SINGLE TRANSFERABLE VOTES WITH TAX-CUTS

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Abstract Some tally methods for preferential elections are discussed from the following point of view: how well do they respect a wish from the voter that subsidiary votes in the ballot cannot hurt the chances of the ballot's top-ranked candidate? The tally method of Single Transferable Votes, STV, is constructed to obey this principle without exception, but other defects show up, in particular nonmonotonicity, premature eliminations, and free rides. Various modifications of the STV are suggested to reduce the election method's weaknesses without losing too much of its strengths.

Key words Election systems, Single Transferable Votes, monotonicity.

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Introduction The STV, defined in (1.3) and used in many political elections, is one of the election methods which allow voters to express their opinions by ranking the candidates. Is the STV truly democratic? It is nonmonotonic. Thus there may be situations where a voter, i, may hurt candidate x by giving top-rank to x rather than to z. Similarly the supporters of x may help x to be elected by a temporary sacrifice of top-ranks, giving some of them to z instead of x. The nonmonotonicity is linked to eliminations of candidates that occur in the tally procedure of STV. Voter i hurts x by causing z to be eliminated instead of a third candidate y. The smart supporters of x will recover their sacrifice by having y eliminated. It is an understandable thought that the nonmonotonicity casts doubt upon the method's democratic character.

The purpose of eliminating a candidate, who is considered chanceless, is to allow the voters who support this candidate to transfer their support to their second-ranked candidates. One main idea behind STV is to avoid wasting of votes. It is equally understandable that this purpose is considered truly democratic.

Moreover, in comparison with other elections, it is known to be very hard for a voter group to manipulate the result of an STV-election by exploiting the nonmonotonicity (Nurmi 1992), because this requires accurate knowledge of the preference profile and accurate execution, as illustrated below (Example 3). This is linked to another main idea behind STV: it is designed to take the ballot rankings very seriously.

The STV may be seen as a modification of the wide-spread plurality method. The Jenkins Commision recommends to replace the plurality method in British elections by a new system with STV in single-seat constituencies as the main component. Presumably the over-all effect of a change was considered to improve the voting system's democratic character, despite the nonmonotonicity.

The democratic process does not start with the formal election; it starts long before and includes nomination processes. Also in other election systems, e.g. the Borda count, it may well happen that a race with candidates A, B and C will be won by A, while a race with A, B and D will be won by B. Both C and D may be quite chanceless. Nevertheless, if a small party has to decide between eliminating C or eliminating D in its nomination process, that decision does in fact

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decide between A and B in the election. This gives a manipulative power which may also cast doubt upon the democratic character of the combined process of nomination and election.

The frame of reference must be even wider when the task is to choose an election system. The performance of various voting systems should be compared on background of the entire political landscape and tradition where the systems are supposed to serve. The recommendation of STV in the Jenkins report (Jenkins 1998) is based on such considerations. Another example may be the approval voting (Example 2) with very special ballots; it appears that the main type of arenas for this method is elections of officers in large professional organizations (Brams and Fishburn 1992).

Also in such a wide frame of reference, shortcomings of the method used in the final formal election may be relevant. In this broader view, however, the worst consequence of the nonmonotonicity of STV may well be on the psychological level; it does not sound too good that a voter does not necessarily support his favorite candidate the best possible way by giving that candidate top-rank.

STV is indeed a theme that allows many "embellishments and modifications" (Saari 1994, p.278). Although many such exist, it should still be worth looking for variations of STV that avoid or reduce some obvious weaknesses in the standard procedure.

Section 1 has another look at the classical methods of Borda, Condorcet and Nanson. These methods are naturally defined in a domain allowing all reflexive relations as ballots. The election theory deals mainly with their properties when they are restricted to linear (transitive, complete, and antisymmetric) or to complete and transitive ballots, but in itself such a restriction is quite artificial. They differ from the STV-variations in the way they treat linear ballots. In contrast, the STV is designed just for linear ballots, and the tally is governed by the ballot rankings. The standard STV is seen as very strict in its respect for the voters' rankings.

This strictness may cause more weaknesses than necessary. Section 2 suggests to reduce some recognized weaknesses at the cost of some theoretical relaxation of the strictness. The tax-cut method, in particular, is designed to reduce a free-riding problem which is ubiquitous in standard STV.

In some multi-seat elections there is an additional requirement to fill specified minima of seats with candidates from specified candidate subsets. A situation with 3 seats to be filled with at least one "A-candidate" and one "B-candidate" is illustrated with data from an opinion poll; it raises the problem of how to elect without unnecessarily distorting the political composition of the 3-seat bench. The method of "intermediate tallies with tax-cuts" also gives a way to arrange an STV-election with such a restriction.

The method of "intermediate tallies with tax-cuts" may also be used on the reversed ballot rankings. Then only candidates who get elected leave the race. By avoiding other eliminations at least the main source of nonmonotonicity is removed. This STV-variation may also be run with a single tally, because the transfer mechanism will channel surplus voting power towards the strongest candidates, and moreover, it can be modified (with "dummy candidates") so that the monotonicity axiom is satisfied.

1 Preference elections

1.1 The preference relations

Let λ be a reflexive binary relation in a set C and define

$$P(\lambda,x,y) = 1 \text{ if } x \, \lambda \, y, \; P(\lambda,x,y) = 0 \text{ otherwise}, \quad x, y \in C. \tag{1}$$

The associated binary relations $\bar{\lambda}$ og λ^o are defined by

$$x \, \lambda^o \, y \text{ if } P(\lambda,x,y) = 1 \text{ and } P(\lambda,y,x) = 0, \; x \, \bar{\lambda} \, y \text{ if } P(\lambda,x,y) = 1 \text{ and } P(\lambda,y,x) = 1 \tag{2}$$

Consider an election where voter i, i=1, 2, ... , v submits a ballot which is a binary relation ϕ_i in the set C of candidates 1, 2, ... , p. The tallying procedure defines a map Φ which determines the electorate's final binary relation ϕ in C:

$$\phi = \Phi(\phi_1, ..., \phi_v) \tag{3}$$

The interpretations of $P(\phi_i,x,y) = 1$, $P(\phi_i^o,x,y) = 1$, $P(\bar{\phi}_i,x,y) = 1$ are, respectively, that voter i assesses candidate x to be at least as good as, strictly better than, equally good as candidate y in their pairwise comparison or encounter. In election theory it is usually assumed that each ϕ_i is one of the p! linear orderings of the candidate set, but other situations may occur.

Example 1: An election with p candidates is arranged through a series of $p \cdot (p-1)/2$ debate duels; after the debate between x and y there is a vote on who was best, or if it was a draw. For all debates together, a voters reactions give one of $4^{p(p-1)/2}$ possible ballots. $3^{p(p-1)/2}$ ballots are complete [i.e. $P(\phi_i,x,y) + P(\phi_i,y,x) > 0$ for all i and for each duel x vs y], which means that all voters expressed an opinion on all duels. $3^{p(p-1)/2}$ ballots are antisymmetric [i.e. $P(\phi_i,x,y) + P(\phi_i,y,x) < 2$ for all i and for each duel x vs y], which means that no voter assessed any duel as drawn. $2^{p(p-1)/2}$ ballots are both antisymmetric and complete. A ballot ϕ_i which contains a cycle does not necessarily indicate a confused voter.

