

# Optimal Pension Insurance Design\*

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## Abstract

In this paper we provide a framework for how the traditional life and pension contracts with a guaranteed rate of return can be optimized to increase customers' welfare. Given that the contracts have to be priced correctly, we use individuals' preferences to find the preferred design. Assuming CRRA utility, we cannot explain the existence of any form of guarantees. Through numerical solutions we quantify the difference (measured in security equivalents) to the preferred Merton solution of direct investments in a fixed proportion of risky and risk free assets. The largest welfare loss seems to come from the fact that guarantees are effective by the end of each year, not only by the expiry of the contract. However, the demand for products with guarantees may be explained through behavioral models accounting for loss aversion, e.g. cumulative prospect theory. In this case, the optimal design seems to be a simple contract with a life-time guarantee.

**Keywords:** Household Finance; Portfolio Choice; Life and Pension Insurance; Prospect Theory

**JEL Classification:** G11, G13, G22

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# 1 Introduction

In this paper we combine previous work on valuation on life and pension insurance products with well-developed theories on individuals' preferences in order to optimize customers' utility. We analyze the welfare effects of different components of pension insurance contracts, including annual guarantees. We find that contracts that are closest to a linear payout function give highest welfare. The annual guarantee seems to move away from linearity and hence lower the customers' welfare. Finally, we show that a behavioral model accounting for loss aversion may explain the existence of some form of guarantee. A simple model with life-time guarantee seems to work best in this case.

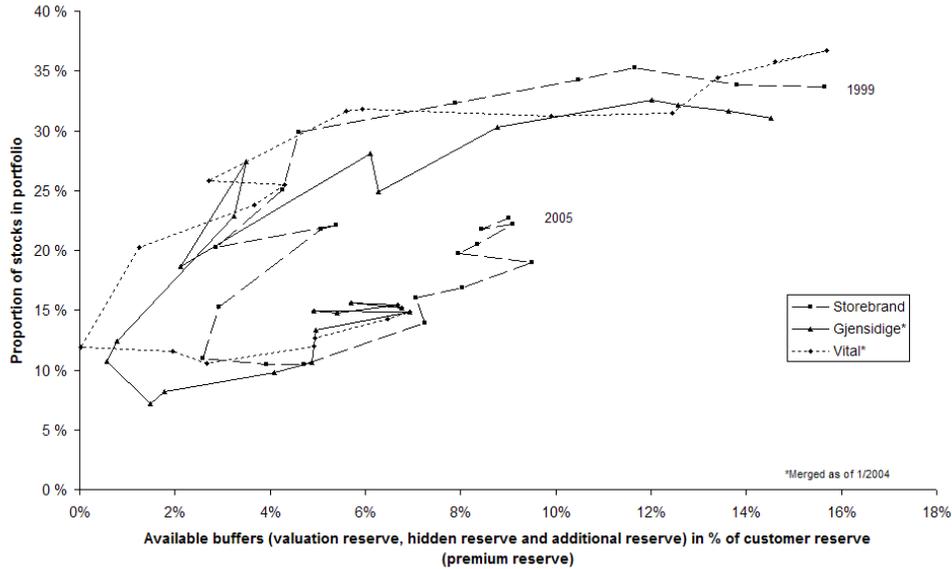
Our paper contributes to the field of household finance, defined by Campbell (2006) as how households use financial instruments to attain their objectives. Assuming that all prices are correct, i.e. that companies do not make profit nor losses, we define a class of contracts, from which the customer can choose. Based on a set of preferences, the customer will then select his optimal contract. We assume that the customers' preferences can be described using the von Neumann and Morgenstern (1944) framework of expected utility. Furthermore, we use the conventional constant relative risk-aversion (CRRA) as our main representation of preferences.

From the Borch (1962) condition we know that any utility function within the broader class of hyperbolic absolute risk aversion(HARA) utility function (including CRRA) induces linear sharing rules, meaning that each individual will get a fixed proportion of total wealth in any state of the economy. In our case, any kind of guarantee will inevitably lead to the customer receiving a higher proportion of total wealth in the states where the guarantee is effective. According to Borch, such a non-linear sharing rule will not be optimal.

