Optimal Management of renewable resources: A General Feedback Approach

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Abstract

Analytical solutions for optimal exploitation of renewable capital stocks are derived as feedback rules for a quite general dynamic optimization problem. By feedback rules is meant that optimal exploitation is given as an explicit function of the capital stock. The value of analytical expressions for the optimal control will be appreciated by all who have tried to determine the separatrix solution numerically and experienced problems. The method described here can be applied to fisheries management, animal stock conservation and conservation of the environment in general. The operationality of the method outlined here is illustrated by two simple examples, one related to fisheries management and one related to pollution control.

INTRODUCTION

Fany, if not most, of the world's renewable resources are subject to excessive exploitation and should therefore be managed carefully. The reason for the excessive exploitation is the well-known tragedy of the commons (Scott, 1955; Hardin, 1968), and is often reinforced by unfortunate circumstances such as government's subsidies, etc. This is, for example, very often the case in the fisheries sector. This emphasizes the need for management based on realistic models that include both biological and economic aspects. Very often, however, such models are made either by biologists or by economists and therefore lack one of these aspects, biology or economy, but are quite sophisticated with respect to the other aspect. As an example can be mentioned bio-economic models based on intricate year-class models for population dynamics combined with very simple economics, that is constant prices and constant costs per unit harvest. Another example is pollution models where the decay of pollution is either constant or linear whereas in reality the decay may be a very complex process. It has been shown that the time path of optimal carbon taxes is strongly dependent upon the shape of the decay function (Sandal and Steinshamn, 1998)

The aim of this article is to develop a model that combines both economic and biological aspects in resource management. The model can in principle be applied to many different types of renewable resources, but it is here exemplified with fisheries management and pollution control. Both the objective function and the dynamic constraint may be *completely general* functions in the control variable as well as the state variable. The only requirement is that the model satisfies the usual conditions for existence of an optimum. The outcome of the model is an analytical expression for the optimal exploitation level of the resource given as a feedback rule. As such this article is a generalization of Sandal and Steinshamn (1997a). A feedback rule means that the optimal control is a function of the state variable. The value of analytical expressions for the optimal control will be appreciated by all who have tried to find such rules numerically and experienced problems. Feedback rules also represent truly adaptive management as the control variable changes immediately when new knowledge about the state variable is available.

The difference between the model presented here and the model presented in Sandal and Steinshamn (1997a) which was based on a quadratic welfare function, is that with a quadratic function the optimal feedback control can be solved explicitly whereas with a general model the optimal control will usually be given implicitly. A version of the quadratic model including a general stochastic process in the biological submodel can be found in Sandal and Steinshamn (1997b). The generality of the method is revealed in Sandal and Steinshamn (1998) where a similar approach is applied to the analysis of carbon taxes.

In the following the optimal feedback rule is derived with and without discounting, and the model is also illustrated by two examples to emphasize the operationality of this method.

MODEL

The problem analyzed in this article is a quite general dynamic optimization problem that can be formulated

$$\max \int_0^\infty e^{-\delta t}\Pi(x,y)dt$$

subject to

$$\dot{x} = f(x,y)$$

where x is the state variable, that is the size of the resource, and y is the control variable, that is the exploitation of the resource. Further, t denotes time, δ is a constant discount rate and dots are used to denote time derivatives. The net revenue or welfare function, Π, and the function, f, may both be completely general functions in x and y as long as they satisfy the conditions for existence of a unique solution. All variables are in current values unless otherwise is stated explicitly. The time horizon can, of course, be finite, but for the management of renewable resources it is natural to emphasize an infinite horizon.

The current value Hamiltonian for this problem is given by

$$\mathcal{H} = \mathcal{H}(x, y, m) = \Pi(x, y) + mf(x, y)$$

where m is the costate variable which in optimum is interpreted as the shadow value of the resource. The first-order conditions for the maximization problem are the usual

$$\begin{aligned} \mathcal{H}_y = 0, \quad \dot{x} = f(x, y) \\ \dot{m} = \delta m - \mathcal{H}_x, \quad \dot{\mathcal{H}} = \delta m \dot{x} \end{aligned} \tag{1}$$

The last of these equations follows from the three previous ones. The maximum condition, $\mathcal{H}_y = 0$, implies $\Pi_y + mf_y = 0$. From this follows that the optimal shadow value of the resource can be written as a function:

$$m = M(x, y) = -\frac{\Pi_y}{f_y}. \tag{2}$$

As this is a known function when we know Π and f, it can be used to eliminate the shadow value. After the shadow value, m, has been eliminated, we get a new function which is always equal to the Hamiltonian in value along an optimal trajectory but it is different as a function. This new function can be defined as

$$P(x, y) = \mathcal{H}(x, y, M(x, y)).$$

Remembering the usual interpretation of the Hamiltonian as the rate of increase of total assets, it is natural to name the new function the total economic rent for short (Clark, 1990).

