

Perspectives of Risk Sharing

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Abstract

In this paper we present an overview of the standard risk sharing model of insurance. We discuss and characterize a competitive equilibrium, Pareto optimality, and representative agent pricing, including its implications for insurance premiums. We only touch upon the existence problem of a competitive equilibrium, primarily by presenting several examples. Risk tolerance and aggregation is the subject of one section. Risk adjustment of the probability measure is one topic, as well as the insurance version of the capital asset pricing model.

The competitive paradigm may be a little demanding in practice, so we alternatively present a game theoretic view of risk sharing, where solutions end up in the core. Properly interpreted, this may give rise to a range of prices of each risk, often visualized in practice by an ask price and a bid price. The nice aspect of this is that these price ranges can be explained by "first principles", not relying on transaction costs or other frictions.

We end the paper by indicating the implications of our results for a pure stock market. In particular we find it advantageous to discuss the concepts of incomplete markets in this general setting, where it is possible to use results for closed, convex subspaces of an L^2 -space to discuss optimal risk allocation problems in incomplete financial markets.

KEYWORDS: Reinsurance Model, Equilibrium, Pareto Optimality, Core Solution, Stock Market, Complete Model

1 Introduction

This paper is primarily a review paper, where we present the standard risk sharing model of reinsurance markets. The model considered starts with a set of I agents, interpreted as (re)insurers, each endowed with a random payoff X_i for agent i , $i = 1, 2, \dots, I$. Supposing the agents can negotiate any affordable contracts among themselves, resulting in a final portfolio Y_i , one essential objective is to characterize these random variable Y_i most preferred by agent i . Other applications are manifold, since this model is indeed very general. For instance, X_i might represent randomly varying water endowments, could stand for a nation's quota in producing diverse pollutants, could be the initial endowment of

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shares in a stock market, e.t.c. This latter application we discuss in detail in the last section of the paper.

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The competitive paradigm may be a little demanding in practice, so we alternatively present a game theoretic view of risk sharing, where solutions end up in the core. Properly interpreted, this may give rise to a range of prices of each risk, often visualized in practice by an ask price and a bid price. The nice aspect of this is that these price ranges can be explained by “first principles”, not relying on transaction costs or other frictions.

We end the paper by indicating the implications of our results for a pure stock market. In particular we find it advantageous to discuss the concepts of incomplete markets in this general setting, where it is possible to use results for closed, convex subspaces of an L^2 -space to discuss optimal risk allocation problems in incomplete financial markets.

The paper is organized as follows: In section 1 we present the basic risk-exchange model, in section 2 we characterize a competitive equilibrium, in section 3 we characterize a Pareto optimum, in section 4 we introduce the representative agent, and in section 5 we discuss existence problems. Section 6 is devoted to risk tolerance and aggregation, section 7 to insurance premiums and section 8 to risk adjustments of the given probability measure. In section 9 we present the capital asset pricing model in insurance terms. Section 10 is a game theoretic approach to the risk allocation problem, and we end the paper in section 11, where the implications for a stock market of efficient allocation of risks is discussed.

2 The Basic Risk-Exchange Model

In this article we study the following model: Let $\mathcal{I} = \{1, 2, \dots, I\}$ be a group of I reinsurers, simply termed agents for the time being, having preferences \succeq_i over a suitable set of random variables, or gambles with realizations (outcomes) in some $A \subseteq R$. These preferences are represented by von Neumann-Morgenstern expected utility, meaning that there is a set of continuous utility indices $u_i : R \rightarrow R$, such that $X \succeq_i Y$ if and only if $Eu_i(X) \geq Eu_i(Y)$. We assume monotonic preferences, and risk aversion, so that, granted enough smoothness, we have $u'_i(w) > 0$, $u''_i(w) \leq 0$ for all w in the relevant domains.¹ Sometimes we shall also require strict risk aversion, meaning strict concavity for some u_i . Each agent is endowed with a random payoff X_i called his initial portfolio. More precisely, there exists a probability space (Ω, \mathcal{F}, P) such that i is entitled to payoff $X_i(\omega)$ when $\omega \in \Omega$ occurs. This means that uncertainty is objective and external. And there is no informational asymmetry. All parties agree upon (Ω, \mathcal{F}, P) as the probabilistic description of the stochastic environment, the

¹Note that the concepts of monotonicity and risk aversion make perfectly sense without assuming the existence of these derivatives.

latter being unaffected by their actions. It will be convenient to posit that both expected values and variances exist for all these initial portfolios, which means that all $X_i \in L^2(\Omega, \mathcal{F}, P)$, or just $X_i \in L^2$ for short.

We suppose the agents can negotiate any affordable contracts among themselves, resulting in a new set of random variables $Y_i, i \in \mathcal{I}$, representing the possible final payout to the different members of the group, or final portfolios. The transactions are carried out right away at “market prices”, where $\pi(Y)$ represents the market price for any $Y \in L^2$, i.e., it signifies the group’s valuation of the random variable Y relative to the other random variables in L^2 . The essential objective is then to determine:

(a) The market price $\pi(Y)$ of any “risk” $Y \in L^2$ from the set of preferences of the agents and the joint probability distribution $F(x_1, x_2, \dots, x_I)$ of the random vector $X = (X_1, X_2, \dots, X_I)$.

(b) For each i , the final portfolio Y_i most preferred by him among those satisfying his budget constraint $\pi(Y_i) \leq \pi(X_i)$.

Some observations are in order. First, observe that the possible events $\mathcal{F} = \mathcal{F}^X := \sigma(X_1, X_2, \dots, X_I)$ is the sigma-field generated by the initial random variables X , so that any random variable can be written in the form $Y = f(X_1, X_2, \dots, X_I)$ for f a suitable Borel-measurable function.² This means that the optimal final portfolios $Y_i = f_i(X_1, X_2, \dots, X_I)$ for some appropriate functions f_i . In order to avoid trivialities, we assume that \mathcal{F}^X is complete, i.e., augmented with all the P -null sets.

Second, unless the functional π on L^2 is *linear*, arbitrage would be possible. To see this, consider the case where e.g., $\pi(Z + Y) > \pi(Z) + \pi(Y)$ for any two random variables Z and Y in L^2 . Since we assume infinite divisibility of any portfolio, a reinsurer could insure the bundle $(Z + Y)$, and then reinsure Z and Y separately. The cash flows from these trades would be

$$\pi(Z + Y) - (\pi(Z) + \pi(Y)) > 0$$

at time 0, and $-(Z + Y)(\omega) + Z(\omega) + Y(\omega) = 0$ at time 1 for any $\omega \in \Omega$. Thus the reinsurer has made a risk-free profit whatever the state of nature, which should not be possible in any consistent model of this market. Thus it must be the case that π is linear, i.e., it satisfies

$$\pi(aZ + bY) = a\pi(Z) + b\pi(Y)$$

for any constants $a, b \in R$ and random variables $Z, Y \in L^2$.

Third, the pricing functional π should be *positive*, meaning simply that $\pi(Z) \geq 0$ for any $Z \geq 0$ P-a.s. In other words, a random variable that is non-negative with probability 1, should have a non-negative market price.

From functional analysis it is known that a positive, linear functional on an L^p -space is bounded ($1 \leq p < \infty$), and hence also continuous, in which case we can use the Riesz representation theorem and conclude that there exists a unique random variable $\xi \in L^2$ such that

$$\pi(Z) = E(Z\xi) \quad \text{for all } Z \in L^2.$$

This random variable, the Riesz representation, we shall sometimes refer to as the *state-price deflator*. At the moment we can only conclude that there exists

²This is a result that is known from measure theory, e.g., Tucker (1967), Theorem 1.1.

a Borel-measurable function f such that $\xi = f(X_1, X_2, \dots, X_I)$ holds for the Riesz representation ξ . Our aim is now to characterize this particular f , and also the f_i -functions corresponding to the optimal $Y_i, i \in \mathcal{I}$. The following notational convention will be used: If X and Y are two random variables, then by $X \leq Y$ we mean that $(Y - X) \geq 0$ P-a.s., i.e., the random variable $(Y - X)$ is non-negative almost surely.

Definition 1 *An allocation $Z = (Z_1, Z_2, \dots, Z_I)$ is called feasible if*

$$\sum_{i=1}^I Z_i \leq \sum_{i=1}^I X_i := X_M.$$

The problem each agent is supposed to solve is the following:

$$\sup_{Z_i \in L^2} Eu_i(Z_i) \quad \text{subject to} \quad \pi(Z_i) \leq \pi(X_i). \quad (1)$$

An important issue is, of course, existence (and uniqueness) of solutions to (1). We shall not elaborate on this here, suffice it is to note the following: If

$$\{Z_i \in L^2 : Eu_i(Z_i) < \infty, \quad \pi(Z_i) \leq \pi(X_i)\}$$

is bounded (in L^2 -norm), then existence is guaranteed.³ Also, a strictly concave u_i suffices for uniqueness.

Definition 2 *A competitive equilibrium is a collection $(\pi; Y_1, Y_2, \dots, Y_I)$ consisting of a price functional π and a feasible allocation $Y = (Y_1, Y_2, \dots, Y_I)$ such that for each i , Y_i solves the problem (1) and markets clear; $\sum_{i=1}^I Y_i = \sum_{i=1}^I X_i$.*

We close the system by assuming *rational expectations*. This means that the market clearing prices π implied by agent behavior is assumed to be the same as the price functional π on which agent decisions are based. The main analytic issue is then the determination of equilibrium price behavior.

In the microeconomic literature there are colorful descriptions of how such an equilibrium might result, involving e.g., the *Walrasian auctioneer*, in the case of no uncertainty. In the reinsurance market it is perhaps more realistic to think of bilateral trades between reinsurers.

We notice that the concept of Walrasian equilibrium is widely employed in consumer theory, although the analysis can be hard and the conclusions require consumers who are extraordinarily sophisticated. There is, however, a lot of experimental evidence, where a number of researchers have attempted to see whether markets perform under controlled conditions in the way economists assume they do. The results obtained are usually striking in their support of Walrasian equilibrium.

When an insurer is invited to cover a large risk, he may decide that he cannot, or does not want to do so entirely. He may rather cover merely part of the risk, say a fraction, against the corresponding part of the premium. This

³By i.a., the Banach-Alaogher Theorem

⁴Market clearing is usually defined by $\sum_{i=1}^I Y_i \leq \sum_{i=1}^I X_i$. Since we have strictly monotonic preferences, equality will result in equilibrium.

leaves the insurer to seek other insurers in the market who are willing to accept the rest of the risk. From the 1680's he knew that he could find these other insurers at the coffee house of Edward Lloyd in London.

Lloyd's of London still operates in this way. To buy insurance at Lloyd's one has to contact a broker who is accredited at Lloyd's. The broker takes a "slip", which contains all relevant information about the risk, to one or more underwriters who specializes in risks of this type. The underwriter who offers the best terms, will set a rate and accept to cover a certain part of the risk. The broker will next contact other underwriters until the slip is filled. Usually these underwriters will follow the rate set by the "leading underwriter", but that may not be the case.

The procedure described above may seem cumbersome, and it can be costly. It serves, however, to illustrate how the competitive equilibrium (CE) of Definition 2 may result, or be well approximated, in practice for a reinsurance market. One is lead to believe that the notion of a CE may be especially fruitful for this type of markets, and gives reasonable predictions of what prices "ought" to be.

Finally let us comment on the assumption of homogeneous beliefs. This assumption seems reasonable for a reinsurance market, where trade is traditionally supposed to take place under the conditions of *umberrimae fidei*, and no information is supposed to be hidden.

Premiums of risks in reinsurance markets are likely to influence premiums in the direct market for insurance, where this assumption seems less realistic. The cause for this may be that the different agents have different information about the risks. It seems likely that the buyers of insurance possess more information about the risk that they try to get rid of, than the insurers. This potential asymmetric information gives rise to the *selection problem* or *adverse selection*. In addition, the buyers may often directly, or indirectly be able to influence events so that the probability distributions of the insured risks are altered. This may happen because the insurer is usually unable to perfectly monitor all the actions of the insured, a phenomenon giving rise to *moral hazard*.

Whereas the problem of moral hazard does not seem of particular importance in a reinsurance market, the problem of adverse selection may occur since a ceding company usually has more detailed information about the risks it has underwritten, and subsequently tries to get rid of in the reinsurance market, than the reinsurers. It may of course be tempting for a direct insurer to get rid of some "bad risks". For this reason the reinsurance industry makes use of a detailed rating system for insurance companies, through e.g., Insurance Solvency International, which may penalize such actions. If an insurer gets a bad reputation, he may get a low classification by such rating agencies, implying that he will face tougher conditions in the reinsurance market, like higher premiums. The very existence of such rating companies is an indication of the severity of the selection problem. In any case, we shall abstract from both these problem areas.

The above model is formulated in terms of a reinsurance syndicate, but other applications are manifold, since the model is indeed very general. For instance,

- X_i might be the randomly varying water endowments of agricultural region (or hydro-electric power station) i ;
- X_i could stand for nation i 's state-dependent quotas in producing diverse pollutants (or in catching various fish species);

- X_i could account for uncertain quantities of different goods that transportation firm i must bring from various origins to specified destinations;
- X_i could be the initial endowments of shares in a stock market, in units of a consumption good.

This latter application we will return to in some detail later. For instance, the present formulation allows us to emphasize and study the concept of *complete financial markets*, and the economic value, or rather the rationale behind *contingent claims*, such as e.g., options and futures contracts.

3 The characterization of a competitive equilibrium

In this section we characterize a CE assuming that it exists. In the literature cited at the end the reader will find several references to the existence issue.⁵ We take it that the initial portfolios are not identically equal to zero, and that a unique equilibrium exists. We also assume quite naturally that $\pi(X_i) > 0$ for each i . In fact, it seems reasonable that each agent is required to bring to the market an initial “endowment” of positive value.⁶ In this case we have the following:

Theorem 1 *Suppose the preferences of the agents are monotonic, i.e., $u'_i > 0$ for all $i \in \mathcal{I}$. The equilibrium is then characterized by the existence of positive constants α_i , $i \in \mathcal{I}$, such that for the equilibrium allocation (Y_1, Y_2, \dots, Y_I)*

$$u'_i(Y_i) = \alpha_i \xi, \quad \text{a.s.} \quad \text{for all} \quad i \in \mathcal{I}, \quad (2)$$

where ξ is the Riesz representation of the pricing functional π .

Proof Recall that $\max_{Z_i} Eu_i(Z_i)$ s.t. $h(Z_i) \leq 0$, where $h(Z_i) := \pi(Z_i) - \pi(X_i)$, is a nice optimization problem: The objective is concave and the constraint function h (the feasible set) is convex. For such problems the *Kuhn-Tucker Theorem* says that, granted a suitable constraint qualification, any optimal solution Y_i will be supported by a Lagrange multiplier α_i : That is, there exists $\alpha_i \geq 0$ such that the Lagrangian

$$\mathcal{L}_i(Z_i; \alpha_i) = Eu_i(Z_i) - \alpha_i h(Z_i)$$

is maximal in Z_i at $Z_i = Y_i$. Moreover, complementary slackness holds: $\alpha_i h(Y_i) = 0$. The said qualification could be $h(Z_i^0) < 0$ for some Z_i^0 . (This is the so called Slater condition.) Here let $Z_i^0 = \frac{1}{2}X_i$.

Next we explore what maximality of $\mathcal{L}_i(\cdot, \alpha_i)$ at Y_i means. For that purpose define a variation $\tilde{Y}_i := Y_i + tZ$ where Y_i is the optimal solution of (1), $t \in R$ is a scalar dummy variable and $Z \in L^2$ is an arbitrary random variable. According to our conditions the function $f(t, Z) := \mathcal{L}_i(\tilde{Y}_i; \alpha_i)$ attains its maximum for $t = 0$ for all $Z \in L^2$, and consequently must

$$f'(0, Z) = E\{Z(u'_i(Y_i) - \alpha_i \xi)\} = 0 \quad \text{for all} \quad Z \in L^2, \quad (3)$$

⁵Existence of Arrow-Debreu equilibria in infinite-dimensional settings seems to have been first treated in Bewley (1972).

⁶This is of course a weaker requirement than the positivity assumption $X_i \geq 0$ P-a.s. for all i found in consumer theory.

which implies that $u'_i(Y_i) - \alpha_i \xi = 0$ a.s.