However, it is likely that actual behavior will not coincide with the standard theories on optimal behavior as described above. As Campbell (2006) writes, "household finance poses a particular challenge to this agenda, because many households seek advice from financial planners and other experts, and some households make decisions that are hard to reconcile with this advice or with any standard model. One response to this is to maintain the hope that actual and ideal behavior coincide, but to consider non-standard behavioral models of preferences incorporating phenomena such as loss aversion and mental accounting." We alternatively explain the existence of guaranteed pension products by introducing behavioral models. We show that both a behavioral model within the expected utility framework (utility function with loss aversion) and outside, cumulative prospect theory (CPT), rationalize guaranteed features of the contract. We focus on CPT, initiated by Tversky and Kahneman (1992), since this model is the most developed and thoroughly investigated.

The main function of most modern life and pension insurance contracts is that of a savings product, distributing financial market risk between customers and shareholders of the life insurance company. Despite the fact that there are no international standard contracts, a

number of common properties determine the risk sharing function, e.g. asset allocation, guaranteed interest rate, the profit sharing and the capital structure of the company.



Proportion of stocks in portfolio versus available buffers in % of customer reserve. Sources: Quarterly reports and analyst presentations.

Figure 1: Quarterly development of the asset allocation of Norwegian life insurers - 1999-2005.

Companies in the same market tend to follow each other closely when it comes to asset allocation (see figure 1). Companies diverging from the "market standards" risk losing customers if their bet does not work as planned, while the upside is more limited. We use the conventional method of fixing asset allocation at the start of the contract (as used e.g. by Grosen and Jørgensen (2000), Hansen and Miltersen (2002), and Miltersen and Persson (2003)), but one could also consider more general versions, allowing for time-dependent, but deterministic allocations, or even allocations being a function of some stochastic process.<sup>1</sup>

Guaranteed rates of return are normally defined in pension contracts as an annual property. Companies are obliged to grant a guaranteed amount in one year, and bonuses given cannot be recalled and used as guaranteed return. However, as we will describe later, we also show a simplified contract, where guarantees are only effective at the expiry of the contract.

The return above the guarantee is shared between the company and the customer. In different countries this profit sharing is regulated by a number of different procedures, ranging from predetermined sharing rules to full company discretion from year to year (limited only by competitive pressure). As the market pressure is hard to assess in a theoretical model, we find it useful letting profit sharing be determined by a set of fixed rules, as in e.g. Briys and

<sup>1</sup>The Hansen and Hansen (2003) method of continuous optimization of the asset allocation with respect to the customers' preferences will typically be unrealistic when the company decides the asset allocation.

Company	Country	Year	Description	Consequences
Executive Life	USA	1991	At the end of the 1980s several US life insurers faced financial distress due to losses on real estate and junk bonds. Among the public, this led to a lack of confidence, causing a flood of surrenders. Executive Life had to file for chapter 11 protection after 1990 surrenders of more than 10 times the levels 4 years earlier.	Administration and run-off of the portfolio was administrated by the insurance supervisory authorities of the state of California and a new company, Aurura was set up by a French consortium. Still, policyholders who didn't surrender before the bankruptcy lost part of their promised amounts.
Nissan Mutual Life	Japan	1997	Nissan Mutual Life collapsed under the combination of high guarantees (70% of the liabilities yielded 5.5%) and low investment yield, due to both low interest rates and a bearish equity market. It was the first Japanese life insurer to go bankrupt after WW2. The equity of company had likely been negative for several years.	Around 2/3s of the net losses to customers were covered by the policyholder protection program, a mandatory program for all Japanese insurers. A run-off-company, Aoba Life was established, and later acquired by the French company Artemis.
Equitable Life	UK	2002	The oldest mutual company in the world went down due to a combination of very high guarantees and wrong assessments of longevity risk in pension products. Failure to meet the guarantees and a lost court appeal to reduce guarantees almost caused Equitable to file for bankruptcy.	Customers faced large losses that despite complaints against supervisory authorities have not been compensated by the government. The active part (salesforce etc) of Equitable Life was sold to Halifax. In a compromise deal customers voted in favor of a rescue operation including a cut in payments to customers by appr. 20%.
Mannheimer	Germany	2003	In the first default scenario of a German insurer for more than 50 years, Mannheimer had to close acquisition of new business following large losses on the equity market after the millennium bubble. The group's non-life business also came under pressure.	Customers' claims were saved due to an issue of new capital by the Austrian insurer Uniqa who acquired a majority of the shares of Mannheimer.

Sources: Press clipping, annual reports, and Briys and de Varenne (2001), chapter 3.

Table 1: Overview of large life insurance financial distress situations.

de Varenne (2001). Again, more general versions will be allowing for time-dependent, but deterministic sharing rules or sharing rules being a function of some stochastic process.

