Optimal control of a renewable natural resources and the "stochastically induced critical depensation"

Arne-Christian Lund*

December 20, 2002

Abstract

This paper focus aspects connected to the optimal control of a renewable resource modelled by a stochastic differential equation. The main point is to show how small changes of the problem may cause severe changes in the properties of the control. In particular we show how the introduction of uncertainty may lead to a less conservative policy. In this context we introduce the notion "induced critical depensation". We also demonstrate how a stochastic process may be analysed by comparison with a simpler process.

*Institute of Finance and Management Science, Norwegian School of Economics and Business Administration, Helleveien 30, 5045 Bergen, Norway. Email arne-christian.lund@nhh.no.
The author wish to thank Leif K. Sandal for valuable comments.

1 Introduction

In this paper we study the optimal control of a renewable resource. We assume that a sole owner manages a resource modelled as a continuous time stochastic process. The optimal exploitation rate is identified, when given a function describing the instantaneous profit from the harvest. It is not the scope of this paper to describe the optimal policy for a specified natural resource. We try to illuminate some important facts a real world stock optimiser must keep in mind.

It is well known how seemingly small perturbations in the specification of a problem may alter the optimal control substantially. This may in turn lead to the extinction of the resource, if the cost of extinction is small. In particular, the effects of a large discount factor is well recognised. The point is that with large discounting the optimal program tend to move consumption closer to the present. The resource is not allowed to grow as much as with lower discounting. If the natural growth rate of the resource is small (e.g. close to zero) it may be economically optimal to "mine" the resource, that is, harvest the total resource, sell it and let the money grow in the bank. This is illustrated later in an example.

In the classical book by Clark [2] the dynamics of continuous deterministic models is categorised as "depensation" and "compensation" models. The growth function is called a depensation curve if it is convex for small levels of the stock. If not it is called a compensation curve. Further, if the growth rate is negative for small stock levels, we say that the resource possess "critical" depensation. For such processes the stock will go extinct if it is reduced below a certain level. With critical depensation it may be optimal to harvest the resource in this region, the situation from a stock conservational view is hopeless anyway. Here extreme care must be taken by the practitioner. What if there is no critical depensation in the real world stock? Introducing this seemingly conservative attitude in the model may then actually lead to the extinction of a reproducing stock! This point can not be emphasised enough, and has not been given the attention it deserves in the literature.

In this work we mainly focus on another important aspect of bio economic resource models with a stochastic resource. Our main point is to show that the introduction of uncertainty into a biological model may or may not lead to a more conservative attitude. We also want to demonstrate the somewhat contra-intuitive fact that harvest policies may be non monotone in the noise. This is to the best of our knowledge not documented earlier in the literature. In the paper we first demonstrate this on an example model. Then we show why this is a natural consequence of the process specification. This is followed by sections demonstrating how the problem may be analysed analytically to explain the inherent properties of the problem.

1.1 The setup

The optimisation problem is formulated as a continuous control problem over an infinite horizon. See e.g. Fleming and Soner [3], Bardi and Capuzzo-Dolcetta [1] or Kamien and Schwartz [4] for thorough presentations of such problems. The stock is described by a stochastic process of the form

$$dx_t = (f(x_t) - u_t)dt + \sigma(x_t)dB_t,$$

where u_t denotes the harvest to be optimised. We assume that this process describes the dynamics of the aggregated stock biomass. No breeding or diffusion from another stock is possible. Therefore

$$x(\hat{t}) = 0 \Rightarrow x(t) = 0, \quad \forall \quad t \geq \hat{t}.$$

The object is to maximise the present value

$$\max_{u \geq 0} \int_0^\infty e^{-\delta t} \Pi(x_t, u_t) dt. \tag{1}$$

The value given $x_0 = x$ and the optimal control can be found by solving the Hamilton-Jacobi-Bellman equation

$$-\delta V + f(x)V_x + \frac{1}{2}\sigma^2(x)V_{xx} + \max_{u \geq 0}\{\Pi(x, u) - uV_x\} = 0$$

We assume that the profit function Π is on the form

$$\Pi(x, y) = \frac{a \cdot y}{\left(\frac{y}{40}\right)^c + b} - 32 \cdot 10^{-5} \cdot \frac{y^2}{x^d}$$

where $a = \frac{11}{6}$, $b = \frac{5}{6}$, $c = \frac{6}{10}$ and $d = \frac{5}{100}$. This is a simplifying approximation of the profit function used in the paper [11]. For positive y observe that this function approach infinity slowly[1] as $x \to 0$. Maximising Π as a function of y gives the "Bliss" control. At this level the instantaneous profit is maximised, without considerations about the long term effects on the stock. Normally this constitutes an upper limit for the optimal control.

We solve the optimisation problem numerically. See Kushner and Dupuis [7] for an excellent presentation of how such problems may be solved with the computer. Fleming and Soner [3] also gives a condensed presentation based on Kushner's earlier work. Both books show how the convergence of numerical schemes may be proved. The book by Bardi and Capuzzo-Dolcetta [1] also has a large section written by M. Falcone with focus on the solution of deterministic control problems. The focus of the present paper is on the control implications the optimisation gives. We therefore skip the technical details concerning the numerics.