Finally, since $u'_i > 0$ for all i , the shadow price $\xi > 0$ a.s., otherwise the problem (1) can not have a solution, contrary to our assumption that an equilibrium exists. From the first order condition (2) it then follows that $\alpha_i > 0$ of all i . \square

Notice that in an equilibrium of the above type only relative prices are determined. We get

$$\frac{u'_i(Y_i(\omega))}{u'_i(Y_i(\omega'))} = \frac{\xi(\omega)}{\xi(\omega')} \quad \text{for almost all } \omega, \omega' \in \Omega.$$

Thus the rate of substitution between states of nature is constant across the agents.

Consider an equilibrium where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_I)$ are the associated positive constants. Then the same equilibrium is obtained for the ray $\hat{\alpha} = c\alpha$ for $c > 0$ a positive scalar. In the latter case all the prices are obtained from the former after multiplication by the constant $1/c$. Thus the equilibrium allocation (Y_1, Y_2, \dots, Y_I) remains invariant to multiplication of the ray α by a normalizing constant c . In general prices are determined by a unique equilibrium only modulo a normalization.

One should perhaps not loose touch with the situation of the more familiar Euclidean space. If the set of states of the world Ω is finite, we are basically back in finite dimensional Euclidean space, if we take proper care of the state probabilities. The result of this theorem is then analogous to the geometrical interpretation that the state-price vector is a suitable normalized positive vector orthogonal to the budget sets of the agents. More precisely, for almost every state the realized marginal utility is (ex-post) perpendicular to the realized budget set. There is an important point here. The utility maximization within budget yields an *expected* marginal utility $Eu'_i(Y_i)$ which is normal to the expected budget set, i.e., $Eu'_i(Y_i) = \alpha_i \pi(1)$ for some $\alpha_i \geq 0$. The more interesting fact is that this condition disintegrates to have $u'_i(Y_i) = \alpha_i \xi$ almost surely.

One should observe that several main features of the geometrical interpretations from Euclidean spaces carry over to the present *Hilbert space* L^2 . For example, the argument leading to equation (3) is very simple and intuitive, and can of course be more formally explained in terms of *directional derivatives*: We define

$$\nabla \mathcal{L}_i(Y_i, Z) = \lim_{t \downarrow 0^+} \frac{\mathcal{L}_i(Y_i + tZ; \alpha_i) - \mathcal{L}_i(Y_i; \alpha_i)}{t},$$

where $\nabla \mathcal{L}_i(Y_i, Z)$ is called the directional derivative of $\mathcal{L}_i(Y_i; \alpha_i)$ in the direction Z . \mathcal{L}_i is differentiable at Y_i means that $\nabla \mathcal{L}_i(Y_i, Z)$ exists for all $Z \in L^2$, and the functional $Z \rightarrow \nabla \mathcal{L}_i(Y_i, Z)$ is linear. This functional, the gradient of \mathcal{L}_i at Y_i , is denoted by $\nabla \mathcal{L}_i(Y_i)$. It can here be shown to be given by

$$(\nabla \mathcal{L}_i(Y_i))(Z) = E\{(u'_i(Y_i) - \alpha_i \xi)Z\}. \quad (4)$$

A necessary condition for a maximum of \mathcal{L}_i at Y_i is that the linear functional in equation (4) is zero in all directions Z , which leads directly to the condition (2).

We now present an example:

Example 1. Consider the case with negative exponential utility functions, with marginal utilities $u'_i(z) = e^{-z/a_i}, i \in \mathcal{I}$, where a_i^{-1} is the *absolute risk aversion* of agent i , or a_i is the corresponding *risk tolerance*. Using the characterization (2), we get

$$\lambda_i e^{-Y_i/a_i} = \xi, \quad a.s., \quad \text{where} \quad \lambda_i = a_i^{-1}, \quad i \in \mathcal{I}.$$

After taking logarithms in this relation, and summing over i , market clearing implies

$$\xi = e^{(K-X_M)/A}, \quad a.s. \quad \text{where} \quad K := \sum_{i=1}^I a_i \ln \lambda_i, \quad A := \sum_{i=1}^I a_i.$$

Furthermore, from the same first order conditions we also get that the optimal portfolios can be written

$$Y_i = \frac{a_i}{A} X_M + b_i, \quad \text{where} \quad b_i = a_i \ln \lambda_i - a_i \frac{K}{A}, \quad i \in \mathcal{I}.$$

Thus the reinsurance contracts involve optimal sharing rules which are *affine* in X_M . Contracts of this type belong to the class of *proportional* reinsurance. The constants of proportionality a_i/A are simply equal to each agent's risk tolerance, measured relative to the market. In order to compensate for the fact that the least risk-averse reinsurer will hold the larger proportion of the market, zero-sum side payments occur between the reinsurers, here represented by the terms b_i . Without these side payments an agent, with a "small" initial endowment but with a large risk tolerance, would end up with a "large" final endowment, but this could not possibly be consistent with his budget constraint. This kind of treaty seems common in reinsurance practice.

In order to determine the ray $\lambda = (\lambda_1, \dots, \lambda_I)$, we employ the budget constraints:

$$E(Y_i e^{(K-X_M)/A}) = E(X_i e^{(K-X_M)/A}), \quad i \in \mathcal{I},$$

which give that

$$b_i = \frac{E\{X_i e^{-X_M/A} - \frac{a_i}{A} X_M e^{-X_M/A}\}}{E\{e^{-X_M/A}\}}, \quad i \in \mathcal{I}.$$

Hence the optimal sharing rules Y_i are completely determined in terms of the given primitives of the model. Now the ray λ can also be determined modulo a normalization. Letting $K = \sum_{i=1}^I a_i \ln \lambda_i$ denote this normalization, then

$$\lambda_i = e^{b_i/a_i} e^{K/A}, \quad i \in \mathcal{I}.$$

If we impose the normalization $E\{\xi\} = 1$ of the state price deflator, we obtain $e^{-K/A} = E\{e^{-X_M/A}\}$, in which case the constants λ are given by

$$\lambda_i = \frac{e^{b_i/a_i}}{E\{e^{-X_M/A}\}}, \quad i \in \mathcal{I}.$$

Through this example we discovered a “pricing principle”, since market prices are now given by

$$\pi(Z) = \frac{E\{Z \cdot e^{-X_M/A}\}}{E\{e^{-X_M/A}\}}, \quad \text{for any } Z \in L^2. \quad (5)$$

Prices given by an expression like (5) are sometimes referred to as the “Esscher principle” in actuarial mathematics, but then with the important distinction that the aggregate market index X_M in (5) is substituted by the risk Z itself. For this latter principle the price rule is of course no longer a linear functional, which then can, unfortunately, lead to arbitrage possibilities and other anomalies. \square

In the above example we were able to completely specify the equilibrium, given that the relevant expectations are well defined. We may safely conjecture that if the side payments b_i 's can be computed, an equilibrium exists and is unique.

We notice that both the optimal, final portfolios Y_i and the state-price deflator ξ depend upon the initial portfolios X_i only through the aggregate $X_M = \sum_{i=1}^I X_i$. In other words, $\xi = f(X_1, X_2, \dots, X_I) = g(X_M)$, and $Y_i = f_i(X_1, X_2, \dots, X_I) = g_i(X_M)$ for some functions g and g_i . One may wonder how general this is. In this particular example g turned out to be smooth, and the g_i -functions are even linear. We know that *non-proportional* reinsurance is another of the main classes of contracts prevailing in real reinsurance markets, so the linearity of the contracts may not be all that general.

Before we investigate these matters any further, we introduce the concept of (strong) *Pareto optimality* of an allocation.

Definition 3 *A feasible allocation $Y = (Y_1, Y_2, \dots, Y_I)$ is called Pareto optimal if there is no feasible allocation $Z = (Z_1, Z_2, \dots, Z_I)$ with $Eu_i(Z_i) \geq Eu_i(Y_i)$ for all i and with $Eu_j(Z_j) > Eu_j(Y_j)$ for some j .*

A famous neoclassical result is that any competitive equilibrium is Pareto optimal, sometimes also termed *efficient*. Not surprisingly, the same result obtains here:

Theorem 2 *Suppose (Y_1, Y_2, \dots, Y_I) is a competitive equilibrium allocation. Then it is Pareto optimal.*

Proof. Let $(\xi; Y_1, Y_2, \dots, Y_I)$ denote the equilibrium, and suppose that Z is a Pareto dominating allocation. Since $Eu_j(Z_j) > Eu_j(Y_j)$ for some j , it must be the case that $\pi(Z_j) > \pi(Y_j)$ for these j . Consider the other i where we only have equality in expected utilities. It must be the case that $\pi(Z_i) \geq \pi(Y_i)$ also for these i . Suppose the opposite. Then by *local insatiability* (and a fortiori by strict monotonicity) any such agent i would be able to achieve a larger expected utility than $Eu_i(Y_i)$ by using all the available budget $\pi(Y_i)$, implying that the resulting expected utility would be strictly larger than $Eu_i(Y_i)$, and the corresponding allocation meets the budget constraint, a contradiction to the optimality of Y_i . Accordingly we have that $\pi(Z_i) \geq \pi(Y_i)$ for all i , and with strict inequality for some j . But then

$$\pi\left(\sum_{i=1}^I X_i\right) \geq \pi\left(\sum_{i=1}^I Z_i\right) = \sum_{i=1}^I \pi(Z_i) > \sum_{i=1}^I \pi(Y_i) = \pi\left(\sum_{i=1}^I Y_i\right) = \pi\left(\sum_{i=1}^I X_i\right),$$

a clear contradiction. The first inequality in the above string follows since Z is feasible and π is positive and linear, the strict inequality follows from what has just been demonstrated, and the last equality follows since Y clears the market. Hence Y must be Pareto optimal. \square

In consumption theory the preceding theorem is known as *First Welfare Theorem*.

4 The characterization of a Pareto optimum

A consequence of the last theorem is that Pareto optima are also characterized by the equations (2), at least those allocations that are also equilibria. It turns out that this include most Pareto optima. Before we show this, we turn to another useful characterization of Pareto optimum. Here we shall employ a version of one of the most useful mathematical tools in microeconomics, *The Separating Hyperplane Theorem*: Suppose X and Y are convex, disjoint subsets of R^I . Then there exists a non-trivial linear functional π on R^I such that $\pi(x) = \sum_{i=1}^I \lambda_i x_i \leq \sum_{i=1}^I \lambda_i y_i = \pi(y)$ for all $x \in X$ and $y \in Y$. Moreover, if $x \in \text{int}(X)$ or $y \in \text{int}(Y)$ then $\pi(x) < \pi(y)$. In the following we assume that all the portfolios $Z \geq c$, where c is some constant. In a one-period model, if we interpret the portfolio of an agent as “wealth”, it may sometimes be difficult to give any meaning to negative wealth, which then necessitates an assumption of this kind where $c = 0$. We now show the following.

Theorem 3 *Suppose u_i are concave and increasing for all i . Then Y is a Pareto optimal allocation if and only if there exists a nonzero vector of agent weights $\lambda \in R_+^I$ such that $Y = (Y_1, Y_2, \dots, Y_I)$ solves the problem*

$$\sup_{(Z_1, \dots, Z_I)} \sum_{i=1}^I \lambda_i E u_i(Z_i) \quad \text{subject to} \quad \sum_{i=1}^I Z_i \leq \sum_{i=1}^I Y_i = X_M. \quad (6)$$

Proof. First, assume (Y_1, Y_2, \dots, Y_I) is Pareto optimal, and define two sets A and B in R^I as follows:

$$A := \{a \in R^I : a_i \leq E u_i(Z_i) - E u_i(Y_i), i \in \mathcal{I}, Z \in \mathcal{Z}\}$$

where \mathcal{Z} denotes the set of feasible allocations $Z = (Z_1, \dots, Z_I)$ such that $\sum Z_i \leq X_M$, and $B := \{b \in R_+^I : b \neq 0\}$. Then the set A is convex, since all the u_i are assumed concave, and $A \cap B = \emptyset$, since Y is Pareto optimal. Thus we know that there exists a separating hyperplane, i.e., there exists a vector $\lambda \in R^I$, $\lambda \neq 0$ such that $\lambda \cdot a \leq \lambda \cdot b \forall a \in A$ and $b \in B$. Given the nature of B , λ can not have a negative coordinate, hence $\lambda \geq 0$. Since $0 \in \text{cl}(B)$ we have that $\lambda a \leq 0 \forall a \in A$, thus

$$\sum_{i=1}^I \lambda_i E u_i(Y_i) \geq \sum_{i=1}^I \lambda_i E u_i(Z_i), \quad \forall Z \in \mathcal{Z},$$

which is the conclusion.

The other direction is easy to show. \square

The fact that some of the weights λ_i may be zero in the characterization of Theorem 3 may be illustrated as follows: Imagine sharing a cake between I

persons having increasing utilities across the whole cake. Then *any* split of the cake is in fact Pareto optimal, including the “sharing” giving the whole cake to one single person. This corresponds to only the weight of this person being positive, all the other weights being zero. In this case the concept of Pareto optimality is void, but that is not the typical case with multiple goods and/or many states of the world.

5 Representative agent pricing

In this section we introduce the representative agent, and demonstrate what implications he has for the pricing of insurance contracts, as well as for how the optimal contracts are obtained.

We have already briefly met this agent in equation (6) of Theorem 3: Consider for each nonzero vector $\lambda \in R_+^I$ of agent weights the function $u_\lambda(\cdot) : R \rightarrow R$ defined by

$$u_\lambda(v) := \sup_{(z_1, \dots, z_I)} \sum_{i=1}^I \lambda_i u_i(z_i) \quad \text{subject to} \quad \sum_{i=1}^I z_i \leq v. \quad (7)$$

As the notation indicates, this function depends only on the variable v , meaning that if the supremum is attained at the point (y_1, \dots, y_I) , all these $y_i = y_i(v)$ and $u_\lambda(v) = \sum_{i=1}^I \lambda_i u_i(y_i(v))$. It is a consequence of the *Implicit Function Theorem* that under our assumptions, the function $u_\lambda(\cdot)$ is two times differentiable in v . In particular it follows that $u'_\lambda(v) = \sum_{i=1}^I \lambda_i u'_i(y_i(v)) y'_i(v)$, and hence that all the functions $y_i(v)$ are also differentiable in v . More importantly in the present situation, we want to show that for appropriate λ the function $u'_\lambda(v) = g(v) = \xi(v)$, i.e., there is a direct connection to the state-price deflator.

Accordingly we are interested in the problem

$$Eu_\lambda(V) := \sup_{(Z_1, \dots, Z_I)} \sum_{i=1}^I \lambda_i Eu_i(Z_i) \quad \text{subject to} \quad \sum_{i=1}^I Z_i \leq V. \quad (8)$$

where $Z_i \in L^2$ for all i .

Theorem 4 *Assume $u'_i > 0, u''_i \leq 0$ for all i , and suppose $(\pi; Y_1, Y_2, \dots, Y_I)$ is a competitive equilibrium. Then*

(i) *There exists a nonzero vector of agent weights $\lambda = (\lambda_1, \dots, \lambda_I)$, $\lambda_i \geq 0$ for all i , such that the equilibrium allocation (Y_1, Y_2, \dots, Y_I) solves the allocation problem (8) at $V = X_M = \sum_{i=1}^I X_i$ in which case $Eu_\lambda(X_M) = \sum_{i=1}^I \lambda_i Eu_i(Y_i)$.*

(ii) *There exists a nonzero vector of agent weights $\lambda = (\lambda_1, \dots, \lambda_I)$, where $\lambda_i \geq 0$ for all i , such that $(\pi; X_M)$ is an equilibrium in the single-agent economy $(u_\lambda; X_M)$. The linear pricing functional π is then given by*

$$\pi(Z) = E(u'_\lambda(X_M) \cdot Z) \quad \forall Z \in L^2,$$

that is $u'_\lambda(X_M) = \xi$ a.s.

Remarks: 1) The equilibrium in the single agent economy must be understood as a consistency requirement, since “the representative agent” has no one to trade with.

2) The importance of the single agent theory in our setting is that this construction enables us to find the prices in the original economy, since π is the same in these two economies. The convenience of accommodating a representative agent is related to the fact that an equilibrium problem thus reduces to an optimization problem.

3) We now see that the Riesz representation, the state price deflator, or the shadow price $\xi = u'_\lambda(X_M)$, so $\xi = f(X_1, X_2, \dots, X_I) = g(X_M)$ is true in general, for $X_M = \sum_{i=1}^I X_i$, and the function $g(x) = u'_\lambda(x)$, $x \in R$, is determined from (7).