An explicit feedback rule is searched for by setting $y = y(x)$. This inserted into the last of the equations in (1) yields a first-order differential equation that can be used to determine the feedback control. This equation is given by[1]

$$\frac{dP}{dx} = \delta \cdot M(x, y), \tag{3}$$

and this is the basic equation that will be used in the following in order to derive optimal feedback control rules. The analysis is divided in two parts. First the case without discounting is investigated, then the case with discounting.

The feedback rule without discounting

As has been shown in numerous studies, the positions of optimal paths are not very sensitive to changes in the discount rate as long as the discount rate is dominated by the maximum growth rate of the resource (Sandal and Steinshamn, 1997a; Mendelssohn, 1982). The present value of the welfare on the other hand is, of course, highly sensitive to changes in the discount rate. This is the main reason for emphasizing the case without discounting. Another reason is that discounting itself puts a strain on the resource as it implies less emphasis on the future relative to the present. For completeness, however, the case with discounting will also be analyzed.

Without discounting it is immediately seen from (3) that P is constant, $P = P_0$. This is the well-known result that without discounting the value of the Hamiltonian is constant, or, in other words, the total economic rent is preserved. This, combined with the expression for the costate variable given in (2), yields an operational feedback rule to determine optimal harvest when we have decided the constant P_0. The feedback rule, y as a function of x, is given implicitly by

$$\Pi(x, y) + M(x, y) f(x, y) = P_0.$$

[1]Note that $\frac{dP}{dx}$ refers to the total derivative, $\frac{dP}{dx} \equiv \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\frac{dy}{dx}$.

The only question that remains is how to determine the constant P_0. With an infinite time horizon P_0 should be equal to the maximum value of the sustainable economic rent defined as

$$S(x) = \Pi(x, y)|_{f(x,y)=0}$$

which is a function in x only, see Sandal and Steinshamn (1997a).

The optimal steady state is given by $x^* = \arg\max S(x)$, and the corresponding exploitation level can be found by solving $f(x^*, y) = 0$ with respect to y. In other words, $S'(x) = 0$ is a necessary condition for an interior solution with respect to the optimal steady state. The interpretation of this is the usual that a marginal change in sustainable revenue should equal the best alternative rate of return. With a zero discount rate the best alternative rate of return is zero, and hence the sustainable revenue ought to be maximized. This result is thoroughly discussed in Sandal and Steinshamn (1997c).

In the case of multiple solutions with respect to the optimal path, one should choose the one that has appropriate dynamic characteristics. In other words, choose y such that $\dot{x} > 0$ for $x < x^*$ and $\dot{x} < 0$ for $x > x^*$.

An interesting property of the S-curve is that it can be used to characterize the dynamic equilibrium points in the model. The stationary points on the S-curve, $S'(x) = 0$, corresponds to equilibrium points in the phase-space. A local minimum corresponds to a centre, a local maximum corresponds to a saddle point and an inflection point corresponds to a cusp. This property is quite useful as the S-curve represents a function in one variable only and therefore can be easily plotted. Hence all equilibrium points in the phase-space can be found by visual inspection of the S-curve no matter how complex the phase-space might be.

The feedback rule with discounting

Equation (3) is in general a highly non-linear ordinary differential equation (ODE). Therefore one can normally not expect to find a closed form solution. This does not pose a serious problem, however, because when we have the exact solution in the case of zero discounting, it is relatively easy to find very good approximative solutions using perturbation theory, see Nayfeh (1973). The approximation will in all practical applications be so good that the deviation by far is outweighed by measurement errors in the stock level, etc. We use a straightforward perturbation scheme in the control variable where the discount rate is used as perturbation parameter. More precisely, the perturbation parameter is the ratio between the discount rate and the maximum (intrinsic) growth rate of the stock. This requires that this ratio is relatively small, otherwise an alternative perturbation scheme must be applied. The perturbation scheme is given by

