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Annuity factors, duration, and convexity:

Insights from a financial engineering perspective

By

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Abstract

This paper applies a unified and integrative financial engineering perspective to key derived concepts in traditional fixed income analysis, with the purpose of enhancing conceptual insights and motivating computational applications. The emphasis on annuity factors and their impact on duration and convexity differs from the focus prevailing in related discussions. By decomposing the cashflow streams of a coupon bond into different, specific, and clearly defined portfolios of component bonds with known duration and convexity measures, equivalent but appearently different expressions for the coupon bond's duration and convexity are obtained as particular weighted averages. One such convexity formula closely corresponds to Babcock's (1985) formula for duration. The Fabozzi (1993) shortcut duration formula does not immediately carry over to convexity, but the required modifications are derived. The interrelationships between various durations, convexities, and annuity factors or transformations thereof are also exhibited. Throughout the paper the results are illustrated numerically, for a particular coupon bond discussed elsewhere in the literature.

Keywords: annuity factors, duration, convexity, closed-form solutions, decomposition, Babcock's formula, Fabozzi's shortcut.

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Introduction

Traditional fixed income analysis relates the current market price of a standard coupon bond to basic terms such as the bond's face value, its coupon rate, its time to maturity, and its inferred yield to maturity. This paper applies a unified and integrative financial engineering perspective to key derived concepts in traditional fixed income analysis, such as annuity factors, (Macaulay) duration, modified duration, and convexity.

Annuity factors simplify valuation of level cash flows, such as the coupon payments of bonds. They also enter into measures for interest rate risk, as shown in this paper.

Duration, modified duration, and convexity reflect the bond's exposition to interest rate risk and are used as tools for risk management purposes. In particular, these measures can be used for approximation of a bond's sensitivity to changes in the term structure of interest rates.

Financial engineering views a complex financial asset as being constructed from a set of more simple component financial instruments. The properties of the complex financial asset are developed from more easily obtainable properties of its components.

The contributions of the paper may be briefly summarized as follows: The emphasis on annuity factors and their impact on duration and convexity differs from the focus prevailing in related discussions. By decomposing the cashflow streams of a coupon bond into different, specific and clearly defined portfolios of component bonds with known duration and convexity measures, equivalent but appearently different expressions for the coupon bond's duration and convexity are obtained as particular weighted averages. One such convexity formula closely corresponds to Babcock's (1985) formula for duration. The Fabozzi (1993) shortcut duration formula does not immediately carry over to convexity, but the required modifications are derived. The interrelationships between various durations, convexities, and annuity factors or transformations thereof are also exhibited. Throughout the paper the results are illustrated numerically, for a particular coupon bond discussed elsewhere in the literature.

In contrast, casual observations suggest that analysts eager to examine the bond price sensitivity to yield changes, are left with the choice between the following alternative procedures for obtaining the required values of auxiliary parameters: (i) Plug into available alternative closed-form formulas for computing duration and/or convexity. There are several complex formulas that look quite differently, even when differing notation has been accounted for[1]. Generally, the economic content of such formulas is not very transparent, even term by term, and some published formulas contain typographical errors which may go unspotted[2]. (ii) Use existing functions in either a standard spreadsheet application such as EXCEL, or leave the computations to specialized software[3]. Neither alternative offers much insight, neither in the economics nor in the computational aspects. (iii) Use "brute force" to do extensive period by period computations according to summation formulas for duration and convexity, presumably by help of a self prepared spreadsheet[4].

To simplify the exposition in the main text, various background materials, general expressions, and detailed derivations are relegated to appendices at the back. Appendix A reviews properties of the annuity factors. Appendix B summarizes the basic general relationsships between price, duration, and convexity for arbitrary bonds. Appendix C derives the duration and convexity expressions for par bonds and for annuity bonds. Appendix D shows how portfolio duration and portfolio convexity can be written as general weighted averages of the components' measures, assuming a common yield to maturity. Appendix E reconciles the Babcock, Fabozzi, and decomposition approaches. Appendix F discusses the applicability in case of a non-flat term structure.

[1] See Babcock (1985), Blake and Orszag (1996), Brooks and Livingston (1989, 1992), Hasager and Jensen (1990), Nawalka and Lacey (1988, 1991), Nawalkha, Lacey, and Schneeweis (1990), and Smith (1998), as well as textbooks like Fabozzi (1993), Jorion and Khoury (1996), and Taggart (1996).
[2] E.g., see expression (10) in Nawalka and Lacey (1988), expression (12) in Nawalka and Lacey (1991), expression (5) in Brooks and Livingston (1992), and expression (15') in Kane and Kane (1995).
[3] EXCEL offers many financial functions, including DURATION and MDURATION for Macaulay duration and modified duration, respectively. Both functions are slightly tricky to implement, as they require a DATE serial number for both settlement and maturity. EXCEL does not provide any CONVEXITY function. The annuity factor has to be computed by means of the PV function.
[4] See equations (B2), (B7), and (B8) of appendix B.

Coupon bonds and annuity factors

Let P denote the current price[5] of an arbitrary bond. For a standard coupon bond, denote the bond's face value by F, its coupon rate by c, its time to maturity by n, and its inferred yield to maturity by y. There are n periodic coupon payments, each in the constant amount cF, taking place at the points in time $t = 1,2,...,n$, as well as repayment of the principal F at time n. The n-period annuity factor at an interest rate y is defined as the sum of the first n consecutive t-period discount factors using that constant interest rate, and may be written as

(1) $$A_{n,y} = \left(\frac{1}{y}\right)\left[1-(1+y)^{-n}\right]$$

Properties of the annuity factors are discussed in appendix A.

Use of the annuity factor alleviates the need for period by period discounting of the constant coupon payments cF. The price at an ex-dividend date of a n-period coupon bond with a coupon rate c may be written as

(2) $$P = cFA_{n,y} + F(1+y)^{-n}.$$

The n-period discount factor can be expressed in terms of the corresponding n-period annuity factor as

(3) $$(1+y)^{-n} = 1 - yA_{n,y}$$

Substitution of (3) into (2) gives the alternative form for the present value of a coupon bond,

(4) $$P = F[1-(y-c)A_{n,y}]$$

which confirms that a coupon bond is priced at par if and only if its coupon rate equals the yield to maturity, i.e., $c = y$.

Appendix B defines the concepts duration, modified duration, and convexity. It also provides explicit computational formulas for the case of a bond with an arbitrary cashflow. Furthermore, it shows how measures between coupon dates relate to the corresponding (hypothetical) measures at the last preceeding scheduled payment date of the bond.

[5] Market price, imputed price, or gross present value, depending on the context.

