (1992) bases his derivation on on the slightly stronger concept of continuous differentiability, resulting in simpler proofs. Stigum's axioms also provide a bit too much, since it gives a strictly concave utility function. In general, we would prefer to keep the question of weather an agent is an expected utility maximizer distinct from his risk attitude.

In section 3 we follow Lensberg, by and large, but avoid the quasi-concavity of preferences inherent in his approach. We generalize both Hens' and Lensberg's treatments from a euclidean space of consequences, to a separable and complete metric space Z, the appropriate setting for our approach.

The paper is organized as follows. In section 2 we give a brief account of the model and the notation, and present some simple examples. In section 3 the theory with a finite number of states of the world and a continuum of consequences is briefly presented. In section 4 we present the general representation theory, based on weak convergence of probability measures and the concept of tightness. Section 5 concludes.

2 Model and notation

Let Ω = the set of states and $\mathcal{F}$ = the set of events, a σ-algebra in Ω. Let Z = set of consequences, or prizes, $\mathcal{B}$ = collection of Borel sets in Z. We assume Z is a separable and complete metric space. H = set of all measurable mappings $h : \Omega \to Z$.

Example 1. Consider the following risk sharing model. We have a group of I agents, having preferences $\succeq_i$ over a suitable set of random variables with realizations (outcomes) in some $A \subseteq R$. Each agent is endowed with a random payoff w_i called his initial portfolio. Let $w = (w_1, w_2, \dots, w_I)$.

To wit, there exists a probability space $(\Omega, \mathcal{F}, P)$ such that i is entitled to payoff $w_i(\omega)$ when $\omega \in \Omega$ occurs. The possible events $\mathcal{F} = \mathcal{F}^w =: \sigma(w_1, w_2, \dots, w_I)$ is the σ-field generated by the initial random vector w, so that any random variable h in this model can be written in the form $h = f(w_1, w_2, \dots, w_I)$ for f a suitable real, Borel-measurable function.

The preferences are represented by expected utility, meaning that for any two $\mathcal{F}$-measurable random variables h and g, there is a set of continuous utility indices $u_i : R \to R$, such that $h \succeq_i g$ if and only if $Eu_i(h) \geq Eu_i(g)$. $\square$

This is a very common set-up, but it is rather unprecise. The question is, do we here mean von Neumann-Morgenstern expected utility, or do we mean Savage expected utility?

To be considered as the former, the preference relation should indeed be defined on a set of probability measures, not on a set of random variables. Furthermore, different random variables may have the same probability distribution. To be the latter, the probability measure should be atomless and finitely additive, but we have here a σ-field $\mathcal{F}$, we have genuine random variables, not just merely acts, and we have said nothing about the probability distribution of the random vector w. Furthermore, we usually think of P as being countably additive, and employ the axioms of probability theory that depend on this assumption.

We could also have one probability measure P_i for each agent. Assuming that the agents have common beliefs, the above description can, however, be seen to be in accordance with the representation theorem we develop in this paper.

Why is the subjective probability theory relevant in economics? Let us consider a very simple example.

Example 2. In a choice between von Neumann-Morgenstern expected utility and the Savage type expected utility, one may consider the scenario:
h - If Manchester United wins the European cupwinner cup next year, you receive 1000, otherwise you get zero.
g - You win 1000 if a fair coin gives heads four times in a row, otherwise you get nothing.

You get to choose between h and g. In the von Neumann-Morgenstern theory this type of situation is not an interesting problem of choice, depending upon a utility function u of the agent (as long as he prefers more to less):

$$Eu(h) = pu(1000) + (1-p)u(0) = p$$

$$Eu(g) = (\frac{1}{2})^4 u(1000) + (1 - (\frac{1}{2})^4)u(0) = \frac{1}{16},$$

assuming we normalize to $u(1000) = 1, u(0) = 0$, which we always can, since u is unique modulo an affine transformation. Here p is objectively known, so every decision maker in the von Neumann-Morgenstern world must arrive at the same conclusion, regardless of his/her utility function u. $\square$

Let $\succeq$ be a preference relation on H, i.e., a complete and transitive binary relation, and let $\succeq_z$ be a preference ordering on Z, such that
1) $\succeq$ on H is consistent with $\succeq_z$ on Z (meaning that if h_z is the function $h_z(\omega) = z$ for all $\omega \in \Omega$, then for arbitrary $z, z' \in Z$ and $h_z, h_{z'} \in H$, $h_z \succeq h_{z'}$ if and only if $z \succeq_z z'$).
2) The preferences are "smooth" (indifference curves exist and are continuously differentiable).