$$
\begin{aligned}
y(x) &= y_0(x) + \delta y_1(x) + O(\delta^2),\\
P(x,y) &= P(x, y_0(x)) + \delta \left[\frac{\partial}{\partial y} P(x, y_0(x)) \cdot y_1(x)\right] + O(\delta^2)\\
M(x,y) &= M(x, y_0(x)) + \delta \left[\frac{\partial}{\partial y} M(x, y_0(x)) \cdot y_1(x)\right] + O(\delta^2),
\end{aligned}
$$

which is then substituted into (3). This represents an asymptotic series as opposed to convergent series. An important difference between these is that in asymptotic series the optimal number of correction terms is small, often one or two, whereas convergent series become better the more correction terms are added. In this model it is for all practical purposes sufficient to include only one correction term. The first order correction term to the feedback rule is given by

$$
y_1(x) = \frac{\int_{x^*}^{x} M(x, y_0(x))dx}{\frac{\partial}{\partial y} P(x, y_0(x))}, \quad P(x, y_0(x)) = P_0,
$$

where P_0 is constant. Using only one correction term the optimal feedback rule with discounting is then

$$y(x) = y_0(x) + \delta y_1(x)$$

which is a fully operational expression for optimal exploitation for any stock level. Again the feedback rule with the appropriate dynamic characteristics should be chosen in the case of multiple solutions; that is, $\dot{x} > 0$ when x is below the optimal steady state and vice versa.

The optimal steady state, however, has changed in the case of discounting. This is now given by solving

$$S'(x) = -\delta \cdot \left. \frac{\Pi_y}{f_y} \right|_{f(x,y)=0} = \delta \left. M(x,y) \right|_{f(x,y)=0} \tag{4}$$

with respect to x. Note that $\left. \frac{\Pi_y}{f_y} \right|_{f(x,y)=0}$ is a function in x only. By dividing (4) by δ it can be seen that this has the usual interpretation that the present value of all future benefit (left-hand side) should equal the instantaneous benefit (right-hand side). It is readily seen that $S'(x) = 0$ is a special case of (4) when $\delta = 0$. A thorough analysis of this expression can be found in Sandal and Steinshamn (1997c).

NUMERICAL EXAMPLE

In this section the feedback rule is illustrated by two numerical examples, one based on fisheries management and one based on pollution control. The numbers chosen here are only meant for illustrative purposes and are not meant to represent any real resource management problem.

Fisheries management

In the first example some stylized facts pertaining to Atlantic cod may be recognized. The net revenue function is given by

$$\Pi = p(y)y - c(x,y)$$

where

$$p(y) = \underline{p} + (\overline{p} - \underline{p})e^{-ay} \quad \text{and} \quad c(x,y) = k\frac{y}{x}.$$

The parameters $\underline{p}$ and $\overline{p}$ in the demand function represent minimum price and maximum price respectively. These have been specified to $\underline{p} = 4$ and $\overline{p} = 10$, and the parameter $a = 0.001$. The parameter k in the cost function has been specified to $k = 10,000$. The dynamic constraint is given by

$$f(x,y) = g(x) - y$$

where g may be any function of x. In this example we use a Gompertz' function of the form

$$g(x) = rx\ln(\frac{K}{x})$$

where $r = 0.25$ and the carrying capacity $K = 8,000$. From (2) we have

$$m(x,h) = -\frac{\Pi_y}{f_y} = \underline{p} + (\overline{p} - \underline{p})(1 - ay)e^{-ay} - \frac{k}{x}.$$

Substituting this into the Hamiltonian we get

$$\left[\underline{p} + (\overline{p} - \underline{p})e^{-ay}\right] y - k\frac{y}{x} + \left(\underline{p} + (\overline{p} - \underline{p})(1 - ay)e^{-ay} - \frac{k}{x}\right)(g(x) - y) = P_0 \qquad (5)$$

with zero discounting. Eq. (5) defines the optimal harvest as a feedback control law although it is not possible to solve this equation explicitly. The optimal steady state,

x^*, corresponds to the global maximum of $S(x)$, and the constant P_0 is given by $P_0 = S(x^*) = \max S(x)$. The optimal harvest, y, as a function of the stock level, x, is plotted in Figure 1 together with the f-function for zero and five per cent discounting.

The optimal steady state with zero discounting is $x^* = 4529$, and the corresponding harvest $y^* = 644$. This corresponds to $P_0 = 3184$.

