American Option Pricing with Transaction Costs

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Abstract

In this paper we examine the problem of finding investors' reservation option prices and corresponding early exercise policies of American-style options in the market with proportional transaction costs using the utility based approach proposed by Davis and Zariphopoulou (1995). We present a model, where investors have a CARA utility, and derive some properties of reservation option prices. We discuss the numerical algorithm and propose a new formulation of the problem in terms of quasi-variational HJB inequalities. Based on our formulation, we suggest original discretization schemes for computing reservation prices of American-style option. The discretization schemes are then implemented for computing prices of American put and call options. We examine the effects on the reservation option prices and the corresponding early exercise policies of varying the investor's ARA and the level of transaction costs. We find that in the market with transaction costs the holder of an American-style option exercises this option earlier as compared to the case with no transaction costs. This phenomenon concerns both put and call options written on a non-dividend paying stock. The higher level the transaction costs is, or the higher risk avers the option holder is, the earlier an American option is exercised.

Key words: option pricing, transaction costs, stochastic control, optimal stopping, Markov chain approximation.

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1 Introduction

The break-trough in option valuation theory starts with the publication of two seminal papers by Black and Scholes (1973) and Merton (1973). In both papers authors introduced a continuous time model of a complete friction-free market where a price of a stock follows a geometric Brownian motion. They presented a self-financing, dynamic trading strategy consisting of a riskless security and a risky stock, which replicate the payoffs of an option. Then they argued that the absence of arbitrage dictates that the option price be equal to the cost of setting up the replicating portfolio.

In the presence of transaction costs in capital markets the absence of arbitrage argument is no longer valid, since perfect hedging is impossible. Due to the infinite variation of the geometric Brownian motion, the continuous replication policy incurs an infinite amount of transaction costs over any trading interval no matter how small it might be. A variety of approaches have been suggested to deal with the problem of option pricing and hedging with transaction costs. We maintain that the utility based approach, pioneered by Hodges and Neuberger (1989), produces the most "optimal" polices. The rationale under this approach is as follows: Since entering an option contract involves an unavoidable element of risk, in pricing and hedging options one must consider the investor's attitude toward risk. The other alternative approaches are mainly preference-free and concerned with the "financial engineering" problem of either replicating or super-replicating option payoffs. These approaches are generally valid only in a discrete-time model with a relatively small number of time intervals.

The key idea behind the utility based approach is the indifference argument. The writing price of an option is defined as the amount of money that makes the investor indifferent, in terms of expected utility, between trading in the market with and without writing the option. In a similar way, the purchase price of an option is defined as the amount of money that makes the investor indifferent between trading in the market with and without buying the option. These two prices are also referred to as the investor's *reservation write price* and the investor's *reservation purchase price*. In many respects a reservation option price is determined in a similar manner to a *certainty equivalent* within the expected utility framework, which is a well grounded pricing principle in economics.

The starting point for the utility based option pricing approach is to consider the optimal portfolio selection problem of an investor who faces transaction costs and maximizes expected utility of terminal wealth. The introduction of transaction costs adds considerable complexity to the utility maximization problem as opposed to the case with no transaction costs. The problem is simplified if one assumes that the transaction costs are proportional to the amount of the risky asset traded, and there are no transaction costs on trades in the riskless asset. In this case the problem amounts to a *stochastic singular control* problem that was solved by Davis and Norman (1990). Shreve and Soner (1994) studied this problem applying the theory of viscosity solutions to Hamilton-Jacobi-Bellman (HJB) equations (see, for example, Fleming and Soner (1993) for that theory).

First, the utility based option pricing approach was applied to the pricing and hedging of European-style options assuming that the investor's utility function exhibits constant absolute risk aversion (CARA investor). Hodges and Neuberger (1989) introduced the approach and calculated numerically optimal hedging strategies and reservation prices using a binomial lattice, without really proving the convergence of the numerical method. Davis, Panas, and Zariphopoulou (1993) rigorously analyzed the same model, showed that the value function of the problem is a unique viscosity solution of a fully nonlinear variational inequality. They proved the convergence of discretization schemes based on the binomial approximation of the stock price, and presented computational results for the reservation write price of an option. Clewlow and Hodges (1997) extended the earlier work of Hodges and Neuberger (1989) by presenting a more efficient computational method, and a deeper study of the optimal hedging strategy. Andersen and Damgaard (1999) and Damgaard (2000b) computed the reservation prices of European-style call options for an investor with a constant relative risk aversion (CRRA investor).

Davis et al. (1993) suggested that the utility based option pricing approach could be also applied to the pricing of American-style options. The problem of pricing American-style options using this approach is both interesting and tricky. This problem is interesting from a stochastic control point of view in that it combines *singular control and optimal stopping*. The problem is tricky, because it is the buyer of option who chooses the optimal exercise policy. Therefore, the writer's problem must be treated from both the writer's and the buyer's perspective simultaneously. The problem of pricing American-style options using the utility based option pricing approach was for the first time treated in Davis and Zariphopoulou (1995). The main technical result of that work is that the value function of the singular stochastic control problem with optimal stopping is the unique viscosity solution of the corresponding HJB equation. This means that the solution can be computed by standard discretization methods. Damgaard (2000a) was the first to calculate the reservation purchase prices of American-style call options for the case of a CRRA investor and proportional transaction costs. The problem of finding the reservation purchase price is simpler than that of reservation write price, since it suffices to consider the buyer's problem alone. Unfortunately, using the CRRA utility the calculations are highly time-consuming and were implemented for a 20-period model only. The author studied only the difference between reservation purchase prices of an American call option and its European counterpart for different levels of the investor's initial wealth and found that it is sometimes optimal to exercise a call option prior to maturity. The analysis of the early exercise policy was not given.

In this paper we extend the works of Davis and Zariphopoulou (1995) and Damgaard (2000a) in a number of ways. First, we formulate the option pricing problem for the CARA investor in the market with proportional transaction costs. For this type of investor, the option price and exercise policy are independent of the investor's holdings in the risk-free asset and the computational complexity is dramatically reduced. This allows to increase the precision of calculations considerably and to interpret the early exercise policy much more easily. We consider both the buyer's and the writer's problems and derive some properties of the reservation prices of American-style options. Then we suggest discretization schemes for computing reservation prices of American-style options. Our numerical schemes differ from those suggested first in Davis et al. (1993) and further used by Damgaard (2000a). We argue that the variational HJB inequalities of a singular stochastic control problem with a nature similar to those of Davis and Norman (1990) cannot be implemented in a numerical method for the computation of reservation option prices of American-style options. We prove that the investor's value function can alternatively be characterized as the unique viscosity solution of *quasi*-variational HJB inequalities, with a nature similar to those used in *stochastic impulse controls* theory (see, for example, Øksendal and Sulem (2002)), and maintain that these inequalities provide the most natural way to construct numerical schemes upon. The discretization schemes were implemented for computing reservation purchase and write prices of American-style put options and reservation purchase prices of American-style call options. We examine the effects on the reservation option prices and the corresponding optimal exercise policies of varying the investor's level of absolute risk aversion (ARA) and the level of transaction costs. We find that in the market with transaction costs the holder of an American-style option exercises this option earlier as compared to the case with no transaction costs. This phenomenon concerns both put and call options written on a non-dividend paying stock. We carry out the detailed analysis of the early exercise policy. In short, the main result of our analysis can be expressed as follows: The higher level the transaction costs is, or the higher risk avers the option holder is, the earlier an American option is exercised.

It is known that in the presence of proportional transaction costs the investor's portfolio space, in the utility maximization problem without options, is divided into three disjoint regions, which can be specified as the Buy region, the Sell region, and the no-transaction (NT) region. The boundaries of the NT region are reflecting barriers, such that the investor refrains from transactions as long the portfolio lies inside the NT region. If a portfolio lies in the Buy or Sell region, then the investor performs the minimum transaction required to reach the closest NT boundary[1]. Our numerical calculations show that the optimal strategy of the buyer of an American option is rather complicated as compared to the case without options. Generally, every region (Buy, Sell, and NT) consists of two sub-regions, and not all the boundaries of the NT sub-regions are reflecting barriers. When a non-reflecting barrier is hit, the investor performs the minimum transaction required to reach the closest boundary of the other NT sub-region.

Moreover, in contrast to the case with no transaction costs where the early exercise boundary depends on the stock price and time, the option holder exercise policy generally depends on his holdings in the stock. However, two bounds on the early exercise boundary could be provided (in the two dimensional space: stock price - time): the upper bound and the lower

[1]The same description of the optimal strategy applies for the buyer/writer of a European option and the writer of an American option.

bound. These two bounds split the option buyer's exercise space into three disjoint regions, which could be specified as the Exercise region, the Keep region, and the region where the exercise policy is not uniquely defined. The Exercise region is the region where the exercise policy clearly dominates the keep policy, and the Keep region is the region where the keep policy dominates the exercise policy for any holdings in the stock. In the remaining region the exercise policy depends on the option holder's holding in the stock.

The rest of the paper is organized as follows. Section 2 presents the continuous-time model and the basic definitions. In Section 3 we derive some important properties of the reservation option prices. Section 4 is concerned with the reformulation of the model and the construction of a discrete time approximation of the continuous time price processes used in Section 2, and the solution method. The numerical results for American-style put and call options are presented in Section 5. Section 6 concludes the paper and discusses some possible extensions.

2 The Formulation of the Model

Here we formulate the continuous time problem within the stochastic singular control framework presented in Davis and Zariphopoulou (1995). We consider a continuous-time economy with one risky and one risk-free asset. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a given filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$. The risk-free asset, which we will refer to as the bank account, pays a constant interest rate of $r \ge 0$, and, consequently, the evolution of the amount invested in the bank, x_t, is given by the ordinary differential equation

$$dx_t = rx_t dt. \tag{1}$$

We will refer to the risky asset to as the stock, and assume that the price of the stock, S_t, evolves according to a geometric Brownian motion defined by

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{2}$$

where μ and σ are constants, and B_t is a one-dimensional $\mathcal{F}_t$-Brownian motion.

The investor holds x_t in the bank account and the amount y_t in the stock at time t. We assume that a purchase or sale of size ξ of the stock incurs transaction costs $\lambda|\xi|$ proportional to the transaction ($\lambda \geq 0$). These costs are drawn from the bank account.

If the investor has the amount x_t in the bank account, and the amount y_t in the stock at time t, his *net wealth* is defined as the holdings in the bank account after either selling of all shares of the stock or closing of the short position in the stock and is given by

$$X_t(x,y) = \begin{cases} x_t + y_t(1-\lambda) & \text{if } y_t \geq 0, \\ x_t + y_t(1+\lambda) & \text{if } y_t < 0. \end{cases} \tag{3}$$

We suppose that at any time the investor can decide to transfer money from the bank account to the stock and conversely. The evolution equations for the system (x_t, y_t) are

$$\begin{aligned} dx_t &= rx_t dt - (1+\lambda)dL_t + (1-\lambda)dM_t, \\ dy_t &= \mu y_t dt + \sigma y_t dB_t + dL_t - dM_t, \end{aligned} \tag{4}$$

where L_t, M_t represent cumulative purchase and sale, respectively, of the stock up to time t. Both L_t and M_t are right-continuous with left-hand limits (RCLL) nonnegative and nondecreasing $\{\mathcal{F}_t\}$-adapted processes. By convention, $L_0 = M_0 = 0$.

We consider an investor with a finite horizon $[0,T]$ who has utility only of terminal wealth. It is assumed that the investor has a constant absolute risk aversion. In this case his utility function is of the form

$$U(\gamma, W) = -\exp(-\gamma W); \quad \gamma > 0, \tag{5}$$

where γ is a measure of the investor's absolute risk aversion, which is independent of the investor's wealth.

2.1 Utility Maximization Problem without Options

The investor's problem is to choose an admissible trading strategy to maximize $E_t[U(\gamma, X_T)]$, i.e., the expected utility of his net terminal wealth, subject to (4). We define the value function at time t as

$$V(t,x,y) = \sup_{(L,M)\in\mathcal{A}(x,y)} E_t^{x,y}[U(\gamma, X_T)], \tag{6}$$

where $\mathcal{A}(x, y)$ denotes the set of admissible controls available to the investor who starts at time t with an amount of x in the bank and y holdings in the stock.

It is known (see Davis and Norman (1990) and Shreve and Soner (1994)) that in the presence of proportional transaction costs the investor's portfolio space is divided into three disjoint regions, which can be specified as the Buy region, the Sell region, and the no-transaction (NT) region. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the boundary between the Buy region and the NT region, while if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the boundary between the Sell region and the NT region. If a portfolio lies in the no-transaction region, it is not adjusted at that time.

Now, by giving heuristic arguments, we intend to characterize the value function and the associated optimal strategy: If for some initial point (t, x, y) the optimal strategy is to not transact, the utility associated with this strategy is $V(t, x, y)$. The necessary conditions for the optimality of the no transaction strategy is $V_y \leq (1+\lambda)V_x$ and $V_y \geq (1-\lambda)V_x$. That is, it is not possible for the investor to increase his indirect utility function by either buying or selling some amount of the stock at the expense of lowering or increasing, respectively, the holdings in the bank account. One of these inequalities holds with equality when it is optimal to rebalance the portfolio. The set of (x, y) points for which $V_y = (1+\lambda)V_x$ defines the Buy region. Similarly, the equation $V_y = (1-\lambda)V_x$ defines the Sell region. Moreover, in the NT region, the application of the dynamic programming principle gives $\mathcal{L}V(t, x, y) = 0$, where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}V(t, x, y) = \frac{\partial V}{\partial t} + rx\frac{\partial V}{\partial x} + \mu y\frac{\partial V}{\partial y} + \frac{1}{2}\sigma^2 y^2\frac{\partial^2 V}{\partial y^2}. \tag{7}$$

The subsequent theorem formalizes this intuition.