By Debreu (1959a) and further by Mas-Colell (1985) there then exists a continuously differentiable utility function $V : H \to R$ numerically representing $\succeq$ on H: $h \succeq g$ if and only if $V(h) \geq V(g)$ for all $h, g \in H$.
3) *The sure thing principle.* Suppose we are comparing h and $g \in H$. Suppose moreover that the set of states Ω contains a subset B on which h and g are identical; for every $\omega \in B$, $h(\omega) = g(\omega)$. Then how the agent feels about h compared to g depends only on how h and g compares on states that are not in B.

Formally, if $h \succeq g$ and h' and g' are two other acts such that
(a) h is identical with h', and g is identical with g' on B^c.
(b) h' and g' are identical on B, h and g are identical on B,
then $h' \succeq g'$ must follow.

This principle says that preferences between acts should not depend on those states that have identical consequences for the two acts, or, the ordering of h and g does not depend on the specific way that they agree on B - that they agree is enough.

In our differentiable setting the sure thing principle implies that for every good, the marginal rate of substitution between two states is independent of the amount of goods consumed in other states.

This principle is found reasonable by many individuals provided that the state that obtains does not depend on the act that is actually implemented.

Let D be the risk free acts, i.e., those that are constant on Ω: $h \in D$ if and only if $h(\omega) = d$ for all $\omega \in \Omega$, for some $d \in Z$.

4) Along D the indifference curves are constant.

A *bet* on an event is an act offering a specific consequence if one of the states in the event prevails, and offering a less preferred consequence in the remaining states.

As an example, consider a bet h giving USD 2 in state 1, and a loss of USD 1 in state 2. This is a bet on state 1, and the "odds" are 2:1. Suppose the decision maker can choose if he wants to bet on state 1 or state 2. Consider the risk free initial position d, and suppose that the decision maker prefers to bet on state 1, in other words, that $d + h \succ d - h$. Next we donate the decision maker an amount so that his risk free position changes to d', and ask if he still prefers to bet on state 1. If the decision maker claims that he now prefers to bet on state 2, it is natural to say that he has changed his mind with regard to whether the odds of 2:1 in favor of state 1 is still favorable. Savage excludes this type of inconsistent attitude towards bets.

This means that if a bet on h is preferred to a bet on $-h$ at level of wealth d, we will assume that h will also be preferred to to $-h$ at any other level of wealth d': Formally, $d + h \succeq d - h$ then $d' + h \succeq d' - h$ for all $d, d' \in D$ and $h \in H$ such that $d + h \in H$, $d - h \in H$, and similarly for d'.

3 The finite representation

Let Ω_n contain a finite number of points, $\Omega_n = \{\omega_1, \omega_2, \dots, \omega_n\}$. For simplicity of notation we refer to these as $\Omega_n = \{1, 2, \dots, n\}$. Let us first hold n fixed, and demonstrate that the numerical representation V has the required representation under the above assumptions on preferences, i.e., that there exists a function $u : Z \to R$, and probabilities p_i, $i = 1, 2, \dots, n$ such that $V(h) = \sum_{i=1}^{n} p_i u(h_i)$.