By switching from a bond with an arbitrary cashflow to a coupon bond, the tedious period by period calculations for duration and convexity are avoided, just as it was avoided for price computations. The coupon bond's duration is

$$(5) \qquad D = (1+y)\frac{A_{n,y} + (y-c)\dfrac{\partial A_{n,y}}{\partial y}}{1-(y-c)A_{n,y}}$$

from combining the coupon bond price equation (4) with the general duration equation (B3). The modified duration of a coupon bond is therefore

$$(6) \qquad D^* = \frac{A_{n,y} + (y-c)\dfrac{\partial A_{n,y}}{\partial y}}{1-(y-c)A_{n,y}}$$

by calling on (B4).

Similarly, from combining the coupon bond price relation (4) with the general convexity equation (B5), the convexity of a coupon bond is

$$(7) \qquad C = -\frac{2\dfrac{\partial A_{n,y}}{\partial y} + (y-c)\dfrac{\partial^2 A_{n,y}}{\partial y^2}}{1-(y-c)A_{n,y}}$$

Observe that the first and second derivatives of the annuity factor with respect to the yield enter the expressions for duration, modified duration, and convexity. By differentiating the annuity factor, say, according to (1),

$$(8) \qquad \frac{\partial A_{n,y}}{\partial y} = \left(\frac{1}{y}\right)\left[n(1+y)^{-(n+1)} - A_{n,y}\right]$$

Another differentiation shows that

$$(9) \qquad \frac{\partial^2 A_{n,y}}{\partial y^2} = \left(\frac{2}{y^2}\right)\left[A_{n,y} - n(1+y)^{-(n+1)}\right] - \left(\frac{1}{y}\right)n(n+1)(1+y)^{-(n+2)}$$

The set of equations (5) through (9) gives semi closed-form equations for the duration, modified duration, and convexity of a coupon bond. The reader will recognize the primitive parameters y, c, and n, as well as the discount factor $(1+y)^{-n}$ and the annuity factor $A_{n,y}$. If so desired, the discount factor may be replaced by its annuity factor representation (3), see equations (A9) and (A10) of appendix A for alternative formulations of the first and second partial derivative of the annuity factor. Duration and convexity measures are invariant to scale, consistent with the face value parameter F dropping out.

For illustrative purposes throughout the paper, consider the 8% five-year bond, with semiannual coupon payments, and priced to give an annual nominal yield of 10%, as introduced by Fabozzi (1993) and also used by Blake and Orszag (1996). With the current notation, and switching to periods of length one half-year, the time to maturity n =10, the periodic coupon rate c =0.04, the periodic yield is y =0.05, and the arbitrary face value is set at F =100. Exhibit 1 shows the characteristics of this coupon bond.

Exhibit 1

A Standard Coupon Bond

Parameter/function	Equation	Notation	Value
Input parameters:			
Coupon rate		c	0,04
Time to maturity		n	10
Yield to maturity		y	0,05
Face value		F	100,00
Computed auxiliary parameters:			
Annuity factor	(1)	$A_{n,y}$	7,72173
Partial derivative of annuity factor	(8)	$\frac{\partial A_{n,y}}{\partial y}$	-37,49884
Second partial derivative of annuity factor	(9)	$\frac{\partial^2 A_{n,y}}{\partial y^2}$	274,91131
Computed properties:			
Price	(4)	P	92,27827
Duration	(5)	D	8,35959
Modified duration	(6)	$D*$	7,96151
Convexity	(7)	C	78,29424

Special bonds and annuity factors

As a preamble to the following sections, this section will review properties of four classes of bond cashflow patterns, with special emphasis on formulations involving annuity factors[6]. Exhibit 2 provides the appropriate formulas in an annuity factor context. Exhibit 3 shows the corresponding numerical values, based on the same parameter values as in Exhibit 1.

A n-period unit zero coupon bond pays one dollar at time n, and is referred to by the subscript $zero$. Its price equals the n-period discount factor corresponding to the yield, and may be reformulated in terms of the annuity factor by using (3). For a single payment, the period by period summations drop out of (B2), (B7), and (B8), resulting in well-known expressions for duration, modified duration, and convexity.

A unit perpetuity pays one dollar at times $t = 1,2,....,\infty$, and is referred to by the subscript $perp$. Letting the horizon n go to infinity, the second factor of (1) approaches one, and the perpetual annuity factor is simply $A_{\infty,y} = \left(\frac{1}{y}\right)$, which equals the perpetuity's price for a fixed yield y. The general expressions (B3) through (B5) then give its characteristics.

A unit n-period par bond, priced at one dollar, with a face value of 1 and some constant coupon payment at times $t = 1,2,....,n$, is referred to by the subscript par. Recall from (4) that a coupon bond is priced at par if and only if $y = c$. Hence, the derivative of the annuity factor with respect to the yield drops out of the expressions (5) for duration and (6) for modified duration. The latter equation shows that the modified duration of a par bond is simply the corresponding annuity factor. Furthermore, the second derivative of the annuity factor with respect to the yield drops out of the convexity expression (7). See Appendix C for additional details.

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[6] Duration and convexity of special cashflows are also available from Nawalka and Lacey (1991) and from Brooks and Livingston (1992). This paper will provide reformulations in term of annuity factor, see Exhibit 2. An appendix of Hasager and Jensen (1990) contains an encyclopedic listing of semi closed-form expressions for duration and convexity of a wide variety of bond types.

Exhibit 2

General properties of special bonds

(i) *A n-period unit zero coupon bond* :

Price $P_{zero} = (1+y)^{-n} = 1 - yA_{n,y}$

Duration $D_{zero} = n$

Modified duration $D^*_{zero} = \dfrac{n}{1+y}$

Convexity $C_{zero} = \dfrac{n(n+1)}{(1+y)^2}$

(ii) *A unit perpetuity bond* :

Price $P_{perp} = \dfrac{1}{y} = A_{\infty,y}$

Duration $D_{perp} = \dfrac{1+y}{y} = (1+y)A_{\infty,y}$

Modified duration $D^*_{perp} = \dfrac{1}{y} = A_{\infty,y}$

Convexity $C_{perp} = \dfrac{2}{y^2} = 2A^2_{\infty,y}$

(iii) *A n-period unit par bond* :

Price $P_{par} = 1$

Duration $D_{par} = (1+y)A_{n,y}$

Modified duration $D^*_{par} = A_{n,y}$

Convexity $C_{par} = \dfrac{2}{y}\left[A_{n,y} - n(1+y)^{-(n+1)}\right] = \dfrac{2}{y}\left[A_{n,y} - \dfrac{n}{1+y}(1 - yA_{n,y})\right]$

(iv) *A n-period unit annuity bond* :