Remark 1.1. In most of this paper we consider problems with "zero discounting". When $\delta \equiv 0$ the integral in (1) may be infinite, and the optimisation problem is not well posed. We are here interested in the optimal control policy, not the value. This policy is continuous as $\delta \to 0$. We can therefore approximate the zero discounting control using a small δ. The term "zero discounting" may in the following be interpreted as "negligible discounting".

[1] This is done to simplify the presentation. The effects we present here may occur in a model where the cost of harvest close to zero stock is large, but not too large.

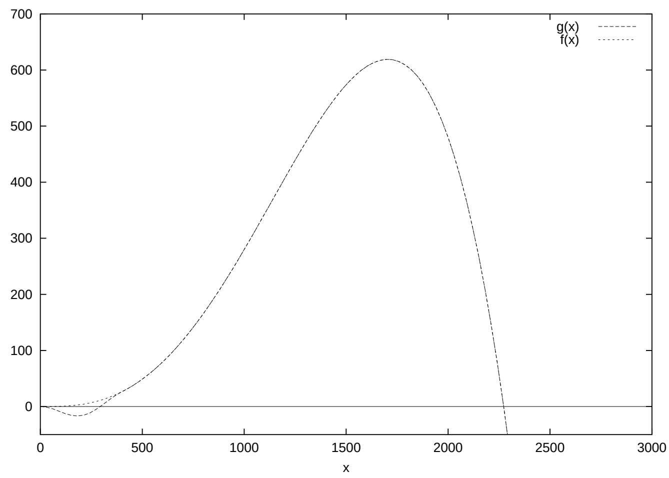

Figure 1: The growth functions.

2 The deterministic case

In the first part of this paper we focus on deterministic models (let $\sigma \equiv 0$ in the stock process). Such models are simpler and important as reference for the stochastic models we study later. We first present some facts about the effects of discounting. These facts are well known in the literature. We still feel that they are relevant for the paper, because this demonstrates how seemingly small changes of a model may have large consequences for the stock process. This is the main point also in the later parts of the paper. Secondly we show how we get similar control implications when we introduce critical depensation. We are then ready to study stochastic models.

Suppose that the growth function is estimated[2] to be of the form

$$f(x) = \alpha x^3 + \beta x^4$$

where $\alpha = 5.0 \cdot 10^{-7}$ and $\beta = -2.2 \cdot 10^{-10}$, see figure 1. This is a depensation curve with a very low growth rate for small stock levels. We have calculated the optimal control as a function of x for four different levels of the discount factor. The result is given in figure 2. Observe that the stock may be driven to extinction when discounting is introduced. If we compare with the case without discounting, we see that a small "shark fin" has appeared close to zero. This is the mining effect due to the low return of the stock in this regime. For this reason discounting is unpopular amongst conservationists[3]. We have observed the dramatic consequence of low growth in combination with discounting.

Suppose that we satisfy the conservationists and keep $\delta = 0$. To be careful we even adjust the original growth function slightly and introduce critical

[2]This functional form has been found to fit the growth data for Namibian pilchard very well. See the paper by S.I. Steinshamn and L.K. Sandal [10] and also S.I. Steinshamn, A.C. Lund and L.K. Sandal [11].

[3]This is, in our opinion, due to a misunderstanding. Discounting must be used, but the problem should be changed to remove ecologically unacceptable harvest policies. This can be done by introducing some kind of social cost of loosing a species.

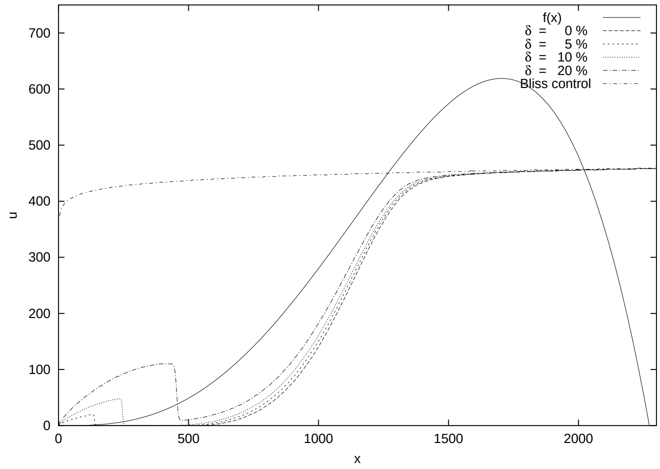

Figure 2: The optimal harvest.