Example 2: In approval voting voter i partitions the candidate set, $C = G_i \cup H_i$, and votes $P(\phi_i,x,y)=1$ if and only if $x = y$ or $x \in G_i$, $y \in H_i$, otherwise $P(\phi_i,x,y) = 0$. With p candidates there are 2^p possible ballots.

1.2 Methods which allow all ballots

Some tallying procedures are naturally restricted to certain types of binary ϕ_i. However, the Condorcet pairing and the Borda count are naturally defined in the most general setting.

Condorcet Set $C(x,y) = \Sigma_i P(\phi_i^o,x,y)$. The Condorcet procedure determines $\phi = \gamma$ where

$$x \gamma y \quad \text{if and only if} \quad C(x,y) \geq C(y,x) \tag{4}$$

The possibility of the Condorcet paradox, i.e. that γ may become nontransitive even if each ϕ_i is chosen as a linear ordering (transitive, complete, antisymmetric) is still central to much work on election theory. In order to call a Condorcet cycle

$$x_1 \gamma x_2,\; x_2 \gamma x_3, \ldots,\; x_k \gamma x_1$$

a paradox, it may be natural to demand that at least one γ can be replaced by γ^o. Define the equivalence relation σ by $x \sigma y$ if and only if x=y or if x and y occur together in a Condorcet cycle. Let (x) be the σ-equivalence class of x. The relation γ induces a well defined linear ordering γ^* among these equivalence classes: then $(x) \gamma^* (y)$ if and only if $x' \gamma y'$ for some $x' \in (x)$ and $y' \in (y)$. If the Condorcet principle is approved, and a complete and transitive ϕ is required, the best one can hope for is a Condorcet extension, i.e. a relation ϕ such that

$$(x) \gamma^* (y) \text{ and } (x) \neq (y) \text{ implies } x \phi^o y.$$

The IIA-axiom, *"Independence of Irrelevant Alternatives"*, means that whether $x \phi y$ or not is well defined by the two voter sets $\{i \mid x \phi_i y\}$ and $\{i \mid y \phi_i x\}$. The Condorcet pairing is only slightly generalized by the IIA-axiom. When restricted to linear ϕ_i any other non-constant Φ satisfying the IIA-axiom would have to violate at least one of three other conditions:

monotonicity (if $x \phi y$ and candidate x is moved upwards on a ballot, we still have $x \phi y$), *neutrality* (switching candidates x and y in all ballots ϕ_i leads to switching them in the final relation ϕ), and
anonymity (if voters i and j switch ballots, the final relation ϕ remains unchanged).

Although IIA generalizes γ, the new possibilities are of limited interest. A method for general political elections will hardly be accepted unless it is neutral and anonymous. The IIA-axiom codifies a certain respect for individual ballots; $P(\phi,x,y)$ depends precicely on the ballot statements about x and y: $x \phi_i y$ or $y \phi_i x$. Therefore the axiom has didactic value, but for most practical uses society must search among election methods which violate the IIA.

Methods that violate monotonicity are actually in common use, although they are sometimes, and understandably, criticized for this shortcoming (Brams and Fishburn 1991). Nonmonotonicity gives rise to effects which are considered undemocratic, and also may be exploited for manipulation. Instead of the IIA one may try other ways to pay reasonable respect to the ballot statements. How can the tallying procedure respect the individual ballots and still achieve monotonicity?

Borda Set $B(i,x) = \Sigma_z P(\phi_i^o,x,z)$. The Borda count determines $\phi = \beta$ where

$$x \beta y \quad \text{if and only if} \quad \Sigma_i B(i,x) \geq \Sigma_i B(i,y) \tag{5}$$

Hence β is necessarily transitive and complete, no matter how the voters choose their ϕ_i. The same is true of any point-awarding method defined by a nondecreasing function F, i.e. a ranking of the candidates according to the values of $\Sigma_i F(B(i,x))$. The wide-spread plurality method corresponds to $F(p-1) = 1$, $F(k)=0$ for $k<p-1$, i.e. giving candidate x 1 point for each ballot where $x \phi_i^o y$ for every other candidate y. The Borda count is determined by the C(x,y) because

$$\Sigma_i B(i,x) = \Sigma_i \Sigma_z P(\phi_i,x,z) = \Sigma_z \Sigma_i P(\phi_i,x,z) = \Sigma_z C(x,z) \qquad (6)$$

Usually the Borda count is combined with a demand that each ϕ_i is chosen as a linear ordering. By ranking x as no. 1 and y as no. 2 a voter may unwillingly contribute to the outcome $y \beta^o x$, while ranking y as no. 5 (say) instead would suffice to ensure $x \beta^o y$. It may well be called an abuse of a ballot when the tallying procedure lets it contribute towards a result contrary to what the ballot expresses. The incentive to vote "strategically", and to suspect that others do so, is also obvious. The Borda count obviously satisfies monotonicity, but also in other ways there is a clear need to protect against ballot abuse and associated strategic voting.

The Borda count was used in figure skating contests (for linear ϕ_i), but according to rule 371 of the International Skating Union from 1994 the contestants are now ranked by the median B(i,x) over the judges i=1, ..., v (supplemented by various tie-break rules), which is more robust to strategic voting. The median principle is compared with other principles by Truchon (1998), who advocates the Condorcet extension due to Kemeny (1959).

Nanson The well-known method of Nanson (1882) is another Condorcet extension. It makes use of another fundamental idea: *elimination* of a candidate who comes last according to some criterion. Nanson's method defines $\phi = \nu$ as follows: eliminate a candidate x with $\Sigma_i B(i,x) \leq \Sigma_i B(i,z)$ for all z (a tie-break may be necessary). Recalculate the B(i,z) for the remaining z, and eliminate again, etc. The last remaining candidate (u) is the (Nanson) winner. Remove u and do the whole procedure over again to get no. 2 in the Nanson ranking, etc. Clearly ν is linear.

Nanson's method is a compromise between the Borda count and the Condorcet pairing when all ballots are antisymmetric and complete. Then $P(\phi_i^o,x,y)+P(\phi_i^o,y,x) = 1$ and $C(x,y)+C(y,x) = v$ for $x \neq y$. Hence w is a Condorcet winner ($w \gamma z$ for all z) if and only if $C(w,z) \geq v/2$ for all $z \neq w$. Then

$$\Sigma_i B(i,w) = \Sigma_z C(w,z) \geq v\cdot(p-1)/2 \qquad (7)$$

Under our assumption $v\cdot(p-1)/2$ is the average Borda sum. Hence by (7) w can never be eliminated unless $\Sigma_i B(i,x) = v\cdot(p-1)/2$ for all x. In that case assume the tie-break does not eliminate a unique Condorcet-winner. If γ does not determine a unique (Condorcet) winner, there is a class (x) winning under γ^*. All members of (x) cannot be eliminated, because the last of them would become a unique Condorcet-winner.