Theorem 1. *The value function V defined by (6) is a unique viscosity solution of the Hamilton-Jacobi-Bellman inequalities:*

$$\max\left\{\mathcal{L}V, \quad -(1+\lambda)\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}, \quad (1-\lambda)\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y}\right\} = 0 \tag{8}$$

with the boundary condition

$$V(T,x,y) = U(\gamma, X_T).$$

The proof can be made by following along the lines of the proofs of Theorems 2 (the existence result) and 3 (the uniqueness result) in Davis et al. (1993).

The amount of x_T is given by

$$x_T = \frac{x}{\delta(T,t)} - \int_t^T \frac{(1+\lambda)}{\delta(T,s)} dL_s + \int_t^T \frac{(1-\lambda)}{\delta(T,s)} dM_s, \tag{9}$$

where $\delta(T,t)$ is the discount factor defined by

$$\delta(T,t) = \exp(-r(T-t)). \tag{10}$$

Note that the net wealth at time T could be written as

$$X_T = x_T + h(y_T), \tag{11}$$

where $h(\cdot)$ is some function. Therefore, taking into consideration the investor's utility function defined by (5), we can write

$$\begin{aligned} V(t,x,y) &= \sup_{(L,M)\in\mathcal{A}(x,y)} E_t^{x,y}[-\exp(-\gamma X_T)] \\ &= \sup_{(L,M)\in\mathcal{A}(x,y)} E_t^{x,y}[-\exp(-\gamma(x_T + h(y_T)))] = \exp(-\gamma\frac{x}{\delta(T,t)})Q(t,y), \end{aligned} \tag{12}$$

where $Q(t,y)$ is defined by $Q(t,y) = V(t,0,y)$. It means that the dynamics of y through time is independent of the total wealth. In other words, the choice in y is independent of x. This representation suggests transformation of (8) into the following HJB for the value function $Q(t,y)$:

$$\max\Big\{\mathcal{D}Q(t,y), \quad \frac{\gamma(1+\lambda)}{\delta(T,t)}Q + \frac{\partial Q}{\partial y}, \quad -\frac{\gamma(1-\lambda)}{\delta(T,t)}Q - \frac{\partial Q}{\partial y}\Big\} = 0, \tag{13}$$

with a proper boundary condition and where the operator $\mathcal{D}$ is defined by

$$\mathcal{D}Q(t,y) = \frac{\partial Q}{\partial t} + \mu y \frac{\partial Q}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 Q}{\partial y^2}. \tag{14}$$

This is an important simplification that reduces the dimensionality of the problem. Note that the function $Q(t, y)$ is evaluated in the two-dimensional space $[0, T] \times \mathbb{R}$.

For fixed values of μ, σ, r, γ, and λ the NT boundaries are functions of the investor's horizon only and do not depend on the investor's holdings in the bank account, so that a possible description of the optimal policy may be given by

$$\begin{aligned} y &= y_l(t) \\ y &= y_u(t). \end{aligned} \tag{15}$$

The equations describe the lower and the upper no-transaction boundaries respectively. If the function $Q(t, y)$ is known in the NT region, then

$$Q(t,y) = \begin{cases} \exp\left(-\gamma \frac{(1-\lambda)(y-y_u)}{\delta(T,t)}\right) Q(t, y_u) & \forall y(t) \geq y_u(t), \\ \exp\left(-\gamma \frac{(1+\lambda)(y-y_l)}{\delta(T,t)}\right) Q(t, y_l) & \forall y(t) \leq y_l(t). \end{cases} \tag{16}$$

This follows from the optimal transaction policy described above. That is, if a portfolio lies in the Buy or Sell region, then the investor performs the minimum transaction required to reach the closest NT boundary.

2.2 Utility Maximization Problem for a Buyer of American Options

Now, in addition to the risk-free asset and the risky stock, we introduce a new asset, a cash settled American-style option contract with expiration time T and exercise payoff $g(S_\tau)$ at time τ when the option is exercised. For the sake of simplicity, we assume that these options may be bought only at (initial) time t. This means that there is no trade in options thereafter, between times t and T.

Consider an investor who trades in the riskless and the risky assets and, in addition, buys $\theta > 0$ options (θ is a constant) at time t. This investor we will refer to as the buyer of options. We assume that if the buyer chooses to exercise, he is required to exercise all of his θ options simultaneously. If the buyer chooses to exercise the options, he receives the exercise payoff and then faces the utility maximization problem without options. That is, his value function, given he exercises the options in the state (τ, x, y, S), is given by

$$V^b_{ex}(\tau, x, y, S, \theta) = V(\tau, x+\theta g(S), y) = \sup_{(L,M)\in\mathcal{A}(x+\theta g(S),y)} E_\tau^{x+\theta g(S),y}[U(\gamma, X_T)]. \tag{17}$$

The buyer's problem is to choose an admissible trading strategy and a time of exercise to maximize (17) subject to (4). We define his value function at time t as

$$J^b(t, x, y, S, \theta) = \sup_{(L,M)\in\mathcal{A}^b_\theta(x,y),\tau} E_t^{x,y}[V^b_{ex}(\tau, x_\tau, y_\tau, S_\tau, \theta)], \tag{18}$$

where $\mathcal{A}^b_\theta(x, y)$ denotes the set of admissible controls available to the buyer who starts at time t with an amount of x in the bank and y holdings in the stock, and τ is a stopping time in $[t, T]$. If the buyer never exercises the options before T, we set $\tau = T$.

Definition 1. The unit reservation purchase price of θ American-style options is defined as the price P^b_θ such that

$$V(t, x, y) = J^b(t, x - \theta P^b_\theta, y, S, \theta). \tag{19}$$

In other words, the reservation purchase price, P^b_θ, is the highest price at which the investor is willing to buy options, and where the investor is indifferent between the two alternatives: (i) a utility maximization problem where he trades in the riskless and risky assets only, and (ii) a utility maximization problem where the investor, in addition, buys options at price P^b_θ.

Theorem 2. *The value functions J defined by (18), assuming it is continuous, is a unique viscosity solution of the Hamilton-Jacobi-Bellman variational inequalities:*

$$\max\left\{V_{ex} - J, \quad \overline{\mathcal{L}}J, \quad -(1+\lambda)\frac{\partial J}{\partial x} + \frac{\partial J}{\partial y}, \quad (1-\lambda)\frac{\partial J}{\partial x} - \frac{\partial J}{\partial y}\right\} = 0, \tag{20}$$

where the operator $\overline{\mathcal{L}}$ given by

$$\overline{\mathcal{L}}J = \frac{\partial J}{\partial t} + rx\frac{\partial J}{\partial x} + \mu y\frac{\partial J}{\partial y} + \mu S\frac{\partial J}{\partial S} + \frac{1}{2}\sigma^2 y^2\frac{\partial^2 J}{\partial y^2} + \sigma^2 yS\frac{\partial^2 J}{\partial y\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 J}{\partial S^2}. \tag{21}$$

The heuristic arguments for this, as well as the proof of the Theorem, is given in Davis and Zariphopoulou (1995) (The existence result is proved in

Theorem 4.1, and the uniqueness result is proved in Theorem 4.2). Note that if $J^b = V_{ex}$ then the value function satisfies the variational HJB inequalities (8). Furthermore note that the variational HJB equation for the buyer of an American option is similar to that of the buyer of a European option (see, for example, equation (4.20) in Davis et al. (1993)) except for the obstacle constraint, $V_{ex} - J \leq 0$.

As in the case of the optimal portfolio selection problem without options, we can show that the dynamics of y through time is independent of the total wealth, and, in particular, of the initial x. Indeed, the amount of x_T is given by

$$x_T = \frac{x_t}{\delta(T,t)} + \frac{\theta g(S_\tau)}{\delta(T,\tau)} - \int_t^T \frac{(1+\lambda)}{\delta(T,s)} dL_s + \int_t^T \frac{(1-\lambda)}{\delta(T,s)} dM_s. \tag{22}$$

Therefore

$$J^b(t,x,y,S,\theta) = \exp(-\gamma \frac{x}{\delta(T,t)}) H^b(t,y,S,\theta), \tag{23}$$

where $H^b(t,y,S,\theta)$ is defined by $H^b(t,y,S,\theta) = J^b(t,0,y,S,\theta)$. This also suggests transformation of (20) into the following HJB for the value function $H^b(t,y,S,\theta)$ (we suppress the superscript b for the easy of notation):

$$\max\left\{ \exp\left(-\gamma \frac{g(S)}{\delta(T,t)}\right) Q - H,\ \overline{\mathcal{D}}H,\ \frac{\gamma(1+\lambda)}{\delta(T,t)} H + \frac{\partial H}{\partial y},\ -\frac{\gamma(1-\lambda)}{\delta(T,t)} H - \frac{\partial H}{\partial y} \right\} = 0, \tag{24}$$

where the operator $\overline{\mathcal{D}}$ is defined by

$$\overline{\mathcal{D}}H = \frac{\partial H}{\partial t} + \mu y \frac{\partial H}{\partial y} + \mu S \frac{\partial H}{\partial S} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 H}{\partial y^2} + \sigma^2 y S \frac{\partial^2 H}{\partial y \partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2}. \tag{25}$$

Thus, we have reduced the dimensionality of the problem by one. Note that the function $H^b(t,y,S,\theta)$ is evaluated in the three-dimensional space $[0,T] \times \mathbb{R} \times \mathbb{R}^+$. Consequently, after all the simplifications, the unit reservation purchase price of θ options is given by (follows from (12), (19), and (23))

$$P^b_\theta(t,S) = \frac{\delta(T,t)}{\theta\gamma} \ln\left(\frac{Q(t,y)}{H^b(t,y,S,\theta)}\right). \tag{26}$$

Clearly, the unit reservation purchase price depends somehow on the initial holdings in y. In practical applications one usually assumes that the investor has zero holdings in the stock at the initial time t. The solution to equation (19) provides the unique reservation purchase price P^b_θ, and the buyer's optimal stopping time τ^*.

The numerical calculations show that, in contrast to the no transaction costs case where the exercise policy depends on (t, S), in the presence of transaction costs the buyer's exercise policy depends on (t, y, S). That is, the exercise policy generally depends on the holdings in the stock account. However, in the (t, S) plane we can provide two bounds on the early exercise boundary: the upper bound $S_u(t)$ and the lower bound $S_l(t)$. Generally, these two bounds split the option buyer's exercise space[2] into three disjoint regions, which could be specified as the Exercise region, the Keep region, and the region where the exercise policy is not uniquely defined. The Exercise region is the region where the Exercise policy dominates the Keep policy for any y. For a put option, the Exercise region lies below $S_l(t)$. The Keep region is the region where the Keep policy dominates the Exercise policy for any y. For a put option, the Keep region lies above $S_u(t)$. Figure (1) illustrates the early exercise policy for a buyer of an American put option.

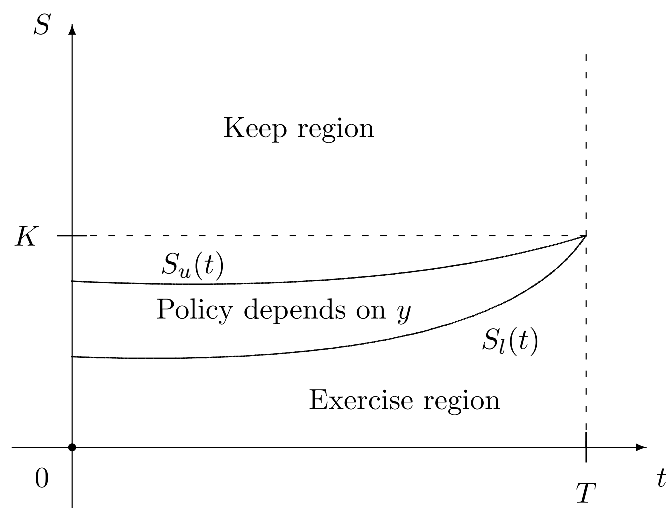

Figure 1: Two bounds on the early exercise policy for a buyer of American put option with strike K.

Moreover, the numerical calculations show that the buyer's optimal strategy is rather complicated as compared to the case without options. Roughly, at any time t the buyer has two alternatives: either to follow the Exercise

[2]In fact, this is a projection of the three-dimensional space (t, y, S) into the two-dimensional space (t, S).

strategy or to follow the Keep strategy. In either of the two strategies, the portfolio space is divided into three disjoint regions: the Buy, the Sell, and the NT region. The Exercise strategy is described by the two NT boundaries: y_u^{ex} - the upper boundary, and y_l^{ex} - the lower boundary of the NT_{ex} region. Similarly, the Keep strategy is described by the following two NT boundaries[3]: y_u^{keep} - the upper boundary, and y_l^{keep} - the lower boundary of the NT_{keep} region. The buyer's problem is to choose the strategy which gives the highest expected utility.