4) Given the probability distribution function F , the optimal equilibrium allocations Y_i depend on the initial portfolios (X_1, X_2, \dots, X_I) only through the aggregate $X_M = \sum_{i=1}^I X_i$ as well, or $Y_i = f_i(X_1, X_2, \dots, X_I) = g_i(X_M)$ is true in general, since $Y_i = (u'_i)^{-1}(\alpha_i \xi)$ follows directly from the characterization in Theorem 1, and ξ depends only on the aggregate risk X_M as just noticed.⁷

Proof of Theorem 4 It follows from Theorem 1 that there exist Lagrange multipliers $\alpha_i > 0$ such that Y_i solves the problem

$$\sup_{Z \in L^2} E\{u_i(Z) - \alpha_i \xi(Z - X_i)\}, \quad (9)$$

and the budget conditions thus hold with equality, i.e., $E(\xi Y_i) = E(\xi X_i)$, $\forall i$. Now choose $\lambda_i = \frac{1}{\alpha_i}$, $\forall i$. For any feasible (Z_1, \dots, Z_I) we then have

$$\begin{aligned} \sum_{i=1}^I \lambda_i E u_i(Y_i) &= \sum_{i=1}^I \lambda_i (E\{u_i(Y_i) - \alpha_i \xi(Y_i - X_i)\}) \geq \\ \sum_{i=1}^I \lambda_i (E\{u_i(Z_i) - \alpha_i \xi(Z_i - X_i)\}) &= \sum_{i=1}^I \lambda_i E u_i(Z_i) - \sum_{i=1}^I E\{\xi(Z_i - X_i)\} \geq \\ &\sum_{i=1}^I \lambda_i E u_i(Z_i). \end{aligned}$$

The first inequality follows from (9), and the second follows from the feasibility of (Z_1, \dots, Z_I) and the positivity of ξ a.s. Thus we have found a set of strictly positive agent weights λ_i such that (Y_1, \dots, Y_I) solves allocation problem (8) at $V = X_M = \sum_{i=1}^I X_i$.

Next, in order to prove (ii) we must show that no trade is optimal in the single agent economy, where the agent has utility index $u_\lambda(\cdot)$ and initial portfolio X_M . If this were not the case, there would $\exists Z_M \neq X_M$ such that

$$E u_\lambda(Z_M) > E u_\lambda(X_M) \quad \text{and} \quad E(\xi Z_M) \leq E(\xi X_M).$$

From the definition of u_λ , this would imply the existence of an allocation (Z_1, Z_2, \dots, Z_I) with $\sum Z_i \leq Z_M$ such that

$$\sum_{i=1}^I \lambda_i E u_i(Z_i) > \sum_{i=1}^I \lambda_i E u_i(Y_i)$$

⁷The function $(u'_i)^{-1}(x)$ denotes the inverse function of $u'_i(x)$, which exists for all i according to our assumptions.

and

$$\sum_{i=1}^I \lambda_i \alpha_i E(\xi Z_i) = E(\xi \sum Z_i) \leq E(\xi Z_M) \leq E(\xi X_M) = \sum_{i=1}^I \lambda_i \alpha_i E(\xi X_i).$$

Putting these two inequalities together we get

$$\sum_{i=1}^I \lambda_i [Eu_i(Z_i) - \alpha_i E\{\xi(Z_i - X_i)\}] > \sum_{i=1}^I \lambda_i [Eu_i(Y_i) - \alpha_i E\{\xi(Y_i - X_i)\}],$$

which contradicts the fact that (Y_i) solves the problem (9).

It remains to show that $\xi = u'_\lambda(X_M)$. From (ii) we know that X_M is the solution of the problem

$$\sup_{Z \in L^2} Eu_\lambda(Z) \quad \text{subject to} \quad \pi(Z) \leq \pi(X_M),$$

where the Lagrangian is given by

$$\mathcal{L}(Z; \alpha) = Eu_\lambda(Z) - \alpha(E(\xi Z) - E(\xi X_M)).$$

By the Kuhn-Tucker Theorem a necessary (and sufficient) condition for optimality of X_M is given by the first order condition

$$u'_\lambda(X_M) = \alpha \xi, \quad a.s.,$$

which now follows precisely as in the proof of Theorem 1. Notice that $u'_\lambda(X_M) > 0$ a.s. follows from strict monotonic preferences of all the reinsurers, and $\xi > 0$ a.s. must hold since the present optimization problem is known to have a solution. Hence $\alpha > 0$, $\xi = \frac{1}{\alpha} u'_\lambda(X_M)$ a.s., and by a renormalization we now have that $u'_\lambda(X_M) = \xi$ \square

A consequence of Remark 4) above is that the reinsurers can hand in all their initial portfolios X_i to a pool, and after $\omega \in \Omega$ is realized, let the pool's clerk distribute parts of the total $X_M(\omega)$ back to the syndicates members according to the optimal sharing rules $Y_i(\omega) = g_i(X_M(\omega))$. In this respect the competitive solution contains, perhaps surprisingly, an element of *cooperation*, i.e., that of pooling.

6 The existence of optimal allocations

In this section we provide conditions for the existence of a Pareto optimum, and we also briefly study the existence of a competitive equilibrium.

The extent to which a Pareto optimal allocation can also be considered as a competitive equilibrium is the contents of our first theorem. In the theory of consumption it is known as *The Second Welfare Theorem*:

Theorem 5 *Under the assumptions of Theorem 3, let (Y_1, Y_2, \dots, Y_I) be a Pareto optimal allocation. Then there exists a re-allocation $(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_I)$, satisfying $\sum \tilde{X}_i = \sum Y_i = X_M$, such that Y_i solves*

$$\sup_{Z_i \in L^2} Eu_i(Z_i) \quad \text{subject to} \quad E(Z_i u'_\lambda(X_M)) \leq E(\tilde{X}_i u'_\lambda(X_M)) \quad (10)$$

for all i , where the function u'_λ is defined through (8) with $V = X_M$ a.s., and the nonzero weights λ follow from the characterization in Theorem 3.

Proof. Under our assumptions we know that there exists a nonzero vector $\lambda \in R_+^I$ of agent weights such that

$$Eu_\lambda(X_M) = \sum_{i=1}^I \lambda_i Eu_i(Y_i). \quad (11)$$

(These weights are used to define the function u'_λ in (10).) We have to show that (Y_1, Y_2, \dots, Y_I) satisfies (10). Suppose the opposite, i.e., for each feasible $(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_I)$, $\sum \tilde{X}_i = X_M$, there exists a feasible allocation (Z_1, Z_2, \dots, Z_I) , satisfying the budget constraints in (10), and which is not equal to (Y_1, Y_2, \dots, Y_I) a.s., such that for all i

$$Eu_i(Z_i) - \alpha_i E\{u_\lambda(X_M)(Z_i - \tilde{X}_i)\} \geq Eu_i(Y_i) - \alpha_i E\{u_\lambda(X_M)(Y_i - \tilde{X}_i)\}, \quad (12)$$

for all α_i , where the inequality is strict for at least some j . In particular these inequalities hold for $\alpha_i = 1/\lambda_i$ (we may use the usual convention that $\infty \cdot 0 = 0$).⁸ Now we have that

$$\begin{aligned} \sum_{i=1}^I \lambda_i \alpha_i E(u'_\lambda(X_M)Z_i) &= E(u'_\lambda(X) \sum_{i=1}^I Z_i) \leq \\ E(u'_\lambda(X_M) \sum_{i=1}^I \tilde{X}_i) &= \sum_{i=1}^I \lambda_i \alpha_i E(u'_\lambda(X_M)\tilde{X}_i) \end{aligned} \quad (13)$$

Further, for the same λ -vector we have

$$\begin{aligned} \sum_{i=1}^I \lambda_i Eu_i(Z_i) &= \sum_{i=1}^I \lambda_i \left[E(u_i(Z_i) - \alpha_i E\{u'_\lambda(X_M)(Z_i - \tilde{X}_i)\}) \right] > \\ \sum_{i=1}^I \lambda_i \left[E(u_i(Y_i) - \alpha_i E\{u'_\lambda(X_M)(Y_i - \tilde{X}_i)\}) \right] &= \sum_{i=1}^I \lambda_i Eu_i(Y_i) \end{aligned}$$

for all market clearing \tilde{X} -allocations. The first equality follows since both the Z - and \tilde{X} - allocations are feasible with equality, the inequality follows from the two inequalities (12) and (13) put together, and the last equality follows since the Y -allocation is feasible with equality, i.e., $\sum Y_i = \sum \tilde{X}_i = \sum Z_i = X_M$, and $\lambda_i \alpha_i = 1$ for all i . But this is contrary to the fact that (Y_1, Y_2, \dots, Y_I) is Pareto optimal. \square

Remark. Let us note here that Karl Borch (1960, 62) used a slightly different definition of Pareto optimality than our Definition 3. In his definition no exchange is to be carried out unless all reinsurers gain from it:

Definition 4 *A feasible allocation $Y = (Y_1, Y_2, \dots, Y_I)$ is (weakly) Pareto optimal if there is no feasible allocation $Z = (Z_1, Z_2, \dots, Z_I)$ with $Eu_i(Z_i) > Eu_i(Y_i)$ for all i .*

⁸Note that if we avoid ‘‘corner allocations’’, i.e., situations where some $Y_i = 0$ a.s., we may safely assume that $\lambda_i > 0$ for all i .

Borch then showed that, under our conditions $u'_i > 0, u''_i < 0$ for all i , an allocation (Y_1, Y_2, \dots, Y_I) is (weakly) Pareto optimal in the sense of Definition 4 if and only if

$$u'_i(Y_i) = k_i u'_1(Y_1), \quad a.s. \quad \text{for all} \quad i \in \mathcal{I}, \quad (14)$$

where $k_1 = 1$ and $k_i > 0$ for all i ,⁹ or equivalently, if and only if (2) holds with constants $\alpha_i > 0$ for all i . Matti Ruohonen (1979) has further shown that, under our conditions on the u_i -functions, this theorem is also true for (strong) Pareto optimality of our original Definition 3. Thus these two definitions are equivalent under our conditions. This equivalence fails when not all the u_i are strictly monotonic, while the theorem remains valid for the (strong) Pareto optimality of Definition 3. \square

Using theorems 3 and 5 we may now say something about the existence of a competitive equilibrium. We note that the allocation problem (6) is also a nice optimization problem. According to the *Saddle Point Theorem*, granted a suitable constraint qualification, any optimal solution Y will be supported by a (stochastic) Lagrange multiplier $\lambda(X_M) \in L^2$. That is, there exists a random variable $\lambda(X_M)$ with finite variance, $\lambda(X_M) \geq 0$ a.s., such that the Lagrangian

$$\mathcal{L}(Z_1, Z_2, \dots, Z_I; \lambda(X_M)) = E\left\{\sum_{i=1}^I \lambda_i u_i(Z_i) - \lambda(X_M) \sum_{i=1}^I (Z_i - X_i)\right\}$$

is maximal in Z at $Z = Y$. Moreover, complementary slackness holds. The first order conditions for this optimization problem are:

$$\lambda_i u'_i(Y_i) = \lambda(X_M) \quad a.s. \quad \forall i, \quad (15)$$

which are seen to be identical to the first order conditions (2) of Theorem 1 with some reinterpretations. Here the Lagrange multiplier $\lambda(X_M)$ associated with the problem (6) can be seen to be the same as the Riesz representation $\xi(X_M)$ in the pricing representation for a competitive equilibrium, or, what we have also called the state price deflator, and, as usual $\lambda_i = 1/\alpha_i$. This explains Karl Borch's characterization of a Pareto optimal solution: Given the existence of a solution to the allocation problem (6), a necessary and sufficient condition for a Pareto optimum is given, under our assumptions, by the conditions in (15). We argue in terms of directional derivatives: Define

$$\begin{aligned} & \nabla \mathcal{L}((Y_1, \dots, Y_I), (Z_1, \dots, Z_I)) = \\ & \lim_{t \downarrow 0^+} \frac{\mathcal{L}(Y_1 + tZ_1, \dots, Y_I + tZ_I; \lambda(X_M)) - \mathcal{L}(Y_1, \dots, Y_I; \lambda(X_M))}{t}, \end{aligned}$$

where $\nabla \mathcal{L}(Y, Z)$ is the directional derivative of $\mathcal{L}(Y; \lambda(X_M))$ in the direction $Z = (Z_1, \dots, Z_I)$. \mathcal{L} is differentiable at $Y = (Y_1, \dots, Y_I)$ now means that $\nabla \mathcal{L}(Y, Z)$ exists for all $Z_i \in L^2, i = 1, 2, \dots, I$, and the functional $Z \rightarrow \nabla \mathcal{L}(Y, Z)$ is linear. This functional, the gradient of \mathcal{L} at Y , we denote by $\nabla \mathcal{L}(Y)$. It is given by

$$(\nabla \mathcal{L}(Y))(Z) = E\left\{\sum_{i=1}^I (\lambda_i u'_i(Y_i) - \lambda(X_M)) Z_i\right\}. \quad (16)$$

⁹A detailed technical proof of this theorem is provided by DuMouchel (1968). Note that these authors have disregarded corner solutions.

A necessary condition for a maximum of \mathcal{L} at Y is that the linear functional in equation (16) is zero in all directions Z , which leads directly to the condition (15).

One may now wonder if there exist Pareto optimal solutions to the risk exchange problem in the first place. This problem has been studied by DuMouchel (1968), who has shown that if all $u'_i(x)$ are continuous and the ranges of the functions $\lambda_i u'_i(x)$ have a common, non-empty intersection, then this problem has a solution. These conditions for the existence of a Pareto optimal solution are very weak indeed. In particular, in the case treated here - where all the utility functions are strictly monotonic - we can always choose the $\lambda_i > 0$, provided we stay away from corner solutions, such that there is a Pareto optimal solution. Thus there will also exist a competitive equilibrium, possibly after a re-allocation of the initial portfolios X_i .

6.1 The existence of an equilibrium

Given an initial allocation $X = (X_1, \dots, X_I)$, one would presume that each reinsurer would require at least *individual rationality*, i.e.,

$$Eu_i(Y_i) \geq Eu_i(X_i), \quad \forall i, \quad (17)$$

for the final allocations Y_i , $i = 1, 2, \dots, I$. This requirement will naturally exclude many of the Pareto optimal points, which do not really take into account improvements from the initial portfolios X_i , only taking as its point of reference the aggregate X_M .

A competitive equilibrium satisfies individual rationality, and we now turn to the existence of an equilibrium for the given initial portfolios. This subject happens to be a rather delicate matter, usually requiring fix-point theorems or other rather technical, mathematical machinery. Matters are further complicated by the infinite dimensionality of the space L^2 . Since the interior of L^2_+ is empty, we will usually have problems to find a non-zero pricing functional using separation arguments, since e.g., the separating hyperplane cannot be used directly in this situation. Note, however, that we have not insisted that our portfolio space is L^2_+ . We will not elaborate on this issue here, but shall be content with referring to one theorem in this regard.

Mas-Colell (1986) has come up with a concept called *properness* which can be used in the present model. Returning to our conditions behind Theorem 1, the following has been shown (Aase (1993a)), which we present without proof:

Theorem 6 *Suppose $u'_i > 0$, $u''_i < 0$, and $\pi(X_i) > 0$ for all i . If $X_M > 0$ a.s., and there exists an allocation Z , $Z_i \geq 0$ a.s., with $\sum_{i=1}^I Z_i = X_M$ a.s. and $E(u'_i(Z_i))^2 < \infty$ for all i , then there exists a competitive equilibrium.*

It seems natural to check the initial portfolio X if it satisfies the above requirements. Note that it follows from the above theorem and from Theorem 1 that if $X_i \geq 0$ a.s. and $E(u'_i(X_i))^2 < \infty$, for all i , then an equilibrium allocation Y exists such that $E(u'_i(Y_i))^2 < \infty$ for all i , since we know that $\xi \in L^2$. Let us consider some examples.

Example 2. We return to the situation in Example 1, and assume that each X_i is exponentially distributed with parameter θ_i , $i \in \mathcal{I}$. Since $X_M = \sum X_i > 0$

a.s., the requirements for the existence of an equilibrium are satisfied since $u'_i(X_i) = \exp(-X_i/a_i)$ and

$$E(u'_i(X_i))^2 = E\left(e^{-\frac{2}{a_i}X_i}\right) = \frac{\theta_i}{\theta_i + 2/a_i} < \infty \quad \text{for all } i$$

for the risk tolerance parameters $a_i > 0$.