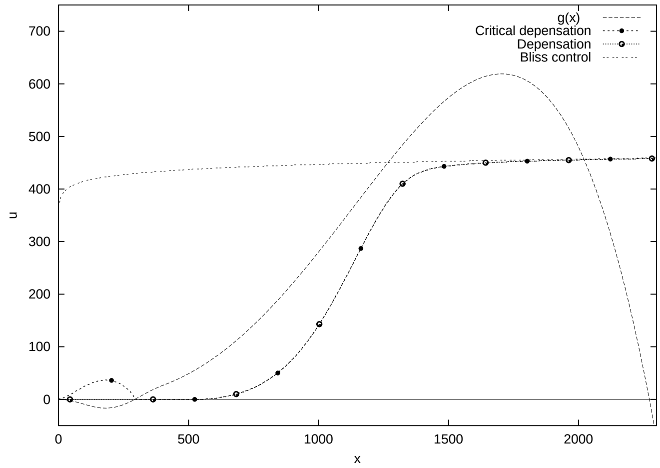

Figure 3: The optimal harvest with critical depensation.

depensation. We use the growth function

$$g(x) = \alpha x^3 + \beta x^4 - 20 I_{x \le 400}(x) \sin^2(\pi x/400), \tag{2}$$

see figure 1. This function is a small perturbation of the original function f, but what about the control? In figure 3 we present the zero discounting control for the two problems using f and g respectively. Observe how the optimal control is increased for small stock levels when critical depensation is used. *Also observe that this control would deplete the real world stock also if it was following a depensation process with low growth rate close to zero* (as for instance the growth rate f). We feel that this example clearly illustrates why a model with critical depensation may be dangerous if used unconsciously.

In the following we use the original growth function defined by f. We relax the assumption $\sigma \equiv 0$, that is, we study the effects of uncertainty. To simplify the presentation, we only focus on the model with depensation. See remark (3.2) on page 9 for a comment on how the introduction of stochasticity affects the model with critical depensation.

3 Stochasticity and induced critical depensation

Suppose that we want to extend our deterministic model and include uncertainty. We let the stock develop according to a stochastic process. It is natural to choose noise proportional to the stock size, that is

$$\sigma(x) = \sigma_0 x.$$

What are the implications of this extension of the model? It seems natural to expect that the introduction of noise lead to a more precautionary harvest policy. We will show that this may be false. We also demonstrate why this is false.

Remark 3.1. We do not want to mislead the reader. It is important to keep in mind that many of the following observations is actually a consequence of the choice of volatility function. If we use a volatility which goes to zero faster, the effects may disappear. Our point is to demonstrate why the choice of σ may be extremely important for the policy implications, and that a completely intuitive σ-choice may give contra-intuitive effects. Hopefully, this will be an important observation for people dealing with different aspects regarding the management of renewable resources.

3.1 Basic observations

Suppose that a deterministic model with the growth function f is extended to include uncertainty. Assume that observations of the stock indicate that the system is extremely volatile, and that the noise seems to be stock proportional. We model the system as a stochastic process with $\sigma = \sigma_0 x$ where $\sigma_0 = 0.7$. In figure 4 we again show the deterministic optimal control in combination with the control for the stochastic control problem. We see, surprisingly, that the new control is more aggressive for small stock levels. It seems contra-intuitive that when uncertainty is taken into account; an apparently more risky behaviour is optimal!

Let us now perturb the deterministic model in a more continuous manner. We solve the same control problem, now with $\sigma_0 = \{0.0, 0.15, \ldots, 1.2\}$. The results are given in figure 5. We see that the policy is increasingly more conservative *as long as the noise is moderate*! If the uncertainty is raised further; the harvest for all stock levels is increased, leading to the possible extinction of the stock. This may also seem contra-intuitive, but will make sense for the reader in short.

3.2 Simulation of the process

We now show how the process described above could develop without harvest. Since the process is stochastic we show several replicates of the process. A

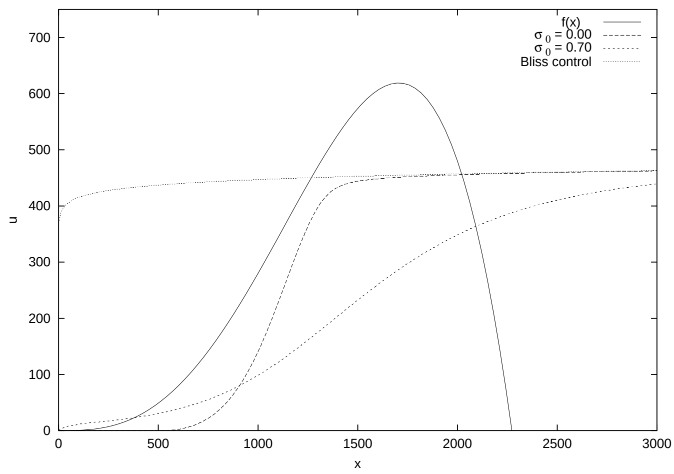

Figure 4: The optimal harvest for a stochastic model compared with the deterministic control.

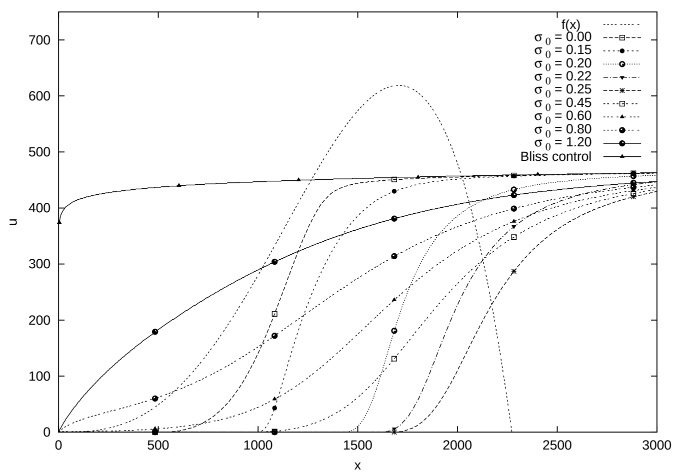

Figure 5: The optimal harvest with zero discounting.