We assume that the capital structure of the company is fixed only at time zero. In line with Miltersen and Persson (2003) we do not allow for dividend payments, nor any form of capital issues. The company will default at the time where book equity is negative after guarantees are met. However, as we describe in section 2, we also show simpler contracts, where bankruptcy (and guarantees) are only effective at expiry, or where shareholders will always pick up losses (unlimited responsibility). At that time the customers will take over all of the company's assets. Further compensation (rescue operations) from the government is not included. While in property & casualty insurance there frequently exist government supported guarantee funds, such funds are rarely seen in life and pension insurance. Practice shows that such rescue operations can hardly be counted on, as in most of the larger recent

defaults of life companies, governments have chosen not to intervene (see table 1 for details).

In line with most literature on this topic (e.g. Grosen and Jørgensen (2000), Hansen and Miltersen (2002), and Miltersen and Persson (2003)) we will not cover pure actuarial risk elements, like mortality risk, disability risk, longevity risk, etc, or any type of administrative costs. Neither will we cover any part of the premium set aside to cover such elements, which means that we assume that the full initial payment from customers go into a form of savings account.

In order to make sure that all contracts have the same value to the company, we ensure that pricing is correct by assigning a profit sharing that fits the other parameters. Individuals are then allowed to choose from the set of correctly priced contracts. As previously explained, we then use CRRA preferences to evaluate the contracts from the individuals' perspective.

There has been limited focus on whether L & P contracts are suited to satisfy customers' welfare. Previous research has focused on pricing life and pension insurance contracts. Only a few papers have used similar models to analyze welfare effects of guaranteed products. Brennan (1993) elaborates on the classical point made by Borch (1962) that guaranteed products will lead to a welfare loss, but without quantifying the effect further. Jensen and Sørensen (2001), and Consiglio, Saunders, and Zenios (2006) builds on this point by quantifying the effects in various cases of life-time interest rate guarantees. We elaborate further the welfare effects of different contract design. To our knowledge, no one has previous investigated the value of contract design in a behavioral framework.

Our paper is organized as follows: In section 2 we present the different features of our model. The numerical examples in section 3 illustrate the efficiency loss of the different components of the contract. Section 4 consists of the same analyzes as section 3 except that we use Cumulative Prospect Theory (CPT) instead of standard expected utility. Finally we conclude.

## 2 The Model

We assume a standard no-arbitrage economy with two assets, a risk free bank account,  $D_t$  and a risky equity index,  $S_t$ . The dynamics of the asset classes are given by:

$$dD_t = rD_t dt, \quad D_0 = d \tag{1}$$

$$dS_t = \mu S_t dt + \sigma S_t dZ_t, \quad S_0 = s \tag{2}$$

where  $r$  is the constant risk-free interest rate,  $\mu$  is the constant expected return on the equity index,  $\sigma$  is the constant volatility of the equity index, and  $Z_t$  is a standard Brownian motion. A proportion  $\theta_t$  is invested in the equity index. We will assume that the proportion of the equity index is fixed, i.e. that  $\theta_t = \theta$ . The dynamics of the total asset portfolio  $A_t$  under the real probability measure  $P$  is then given by

$$dA_t = (rA_t + \theta(\mu - r)A_t)dt + \theta A_t \sigma dZ_t, \quad A_0 = a. \quad (3)$$

Design of "fair contracts" is done under the equivalent martingale measure  $Q$  (Harrison and Kreps, 1979), given by

$$dA_t = rA_t dt + \theta A_t \sigma dZ_t^Q, \quad A_0 = a \quad (4)$$

where  $Z_t^Q$  is the standard Brownian motion under  $Q$ .

### Saving Vehicles

Assets	Liabilities
$A_0$	$E_0 = \alpha A_0$ $B_0 = 0$ $L_0 = (1 - \alpha)A_0$

The table shows the balance sheet of the insurer at the start of the contract.

Table 2: Balance sheet at time  $t = 0$

We describe the following alternative saving vehicles:

1. The customers directly choose the asset allocation, i.e. Merton's problem (Merton, 1971).
2. The customer return has a floor similar to a put option, e.g. index-linked bonds.
3. The customer return has a floor, however face the risk of the company defaulting, e.g. the simple life insurance problem of Briys and de Varenne (1994).
4. The guarantees embedded in the product are realized on an annual basis, i.e. annual guarantees.