Now consider the normal distribution, and assume that each X_i is $\mathcal{N}(\mu_i, \sigma_i)$ -distributed, and furthermore that X is jointly normal. In this case

$$E(u'_i(X_i))^2 = E\left(e^{-\frac{2}{a_i}X_i}\right) = \exp\left(2\left(\frac{\sigma_i}{a_i}\right)^2 - 2\frac{\mu_i}{a_i}\right) < \infty \quad \forall i.$$

However, the positivity requirements are not met. Still all the computations of the equilibrium are well defined, the state-price deflator $\xi(X_M)$ is a strictly positive element of L^2_+ , and prices can readily be computed. We conclude that an equilibrium exists even if the positivity requirements are not satisfied. It may admittedly be unclear what negative wealth should mean in a one period model, but aside from this there are no formal difficulties with this case as long as utility is well defined for all possible values of wealth.

Suppose that each X_i is Pareto distributed with probability density function (see e.g., Johnson et. al. (1994))

$$f_{X_i}(x) = \frac{\alpha_i c_i^{\alpha_i}}{x^{1+\alpha_i}}, \quad c_i \leq x < \infty, \quad \alpha_i, c_i \in (0, \infty).$$

This is known as the *Pareto distribution of the first kind*, also borrowing its name from the Italian-born Swiss professor of economics, Vilfredo Pareto (1848-1923). In this case EX_i exists only if $\alpha_i > 1$, and $\text{var}X_i$ exists only if $\alpha_i > 2$, etc. The moment generating functions $\varphi_i(\theta) = Ee^{\theta X_i}$ of these distributions exist for $\theta \leq 0$, so the above criteria are met for $Z = X$. Accordingly, for these distributions a competitive equilibrium exists. \square

We now turn to the case where the relative risk aversions of all the reinsurers are constants:

Example 3. Consider the case of power utility, where $u_i(x) = (x^{1-a_i} - 1)/(1 - a_i)$ for $x > 0$, $a_i \neq 1$ and $u_i(x) = \ln(x)$ for $x > 0$ and $a_i = 1$, where the natural logarithm results as a limit when $a_i \rightarrow 1$. This example only makes sense in the no-bankruptcy case where $X_i > 0$ a.s. for all i . The parameters $a_i > 0$ are then the *relative risk aversions* of the agents, which are given by positive constants for this class of preferences.

Consider first the case where $a_1 = a_2 = \dots = a_I = a$. Here all the marginal utilities are given by $u'_i(x) = x^{-a}$, and using Theorem 1 we get

$$u'_i(Y_i(X_M)) = \alpha_i \xi(X_M), \quad \text{a.s.} \quad \text{for all } i,$$

which implies that $Y_i(X_M) = \alpha_i^{-1/a} \xi(X_M)^{-1/a}$, a.s., and using the market clearing $X_M = \sum_{i \in \mathcal{I}} Y_i(X_M)$, a.s., we get

$$u'_\lambda(X_M) = \xi(X_M) = \left(\sum_{i \in \mathcal{I}} \lambda_i^{1/a}\right)^a X_M^{-a} \quad \text{a.s.},$$

where $\lambda_i = 1/\alpha_i$, showing that the marginal utility of the representative agent is of the same type as that of the individual agents. The optimal sharing rules are linear, and given by

$$Y_i(X) = \frac{\lambda_i^{1/a}}{\sum_{j \in \mathcal{I}} \lambda_j^{1/a}} X_M \quad a.s. \quad \text{for all } i.$$

The weights λ_i are determined by the budget constraints, implying that

$$\lambda_i = k \left(\frac{E(X_i X_M^{-a})}{E(X_M^{1-a})} \right)^a, \quad i \in \mathcal{I},$$

or, λ_i is determined modulo the proportionality constant $k = (\sum_{j \in \mathcal{I}} \lambda_j^{1/a})^a$ for each i .

If we normalize such that $E(u'_\lambda(X_M)) = 1$ we find that $k = 1/E(X_M^{-a})$ and the ‘‘pricing principle’’

$$\pi(Z) = \frac{E(Z \cdot X_M^{-a})}{E(X_M^{-a})}, \quad \text{for any } Z \in L^2 \quad (18)$$

results.

When it come to existence, let us check our criterion in the case where all the X_i are exponentially distributed. In this case we have to check the integrals

$$E(X_i^{-2a_i}) = \int_0^\infty x^{-2a_i} \theta_i e^{-\theta_i x} dx < \infty,$$

which converge (near zero) when $a_i < 1/2$. An equilibrium may still exist outside this region depending upon the stochastic interdependence between the initial portfolios. Empirical studies suggest that the interesting values of a_i may be in the range between one and three, say.

Let us consider a situation where there exists a feasible allocation Z as in Theorem 6, where the Z_i components are i.i.d. exponentially distributed with parameter θ . Let $X = AZ$ where A is an $I \times I$ -matrix with elements $a_{i,j}$ satisfying $\sum_i a_{i,j} = 1$ for all j , so that $X_M = \sum_{i=1}^I Z_i := Z_M$. This yields an initial allocation X of dependent portfolios, which we must require in a realistic model of a reinsurance market, and it means that the X_i portfolios are mixtures of exponential distributions with a fairly arbitrary dependence structure. Now it turns out that we can still compute the λ_i -weights in the region $a < I$. In this case X_M has a Gamma distribution with parameters I and θ , and the expectations $E(X_M^{1-a})$ and $E(Z_i X_M^{-a})$ both exist for $a < I - 1$. In order to verify this, we note that the joint distribution of Z_i and X_M is given by the probability density

$$f(z_i, x) = \theta^2 e^{-\theta x} \frac{(\theta(x - z_i))^{I-2}}{(I-2)!}, \quad z_i \leq x < \infty, 0 \leq z_i < \infty.$$

So we have to check the integral

$$E(Z_i X_M^{-a}) = \int_0^\infty \int_{z_i}^\infty z_i x^{-a} \theta^2 e^{-\theta x} \frac{(\theta(x - z_i))^{I-2}}{(I-2)!} dz_i dx.$$

The possible convergence problem is seen to occur around zero, and the standard test yields that when $(1 - a + I - 2) > -1$, this integral is finite. From this it is obvious that the expectations $E(X_i X_M^{-a})$ also converge in the same region, by the linearity of expectation, since the $X_i = \sum_j a_{i,j} Z_j$.

Similarly we have to check the following expectation:

$$E(X_M^{1-a}) = \int_0^\infty x^{1-a} \theta e^{-\theta x} \frac{(\theta x)^{I-1}}{(I-1)!} dx.$$

Near zero the possible problem again occurs, and the standard comparison test gives convergence when $(1 - a + I - 1) > -1$. So when $I > \max\{a, a - 1\} = a$, both expectations exist, suggesting that an equilibrium will also exist in the interesting region for the parameter a when the number of reinsurers $I \geq 4$.

Let us consider the case of Pareto distributions as well. Now the integrals

$$E(X_i^{-2a_i}) = \left(c_i^{2a_i} \left(1 + \frac{2a_i}{\alpha_i} \right) \right)^{-1} < \infty.$$

Since $\min_{i \in \mathcal{I}} \alpha_i > 0$ there are no problems with convergence, and an equilibrium exists in this case regardless of the values of the relative risk aversion parameters. In this latter case all the portfolios are bounded away from zero which helps on the existence problem for power utility, while the exponential distribution has more probability mass near zero, potentially causing some problems with existence of equilibrium. \square

When sharing rules are affine, it is possible to reach a Pareto optimum by an exchange of fractions of the initial portfolios, sometimes also with zero-sum side payments. Affine sharing rules are optimal when the individual utility indices are members of the *Hyperbolic Absolute Risk Aversion* (HARA) class. In a reinsurance market this means that there should be no need for more than the standard proportional reinsurance contract when this is true. Applied to a stock market the assumption means that there should be no need for trading other securities than ordinary shares (common stock). Non-proportional reinsurance and securities such as contingent claims (e.g., options) both exist and are important, so we must conclude that the preferences of the decision makers are at least so diverse that they can not be represented by HARA-utility functions only. For some reason many economists used to refer to a market in which it is impossible to reach a Pareto optimum through an exchange of proportions of the initial portfolio as an “incomplete market”.

Our next example illustrates a situation where the Pareto optimal sharing rules are not affine:

Example 4. Consider power utility when the exponents are not equal, e.g., $u_i(x) = x^{a_i}$, $a_i \in (0, 1)$, $i \in \mathcal{I}$. The first order conditions give

$$Y_i(X) = \left(\frac{u'_\lambda(X_M)}{\lambda_i a_i} \right)^{\frac{1}{(a_i-1)}} \quad a.s. \quad i \in \mathcal{I},$$

where the state-price deflator is implicitly determined by the market clearing condition, and the budget constraints determine the agent weights modulo a normalizing constant.

Consider the special case where $I = 2$, $a_1 = 1/2$, $a_2 = 3/4$. The marginal utility of the representative agent equals

$$u'_\lambda(X_M) = \left(\frac{\sqrt{h} + \sqrt{h + 4X_M}}{2X_M} \right)^{1/2} \quad a.s.$$

where we have arbitrarily set $\lambda_2 = 3/4$, which we can do since only the ratio of the two weights matters. Here

$$h = \left(\frac{a_1 \lambda_1}{a_2 \lambda_2} \right)^4.$$

In this case the optimal sharing rules are

$$Y_1(X_M) = \frac{1}{2} \left(\sqrt{h^2 + 4hX_M} - h \right), Y_2(X) = X_M + \frac{1}{2} \left(h - \sqrt{h^2 + 4hX_M} \right),$$

a.s. Finally, one of the budget constraints is now enough to determine the remaining unknown constant h , in which case everything is determined in terms of the primitives of the model. \square

It should be clear that this Pareto optimum can not be achieved by an exchange of proportional reinsurance contracts.

7 Risk tolerance and aggregation

The risk tolerance function of an agent $\rho(x) : R \rightarrow R_+$, is defined by the reciprocal of the absolute risk aversion function $R(x) = -\frac{u''(x)}{u'(x)}$, or $\rho(x) = 1/R(x)$. There is a neat result connecting the risk tolerances of all the agents in the market to the risk tolerance of the representative agent in a Pareto optimal allocation. It goes as follows: In a Pareto optimum we know that

$$u'_i(Y_i(x)) = \alpha_i u'_\lambda(x), \quad x \in R.$$

Because of our smoothness assumptions, both sides of the above equation are real, differentiable functions a.e. (the right-hand-side because of the implicit function theorem), so taking derivatives of both sides gives

$$u''_i(Y_i(x))Y'_i(x) = \alpha_i u''_\lambda(x), \quad x \in R.$$

Dividing the second equation by the first, we obtain the following non-linear differential equation for the Pareto optimal allocation function $Y_i(x)$:

$$Y'_i(x) = \frac{R_\lambda(x)}{R_i(Y_i(x))}, \quad x \in R, \quad (19)$$

where $R_\lambda(x) = -\frac{u''_\lambda(x)}{u'_\lambda(x)}$ is the absolute risk aversion function of the representative agent, and $R_i(Y_i(x)) = -\frac{u''_i(Y_i(x))}{u'_i(Y_i(x))}$ is the absolute risk aversion of agent i at the Pareto optimal allocation function $Y_i(x)$, $i \in \mathcal{I}$. Since $\sum_{i \in \mathcal{I}} Y'_i(x) = 1$, we now get by summation in (19)

$$\rho_\lambda(X) = \sum_{i \in \mathcal{I}} \rho_i(Y_i(X_M)) \quad a.s., \quad (20)$$

or in words, the risk tolerance of the market equals the sum of the risk tolerances of the individual agents in a Pareto optimum. The above result has been found by Borch (1985); see also Bhlmann (1980) for the special case of exponential utility functions.

Example 5. Returning to Example 1 where $u'_i(x) = e^{-x/a_i}$ for all $i \in \mathcal{I}$, we get that $\rho_i(x) = a_i$ for all $x \in R$, i.e., the risk tolerance function of each agent is a constant. Using the result (20), we get that $\rho_\lambda(x) = \sum_{i \in \mathcal{I}} a_i = A$ for all x , also a constant. That $\rho_\lambda(x) = A$ can easily be verified by going back to Example 1, where we showed that $u'_\lambda(x) = \xi = \exp((K - x)/A)$. \square

Imagine that agent j is risk neutral, meaning that $\rho_j(Y_j) = \infty$, while the others are risk averse. From the result (20) it follows that $\rho_\lambda = \infty$ as well, i.e., the representative agent is then also risk neutral. From the relation (19) it may be seen that this implies that $Y'_j(x) = 1$ for all x , meaning that agent j will then carry all the risk in the market. In other words, we have shown that *in a Pareto optimum all risk should be carried by the risk neutral participant*.

Example 6. In order to illustrate this last point, consider a case where $u_1(x) = x$ and $u_2(x) = 2\sqrt{x}$, and $I = 2$. Here agent 1 is risk neutral. The first order conditions give

$$1 = \alpha_1 \xi, \quad \frac{1}{Y_2(X_M)} = \alpha_2 \xi, \quad a.s.$$

implying that $\xi = \frac{1}{\alpha_1}$, a constant, and $\sqrt{Y_2(x)} = \frac{\alpha_1}{\alpha_2} = \frac{\lambda_2}{\lambda_1}$, another constant. The optimal sharing rules are thus

$$Y_1(X_M) = X_M - \left(\frac{\lambda_2}{\lambda_1}\right)^2, \quad Y_2(X_M) = \left(\frac{\lambda_2}{\lambda_1}\right)^2, \quad a.s.$$

and the utility function of the representative agent is given by

$$u_\lambda(x) = \lambda_1 Y_1(x) + \lambda_2 2\sqrt{Y_2(x)} = \lambda_1 x + \frac{\lambda_2^2}{\lambda_1}.$$

Thus from two risky projects brought to the market having payoffs X_i , $i = 1, 2$, the risk neutral agent takes all the risk, leaving a fixed amount, or a deterministic salary, to the risk averse agent. The representative agent is seen to be risk neutral in accordance with the above theory, and the state-price deflator $\xi = u'_\lambda(X_M) = \lambda_1$, a constant. The budget constraints determine the ratios between the agent weights as follows:

$$\frac{\lambda_2}{\lambda_1} = \sqrt{E(X_2)}.$$

If we normalize such that $E u'_\lambda(X_M) = 1$, then since $\xi = u'_\lambda(X_M) = \lambda_1$, $\lambda_1 = 1$ and $\lambda_2 = \sqrt{E(X_2)}$. \square

One may wonder what happens when more than one agent is risk neutral. In the above example, if both agents are risk neutral they can not both assume all the risk. In this case the risk neutral agents as a group presumably end up with all the risk, where they are indifferent to any split of the total risk among them that does not change each individual's expected payoff.

8 Insurance premiums

The foregoing has been formulated in terms of portfolios and market values of net reserves. To obtain market premiums of insurance contracts, we note the net reserves of insurer i consists of assets a_i less of liabilities Z_i under the insurance contracts held by the insurer. Assume for simplicity that the assets a_i are riskless. Then we may apply the foregoing theory to

$$X_i = a_i - Z_i, \quad i \in \mathcal{I}.$$

We note that the market values of the initial portfolios can be written

$$\pi(X_i) = a_i - \pi(Z_i) = a_i - E(u'_\lambda(a - Z_M)Z_i),$$

where $a = \sum a_i$ and $Z_M = \sum Z_i$. We may define the *market disutility* of claim payments by the function $v_\lambda(z)$, where $v'_\lambda(z) = u'_\lambda(a - z)$. From our assumption it follows that $v_\lambda(x)$ is a decreasing function in z and $v''_\lambda(z) = -u''_\lambda(a - z) > 0$. The above formula simply says that the market value of the insurer's portfolio is equal to his riskless assets less the market premium for insurance of the liabilities. This formula makes it easy to translate results expressed in terms of values of net reserves into insurance premiums. Notice in particular that if for some portfolio X_i the market value $\pi(X_i) < E(X_i)$, then we get from the above formula that the corresponding insurance premium $\pi(Z_i) > E(Z_i)$ so that the economic *risk premium* ($\pi(Z_i) - E(Z_i)$) of this insurance contract is positive.

Using the normalization $E v'_\lambda(a - Z_M) = 1$, (meaning that the risk-free interest rate equals zero), we find that the risk premium can in general be written as follows:

$$\pi(Z_i) - E(Z_i) = \text{cov}(Z_i, v'_\lambda(Z_M)). \quad (21)$$

Since the marginal disutility of the representative agent is an increasing function of z , from (21) one may be led to believe that for claims Z_i that are positively correlated with the aggregate claims Z_M in the market, the risk premium is positive, and for claims that are negatively correlated with Z_M the risk premium is negative. This is, however, only true in general when (Z_1, \dots, Z_I) is multinormally distributed. There exist joint distributions for the claims where this may not be true. Here one has to remember that covariance is a measure of linear statistical dependence, and can accordingly only be considered as a good measure of "stochastic association" under multinormality.