Price $P_{ann} = A_{n,y}$

Duration $D_{ann} = \dfrac{1+y}{y} - n\left[\dfrac{1}{yA_{n,y}} - 1\right] = \dfrac{1+y}{y} - \dfrac{n}{(1+y)^n - 1}$

Modified duration $D^*_{ann} = \dfrac{1}{y} - \dfrac{n}{1+y}\left[\dfrac{1}{yA_{n,y}} - 1\right] = \dfrac{1}{y} - \dfrac{n}{1+y}\dfrac{1}{(1+y)^n - 1}$

Convexity

$$C_{ann} = \frac{2}{y^2} - \left[\frac{2}{y}\frac{n}{1+y} + \frac{n(n+1)}{(1+y)^2}\right]\left[\frac{1}{yA_{n,y}} - 1\right] = \frac{2}{y^2} - \left[\frac{2}{y} + \frac{n+1}{1+y}\right]\frac{n}{1+y}\frac{1}{(1+y)^n - 1}$$

Exhibit 3

Special Bonds: Numerical Values of Properties

Bond	Price P	Duration D	Modified duration $D*$	Con-vexity C
A n-period unit zero coupon bond	0,61391	10,00000	9,52381	99,77324
A unit perpetuity bond	20,00000	21,00000	20,00000	800,00000
A n-period unit par bond	1,00000	8,10782	7,72173	74,99768
A n-period unit annuity bond	7,72173	5,09909	4,85627	35,60227

Input parameters as in Exhibit 1

A n-period unit annuity bond pays one dollar at times $t = 1,2,....,n$, and is referred to by the subscript ann. The n-period unit annuity bond has a price equal to the corresponding annuity factor. Its duration then follows from substituting the partial derivative of the annuity factor (8) into the general duration equation (B3). Furthermore, the discount function may be eliminated by (3). Similarly, substituting the partial second derivative of the annuity factor (9) into the general convexity equation (B5) provides the annuity bond convexity. More details are found in Appendix C.

These measures for the different assets are interrelated. As an example, the duration of the annuity bond is

$$D_{ann} = D_{perp} - D_{zero} \cdot [D*_{perp} \cdot \left(\frac{1}{D*_{par}} \right) - 1]$$

The convexity of the annuity bond is

$$C_{ann} = C_{perp} - [2D*_{perp} \cdot D*_{zero} + C_{zero}][D*_{perp} \cdot \left(\frac{1}{D*_{par}} \right) - 1]$$

These relations are qualitatively rather similar, but going from duration to convexity involves the additional subtraction of the product $[2D*_{perp} \cdot D*_{zero}][D*_{perp} \cdot \left(\frac{1}{D*_{par}} \right) - 1]$.

Duration and convexity of portfolios of two bonds

For a portfolio consisting of multiple securities with identical yields, the portfolio duration, modified duration, and convexity are linear combinations of the components' duration, modified duration, and convexity, with the components' proportion of total portfolio value as weights (see appendix D). Provided only properties of the portfolio itself are of ultimate interest, the procedure below (and in appendix D) may nevertheless be applied, even when the components have non-identical yields to maturity. The components are then treated as if they have the common yield y, resulting in hypothetical prices (or gross present values) P_j. These hypothetical component prices then form the basis for computing hypothetical values of the components' duration, modified duration, and convexity measures.

In particular, when there are just two components indexed by 1 and 2, whereas the portfolio remains unindexed, the portfolio price is $P = P_1 + P_2$, the portfolio duration $D = D_1 \frac{P_1}{P} + D_2 \frac{P_2}{P}$, the portfolio modified duration $D^* = D^*_1 \frac{P_1}{P} + D^*_2 \frac{P_2}{P}$, and the portfolio convexity $C = C_1 \frac{P_1}{P} + C_2 \frac{P_2}{P}$.

The trick is just exactly how to decompose a non-par coupon bond into different pairs of securities. The two securities must together provide a combined cashflow element of cF at points in time $t = 1,2,...,n-1$ and also the cashflow element of $(1+c)F$ at time $t = n$, and their combined present (or market) value must equal P. Three interesting pairs of decomposed bonds are examined next.

First decomposition: Par bond and zero coupon bond

For the first component, consider a n-period bond with a face value of $\frac{c}{y}F$ and a periodic coupon payment amount cF, priced to yield y. As the yield on the principal equals the

Exhibit 4

First Decomposition: Par Bond and Zero Coupon Bond

	Equation	Notation	Par bond	Zero coupon bond	Coupon bond
Cashflow, periods *t = 1,2,,,n-1*		x_t	4,00000		4,00000
Cashflow, period *t = n*		x_n	84,00000	20,00000	104,00000
Price		P	80,00000	12,27827	92,27827
Price proportion (weight)		P_j / P	0,86694	0,13306	1,00000
Duration	(10)	D	8,10782	10,00000	8,35959
Modified duration	(11)	$D*$	7,72173	9,52381	7,96151
Convexity	(12)	C	74,99768	99,77324	78,29424

Input parameters as in Exhibit 1

coupon payment, $y\frac{c}{y}F = cF$, the first bond is a par bond with price $P_1 = \frac{c}{y}FP_{par} = \frac{c}{y}F$, duration $D_1 = D_{par} = (1+y)A_{n,y}$, modified duration $D*_1 = D*_{par} = A_{n,y}$, and convexity

$$C_1 = C_{par} = \frac{2}{y}[A_{n,y} - n(1+y)^{-(n+1)}] = \frac{2}{y}\left[A_{n,y} - \frac{n}{1+y}(1-yA_{n,y})\right].$$

To preserve the cashflow restriction at the horizon, the second component must be a zero coupon bond, having a face value of $\left(1-\frac{c}{y}\right)F$, with price

$P_2 = \left(1-\frac{c}{y}\right)FP_{zero} = \left(1-\frac{c}{y}\right)F(1+y)^{-n}$, duration $D_2 = D_{zero} = n$, modified duration $D*_2 = D*_{zero} = \frac{n}{1+y}$, and convexity $C_2 = C_{zero} = \frac{n(n+1)}{(1+y)^2}$.

The price proportion of the par bond can be written as $\frac{P_1}{P} = \frac{c}{y}\frac{F}{P}$. By value additivity, the price proportion of the zero bond is $\frac{P_2}{P} = 1 - \frac{c}{y}\frac{F}{P}$.