simple Euler scheme is used to compute numerical realisations of the stock development. See Kloeden and Platen [6] for an extensive discussion about the numerical solution of stochastic differential equations. In figure 6 we give the growth when $\sigma_0 = 0$, i.e. the deterministic development. Each of the following figures shows 10 possible replicates for increasing levels of σ_0, all starting at $x_0 = 500$. We make some observations:

- First, observe that the stock volatility is extreme when x and σ_0 is large.
- Second, and more important, observe that if σ_0 is "to large" the expected level of the stock is zero as time gets large. This is the dangerous effect of low growth combined with proportional noise. We may call this critical depensation induced by the uncertainty, i.e. "stochastically induced critical depensation".

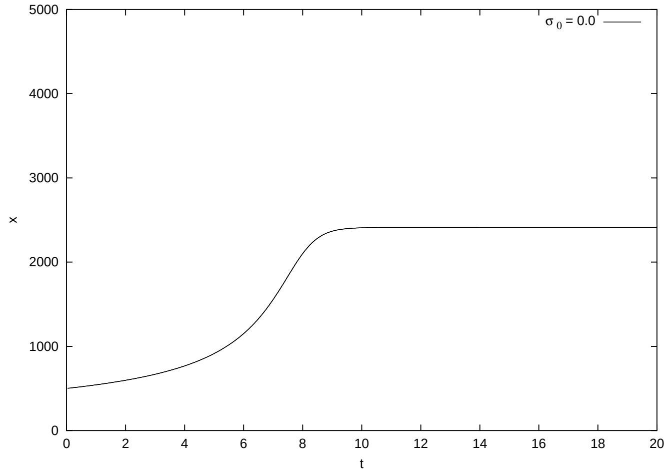

Figure 6: Simulation of replicates, $\sigma_0 = 0.0$.

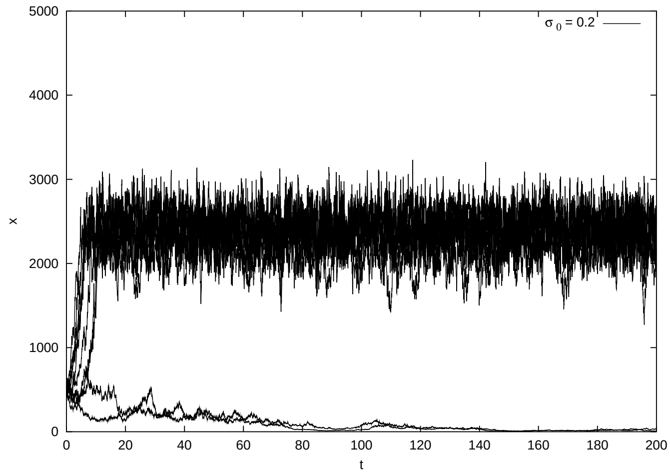

Figure 7: Simulation of replicates, $\sigma_0 = 0.2$.

With this knowledge it is not at all surprising that the introduction of risk to the model may actually cause a very (biologically) risky behaviour! Still this may be correct in some cases[4], the point is that the modeler must be aware of these effects.

It is important not to believe that depensation alone is the reason for this behaviour. It is more due to the combination of low growth, large volatility close to zero. This is illustrated in the figures 9 and 10 where an adjusted growth function (still with depensation) of the form

$$h(x) = 0.2x + \hat{\alpha}x^3 + \hat{\beta}x^4$$

is used, with $\hat{\alpha} = 3.2 \cdot 10^{-7}$ and $\hat{\beta} = -1.6 \cdot 10^{-10}$. As we see from the simulation with $\sigma_0 = 0.6$, this process recovers from weak periods. The optimal control

[4] A stock process as in figure 8 is though unlikely since then the species would have disappeared a long time ago!

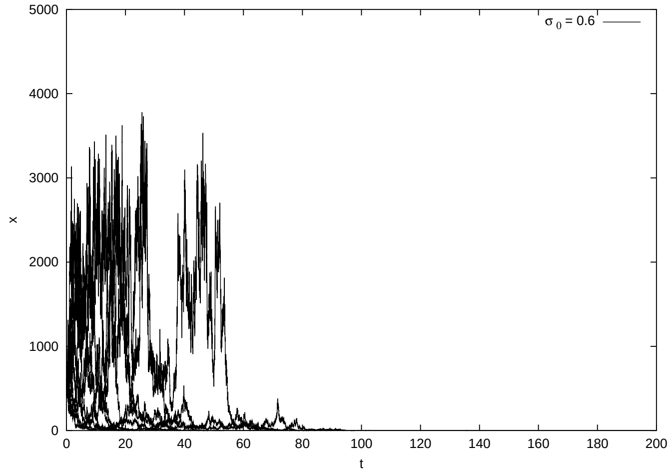

Figure 8: Simulation of replicates, $\sigma_0 = 0.6$.