The liability side of the insurer's balance sheet consists of  $E_t$  being the equity of the company,  $L_t$  being the reserves (customers' funds) and  $B_t$  being the bonus account (to be further described in section 2.4). In the Merton problem (1 above) we define  $E_t = 0$  and  $B_t = 0$  for all  $t$  and in the cases of index-linked bonds and simple life insurance (2 and 3 above) we define  $B_t = 0$  for all  $t$ .

The initial balance sheet of the insurer (at time  $t = 0$ ) is shown in table 2, where  $\alpha$  is defined as the proportion of equity to total assets at time  $t = 0$ .

### 2.1 The Merton Problem

The Merton problem is the one of an individual investor who make direct investments in the two assets described above. We formulate the problem definition

$$\max_{\theta} U = \max_{\theta} E(u(L_T)) \quad (5)$$

where  $u$  is the customer's utility function, with the usual assumptions that  $u' > 0$  and  $u'' < 0$ . Under CPT the U-function will be replaced with a V-function to be defined section 4.1.

## 2.2 Index-linked Bonds

In the index-linked bond contract the customer has an absolute guarantee on receiving a promised amount at the expiry of the contract. If the assets in the company is insufficient to cover the guarantee, the customer has the right to extract the missing amount from the owners of the company. This is close to the type of situation faced by single companies or products in a larger group setting, e.g. index-linked bonds as part of a wide menu of products in a financial conglomerate.

At date zero the company receives an initial amount of assets  $A_0$  which they invest in a risk free asset (bank account -  $D$ ) and a equity index ( $S$ ). The investment comes from customers, providing an amount  $L_0$ , and owners, providing an amount  $E_0$ . The ratio of capital provided by owners ( $E_0/A_0$ ) is noted  $(1 - \alpha)$ .

At the payout date  $T$ , the assets of the company are split according to the following rules:

1. The customer has a claim of his initial investment capitalized by a guaranteed rate  $g$ , in total amounting to  $L_0 e^{gT}$ .
2. The owner has a second priority claim of his proportion  $(1 - \alpha)$  of the total assets at time  $T$ , amounting to  $(1 - \alpha)A_T$ .
3. The remaining profit is split with a proportion  $\delta$  to the customer and  $(1 - \delta)$  to the owners.

The payout structure is illustrated in the forthcoming figure 2. Rearranged and formalized

$$L_0 = \alpha A_0 \quad (6)$$

$$L_T = \begin{cases} L_0 e^{gT} & \text{if } A_T < \frac{1}{\alpha} L_0 e^{gT} \\ L_0 e^{gT} + \alpha \delta (A_T - \frac{1}{\alpha} L_0 e^{gT}) & \text{if } A_T \geq \frac{1}{\alpha} L_0 e^{gT}. \end{cases}$$

We search for "fair contracts", i.e. solutions where

$$L_0 = \alpha A_0 = e^{-rT} E^Q(L_T) \quad (7)$$

where

$$L_T = L_0 e^{gT} + \alpha \delta (A_T - \frac{1}{\alpha} L_0 e^{gT})^+ \quad (8)$$

or, equivalently

$$E_0 = (1 - \alpha)A_0 = e^{-rT} E^Q(A_T - L_T) \quad (9)$$

where  $A_T - L_T = E_T = A_T - L_0 e^{gT} - \alpha \delta (A_T - \frac{1}{\alpha} L_0 e^{gT})^+$ .

Following Black and Scholes (1973) we can define date  $t$  market value  $M_t(L_T)$  of the customers contract:

$$\begin{aligned} M_t(L_T) &= M_t[L_0 e^{gT} + \alpha \delta (A_T - \frac{1}{\alpha} L_0 e^{gT})^+] \\ &= L_0 e^{gT} e^{-r(T-t)} + \alpha \delta (A_t N(d_1) - \frac{1}{\alpha} L_0 e^{gT} e^{-r(T-t)} N(d_2)), \end{aligned} \quad (10)$$

where

$$d_1 = \frac{(r - g + \sigma_A^2/2)(T - t)}{\sigma_A \sqrt{(T - t)}}, \quad d_2 = d_1 - \sigma_A \sqrt{(T - t)}, \quad (11)$$

$$\sigma_A = \theta \sigma,$$

and,

$$N(d) = \int_{-\infty}^d \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx. \quad (12)$$

### Delta, $\delta$

In our model we assume all parameters are set at time 0, therefore no time index on the control variables  $g$ ,  $\theta$ , and  $\alpha$ . We let  $\delta$  be the residual parameter that makes the contract fair, thus solve equation (10) at time  $t = 0$  with respect to  $\delta$ :

$$\delta = \frac{1 - e^{gT} e^{-rT}}{N(d_1) - e^{gT} e^{-rT} N(d_2)}. \quad (13)$$

We find that  $\delta$  is independent of  $L_0$ ,  $\alpha$  and  $A_0$ .