It is known that in the p=3 case, the Borda count is the only member of a family of point-awarding methods which guarantees that a Condorcet-winner is not bottomranked (Saari 1994, p.192).

Tallying methods which disregard ballot transitivity The tallies of Borda, Condorcet and Nanson are naturally defined for the most general setting with $2^{p(p-1)}$ possible ballot types. Borda and Nanson always define complete and transitive relations, but they are usually studied for elections where all ballots are linear. That does not mean that they are well designed to interprete and respect a linear ballot as any kind of voter instruction, e.g. that the ballot statement x ϕ_i^o y should be treated as an instruction not to reckon voter i as supporting candidate y at the cost of candidate x.

The Borda count is based on a higher aggregation of the ballot data than the Condorcet pairing (double sums compared to single sums) and it obliterates more of the profile structure. As a ranking-by-points system, the Borda count cannot be sensitive enough to pick up a cyclical preference in situations like example 1 even if an overwhelming majority shares it. It is remarkable that the Borda count preserves enough profile information to define a Condorcet extension, and that it does so even under weaker assumptions on ballots than in standard theory (antisymmetry, completeness, but not neccesarily transitivity).

In normal political elections with a large electorate, Condorcet cycles are rare, so Nanson's ν will mostly coincide with Condorcet's γ and therefore avoid the mentioned abuse of ballots. By stating x ϕ_i y in the ballot, a voter has then done as much as practically possible to avoid the result y ϕ x. However, with a small number of voters or voter blocks as in a committee or a political assembly, it will happen more often that γ is nontransitive.

A very different philosophy is behind election methods with the following property:

Respect for ballot rankings If ϕ_i ranks candidates c_1, c_2, c_3, ... as no. 1, 2, 3, ... , then for all r the tallying procedure decides the final ranking of candidate c_r before it takes into consideration the further rankings c_{r+1}, c_{r+2}, ... of ϕ_i.

Such a tally may be done by an official who puts one candidate at a time on the first or the last vacant place in the final list. The official asks each voter to name the candidate ranked first among those not yet listed, then places one candidate on the final list, and repeats the question.

1.3 Tallying methods for linear ballots

Consider an election with candidates from two equally strong political wings and from a small center. Condorcet's method, and therefore also Nanson's, will generally let a candidate from the political center win over any wing candidate with subsidiary support from the opposite wing. This, of course, is not necessarily a good property; the widespread acceptance of plurality in single-seat constituencies in elections for political assemblies is certainly based on its ability to create an assembly with a majority fit for government and a minority fit to oppose it.

However, even in a multiseat constituency the Condorcet/Nanson methods let a dominant voter group grab much more than its proportional share of seats. In addition to respecting ballot rankings as formulated above, an election method should also give an acceptable distribution of the seats in a multiseat constituency.

The STV-tally (Single Transferable Vote, often called Hare's method) is designed to fill s seats in a roughly proportional way. Unlike Condorcet's method, or the derived methods of Borda and Nanson, STV requires each ballot ϕ_i to be a linear ordering of the candidates. The STV-tally is performed in several rounds. Each round either ends with the election of a candidate who satisfies a specified election criterion or, if no candidate satisfies the criterion, ends with the elimination of a candidate who is ranked last according to some specified elimination criterion.

STV, as defined below, respects ballot ranking. It has the elimination idea in common with Nanson's method, but it also adopts another fundamental idea:

The ballot weight The weight w(i,r) defines the influence of voter i in round r. It is reduced whenever voter i is reckoned as having received some satisfaction by the election of a candidate the voter supported. The voters who gave top rank in round r to a candidate that left the race, transfer their support w(i,r+1) to their new top ranked candidate in round r+1. Thus

$$1 = w(i,1) \geq w(i,2) \geq \ldots. \geq w(i,r) \geq \ldots \geq 0, \tag{8}$$
$$\Sigma_i\, w(i,r) = v - t{\cdot}v/(s+1), \quad i=1, 2, ..., v \tag{9}$$

where t is the number of candidates elected in previous rounds. Round r consists of 3 steps:

1) Define $V(x,r) = \{i|x\phi_i y$ for all remaining candidates $y\}$ and calculate

$$W(x,r) = \Sigma_i\, w(i,r), \quad i \in V(x,r)$$

2) Choose two remaining candidates x and y so that

$$W(x,r) \geq W(z,r) \geq W(y,r) \text{ for all remaining z.}$$

3a) If $W(x,r) > v/(s+1)$, elect candidate x to seat no. t+1 and calculate the new weights:

$$w(i,r+1) = w(i,r).[1-W(x,r)^{-1}.v/(s+1)] \quad \text{if} \quad i \in V(x,r),$$
$$w(i,r+1) = w(i,r) \quad \text{if} \quad i \notin V(x,r)\ .$$

3b) If $W(x,r) \leq v/(s+1)$, eliminate candidate y from the race and keep the weights:

$$w(i,r+1) = w(i,r) \text{ for all i.}$$

Since all ϕ_i are linear, each voter belongs to a unique set V(x,r) in round r. By (9), the weight reductions sum up to v/(s+1). They may be distributed among the voters in other ways. The idea of 3a is to impose a "tax" with the same tax rate $W(x,r)^{-1}.v/(s+1)$ for all voters in V(x,r). These voters have received a satisfaction in round r by having their top ranked x elected. The taxation idea turns out to be advantageous in dealing with the free rider problem discussed below.

Other elimination criteria than 3b may be contemplated. Coomb's elimination criterion eliminates the candidate with the highest number of bottom-rankings instead of y. It favors compromise candidates. A controversial candidate with 49% topranks, 2 % secondranks and 49 % bottomranks is likely to win a single-seat standard STV election after 3b-eliminations of other candidates, but is also likely to be the first one eliminated under Coomb's criterion. The Coomb's STV does not respect ballot ranking, but it does give a weaker protection against ballot abuse, equivalent to an election official who may ask voters for their top-ranked and bottom-ranked candidate.

With multiseat constituencies weight reductions tend to give a roughly proportional representation for voter groups with sufficiently high weight sums, i.e. size, even if a group's first-ranks may be split on several candidates. In a situation as described above, STV will tend to eliminate candidates from the small center, but compensate for this by giving the center voters influence on the choice between the two wings.

One essential difference between STV and plurality in single-seat constituencies is that a split majority does not lose to a united minority, and another that the voters who supported a candidate in vain still may count through their further rankings. By and large however, the methods lead to similarly composed assemblies. It is understandable that the Jenkins Commission recommends single-seat STV (in the report called AV - the Alternative Vote) in combination with other remedies to avoid too large underrepresentation of smaller groups. The committee states in section 81 of the report about the AV: "It would fully maintain the link between between MPs and a single geographical constituency. It would increase voter choice in the sense that it would enable voters to express their second and sometimes third or fourth preferences, and thus free them from a bifurcating choice between realistic and ideological commitment or, as it is sometimes called, voting tactically. There is not the slightest reason to think that AV would reduce the stability of government; it might indeed lead to larger parliamentary majorities."

The Jenkins report gives the impression that strict obedience to axioms like IIA, respect for ballot linearities, or monotonicity, is a lesser concern for politicians than for election theorists.