The weighted average representations for the original coupon bond are thus:

(10) $$D = (1+y)A_{n,y}\frac{cF}{yP} + n\left(1 - \frac{cF}{yP}\right)$$

(11) $$D^* = A_{n,y}\frac{cF}{yP} + \frac{n}{1+y}\left(1 - \frac{cF}{yP}\right)$$

(12) $$C = \frac{2}{y}\left[A_{n,y} - \frac{n}{1+y}(1 - yA_{n,y})\right]\frac{cF}{yP} + \frac{n(n+1)}{(1+y)^2}\left(1 - \frac{cF}{yP}\right)$$

for duration, modified duration, and convexity, respectively.

Exhibit 4 illustrates the computations.

Second decomposition: Annuity bond and zero coupon bond

The first component now consists of a n-period annuity bond, corresponding to the level interest amount cashflow stream cF. This bond has market value $P_1 = cFP_{ann} = cFA_{n,y}$. Its duration is $D_1 = D_{ann} = \frac{1+y}{y} - n\left[\frac{1}{yA_{n,y}} - 1\right]$, modified duration $D^*_1 = D^*_{ann} = \frac{1}{y} - \frac{n}{1+y}\left[\frac{1}{yA_{n,y}} - 1\right]$, and convexity

$$C_1 = C_{ann} = \frac{2}{y^2} - \left[\frac{2}{y}\frac{n}{1+y} + \frac{n(n+1)}{(1+y)^2}\right]\left[\frac{1}{yA_{n,y}} - 1\right].$$

The n-periodic zero coupon bond now has face value F, with a corresponding price $P_2 = FP_{zero} = F(1+y)^{-n} = P - cFA_{n,y}$. Its duration, modified duration, and convexity are as under the first decomposition .

Exhibit 5

Second Decomposition: Annuity Bond and Zero Coupon Bond

	Equation	Notation	Annuity bond	Zero coupon bond	Coupon bond
Cashflow, periods *t = 1,2,,,n-1*		x_t	4,00000		4,00000
Cashflow, period *t = n*		x_n	4,00000	100,00000	104,00000
Price		P	30,88694	61,39133	92,27827
Price proportion (weight)		P_j / P	0,33472	0,66528	1,00000
Duration	(13)	D	5,09909	10,00000	8,35959
Modified duration	(14)	D^*	4,85627	9,52381	7,96151
Convexity	(15)	C	35,60227	99,77324	78,29424

Input parameters as in Exhibit 1

Substitution into the appropriate weighted averages now provides the following expressions, for, respectively, duration, modified duration, and convexity:

(13) $$D = \left\{ \frac{1+y}{y} - n\left[\frac{1}{yA_{n,y}} - 1 \right] \right\} \frac{cFA_{n,y}}{P} + n\left(1 - \frac{cFA_{n,y}}{P} \right)$$

(14) $$D^* = \left\{ \frac{1}{y} - \frac{n}{1+y}\left[\frac{1}{yA_{n,y}} - 1 \right] \right\} \frac{cFA_{n,y}}{P} + \frac{n}{1+y}\left(1 - \frac{cFA_{n,y}}{P} \right)$$

(15) $$C = \left\{ \frac{2}{y^2} - \left[\frac{2}{y}\frac{n}{1+y} + \frac{n(n+1)}{(1+y)^2} \right]\left[\frac{1}{yA_{n,y}} - 1 \right] \right\} \frac{cFA_{n,y}}{P} + \frac{n(n+1)}{(1+y)^2}\left(1 - \frac{cFA_{n,y}}{P} \right)$$

Exhibit 5 provides numerical illustrations.

Exhibit 6

Third Decomposition: Annuity Bond and Par Bond

	Equation	Notation	Annuity bond	Par bond	Coupon bond
Cashflow, periods $t = 1,2,,,n\text{-}1$		x_t	-1,00000	5,00000	4,00000
Cashflow, period $t = n$		x_n	-1,00000	105,00000	104,00000
Price		P	-7,72173	100,00000	92,27827
Price proportion (weight)		P_j / P	-0,08368	1,08368	1,00000
Duration	(16)	D	5,09909	8,10782	8,35959
Modified duration	(17)	$D*$	4,85627	7,72173	7,96151
Convexity	(18)	C	35,60227	74,99768	78,29424

Input parameters as in Exhibit 1

Third decomposition: Annuity bond and par bond

The annuity bond now corresponds to the level cashflow stream $(c-y)F$ over the n periods. The current market price of this annuity bond is $P_1 = (c-y)FP_{ann} = (c-y)FA_{n,y}$. Its duration, modified duration, and convexity are listed under the second decomposition.

The par bond must catch up the remaining coupon payments yF as well as the principal F , and is therefore priced at $P_2 = FP_{par} = F = P-(c-y)FA_{n,y}$. Its duration, modified duration, and convexity can be found under the first decomposition.

Note that $P_1 = P - F$, such that the weight applicable to the annuity bond is $\left(1-\frac{F}{P}\right)$.

The weighted averages then imply the following semi closed-formed formulations for duration, modified duration, and convexity

$$(16) \qquad D = \left\{\frac{1+y}{y} - n\left[\frac{1}{yA_{n,y}}-1\right]\right\}\left(1-\frac{F}{P}\right) + (1+y)A_{n,y}\frac{F}{P}$$

$$(17) \qquad D* = \left\{\frac{1}{y} - \frac{n}{1+y}\left[\frac{1}{yA_{n,y}}-1\right]\right\}\left(1-\frac{F}{P}\right) + A_{n,y}\frac{F}{P}$$

$$(18)\quad C=\left\{\frac{2}{y^2}-\left[\frac{2}{y}\frac{n}{1+y}+\frac{n(n+1)}{(1+y)^2}\right]\left[\frac{1}{yA_{n,y}}-1\right]\right\}\left(1-\frac{F}{P}\right)+\left\{\frac{2}{y}\left[A_{n,y}-\frac{n}{1+y}(1-yA_{n,y})\right]\right\}\frac{F}{P}$$

respectively.

For the running numerical example, see Exhibit 6.

Extending Babcock's approach

Babcock (1985) showed that duration could be written as a weighted average of the annuity factor times $(1+y)$ and the number of periods n to maturity. The duration equation (10) from the first decomposition does in fact coincide with Babcock's formula. However, the expression was found in quite a different way, as Babcock based his derivation on Chua's (1984) closed-form duration expression. Babcock also interpreted the weight $\frac{cF}{yP}$ applied to the annuity factor as "the unusual but simple ratio of current yield to maturity yield", i.e., in the current notation the ratio of $\frac{cF}{P}$ to y, rather than as the price $\frac{cF}{y}$ of an appropriately decomposed par bond component relative to the price P of the whole coupon bond.

The convexity expression (12) from the first decomposition is the obvious extension of Babcock's duration formula, recognizing the suitably decomposed par bond and zero coupon bond weights as proper price proportions. This extension has probably not been reported elsewhere in the literature.