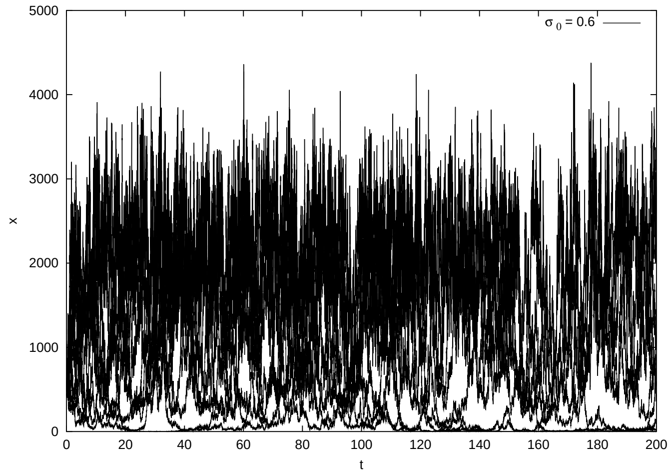

Figure 9: Simulation of replicates, new growth function. $\sigma_0 = 0.6$.

also develops in a more intuitive manner, disregarding the case $\sigma_0 = 1.2$ where the model breaks down again. Observe that the control is more aggressive compared with figure 5. This is because the possibility of self extinction is removed with this adjusted growth function, at least for moderate volatility levels. For the same reason, the control is now less sensitive to an increase in the volatility.

We can now give an intuitive explanation for the changes in the control curves in figure 5 as the volatility increases. When σ_0 is small and the harvest is zero, the stock will develop as in figure (7). The stock is safe as long as it is kept sufficiently large. This is reflected by the control policies. When the volatility is increased further, there is a trade of between the expected life time and the profit. This leads to the tilting of the curves.

Remark 3.2. With uncertainty, the critical depensation model defined in equation (2) on page 5 prescribes a more conservative harvest for large stock levels

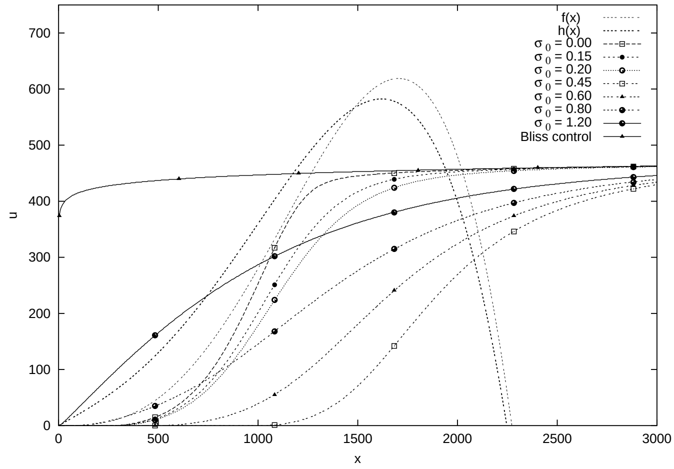

Figure 10: Optimal harvest, new growth function.

compared to the depensation model. As the volatility grows, the "shark fin" close to zero becomes smaller. This is because there is a positive probability that the stock may get out of the "trap" in this case.

The lesson learned from this is that the qualitative characteristics of the process should be examined carefully either analytically or numerically before a process is used for optimal control calculations. The figure 7 on page 8 illustrates how simulation may be used to estimate the self extinction probability when x_0 and σ_0 is given. From the plot we see that 2 out of 10 curves go to zero, indicating that this probability is approximately 0.2 when $x_0 = 500$. Increasing the number of replicates improves the estimate. From this figure we also see that the stock seems to be safe if it is sufficiently large. We must however keep in mind that we try to characterise the properties of the process for all $t > 0$ based on finite horizon simulations. This may be problematic, especially since the stochastic differential equation is solved by discretisation. For the process in figure 7, there may be a small probability that the stock can escape the high level "trap". If so, the self extinction probability may actually be 1! Still, if we observed this in the plot, this could be due to discretisation errors. It is therefore clearly important to find some analytical estimates.

4 Analytic examination

To our knowledge the given process does not possess a closed form solution. It is still possible to make some statements about its future behaviour. Our first idea is to compare the process with a simpler process. Suppose we study two stochastic process of the form

$$\begin{aligned} dx_t &= b_1(t, x_t)dt + \sigma(t, x_t)dB_t \\ dy_t &= b_2(t, y_t)dt + \sigma(t, y_t)dB_t \end{aligned}$$

It can be proved, under weak technical conditions on b_i and σ, that if $x_0 = y_0$ and

$$b_2(t,x) \geq b_1(t,x)$$

for $0 < t < \infty$ and all[5] $x \in \mathbb{R}$, then $P(x_t \leq y_t, 0 \leq t < \infty) = 1$. See proposition 2.18 in the book by Karatzas and Shreve [5]. We now study the example process

$$dx_t = f(x_t)dt + \sigma x_t dB_t \tag{3}$$

where $f(x) = \alpha x^3 + \beta x^4$ as before. The function ax with $a \approx 0.383$ is the smallest linear function dominating f for all $x \geq 0$. We know that the equation

$$dy_t = ay_t dt + \sigma y_t dB_t. \tag{4}$$

has solution

$$y_t = y_0 e^{(a-\frac{1}{2}\sigma^2)t+\sigma B_t}.$$

It is well known that $Ey_t = y_0 e^{at}$. Further, we know that

$$\lim_{t\to\infty} y_t = 0 \text{ a.s. } \Leftrightarrow \sigma > \sqrt{2a}.$$

The process is assumed to model a natural resource. We will therefore say that the process possess self extinction[6]. Using comparison we can conclude that

$$\lim_{t\to\infty} x_t = 0$$

almost surely when $\sigma > 0.875$. This result is independent of the level of x_0. This is however a rather weak estimate of the critical σ. More can be said if the initial stock is small, see the following subsections.