2 Variations on the STV-theme

2.1 STV with intermediate tallies

Pragmatic politicians aside, election theorists will look at the elimination rule 3b as a mixed blessing. There are two interrelated problems:

The problem of nonmonotonicity STV is actually non-monotonic, and it is easy to concoct examples where strategic voting based on nonmonotonicity may succeed:

Example 3 Consider an election with candidates A, B, C for a single seat, with profile

A>B>C : 20, A>C>B : 12, C>A>B : 20, C>B>A : 3+u, B>C>A : 2, B>A>C : 43-u,

where > means "strictly preferred to". A, B and C have, respectively, 32, 45-u and 23+u topranks. No candidate passes 50% support if u<28. For u=0, ..., 8, C is eliminated and A wins 52 - 48 against B. For u=10, 11, 12, A is eliminated and B wins (65-u) - (35+u) against C. For u=14, ... , 27 B is eliminated and the runoff score is (75-u) - (25+u) for A against C. Increasing u from 8 to 10 means reduces the support for B but still turns B into a winner, which shows the nonmonotonicity.

Suppose that initially u=0. Then the B>A>C voters may discover that they can make B a winner by letting 10, 11 or 12 of them vote C>B>A instead of B>A>C. B becomes a winner instead of A. By making C stronger than A they exploit rule 3b to get A eliminated instead of C, and prevent A from getting 20 new votes transferred from C.

A non-monotonic ϕ creates the possibility for strategic voting in example 3. Unfortunately, such a possibilitiy is a consequence of the method of eliminations and vote transfer. The scenario is that the voters supporting candidate x see that x may get elected after a large transfer of votes from y if y is eliminated, while y may get elected with a large transfer from z if z is eliminated. The supporters of x may therefore temporarily sacrifice some x-votes to keep z in the race and capture the support from the y-voters (example 3).

Strategic possibilities may occur in all nondictatorial systems, but in the case of STV they are unlikely to occur and are hard to exploit. Investigations of how robust various voting methods are to strategic voting, gives the best marks to STV (Nurmi 1992). Still the STV may be improved. The non-monotonicity in example 3 is linked to a 3b elimination in one of the tally rounds. This particular mechanism can be avoided through eliminating the need for 3b eliminations. The use of intermediate tallies explained below eliminates the need for 3b-eliminations.

The problem of premature eliminations One of the good intentions behind STV is that votes should not be wasted; a group of voters who is split on several candidates will gradually concentrate their support on a compromise candidate who gets elected if the group is not too small. However, STV does not always function as intended, because the strongest compromise candidate may be eliminated prematurely:

Example 4 With 7 candidates and 100 voters, assume

u voters say a>c> ... ; 40-u voters say a>b> ... ;
t voters say b>c> ... ; 26-t voters say b>a> ... ;
2 voters say c> ... ; 10 voters say d>c>b> ... ; 9 voters say e>c>b> ... ;
7 voters say f>c>b ... ; 6 voters say g>c>b> ... .

There are 3 main voter groups; 40 voters support candidate a, 26 support candidate b, and 34 are split on candidates c, d, e, f, g but have c as a strong compromise candidate. Depending on u and t, candidate c may also appear as a good compromise between a and b.

In an election for a single seat the election criterion is to obtain >50 votes. In round 1 none is elected, and c is eliminated before the 34-voter group is allowed to concentrate their votes on c. In the following rounds subsequent eliminations of g, f, e, d transfer votes to b. Finally b is elected with at least 26+6+7+9+10=58 topranks.

In an election for two seats the election criterion is to obtain >100/3 votes. In round 1 a is elected to the first seat with a surplus of 40 - 100/3 = 40/6 votes; voter i for candidate a keeps a weight w(i,2)= 1/6 for round 2, making the supports for b, c and g, respectively, (196-u)/6, (12+u)/6 and 36/6. None are elected. If u<24, then c is eliminated, and in round 3 or 4 b will get enough support to be elected. If u>24, then g is eliminated, and in the following rounds f, e, and d get eliminated too, transferring all their support to c. Finally c is elected to the second seat.

The pairwise encounter c vs a tallies at (34+t)-(66-t). For t>16 c actually defeats a, who is elected ahead of c in a two-seat election.

Notice moreover that a candidate who wins in a single-seat election is not necessarily elected in a two-seat election.

With lower s, the election criterion v/(s+1) is raised, and a premature elimination of a strong compromise candidate will occur more often. There will also be more opportunities for strategic exploitation of nonmonotonicity because a higher stake in the elimination game justifies a higher initial sacrifice. These weaknesses of the standard STV will be reduced in STV with intermediate tallies:

DEFINITION STV with intermediate tallies If the purpose is to choose s candidates out of p, then run a sequence of p-s intermediate tallies. Tally first for p-1 seats, among the p-1 successful candidates tally for p-2 seats and so on until s candidates remain. Then each round within an intermediate tally ends with a 3a (temporary) gain of a seat, a weight reduction and transfer of the surplus. After p-s intermediate tallies s candidates remain. To get a linear ordering of the s elected candidates, one may continue as for s=1. The candidate left behind after intermediate tally no. t is the one who could not get past the v/(s+2-t)-mark of 3a even with a maximum length of 3a type vote transfers. This candidate is then eliminated in what may be called a 3a elimination, and a new intermediate tally is done with one candidate less, and with all initial voter weights reset at 1.

Let STV with intermediate tallies for s_1 and s_2 seats choose candidate sets C_1 and C_2 respectively. Since the tally sequence for s_1 seats begins with the tally sequence for s_2 seats, it is clear that

$$\text{if } s_1 < s_2 \text{, then } C_1 \subset C_2.$$

As example 4 shows, this property is not shared by the standard STV.

In order to see how STV with intermediate tallies may handle the problems of nonmonotonicity and premature eliminations, take another look at the profiles of examples 3 and 4:

Example 5 With the profile of example 3, the intermediate 2-seat tally for u= .. ,7, 8, 9, 10, 11, 12, 13, 14, 15, .. leads to elimination of .. ,C, C, C, C, A, A, B, B, B, .. respectively. The right hand column in Table 1 shows the support of the eliminated candidate. The opportunity to exploit the non-monotonicity is clearly reduced. The set of u-values for successful strategic voting for B is reduced from {10, 11, 12} (with ties at u=9 and u=13) to {11, 12}.

Example 6 With the profile of example 4, the intermediate tallies start with election criterion >100/7. Candidates a and b are assured "temporary seats" which spend 200/7 votes, leaving a surplus of 40+26 - 200/7 for c who gets 39 + 3/7 votes. In subsequent rounds the transfer after eliminations from g, f, e, d gives candidate c sufficient compensation for the reduced surplus from a and b. When a, b and c remain, the profile has become:

40-u: a>b>c ; u: a>c>b ; h: c>a>b ; 34-h: c>b>a ; 26-t: b>c>a ; t: b>a>c with h=0,1, or 2.

By now the election criterion has become >100/3 votes, and so a and c are elected for all parameter values h, t, u. For t>16 candidate c will even win a single-seat election.