### Effective theta, $\Theta_t$

The effective proportion of the customers' wealth held in the equity index at time  $t$ ,  $\Theta_t$ , is given by:

$$\begin{aligned} \Theta_t &\equiv \frac{\frac{\partial L_t}{\partial S_t}}{L_t} \\ &= \frac{\theta A_t \alpha \delta N(d_1)}{L_0 e^{gT} e^{-r(T-t)} + \alpha \delta (A_t N(d_1) - \frac{1}{\alpha} L_0 e^{gT} e^{-r(T-t)} N(d_2))}. \end{aligned} \quad (14)$$

The exposure,  $\Theta_t$ , vary over time and as a function of wealth. Hence a fixed equity

index exposure cannot be achieved by an index-linked bond contract with fixed parameters. However, in the numerical results we will show that the effective theta,  $\Theta_t$ , is closer than the company's asset allocation ( $\theta$ ) to the asset allocation of the Merton solution.

The date 0 effective theta,  $\Theta_0$  is given by:

$$\begin{aligned}\Theta_0 &= \frac{\frac{\partial L_0}{\partial S_0}}{L_0} \\ &= \frac{\theta \delta N(d_1)}{e^{gT} e^{-rT} + \delta(N(d_1) - e^{gT} e^{-rT} N(d_2))}.\end{aligned}\tag{15}$$

The date 0 exposure,  $\Theta_0$ , is independent of  $L_0$ ,  $\alpha$  and  $A_0$ .

### Maximization Problem

Rearranging equation (8)

$$L_T = \begin{cases} L_0 e^{gT} & \text{if } A_T < A_0 e^{gT} \\ L_0 e^{gT} + \delta L_0 (e^{\hat{r}T} - e^{gT}) & \text{if } A_T \geq A_0 e^{gT} \end{cases}$$

where  $\hat{r} = \ln(A_T / (T * A_0))$  is the realized return on the investment portfolio. The problem is now independent of  $\alpha$ . Due to the assumption that owners' capital is callable, even if outside the company, a high  $\alpha$  is not required in order to secure the guaranteed amounts to customers and hence  $\alpha$  is redundant. The maximization problem can be formulated as:

$$\max_{\theta, g} E(u(L_T))\tag{16}$$

subject to the restrictions above and the restriction that the contracts are fair, i.e. that

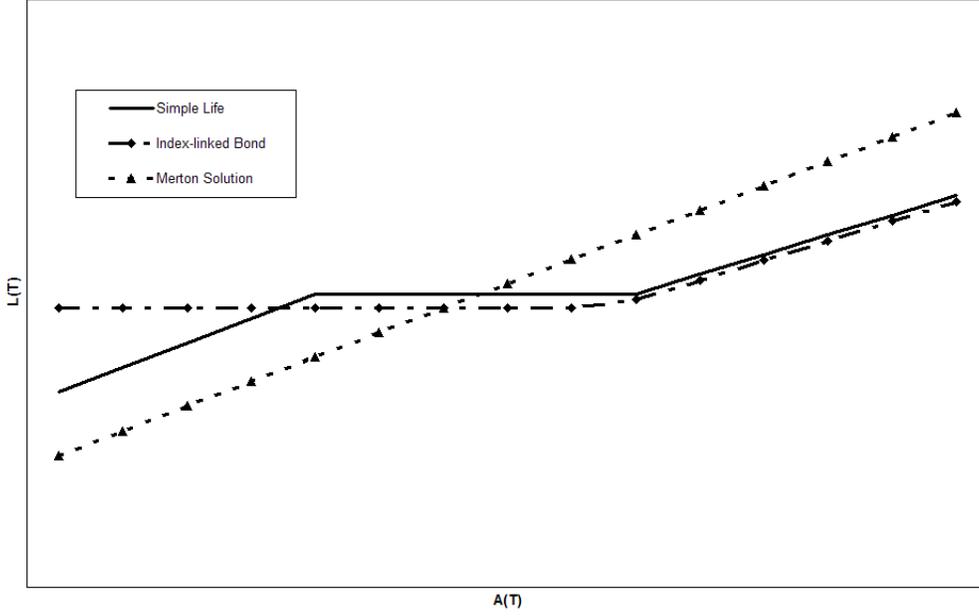
$$e^{-rT} E^Q(A_T - L_T) = E_0.\tag{17}$$

We solve this problem numerically, see appendix A for further details.