One can of course argue that in insurance an assumption of joint normality is not very realistic, since for once claims can only be non-negative. We may therefore be reluctant to use the nice theoretical results obtainable from this assumption in insurance. Here we must remember, however, that the normal distribution is commonly used with great success to model a number of quantities, like the heights, or weights of recruits, and many other quantities which are clearly non-negative. The point is that the resulting parameter estimates will usually yield a completely negligible probability of falling in the forbidden regions. This is one of the reasons why we still find it fruitful to return to the situation with a multinormal distribution for the net reserves in the next section.

Although the present reformulation is straight-forward, one has to be careful when modeling claim size distributions. In practice insurance claims are always

finite, and models where claim sizes are bounded seem natural, but it is often convenient to use standard continuous probability distributions with known properties on unbounded supports, as we have just argued.

For example, if we let Z_i be Pareto distributed as in Example 2, where we have negative exponential utility functions, our test cannot guarantee existence of equilibrium. If the claim sizes are instead exponentially distributed, existence is only guaranteed if $a_i > 2\theta_i$ for all i , while for normally distributed claims existence is more or less guaranteed in this situation. This is another reason for studying the multinormal case separately.

For power utility none of these distributions can be employed directly, unless we set the utility equal to zero when the argument becomes negative, i.e., substitute the argument x in $u(x)$ by $|x|^+$. In the latter case we may run into existence problems if too much probability is attached to the zero point.

As a conclusion to our equilibrium pricing theory so far, we note the following: Premiums of a risk Z in a reinsurance market must typically depend on: (i) The stochastic properties of the risk itself. (ii) The stochastic relationship between the particular risk Z and claims in the market as a whole, described by the covariance between $v'_\lambda(Z_M)$ and Z . (iii) The attitude towards risk in the market as a whole, represented by $v'_\lambda(Z_M)$. (iv) The total assets of all the insurers in the market, represented by a .

A realistic theory of insurance premiums must of course take all these four elements into account. This is, however, rarely done in actuarial risk theory. Several books have been written on insurance premium principles, some even recent, where only the first of these four elements are covered. This is also the case for current articles published in scientific journals dealing with actuarial theory.

9 Risk adjustments of the probability measure

In the contemporary literature one often encounters market prices computed as a discounted, expected value of the final payoffs of an asset, the expectation being taken with respect to another probability measure than the one originally given. This is particularly true in many financial models, where the setting is not that of risk neutrality. We now relate our pricing results to this tradition, and investigate if there is anything to be gained by making this transformation.

In the formulation of our theory we have assumed that there exists a riskless security, X_1 say, such that $X_1(\omega) = 1$ a.s. with market price $\pi(X_1) = Eu'_\lambda(X_M) := d := 1/(1 + r_f)$, defining the equilibrium interest rate r_f through the discount factor d . Consider $\xi := u'_\lambda(X_M)/d$. Clearly $E(\xi) = 1$ and the pricing rule can be written:

$$\pi(Z) = \frac{1}{1+r} E(Z \cdot \xi) \quad \text{for any } Z \in L^2, \quad (22)$$

Under our assumptions $P[u'_\lambda(X_M) > 0] = 1$ and $d > 0$, and we define a set function Q as follows:

$$Q(A) = \frac{1}{d} \int_A u'_\lambda(X_M(\omega)) dP(\omega), \quad A \in \mathcal{F}.$$

It is then clear that the following three properties hold for Q : (a) $Q(\Omega) = 1$, (b) $Q(\cup_i A_i) = \sum_i Q(A_i)$ whenever $\{A_i\}$ is a disjoint countable collection of

members of \mathcal{F} , and (c) $Q(A) \geq 0$ for all $A \in \mathcal{F}$. Property (a) follows since $E(\xi) = 1$, property (b) is a consequence of the corresponding property of the abstract integral which defines Q , and (c) follows from the almost sure positivity of ξ . Thus Q is formally a probability measure which is mutually absolutely continuous with respect to the given measure P , or in standard mathematical notation, $Q \ll P$ and $P \ll Q$, or $Q \sim P$. This means that if $P(B) = 0$ for any $B \in \mathcal{F}$, then $Q(B) = 0$ and if $Q(A) = 0$ for any $A \in \mathcal{F}$, then $P(A) = 0$. Here ξ is the Radon Nikodym derivative of Q with respect to P , or

$$\xi(\omega) = \frac{1}{d} u'_\lambda(X_M(\omega)) = \frac{dQ(\omega)}{dP(\omega)}. \quad (23)$$

Notice that in our setting Q is only formally a probability measure, meaning that if $A \in \mathcal{F}$ is any event such that $Q(A) > 0$, then $Q(A)$ is not really the probability of the event A if $Q(A) \neq P(A)$, since we have assumed that all the agents agree on the probability measure P . The measure Q can instead be thought of as a risk adjusted probability measure. The pricing functional π can be expressed as a discounted expected value under the measure Q :

$$\pi(Z) = \frac{1}{1+r} E\{Z \cdot \xi\} = \frac{1}{1+r} \int_{\Omega} Z(\omega) \xi(\omega) dP(\omega).$$

Using (23), this can be written

$$\pi(Z) = \frac{1}{1+r} \int_{\Omega} Z(\omega) dQ(\omega) = \frac{1}{1+r} E^Q(Z), \quad Z \in L^2, \quad (24)$$

where the symbol E^Q obviously signifies the expectation operator under the measure Q . After having adjusted for the time depreciation of money, market prices are computed as expectations under Q . This would correspond to a world of only risk neutral agents having probability assessments given by Q . In order to calculate market prices we thus carry out two adjustments: *First* we deflate all risks by the discount factor d adjusting for the time depreciation of money, i.e., one dollar received today is preferred to one dollar received tomorrow. *Second* we deflate all risks by ξ , meaning adjusting for risk aversion.¹⁰ The expected value of the final result is then the market value.

Now, are there any advantages to introducing the risk adjusted measure Q ? In our one-period model the answer is no. In general the expression in (22) will suffice to compute prices, i.e., discount by the state-price deflator $u'_\lambda(X_M)$ and take expectation, but there may be situations where it is relatively easy to find the probability distribution of the risk Z under the measure Q , in which case it may simply be more convenient to use the expression (24), since we are usually well trained in taking expectations of random variables.

This is in particular true for certain multiperiod models, e.g., time continuous models of financial economics, when the primitive risks X are modeled by Ito-diffusions. In this situation one has a powerful theorem due to Girsanov giving a recipe how to find the state prices using Q , and in particular how these can be linked to the primitives of the model, in which case there is also a clear conceptual advantage to the introduction of Q .

¹⁰Naturally these two steps are the same as just deflating once by the state-price deflator $u'_\lambda(X_M)$.

10 An Insurance version of the Capital Asset Pricing Model

We may now derive some very simple and elegant results based on two assumptions: (i) An interior equilibrium exists; and (ii) $X = (X_1, X_2, \dots, X_I)$ is multnormally distributed.

Let us quickly recall the properties we make use of. The random vector X is said to be multnormally distributed if any linear combination $Y = \sum_{i \in \mathcal{I}} a_i X_i$ of elements of X is normally distributed, for any set of real constants a_1, a_2, \dots, a_I .

Suppose X is multnormally distributed, and consider any two linear combinations $Y = \sum_{i \in \mathcal{I}} a_i X_i$ and $Z = \sum_{i \in \mathcal{I}} b_i X_i$. Then (Y, Z) is binormally distributed.

Consider the linear relation $X = \alpha + \beta Y + U$, where X, Y and U are random variables and α and β real constants. If (X, Y) is binormally distributed, then $U = X - (\alpha + \beta Y)$ is normally distributed according to the above definition. Also, by the above result (U, Y) is binormally distributed, and so is (U, X) .

We will also need a result called Stein's lemma after Charles Stein. It goes as follows: Suppose that (X, Y) is binormally distributed, and let $g : R \rightarrow R$ be a real function such that $E\{g(Y)\}$ exists. Then

$$(a) \quad \text{cov}(X, g(Y)) = \frac{\text{cov}(Y, g(Y))}{\text{var}Y} \text{cov}(X, Y).$$

If in addition the derivative $g'(\cdot)$ exists for all reals and $E |g'(Y)| < \infty$, then

$$(b) \quad E(g'(Y)) = \frac{\text{cov}(Y, g(Y))}{\text{var}Y}.$$

A very simple proof of this result can be found in Aase (1993a). Since it builds upon the above observations, let us present a version here:

Proof of Stein's lemma: Between any two random variables X and Y , possessing the appropriate moments, the following relationship holds:

$$X = \alpha + \beta Y + U$$

where $\text{cov}(Y, U) = 0$ and

$$\beta = \frac{\text{cov}(X, Y)}{\text{var}Y}, \quad \alpha = EX - \beta EY.$$

Thus we have quite generally

$$\text{cov}(Y, g(Y)) = \beta \text{cov}(Y, g(Y)) + \text{cov}(U, g(Y)).$$

By the above remarks, since (X, Y) is binormally distributed, so is (Y, U) and $\text{cov}(Y, U) = 0$. Because of the binormality, this implies that Y and U are indeed *independent* random variables, which implies that U and $g(Y)$ are also independent, and thus $\text{cov}(U, g(Y)) = 0$. Accordingly $\text{cov}(X, g(Y)) = \beta \text{cov}(Y, g(Y))$, which is the conclusion in (a).

The (b) part of the result is shown using integration by parts, where the functional form of the normal probability density $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y}} e^{-\frac{1}{2\sigma_Y^2}(y-\mu_Y)^2}$ of Y must explicitly be employed. \square

Notice that the “covariance orthogonality” between the random variable Y and the remainder term U is not quite enough to conclude that the term $cov(U, g(Y)) = 0$. This is precisely where the (bi)normality is vital for the conclusion.

Returning to the equilibrium characterization, we maintain our previous assumptions regarding the smoothness of the utility functions of the agents. Denote $E\{u'_\lambda(X_M)\} = d = \frac{1}{1+r}$. Then our general equilibrium pricing result, applied to the initial portfolios X_i , can be written

$$\pi(X_i) = dEX_i + cov(X_i, u'_\lambda(X_M)), \quad i = 1, 2, \dots, I.$$

From Stein’s lemma it follows that

$$\pi(X_i) = dEX_i + E(u''_\lambda(X_M))cov(X_i, X_M), \quad i = 1, 2, \dots, I.$$

Summation over the agents gives $\pi(X_M) = dEX_M + E(u''_\lambda(X_M))varX_M$, and elimination of the term $E(u''_\lambda(X_M))$ in these two equations finally yields the CAPM:

$$\pi(X_i) - dEX_i = \frac{cov(X_i, X_M)}{varX_M}(\pi(X_M) - dEX_M), \quad i = 1, 2, \dots, I. \quad (25)$$

The interpretation of this relationship is: The risk premium of the portfolio X_i equals the portfolio’s “beta” times the risk premium of the market portfolio X_M .

Note that this insurance version of the CAPM is in general only valid for the *given* initial portfolios. This is in contrast to the CAPM in a stock market, where the corresponding relationship is also valid for any portfolio of stocks. If, however, the final optimal sharing rules Y_i are affine, then the CAPM will also be valid for these, since multinormality is maintained under affine transformations. In this case we get

$$\pi(Y_i) - dEY_i = E(Y'_i(X_M))(\pi(X_M) - dEX_M), \quad i = 1, 2, \dots, I, \quad (26)$$

where

$$Y'_i(X_M) = \frac{R_\lambda(X_M)}{R_i(Y_i(X_M))} > 0 \quad a.s.$$

under our assumptions. Thus, if the optimal sharing rules are affine, the corresponding betas are strictly positive, which is the analogue of the following result in a stock market: “In the CAPM efficient portfolios have positive betas”. When only affine sharing rules result, it also corresponds to the observation: “Investors hold efficient portfolios in CAPM”, efficient here referring to portfolios on the portfolio frontier above the minimum variance portfolio.

Notice that the beta of the original portfolios may be both positive and negative. What would a negative beta mean? Because of risk aversion the quantity $(\pi(X_M) - dEX_M) < 0$, so the market would find such a portfolio, say X_i , so valuable, that it would accept a negative expected return in equilibrium: $(EX_i - \pi(X_i))/\pi(X_i) < 0$. Such an asset would come in handy when really needed, namely when the rest of the market is down, which accounts for its relatively high market value. It is noteworthy that our relatively simple theory can capture this kind of wisdom.

11 A Game Theoretic Approach to the Risk Allocation Problem

11.1 Introduction

Game theory was created to generalize the behavioral assumptions usually made in neo-classical economic theory. Some of these assumptions may appear unrealistic, or at least seem to require a large degree of sophistication on behalf of the reinsurers. The assumption of rational expectations leaves the reinsurers as “price takers”, yet it is their very actions that determine the prices. A justification for price taking is usually that there is a very large number of participants in the market, and none of them acting alone can influence the prices to any significant degree. Example 3 showed a situation where an equilibrium existed only if the number of participants exceeded a certain fixed number. In the following we shall give a brief and oversimplified presentation of the essential elements in the game theoretical approach.

Assume that the game has I players, let $\mathcal{I} = \{1, 2, \dots, I\}$ and let \mathcal{S} be an arbitrary subset of \mathcal{I} . The *characteristic function* of the game, $v(\mathcal{S})$ is a real-valued function defined for any $\mathcal{S} \subseteq \mathcal{I}$. The function $v(\mathcal{S})$ gives the total payoff which the players in \mathcal{S} - belonging to the “coalition” \mathcal{S} - can obtain by cooperating.

The characteristic function is superadditive, i.e., $v(\mathcal{S} \cup \mathcal{T}) \geq v(\mathcal{S}) + v(\mathcal{T})$, where \mathcal{S} and \mathcal{T} are disjoint subsets of \mathcal{I} . This means that the players can not lose by cooperation.

Let z_i be the payoff to player i in the outcome of the game. The relevant behavioral assumptions in terms of game theory are:

$$\sum_{i=1}^I z_i = v(\mathcal{I}). \quad (27)$$

This represents “collective rationality”, which we usually refer to as *efficiency*, and implies that the players will cooperate so that they obtain the maximum total payoff. The assumption corresponds to Pareto optimality in our reinsurance market, i.e., there exists a non-negative vector of agent weights $\lambda \neq 0$ such that the optimal solution Y solves

$$\sum_{i=1}^I \lambda_i E u_i(Y_i) = E u_\lambda(X_M).$$

Next consider the condition

$$z_i \geq v(\{i\}). \quad (28)$$

This represents “individual rationality”, and implies that no player will participate in the game if he can do better in splendid isolation. The assumption corresponds to $E u_i(Y_i) \geq E u_i(X_i)$ in the reinsurance market. The two assumptions define the set of payoff vectors which constitute the “imputations” of the game.

It is natural to assume that the corresponding rationality assumptions hold for all coalitions, not just for the one-player coalition, and for the coalition of

all players. This suggests the following assumption

$$\sum_{i \in \mathcal{S}} z_i \geq v(\mathcal{S}) \quad (29)$$

for all $\mathcal{S} \subseteq \mathcal{I}$. The latter condition we may refer to as *social stability*. It means that no coalition $\mathcal{S} \subset \mathcal{I}$ could improve its members' outcome by splitting away from the others. In our reinsurance market the condition would correspond to a further restriction on the investor weights $\lambda \neq 0$ such that

$$\sum_{i \in \mathcal{S}} \lambda_i Eu_i(Y_i) \geq Eu_\lambda(X_{\mathcal{S}})$$

where

$$Eu_\lambda(X_{\mathcal{S}}) := \sup_{Z_i} \sum_{i \in \mathcal{S}} \lambda_i Eu_i(Z_i) \quad s.t. \quad \sum_{i \in \mathcal{S}} Z_i \leq \sum_{i \in \mathcal{S}} X_i := X_{\mathcal{S}}.$$

The set of payoff vectors which satisfies (29) is called the *core* of the game, a concept introduced by Gillies (1959). The core appears as a very attractive solution concept for a general game, but it has the unpleasant property of being empty for large classes of games. For a three-person game let us make a transformation of the origin so that $v(\{i\}) = 0$ for $i = 1, 2$ and 3 . The core is then defined by nonnegative solutions of

$$\begin{aligned} z_1 + z_2 &\geq v(\{1, 2\}) \\ z_1 + z_3 &\geq v(\{1, 3\}) \\ z_2 + z_3 &\geq v(\{2, 3\}) \\ z_1 + z_2 + z_3 &= v(\{1, 2, 3\}) \end{aligned}$$

We see that this system has a solution only if

$$2v(\{1, 2, 3\}) \geq v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\}). \quad (30)$$

Note that mere stability is easy to achieve: Simply let the numbers z_i be so large that $\sum_{i \in \mathcal{S}} z_i \geq v(\mathcal{S})$ for all $\mathcal{S} \subset \mathcal{I}$. Thus, not very surprising, the essential difficulty resides in the requirement that the total payoff be efficient and not distributed excessively.