Until candidate z is eliminated the order of the candidates after z in any ballot has not yet influenced the tallying process. This is true for the standard STV and for the STV with intermediate tallies. In the intermediate tally STV the elimination of z has been influenced by the information in the values of $P(\phi_i,x,z)$ for all x. The true spirit of STV, however, is to pay full attention also to a ballot's full ranking after candidate z in the race between the other contenders, including those who would have left the race (through election or elimination) before z under standard STV. That is done in the intermediate tally STV by running another intermediate tally with the reduced candidate set, resetting all initial weights as 1.

2.2 STV with tax cuts

With the technical meaning of "respecting ballot ranking" introduced above, ballot ranking is clearly respected within each intermediate tally, but not in the total tallying process. On the other hand, the problems of nonmonotonicity and premature eliminations seem to be reduced in STV with intermediate tallies.

Nonmonotonicity is, however, still possible. A precice definition of what is a premature elimination may be a matter of opinion, but the examples 4 and 6 indicate that the barriers to succesful gathering of a voter group around a compromise candidate are much reduced. But another problem still remains:

The problem of free riding An obvious weakness of the STV is a free rider problem. Consider two voters: voter i ranks x as no.1 and y as no.2, voter j ranks y as no.1 and x as no.2. Suppose x is "elected" first in an intermediate tally, then y. Voter i gets his weight reduced twice, and voter j only once. After two rounds they have obtained the same satisfaction; their two top ranked candidates are among those succesful in the intermediate tally, but $w(i,3)<w(j,3)$. Voter j has become a free rider, receiving the same satisfaction as voter i, but the weight of j is taxed less than the weight of i, and voter j has more influence in the next round.

The taxation of the voters' weights in STV obviously brings up a more general free riding problem. Suppose voter i sincerely would give candidate x toprank, but x is very popular and will be elected anyway. Voter i may well get the idea that it is better to rank x very low, in order to avoid unneccesary taxation and to keep more weight for other candidates.

Free riding can be reduced with the tax-cut variation of STV introduced below.

DEFINITION **STV with tax cuts** This method is also a sequence of intermediate tallies, and each intermediate tally ends with the elimination of one candidate. It is, however, not performed round by round. Assume there are n candidates in an intermediate tally, $n \leq p$, and that B(i,j) is candidate no.j in the ranking of voter i. Let $T(x) \in (0,1]$ and set

$$S(B(i,k)) = T(B(i,k)) \cdot \prod_{j=1}^{k-1} [1-T(B(i,j))] \qquad (10)$$

S(B(i,k)) is the support from voter i to candidate B(i,k) after the candidates B(i,1), ... , B(i,k-1) have collected their taxes; T(x) is the tax rate used by candidate x. The total support for candidate x is

$$S(x) = \sum_{B(i,k)=x} S(B(i,k)) \qquad (11)$$

There exist unique tax rates which make $S(x) = v/n$ for all x. These tax rates are determined iteratively, starting at $T(x)=1$ for all x. Thus at the start $S(x)=W(x,1)$ = the number of firstranks for x. If $S(x)>v/n$, then x will cut the tax so that $S(x) := v/n$, i.e.

$$T(x) := T(x) \cdot S(x)^{-1} \cdot v/n \qquad (12)$$

Unless all candidates have support v/n, at least one of them will have to make a tax cut. When candidate x cuts the tax, any other candidate y gets increased or unchanged support.

THEOREM 1 Consider a process which involves infinitely many tax cuts given by (12) from all candidates. The process converges:

$$S(x) \rightarrow v/n \text{ for all x and } T(x) \rightarrow T^*(x)$$

and the limit tax rate vector [T*(x)] does not depend on the order in which the candidates cut their taxes.

Proof: No candidate can afford a tax rate below 1/n, so the convergence is obvious. To see the uniqueness, it is convenient to include tax cutting processes where a smaller tax cut is allowed provided the candidate later brings the tax down as far as indicated by (12). Let process I start with x cutting tax from $T(x)_1$ to $T(x)_2$ and process II with y cutting tax from $T(y)_1$ to $T(y)_2$. The tax cut by x from $T(x)_1$ to $T(x)_2$ allows y to follow up with the cut from $T(y)_1$ to $T(y)_2$ (or even a larger cut), and vice versa. After the opening cut of one process it is therefore still possible to break off and reach a stage in the other process. Repeated application of this shows that after any finite number of steps in one process one may still reach a stage of the other process. Therefore the two processes cannot have different limit tax rate vectors.

Let $T^*(y) \geq T^*(x)$ for all x; then $T^*(y)=1$. Except for rare cases where a tie-break is needed, y is uniquely determined. Unless a tie-break is needed y appears as the unique candidate whose support never passes the v/n - mark despite collecting all transferred surpluses from all other candidates. The intermediate tally ends with the elimination of y. In practice it suffices to do the iterations until only one candidate is left with tax rate 1.

The following table illustrates with the constructed data from examples 3 and 5 how the eliminations depend on the parameter u for standard STV, STV with intermediate tallies, and its tax-cut variety.

Table 1

Profiles from examples 3 and 5; eliminated candidate under various STV-versions

u	A's limit tax	B's limit tax	C's limit tax	eliminated, tax-cut STV	eliminated, standard STV	eliminated, intermed. STV
7	.866	.819	1.000	C	C	C (30 14/57)
8	.898	.854	1.000	C	C	C(31 32/111)
9	.937	.894	1.000	C	tie A-C	C(32 4/27)
10	.986	.944	1.000	C	A	C(33 2/21)
11	1.000	.972	.978	A	A	A(32 32/51)
12	1.000	.988	.952	A	A	A (32 20/21)
13	.995	1.000	.924	B	tie A-B	B(32 5/27)
14	.977	1.000	.894	B	B	B (32 76/111)
15	.962	1.000	.867	B	B	B (32 4/19)

The set of u-values which correspond to successful strategic voting is reduced from the standard STV-case (ties at u=9 and u=13) to much shorter intervals. The last column also shows the support for the eliminated alternative.
Clearly each intermediate tally in the tax-cut STV satisfies a weaker version of the "respect for ballot ranking" axiom:

Weak respect for ballot rankings If ϕ_i ranks candidates c_1, c_2, c_3, ... as no. 1, 2, 3, ... , the tallying procedure must have decided that candidate c_r will not be eliminated before it takes into consideration the further rankings c_{r+1}, c_{r+2}, ... of ϕ_i .

Dummy candidates This version of tax-cut STV include a number of dummy candidates. We make the technical assumption, in the proof of theorem 2, that at least half of the candidates are dummies, and that they come after all real candidates in every ballot. With the limit tax rate vector [T*(x)] each voter has been taxed at most 50% before coming to the dummy candidates.

A monotonicity property If voter i decides to interchange two consecutive candidates x and y, originally ranked as no. j and no. j+1 in ballot ϕ_i , y will collect more tax whatever the tax-rates are. However, it is not obvious that this transposition never can hurt y in the end, because most likely all the limit tax-rates will change as a consequence of this neighbor transposition in a single ballot. Fortunately, the natural result is true, at least under the mentioned condition:

THEOREM 2 Let at least half of the candidates be dummies, ranked after the real candidates in every ballot. Then each intermediate tax-cut tally is monotonic on the real candidates.