### 2.3 Simple Life Insurance

Contrary to the previous section, the simple life contract allows the company to default without any obligation for the owners to insert more capital. This is typical for a public company where life insurance is the main or only business. This type of contract was first described by Briys and de Varenne (1994). At the payout date T, the assets of the company are split according to the following rules:

1. The customer has a claim of his initial investment capitalized by a guaranteed rate  $g$ , in total amounting to  $L_0 e^{gT}$ .



The figure illustrates payoff patterns of different contracts at time  $T$ . Customer reserve ( $L_T$ ) versus value of company ( $A_T$ ).

Figure 2: Payoff patterns

2. The owner has a second priority claim of his proportion  $(1 - \alpha)$  of the total assets at time  $T$ , amounting to  $(1 - \alpha)A_T$ .
3. The remaining profit is split with a proportion  $\delta$  to the customer and  $(1 - \delta)$  to the owners.

In figure 2 we give a comparison of the form of  $L_T$  as a function of  $A_T$  for the three contracts given (Merton problem, index-linked bonds, and simple life). Rearranged and formalized

$$L_0 = \alpha A_0 \quad (18)$$

$$L_T = \begin{cases} A_T & \text{if } A_T \leq L_0 e^{gT} \\ L_0 e^{gT} & \text{if } L_0 e^{gT} \leq A_T \leq \frac{1}{\alpha} L_0 e^{gT} \\ L_0 e^{gT} + \delta(\alpha A_T - L_0 e^{gT}) & \text{if } A_T \geq \frac{1}{\alpha} L_0 e^{gT} \end{cases}$$

We search for solutions where

$$L_0 = \alpha A_0 = e^{-rT} E^Q(L_T) \quad (19)$$

where

$$L_T = A_T - (A_T - L_0 e^{gT})^+ + \alpha \delta (A_T - \frac{1}{\alpha} L_0 e^{gT})^+. \quad (20)$$

Following Black and Scholes (1973) we can again define date  $t$  market value  $M_t(L_T)$  of the customers contract:

$$\begin{aligned} M_t(L_T) &= M_t[L_0e^{gT} - (L_0e^{gT} - A_T)^+ + \alpha\delta(A_T - \frac{1}{\alpha}L_0e^{gT})^+] \\ &= L_0e^{gT}e^{-r(T-t)}N(d'_2) + A_t(1 - N(d'_1)) + \alpha\delta(A_tN(d_1) - \frac{1}{\alpha}L_0e^{gT}e^{-r(T-t)}N(d_2)) \end{aligned} \quad (21)$$

where

$$\begin{aligned} d_1 &= \frac{(r - g + \sigma_A^2/2)(T - t)}{\sigma_A\sqrt{(T - t)}}, & d_2 &= d_1 - \sigma_A\sqrt{(T - t)} \\ d'_1 &= d_1 - \frac{\ln\alpha}{\sigma_A\sqrt{(T - t)}}, & d'_2 &= d'_1 - \sigma_A\sqrt{(T - t)} \\ \sigma_A &= \theta\sigma \end{aligned} \quad (22)$$

### Delta, $\delta$

We let  $\delta$  be the residual parameter that makes the contract fair, thus solve equation (21) at time  $t = 0$  with respect to  $\delta$ :

$$\delta = \frac{\alpha - \alpha e^{gT}e^{-r(T-t)}N(d'_2) - 1 + N(d'_1)}{\alpha(N(d_1) - e^{gT}e^{-r(T-t)}N(d_2))} \quad (23)$$

where  $\delta$  is independent of  $L_0$  and  $A_0$ .

### Effective theta, $\Theta_t$

The effective proportion of the customers's wealth held in the equity index at time  $t$ ,  $\Theta_t$ , is given by:

$$\Theta_t = \frac{\theta A_t[1 - N(d'_1) + \alpha\delta N(d_1)]}{L_0e^{gT}e^{-r(T-t)}N(d'_2) + A_t(1 - N(d'_1)) + \alpha\delta(A_tN(d_1) - \frac{1}{\alpha}L_0e^{gT}e^{-r(T-t)}N(d_2))}. \quad (24)$$

The proportion,  $\Theta_t$ , vary over time and as a function of wealth. Hence a fixed equity index exposure cannot be achieved by a life insurance contract with fixed parameters.