To this end, suppose the decision maker chooses an act h, and let $V_i'(h) = \frac{\partial V}{\partial h_i}(h)$, where $h = (h_1, h_2, \dots, h_n)$. The marginal rate of substitution of prizes between states i and j is defined as

$$m^{ij}(h) = \frac{V_i'(h)}{V_j'(h)}.$$

According to the sure thing principle it follows that $m^{ij}(h) = m^{ij}(h_i, h_j)$. Consider a third state k. Then we may write $m^{ij}(h_i, h_j)$ as follows

$$m^{ij}(h_i, h_j) = \frac{V_i'(h)}{V_j'(h)} = \frac{V_i'(h)/V_k'(h)}{V_j'(h)/V_k'(h)}) = \frac{m^{ik}(h_i, h_k)}{m^{jk}(h_j, h_k)}.$$

From Leontif's separability theorem (see e.g., Debreu (1959b)[1]) we then know that the preference relation can be represented by an additive function, i.e., there exist functions $u_i : Z \to R$, $i = 1, 2, \ldots, n$ such that

$$h \succsim g \Leftrightarrow \sum_{i=1}^{n} u_i(h_i) \geq \sum_{i=1}^{n} u_i(g_i)$$

for any h and $g \in H$.

By the consistency requirement 1), since each u_i is a numerical representation of $\succsim_z$ on Z, these functions must be identical up to strictly increasing rescalings $\vartheta_i : R \to R$, that is, there exists a *state utility function* $v : Z \to R$ representing the agent's basic tastes over consequences, such that $u_i(h) = \vartheta_i(v(h))$, $i = 1, 2, \ldots, n$. Thus the relation $\succsim$ on H can be written

$$h \succsim g \Leftrightarrow \sum_{i=1}^{n} \vartheta_i(v(h_i)) \geq \sum_{i=1}^{n} \vartheta_i(v(g_i)).$$

Next we want to establish the relation (1). To this end, define for each ω_i the function $P_i(h) : H \to R$ by

$$P_i(h) =: \frac{\vartheta_i'(v(h_i))}{\sum_{j \in \Omega_n} \vartheta_j'(v(h_j))}. \tag{2}$$

This function is proportional to the gradient of $V(h)$, and $\sum_{j \in \Omega_n} P_j(h) = 1$ for all $h \in H$.[2] Now requirement 4) implies that this function is constant on D. Thus $P_i(h_d) = p_i$ for $i = 1, 2, \ldots, n$ and for all $h_d \in D$. Consequently

$$\frac{\vartheta_i'(v(h_d))}{\vartheta_j'(v(h_d))} = \frac{p_i}{p_j},$$

[1]Not based on derivatives.

[2]Note the similarity between (2), and the likelihood ratio between the "risk adjusted" probability and the given measure in models of exchange equilibria, where the latter quantity is given by the marginal utility (of the representative agent).

or, for some differential function ϑ it must be the case that

$$\frac{\vartheta_1'(v(h_d))}{p_1} = \frac{\vartheta_2'(v(h_d))}{p_2} = \ldots = \frac{\vartheta_n'(v(h_d))}{p_n} = \vartheta'(v(h_d))$$

for all $h_d \in D$, independent of the state i. This means that $p_i \vartheta'(v(h_i) = \vartheta_i'(v(h_i)$ for all i and any $h_i \in Z$. Integrating over the range of the function v gives

$$\vartheta_i(v(h_i)) = p_i \vartheta(v(h_i)) + k_i$$

for appropriate integration constants k_i, $i = 1, 2, \ldots, n$. This gives

$$V(h) = \sum_{i=1}^{n} \vartheta_i(v(h_i)) = \sum_{i=1}^{n} (p_i \vartheta(v(h_i)) + k_i) = \sum_{i=1}^{n} p_i \vartheta(v(h_i)) + k$$

for $k = \sum_i k_i$. Choosing $u(\cdot) = \vartheta(v(\cdot))$, and using the fact that expected utility is unique modulo an affine transformation, we get the required representation (1).

4 The general representation

We now want to demonstrate the following:

Theorem 1 *Suppose the preference relation $\succeq$ on H satisfies 1)-4). Then it is possible to construct a numerical representation of $\succeq$ given by*

$$V(h) = \int_{\Omega} u(h(\omega))\, dP(\omega), \qquad h \in H \tag{3}$$

where $u : Z \to R$ is a Bernoulli utility function, and P is a countably additive probability measure on Ω.