Proof: Consider first a situation where real candidate q has total support v/n + Δ, dummy candidate k has v/n - Δ, and all others have v/n. Assume Δ is small, so that the tax-cuts are negligible in comparison to the tax-rates. Consider a "basic step" where q starts with a maximal tax-cut to keep a support of v/n and then all candidates except q and w, w being a real candidate different from q, are allowed to cut taxes. In the limit

q has accumulated a new surplus $\alpha\Delta$ and w has $\beta\Delta$ with $\alpha + \beta < 1$,

whereas $(1- \alpha - \beta)\Delta$ has been used to reduce the deficit of candidate k.

After q's initial tax-cut at least $\Delta/2$ will become additional support for the dummies, and tax reduction from a dummy candidate can give increased support only to other dummy candidates. Therefore the inequality is sharpened to

$$1 - \alpha - \beta > 1/2, \text{ i.e. } \alpha + \beta < 1/2$$

Consider next a process consisting of basic steps where q through tax-cuts get rid of the surplus Δ. This process transfers to w an additional support of

$$(1 + \alpha + \alpha^2 + \alpha^3 + \dots\ ) \cdot \beta \cdot \Delta = \frac{\beta}{1-\alpha} \cdot \Delta$$

and candidate q has been able to cut taxes corresponding to a surplus amounting to

$$(1 + \alpha + \alpha^2 + \alpha^3 + \dots\ ) \cdot \Delta = \frac{1}{1-\alpha} \cdot \Delta$$

If instead w has a surplus v/n + Δ', k has v/n - Δ', and all others have v/n, then there is a similar basic step after which q has v/n + $\delta\Delta'$, and w has accumulated a new surplus $\gamma\Delta'$, $\gamma + \delta < 1/2$. After a process of such basic steps

q has additional support $\frac{\delta}{1-\gamma} \cdot \Delta'$, and w has cut taxes corresponding to a surplus $\frac{1}{1-\gamma} \cdot \Delta'$

Consider finally a sequence of processes which shuffles a diminishng surplus back and forth between q and w, thus gradually transferring Δ from q to k. After the first process w has surplus $\Delta' = \frac{\beta}{1-\alpha} \cdot \Delta$, after the second q has $\frac{\delta}{1-\gamma} \cdot \frac{\beta}{1-\alpha} \cdot \Delta$ etc. During the sequence q and w have cut taxes corresponding to total surpluses of

$$\frac{1}{1-\alpha}\cdot\left(1+\frac{\beta\cdot\delta}{(1-\alpha)\cdot(1-\gamma)}+\left(\frac{\beta\cdot\delta}{(1-\alpha)\cdot(1-\gamma)}\right)^2+\ldots\right)\cdot\Delta = \frac{1}{1-\alpha}\cdot\frac{(1-\alpha)(1-\gamma)}{(1-\alpha)(1-\gamma)-\beta\delta}\cdot\Delta \quad \text{and}$$

$$\frac{1}{1-\gamma}\cdot\frac{(1-\alpha)(1-\gamma)}{(1-\alpha)(1-\gamma)-\beta\delta}\cdot\frac{\beta}{1-\alpha}\cdot\Delta = \frac{1}{1-\alpha}\cdot\frac{(1-\alpha)(1-\gamma)}{(1-\alpha)(1-\gamma)-\beta\delta}\cdot\Delta\cdot\frac{\beta}{1-\gamma} \quad \text{, respectively.}$$

Observe that candidate q, who posesses the initial surplus Δ, has an advantage over w: during the final sequence candidate w gets rid of surplusses adding up to $\beta/(1-\gamma)$ times the surplus total that candidate q gets rid of, and because $\beta+\gamma < \alpha+\beta+\gamma+\delta < 1$, we have

$$\beta/(1-\gamma) < 1.$$

Assume now the theorem is false. Then there exists a profile where voter 1 (say) ranks x [y] as no. j [j+1] and y defeats z, but transposing x and y in ballot 1 makes z defeat y. Allow voter 1 to give weight 1-s [s] to the original ballot [the ballot after transposition], $s \in [0, 1]$. At the "s-profile", for an arbitrary tax vector [T(c)] voter 1 supports

x with $(1-s)\cdot T(x)\cdot P + s\cdot T(x)\cdot[1-T(y)]\cdot P = T(x)\cdot P - s\cdot T(x)\cdot T(y)\cdot P$ and

y with $(1-s)\cdot T(y)\cdot[1-T(x)]\cdot P + s\cdot T(y)\cdot P = T(y)\cdot[1-T(x)]\cdot P + s\cdot T(x)\cdot T(y)\cdot P$

where $P = \Pi[1-T(B(1,u))]$, u=1..j-1. For $(i,c) \notin \{(1,x,),(1,y)\}$ the support from i to c does not depend on s.

Choose $\bar{s} \in (0,1)$ to make a tie between y and z, i.e. so that $T^*(y)=T^*(z)$ at the $\bar{s}$-profile, and choose a small ε so that

$$T^*(y)<T^*(z) \text{ at } \bar{s}-\varepsilon/2 \quad \text{while} \quad T^*(z)<T^*(y) \text{ at } \bar{s}+\varepsilon/2 .$$

We want to derive a contradiction to these inequalities. Modify first the tax-cut process so that the support for x [y] is calculated at $\bar{s}+\varepsilon/2$ [$\bar{s}-\varepsilon/2$], i.e. in the least favorable way. For ε small enough, the limit tax vector [$\overline{T}(c)$] of the modified process is as close as one wants to the limit tax vector [T*(c)] at $\bar{s}$, and we may assume that

at $\bar{s}-\varepsilon/2$ x has total support $v/n + \varepsilon\cdot\overline{T}(x)\cdot\overline{T}(y)\cdot\overline{P}$ while y has v/n (situation L_1), and

at $\bar{s}+\varepsilon/2$ y has total support $v/n + \varepsilon\cdot\overline{T}(x)\cdot\overline{T}(y)\cdot\overline{P}$ while x has v/n (situation L_2);

here n is the number of candidates in the intermediate tally and $\overline{P}$ is similar to P above. Moreover,

any other candidate c with limit tax rate $T^*(c)<1$ at $\bar{s}$ has $\overline{T}(c)<1$ and total support v/n.

The surplus support $\Delta = \varepsilon\cdot\overline{T}(x)\cdot\overline{T}(y)\cdot\overline{P}$ that must be removed by further tax-cuts is the same at $\bar{s}-\varepsilon/2$ and at $\bar{s}+\varepsilon/2$, but we shall see that it benefits y most to be the one who possesses the surplus.

In order to do this comparison, treat the situation L_1 at profile $\bar{s}-\varepsilon/2$ as follows: Candidate x starts with a tax-cut that gets rid of the surplus Δ, and thereafter all candidates except y and z are allowed to cut taxes. In the limit situation (L_3 say), y [z] posesses a surplus $A\Delta$ [$B\Delta$] with $A+B < 1$, and no other candidate has a surplus.