Studying the probabilities gives more information. We know that

$$P(y_t < \bar{x}) < P(x_t < \bar{x}).$$

Since we can calculate $P(y_t < \bar{x})$ by integrating

$$\int_0^{\bar{x}} \frac{1}{\sqrt{2\pi t}\sigma u} e^{-\frac{\left(\ln u - \ln y_0 - (a-\frac{1}{2}\sigma^2)t\right)^2}{2\sigma^2 t}} du$$

we can get an idea of the self extinction probability for the process in question. This probability is obviously a function of the initial stock level, as well as all the other parameters. With this tool in hand we can find lower bounds for the probability of the process x being less than $\bar{x}$ at any time t. More precise estimates may be found with simulation techniques or by numerical solution of the Kolmogorov forward equation.

Suppose that $\sigma > \sqrt{2a}$. We then know that $\lim_{t\to\infty} y_t = 0$ a.s. But what is the probability for y_t ever reaching $\bar{x} > y_0$? Define

$$\begin{aligned} \tau_\epsilon &= \inf\{t > 0; y_t = \bar{x} \text{ or } y_t = \epsilon\} \\ \tilde{p} &= \lim_{\epsilon\to 0} P[y_{\tau_\epsilon} = \bar{x}] \end{aligned}$$

[5] Here we study a nonnegative process. It is therefore sufficient to consider all $x \geq 0$.

[6] We do not discuss the realism of such a process. It may be realistic after some kind of environmental change, such as the introduction of a new species.

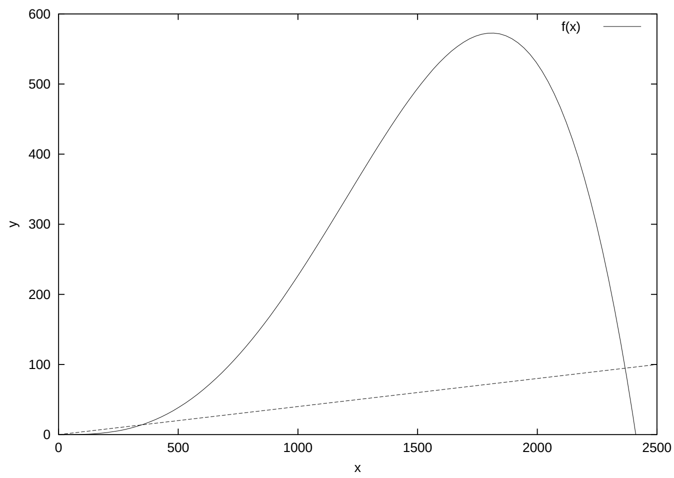

Figure 11: Original and approximating drift functions.

Using Dynkin's formula (B. Øksendal [9] and also appendix A) it can be shown that

$$\tilde{p} = \left(\frac{y_0}{\bar{x}}\right)^{1-\frac{2a}{\sigma^2}}. \quad (5)$$

This gives us a tool to find better estimates for the extinction probabilities for the original process.

4.1 A first estimate of the extinction probability

We study the properties of the process (3) with $f(x) = \alpha x^3 + \beta x^4$. Again x_t is compared with the process (4). Assume that we know the present stock level x_0, where x_0 is small. We further assume that the parameters α, β and σ are estimated. We want to give an approximate probability for the self extinction of the stock. For the parameters used earlier we know that all paths goes to zero when $\sigma > 0.875$. What if σ is more reasonable, say 0.3?. For the y-process the critical a is then $a^* = \frac{1}{2}\sigma^2 = 0.045$. If we for example chose $a = 0.04$ the process is self extinctive. For this a the drift of the process y_t dominates the x_t drift when $x < \bar{x}$ where $\bar{x}$ is the smallest positive solution of

$$\alpha x^3 + \beta x^4 = ax, \quad (6)$$

see figure 11. In this case we have $\bar{x} = 303.89$. We know that the approximating process does possess self-extinction, but this does not imply that the original process does. Since y_t goes extinct and dominates x_t in the region $[0, \bar{x}]$, we know that a replicate of x_t goes to zero if it cannot get above $\bar{x}$. It is not possible to calculate the probability that this will happen explicitly. We know however that this probability is smaller than the probability that $y_{\hat{t}} > \bar{x}$ for some $\hat{t}$. Let us therefore assume that $x_0 < \bar{x}$. Using the expression (5) we can now calculate an upper bound for the probability of x passing $\bar{x}$. If we let $y_0 = x_0 = 50$, we find that