Note that P need not be atomless, nor discrete only.

The construction is carried out as follows: Let Ω_n contain a finite number of points of Ω, and $\Omega_n \subseteq \Omega_{n+1} \subseteq \ldots \subseteq \Omega$. Here we assume that $\Omega = \cup_{n=1}^{\infty} \Omega_n$, $\Omega_n \in \mathcal{F}$. Let $\mathcal{F}_n$ be the corresponding restriction of $\mathcal{F}$ to Ω_n. Any function $h \in H$ is automatically defined on Ω_n by its natural restriction to this set.

The function $u : Z \to R$ is continuous, and can also be shown to be bounded. The latter we prove as follows: For fixed n consider the representation

$$V(h) = \int_Z u(z)d(P_n h^{-1})(z) =: U(P_n h^{-1}).$$

Since P_n is a countably additive probability measure on $(\Omega, \mathcal{F}_n)$, and h is a function in H, $P_n h^{-1}(\cdot) =: \mu_h(\cdot)$ is a countably additive probability measure on $(Z, \mathcal{B})$. As h varies in H, the functional $U(\cdot)$ is continuous in the topology generated by weak convergence of probability measures. Moreover $U(\cdot)$ is also linear, i.e.,

$$U(a\mu_h + b\mu_g) = aU(\mu_h) + bU(\mu_g).$$

Hence $U(\cdot)$ is a continuous, linear functional on the set of probability measures $\{\mu_h : h \in H\}$, and is consequently bounded on this set. For every $z \in Z$, let h_z be the act in H such that $\mu_{h_z} = \delta_z$, where δ_z is the probability measure in $\{\mu_h : h \in H\}$ that puts all its probability mass in the point z, i.e., $h_z(\omega_1) = h_z(\omega_2) = \ldots = h_z(\omega_n) = z$. Then, from the above representation it follows that $U(P_n h_z^{-1}) = u(z)$ for all $z \in Z$, and this implies that $u(\cdot)$ is bounded on Z.

The probability measure we have just constructed on Ω_n we denote by P_n. We now increase n and consider the effect of any given act h along the sequence of probability measures P_n. Since the sequence $P_n h^{-1}$ of probability measures is defined on the measurable space Z, we may utilize the nice mathematical structure on the this space: First notice that

$$V(h) = \sum_{\omega \in \Omega_n} u(h(\omega))P_n(\omega) = \int_{\Omega_n} u(h(\omega))dP_n(\omega)$$

$$= \int_Z u(z)d(P_n h^{-1})(z).$$

The first equality is just a matter of notation. The second involves a change of variable. Let us now suppose that this probability measure is *tight*, i.e., for all $\epsilon > 0$ there exists a compact K in Z such that $P_n h^{-1}(K) > 1 - \epsilon$. This seems like an innocuous assumption for any *given* probability measure on Z, and it turns out that it really is: If Z is separable and complete (as we assume it is), then *every* probability measure on $(Z, \mathcal{B})$ is tight. Now we make the following assumption:

Assumption 1 *The collection of probability measures* $\Gamma_h := \{P_n h^{-1} : n = 1, 2, \dots\}$ *is tight, i.e., for all* $\epsilon > 0$ *there exists a compact* $K \in \mathcal{B}$ *such that* $P_n h^{-1}(K) > 1 - \epsilon$ *for all* $n = 1, 2, \dots$

The above assumption is considered very weak in general. That this involves a restriction, can be seen from the following examples:

Example 3. Let $P_n h^{-1}(-n) = \frac{1}{2}$, and $P_n h^{-1}(n) = \frac{1}{2}$. Then this sequence of probability measures is not tight: The probability mass runs out to infinity as n grows. Another example is the following sequence of "uniform" distributions: $P_n h^{-1}(1) = P_n h^{-1}(2) = \dots = P_n h^{-1}(n) = 1/n$. Here the rightmost tails of the distributions become too heavy as n grows. $\square$

In the above examples, for each h, note that $(P_n h^{-1})$ is a probability distribution on the set of *consequences* Z. Consider e.g., the above uniform example: According to equation (2), describing how probabilities are assessed by the decision maker, the marginal utility of large consequences are not decreasing sufficiently fast, relative to the marginal utility of more moderate $z-$values. Most people would agree to some sort of decreasing marginal utility, in which case these type of examples are effectively ruled out.