The fact that the core often does not exist may limit its usefulness in general game theory, but the concept has proved useful in economic applications of game theory.

The presentation of some elements of game theory assumes side-payments, and inter-person comparability of utility. These assumptions are very strong, but they can be relaxed at the cost of a more cumbersome notation.

A market of pure exchange can be interpreted as a game, as we have indicated above. The players enter the game with an initial allocation of risks, or goods, exchange these risks (goods) in the market and end up with a final allocation which has a higher utility. One of the original objectives of game theory was to analyze markets with so few participants that the assumptions behind the neo-classical competitive equilibrium appear unreasonable.

Debreu and Scarf (1963) have proved that the core of a market game is non-empty, and that it contains the allocation corresponding to the competitive equilibrium in the market, if any.

They further proved that as the number of players increase to infinity, the core will, under certain assumptions, shrink to the competitive equilibrium. This means that the heroic neo-classical behavioral assumptions used to determine the competitive equilibrium in an economy may not be necessary. The result can be reached from the assumptions of rational behavior behind game theory, i.e., it is not necessary to assume that the agents are “price takers”. This really offers us two ways to the market equilibrium, the conventional one, and the avenue via the limit of the core in a market game.

Baton and Lemaire (1981a) have determined the core for a special case of a reinsurance market, in the situation of Example 1 with negative exponential utilities, assuming in addition that the initial portfolios X_1, X_2, \dots, X_I are independent. We will return to this example, but we drop their independence assumption.

Another solution concept which may be useful in the analysis of reinsurance is the *bargaining sets* introduced by Aumann and Maschler (1964). The bargaining set contains the core, if it is not empty, and a number of other allocations, which may occur if the players for some reason fail to form the all-player coalition.

The starting point of the different bargaining sets is a “payoff configuration”, which consists of a partition $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_m$ of the set \mathcal{I} of all players, and a payoff vector (z_1, z_2, \dots, z_I) . A payoff configuration is individually rational if

$$\sum_{r \in \mathcal{I}_s} z_r = v(\mathcal{I}_s), \quad s = 1, 2, \dots, m$$

$$z_r \geq v(\{r\}).$$

The simplest bargaining set consists of all stable individually rational payoff configurations.

Baton and Lemaire (1981b) have determined the bargaining set for some special cases of a reinsurance market. Their paper seems to be the first to apply the theory of bargaining sets to insurance, and the approach may be promising, for instance if there is some segmentation of the market.

11.2 The core of a reinsurance market

In many competitive markets one is not confronted with a definitive set of market prices for an arbitrary collection of risks. Instead one often faces a range of “rational” prices that each may be accepted. This is so in financial markets, but also in many other markets there seems to be such a “bid-ask spread”. A negotiation process is then needed in order to obtain a final transaction at one of these prices. Thus the competitive paradigm may be a bit too strong, giving sharper predictions of market behavior than is actually observed.

The usual explanation of such bid-ask spreads are transaction costs, or possibly asymmetric information, i.e., deviations from our standard model. Let us mention a specific event: During the summer of 1999 Reuter arranged for euro-dollar trade with no transaction costs. It was puzzling to some participants to observe that a bid-ask spread still persisted.

The derivation we offer below is not intended to fully explain such observations, but we think it is of interest since it is developed entirely within the neo-classical paradigm. Let us start by an example:

Example 7. We return to Example 1, where the reinsurers have negative exponential utility functions of the form

$$u_i(x) = (1 - a_i e^{-x/a_i}), \quad x \in R, \quad i \in \mathcal{I}.$$

The initial portfolios are X_1, X_2, \dots, X_I , and the “market portfolio” X_M we here denote by $X_{\mathcal{I}} = \sum_{i \in \mathcal{I}} X_i$. The all-player coalition results in the Pareto optimal allocations

$$Y_i = \frac{a_i}{A_{\mathcal{I}}} X_{\mathcal{I}} + b_i, \quad \text{where} \quad b_i = a_i \ln(\lambda_i) - a_i \frac{K_{\mathcal{I}}}{A_{\mathcal{I}}},$$

$$A_{\mathcal{I}} = \sum_{i \in \mathcal{I}} a_i, \quad \text{and} \quad K_{\mathcal{I}} = \sum_{i \in \mathcal{I}} a_i \ln(\lambda_i).$$

Let $K_{\mathcal{I}}$ denote a normalization constant.

The “investor weights” λ_i are arbitrary positive constants, and our aim is to further constrain the value sets of these constants, or equivalently, to impose constraints on the zero-sum side payments b_i . Notice that we have also found the characteristic function of the game, here given by the expected utility of the of the “representative agent” restricted to any subset \mathcal{S} of \mathcal{I} :

$$Eu_{\lambda}(X_{\mathcal{S}}) = E \left(\sum_{i \in \mathcal{S}} \lambda_i - A_{\mathcal{S}} e^{(K_{\mathcal{S}} - X_{\mathcal{S}})/A_{\mathcal{S}}} \right) \quad \text{for any} \quad \mathcal{S} \subseteq \mathcal{I}.$$

First consider individual rationality:

$$Eu_i(Y_i) \geq Eu_i(X_i), \quad i \in \mathcal{I}.$$

This is equivalent to

$$\lambda_i \geq \frac{E(e^{-X_{\mathcal{I}}/A_{\mathcal{I}}})}{E(e^{-X_i/a_i})} e^{K_{\mathcal{I}}/A_{\mathcal{I}}}, \quad i \in \mathcal{I}$$

or

$$b_i \geq a_i \left\{ \ln(E[e^{-X_{\mathcal{I}}/A_{\mathcal{I}}})] - \ln(E[e^{-X_i/a_i}]) \right\}$$

since $\lambda_i = e^{b_i/a_i} e^{K_{\mathcal{I}}/A_{\mathcal{I}}}$.

Next consider social stability. Let us restrict attention to the case $I = 3$. The core is then characterized by the following inequalities in b_1, b_2 and b_3 :

$$\begin{aligned} (a_1 + a_2) E \left(e^{-(X_1 + X_2)/(a_1 + a_2)} \right) \exp\{(b_1 + b_2 + 2K/A)/(a_1 + a_2)\} & \quad (31) \\ \geq a_1 E \left(e^{-X_M/A} \right) e^{-b_1/a_1} + a_2 E \left(e^{-X_M/A} \right) e^{-b_2/a_2}, \end{aligned}$$

$$\begin{aligned} (a_1 + a_3) E \left(e^{-(X_1 + X_3)/(a_1 + a_3)} \right) \exp\{(b_1 + b_3 + 2K/A)/(a_1 + a_3)\} & \quad (32) \\ \geq a_1 E \left(e^{-X_M/A} \right) e^{-b_1/a_1} + a_3 E \left(e^{-X_M/A} \right) e^{-b_3/a_3}, \end{aligned}$$

$$\begin{aligned}
& (a_2 + a_3)E \left(e^{-(X_2+X_3)/(a_2+a_3)} \right) \exp\{(b_2 + b_3 + 2K/A)/(a_2 + a_3)\} \quad (33) \\
& \geq a_2E \left(e^{-X_M/A} \right) e^{-b_2/a_2} + a_3E \left(e^{-X_M/A} \right) e^{-b_3/a_3},
\end{aligned}$$

and

$$\begin{aligned}
& 2AE\{e^{-X_M/A}\} \leq \\
& (a_1 + a_2) \exp\{(b_1 + b_2 + K/A)/(a_1 + a_2)\}E \left(e^{-(X_1+X_2)/(a_1+a_2)} \right) \quad (34) \\
& (a_1 + a_3) \exp\{(b_1 + b_3 + K/A)/(a_1 + a_3)\}E \left(e^{-(X_1+X_3)/(a_1+a_3)} \right) \\
& (a_2 + a_3) \exp\{(b_2 + b_3 + K/A)/(a_2 + a_3)\}E \left(e^{-(X_2+X_3)/(a_2+a_3)} \right),
\end{aligned}$$

where the last inequality corresponds to (30). Finally $\sum_{i \in \mathcal{I}} b_i = 0$. \square

In general the core will be characterized by the Pareto optimal allocations corresponding to investor weights λ_i in some region restricted by inequalities of the above kind, in general a polyhedron $\Lambda \subset \text{int}(R_+^I)$. Let $u'_\lambda(X_M)$ correspond to the state-price deflator for some $\lambda \in \Lambda$. For an arbitrary risk $Z \in L^2$ this will give rise to a market premium $\pi(Z) = E\{Z \cdot u'_\lambda(X_M)\}$ as we have seen earlier. As λ varies in the region Λ we obtain a set of “rational” prices, and it seems natural to consider the largest and the smallest of these, the “ask” price and the “bid” price:

$$\pi_b(Z) = \inf_{\lambda \in \Lambda} E\{Z \cdot u'_\lambda(X_M)\}, \quad \pi_a(Z) = \sup_{\lambda \in \Lambda} E\{Z \cdot u'_\lambda(X_M)\}.$$

We now illustrate this by the following situation, describing efficient risk allocation between an insurer and an insurance buyer.

Consider a policy holder having initial capital w_1 , a positive real number, and facing a risk X , a non-negative random variable. The insured has utility function u_1 , where $u'_1 > 0$, $u''_1 < 0$. The insurer has utility function u , $u' > 0$, $u'' \leq 0$, and initial fortune w , also a positive real number. These parties can negotiate an insurance contract, stating that the indemnity $I(x)$ is to be paid by the insurer to the insured if claims amount to $x \geq 0$. It seems reasonable to require that $0 \leq I(x) \leq x$ for any $x \geq 0$, and also that no payments should take place if there are no claims, i.e., $I(0) = 0$. The premium p for this contract is payable when the contract is initialized. Using our established theory for generating Pareto optimal contracts, we easily deduce that the optimal contract satisfies the following differential equation:

$$\frac{\partial I(x)}{\partial x} = \frac{R_1(w_1 - p - x + I(x))}{R_1(w_1 - p - x + I(x)) + R(w + p - I(x))}, \quad (35)$$

where the functions $R_1 = -\frac{u''_1}{u'_1}$, and $R = -\frac{u''}{u'}$ are the absolute risk aversion functions of the insured and the insurer, respectively.

Some conclusions immediately follow from this equation: If $u'' < 0$, we see that $0 < I'(x) < 1$ for all x , and together with the boundary condition $I(0) = 0$, by the mean value theorem we get that

$$0 < I(x) < x, \quad \text{for all } x > 0,$$

stating that full insurance is not Pareto optimal. We notice that the natural restriction $0 \leq I(x) \leq x$ is not binding at the optimum for any $x > 0$.

We also notice that contracts with a deductible d can not be Pareto optimal either, since such a contract means that $I_d(x) = x - d$ for $x \geq d$, and $I_d(x) = 0$ for $x \leq d$ for $d > 0$ a positive real number. Thus either $I'_d = 1$ or $I'_d = 0$, contradicting $0 < I'(x) < 1$ for all x .

However, when $u'' = 0$ we notice that $I(x) = x$ for all $x \geq 0$, full insurance is optimal and the risk-neutral part, the insurer, assumes all the risk. Clearly, when R is uniformly much smaller than R_1 , this will approximately be true even if $R > 0$.

The fact that the classical model can not explain contracts with deductibles has led some writers, like Gerber (1978) and Bühlmann and Jewell (1979), to consider so-called “constrained” Pareto optimal risk exchanges, where exogenous constraints have been imposed. This may lead to contracts we observe in real life. It would of course be more desirable to obtain contracts of, say, the stop loss type, or XL-reinsurance treaties from more fundamental assumptions, and it is by now well known that the introduction of transactions costs, or including moral hazard may lead to non-trivial deductibles, and coinsurance above the deductible. We will not, however, discuss these theories in this paper.

Returning to the above situation, we can find the “competitive equilibrium” if it exists. Here $X_M = w + w_1 - X$, and the budget constraint $\pi(Y_1) = \pi(X_1)$, i.e.,

$$E\{(w_1 - p - X + I(X))u'_\lambda(X_M)\} = E\{(w_1 - X)u'_\lambda(X_M)\},$$

implies that the competitive equilibrium premium $p = p_{ce}$ is given by

$$p_{ce} = \frac{E\{I(X)u'_\lambda(X_M)\}}{E\{u'_\lambda(X_M)\}}. \quad (36)$$

As a competitive equilibrium may seem a bit artificial in the present situation, let us instead determine the core. Here the typical core element is given by $(Eu(w + p - I(X)), Eu_1(w_1 - p - X + I(X)))$, where the parameter p is constrained by the individual rationality requirements, and the indemnity function I satisfies (35). Let us determine the relevant interval of p -values: The largest premium p_a that the insured will accept is given by

$$Eu_1(w_1 - p_a - X + I_{p_a}(X)) = Eu_1(w_1 - X),$$

a premium that could result if the insurer is a monopolist. Here $I_{p_a}(x)$ is the indemnity function satisfying (35) for $p = p_a$. The smallest premium p_b that could result in this situation is given by

$$Eu(w + p_b - I_{p_b}(X)) = u(w).$$

One situation where this premium could result is the case with several identical insurers and many identical customers. Then the price p_{ce} would not be stable if $p_{ce} > p_b$, as one insurer could attract all the customers by offering insurance at a slightly smaller price. Other insurers could then repeat this until the premium p_b was reached, and further reductions would not be rational as it would lead to a loss (in expected utility).

Between these two prices the price p_{ce} must lie, i.e., if p_{ce} exists, then $p_{ce} \in [p_b, p_a]$. Let us now illustrate this by an example.

Example 8. Consider the case where $u_1(x) = 1 - e^{-ax}$ and $u(x) = 1 - e^{-bx}$, where a and b are the absolute risk aversion parameters of the insurance customer and the insurer respectively. In this case we can solve the differential equation (35), and the solution is $I(x) = \frac{a}{a+b}x$. Let $p_{af} = E(I(X))$ be the “actuarially fair” premium, which may be of interest for comparisons. Then the prices p_b, p_a, p_{ce} and p_{af} are given by

$$p_a = \frac{1}{a} \ln \left(\frac{E(e^{aX})}{E(e^{\frac{ab}{a+b}X})} \right), \quad p_b = \frac{1}{b} \ln \left(E(e^{\frac{ab}{a+b}X}) \right)$$

and

$$p_{ce} = \frac{a}{a+b} \frac{E(Xe^{X/A})}{E(e^{X/A})}, \quad p_{af} = E\{I(X)\} = \frac{a}{a+b} E\{X\},$$

where $A = 1/a + 1/b$.

Assume now that X is exponentially distributed with parameter θ . Then the expected utilities of the relevant contracts are well defined if $\theta > \max\{a, \frac{ab}{a+b}\}$. In this case we get that the above prices can be written

$$p_a = \frac{1}{a} \ln \left(\frac{1 - \frac{ab}{a+b} \frac{1}{\theta}}{1 - \frac{a}{\theta}} \right), \quad p_b = -\frac{1}{b} \ln \left(1 - \frac{ab}{a+b} \frac{1}{\theta} \right)$$

and

$$p_{ce} = \frac{a}{(a+b)(\theta - 1/A)}, \quad p_{af} = \frac{a}{(a+b)\theta},$$

Here we notice that the indifference premiums p_a and p_b both exist if $\theta > \max\{a, \frac{ab}{a+b}\}$, i.e., they exist if the model is well defined. The premium p_{ce} exists when $\theta > 1/A$, which also holds if $\theta > \frac{ab}{a+b}$ since $1/A = \frac{ab}{a+b}$. Thus whenever the core is well defined, so is the price p_{ce} , but note that this is peculiar for our example, and may not be the case in general.

Let us illustrate by a numerical example. First we choose $\theta = 1$, $a = 1/2$ and $b = 1/8$. Here $A = 10$ and the model is well defined. We obtain that $p_a = 1.176$, $p_b = 0.843$, $p_{ce} = 8/9 = 0.889$, and $p_{af} = 4/5 = 0.80$, i.e.,

$$p_{ce} = 8/9 \in [0.843, 1.176] = [p_b, p_a].$$

Notice that the actuarially fair premium p_{af} is not in the core.