Extending Fabozzi's shortcut approach

Fabozzi (1993) has presented a shortcut formula for computing duration. In the current notation, his result is that duration

(19) $$D = \left[\frac{1+y}{y}\right]H + \left[1-\frac{c}{y}\right]n(1-H)$$

where Fabozzi defines H as the ratio of the present value of the coupon payments to the price of the coupon bond,

(20) $$H \equiv \frac{cFA_{n,y}}{P}$$

The term $\left[1-\frac{c}{y}\right] = \frac{y-c}{y}$ may be interpreted as a relative excess yield beyond the coupon rate.

The modified duration is then

(21) $$D^* = \frac{1}{y}H + \left[1-\frac{c}{y}\right]\frac{n}{1+y}(1-H)$$

At first glance, these formulas seem somewhat surprising. The duration equation (19) can be rewritten as

$$D = D_{perp}H + \left[1-\frac{c}{y}\right]D_{zero}(1-H)$$

and the modified duration equation (21) correspondingly. The first product on the RHS is the duration of a perpetuity multiplied by the price proportion of a finite n-period annuity. Also, it is not immediately obvious whether the relative excess yield term $\left[1-\frac{c}{y}\right]n$ has any duration interpretation. Whereas n by itself is a recognized duration, it would imply a second weight of $\left[1-\frac{c}{y}\right](1-H)$, and the two weights would not sum to unity.

Fortunately, though, Fabozzi's duration result may be reconciled with Babcock's formula as well as the three decomposition formulas derived earlier. Appendix E contains the derivations.

Still, it is far from obvious how to generalize Fabozzi's duration result into a corresponding convexity result as well. In particular, the first guess based on simply replacing duration by convexity in (19) is dead wrong, as $C \neq C_{perp} H + \left[1 - \frac{c}{y}\right] C_{zero}(1-H)$.

Exhibit 7

An Extended Fabozzi's Shortcut Approach

Parameter/function	Equation	Notation	Value
Input parameters:			
Coupon rate		c	0,04
Time to maturity		n	10
Yield to maturity		y	0,05
Face value		F	100,00
Computed auxiliary parameters:			
Annuity factor	(1)	$A_{n,y}$	7,72173
Price ratio of coupon payments to full bond	(20)	H	0,33472
Relative excess yield		$\left[1-\frac{c}{y}\right]$	0,20000
Computed properties:			
Price	(2)	P	92,27827
Duration	(19)	D	8,35959
Modified duration	(21)	$D*$	7,96151
Convexity	(22)	C	78,29424

Appendix E shows how a Fabozzi convexity result may be obtained from decomposition, providing the desired expression

(22) $$C = \frac{2}{y^2} H + \frac{n(n+1)}{(1+y)^2}\left[1 - \frac{c}{y}\right](1-H) - \frac{2}{y^2}\frac{n}{1+y} c(1-H)$$

This extended Fabozzi convexity expression appears to be a new result in the literature.

To show the interrelationships, write

$$C = C_{perp} \cdot H + C_{zero} \cdot \left[1 - \frac{c}{y}\right](1-H) - C_{perp} \cdot D^*_{zero} \cdot c(1-H)$$

Thus, a further product of four factors has to be deducted from the duration based guess, to obtain a Fabozzi like convexity expression.

Numerical illustrations are available in Exhibit 7.

Concluding remarks

This paper has presented a unified approach to bond valuation and risk assessment, building on and integrating key fixed income analysis concepts such as annuity factors, duration, and convexity. It has demonstrated that there are indeed "many roads leading to Rome": A set of semi closed-form equations, three particular decomposition procedures, and two new extensions to convexity of previous duration formulas by Babcock and by Fabozzi.

Appendix F discusses the question of whether the procedures of this paper are of interest only in case of a flat term structure. It shows how the concepts and computations are applicable under less restrictive assumptions, but extensions to immunization strategies are troublesome under non-flat term structures. Swithching to alternative duration and convexity measures, tailor-made for the assumed underlying stochastic term structure process, may be a suitable remedy[7].

An underlying premise for this paper is that a "back to basics" and "step by step" approach, consistent with financial engineering, is useful both for providing conceptual insights and motivating computational applications. It builds on the economic assumption of value additivity (even in the case of hypothetical valuations), combined with interpreting complex bonds as portfolios of simpler components with known properties. Needs for advanced mathematical tools are avoided through constructive applications of elementary calculus and simple algebraic manipulations. The running numerical examples should be helpful

[7] See Jorion and Khoury (1996, pp. 104-109) for discussions of alternative duration measures for additive, multiplicative, and log stochastic term structure processes.

supplements to the economic arguments and formal analyses. Numerous equations are handy for reference purposes, but they are more intended as derived economic results motivating applications rather than as ingredients in cookbook receipes for "black box" computations. Such advantages of a financial engineering perspective are expected to provide substantial value added, even when aspects of most issues discussed and results presented may be found scattered around in the finance literature.

Appendix A:
Properties of annuity factors

With a flat term structure of interest over the horizon ending at time n, the periodic yields y_t for different maturities t are equal, such that $y_t = y$ for $t = 1,2,...,n$. Alternatively, with a non-flat term structure, interpret y as the implied constant yield to maturity of an arbitrary financial asset with cashflow vector $x = (x_1, x_2,..., x_n)$, where x_t is the cashflow element at point in time $t = 1,2,...,n$. The implied yield to maturity is then implicitly defined through the condition $P = \sum_{i=1}^{n} x_t (1+y)^{-t} = \sum_{i=1}^{n} x_t (1+y_t)^{-t}$. In either case, the sum of the first n discount factors corresponding to the constant interest rate y is termed the n-period annuity factor $A_{n,y}$, that is,

(A1) $$A_{n,y} \equiv \sum_{t=1}^{n} (1+y)^{-t}$$

It is also referred to by various acronyms such as PVIFA(n,y), corresponding to Present Value of Interest Factor.

Using the formula for the sum of a convergent geometric series, the n-period annuity factor can be simplified as

(A2) $$A_{n,y} = \left(\frac{1}{y}\right)\left[1-(1+y)^{-n}\right]$$

which is stated as equation (1) in the text. An equivalent annuity factor formulation is

(A3) $$A_{n,y} = \frac{(1+y)^n - 1}{y(1+y)^n}.$$

Letting the horizon n go to infinity, the second factor of (A2) approaches one, and the perpetual annuity factor is simply

(A4) $$A_{\infty,y} = \left(\frac{1}{y}\right)$$

The first and second derivatives of the annuity factor with respect to the yield enter the expressions for duration and convexity. By differentiating, say, (A2), (A5) $$\frac{\partial A_{n,y}}{\partial y} = \left(\frac{1}{y}\right)\left[n(1+y)^{-(n+1)} - A_{n,y}\right]$$

which is equation (8) of the text. Another differentiation shows that

(A6) $$\frac{\partial^2 A_{n,y}}{\partial y^2} = \left(\frac{1}{y^2}\right)\left[2A_{n,y} - 2n(1+y)^{-(n+1)}\right] - \left(\frac{1}{y}\right)n(n+1)(1+y)^{-(n+2)}$$

listed as equation (9) in the text.