To illustrate the optimal trading strategy, let's consider the buyer of an American put option. In this case, the NT_{keep} region is located above the NT_{ex} region. If these two NT regions do not overlap, we have five possible situations:

1. The Keep strategy dominates the Exercise strategy for all y. Thus, the buyer's optimal strategy is divided into three disjoint regions: the Buy, the Sell, and the NT, and the optimal strategy is described by y_u^{keep} and y_l^{keep}, such that

$$
\begin{aligned}
Sell &= \{y : y \in (\infty, y_u^{keep})\} \\
NT &= \{y : y \in [y_u^{keep}, y_l^{keep}]\} \\
Buy &= \{y : y \in (y_l^{keep}, -\infty)\}.
\end{aligned}
$$

2. There is no dominant strategy. Instead, there is a boundary y^* such that if $y > y^*$ then the Keep strategy dominates the Exercise strategy, and if $y < y^*$ then the Exercise strategy dominates the Keep strategy. The boundary y^* lies in between the NT_{keep} region and the NT_{ex} region, i.e., $y_l^{keep} < y^* < y_u^{ex}$. In this case every region (Buy, Sell, and NT) consists of two sub-regions. That is,

$$
\begin{aligned}
Sell &= \{y : y \in (\infty, y_u^{keep}) \cup (y^*, y_u^{ex})\} \\
NT &= \{y : y \in [y_u^{keep}, y_l^{keep}] \cup [y_u^{ex}, y_l^{ex}]\} \\
Buy &= \{y : y \in (y_l^{keep}, y^*] \cup (y_l^{ex}, -\infty)\}.
\end{aligned}
$$

   In between the two NT sub-regions, the strategy depends on whether the investor's y lies above or below y^*. If $y > y^*$, the optimal strategy

[3] The real description of the Keep strategy is more complicated, see Section 4. For illustration purposes we simplify things a bit.

is to buy the risky asset until the portfolio reaches y_l^{keep}. On the contrary, if $y < y^*$, the optimal strategy is to exercise the option and sell the risky asset until the portfolio reaches y_u^{ex}. Figure (2) illustrates this case.

3. The same as the previous case, but here the boundary y^* lies inside the NT_{keep} region, i.e., $y_u^{keep} < y^* < y_l^{keep}$. In this case the Sell and the NT region consist of two sub-regions such that

$$
\begin{aligned}
Sell &= \{y : y \in (\infty, y_u^{keep}) \cup (y^*, y_u^{ex})\} \\
NT &= \{y : y \in [y_u^{keep}, y^*] \cup [y_u^{ex}, y_l^{ex}]\} \\
Buy &= \{y : y \in (y_l^{ex}, -\infty)\}.
\end{aligned}
$$

Note that in this case y^* is *not a reflecting boundary* of the upper NT sub-region. As soon as this boundary is hit, the optimal strategy is to exercise the option and sell the risky asset (we can interpret it as liquidating the hedge) until the portfolio reaches the closest boundary of the lower NT sub-region. Figure (3) illustrates this case.

4. The same as the previous case, but here the boundary y^* lies inside the NT_{ex} region, i.e., $y_u^{ex} < y^* < y_l^{ex}$. In this case the Buy and the NT region consist of two sub-regions such that

$$
\begin{aligned}
Sell &= \{y : y \in (\infty, y_u^{keep})\} \\
NT &= \{y : y \in [y_u^{keep}, y_l^{keep}] \cup [y^*, y_l^{ex}]\} \\
Buy &= \{y : y \in (y_l^{keep}, y^*) \cup (y_l^{ex}, -\infty)\}.
\end{aligned}
$$

5. The Exercise strategy dominates the Keep strategy for all y. Therefore, the buyer's optimal strategy is described by y_u^{ex} and y_l^{ex}, such that

$$
\begin{aligned}
Sell &= \{y : y \in (\infty, y_u^{ex})\} \\
NT &= \{y : y \in [y_u^{ex}, y_l^{ex}]\} \\
Buy &= \{y : y \in (y_l^{ex}, -\infty)\}.
\end{aligned}
$$

If the two NT sub-regions, the NT_{keep} and the NT_{ex}, overlap, then we have four possible situations. They are similar to ones described above except the second case, where the boundary y^* lies in between the NT_{keep}

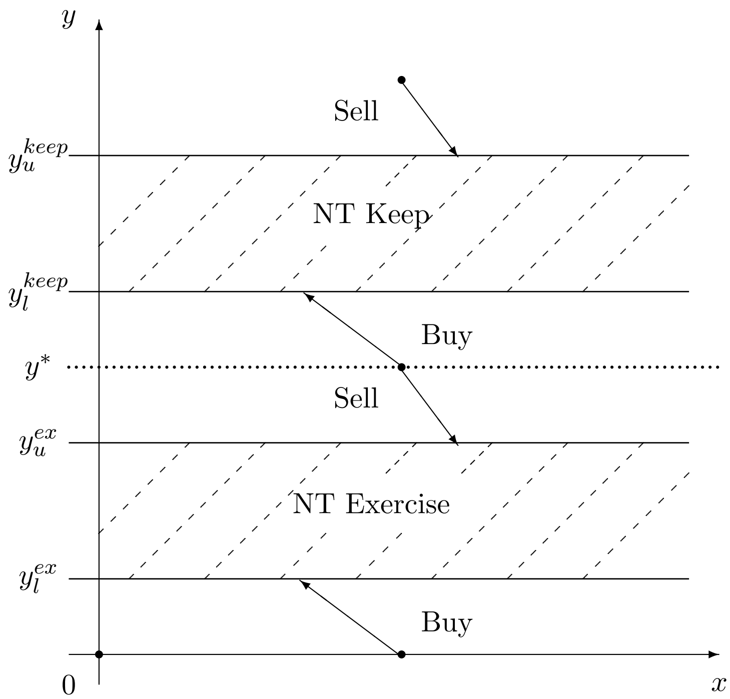

Figure 2: Illustration of the optimal transaction strategy when the division boundary lies in between NT_{keep} and NT_{ex} regions.

region and the NT_{ex} region.

We define the *outer lower boundary* y_l^* by

$$y_l^* = \min\{y_l^{ex}, y_l^{keep}\}, \tag{27}$$

and the *outer upper boundary* y_u^* by

$$y_u^* = \max\{y_u^{ex}, y_u^{keep}\}. \tag{28}$$

The reason to introduce these definitions is the following claim:

Proposition 1. *If one of the two possible strategies (keep, exercise) does not dominate the other for all* y*, then the division boundary* y^* *could lie only in between the outer boundaries, that is*

$$y_l^* < y^* < y_u^*. \tag{29}$$

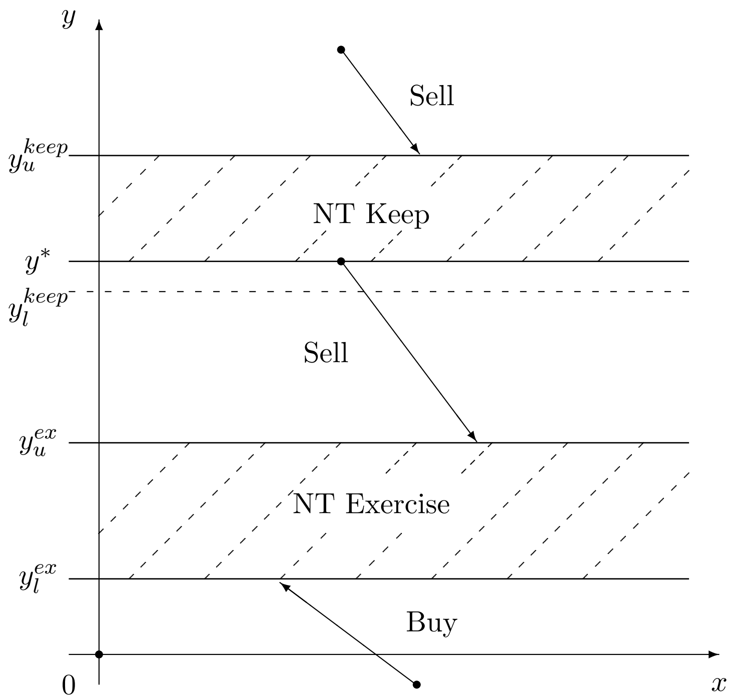

Figure 3: Illustration of the optimal transaction strategy when the division boundary lies inside NT_{keep} region.

Proof. First we prove that the division boundary cannot lie below the outer lower boundary.

Consider a point (x, y) such that $y < y_l^*$. The Keep strategy mandates to transact to $(x-(1+\lambda)(y_l^{keep}-y), y_l^{keep})$, and the Exercise strategy mandates to exercise the option and to transact to $(x + g(S_t) - (1 + \lambda)(y_l^{ex} - y), y_l^{ex})$. We write the latter as $(x + g(S_t) - (1 + \lambda)((y_l^{ex} - y_l^{keep}) + (y_l^{keep} - y)), y_l^{ex})$. This means in particular

$$V^{ex}(t, x, y) = e^{-\frac{\gamma}{\delta(t,T)}(x-(1+\lambda)(y_l^{keep}-y))} e^{-\frac{\gamma}{\delta(t,T)}(g(S_t)-(1+\lambda)(y_l^{ex}-y_l^{keep}))} Q(t, y_l^{ex})$$
$$J^{keep}(t, x, y) = e^{-\frac{\gamma}{\delta(t,T)}(x-(1+\lambda)(y_l^{keep}-y))} H(t, y_l^{keep}).$$

It is easy now to see that the inequality relation between these two value functions does not depend on a particular value of y as long as it lies below the outer lower boundary. In other words, their ratio for any $y < y_l^*$,

$$\frac{V^{ex}(t, x, y)}{J^{keep}(t, x, y)} = \frac{e^{-\frac{\gamma}{\delta(t,T)}(g(S_t)-(1+\lambda)(y_l^{ex}-y_l^{keep}))} Q(t, y_l^{ex})}{H^{keep}(t, y_l^{keep})},$$

does not depends neither on y nor on x.

Similarly we can prove that the division boundary cannot lie above the outer upper boundary. □

Proposition 1 is very useful for numerical calculations as it precisely defines the region where a division boundary could be located.

2.3 Utility Maximization Problem for a Writer of American Options

Consider now an investor who trades in the riskless and the risky assets and, in addition, writes $\theta > 0$ (recall that θ is a constant) American-style options. This investor we will refer to as the writer of options. The problem of finding the reservation write price of an American-style option is somewhat tricky, because it is the buyer of options who chooses the optimal exercise policy. Therefore, the writer's problem must be treated from both the writer's and the buyer's perspective simultaneously. Here we assume that the buyer's problem has already been solved and the writer knows the buyer's optimal stopping time τ^*, or, equivalently, the buyer's optimal exercise policy. If the buyer chooses to exercise the options, the writer pays him the exercise payoff and then faces the utility maximization problem without options. That is, his value function, given the buyer exercises the options in the state (τ^*, x, y, S), is defined by

$$V^w_{ex}(\tau^*, x, y, S, \theta) = V(\tau^*, x - \theta g(S), y) = \sup_{(L,M)\in\mathcal{A}(x-\theta g(S),y)} E^{x-\theta g(S),y}_{\tau^*}[U(\gamma, X_T)]. \tag{30}$$

The writer's problem is to choose an admissible trading strategy to maximize (30) subject to (4) and the buyer's optimal stopping time τ^*. We define his value function at time t as

$$J^w(t, x, y, S, \theta) = \sup_{(L,M)\in\mathcal{A}^w_\theta(x,y)} E^{x,y}_t[V^w_{ex}(\tau^*, x_{\tau^*}, y_{\tau^*}, S_{\tau^*}, \theta)], \tag{31}$$

where $\mathcal{A}^w_\theta(x, y)$ denotes the set of admissible controls available to the writer who starts at time t with an amount of x in the bank and y holdings in the stock. Note that if the buyer never exercises the options before T, we set $\tau^* = T$.

Definition 2. The unit reservation write price of θ American-style options is defined as the compensation P_θ^w such that

$$V(t,x,y) = J^w(t, x + \theta P_\theta^w, y, S, \theta). \tag{32}$$

That is, the reservation write price, P_θ^w, is the lowest price at which the investor is willing to sell options, and where the investor is indifferent between the two alternatives: (*i*) a utility maximization problem where he trades in the riskless and risky assets only, and (*ii*) a utility maximization problem where the investor, in addition, writes options at price P_θ^w.

The solutions to problem (32) provide the unique reservation write price and the optimal trading strategy.

Theorem 3. *The value functions J defined by (31), assuming it is continuous, is a unique viscosity solution of the Hamilton-Jacobi-Bellman variational inequalities:*

$$\max\left\{\overline{\mathcal{L}}J, \quad -(1+\lambda)\frac{\partial J}{\partial x} + \frac{\partial J}{\partial y}, \quad (1-\lambda)\frac{\partial J}{\partial x} - \frac{\partial J}{\partial y}\right\} = 0, \tag{33}$$

where the operator $\overline{\mathcal{L}}$ given by (21).

The heuristic arguments for this, as well as the proof of the Theorem, are similar to that of Theorem 2. Note that this value function is defined in the region where the option buyer's Keep strategy dominates the Exercise strategy. Outside of this region the value function satisfies the variational HJB inequalities (8). Furthermore note that the variational HJB equation for the writer of an American option is completely similar to that of the writer of a European option. As to numerical computations of a reservation write price of an American option, they are only slightly more difficult than those of its European counterpart: instead of a single "exercise" time T we have a known exercise boundary τ^*.