Date 0 effective theta,  $\Theta_0$ :

$$\Theta_0 = \frac{\theta[1 - N(d'_1) + \alpha\delta N(d_1)]}{\alpha e^{gT}e^{-rT}N(d'_2) + 1 - N(d'_1) + \alpha\delta(N(d_1) - e^{gT}e^{-rT}N(d_2))}. \quad (25)$$

In this case,  $\Theta_0$  is independent of  $L_0$  and  $A_0$ .

### Maximization Problem

The problem can now be formulated as follows:

$$\max_{\alpha, \theta, g} E(u(L_T)) \quad (26)$$

subject to the restriction that the contracts are fair. We solve this problem numerically, see appendix A for further details.

## 2.4 Annual Guarantees

As mentioned in the introduction, the existence of annual guarantees calls for a different treatment of contracts. We solve this by doing year-by-year-simulations and by declaring bankruptcy if book equity at the end of year turns out to be negative. In addition, bonuses are calculated at the end of each year and credited to the reserve. However, in order to keep the model as simple as possible, we do not allow for the company neither to pay dividends nor to issue new equity. Neither do we allow companies to run at negative equity for a period of time, even though this is commonly seen in practice.<sup>2</sup>

Annual dynamics of the contracts are performed in a similar way as in section 2.3 with the exception of the bankruptcy possibility which also have consequences for coming years:

$$\begin{aligned} L_0 &= \alpha A_0 & E_0 &= (1 - \alpha)A_0 & (27) \\ L_t &= \begin{cases} A_t & \text{if } A_t \leq L_{t-1}e^g \\ L_{t-1}e^g & \text{if } L_{t-1}e^g \leq A_t \leq \frac{1}{\alpha}L_{t-1}e^g \\ L_{t-1}e^g + \delta\alpha(A_t - L_{t-1}e^g) & \text{if } A_t \geq \frac{1}{\alpha}L_{t-1}e^g \end{cases} \\ E_t &= A_t - L_t. & (28) \end{aligned}$$

In the case of a bankruptcy, customers will receive the full value of the company's assets (we assume no bankruptcy costs). We assume this is invested in the risk free asset, such that:

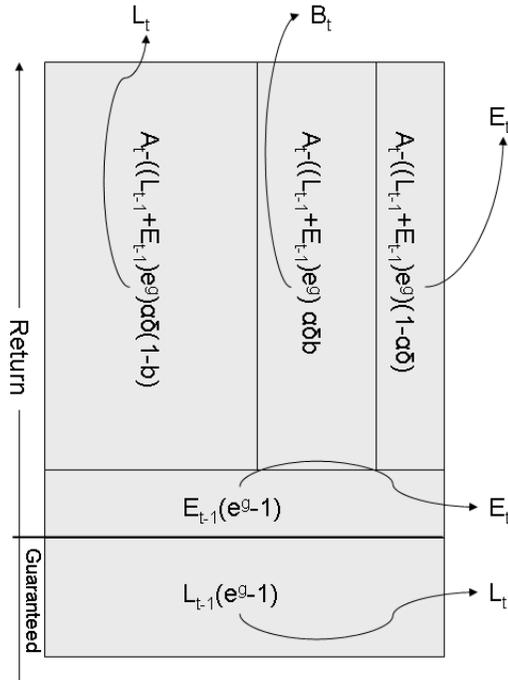
$$\begin{aligned} E_T &= 0 \\ L_T &= A_\tau e^{r(T-\tau)} \end{aligned} \quad (29)$$

where  $\tau$  is the (stochastic) time of bankruptcy. The assumption that investments after bankruptcy is done solely in the risk free asset may give a penalty that is unrealistic. However, our assumption of no bankruptcy costs will (at least partly) offset this.

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<sup>2</sup>See Briys and de Varenne (2001), page 59 for anecdotal evidence.

## The Bonus Reserve



The figure illustrates how return is split between different types of capital. The first part of the return (guarantee) is allocated to the customer reserve, then a part is allocated to equity, while return above is split between customer reserve, bonus reserve, and equity.

Figure 3: Contract design

In order to provide buffers for companies to meet bad years in the security markets, regulators in most countries allow for (and to a certain extent require) the build up of buffers of capital that are yet to be allocated to customers' reserves. These buffers have different forms, importance and names from country to country, e.g. bonus reserves, value adjustment reserves, unrealized gains (reserves), fund for future appropriations, etc. We name them bonus reserves,  $B_t$ . Bonus reserves can be used if the achieved return is not sufficient of covering guaranteed returns.