Useful terms for derivations and interpretations can by obtained from elementary manipulations of the equations defining the annuity factor. By multiplying (A2) on both sides by y,

(A7) $$yA_{n,y} = 1-(1+y)^{-n}$$

An alternative expression for the n-period discount factor with a constant yield is obtained from (A7) as

(A8) $$(1+y)^{-n} = 1 - yA_{n,y}$$

to be found as equation (3) in the text. By substitution of (A8) into (A5), the n-period discount factor may be replaced from the expression for the derivative of the annuity factor,

(A9) $$\frac{\partial A_{n,y}}{\partial y} = \frac{1}{y}\left[\frac{n}{1+y}(1-yA_{n,y}) - A_{n,y}\right]$$

from which the convexity C_{par} of a par bond will be obtained by multiplying by –2. A similar substitution of (A8) into (A6) removes the discount factor from the second derivative of the annuity factor, leading to

(A10) $$\frac{\partial^2 A_{n,y}}{\partial y^2} = \left(\frac{2}{y^2}\right)\left[A_{n,y} - \frac{n}{1+y}(1-A_{n,y})\right] - \left(\frac{1}{y}\right)\frac{n(n+1)}{(1+y)^2}(1-yA_{n,y})$$

The fractions $\frac{n}{1+y}$ and $\frac{n(n+1)}{(1+y)^2}$ will later be identified as, respectively, the modified duration D^*_{zero} and the convexity C_{zero} of a n-period zero coupon bond.

By taking the inverse of (A7) and subtracting 1,

(A11) $$\frac{1}{yA_{n,y}} - 1 = \frac{1}{(1+y)^n - 1}.$$

(A11) may be used in reformulations of the duration and convexity of an annuity bond.

Appendix B:
Some important basic relations for arbitrary bonds

Consider an arbitrary bond with cashflow vector $x = (x_1, x_2, ..., x_n)$, such that x_t is the cashflow element at point in time $t = 1,2,...,n$. Furthermore, let the term structure of interest be reflected by a required yield of y_t for a zero coupon bond of maturity $t = 1,2,...,n$. The bond is then priced at $P = \sum_{i=1}^{n} x_t (1 + y_t)^{-t}$. Macaulay (1938) defined duration as a weighted average of the times at which cashflows occur, with the proportion of the present value of a cashflow element relative to the price of the whole bond price as weights. It can be written as

(B1) $$D \equiv \frac{1}{P} \sum_{t=1}^{n} t \cdot x_t \cdot (1 + y_t)^{-t}$$

In terms of the bond's implied yield to maturity of y, the current bond price may be written as $P = \sum_{i=1}^{n} x_t (1 + y)^{-t}$. In practice, the Macaulay duration measure has been reinterpreted as

(B2) $$D \equiv \frac{1}{P} \sum_{t=1}^{n} t \cdot x_t \cdot (1 + y)^{-t}$$

using the constant yield to maturity y rather than a term structure of general form as in (B1).

An alternative approach, originating with Hicks (1939), defines the duration of a bond as its (negative) price elasticity with respect to the yield,

(B3) $$D \equiv -(1 + y) \frac{1}{P} \frac{\partial P}{\partial y}$$

By performing the differentiation, it is easily verified that (B2) and (B3) are equivalent definitions.

Often it is more convenient to work with the modified duration, defined by

(B4) $$D^* \equiv \frac{D}{1 + y}$$

such that $D^* = -\frac{1}{P} \frac{\partial P}{\partial y}$. Convexity is defined as

$$\text{(B5)} \quad C \equiv \frac{1}{P}\frac{\partial^2 P}{\partial y^2}$$

Suppose the bond yield changes instantaneously from its current value y to a new value $(y+\Delta y)$. A standard result in fixed income analysis is that the corresponding change in bond value in terms of modified duration and convexity, is approximately[8]

$$\text{(B6)} \quad \Delta P \approx -PD^*\Delta y + \frac{1}{2}PC(\Delta y)^2,$$

obtained by taking a Taylor expansion of the new price $P(y+\Delta y)$ around the current yield y, and substituting the definitions (B3) through (B5).

For an arbitrary cashflow stream it is well known and follows directly from (B5) that the convexity is

$$\text{(B7)} \quad C = \frac{1}{(1+y)^2}\frac{1}{P}\sum_{t=1}^{n} t(t+1)\cdot x_t \cdot (1+y)^{-t}$$

By (B2) and (B4), modified duration is

$$\text{(B8)} \quad D^* = \frac{1}{1+y}\frac{1}{P}\sum_{t=1}^{n} t\cdot x_t \cdot (1+y)^{-t}$$

With the common yield to maturity framework, the points in time $t = 1,2,...,n$ are equally spaced and may be interpreted as coupon dates, or more generally as the set of dates of the promised fixed payments. Without loss of generality, the computed price P, duration D, modified duration D^*, and convexity C all apply to the point in time t = 0, which for bonds with fixed payments intervals may be interpreted as the moment the bond goes ex coupon and trades without any accrued interest. Suppose to the contrary that the measures all apply at a point in time τ, with $0<\tau<1$, such that τ indicates the time (in fractions of a period) since the last bond payment. Let the subscript 0 indicate the corresponding time t = 0 hypothetical values, if the current yield to maturity had been constant since t = 0. The current "dirty" price, including accrued interest, then satisfies

[8] For the standard numerical example, price, modified duration, and convexity are listed in Exhibit 1. Suppose the yield changes instantaneously to the coupon rate, such that Δy =-0.01. The new price then equals the face value F =100, and the exact price change (subject to roundoff error) is $\Delta P = 7.72173$. The approximated price change from (B6) is $\Delta P = 7.70799$, of which 7.34675 is attributed to duration and 0.36124 is attributed to convexity.

(B9) $P_\tau = P_0 \cdot (1+y)^\tau$

where the hypothetical price P_0 is computed from the current yield y as of $t = \tau$ and may be different from the actual price prevailing at $t = 0$. By time indexing (B3) through (B5) for $t = 0$ and $t = \tau$, and using (B9), relationships between measures at different points in time are established, for duration

(B10) $D_\tau = D_0 - \tau$

for modified duration

(B11) $D^*_\tau = D^*_0 - \tau/(1+y)$

and for convexity

(B12) $C_\tau = C_0 - \frac{\tau}{(1+y)^2}\left[2D_0 + (1-\tau)\right]$

as demonstrated by Smith (1998).