Next, consider the case when the insurer is risk neutral. Letting $b \rightarrow 0$ we obtain that full insurance is optimal, so $I(x) = x$ for all $x \geq 0$, $p_b = p_{af} = E(I(X)) = E(X)$, and

$$p_a = -\frac{1}{a} \ln \left(1 - \frac{a}{\theta} \right).$$

The core is $[1, 1.386]$. Here the actuarially fair premium ($= 1$) just made it to the core, and because the insurer is risk averse, he may be willing to pay more than $p_{af} = 1$. Notice that the competitive equilibrium premium $p_{ce} = E(X) = 1$ as well. \square

Finally we consider the case with constant relative risk aversion.

Example 9. We consider the case of power utility where all the agents have utility functions of the form $u_i(x) = (x^{1-a} - 1)/(1 - a)$ for $x > 0$, where the $a < 1$, and $(1 - a)$ is the coefficient of relative risk aversion. Here the agents are different because the initial portfolios X_i are. From Example 3 we deduce that the characteristic function of the game is

$$Eu_\lambda(X_S) = E \left(\left(\left(\sum_{i \in S} \lambda_i^{1/a} \right)^a X_S^{1-a} - \sum_{i \in S} \lambda_i \right) / (1 - a) \right) \quad \text{for any } S \subset \mathcal{I}.$$

Let $\sum_{i \in \mathcal{I}} \lambda_i^{1/a} = k^{1/a}$, where k is a normalization constant. Then individual rationality gives

$$\lambda_i \geq k \left(\frac{EX_i^{1-a}}{EX_{\mathcal{I}}^{1-a}} \right)^{a/(1-a)}.$$

For $I = 3$ social stability can be written

$$\lambda_1 \geq k(a_{1,2} + a_{1,3} - a_{2,3})^a,$$

$$\lambda_2 \geq k(a_{1,2} + a_{2,3} - a_{1,3})^a,$$

$$\lambda_3 \geq k(a_{1,3} + a_{2,3} - a_{1,3})^a,$$

where

$$a_{i,j} = \left(\frac{E(X_i + X_j)^{1-a}}{E(X_{\mathcal{I}})^{1-a}} \right)^{1/(1-a)} \quad \text{for } (i,j) \in \{(1,2), (1,3), (2,3)\}.$$

Finally using (30) we have that the investor weights λ_i must also satisfy

$$2k \geq (\lambda_1^{1/a} + \lambda_2^{1/a})^a a_{1,2}^{1/(1-a)} + (\lambda_1^{1/a} + \lambda_3^{1/a})^a a_{1,3}^{1/(1-a)} + (\lambda_2^{1/a} + \lambda_3^{1/a})^a a_{2,3}^{1/(1-a)}.$$

By the results of Example 3 the core allocations $Y = (Y_1, Y_2, \dots, Y_I)$ are then given by

$$Y_i = \frac{\lambda_i^{1/a}}{k^{1/a}} X_{\mathcal{I}}, \quad \text{for all } i \in \mathcal{I},$$

and $\lambda_i \in \Lambda$ defined by the 7 inequalities above. \square .

12 Efficient Allocation of Risk: The case of a Stock Market

12.1 Introduction

Much of the foregoing theory of risk allocation can of course be directly applied to a stock market. The principal difference from the risk allocation model we have considered so far, is that only linear risk sharing is then allowed among

the given risks. In certain situations this may also be optimal, but by and large this type of risk sharing can not be Pareto optimal. Still, it is quite plausible that a competitive equilibrium may exist.

In order to improve the risk sharing between the agents, derivative assets may be introduced. If we want to complete a model by introducing new securities, we should make sure that the resulting model really becomes complete, otherwise the situation may not improve very much, demonstrated by Hart (1975), who even found examples where the welfare of the agents went down.

Consider the following model. We are given I individuals having preferences of period one consumption represented by expected utility, where the utility indices are given by u_i , $i \in \mathcal{I}$. There are N securities, where Z_n is the pay-off at time 1 of security n , $n = 1, 2, \dots, N$.

We suppose individual i is initially endowed with shares of the different securities, so his initial, random endowment is

$$X_i = \sum_{n=1}^N \bar{\theta}_n^{(i)} Z_n,$$

where $\bar{\theta}_n^{(i)}$ is the proportion of firm n held by individual i . In other words, the total supply of a security is one share, and the number of shares held by an individual can be interpreted as the proportion of the total supply held. Denote by p_n the price of the security n , $n = 1, \dots, N$, where $p = (p_1, p_2, \dots, p_N)$.

An *equilibrium* for the economy $[(u_i, X_i), Z]$ is a collection $(\theta^1, \theta^2, \dots, \theta^I; p)$ such that given the security prices p , for each individual i , θ^i solves

$$\sup_{\theta} E u_i(Y_i) \tag{37}$$

subject to

$$Y_i = \sum_{n=1}^N \theta_n^{(i)} Z_n \quad \text{and} \quad \sum_{n=1}^N \theta_n^{(i)} p_n \leq \sum_{n=1}^N \bar{\theta}_n^{(i)} p_n, \tag{38}$$

and markets clear:

$$\sum_{i=1}^I Y_i = \sum_{i=1}^I X_i = \sum_{n=1}^N Z_n. \tag{39}$$

Denote by $M = \text{span}(Z_1, \dots, Z_N) := \{\sum_{n=1}^N \theta_n Z_n, \text{ for } \sum_{n=1}^N \theta_n \leq 1\}$ the set of all possible portfolio payoffs. We call M the *marketed subspace* of $L^2(\Omega, \mathcal{F}, P)$, where $\mathcal{F} = \mathcal{F}^Z := \sigma\{Z_1, Z_2, \dots, Z_I\}$ (all the null sets are included). The markets are *complete* if $M = L^2$ and are otherwise *incomplete*.

Here we remark that a common alternative formulation of this model starts out with initial endowments X_i measured in units of the consumption good, but there are no outstanding shares, so that the clearing condition is $\sum_{i=1}^I \theta_n^{(i)} = 0$ for all n . In this case we would have $\mathcal{F} = \mathcal{F}^X$. More generally we could let the initial endowments consist of shares and other types of wealth, in which case $\mathcal{F} = \mathcal{F}^{X,Z}$.

12.2 Arrow securities and complete markets

If there is a finite number of states, so that $\Omega = \{\omega_1, \omega_2, \dots, \omega_S\}$, let us denote the $N \times S$ payout matrix of the stocks by \mathbf{Z} , where

$$\mathbf{Z} = \begin{pmatrix} z_{1,\omega_1} & z_{1,\omega_2} & \cdots & z_{1,\omega_S} \\ z_{2,\omega_1} & z_{2,\omega_2} & \cdots & z_{2,\omega_S} \\ \vdots & \vdots & \ddots & \vdots \\ z_{N,\omega_1} & z_{N,\omega_2} & \cdots & z_{N,\omega_S} \end{pmatrix}$$

and z_{n,ω_s} is the payout of common stock n in state ω_s . If $N = S$ and \mathbf{Z} is nonsingular, then markets are complete. It is sufficient to show that Arrow securities can be constructed by forming portfolios of common stocks. Since \mathbf{Z} is nonsingular we can define

$$\theta^{(\omega_s)} = e^{(\omega_s)} \mathbf{Z}^{-1}$$

where $e^{(\omega_s)} = (0, 0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the s -th place. Then $\theta^{(\omega_s)} \mathbf{Z} = e^{(\omega_s)}$ by construction. The portfolio $\theta^{(\omega_s)}$ tells us how many shares of each common stock to hold in order to create an Arrow security that pays “one unit of account” in state ω_s . It is obvious that as long as \mathbf{Z} is nonsingular, we can do this for each $\omega_s \in \Omega$. Hence a complete set of Arrow securities can be constructed, and then we know that the market structure is complete.

Markets can not be complete if the random payoffs Z have continuous distributions, or we have a infinite and countable number of states, cases that interest us. In the finite case, the market can not be complete if the rank of \mathbf{Z} is strictly less than S , the number of states. Consider such a case and allow individuals to create call and put options on portfolios of common stocks.

Example 10. Suppose

$$\mathbf{Z} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 3 & 2 \end{pmatrix}$$

The payoff of the market portfolio is $(3, 4, 5)$. Let $c_M(k)$ denote the price at date 0 of a European call option on the market portfolio expiring at date 1 with an exercise price k . The payoffs for $c_M(3)$ and $c_M(4)$ are $(0, 1, 2)$ and $(0, 0, 1)$. Putting these payoffs together with the market portfolio, we have the payoff structure

$$\begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 2 & 1 \end{pmatrix}$$

which is a nonsingular matrix. Arrow securities can then be constructed by forming portfolios of the market portfolio and the two call options, so this market structure is complete. \square

This example demonstrates a situation where options can play an allocative role, and thus be welfare improving. More generally one can show the following: In an economy where options can freely be created on portfolios of common stocks, the market is Arrow complete if and only if there exists a portfolio of common stocks whose payoffs are different in each state, or whose payoffs separate.

12.3 Some general pricing principles

We now consider some general pricing principles. Let there be a stock market in a single good, single period economy. Agents have von Neumann-Morgenstern strictly concave and strictly increasing utility functions. Returning to the problem (37), we substitute the first constraint into the objective function and form the Lagrangian of each individual's optimization problem:

$$\mathcal{L}_i(\theta) = E \left\{ u_i \left(\sum_{n=1}^N \theta_n^{(i)} Z_n \right) - \alpha_i \left(\sum_{n=1}^N p_n (\theta_n^{(i)} - \bar{\theta}_n^{(i)}) \right) \right\}.$$

The first order conditions are

$$\frac{\partial \mathcal{L}_i(\theta)}{\partial \theta_n^{(i)}} = E(u'_i(Y_i) Z_n) - \alpha_i p_n = 0,$$

implying that

$$p_n = \frac{1}{\alpha_i} E(u'_i(Y_i) Z_n), \quad n = 0, 1, \dots, N.$$

Defining $R_n = Z_n/p_n$, the return of asset n , we have that for each $i \in \mathcal{I}$

$$\frac{1}{\alpha_i} E(u'_i(Y_i)(R_n - R_m)) = 0, \quad \forall n, m,$$

or, by the definition of covariance,

$$\frac{1}{\alpha_i} E(u'_i(Y_i)) E(R_n - R_m) + \frac{1}{\alpha_i} \text{cov}(u'_i(Y_i), R_n - R_m) = 0 \quad \forall n, m, \quad (40)$$

hold for each $i \in \mathcal{I}$.

Suppose there exists a riskless asset, the 0-th asset, that promises to pay one unit of the consumption good at date 1 in all states $\omega \in \Omega$. This asset is assumed to be in zero net supply. Thus

$$p_0 = \frac{1}{\alpha_i} E(u'_i(Y_i) \cdot 1) := \frac{1}{R_0} := \frac{1}{1 + r_f} \quad \text{for all } i \in \mathcal{I},$$

where r_f denotes the risk-free interest rate. Combining this with equations (40) gives

$$\frac{1}{1 + r_f} E(R_n - R_m) + \frac{1}{\alpha_i} \text{cov}(u'_i(Y_i), R_n - R_m) = 0 \quad \forall n, m, \quad (41)$$

for all $i \in \mathcal{I}$. Set $m = 0$ in this relationship. Then (41) becomes

$$E(R_n) - (1 + r_f) = -(1 + r_f) \text{cov} \left(\frac{u'_i(Y_i)}{\alpha_i}, R_n \right), \quad \forall n, \quad (42)$$

saying that the risk premium of any asset in equilibrium is proportional to the covariance between the return of the asset and the normalized, marginal utility of the equilibrium allocation Y_i for any i of the individuals. This latter quantity one may conjecture to be equal on M across all the individuals in equilibrium. We shall look into this conjecture below, but first we may utilize the relation (42) to derive the capital asset pricing model.

12.4 CAPM derived under multinormality

The results of the previous section can now be utilized to derive the standard CAPM. Two avenues could be chosen: One is to assume that all the individuals possess quadratic utility functions. This we do not find plausible in financial economics, where the utility is taken over final consumption, which in a one period model equals final wealth. It is highly unlikely to have a satiation point when it comes to wealth.

The other is to assume that returns of common stocks are multinormally distributed. Fama (1976) in his book “Foundations of Finance” has repeatedly tested out this hypothesis on US stocks, and found the assumption acceptable under certain conditions. This assumption is frequently employed in theoretical models in finance, such as in the Black and Scholes model, but is frequently refuted in empirical studies. For the moment, let us nevertheless assume that R is multivariate normal, and thus that Z is multivariate normal, since the prices p of the common stocks are all constants at time 0. Using Stein’s lemma, from (42) we get that

$$E(R_n) - (1 + r_f) = -(1 + r_f)E\left(\frac{u_i''(Y_i)}{\alpha_i}\right) \text{cov}(R_n, Y_i), \quad \forall n, i. \quad (43)$$

Let $Z_M := \sum_{n=1}^N Z_n$ and $p_M := \sum_{n=1}^N p_n$ and consider the weights $w_n := p_n/p_M$ for $n = 1, 2, \dots, N$. Clearly $\sum_{n=1}^N w_n = 1$. By the definition of return, $R_M := Z_M/p_M$ signifies the return on the market portfolio, and it follows that this can be written $R_M = \sum_{n=1}^N w_n R_n$, i.e., R_M is the return on the value-weighted market portfolio. Multiplying (43) by w_n and summing over the stocks n we get

$$E(R_M) - (1 + r_f) = -(1 + r_f)E\left(\frac{u_i''(Y_i)}{\alpha_i}\right) \text{cov}(R_M, Y_i), \quad \forall i.$$

Rearranging this equation, summing over the individuals i , and noticing that $\text{cov}(R_M, Z_M) = \text{var}(R_M)/p_M$, we obtain using the market clearing condition (39)

$$(E(R_M) - (1 + r_f)) \sum_{i \in \mathcal{I}} \frac{\alpha_i}{E u_i''(Y_i)} = -(1 + r_f) \frac{\text{var}(R_M)}{p_M}. \quad (44)$$

Returning to equation (43), rearranging and summing over the individuals, using again the market clearing condition (39), we get

$$(E(R_n) - (1 + r_f)) \sum_{i \in \mathcal{I}} \frac{\alpha_i}{E u_i''(Y_i)} = -(1 + r_f) \frac{\text{cov}(R_n, R_M)}{p_M}. \quad (45)$$

Finally, we substitute the term $\sum_{i \in \mathcal{I}} \frac{\alpha_i}{E u_i''(Y_i)}$ from equation (44) into equation (45), and the result is:

$$E(R_n) - (1 + r_f) = \frac{\text{cov}(R_n, R_M)}{\text{var}(R_M)} (E(R_M) - (1 + r_f)), \quad \forall n. \quad (46)$$

The risk premium of any of the given common stocks, $(E(R_n) - (1 + r_f))$, is proportional to the corresponding risk premium of the market, $(E(R_M) -$

$(1 + r_f)$), where the constant of proportionality $\beta_n := \text{cov}(R_n, R_M) / \text{var}(R_M)$ is called the stock's beta. This is the traditional version of the CAPM due to Mossin, Lintner and Sharpe. Note that we needed no completeness assumption for this relationship to hold.

Let $R_\theta = \sum_{n=1}^N \theta_n R_n$ be the return on any portfolio of common stocks, where the portfolio weights satisfy $\sum_{n=1}^N \theta_n = 1$. Then, from the above it is trivial to see that

$$E(R_\theta) - (1 + r_f) = \beta_\theta (E(R_M) - (1 + r_f)), \quad (47)$$

where $\beta_\theta := \text{cov}(R_\theta, R_M) / \text{var}(R_M)$ is the portfolio's beta. Since only portfolio formation can be made in this market, we here see a difference between this version and the corresponding insurance version presented earlier.

12.4.1 Existence of equilibrium

The problem of existence of equilibrium is, perhaps surprisingly, only dealt with fairly recently (Nielsen (1987, 1988, 1990a,b), Allingham (1991), Dana (1999)). Instead of assuming multinormality as we did in the above, a common assumption in this literature is that the preferences of the investors only depend on the mean and the variance, in other words, if $Z \in M$, then a utility function $u_i : M \rightarrow R$ is mean variance if there exists $U_i : R \times R \rightarrow R$ s.t.,

$$u_i(Z) = U_i(E(Z), \text{var}(Z)) \quad \text{for all } Z \in M.$$

The function U_i is assumed strictly concave and C^2 , increasing in its first argument and decreasing in the second.