Again we can show that the dynamics of y through time is independent of the investor's total wealth. Therefore

$$J^w(t,x,y,S,\theta) = \exp(-\gamma\frac{x}{\delta(T,t)})H^w(t,y,S,\theta), \tag{34}$$

where $H^w(t,y,S,\theta)$ is defined by $H^w(t,y,S,\theta) = J^w(t,0,y,S,\theta)$. This also suggests transformation of (33) into the following HJB for the value function

$H^w(t,y,S,\theta)$ (here we suppress the superscript w):

$$\max\left\{\overline{\mathcal{D}}H,\ \frac{\gamma(1+\lambda)}{\delta(T,t)}H+\frac{\partial H}{\partial y},\ -\frac{\gamma(1-\lambda)}{\delta(T,t)}H-\frac{\partial H}{\partial y}\right\}=0, \tag{35}$$

where the operator $\overline{\mathcal{D}}$ is defined by (25). Consequently, the unit reservation write price is given by (follows from (12), (32), and (34))

$$P_\theta^w(t,S)=\frac{\delta(T,t)}{\theta\gamma}\ln\left(\frac{H^w(t,y,S,\theta)}{Q(t,y)}\right). \tag{36}$$

The qualitative description of the writer's optimal policy is similar to that without the options. That is, the writer's portfolio space is always divided into three disjoint regions: the Buy, the Sell, and the NT region. If a portfolio lies either in the Buy or in the Sell region, the optimal strategy is to transact to the nearest NT boundary. Therefore, if the function $H(t,y,S,\theta)$ is known in the NT region, then

$$H^w(t,y,S,\theta)=\begin{cases}\exp\left(\gamma\frac{-(1-\lambda)(y-y_u^*)}{\delta(T,t)}\right)H^w(t,y_u^*,S,\theta) & \forall y(t,S)\geq y_u(t,S),\\ \exp\left(\gamma\frac{(1+\lambda)(y_l^*-y)}{\delta(T,t)}\right)H^w(t,y_l^*,S,\theta) & \forall y(t,S)\leq y_l(t,S),\end{cases} \tag{37}$$

where $y_u(t,S)$ and $y_l(t,S)$ are the upper and the lower boundaries of the writer's NT region.

3 Properties of the Reservation Option Prices

The purpose of this section is to derive some properties of the reservation option prices.

Theorem 4. *In a complete and friction-free market the reservation option prices coincide with the no-arbitrage price.*

If it is not true, an arbitrage opportunity is available in the market.

Conjecture 1. *For an investor with exponential utility function we have that*

1. *The unit reservation purchase price, $P_\theta^b(t,S)$, is a decreasing function of γ.*

2. *The unit reservation write price,* $P^w_\theta(t,S)$*, is an increasing function of* γ*.*

The above conjecture is quite intuitive. When there are transaction costs in the market, holding options involves an unavoidable element of risk. Therefore, the more risk avers investor is, the less he is willing to pay per an option. Similarly, the seller of an option will demand a unit price which increases as the seller's risk aversion increases.

Remark 1. The proof of the conjecture seems to be rather difficult and problematic, because there are no closed-form solutions for the investor's indirect value functions that enter into formulas for reservation option prices (see (26) and (36)). The statements in the conjecture rely solely on the results of numerical calculations of reservation option prices for different sets of model parameters.

Let's for the moment write the investor's value function of the utility maximization problem without options as $V(t,\gamma,x,y)$, and the corresponding value function of the utility maximization problem with options as $J(t,\gamma,x,y,S,\theta)$ (here we suppress the superscripts b and w). By this we want to emphasize that both of the value functions depend on the investor's coefficient of absolute risk aversion.

Theorem 5. *For an investor with the exponential utility function and an initial endowment* (x,y) *we have*

$$V(t,\gamma,x,y) = V(t,\theta\gamma,\frac{x}{\theta},\frac{y}{\theta}), \tag{38}$$

$$J(t,\gamma,x,y,S,\theta) = J(t,\theta\gamma,\frac{x}{\theta},\frac{y}{\theta},S,1). \tag{39}$$

Proof. Both these relationships can be easily established from the form of the exponential utility function. In particular, the portfolio process $\{\frac{x_s}{\theta},\frac{y_s}{\theta}; s>t\}$ is admissible given the initial portfolio $(\frac{x_t}{\theta},\frac{y_t}{\theta})$ if and only if $\{x_s,y_s; s>t\}$ is admissible given the initial portfolio (x_t,y_t). Furthermore, the amounts in the bank and in the stock accounts at time T for the value function V is given by the following integral versions of the state equations (4):

$$\begin{aligned} x_T &= x_t + \int_t^T r x_s ds - (1+\lambda)\int_t^T dL_s + (1-\lambda)\int_t^T dM_s \\ y_T &= y_t + \int_t^T \mu y_s ds + \int_t^T \sigma y_s dB_s + \int_t^T dL_s - \int_t^T dM_s. \end{aligned} \tag{40}$$

For the value function J^b the evolution of the amount invested in the bank is given by

$$\begin{aligned} dx_t &= r x_t dt - (1+\lambda) dL_t + (1-\lambda) dM_t \\ x_\tau &= x_{\tau-} + \theta g(S), \end{aligned} \tag{41}$$

where τ is the stopping time when the buyer chooses to exercise the options. The integral version of the system (41) can be written as equation (22). Similar equations can be written for the amount in the bank account for the value function J^w. The amount in the stock account at time T for both the value functions J^b and J^w is given by the same integral equation as for the value function V. Clearly, for any value function $U(\gamma, X_T) = U(\theta\gamma, \frac{X_T}{\theta})$. □

Corollary 6. *For an investor with the exponential utility function and an initial holding in the stock y we have*

$$Q(t,\gamma,y) = Q(t,\theta\gamma,\frac{y}{\theta}), \tag{42}$$

$$H(t,\gamma,y,S,\theta) = H(t,\theta\gamma,\frac{y}{\theta},S,1). \tag{43}$$

Proof. This follows from Theorem 5 and the definitions of the value functions Q and H. □

Theorem 7. *For an investor with exponential utility function we have that*

1. *An investor with an initial holding in the stock y and ARA coefficient γ has a unit reservation purchase price of θ options equal to his reservation purchase price of one option in the case where he has an initial holding in the stock $\frac{y}{\theta}$ and ARA coefficient $\theta\gamma$. That is,*

$$P^b_\theta(t,S) = \frac{\delta(T,t)}{\gamma}\ln\left(\frac{Q(t,\theta\gamma,\frac{y}{\theta})}{H^b(t,\theta\gamma,\frac{y}{\theta},S,1)}\right). \tag{44}$$

2. *An investor with an initial holding in the stock y and ARA coefficient γ has a unit reservation write price of θ options equal to his reservation write price of one option in the case where he has an initial holding in the stock $\frac{y}{\theta}$ and ARA coefficient $\theta\gamma$. That is,*

$$P_\theta^w(t,S) = \frac{\delta(T,t)}{\gamma} \ln\left(\frac{H^w(t,\theta\gamma,\frac{y}{\theta},S,1)}{Q(t,\theta\gamma,\frac{y}{\theta})}\right). \tag{45}$$

Proof. This follows from Theorem 5, the definitions of the value functions Q and H, Corollary 6, and equations (19) and (32). □

As mentioned above, in the practical applications of the utility based option pricing method one assumes that the investor has zero holdings in the stock at the initial time t, i.e., $y = 0$, hence $\frac{y}{\theta} = 0$ as well. In this case Theorem 7 says that the resulting unit reservation option price and the corresponding optimal trading strategy[4] in the model with the pair of parameters (γ, θ) will be the same as in the model with $(\theta\gamma, 1)$. That is, instead of calculating a model with θ options we can calculate a model with 1 option only. All we need is adjusting one parameter for θ: the absolute risk aversion from γ to $\theta\gamma$.

Corollary 8. *For an investor with exponential utility function and an initial holding in the stock $y = 0$ we have that*

1. *The unit reservation purchase price, $P_\theta^b(t,S)$, is decreasing in the number of options θ.*

2. *The unit reservation write price, $P_\theta^w(t,S)$, is increasing in the number of options θ.*

Note that the unit reservation option price in the model with a pair of parameters (γ, θ) is equal to the unit reservation option price in the model with $(\theta\gamma, 1)$. In other words, an increase in the number of options from 1 to θ is completely equivalent to an increase in the investor's level of risk aversion from γ to $\theta\gamma$. The consequence follows from Conjecture 1. Again, the result in Corollary 8 is quite intuitive. When there are transaction costs in the market, holding options involves an unavoidable element of risk. Therefore, the greater number of options the investor holds, the more risk he takes.

[4] Here, the trading strategy per option. For θ options the strategy must be re-scaled accordingly.

Consequently, the more options the risk averse investor has to buy, the less he is willing to pay per option. Similarly, the seller of options will demand a unit price which is increasing in the number of options. Note, in particular, that the linear pricing rule from the complete and frictionless market does not apply to the reservation option prices.

4 Numerical Procedure

4.1 Reformulation of the Problem

The main objective of this section is to present numerical procedures for computing the investor's value functions and the corresponding optimal trading policies. The starting point for our numerical calculations is the variational HJB inequalities (8). In this subsection we first argue that these inequalities of a singular stochastic control problem with a nature similar to those of Davis and Norman (1990) cannot be implemented in a numerical method for the computation of reservation option prices of American-style options. Then we prove that the investor's value function can alternatively be characterized as the unique viscosity solution of *quasi*-variational HJB inequalities, with a nature similar to those used in *stochastic impulse control* theory (see, for example, Bensoussan and Lions (1984) for that theory), and maintain that these inequalities provide the most natural way to construct numerical schemes upon.

As a beginning of the argument, we would like to mention that when it comes to the numerical computation of the value function and the associated optimal policy, the two inequalities in (8), which describe the Buy and the Sell region, cannot be implemented *explicitly* in a numerical method. The catch is that these two inequalities[5] describe how the value function should behave provided we know the value function at, say, times t and $t + dt$. On the contrary, any numerical method, either a finite-difference or a Markov chain approximation, implements a dynamic programming algorithm where the unknown values at time t is found by using the known values at the next time instant $t + dt$. Thus, these inequalities provide only an implicit indication on how to compute the value function.

Assume we know the value function V at time $t+dt$. How do we proceed

[5]These inequalities may alternatively be called as gradient constraints.

to find the value function at time t? An obvious start is to solve the partial differential equation $\mathcal{L}V(t,x,y) = 0$ between times t and $t+dt$ to find a lower[6] estimate for the value function. Then one finds the no transaction region where both $-(1+\lambda)V_x + V_y < 0$ and $(1-\lambda)V_x - V_y < 0$ are satisfied. Outside the no transaction region the value function is recomputed by using

$$V(t,x,y) = \begin{cases} V(t, x + (1-\lambda)(y - y_u), y_u) & \text{if } (x,y) \in \text{Sell region}, \\ V(t, x - (1+\lambda)(y_l - y), y_l) & \text{if } (x,y) \in \text{Buy region}, \end{cases}$$

where y_u and y_l are points on the upper and lower boundaries, respectively, of the no transaction region. This follows from the optimal transaction policy which mandates to transact to the nearest boundary of the no transaction region if the portfolio lies outside this region.

A serious problem with such an algorithm is that there might be several regions where both $-(1+\lambda)V_x + V_y < 0$ and $(1-\lambda)V_x - V_y < 0$ are satisfied. This could happen in the case when the lower estimate of the value function, after solving the partial differential equation $\mathcal{L}V(t,x,y) = 0$, has multiple local maxima[7] which, in their turn, produce multiple maxima as one transacts along the Buy or Sell direction. In this situation we face the problem of choosing true NT sub-regions and transaction policy. The only way to do it is to perform a search for a global maximum along the direction of transaction. Fortunately, in the optimal portfolio choice problem with either no options or only European-style options this does not happen. However, this is the case for the optimal portfolio choice problem with American-style options.

Let us elaborate on this more specifically. Consider now the value function J (we suppress the superscript b for the ease of notation) of the buyer of American-style options, defined by (20). It is tempting to implement the numerical algorithm using the same sequence of steps as described above: First, we start from the final date and solve $\overline{\mathcal{L}}J(t,x,y,S) = 0$ to find the lower estimate of the value function at the preceding time instant. Then for every (x,S) we find y_l and y_u such that in the points (x,y_l,S) and (x,y_u,S)

[6]That is, we find the expectation of the value function at the next time instant. Generally, the value function must be not less than its expectation. That is, $V(t,x(t),y(t)) \leq E\{V(t+dt, x(t+dt), y(t+dt))\}$.

[7]Note that the conditions $-(1+\lambda)V_x + V_y = 0$ and $(1-\lambda)V_x - V_y = 0$ are nothing else than the first order conditions of a local extremum as one transacts along the Buy or Sell direction, respectively.

the following conditions are satisfied:

$$\begin{aligned} -(1+\lambda)J_x(t,x,y_l,S) + J_y(t,x,y_l,S) = 0, \\ (1-\lambda)J_x(t,x,y_u,S) - J_y(t,x,y_u,S) = 0. \end{aligned} \tag{46}$$

We expect that the first equation give us the lower boundary of the NT region, and the second one gives the upper boundary of the NT region. Afterwards we recompute the value function outside the NT region. Then we repeat the previous steps for the remaining time instants backwards to the initial date.

However, due to the obstacle constraint $V_{ex} - J \leq 0$, the resulting value function often has two local maxima (see the description of the optimal policy of the buyer of an American put option in Section 2). One of them corresponds to the Exercise policy, and the other to the Keep policy. After solving $\overline{\mathcal{L}}J(t,x,y,S) = 0$, the picture remains essentially the same (see Figure (4)). Now we fix (x,S) and implement the search for the first condition in (46) starting from $y = 0$ (assuming it lies in the Buy region) and going upward to some $y = y_{max}$ (assuming it lies in the Sell region). We see that this condition is satisfied in two points, namely in y_1 and y_2. Moreover, it is clearly seen that the second condition in (46) will be also satisfied in two points. That is, it looks like the value function J has two NT sub-regions where the gradient constraints in (20) are satisfied. One is tempted to keep the value function inside these NT sub-regions and recalculate the value function outside of them. But generally this is not a correct solution. To understand it, let's take a look at the problem from another angle. We formulate the question as follows: how do we chose the proper point y to where to transact to if we start from $y = 0$? An obvious answer (recall that the investor's problem is to maximize his expected utility) is to choose the point which maximizes the value function, that is

$$y := \arg\max \big\{ J(t, x - (1+\lambda)y_1, y_1, S), J(t, x - (1+\lambda)y_2, y_2, S) \big\}. \tag{47}$$

In other words, we explicitly have to search for a point where the value function attains its maximum.