Pollution control

The second example is based on pollution control. Assume that pollution, a, is associated with the production of some good, b, such that the change in a is given by

$$\dot{a} = g(b) - f(a) = b^2 - \alpha \cdot \exp\left(-\beta\left(a-\gamma\right)^2\right).$$

Pollution increases exponentially with production, and the decay of pollution follows a bell-shaped curve. Further, the benefit function is given by

$$B(a,b) = \frac{ln(b)}{a+1}.$$

This benefit function has the properties that $B_a < 0$, $B_b > 0$, $B_{bb} < 0$ and $B \to 0$ as $a \to \infty$. In addition to this $B(0,b) = ln(b)$.

From (2) we have

$$m(a,b) = -\frac{B_b}{f_b} = \frac{-1}{2b^2(a+1)}$$

which inserted into the Hamiltonian yields

$$\frac{ln(b)}{1+a} + \frac{-1}{2b^2(a+1)}\left[b^2 - \alpha \cdot \exp\left(-\beta\left(a-\gamma\right)^2\right)\right] = P_0 \qquad (6)$$

in the case with zero discount rate. In this particular case it is possible to find a closed form solution for b from eq. (6):

$$b(a) = \exp\left(\frac{1}{2}W(\Omega(a)) + \frac{1}{2} + P_0(a+1)\right) \qquad (7)$$

where

$$\Omega(a) = -\alpha \cdot \exp\left[-\beta(\gamma - a)^2 - 2P_0(a+1) - 1\right],$$

and W denotes the Lambert$W-$function defined by the expression

$$W(x) \cdot e^{W(x)} = x.$$

The numerical specification of the decay function is as follows: $\alpha = 2.5$, $\beta = 0.08$, $\gamma = 5$. Then the value of P_0 is found in the usual manner to be 0.084 and the corresponding values of a and b are 3.95 and 1.5 respectively. Optimal production as a function of the pollution level, that is equation (7), is illustrated in Figure 2 for zero and five per cent discounting.

SUMMARY

In this paper a general dynamic optimization problem has been solved by eliminating the costate variable from the Hamiltonian, and this has been combined with the fact that the Hamiltonian is constant with zero discounting (autonomous case). This yields an exact feedback solution in the case *without* discounting; that is, the exploitation level (control) is given as a function of the resource stock (state variable). In the case *with* discounting the solution has been expanded using perturbation theory in order to find an approximate solution. The approximation is sufficiently good for all practical purposes, and is usually by far outweighed by other sources of uncertainty.

As optimal exploitation of renewable resources has been found as feedback rules, this represents a truly adaptive approach to resource management. The feedback rules are found analytically even when the model is completely general. By completely general is meant that the objective function as well as the dynamic constraint are general functions of both the control and the state variable. In the general case the feedback rule is given as an implicit expression. It is possible, however, to find closed form solutions for many special cases. Analytical solutions make it possible to perform parameter analysis, comparative dynamics, etc., which makes this approach useful for qualitative as well as quantitative studies. The approach is illustrated by two numerical examples, one based on a fisheries management problem and one based on a pollution control problem.

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REFERENCES

[1] Clark, C.W., 1990, *Mathematical Bioeconomics*, Wiley, New York.

[2] Hardin, G., 1968, The tragedy of the commons, Science 162: 1243:1247.

[3] Mendelssohn, R., 1982, Discount factors and risk aversion in managing random fish populations, *Canadian Journal of Fisheries and Aquatic Sciences* 39: 1252-1257.

[4] Nayfeh, A. H. (1973), *Perturbation Methods*, Wiley, New York.

[5] Sandal, L.K., and Steinshamn, S.I., 1997a, A feedback model for the optimal management of renewable natural capital stocks, *Canadian Journal of Fisheries and Aquatic Sciences* 54: 2475-2482.

[6] Sandal, L.K., and Steinshamn, S.I., 1997b, A Stochastic Feedback Model for the Optimal Management of Renewable Resources, *Natural Resource Modeling* 10: 31-52.

[7] Sandal, L.K., and Steinshamn, S.I., 1997c, Optimal Steady States and the Effects of Discounting, *Marine Resource Economics* 12: 95-105.

[8] Sandal, L.K., and Steinshamn, S.I., 1998, Dynamic Corrective Taxes with Flow and Stock Externalities: A Feedback Approach, *Natural Resource Modeling* 11: xx-xx (forthcoming).

[9] Scott, A.D., 1955, The fishery: The objective of sole ownership, *Journal of Political Economy* 63: 116-124.

FIGURE LEGENDS

Figure 1. Optimal harvest as functions of the stock level for zero discounting (lower curve) and five per cent discounting (upper curve) together with the growth function.

Figure 2. Optimal production as functions of the aggregated pollution level for zero discounting (lower curve) and ten per cent discounting (upper curve).