Consider the following result (Dana (1999)):

Theorem 7 *Assume that $E(X_i) > 0$ for every $i = 1, 2, \dots, I$ and Z_M is a non-trivial random variable (i.e., not equal to a constant a.s.). Then there exists an equilibrium.*

When utilities are linear in mean and variance, we talk about quadratic utility, i.e., $U_i(x, y) = x - a_i y$, $a_i > 0$ for every i . If this is the case, equilibrium both exists and is unique. In the above it was assumed that utilities were strictly concave, so quadratic utility only fits into the above framework as a limiting case.

Let us recall one definition of risk aversion: A preference relation \succeq on a subset M of L^2 is called risk averse if $X \succeq X + Y$ for any $X \in M$ and non-zero Y in L^2 satisfying $X + Y \in M$ and $E(Y | X) = 0$. This means that an agent is risk averse if the addition of a random prospect that has no incremental effect on expected value is undesirable.

A related concept is the following: A preference relation \succeq on a subset M of L^2 is variance averse if $X \succeq X + Y$ whenever X and $X + Y$ are in M and $EY = \text{cov}(X, Y) = 0$. This means that an increase in variance is disliked if it does not affect expected value. In this case quadratic utility is a special case of a variance averse preference relation.

Suppose that the vector space M has a Hamel basis of jointly normally distributed random variables. If \succeq is a risk averse preference relation on M , it follows that \succeq is variance averse. In verifying this, you may notice that if X

and Y are bivariate normally distributed, then $E(XY) = EX = 0$ implies that $E(Y | X) = 0$.

In these two examples variance aversion applies because the agent's preferences are given only in terms of means and variances of an asset, and for a given mean, more variance is worse. However, nothing in the definition of variance aversion requires that preferences depend only on mean and variance.

12.5 Incomplete models and allocation efficiency

In this section we present some financial models that are not complete, but still an equilibrium exists and the optimal allocations are Pareto optimal. First we recall some stylized facts.

The principle of no-arbitrage was introduced shortly in the standard model, where it was the motivation behind a linear pricing functional. In the reinsurance model we then relied on the assumption of arbitrary contract formation. We use the following notation. Let X be any random variable. Then by $X > 0$ a.s. we mean that $P[X \geq 0] = 1$ and the event $\{\omega : X(\omega) > 0\}$ has strictly positive probability. In the present setting, by an *arbitrage* we mean a portfolio θ with $p \cdot \theta \leq 0$ and $\theta \cdot Z > 0$ a.s., or $p \cdot \theta < 0$ and $\theta \cdot Z \geq 0$ a.s. Then we have the following version of "The Fundamental Theorem of Asset Pricing": *There is no arbitrage if and only if there exists a state-price deflator*. This means that there exists a strictly positive random variable $\xi \in L^2$, i.e., $P[\xi > 0] = 1$, such that the market price $p_\theta := \sum_{n=1}^N p_n \theta_n$ of any portfolio θ can be written

$$p_\theta = \sum_{n=1}^N \theta_n E(\xi \cdot Z_n).$$

The proof of this theorem can be found in standard texts, such as e.g., Duffie (1996), and relies on the separating hyperplane theorem for cones. The following result is also useful: *There exists a solution to at least one of the optimization problems (37) of the individuals if and only if there is no arbitrage* (see Ross (1976)).

Remark: The conditions on the utility functional may be relaxed considerably for this result to hold. Consider a strictly increasing utility function $U: L^2 \rightarrow R$. If there is a solution to (37) for at least one such U , then there is no arbitrage. Conversely, if this U is also continuous, then absence of arbitrage implies that there exists a solution to the problem (37) for this U . Note that the utility function U we use is $U(X) = Eu(X)$, i.e., a von Neumann-Morgenstern expected utility function.

Clearly, the no-arbitrage condition is a weaker requirement than the existence of a competitive equilibrium, so if an equilibrium exists, there can be no arbitrage.

Now, consider a model where an equilibrium exists, so that there is no arbitrage, and hence there is a strictly positive state-price deflator $\xi \in L^2$. Recall the optimization problem of Theorem 3:

$$Eu_\lambda(Z_M) = \sup_{(Y_1, \dots, Y_I)} \sum_{i=1}^I \lambda_i Eu_i(Y_i) \quad \text{subject to} \quad \sum_{i=1}^I Y_i \leq Z_M,$$

where $Y_i \in L^2$, $i \in \mathcal{I}$. For u_i concave and increasing for all i , we know that the solution to this problem characterizes the Pareto optimal allocations. Consider the following problem:

$$E\tilde{u}_\lambda(Z_M) := \sup_{(Y_1, \dots, Y_I)} \sum_{i=1}^I \lambda_i E u_i(Y_i) \quad \text{subject to} \quad \sum_{i=1}^I Y_i \leq Z_M, \quad (48)$$

where $Y_i \in M$, $i \in \mathcal{I}$. In the situation where a competitive equilibrium exists, we can proceed along the same lines as in Theorem 4: The first order conditions are

$$E\{(\tilde{u}'_\lambda(Z_M) - \alpha\xi)Z\} = 0 \quad \text{for all} \quad Z \in M,$$

where $\alpha > 0$ is a Lagrange multiplier. This gives rise to the pricing rule

$$\pi(Z) = \frac{1}{\alpha} E(\tilde{u}'_\lambda(Z_M) \cdot Z) = E(\xi \cdot Z) \quad \text{for all} \quad Z \in M.$$

Similarly, for the problem in (37) the first order conditions can be written

$$E\{(u'_i(Y_i) - \alpha_i \xi)Z\} = 0 \quad \text{for all} \quad Z \in M, \quad i = 1, 2, \dots, I,$$

where Y_i are the optimal portfolios in M for agent i , $i = 1, 2, \dots, I$, giving rise to the market value

$$\pi(Z) = \frac{1}{\alpha_i} E(Y_i \cdot Z) = E(\xi \cdot Z) \quad \text{for any} \quad Z \in M.$$

Let us use the notation

$$\tilde{\xi} = \frac{\tilde{u}'_\lambda(Z_M)}{\alpha}, \quad \xi_i = \frac{u'_i(Y_i)}{\alpha_i}, \quad i = 1, 2, \dots, I.$$

Since M is a closed, linear subspace of the Hilbert space L^2 , if $M \neq L^2$ then the model is incomplete. In this case there exists an X in L^2 , $X \neq 0$, such that $E(X \cdot Z) = 0$ for all $Z \in M$. We use the notation $X \perp Z$ to signify $E(X \cdot Z) = 0$, and say that X is orthogonal to Z . Also let M^\perp be the set of all X in L^2 which are orthogonal to all elements Z in M . There exists a unique pair of linear mappings T and Q such that T maps L^2 into M , Q maps L^2 into M^\perp , and

$$X = TX + QX$$

for all $X \in L^2$. The orthogonal projection TX of X in M is the unique point in M closest (in L^2 -norm) to X . If $X \in M$ then $TX = X$, $QX = 0$; if $X \in M^\perp$, then $TX = 0$, $QX = X$.

Using this notation, from the above first order conditions we have that

$$(\tilde{\xi} - \xi) \perp M \quad \text{and} \quad (\xi_i - \xi) \perp M, \quad i = 1, 2, \dots, I.$$

In other words $(\tilde{\xi} - \xi) \in M^\perp$ and $(\xi_i - \xi) \in M^\perp$ for all i and accordingly $T(\tilde{\xi} - \xi) = 0$ and $T(\xi_i - \xi) = 0$ for all i , so the orthogonal projections of ξ , $\tilde{\xi}$ and ξ_i , $i = 1, 2, \dots, I$ on the marketed subspace M are all the same, i.e.,

$$T\xi = T\tilde{\xi} = T\xi_i, \quad i = 1, 2, \dots, I. \quad (49)$$

The conditions $T\xi = T\xi_i$ for all i correspond to the necessary conditions $\xi = \xi_i$ for all i in Theorem 1 of an equilibrium, when trade in all of L^2 is unrestricted, and similarly the condition $T\xi = T\tilde{\xi}$ corresponds to the necessary condition $\xi = \tilde{\xi}$ in Theorem 4 of the corresponding unrestricted, representative agent equilibrium.

If an equilibrium exists and $M = L^2$, then $\xi = \tilde{\xi}$ and the equilibrium allocations Y_1, \dots, Y_I are Pareto optimal. In this situation contingent claims in zero net supply would not have any allocational effects, in other words, such financial instruments would not be welfare improving.

If $M \neq L^2$ the market is incomplete, and two situations can arise:

(a) $E\tilde{u}_\lambda(Z_M) = Eu_\lambda(Z_M)$ or (b) $E\tilde{u}_\lambda(Z_M) < Eu_\lambda(Z_M)$.

In situation (b) the equilibrium allocation is not Pareto optimal, which is likely to be the typical case. Welfare could hence be improved by allowing trade in non-linear financial instruments (in zero net supply). One interesting issue would be to design the minimum set of derivatives required in order to complete the model.

In situation (a) the “welfare function” $E\tilde{u}_\lambda(Z_M)$ is equal to its maximal value, the value it would obtain if trade in all of L^2 was permitted (or possible). By Theorem 3 the equilibrium allocation is then Pareto optimal. Thus, even if the market is incomplete, there is no loss of welfare in restricting attention to the marketed subspace M . If this is the case we call the market *allocationally efficient*. Here we face the same situation as for a complete market: Contingent claims in zero net supply would not have any allocational effects, i.e., would not improve welfare at large (see e.g., Rubinstein (1974), Wilson (1968)).

Let us present a few examples of situation (a). In the first example the individuals have constant absolute risk aversions.

Example 11. Consider the case of negative exponential utility functions, with marginal utilities $u'_i(z) = e^{-z/a_i}$, $i \in \mathcal{I}$, where a_i^{-1} is the *absolute risk aversion* of agent i , or a_i is the corresponding *risk tolerance*. We assume that the payouts of the stocks Z_i are continuously distributed random variables, so that the market is incomplete, and let us assume that an unconstrained equilibrium exists in L^2 . We know from Example 1 that the equilibrium allocations are given by

$$Y_i = \frac{a_i}{A} Z_M + b_i, \quad \text{where} \quad b_i = a_i \ln \lambda_i - a_i \frac{K}{A}, \quad i \in \mathcal{I}.$$

where $\lambda_i = \alpha_i^{-1}$ are the agent weights in the representative agent utility function, the reciprocals of the Lagrangian multiplier α_i of agent i 's individual optimization problem, and where the constants K and A are given by

$$K = \sum_{i=1}^I a_i \ln \lambda_i, \quad A = \sum_{i=1}^I a_i.$$

The constants b_i represented the zero-sum side-payments in the reinsurance application, i.e., $\sum_{i \in \mathcal{I}} b_i = 0$.

The question is now whether these allocations can also result in the marketed subspace $M \subset L^2$. Consider the case where a riskless asset exists, the zeroth security. Then we may write

$$Y_i = \sum_{n=0}^N \theta_n^{(i)} Z_n = b_i \cdot 1 + \sum_{n=1}^N \frac{a_i}{A} Z_n$$

Thus, if individual i puts the same weight a_i/A on each of the common stocks $n = 1, 2, \dots, N$ and invests $\theta_0^{(i)} = b_i$ in the riskless security, he will obtain his unconstrained Pareto optimal equilibrium allocation Y_i . Notice that the more risk tolerant an individual is, the more he holds of each of the risky assets. In order for this to be possible he may borrow or lend the riskfree asset. If, say, a more risk tolerant investor has a low initial endowment X_i , he will finance his optimal portfolio by borrowing, whereas a more risk averse investor will hold less of the risky assets and more of the riskless, i.e., he may be a lender, at least if he is initially well endowed. In equilibrium this just adds up, since $\sum_{i \in \mathcal{I}} \theta_0^{(i)} = \sum_{i \in \mathcal{I}} b_i = 0$.

We notice that the individuals hold varying fractions of the market portfolio Z_M and the riskless asset in equilibrium, called *two fund separation*. \square

In the above example, even if the model is incomplete, the individuals obtain their Pareto optimal allocations by an exchange of common stocks only, so long as riskfree borrowing and lending is unrestricted. We notice that this could lead a more risk tolerant, poorly endowed investor to assume a rather risky position (despite the fact that he is of course risk averse as well).

In the next example we consider the case of constant relative risk aversion. Here it turns out that risk tolerant and poorly endowed individuals may not engage in quite so risky positions as in the previous example, and they will do just fine without a riskfree asset:

Example 12. Here we consider the case of power utility, where $u_i(x) = (x^{1-a} - 1)/(1-a)$ for $x > 0$, $a \neq 1$, $u_i(x) = \ln(x)$ if $a = 1$. The parameter $a > 0$ is the *relative risk aversion* of the agents, here assumed equal for all the individuals. The investors are not equal because their initial endowments X_i may be different. Again we consider continuous distributions so the model is incomplete, and we assume an unconstrained equilibrium exists in L^2 . Then we know from Example 3 that the unconstrained equilibrium allocations are given by

$$Y_i = \frac{\lambda_i^{1/a}}{\sum_{j \in \mathcal{I}} \lambda_j^{1/a}} Z_M \quad a.s. \quad \text{for all } i.$$

where again $\lambda_i = 1/\alpha_i$, and the investor weights λ_i are determined by the budget constraints, implying that

$$\lambda_i = k \left(\frac{E(X_i Z_M^{-a})}{E(Z_M^{1-a})} \right)^a, \quad i \in \mathcal{I},$$

or, λ_i is determined modulo the proportionality constant $k = (\sum_{j \in \mathcal{I}} \lambda_j^{1/a})^a$ for each i . The question again is whether these Pareto optimal equilibrium allocations can be obtained in $M \subset L^2$. Also now the answer is yes. Here agent i may choose the portfolio weights $\theta_n^{(i)}$ such that

$$Y_i = \sum_{n=1}^N \theta_n^{(i)} Z_n = \sum_{n=1}^N \frac{\lambda_i^{1/a}}{\sum_{j \in \mathcal{I}} \lambda_j^{1/a}} Z_n,$$

which means that

$$\theta_n^{(i)} = \frac{\lambda_i^{1/a}}{\sum_{j \in \mathcal{I}} \lambda_j^{1/a}}, \quad n = 1, 2, \dots, N, \quad \theta_0^{(i)} = 0, \quad i \in \mathcal{I}.$$

We see that this equilibrium can be obtained in a market for common stocks only, where riskfree lending or borrowing is not necessary.¹¹ Again the individuals choose the same percentage of each of the stocks, but this time the percentage is a positive linear functional of the initial endowment X_i of each individual i , meaning that someone with a high initial endowment will quite naturally hold more stocks in equilibrium than someone with a lower endowment.

Here we notice that each individual holds a fraction of the market portfolio Z_M in equilibrium. \square

We round off this section with a few comments on pricing principles in general. Suppose there is no arbitrage. Then there exists a state-price deflator $\xi \in L^2$ such that any $X \in L^2$ has market price

$$\pi(X) = E(\xi \cdot X).$$

If there exists an equilibrium in L^2 , we can characterize the state-price deflator as $\xi = u'_\lambda(Z_M)$. If the model is not complete and there exists an equilibrium in the marketed subspace M , we know that $\xi = \tilde{u}'_\lambda(Z_M)$ on M . In this case

$$\pi(X) = E(TX \cdot \tilde{u}'_\lambda(Z_M)) + E(QX \cdot Q\xi).$$

If $X \in M$, then $X = TX$ and $QX = 0$ so the last term in the above pricing formula disappears. Under this pricing rule, in case (a), if a new financial asset in zero net supply is introduced for trade, the original equilibrium in M will not be upset, and no individual will demand this asset. In case (b) the introduction of new financial instruments may change the equilibrium. Consider e.g., the polar case where the resulting market becomes complete. Then we know that the final equilibrium allocations must have changed, since the equilibrium allocations are now Pareto optimal unlike the original allocations. Some agents will hold other assets than those in the original stock market economy, and pricing is now under the first rule above, i.e., ξ on M has changed from $\tilde{u}'_\lambda(Z_M)$ to $u'_\lambda(Z_M)$.

When it comes to existence of equilibrium in L^2 we have already discussed this issue in section 6.1. Existence of equilibrium in M we touched upon in section 12.4 for a special set of preferences. For more general preferences most results in the literature are only treating the finite dimensional case (Hart (1974), Werner (1987), Dana and Van (1999)), which is a little outside our focus of interest. It would be of interest to consider the existence issue in the above infinite dimensional setting.

The idea of restricting attention to the core, instead of requiring a full fledged equilibrium, can be carried out in this financial model along the lines of section 11, the details being left to the reader.

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¹¹There could of course still be a riskfree asset, if say $Z_1 = 1$ a.s.

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