The inequalities for the Buy and the Sell regions in either (8) or (20) tell us that it is impossible to increase the value function by either buying or selling some amount of the stock at the expense of lowering or increasing,

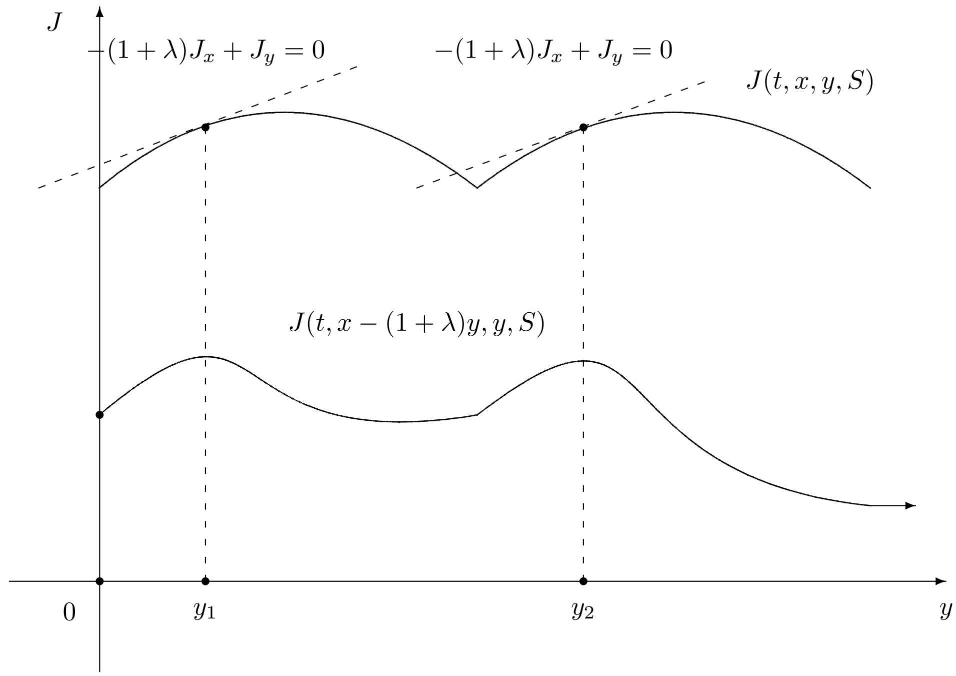

Figure 4: A schematic sketch of the case where the value function $J(t, x(t), y(t), S(t)) = E[J(t + dt, x(t + dt), y(t + dt), S(t + dt))]$ has two local maxima along the direction of transaction.

respectively, the holdings in the bank account. An alternative and more explicit numerical procedure to solve the optimal portfolio selection problem with proportional transaction cost is analogous to that used to solve the optimal portfolio selection problem with both fixed and proportional transaction costs[8]. Consider again the optimal portfolio choice problem without options: As before, we start with solving the partial differential equation $\mathcal{L}V(t, x, y) = 0$ for the no-transaction problem. Then we need to compare the value function at each point (x, y) with the maximum attainable values from either buying or selling some amount of the stock. Mathematically this

[8]The solution to the optimal portfolio selection problem where each transaction has a fixed cost component is based on the theory of stochastic impulse controls (see, for example Øksendal and Sulem (2002)).

procedure is described by the *maximum utility operator* $\mathcal{M}$:

$$\mathcal{M}V(t,x,y) = \sup_{(x',y')\in\mathcal{A}(x,y),(x',y')\neq(x,y)} V(t,x',y'), \tag{48}$$

where $\mathcal{A}(x,y)$ denotes the set of admissible controls available to the investor who starts at time t with an amount of x in the bank and y holdings in the stock, and x' and y' are the new values[9] of x and y after transaction. In other words, $\mathcal{M}V(t,x,y)$ represents the value of the strategy that consists in choosing the best transaction.

Remark 2. Note that in the definition of the maximum utility operator we require that $(x',y') \neq (x,y)$. That is, in finding the best possible transaction we do not consider the initial point and require a non-zero (probably infinitesimal) transaction size.

Now, by giving heuristic arguments, we intend to characterize the value function and the associated optimal strategy by using the notion of the maximum utility operator: If for some initial point (t,x,y) the optimal strategy is to not transact, the utility associated with this strategy is $V(t,x,y)$. Choosing the best transaction and then following the optimal strategy gives the utility $\mathcal{M}V(t,x,y)$. The necessary condition for the optimality of the first strategy is $V(t,x,y) \geq \mathcal{M}V(t,x,y)$. This inequality holds with equality when it is optimal to rebalance the portfolio. Moreover, in the no-transaction region, the application of the dynamic programming principle gives $\mathcal{L}V(t,x,y) = 0$.

The subsequent theorem formalizes this intuition.

Theorem 9. *The value function V defined by (6) is the unique viscosity solution of the quasi-variational Hamilton-Jacobi-Bellman inequalities (QVHJBI, or just QVI):*

$$\max\Big\{\mathcal{L}V, \quad \mathcal{M}V - V\Big\} = 0 \tag{49}$$

with the boundary condition

$$V(T,x,y) = U(\gamma, X_T).$$

The proof can be made by following along the lines of the proofs of The-

[9] That is, $y' = y + \Delta y$ and $x' = x - k - \Delta y - \lambda|\Delta y|$, where Δy is the size of transaction.

orem 3.7 (the existence result) and Theorem 3.8 with subsequent Corollary (the uniqueness result) in Øksendal and Sulem (2002).

Moreover, we can prove that the two different formulations of the same problem, (8) and (49), yield the same result. It suffices to prove the following theorem.

Theorem 10. *For the optimal portfolio selection problem with proportional transaction costs,*

$$\begin{aligned} -(1+\lambda)V_x + V_y \leq 0, \\ (1-\lambda)V_x - V_y \leq 0, \end{aligned} \tag{50}$$

if and only if

$$\mathcal{M}V - V \leq 0. \tag{51}$$

Proof. The first part. Assume (50) holds. Chose any point (x_0, y_0). Suppose that the maximum along the Buy line starting in (x_0, y_0) is attained at the point $(x_0 - (1+\lambda)\alpha, y_0 + \alpha)$, and that the maximum along the Sell line starting in (x_0, y_0) is attained at the point $(x_0 + (1-\lambda)\beta, y_0 - \beta)$. Then for the maximum along the Buy line we have that

$$\begin{aligned} &V(t, x_0 - (1+\lambda)\alpha, y_0 + \alpha) = V(t, x_0, y_0) \\ &+ \int_0^\alpha \big[-(1+\lambda)V_x(t, x_0 - (1+\lambda)s, y_0 + s) + V_y(t, x_0 - (1+\lambda)s, y_0 + s)\big] ds \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq V(t, x_0, y_0). \end{aligned}$$

Similarly, for the maximum along the Sell line we have that

$$\begin{aligned} &V(t, x_0 + (1-\lambda)\beta, y_0 - \beta) = V(t, x_0, y_0) \\ &+ \int_0^\beta \big[(1-\lambda)V_x(t, x_0 + (1-\lambda)s, y_0 + s) - V_y(t, x_0 - (1-\lambda)s, y_0 + s)\big] ds \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq V(t, x_0, y_0). \end{aligned}$$

Consequently, $\mathcal{M}V(t, x_0, y_0) - V(t, x_0, y_0) \leq 0$. Since the point (x_0, y_0) was chosen arbitrary, this holds for every point (x, y) in the domain of V.

The second part. Assume (51) holds. Chose any point (x_0, y_0). Then for any point along the Buy line starting in (x_0, y_0) we have that

$$V(t, x_0 - (1+\lambda)h, y_0 + h) \leq V(t, x_0, y_0),$$

and for any point along the Sell line starting in (x_0, y_0) we have that

$$V(t, x_0 + (1-\lambda)h, y_0 - h) \leq V(t, x_0, y_0),$$

where h is an arbitrary positive real number. Allowing $h \to 0$ we obtain that

$$\lim_{h\to 0} \frac{1}{h}\big[V(t, x_0 - (1+\lambda)h, y_0 + h) - V(t, x_0, y_0)\big]$$
$$= -(1+\lambda)V_x(t, x_0, y_0) + V_y(t, x_0, y_0) \leq 0$$

$$\lim_{h\to 0} \frac{1}{h}\big[V(t, x_0 + (1-\lambda)h, y_0 - h) - V(t, x_0, y_0)\big]$$
$$= (1-\lambda)V_x(t, x_0, y_0) - V_y(t, x_0, y_0) \leq 0$$

Again, since the point (x_0, y_0) was chosen arbitrary, this holds for every point (x, y) in the domain of V. □

Even though the two different formulations of the same problem, (8) and (49), yield the same result, the latter has direct implications for the practical realization of a numerical procedure. Similarly, as for the investor's problem (8), we propose an alternative formulation of the buyer's problem described by (20)

$$\max\Big\{V_{ex} - J, \quad \overline{\mathcal{L}}J, \quad \mathcal{M}J - J\Big\} = 0, \tag{52}$$

and an alternative formulation of the writer's problem described by (33)

$$\max\Big\{\overline{\mathcal{L}}J, \quad \mathcal{M}J - J\Big\} = 0. \tag{53}$$

Consequently, the two following theorems can be proved:

Theorem 11. *The value functions J defined by (18) is a unique viscosity solutions of the QVI (52).*

Theorem 12. *The value functions J defined by (31) is a unique viscosity solutions of the QVI (53).*

4.2 A Markov Chain Approximation of the Continuous Time Problem

In this subsection we turn on to the specific discretization and the solution of the investor's problems applying the method of the Markov chain approximation. The Markov chain approximation method for the solution of continuous-time continuous-space stochastic control problems was suggested by Kushner (see, for example, Kushner and Martins (1991) and Kushner and Dupuis (1992)). First, according to this method, one constructs discrete time approximations of the continuous time price processes used in the continuous time model. Then the discrete time program is solved by using the discrete time dynamic programming algorithm (i.e., backward recursion algorithm).

Consider the partition $0 = t_0 < t_1 < \ldots < t_n = T$ of the time interval $[0, T]$ and assume that $t_i = i\Delta t$ for $i = 0, 1, \ldots, n$ where $\Delta t = \frac{T}{n}$. Let ε be a stochastic variable:

$$\varepsilon = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } 1 - p. \end{cases}$$

We define the discrete time stochastic process of the stock as

$$S_{t_{i+1}} = S_{t_i} \varepsilon, \tag{54}$$

and the discrete time process of the risk-free asset as

$$x_{t_{i+1}} = x_{t_i} \rho. \tag{55}$$

If we choose $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, $\rho = e^{r\Delta t}$, and $p = \frac{1}{2}\left[1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right]$, we obtain the binomial model proposed by Cox, Ross, and Rubinstein (1979). An alternative choice is $u = e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}}$, $d = e^{(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}$, $\rho = e^{r\Delta t}$, and $p = \frac{1}{2}$, which was proposed by He (1990). As n goes to infinity, the discrete time processes (54) and (55) converge in distribution to their continuous counterparts (2) and (1). This is what is called the *local consistency conditions* for a Markov chain.

The following discretization scheme is proposed to find the value function

$V(t,x,y)$ defined by (49)

$$V^{\Delta t}(t_i,x,y) = \max\Big\{ \max_m V^{\Delta t}(t_i, x-(1+\lambda)m\delta y, y+m\delta y), \\ \max_m V^{\Delta t}(t_i, x+(1-\lambda)m\delta y, y-m\delta y), \\ E[V^{\Delta t}(t_{i+1}, x\rho, y\varepsilon)]\Big\}, \quad (56)$$

where m runs through the positive integer numbers ($m = 1, 2, 3, \ldots$), and

$$V^{\Delta t}(t_i, x-(1+\lambda)m\delta y, y+m\delta y) \\ = E\left\{V^{\Delta t}(t_{i+1}, (x-(1+\lambda)m\delta y)\rho, (y+m\delta y)\varepsilon)\right\}, \quad (57)$$

$$V^{\Delta t}(t_i, x+(1-\lambda)m\delta y, y-m\delta y) \\ = E\left\{V^{\Delta t}(t_{i+1}, (x+(1-\lambda)m\delta y)\rho, (y-m\delta y)\varepsilon)\right\}, \quad (58)$$

as at time t_i we do not know yet the value function. In this case we find expectation using the known values at the next time instant t_{i+1}. Here we have discretized the y-space in a lattice with grid size δy, and the x-space in a lattice with grid size δx[10]. This scheme is a dynamic programming formulation of the discrete time problem. The solution procedure is as follows: Start at the terminal date and give the value function values by using the boundary conditions as for the continuous value function over the discrete state space. Then work backwards in time. That is, at every time instant t_i and every particular state (x, y), by knowing the value function for all the states in the next time instant, t_{i+1}, find the investor's optimal policy. This is carried out by comparing maximum attainable utilities from buying, selling, or doing nothing.