$$\tilde{p} = 0.818.$$

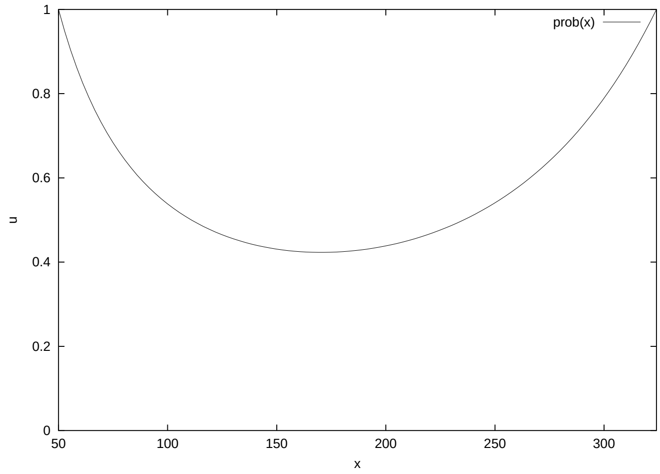

Figure 12: The barrier probability (5) as a function of x.

We can conclude that the probability of self-extinction of the original process is greater than 0.192. This is a weak estimate that can be improved if we chose the a (or $\bar{x}$) parameter better.

4.2 Optimal linear approximating process

As in the last subsection we take the parameters $\alpha, \beta, \sigma, x_0$ as given (observed). We want to find the optimal $\bar{x}$ and corresponding $\bar{a} = \alpha\bar{x}^2 + \beta\bar{x}^3$ such that the probability of the process ever reaching $\bar{x}$ is minimised. The expression (5) is only meaningful for $\bar{x} \geq x_0$ and $a \leq \frac{1}{2}\sigma_0^2$ ($\Rightarrow x \leq \tilde{x}$ where $\tilde{x}$ is the smallest positive solution of $\alpha\tilde{x}^2 + \beta\tilde{x}^3 = \frac{1}{2}\sigma_0^2$). It can be shown that (5) is a convex function of x in this region. See figure 12. We can find the optimal $\bar{x}$ by the first order condition

$$-\frac{2}{\sigma^2}(2\alpha x + 3\beta x^2)\ln(\frac{x_0}{x}) - \frac{1}{x} + \frac{2}{\sigma^2}(\alpha x + \beta x^3) = 0 \text{ for } x \in [x_0, \tilde{x}]$$

In the example case we found $\bar{x} = 170.4$, and $\bar{a} = 0.0134$. With this optimal choice of $\bar{x}$, the probability that the original process ever passes $\bar{x}$ is less than 0.43. Therefore we may conclude that the self-extinction probability is greater than 0.57 in this case.

4.3 Exact calculation

Using the results from the appendix with $x = x_0, \hat{x} = 0$ and $\bar{x} = 170.4$, we can give a quasi explicit expression for the above probability. We have that

$$\tilde{p} = (1 - p) = \frac{r(x_0)}{r(\bar{x})}.$$

where

$$r(x) = \int_0^x e^{-\frac{2}{\sigma^2}(\frac{\alpha}{2}z^2 + \frac{\beta}{3}z^3)} dz.$$

This integral must be evaluated numerically. With the parameters used above this gives that the probability for x_t passing $\bar{x}$ is $p = 0.307$, i.e. the self extinction probability is higher than 0.693.

5 Summary and Conclusion

We have demonstrated that a deterministic model with reasonable drift function may be turned into a process with self-extinction when uncertainty is introduced. We called this induced critical depensation. When compared to the deterministic model, the stochastic optimisation problem can therefore give more aggressive harvest policies. This happens when the noise is large compared to the growth rate for small stock sizes. We do not suggest that this is necessarily wrong. The point is that the noise cannot be introduced in an arbitrary manner, since the control may be a consequence of the functional form of the volatility function! If we really believe in proportional noise for a problem with such a low growth rate it may be optimal to wipe out the resource. This conclusion is completely altered if we e.g. believe in a noise proportional with the growth rate it self. Therefore, the volatility specification does matter.

Due to lack of data it is normally hard to find the functional form of the volatility function for a real world natural resource. It may for the same reason be hard to say anything about the stock development close to zero. In our opinion the modeler should therefore have a clear picture about the properties of the chosen stock process. This can be obtained either by analytical methods, or by simulation. The natural steps of the real world modelling of a natural resource can be as follows:

1. We must have some clear ideas about the fundamental biological effects of the system. What is the natural growth rate close to zero? Is it natural that the self extinction probability is positive? What is the carrying capacity of the resource?

   Ideally the questions should be answered with thorough detail knowledge about the resource and the biological system in question. From this we chose a functional form reflecting our believes.

2. Calibrate the functions to data. This is normally a difficult task. If we have much data and the process is of a form with known distribution, the parameters can be estimated by maximum likelihood. This is often not the case. Another approach using Kolmogorov-Smirnov statistics is developed by D. McDonald and L.K. Sandal in [8].

3. Check the calibrated process by e.g. simulation. Are the properties of this process compatible with our initial assumptions? Observe that this goes both ways: If we believe that the process is self extinctive, is this reflected by the process?