Appendix C:
Duration and convexity of par bonds and of annuity bonds

As stated in the text, equation (4) demonstrates that a coupon bond is priced at par if and only if $y - c = 0$. When this latter condition is inserted into (5), the derivative term in the denominator drops out, whereas the numerator becomes zero. Hence, the par bond has duration

(C1) $D_{par} = (1+y)A_{n,y}$

Similarly, the condition $y - c = 0$ cancels the second partial derivative term in the denominator of (7), and the par bond's convexity is just minus twice the first derivative of the annuity factor with respect to the yield. Substitution from (8) and taking the minus sign inside the bracket term, provides

(C2) $C_{par} = \frac{2}{y}[A_{n,y} - n(1+y)^{-(n+1)}]$

Replacing the discount factor by its equivalent annuity factor representation (3), the par bond convexity may alternatively be formulated as

(C3) $$C_{par} = \frac{2}{y}\left[A_{n,y} - \frac{n}{1+y}(1 - yA_{n,y})\right]$$

For the unit annuity bond, with price $P_{par} = A_{n,y}$, from (B3) its duration is given by

(C4) $$D_{ann} = -(1+y)\frac{1}{A_{n,y}}\frac{\partial A_{n,y}}{\partial y}$$

Substitution from (8) and rearranging leads to $D_{ann} = \frac{1+y}{y} - n\frac{1}{yA_{n,y}}(1+y)^{-n}$. Replacing the discount factor by (3) and then multiplying gives duration as

(C5) $$D_{ann} = \frac{1+y}{y} - n\left[\frac{1}{yA_{n,y}} - 1\right]$$

A further substitution of the bracket term according to (A11) leads to the more familiar traditional duration expression

(C6) $$D_{ann} = \frac{1+y}{y} - \frac{n}{(1+y)^n - 1}$$

The annuity bond's convexity is, from (B5), computed as

(C7) $$C_{ann} = \frac{1}{A_{n,y}}\frac{\partial^2 A_{n,y}}{\partial y^2}$$

By direct substitution from (9), the annuity bond's convexity is

$$C_{ann} = \left(\frac{2}{y^2 A_{n,y}}\right)[A_{n.y} - n(1+y)^{-(n+1)}] - \left(\frac{1}{yA_{n,y}}\right)\frac{n(n+1)}{(1+y)^2}(1+y)^{-n}$$

Cancelling terms, replacing the discount factor by the annuity factor relation (3), and rearranging,

(C8) $$C_{ann} = \frac{2}{y^2} - \left[\frac{2}{y}\frac{n}{1+y} + \frac{n(n+1)}{(1+y)^2}\right]\left[\frac{1}{yA_{n,y}} - 1\right]$$

Using (A11) to replace the last bracketed term, the convexity of the annuity bond may alternatively be stated as

(C9) $$C_{ann} = \frac{2}{y^2} - \left[\frac{2}{y} + \frac{n+1}{1+y}\right]\frac{n}{1+y}\frac{1}{(1+y)^n - 1}$$

which is the more traditionally reported form.

Appendix D:
Duration and convexity as general value weighted averages

For $j = 1,2,...., J$, let x_j be a cashflow vector, with cashflow element x_{jt} at time $t = 1,2,....., n_j$, current market price P_j, yield to maturity y_j, duration D_j, modified duration $D*_j$, and convexity C_j. These cashflow streams may be interpreted as components of a cashflow portfolio, whose cashflow vector

(D1) $x = \sum_{j=1}^{J} x_j$

This portfolio has market price P, yield to maturity y, duration D, modified duration $D*$, and convexity C. By market value additivity, the portfolio's price equals the sum of the components' prices:

(D2) $P = \sum_{j=1}^{J} P_j$.

Suppose all components have a common yield, which therefore is also the yield of the portfolio, i.e.,

(D3) $y_j = y \quad \forall j = 1,..., J$

From (D2), and using the simple fact that the derivative of a sum equals the sum of the derivatives,

(D4) $$\frac{\partial P}{\partial y} = \sum_{j=1}^{j=J} \frac{\partial P_j}{\partial y}$$

On the RHS of (D4), divide and multiply with P_j. Then, divide through by P on both sides of (D4), to get

(D5) $$\frac{1}{P}\frac{\partial P}{\partial y} = \sum_{j=1}^{j=J} \frac{1}{P_j}\frac{\partial P_j}{\partial y}\frac{P_j}{P}$$

From (D5) combined with the general expressions (B3) and (B4) for duration and modified duration, respectively, we obtain the portfolio's duration as

(D6) $$D = \sum_{j=1}^{j=J} D_j \frac{P_j}{P}$$

and modified duration

(D7) $$D^* = \sum_{j=1}^{j=J} D^*_j \frac{P_j}{P}.$$

Similarly, from (D4),

(D8) $$\frac{\partial^2 P}{\partial y^2} = \sum_{j=1}^{j=J} \frac{\partial^2 P_j}{\partial y^2}$$

Dividing and multiplying with P_j on the RHS of (D8), and then dividing through by P on both sides of (D8), shows that

(D9) $$\frac{1}{P}\frac{\partial P}{\partial y} = \sum_{j=1}^{j=J} \frac{1}{P_j}\frac{\partial P_j}{\partial y}\frac{P_j}{P}$$

Substitution of (D9) into the general convexity expression (B5) provides

(D10) $$C = \sum_{j=1}^{j=J} C_j \frac{P_j}{P}$$

Thus, when the components have identical yields, then the portfolio's duration, modified duration, and convexity are all weighted averages of the corresponding terms for the components, with the components' value proportions as weights.

Appendix E:
Reconciliation with the Fabozzi approach

The consistency between the duration formulas (10) of Babcock and (19) of Fabozzi is easily demonstrated. Starting with Babcock's formula, repeated as

(E1) $$D = (1+y)A_{n,y}\frac{cF}{yP} + n\left(1 - \frac{cF}{yP}\right)$$

By a simple reorganization, $D = \frac{1+y}{y}\frac{cFA_{n,y}}{P} + n\left(1 - \frac{cF}{yP}\right)$. Denote the fraction $\frac{cFA_{n,y}}{P}$ as H, according to Fabozzi's definition (20). Furthermore, from the bond price equation (4), the ratio $\frac{F}{P}=\frac{P+F(y-c)A_{n,y}}{P}$, such that the product $\frac{c}{y}\frac{F}{P}=\frac{c}{y}+\frac{y-c}{y}\frac{cFA_{n,y}}{P}$. This latter expression simplifies to

(E2) $$\frac{cF}{yP}=\frac{c}{y}(1-H)+H$$

Substitutions back into (E1) and reorganization give

(E3) $$D=\frac{1+y}{y}H+n\left(1-\frac{c}{y}\right)(1-H)$$

This is the Fabozzi's shortcut duration equation (19).