Theorem 13. *The solution $V^{\Delta t}$ of (56) converges weakly to the unique continuous viscosity solution of (49) as $\Delta t \to 0$.*

For a rigorous treatment of a proof of this type of convergence theorems, we refer the reader to, for example, Kushner and Martins (1991), Davis et al. (1993), and Davis and Panas (1994). Instead of presenting a cumbersome proof of Theorem (13), we want to prove a simple proposition:

[10]It is supposed that $\lim_{\Delta t \to 0} \delta y \to 0$, and $\lim_{\Delta t \to 0} \delta x \to 0$, that is, $\delta y = c_y \Delta t$, and $\delta x = c_x \Delta t$ for some constants c_y and c_x.

Proposition 2. *Assuming* $\lim_{\Delta t \to 0} V^{\Delta t} = V$*, the solution of discrete time program (56) converges to the solution of continuous time quasi-variational inequalities (49) as* $\Delta t \to 0$.

Proof. We choose the choice of u, d, ρ, and p which was proposed by He (1990). This choice clearly satisfies the local consistency conditions. In this case we can approximate the dynamics of the controlled processes as

$$y(t+\Delta t) - y(t) = y(t)\mu\Delta t \pm y(t)\sigma\sqrt{\Delta t},$$
$$x(t+\Delta t) - x(t) = x(t)r\Delta t.$$

Consider the term $E\left\{V^{\Delta t}(t+\Delta t, x(t+\Delta t), y(t+\Delta t))\right\}$. Assuming that $V^{\Delta t}$ is differentiable (in the viscosity sense), using the Taylor expansion of $V^{\Delta t}$ around (t,x,y), and taking the expectation we get

$$\begin{aligned} &E\left\{V^{\Delta t}(t+\Delta t, x(t+\Delta t), y(t+\Delta t))\right\} \\ &\quad = V(t,x,y)^{\Delta t} + (V_t^{\Delta t} + rxV_x^{\Delta t} + \mu y V_y^{\Delta t} + \frac{1}{2}\sigma^2 y^2 V_{yy}^{\Delta t})\Delta t + o(\Delta t), \end{aligned}$$

where $o(\Delta t)$ are error terms containing Δt of order higher than one. Allowing $\Delta t \to 0$ we obtain

$$\lim_{\Delta t \to 0} E\left\{V^{\Delta t}(t+\Delta t, x(t+\Delta t), y(t+\Delta t))\right\} = V(t,x,y) + \mathcal{L}V(t,x,t)dt. \tag{59}$$

Moreover, as $\Delta t \to 0$,

$$\lim_{\Delta t \to 0} \max_m V^{\Delta t}(t_i, x-(1+\lambda)m\delta y, y+m\delta y) = \sup_{\Delta y>0} V(t, x-(1+\lambda)\Delta y, y+\Delta y), \tag{60}$$

$$\lim_{\Delta t \to 0} \max_m V^{\Delta t}(t_i, x+(1-\lambda)m\delta y, y-m\delta y) = \sup_{\Delta y>0} V(t, x+(1-\lambda)\Delta y, y-\Delta y). \tag{61}$$

Note here that Δy can take any positive real value but zero. Allowing Δy to take both positive and negative values, we can combine (60) and (61) and, thus, get

$$\begin{aligned} &\sup_{\Delta y \neq 0} V(t, x-\Delta y - \lambda|\Delta y|, y+\Delta y) \\ &\qquad = \lim_{\Delta t \to 0} \max \begin{cases} \max_m V^{\Delta t}(t_i, x-(1+\lambda)m\delta y, y+m\delta y), \\ \max_m V^{\Delta t}(t_i, x+(1-\lambda)m\delta y, y-m\delta y). \end{cases} \end{aligned} \tag{62}$$

By definition (48), the left hand side of (62) is nothing else than the maximum utility operator, that is

$$\sup_{\Delta y \neq 0} V(t, x - \Delta y - \lambda|\Delta y|, y + \Delta y) = \mathcal{M}V(t, x, y). \tag{63}$$

Therefore in the limit as $\Delta t \to 0$ the discrete time program (56) converges to (using (59), (62), and (63))

$$V(t, x, y) = \max\{V(t, x, y) + \mathcal{L}V(t, x, t)dt, \quad \mathcal{M}V(t, x, y)\},$$

which can be rewritten as

$$\max\{\mathcal{L}V(t, x, t), \quad \mathcal{M}V(t, x, y) - V(t, x, y)\} = 0.$$

This completes the proof. □

We now suggest a Markov chain approximation of the problem of the buyer of an American option. The basic idea here is that the buyer needs to compare the maximum attainable expected utilities from keeping the option and exercising the option. That is, the value function in the state (t_i, x, y, S) is given by

$$J^{b,\Delta t}(t_i, x, y, S) = \max\{J^{b,\Delta t}_{keep}(t_i, x, y, S), \quad V^{b,\Delta t}_{ex}(t_i, x, y, S)\}. \tag{64}$$

This means that if

$$J^{b,\Delta t}_{keep}(t_i, x, y, S) < V^{b,\Delta t}_{ex}(t_i, x, y, S),$$

then it is optimal to exercise the option in the state (t_i, x, y, S). Otherwise, it is optimal to keep the option. The discretization scheme for the value function $V^{b,\Delta t}_{ex}$ is similar to that of the value function $V^{\Delta t}$ and is given by the discrete time version of the equation (17):

$$V^{b,\Delta t}_{ex}(t_i, x, y, S) = V^{\Delta t}(t_i, x + g(S), y). \tag{65}$$

The following discretization scheme is proposed for the value function

$J^{b,\Delta t}_{keep}$

$$J^{b,\Delta t}_{keep}(t_i,x,y,S) = \max\Big\{ \max_m J^{b,\Delta t}_{keep}(t_i, x-(1+\lambda)m\delta y, y+m\delta y, S),$$
$$\max_m J^{b,\Delta t}_{keep}(t_i, x+(1-\lambda)m\delta y, y-m\delta y, S), \quad (66)$$
$$E\{J^{b,\Delta t}(t_{i+1}, x\rho, y\varepsilon, S\varepsilon)\}\Big\},$$

where m runs through the positive integer numbers, and

$$J^{b,\Delta t}_{keep}(t_i, x-(1+\lambda)m\delta y, y+m\delta y, S)$$
$$= E\left\{J^{b,\Delta t}(t_{i+1}, (x-(1+\lambda)m\delta y)\rho, (y+m\delta y)\varepsilon, S\varepsilon)\right\},$$

$$J^{b,\Delta t}_{keep}(t_i, x+(1-\lambda)m\delta y, y-m\delta y, S)$$
$$= E\left\{J^{b,\Delta t}(t_{i+1}, (x+(1-\lambda)m\delta y)\rho, (y-m\delta y)\varepsilon, S\varepsilon)\right\}.$$

The principle behind this scheme is the same as for the discretization scheme (56). As before, we have discretized the y-space in a lattice with grid size δy, and the x-space in a lattice with grid size δx. In addition, we use a binomial tree for the stock price process.

Theorem 14. *The solution $J^{b,\Delta t}$ of (64) converges weakly to the unique viscosity solution of the continuous time problem characterized by (52) as $\Delta t \to 0$.*

The proof follows along similar arguments as in Theorem (13).

We now suggest a Markov chain approximation of the problem of the writer of an American option. His value function in the state (t_i, x, y, S) is given by

$$J^{w,\Delta t}(t_i,x,y,S) = \begin{cases} J^{w,\Delta t}_{keep}(t_i,x,y,S) & \text{if the buyer keeps the option,} \\ V^{w,\Delta t}_{ex}(t_i,x,y,S) & \text{if the buyer chooses to exercise.} \end{cases} \quad (67)$$

The discretization scheme for the value function $V^{w,\Delta t}_{ex}$ is similar to (56) and is given by

$$V^{w,\Delta t}_{ex}(t_i,x,y,S) = V^{\Delta t}(t_i, x-g(S), y). \quad (68)$$

The following discretization scheme is proposed for the QVI (53):

$$
\begin{aligned}
J_{keep}^{w,\Delta t}(t_i, x, y, S) = \max\Big\{ & \max_m J_{keep}^{w,\Delta t}(t_i, x-(1+\lambda)m\delta y, y+m\delta y, S), \\
& \max_m J_{keep}^{w,\Delta t}(t_i, x+(1-\lambda)m\delta y, y-m\delta y, S), \\
& E\{J^{w,\Delta t}(t_{i+1}, x\rho, y\varepsilon, S\varepsilon)\}\Big\},
\end{aligned} \tag{69}
$$

where m runs through the positive integer numbers, and

$$
\begin{aligned}
& J_{keep}^{w,\Delta t}(t_i, x-(1+\lambda)m\delta y, y+m\delta y, S) \\
& \qquad = E\left\{J^{w,\Delta t}(t_{i+1}, (x-(1+\lambda)m\delta y)\rho, (y+m\delta y)\varepsilon, S\varepsilon)\right\}
\end{aligned}
$$

$$
\begin{aligned}
& J_{keep}^{w,\Delta t}(t_i, x+(1-\lambda)m\delta y, y-m\delta y, S) \\
& \qquad = E\left\{J^{w,\Delta t}(t_{i+1}, (x+(1-\lambda)m\delta y)\rho, (y-m\delta y)\varepsilon, S\varepsilon)\right\}.
\end{aligned}
$$

The principle behind this scheme is the same as for the discretization schemes described above. The following theorem can be proved:

Theorem 15. *The solution $J^{w,\Delta t}$ of (69) converges weakly to the unique viscosity solution of the continuous time problem characterized by (53) as $\Delta t \to 0$.*

All the discretization schemes described above are valid for any type of utility function. Note that for a general utility function we need to perform the calculations first in a three-dimensional space (t, x, y) and then in a four-dimensional space (t, x, y, S), and the amount of computations is very high. For the negative exponential utility function the dynamics of y through time is independent of the total wealth in the continuous time and in the discrete time framework as well. Therefore the discrete space equivalents of (12), (23), and (34) can be written as follows:

$$
\begin{aligned}
V^{\Delta t}(t, x, y) &= \exp(-\gamma \tfrac{x}{\delta(T,t)}) Q^{\Delta t}(t, y), \\
J^{b,\Delta t}(t, x, y, S, \theta) &= \exp(-\gamma \tfrac{x}{\delta(T,t)}) H^{b,\Delta t}(t, y, S, \theta), \\
J^{w,\Delta t}(t, x, y, S, \theta) &= \exp(-\gamma \tfrac{x}{\delta(T,t)}) H^{w,\Delta t}(t, y, S, \theta).
\end{aligned} \tag{70}
$$

Thus, the dimensionality of the problem is reduced by one. This dramatically decreases the amount of computations. The discretization scheme for the function $Q^{\Delta t}(t,y)$ is derived from (56) and (70) to be

$$Q^{\Delta t}(t_i,y) = \max\Big\{ \max_m \exp\left(\gamma\frac{(1+\lambda)m\delta y}{\delta(T,t_i)}\right) Q^{\Delta t}(t_i, y+m\delta y), \\ \max_m \exp\left(\gamma\frac{-(1-\lambda)m\delta y}{\delta(T,t_i)}\right) Q^{\Delta t}(t_i, y-m\delta y), \\ E\{Q^{\Delta t}(t_{i+1}, y\varepsilon)\}\Big\}. \tag{71}$$

As in the continuous time case, if the value function $Q^{\Delta t}(t_i,y)$ is known in the NT region, then it can be calculated in the Buy and Sell regions by using the discrete space version of (16):

$$Q^{\Delta t}(t_i,y) = \begin{cases} \exp\left(\gamma\frac{-(1-\lambda)(y-y_u)}{\delta(T,t_i)}\right) Q^{\Delta t}(t_i,y_u) & \forall y(t_i) \geq y_u(t_i), \\ \exp\left(\gamma\frac{(1+\lambda)(y_l-y)}{\delta(T,t_i)}\right) Q^{\Delta t}(t_i,y_l) & \forall y(t_i) \leq y_l(t_i). \end{cases} \tag{72}$$

The practical implementation of the numerical scheme for $Q^{\Delta t}$ is based on the qualitative knowledge of the form of the optimal trading strategy. That is, at every time t the optimal strategy is completely described by the two numbers: $y_l(t)$ - the lower boundary of the NT region, and $y_u(t)$ - the upper boundary of the NT region. Assuming we know the value function at t_{i+1} and that $(y_l(t_i), y_u(t_i)) \in (y_{min}, y_{max})$, the following sequence of steps is performed at time t_i:

1. Starting from y_{min} and going up to y_{max} we perform the search for a maximum along the Buy line:

   $$y_l(t_i) = \arg\max_y E\left\{\exp\left(\gamma\frac{(1+\lambda)(y-y_{min})}{\delta(T,t_{i+1})}\right) Q^{\Delta t}(t_{i+1}, y\varepsilon)\right\},$$

   where $y = y_{min} + m\delta y$, $m = \{1,2,\dots,M\}$, and $M = \frac{y_{max}-y_{min}}{\delta y}$.

2. Similarly, starting from y_{max} and going down to y_{min} we perform the search for a maximum along the Sell line:

   $$y_u(t_i) = \arg\max_y E\left\{\exp\left(\gamma\frac{-(1-\lambda)(y_{max}-y)}{\delta(T,t_{i+1})}\right) Q^{\Delta t}(t_{i+1}, y\varepsilon)\right\}.$$

3. Having determined the boundaries of the NT region, we proceed by computing and storing in the memory[11] the value function inside the

[11]We need to keep the value function at time t_i for at lest one period in order to use it

NT region for every grid step $y = y_l + m\delta y$, $y \in (y_l(t_i), y_u(t_i))$:

$$Q^{\Delta t}(t_i, y) = E\{Q^{\Delta t}(t_{i+1}, y\varepsilon)\}.$$

Outside of the NT region, we can compute the function applying equation (72). Note that we do note require that $y\varepsilon$ lies exactly in a node of the grid for y-space. We estimate $Q^{\Delta t}(t_{i+1}, y\varepsilon)$ on a set of points $(t_{i+1}, m\delta y)$ using some form of interpolation.