   If there seems to be a miss-specification, the process must be reformulated or the parameters estimated differently. One important point is to keep irrelevant noise away from the stock process. Measurements of a marine resource is typically very uncertain. Uncritical use of such data may therefore imply a volatility for the stock process which is unreasonably high. It is important to let the stochasticity in the process model the relevant noise, i.e the noise associated to fluctuations of the stock. This is uncertainty driven by sources such as weather and temperature variations, and biological interaction.

This procedure prevents conclusions which may accidentally lead to the extinction of a species.

The reader could argue that parts of this paper focus on superficial side effects of the stochastic model, effects that occur when the model is pushed to a point where it may cease to be well defined. It is our opinion that it is immaterial whether induced critical depensation is a real world phenomenon or a model property. It is very important to be aware of the effect either way.

A Barrier probabilities

We now show how Dynkin's formula may be used to calculate barrier probabilities. Suppose we study a stochastic differential equation

$$dx_t = \mu(x_t)dt + \sigma(x_t)dB_t$$

with $x_0 = x$. Define the *generator* $\mathcal{A}$ of the x_t-process

$$\mathcal{A}f(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).$$

Suppose we have $\hat{x} < x < \bar{x}$. Define stopping times

$$\begin{aligned} \tau_1 &= \inf\{t > 0; x_t = \hat{x}\} \\ \tau_2 &= \inf\{t > 0; x_t = \bar{x}\} \\ \tau &= \min(\tau_1, \tau_2). \end{aligned}$$

Define the probability

$$p = P[\tau = \tau_1]$$

i.e. the probability that x_t reach $\hat{x}$ before $\bar{x}$.

Proposition 1 *(Barrier probabilities).* Assume that $E^x\tau < \infty$, and that $g(x) = r(x) + s(x)$ is a solution of $\mathcal{A}g = 1$ with $\mathcal{A}r = 0$. Then

$$p = \frac{r(x) - r(\bar{x})}{r(\hat{x}) - r(\bar{x})}.$$

Further,

$$E^x\tau = ps(\hat{x}) + (1-p)s(\bar{x}) - s(x).$$

Proof. Since $\mathcal{A}r(x) = 0$, Dynkin's formula implies that

$$E^x r(x_\tau) = r(x).$$

The left hand side of this expression can, under the assumption $E^x\tau < \infty$, be written as

$$E^x r(x_\tau) = pr(x_{\tau_1}) + (1-p)r(x_{\tau_2}) = pr(\hat{x}) + (1-p)r(\bar{x})$$

giving the first part of the proposition. Since $\mathcal{A}s(x) = 1$, Dynkin's formula gives that

$$E^x s(x_\tau) = s(x) + E^x\tau.$$

Using the same argument we get the second part of the proposition.□

Proposition 2. For the process (3) we can chose

$$r(x) = \int_0^x e^{-\frac{2}{\sigma^2}(\frac{\alpha}{2}z^2+\frac{\beta}{3}z^3)}dz.$$

Further

$$s(x) = u_1(x)r(x) + u_2(x)$$

where u_1, u_2 solves the equations

$$\begin{aligned} u_1'(x) &= \frac{2}{\sigma^2 x^2 r'(x)} \\ u_1'(x) &= -\frac{2}{\sigma^2 x^2}\frac{r}{r'(x)}. \end{aligned}$$

Proof. The function $r(x)$ is the solution of the homogenous differential equation $\mathcal{A}r(x) = 0$. Further, $s(x)$ is the solution of $\mathcal{A}s(x) = 1$, and is found with "variation of parameters".□

Observe that r is positive and increasing when $x > 0$.

References

[1] M. Bardi and I. Capuzzo-Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations.* Birkhäuser, Boston, 1997.

[2] C. W. Clark. *Mathematical Bioeconomics.* John Wiley & Sons, New York, second edition, 1990.

[3] W. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions.* Springer-Verlag, Berlin, 1993.

[4] M. I. Kamien and N. L. Schwartz. *Dynamic Optimization.* Elsevier Science, 2000.

[5] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus.* Springer Verlag, New York, 1997.

[6] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations.* Springer Verlag, 1999. third printing.

[7] H. Kushner and P. Dupuis. *Numerical Methods for Control Problems in Continuous Time.* Springer, second edition, 2001.

[8] A. D. McDonald and L. K. Sandal. Estimating the parameters of stochastic differential equations using a criterion function based on the kolmogorov-smirnov statistic. *J. Statist. Comput. Simul.*, 64:235–250, 1999.

[9] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Springer-Verlag, Berlin/Heidelberg, fourth edition, 1995.

[10] L. K. Sandal and S. I. Steinshamn. Optimal management of renewable resources: A general feedback approach applied to namibian pilchard. Working Paper 32, Centre for Research in Economics and Business Administration, 1998. SNF-project No.5601.

[11] S. I. Steinshamn, A. C. Lund, and L. K. Sandal. A stochastic feedback model for optimal management of namibian pilchard. In U. R. Sumaila, D. Boyer, M. Skogen, and S. I. Steinshamn, editors, *Ecological, Economic and Social Aspects of Namibian Fisheries*. Eburon, Delft, 2003. To appear.