Alternatively, start out from the second decomposition of the coupon bond into an annuity bond and a zero coupon bond. The duration is then given by (13), for convenience repeated below as

(E4) $$D=\left\{\frac{1+y}{y}-n\left[\frac{1}{yA_{n,y}}-1\right]\right\}\frac{cFA_{n,y}}{P}+n\left(1-\frac{cFA_{n,y}}{P}\right)$$

Using Fabozzi's definition (20) of the H-term, duration can be rewritten as

(E5) $$D=\frac{1+y}{y}H-n\frac{cF}{yP}+nH+n(1-H)$$

Fabozzi's duration result is then obtained by substituting (E2) and collecting terms.

There are also several alternatives for deriving the extended Fabozzi convexity result presented as (22). Keeping to the second decomposition, the convexity equation (15) is repeated as

(E6) $$C=\left\{\frac{2}{y^2}-\left[\frac{2}{y}\frac{n}{1+y}+\frac{n(n+1)}{(1+y)^2}\right]\left[\frac{1}{yA_{n,y}}-1\right]\right\}\frac{cFA_{n,y}}{P}+\frac{n(n+1)}{(1+y)^2}\left(1-\frac{cFA_{n,y}}{P}\right)$$

Taking the $\frac{cFA_{n,y}}{P}$-term inside the curly brackets, cancelling, and introducing the H-notation,

(E7) $$C=\frac{2}{y^2}H-\frac{2}{y}\frac{n}{1+y}\left[\frac{c}{y}\frac{F}{P}-H\right]-\frac{n(n+1)}{(1+y)^2}\left[\frac{c}{y}\frac{F}{P}-H\right]+\frac{n(n+1)}{(1+y)^2}(1-H)$$

Substitution of (E2) and rearranging provide

(E8) $$C = \frac{2}{y^2}H - \frac{2}{y}\frac{n}{1+y}\left[\frac{c}{y}(1-H) + H - H\right] - \frac{n(n+1)}{(1+y)^2}\left[\frac{c}{y}(1-H) + H - H - (1-H)\right]$$

By cancelling and further reorganizing,

(E9) $$C = \frac{2}{y^2}H + \frac{n(n+1)}{(1+y)^2}\left[1 - \frac{c}{y}\right](1-H) - \frac{2}{y^2}\frac{n}{1+y}c(1-H)$$

Hence, the extended Fabozzi convexity equation (22) follows from the second decomposition.

Appendix F:
Non-flat term structures

In the literature there appears to be substantial controversy and/or confusion as to whether Macaulay duration is only applicable with a flat term structure of interest, possibly subject to parallel shifts only. Jorion and Khoury (1996, p. 85) flatly state that "Macauley's *[sic]* duration assumes a flat yield curve, because each cash flow is discounted at the same rate". Dumas and Allaz (1996, p. 298) write about Macaulay's duration "As shown by [Ingersoll et al. 1978], this last definition is only a correct measurement of the impact of an infinitesimal variation in interest rates on the price of the security when all spot rates are equal and vary by the same amount in the event of a shock, which corresponds to the case of a flat term structure of rates subject to parallel movements." Actually, Ingersoll et al. (1978) are concerned with an arbitrary set of assets with fixed payments and phrase their results in terms of "all assets" or "any asset", rather than focusing on characteristics of a single, predetermined asset.

In contrast, as observed by Jensen (1998, p. 213), Macaulay duration and related modified duration and convexity measures do not by themselves require any assumptions at all about a flat term structure subject to parallel shifts, as long as the attention is restricted to the one particular asset for which the (implied) yield to maturity y applies. Whether or not there are other assets with the same (initial) yield, and how the yields on other asset change, are of no concern for the definitions (B2) through (B5), the sensitivity approximation (B6), and the alternative characterizations (B7) and (B8). Various different combinations of shifts and twists in the arbitrary term structure may cause the same change in the yield to maturity of the particular asset under examination, as noted by Smith (1998). Regardless of the underlying source of the yield change for that asset, its price sensitivity towards the yield change is approximated by (B6), with (modified) duration and convexity as important risk measures for that purpose and for that particular asset. It may be a challenge, though, to recognize how a particular change in a non-flat term structure of interest rate translates into a fixed numerical value of the change in the yield to maturity of the asset in question.

For a non-flat term structure of interest rates, the implied yield to maturity of a particular financial asset depends not only on the terms structure itself but also on the entire cashflow pattern of that asset. Note that the constant yield to maturity applicable to a particular financial asset does not necessarily provide a correct period by period valuation of the periodic cashflow elements of that asset. The asset as a whole is, however, by definition correctly priced using discounting at its constant yield. In the coupon bond context, with a non-flat term structure, the coupon payments as such will be incorrectly priced by the annuity factor based on the inferred constant yield. However, the pricing error of the coupon payments are exactly offset by the corresponding pricing error from incorrectly discounting the principal at the same constant yield. Thus, the annuity factor approach suggested in this paper does not require a flat yield structure.

Furthermore, decomposing an asset (such as a coupon bond) into components (such as an annuity and a zero coupon bond), the components' computed prices based on the composite asset's yield do not necessarily coincide with the prices consistent with the non-flat yield curve. Similarly, durations, modified durations, and convexity computed for the components based on the composite asset's yield, will not in general be identical to similar characteristics computed using each component's own individual yield to maturity. Fortunately, the decomposition procedures of this paper will still work. The "as if" computed fictitious and hypthetical measures based on a common yield, are fully applicable to the composite asset (or portfolio) under consideration.

However, assumtions about the form of the initial term structure as well as about the form of interest changes are indeed important for assessing the impact of interest changes across different assets. Immunization strategies are interesting cases of interest rate risk management. Unfortunately, Macaulay duration and convexity measures are of limited relevance for immunization purposes, unless in case of a flat initial term structure subject to parallel, infinitesimal, and instantaneous yield changes. Fisher-Weil or other more advanced duration and convexity concepts may then be called on in more complex situations, see, e.g., Christensen and Sørensen (1994). Still, a thorough understanding of the conceptual and computational foundations for the traditional Macaulay duration and convexity will be an important departure point for assessing and implementing more advanced interest rate risk management schemes.

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