In the same manner we can derive from (64), (69), and (70) the discretization schemes for the value function $H^{w,\Delta t}(t, y, S, \theta)$ of the option writer. The option buyer's strategy is more complicated than that of the writer, but, anyway, we can describe his transaction policy by a list of some simple rules.

5 Numerical Results

In this section we present the results of our numerical computations of reservation purchase and write prices and the corresponding exercise policies for American put and call options. In most of our calculations we used the following model parameters: the risky asset price at time zero $S_0 = 100$, the strike price $K = 100$, the volatility $\sigma = 20\%$, the drift $\mu = 10\%$, and the risk-free rate of return $r = 5\%$ (all in annualized terms). The options expire at $T = 1$ year. The proportional transaction costs $\lambda = 1\%$. The discretization parameters of the Markov chain, depending on the investor's ARA, are: $n \in [100, 250]$ periods of trading, and the grid size $\delta y \in [0.001, 0.1]$. When we calculate the prices of call options for investors with high ARA, we cannot increase the number of periods of trading beyond some threshold, as the values of the exponential utility are either overflow or underflow. However, this is not an issue for calculating the prices of put options.

The number of options is always 1 in all our calculations. Recall that, according to Theorem 7, the resulting unit reservation option price and the corresponding exercise policy in the model with the pair of parameters $(\gamma, 1)$ will be the same as in the model with $(\frac{\gamma}{\theta}, \theta)$. This means if, for example, we choose $\gamma = 1$ and $\theta = 1$, then we get the same unit reservation option price as in the model with $\gamma = 0.01$ and $\theta = 100$.

for the computation of the value function at the preceding time t_{i-1}.

In the case with no transaction costs and a put option, the exercise policy depends on (t, S) and can be easily drawn in the (t, S) - plane as an early exercise boundary $S^*(t)$. Recall that the option buyer's exercise policy generally depends on his holdings in the stock account, that is, it depends on (t, y, S). This results in a couple of complications: First, it is rather cumbersome to depict and interpret the early exercise boundary in three dimensions. Second, as we use a recombining binomial tree for representing the evolution of the stock price, in some nodes of the tree, on the basis of knowing only (t, S), we do not know whether the buyer chooses to exercise the option or not to exercise. This means that we cannot uniquely define the exercise policy on this type of tree for the computation of a reservation option price for the writer. We could overcome this problem by using a non-recombining (bushy) tree, but in this case the computations would be highly time consuming and could be implemented for a model with maximum 20-25 periods of trading[12].

We suggest a simple resolution of these complications. Recall that in the (t, S) plane we can provide two bounds on the early exercise boundary, the upper bound $S_u(t)$ and the lower bound $S_l(t)$. Consequently, we provide two bounds on the reservation write price. In the computation of the first one we use the exercise policy $S_u(t)$, and in the computation of the second one we use the exercise policy $S_l(t)$. Note that the reservation purchase price is unique.

5.1 Numerical Results for American Put Options

In this subsection we present the results of our numerical computations of reservation option prices and exercise policies for American put options. We begin our presentation with the study of how reservation option prices depend on the level of the investor's absolute risk aversion γ. The results of the numerical computations are presented in Figure (5). On the basis of studying the Figure, we can make the following observations concerning the reservation option prices: The reservation purchase price is always below the option price in the model with no transaction costs and is a decreasing function of γ. The reservation write price is an increasing function of γ. For

[12]This algorithm grows quadratically in complexity as the number of periods increases, meaning that the calculation of the optimal policy for period $n + 1$ takes approximately the same time as the calculation of the optimal policies for all n previous periods.

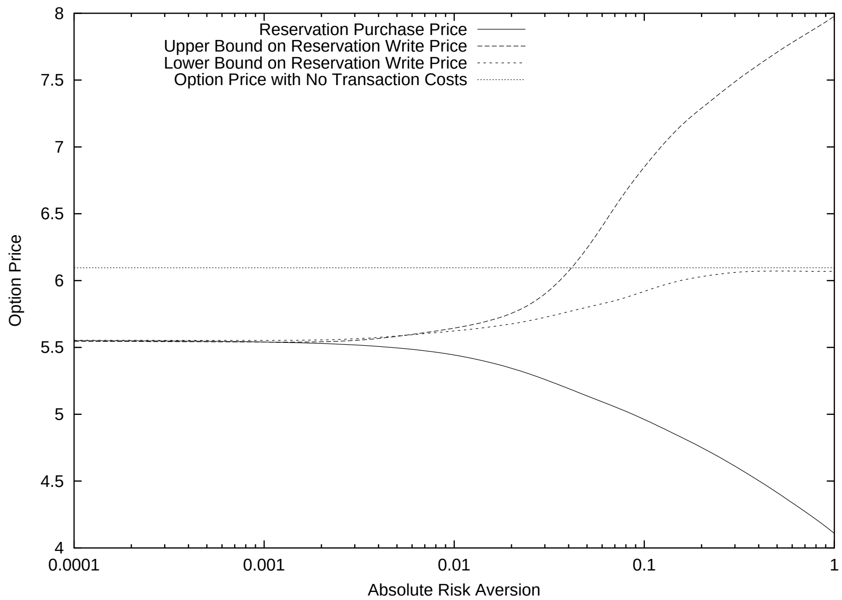

Figure 5: Reservation prices of American options as functions of the level of absolute risk aversion.

high values of γ, the average (of the two bounds) reservation write price is located above the option price in the model with no transaction costs. As γ decreases, all the reservation option prices approach a horizontal asymptote located below the option price in the model with no transaction costs. Here, for low values of γ, the reservation option prices are virtually independent of the choice of γ and are very close to each other.

Now we present the intuition behind the observed dependence of the reservation option prices on the parameter γ.

In the presence of transaction costs the hedging of options is costly. The amount of hedging transaction costs increases when the option holder's risk aversion increases (a more risk avers option holder hedges options more often). These hedging transaction costs reduce the reservation purchase price and increase the reservation write price. That is, the more risk avers the option buyer is, the less he is willing to pay per an option. Similarly, the writer of an option will demand a unit price which increases as the writer's risk aversion increases.

On the other hand, as the option holder's risk aversion decreases, he invests more wealth in the risky stock. At the same time, the frequency of trading caused by hedging decisions decreases, and, eventually, the highly risk tolerant option holder implements mainly a so-called *static* hedge, regardless of how high risk tolerant he is. That is why a reservation option price approaches a horizontal asymptote as γ decreases. Now we turn on to explain why this horizontal asymptote located below the option price in the market with no transaction costs.

In particular, to implement the static hedge of a long option position, the option buyer needs to buy additional number of shares of the stock at time zero and sell them when he chooses to exercise the option. He deducts these extra transaction costs from the price he is willing to pay for the option. On the contrary, to hedge a short option position, the option writer needs to sell short some number of shares of the stock at time zero and liquidate the short position in the stock when the buyer chooses to exercise the option. Consequently, selling options causes the writer to invest less in the stock. Thus, it reduces transaction costs payed in the stock market, and these savings reduce the reservation write price.

It turns out that, for the option holder with low value of γ, the discrepancy between a reservation option price and the option price in the model with no transaction costs is roughly equal to the round trip transaction costs of buying some number (required by the optimal hedge) of shares of the stock. Here, the reservation option price can be approximately calculated using the formula[13]

$$P = P_{BS} - 2\Delta_{BS}(0)S_0\lambda, \tag{73}$$

where P_{BS} and $\Delta_{BS}(0)$ is the option price and the option delta at time zero, respectively, in the model with no transaction costs (Black-Scholes price and delta), and S_0 is the stock price at time zero. Note that the level of proportional transaction costs fully explains the magnitude of the discrepancy between the option prices with and without transaction costs.

We now turn to the analysis of how the early exercise boundary depends on the level of the option holder's absolute risk aversion γ. Figures (6) and (7) show the two bounds on the early exercise boundary for an option holder with $\gamma = 0.001$ and $\gamma = 1$ respectively. The obvious conclusion here is that the more risk avers the option holder is, the earlier he exercises the

[13]This is a conjecture which is confirmed by comparison with the numerically calculated reservation option prices.

option. The similar effect on the early exercise boundary has the level of transaction costs. That is, the higher the transaction costs are, the earlier the option holder exercises the option. The explanation for this is quite intuitive and can also be done in terms of hedging transaction costs. Recall that the amount of transaction costs caused by hedging decisions increases when either the level of transaction costs increases or the option holder's risk aversion increases. When the expected amount of hedging transaction costs becomes greater than the difference between the option value and the exercise payoff, the option does not worth further hedging. That is why options are exercised earlier in the market with transaction costs.

In the region where the exercise policy is not uniquely defined on the basis of knowing only (t, S), there is a simple rule of thumb: The buyer of an option is the more inclined to exercise the option, the less holdings in the stock he has as compared to the optimal hedging position in the stock in a friction-free market. The intuition behind this is as follows: Instead of buying additional number of shares of the stock, in order to bring the hedging position in the stock in correspondence with the optimal amount (thus paying some transaction costs), it is better to exercise the option. Again, this happens when the discrepancy between the option value and the exercise payoff becomes less than the expected amount of hedging transaction costs.

Figures (8) and (9) show the bounds on reservation option prices versus the price of the underlying stock for an option holder with $\gamma = 0.001$ and $\gamma = 1$ respectively. For the option holder with $\gamma = 0.001$, the reservation purchase price and the two bounds on the reservation write price are almost coincide. We observe that the reservation option prices are always below the corresponding prices in the market with no transaction costs. Moreover, the deviation of a reservation option price from the corresponding price in the market with no transaction costs almost does not depend on the underlying stock price (or the moneyness of the option), at least in the chosen interval of the stock prices. For the option holder with $\gamma = 1$, the reservation purchase price is below, and the average of the two bounds on the reservation write price is generally above the corresponding price in the market with no transaction costs. As the option becomes more out-of-the-money, the reservation purchase and write prices converge, because such options tend to be exercised very soon.

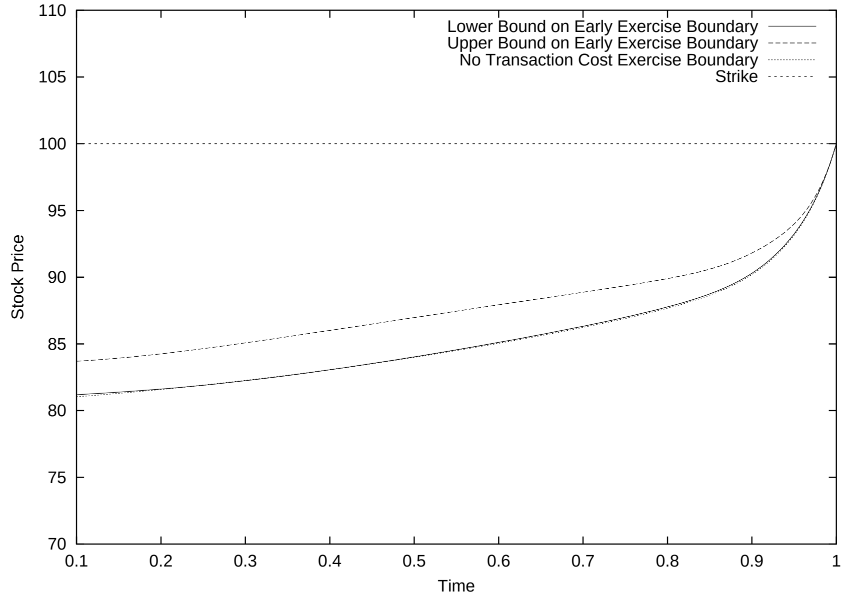

Figure 6: Bounds on the early exercise boundary for an American option holder with $\gamma = 0.001$.

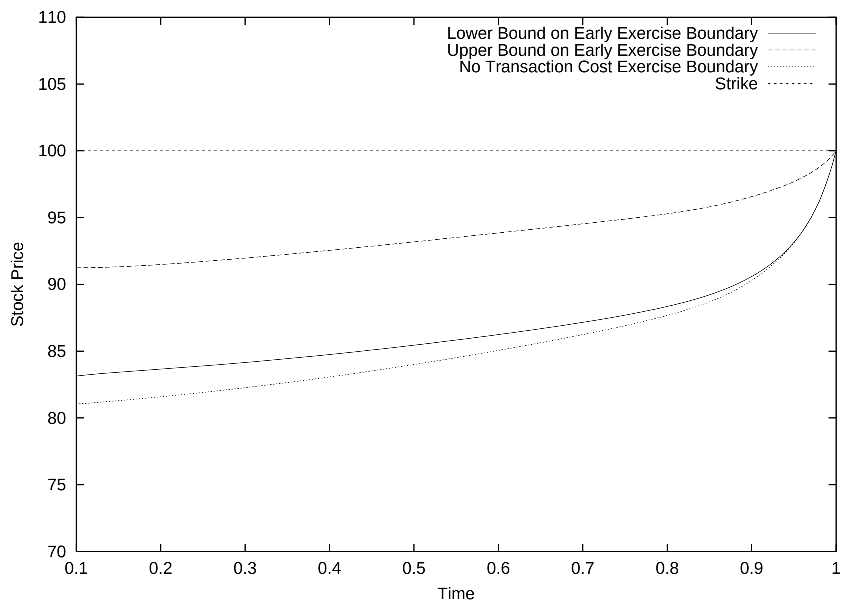

Figure 7: Bounds on the early exercise boundary for an American option holder with $\gamma = 1$.