There are other reasons why these classes of distributions do not make much economic sense, as n grows beyond any limit - one being simply that the resources in the world are limited. If this is not reflected in the range of possible values of consequences, it should certainly be reflected in the corresponding probabilities of extreme values, as compared to more plausible ones.

Definition 1 μ_n *converges weakly to the probability measure* μ *on* $(Z, \mathcal{B})$ *(notation* $\mu_n \Rightarrow \mu$*) if and only if* $\int_Z v(z) d\mu_n(z) \to \int_Z v(z) d\mu(z)$ *as* $n \to \infty$ *for all continuous and bounded real functions* $v : Z \to R$.

Definition 2 *A family* Γ *of probability measures on* $(Z, \mathcal{B})$ *is called relatively compact if every sequence of elements of* Γ *contains a weakly convergent subsequence.*

By *Prohorov's theorem* (see e.g., Billingsley (1968)), if Γ_h is tight, then it is relatively compact, and thus there is a subsequence $\{P_{n'} h^{-1}\}$ such that $P_{n'} h^{-1} \Rightarrow P h^{-1}$, or

$$V(h) = \int_Z u(z) d(P_{n'} h^{-1})(z) \to \int_Z u(z) d(P h^{-1})(z),$$

as $n' \to \infty$, since u is continuous and bounded (and thus will do as a test function v). But

$$\int_Z u(z)d(Ph^{-1})(z) = \int_\Omega u(h(\omega)dP(\omega),$$

by (the now) standard change of variable. Thus we obtain the required representation

$$V(h) = \int_\Omega u(h(\omega)dP(\omega).$$

The measure space Z is assumed to be separable, so the set F of all probability measures on $(Z, \mathcal{B})$, endowed with the topology generated by weak convergence of probability measures in F, is also known to be separable, i.e., any probability measure μ in F can be approximated in the weak sense by simple probability measures with a finite domain. The limiting measure P on $(\Omega, \mathcal{F})$ can accordingly be of arbitrary type, depending on the nature of Ω, and is countably additive. Since the probability measure P is approximated by subjectively derived probability measures P_n, it must therefore itself be derived from preferences.

5 Conclusions

When representing preferences in models of general competitive equilibrium, a numerical representation of preferences cannot be dispensed with. The subjective expected utility representation of Savage is the one that seems most useful among the three representations considered in this paper. This is not to say that there are no problems with this representation, in that it does not always give the right predictions of what people actually do in various situations of choice in a world of uncertainty. These problems have been discussed intensively in the literature, starting already in the 50ties (e.g., Ellsberg (1961)).

The topic of this paper has been some weaknesses of the original mathematical representation: In Savage's original treatment the subjective probability measure is atomless only, and it is also finitely additive. In general we have situations in mind where neither is fulfilled: It is natural, and common, to work with probability measures which are countably additive, since most of the theory of random variables, and stochastic processes, is based on this assumption. Furthermore it is not natural at the outset to limit this probability measure to be atomless only, so that $P(\omega) = 0$ *for all* $\omega \in \Omega$.

In this paper we have presented conditions under which a numerical representation of preferences can be given by expected utility á la Savage, but with respect to an arbitrary, countably additive probability measure P. This we have done by going via a representation, valid for a finite state space, and then increased the number of states. We use a transformation through acts to a sequence of probability measures on a separable, complete metric space, the set of consequences, where we have the required mathematical structure, of tightness and relative compactness, to obtain our representation, using Prohorov's theorem. Our additional assumption about tightness turns out to be a very weak requirement in light of its economic implications. It is also considered a weak mathematical requirement.

Savage has seven axioms in his theory, some of which are rather hard to interpret in economics. One advantage with the theorem of this paper is that, essentially, one has to relate to only four axioms for the general numerical representation to hold, axioms that economists seem more or less familiar with.

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