In the final pairwise tax-cut contest between y and z, they start at tax-rates $\overline{T}(y)$ and $\overline{T}(z)$ respectively, as close to $T^*(y)=T^*(z)$ as we like.

In L_2 at profile $\bar{s}+\varepsilon/2$, y [z] has surplus Δ [0], and

in L_3 at profile $\bar{s}-\varepsilon/2$, y [z] has surplus $A\Delta$ [$B\Delta$] with $A+B < 1$.

From L_2 y cuts tax more than z in getting rid of Δ; from L_3 y gets a smaller advantage from getting rid of $A\Delta$, at the same time as z has an advantage in getting rid of $B\Delta$. So we have the desired contradiction.

Remark It is not clear if one can drop an assumption of dummy candidates in theorem 2. It has only been used to derive the inequalities $\alpha + \beta < 1/2$ or $\gamma + \delta < 1/2$, and they are only used to establish the inequality $\beta + \gamma < 1$. Dummy candidates may actually make a difference in an almost even pairwise contest; in the intermediate tally 1 of table 2 below, FrP beats Krf with 0, 1 or 2 dummies added at the end, while KrF beats FrP with 3 or more dummies.

2.3 An application on real data

174 students at the author's institution ranked 8 parties at the election day 1993. Aspects of these data have been analyzed by B-D. H. Syversten and the author (Stensholt 1996). Taking the data as the profile in an election with one imagined candidate from each party, we get the following result, where the limit tax rates are given in % :

Table 2

NHH-data 1993 (Ap, FrP, H, KrF, RV, Sp, SV, V). Taxes charged in each round are given in %.

taxes	Ap	FrP	H	KrF	RV	Sp	SV	V
int.tal.1	18.754	30.641	15.773	32.633	100.000	42.853	43.042	34.966
int.tal.2	26.814	49.618	20.109	66.431		100.000	84.033	73.056
int.tal.3	32.402	61.482	23.167	86.029			95.353	100.000
int.tal.4	37.613	73.956	27.967	91.018			100.000	
int.tal.5	46.971	99.477	34.059	100.000				
int.tal.6	59.846	100.000	44.114					
int.tal.7	100.000		73.109					

The parties may, in this order, be briefly characterized as social democrat, market liberalist, conservative, christian, socialist, agrarian, socialist, and liberal. H has much higher support in this particular student group than in the nation as a hold. The imagined candidates leave the race in the order RV, Sp, V, SV, KrF, FrP, Ap, and H emerges as the winner. FrP's strength comes mainly from nr. 2-rankings after H. In a multiseat election with several candidates from each party FrP would hardly do as well as it appears because H might gain more seats before transferring any remaining surplus to FrP.

The final ranking is H>Ap>FrP>KrF>SV>V>Sp>RV.

Tables 3a-c show how the support from three selected ballots are distributed on the candidates in each intermediate tally, with the limit taxes of Table 2. Voters 10, 11, and 118 have the preference orders

H>FrP>Ap>V>KrF>SV>Sp>RV,
FrP>H>Ap>KrF>V>Sp>SV>RV, and
RV>SV>V>KrF>Sp>Ap>H>FrP, respectively.

Voter 10 is a main-stream voter; except in intermediate tally no. 3 the support from voter 10 trickles down the line to the last candidate. Voter 118 is an anti-establishment voter, frequently putting all weight behind the last candidate.

Voters 10 and 11 both have Ap in third place, after H and FrP; their support for Ap is therefore always the same even though voter 10 says H>FrP and voter 11 says FrP>H. Similarly their supports for KrF and V together is the same as long as both parties remain in the race.

Table 3a

NHH-data 1993 (Ap, FrP, H, KrF, RV, Sp, SV, V). Distribution of the support from voter 10

voter 10	3) Ap	2) FrP	1) H	5) KrF	8)RV	7) Sp	6) SV	4) V
int.tal.1	.109557	.258078	.157730	.100727	.067685	.050756	.089502	.165961
int.tal.2	.107927	.396401	.201087	.052727		.004254	.022390	.215211
int.tal.3	.095892	.472383	.231673	.000000			.000000	.200050
int.tal.4	.070570	.532711	.279667	.106537			.010513	
int.ele.5	.001618	.655968	.340585	.001827				
int.tal.6	.000000	.558862	.441137					
int.tal.7	.268907		.731092					

Table 3b

NHH-data 1993 (Ap, FrP, H, KrF, RV, Sp, SV, V). Distribution of the support from voter 11

voter 11	3) Ap	1) FrP	2) H	4) KrF	8) RV	6) Sp	7) SV	5) V
int.tal.1	.109557	.306408	.109400	.154884	.067685	.089111	.051147	.111803
int.tal.2	.107927	.496176	.101312	.195694		.026644	.000000	.072244
int.tal.3	.095892	.614821	.089235	.172102			.000000	.027948
int.tal.4	.070570	739535	.072843	.106537			.010513	
int.tal.5	.001618	.994774	.001779	.001827				
int.tal.6	.000000	1.00000	.000000					
int.tal.7	.268907		.731092					

Table 3c

NHH-data 1993 (Ap, FrP, H, KrF, RV, Sp, SV, V). Distribution of the support from voter 118

voter 118	6) Ap	8) FrP	7) H	4) KrF	1) RV	5) Sp	2) SV	3) V
int.tal.1	.000000	.000000	.000000	.000000	1.00000	.000000	.000000	.000000
int.tal..2	.000000	.000000	.000000	.028580		.014442	.840326	.116651
int.tal.3	.000000	.000000	.000000	.000000			.953534	.046465
int.tal.4	.000000	.000000	.000000	.000000			1.00000	
int.tal.5	.000000	.000000	.000000	1.00000				
int.tal.6	.598464	.224403	.177132					
int.tal.7	1.00000		.000000					

Tax-cut STV with constraints Assume the candidate set is partitioned, $C = A \cup B$ (e.g. in male and female candidates) and that the task for an STV election is to fill s seats so that both A and B obtain at least a guaranteed number of seats. With the data from above, let A = {Ap, FrP, H, KrF}, B = {RV, Sp, SV, V}. Let s=3 and assume both A and B must be represented.

If the constraint is disregarded, H, Ap and FrP are elected in this order. To obey the constraint, one may eliminate FrP. However, it is not clear that SV, i.e. the B-candidate who stayed longest in the race, should be elected instead of FrP. That might change the political character of the election result more than necessary. The favorite B-candidate among the voters who supported FrP in vain may well have been eliminated prematurely, i.e. before SV.

Instead one may run a new single-seat election with all B-candidates, and let each voter keep the weight w(i,3) after the ranking of H and Ap ahead of FrP in a 3-candidate race. In particular, voters 10, 11 and 118 start with weights 0.558862, 1.0000, 0.224403, respectively. All voters start with one of these weights or with 0.401535 (= 1 - .59846); the latter being the weight of those with ranking Ap>FrP>H. The weight sum is v/3 = 58, and in the new intermediate tallies the successive criteria are 58/4, 58/3, 58/2.