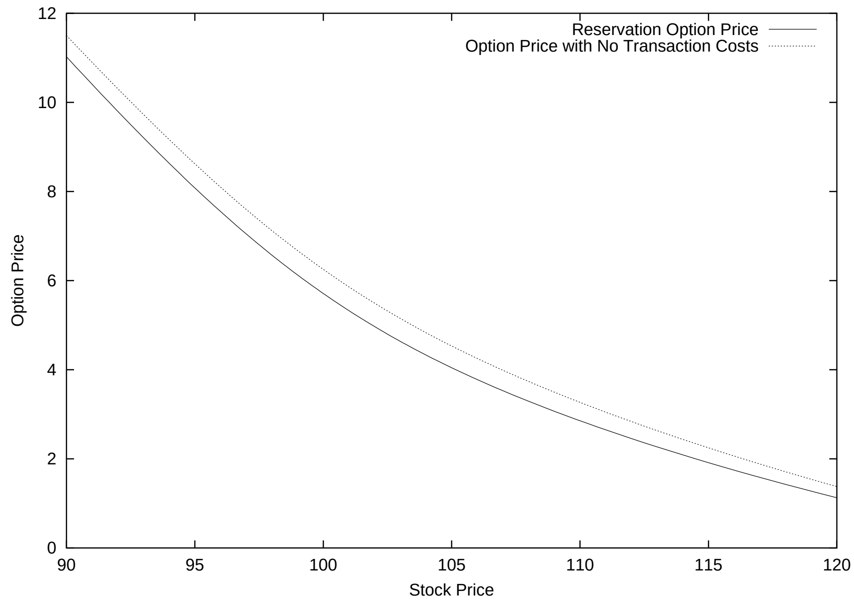

Figure 8: Bounds on reservation prices of American options versus the price of the underlying stock for an option holder with $\gamma = 0.001$.

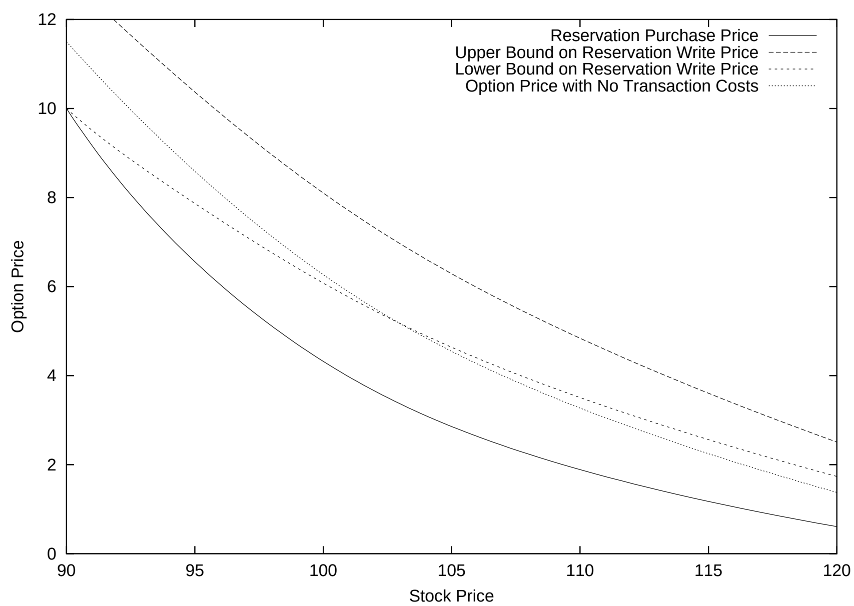

Figure 9: Bounds on reservation prices of American options versus the price of the underlying stock for an option holder with $\gamma = 1$.

5.2 Numerical Results for American Call Options

Our numerical calculations for American call options agree with the findings presented in Damgaard (2000a). That is, in the market with transaction costs the premature exercise of an American call written on a non-dividend paying security may, under some circumstances, be optimal.

Damgaard (2000a) studied the difference between reservation purchase prices of an American call option and its European counterpart for different levels of initial wealth of a CRRA investor. He found that for low levels of wealth the American call option is of more value to the investor than the European call option, whereas for wealth above a certain level the investor perceives the European call option as being just as valuable as its American counterpart. Moreover, he found that the higher level of transaction costs is, the more the holder of an American call option is inclined to exercise the option before maturity. A lower wealth for a CRRA utility corresponds to a higher ARA. This suggests that it is the level of ARA, together with the level of transaction costs, that influence the option holder decision to exercise the option before maturity.

Our results of computing the reservation purchase prices of an American call option and of its European counterpart for different levels of absolute risk aversion are reported in Figure (10). From the Figure we observe that for high levels of absolute risk aversion the price of the American call option is higher than the price of its European counterpart, whereas for low levels of absolute risk aversion the prices of the two options coincide. Furthermore, we found that the discrepancy in the prices of the two options increases as the level of transaction costs increases. Again, as in the case of put options, the premature exercise can be explained in terms of hedging transaction costs: The presence of transaction costs makes hedging expensive. Moreover, the amount of hedging transaction costs increases when either the level of transaction costs increases or the option holder's risk aversion increases.

When the investor with a long position in an American call option wants to hedge away the risk of the option, he faces the tradeoff between the expected amount of hedging transaction costs and the loss in the expected wealth caused by premature exercise. When the expected amount of hedging transaction costs becomes greater than the expected gain from holding the option until maturity, it is optimal for the option holder to exercise. The analysis of the option holder's exercise policy reveals that an Ameri-

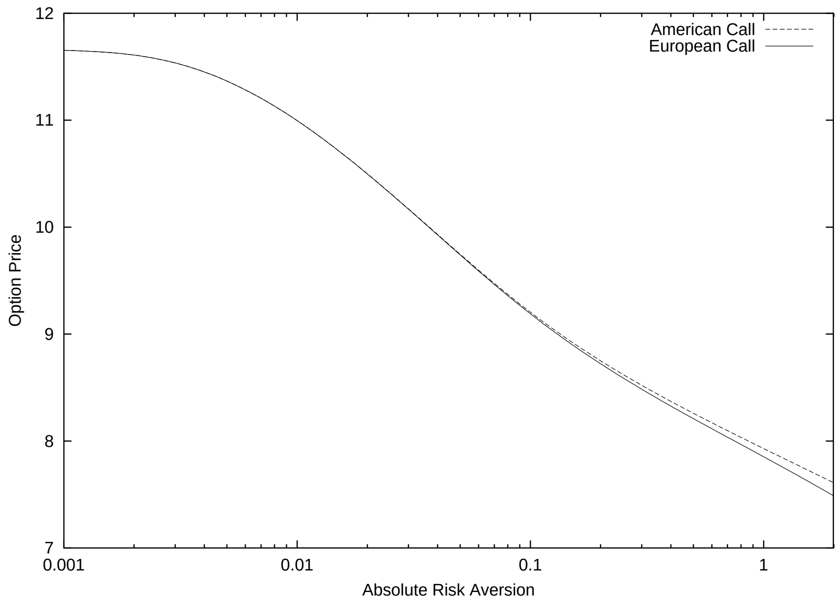

Figure 10: Reservation purchase prices of an American call option and its European counterpart as functions of the level of the buyer's absolute risk aversion.

can call option is exercised at a time close to maturity (see Figure (11)). The intuition here is in that, as time passes, a call option tends to become less valuable (the option theta is negative). Thus, when the discrepancy between the option value and the intrinsic option value becomes less than the expected amount of hedging costs, the option does not worth hedging. Moreover, the option buyer is the more inclined to exercise the option, the higher holdings in the stock he has as compared to the optimal hedging position in the stock in a friction-free market. The intuition behind this is as follows: Instead of selling some number of shares of the stock, in order to bring the hedging position in the stock in correspondence with the optimal amount (thus paying some transaction costs), it is better to exercise the option.

6 Conclusions and Extensions

In this paper we examined the problem of finding investors' reservation option prices and corresponding early exercise policies of American-style options in the market with proportional transaction costs. We formulated the

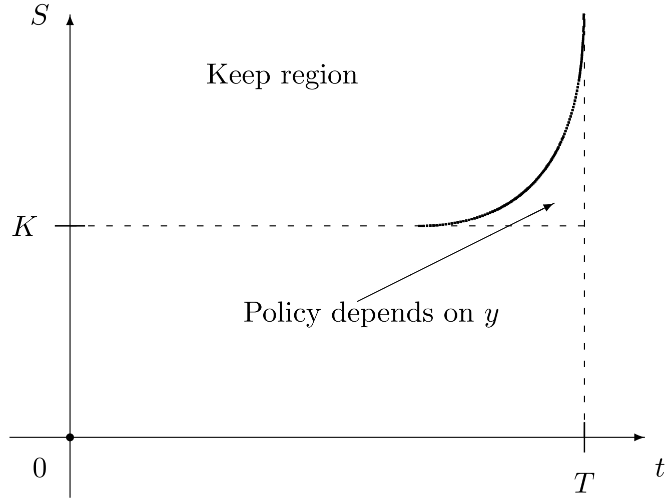

Figure 11: A schematic sketch of the early exercise policy for a buyer of American call option with strike K.

continuous time option pricing problem for the CARA investor with finite horizon. We considered both the buyer's and the writer's problems. Then we derived some important properties of the reservation prices of American-style options. We discussed the numerical algorithm and proposed a new reformulation of these problems in terms of quasi-variational HJB inequalities. Based on our formulation, we suggested original discretization schemes for computing reservation prices of American-style option. The discretization schemes were then implemented for computing reservation purchase and write prices of American-style put options and reservation purchase prices of American-style call options.

We examined the effects on the reservation option prices and the corresponding optimal exercise policies of varying the investor's ARA and the level of transaction costs. We found that in the market with transaction costs the holder of an American-style option exercises this option earlier as compared to the case with no transaction costs. This phenomenon concerns both put and call options written on a non-dividend paying stock. We carried out the detailed analysis of the early exercise policy and found that the higher level the transaction costs is, or the higher risk avers the option holder is, the earlier an American option is exercised.

It is known that in the presence of proportional transaction costs the investor's portfolio space, in the utility maximization problem without options, is divided into three disjoint regions, which can be specified as the Buy region, the Sell region, and the no-transaction (NT) region. The boundaries of the NT region are reflecting barriers, such that the investor refrains from transactions as long the portfolio lies inside the NT region. If a portfolio lies in the Buy or Sell region, then the investor performs the minimum transaction required to reach the closest NT boundary. Our numerical calculations showed that the same description of the optimal policy applies to the option writer, but the option buyer's optimal strategy is rather complicated: Generally, every region (Buy, Sell, and NT) consists of two sub-regions, and not all the boundaries of the NT sub-regions are reflecting barriers. When a non-reflecting barrier is hit, the investor performs the minimum transaction required to reach the closest boundary of the other NT sub-region. Moreover, in contrast to the case with no transaction costs where the early exercise boundary depends on the stock price and time, we found that the option holder exercise policy generally depends on his holdings in the stock.

As an important concluding remark, we shall now shortly discuss the robustness of the utility based option pricing approach to the input parameters of the model. It seems that for the investors with low level of (re-scaled for the number of options) absolute risk aversion the reservation option prices are very robust to all the model parameters. However, we should point out that the reservation option prices generally depend on the initial holdings in the stock (see (26) or (36)). For practical applications one usually assumes that the investor starts with zero holdings in the stock. This implies that the investor's initial stock inventory lies in the Buy region. In this situation a buyer of options will, for example, value a put option less[14] than that one who wishes to sell the stock. The opposite pattern will be obtained if the investor's initial stock inventory lies in the Sell region. That is, the sign of the bias in (73) will be the opposite (this phenomenon was closely studied in Monoyios (2003) for European-style options). Moreover, the reservation option prices depend on the measure of risk aversion for the investors with high level of absolute risk aversion. It should be emphasized, however, that one of the most attractive features of the utility based option pricing approach is that the hedging and early exercise strategies depend almost entirely on the option holder's measure of risk aversion. If we are

[14]Note that hedging of a long put requires buying some number of shares of the stock.

able to specify it, the risk management part of the problem will be solved.

There are several directions in which our work could be extended.

1. Nonexponential utilities. There is no issue of principle here, but only of increase of computational load, since the reduction from four to three dimensions is no longer available. However, as it was conjectured by Davis et al. (1993) and showed in Andersen and Damgaard (1999), the reservation option prices are approximately invariant to the specific form of the investor's utility function, and mainly only the level of absolute risk aversion plays an important role. As a result, it seems to be of a little practical interest to calculate the reservation option prices and optimal hedging and early exercise strategies using other utility functions besides the exponential one. These calculations will be very time consuming, and, moreover, the optimal hedging and early exercise strategies will be difficult to interpret because of their four-dimensional (t, x, y, S)-form.

3. Incomplete market. The utility based option pricing approach can be generalized to cover the case of incomplete market with transaction costs. In particular, this approach could be extended to include jumps in the price of the risky asset.

4. American options with fixed transaction costs. Clearly, we can extend our work to price American options in market with both fixed and proportional transaction costs. There is no issue of principle here, just the increased load of computations.

5. Optimal exercise of several American options. Recall our assumption: if the buyer of several American options chooses to exercise, he is required to exercise all of his θ options simultaneously. We can remove this assumption and solve the problem of optimal exercise of θ American options through time. Again, there is no issue of principle here, just the addition of one more dimension which will result in an increased computational load.

6. Asymptotic analysis for American options. It would be interesting to see if the asymptotic expansion methods pioneered by Whalley and Wilmott (1997) for European options could also be applied for the American options.

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