The elected B-candidate will, according to the voters' opinion, be the best possible political replacement for FrP. Applying tax-cut STV on this 4-candidate the tally is as follows:

Table 4

Percentages charged by the B-candidates when the voters carry weights from the main tally.

taxes	RV	Sp	SV	V
int.tal.1	100.000	44.301	52.159	35.477
int.tal.2		78.028	100.000	58.328
int.tal.3		100.000		81.966

Candidate V emerges as the clear winner, and the 3 seats go to H, Ap and V.

Reversed tax-cut STV The nonmonotonicity in the STV variations discussed above is caused by eliminations. The idea behind the associated strategic voting was to obtain transfer of votes to a candidate x by the elimination of a candidate y, who most likely will be politically close to x. Such eliminations are avoided when the tally is based on the reversed rankings for each voter, because then only elected candidates disappear from the race. Table 5 illustrates this on the real data discussed above.

Table 5

NHH-data 1993 (Ap, FrP, H, KrF, RV, Sp, SV, V). Reversed tax-cut STV.

taxes	Ap	FrP	H	KrF	RV	Sp	SV	V
int.tal.1	73.076	28.907	100.000	34.474	15.054	25.207	23.459	31.122
int.tal.2	100.000	38.673		49.837	17.724	32.922	29.918	43.532
int.tal.3		59.202		100.000	22.429	51.351	45.561	76.744
int.tal.4		74.920			27.495	68.298	62.296	100.000
int.tal.5		96.995			36.512	100.000	90.077	
int.tal.6		100.000			45.282		86.958	
int.tal.7					58.389		100.000	

The final ranking becomes H>Ap>KrF>V>Sp>FrP>SV>RV. Now a high limit tax-rate is better than a low one. The reversed tax-cut STV gives higher priority to avoid the "worst" candidate than to elect the "best". Table 5 indicates that it favors candidates in the political center, as is also the case with the Coombs variation of STV. The results in Table 5 may be compared to those of Table 2. The candidates KrF, V, Sp, generally recognized as the political center, move up 1+2+2 steps, the right wing candidate FrP and the left wing candidate SV move down 3+2 steps.

Starting from the bottom of the ballot, a voter gives "anti-support" to the candidates, and a candidate x starts to cut the tax (i.e. reject more anti-support) when the criterion S(x) = v/p is reached.

As shown in table 5, only elected candidates leave the race; after intermediate tally no. 1 H emerges as the winner, and is removed from the further contest. The strategy of example 3 is not possible with the reversed STV. The main cause for nonmonotonicity has been removed and associated opportunity strategic voting reduced, because when a candidate leaves the race, it is by being elected, and one more seat becomes occupied.

An obvious idea for strategic manipulation is to give a candidate too many bottomranks to remain until the end. A determined avoid-H-group of more than 1/8 of the voters can achieve their goal by strategically ranking H last; that forces H to reduce the tax below 100%, and some other candidate will get first place in the final ranking. Generally, with p candidates and v voters, a group of more than v/p voters could prevent the election of a candidate.

There are also some obvious protective measures. In a single-seat constituency the party H may enter several candidates; it is hard for the conspirators to hit several targets simultaneously, because their voting power gets reduced every time they succeed in preventing the election of one H-candidate.

One may also consider running intermediate tax-cut tallies "from the top" until the number of remaining candidates is small enough to make such manipulation unrealistic, and then pass to the reversed rankings.

Using a single tally In a muliti-seat constituency, the individual candidate would get some protection against strategic bottom-ranking if there was only the first of the planned intermediate tallies, i.e. so that the final ranking would be H>Ap>KrF>V>FrP>Sp>SV>RV. It would take a larger anti-H group to make H’s limit tax-rate so low that H does not gain anyone of several seats.

Finally we remark that the first intermediate tally alone, in comparison with standard STV, avoids premature eliminations, reduces free riding, but still transfers unused voting power, and it can be made monotonic:

Corollary to theorem 2 A reversed tax-cut-STV with a single tally is monotonic on the set of real candidates if at least one half of the candidates are dummies, and each ballot starts with the dummies.

Weight reductions In a constituency with s>1 seats it is also possible to incorporate the idea of weight reductions in the reversed tax-cut STV, electing one candidate at a time. Let w(i,t) be the initial weight of voter i in tally no. t, and let w*(i,t) be the antisupport from voter i to the candidate elected in the tally. Then

$$\Sigma_i \, w^*(i,t) = [p+1-t]^{-1} \cdot \Sigma_i \, w(i,t) \tag{13}$$

where p is the number of candidates. The initial weights are normalized to give the same sum as before election to seat no. t in the standard STV. Then

$$\Sigma_i \, w(i,t) = v \cdot [1-(t-1)/(s+1)] \tag{14}$$

where v is the number of voters. New initial weights will be assigned by means of a formula,

$$w(i,t+1) = a(t) \cdot w^*(i,t) + b(t) \tag{15}$$

with nonnegative a(t) and b(t). The term $a(t) \cdot w^*(i,t)$ should be interpreted as a compensation to voter i proportional to the frustration the election in tally t caused voter i; it will have an effect on the distribution of seats similar to the weight reductions through taxation in standard STV.

Summation over i in (15), substitution of (13) and (14), and multiplication with (s+1)/v yield

$$1 - t/(s+1) = a(t) \cdot [1 - (t-1)/(s+1)] \cdot [p+1-t]^{-1} + b(t) \tag{16}$$

Letting the maximal weight be 1, as in standard STV, we get by (15)

$$1 = a(t) \cdot 1 + b(t) \tag{17}$$

From (15) and (17) we have

$$1 - w(i,t+1) = a(t) \cdot (1 - w^*(i,t)) \tag{18}$$

while (16) and (17) determine

$$a(t) = \frac{t \cdot (p+1-t)}{s \cdot (p+1-t) + p - s - 1} \tag{19}$$

As in standard STV the average w(i,t) decreases by 1/(s+1) when t increases by 1, but it happens that w(i,t+1)>w(i,t). This is the case when w*(i,t) = w(i,t) < 1, i.e. when the bottomranked candidate in ballot i was elected in tally no. t.

In table 6 the same data are used again to elect s=3 candidates with reversed tax-cut STV and weight reductions.

Table 6

NHH-data 1993 (Ap, FrP, H, KrF, RV, Sp, SV, V). 3 seats. Reversed rankings and weight reductions.

taxes	Ap	FrP	H	KrF	RV	Sp	SV	V
int.tal.1	73.076	28.907	100.000	34.474	17.909	25.207	23.459	31.122
int.tal.2	100.000	37.679		49.957	19.110	33.705	30.709	44.146
int.tal.3		57.923		100.000	22.687	52.316	45.694	77.294

In the second tally the voting power increases with the voter's level of frustration after the election of H in the first tally. Maximal frustration is experienced by a voter i who ranked H last: then w*(i,1)=1 in (15). The election of H in the first tally strengthens the opposition to candidate FrP. In the second tally FrP is forced to cut taxes from 38.673 % in Table 5 to 37.679 % in Table 6, and in the third tally from 59.202 % to 57.923 %. All the other candidates are somewhat better off. The example indicates that weight reductions are not likely to change the results in a multi-seat reversed tax-cut STV unless there are two or more candidates with almost equally strong electoral support.

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