Allocation to bonus reserves in practice are done in a number of ways, e.g. through allocating a proportion of bonuses each year, allocating unrealized gains on various types of securities, increasing the funds in the same rate as the other reserves, bringing the bonus reserve to a target level, etc. We shall use a simple allocation mechanism similar to the method described by Miltersen and Persson (2003). More sophisticated methods exist, see e.g. Grosen and Jørgensen (2000) where allocations to the bonus reserves are also a function of a given target level (relative to reserves). However, for our purpose the gain of using such

methods is limited.

In our model we credit the bonus reserves by a proportion of declared bonuses,  $b$ . Figure 3 illustrates the allocation rules. The bottom part of the return covers the guaranteed amount. If returns exceeds the level of the guarantee, an amount will be used to cover a similar return on shareholders' capital. Then, if there still is something left, the remaining return will be split proportionally between equity, reserves, and bonus reserves.

Mathematically we can now show that:

$$\begin{aligned}
L_0 &= \alpha A_0 \\
E_0 &= (1 - \alpha)A_0 \\
B_0 &= 0
\end{aligned} \tag{30}$$

$$L_t = \begin{cases} A_t & \text{if } A_t \leq L_{t-1}e^g \\ L_{t-1}e^g & \text{if } L_{t-1}e^g < A_t \leq L_{t-1}e^g + E_{t-1}e^g + B_{t-1} \\ L_{t-1}e^g + \delta\alpha(1-b)(A_t - (L_{t-1}e^g + E_{t-1}e^g + B_{t-1})) & \text{if } A_t > L_{t-1}e^g + E_{t-1}e^g + B_{t-1} \end{cases}$$

$$B_t = \begin{cases} 0 & \text{if } A_t \leq L_{t-1}e^g + E_{t-1} \\ A_t - L_{t-1}e^g - E_{t-1} & \text{if } L_{t-1}e^g + E_{t-1} < A_t \leq L_{t-1}e^g + E_{t-1} + B_{t-1} \\ B_{t-1} & \text{if } L_{t-1}e^g + E_{t-1} + B_{t-1} < A_t \leq L_{t-1}e^g + E_{t-1}e^g + B_{t-1} \\ B_{t-1} + \delta\alpha b(A_t - (L_{t-1}e^g + E_{t-1}e^g + B_{t-1})) & \text{if } A_t > L_{t-1}e^g + E_{t-1}e^g + B_{t-1} \end{cases}$$

$$E_t = A_t - L_t - B_t. \tag{31}$$

In case of bankruptcy ( $A_t < L_{t-1}e^g$ )

$$\begin{aligned}
E_T &= 0 \\
B_T &= 0 \\
L_T &= A_\tau e^{\tau(T-\tau)}
\end{aligned} \tag{32}$$

where  $\tau$  is the (stochastic) time of bankruptcy.

For annual guarantees, with or without bonus reserves, closed form solutions are unavailable, and we have to rely on numerical solutions by simulation. However, including the bonus reserve, we use the same problem definition as in section 2.3, namely:

$$\max_{\alpha, \theta, g, b} E(u(L_T + B_T)) \tag{33}$$

subject to the restriction that the contracts are fair. We solve this problem numerically, see appendix A for further details.

### 3 Results with Expected Utility

#### 3.1 Power Utility

Within the expected utility framework, we assume that the customer’s utility belongs to the class of CRRA utility functions with a relative risk aversion coefficient  $\gamma$ . Then the utility can be described as a power utility function on the form

$$u(x) = \frac{1}{1-\gamma} x^{1-\gamma}. \tag{34}$$

#### 3.2 Parameters

In the example we use the following parameters:

$$\begin{aligned} A_0 &= 5 & r &= 0.04 \\ \mu &= 0.065 & \sigma &= 0.15 \\ T &= 5 & \gamma &= 3 \\ b &= 0.2 \end{aligned}$$

#### 3.3 The Merton Problem

The standard asset allocation problem, solved by Merton (1971), is to place a share equal to

$$\theta = \frac{\mu - r}{\gamma\sigma^2} \tag{35}$$

in the equity index. Given our parameters, the allocation to the equity index is  $\theta = 37\%$ . We test the numerical algorithm by finding exactly the same answer as in the analytical solution.

#### 3.4 Index-linked Bonds

Guarantee $g$	Optimal $\theta_0$	$\delta$	Effective theta $\Theta_0$	Mean return $\bar{r}_A$
0.0 %	37 %	0.984	35 %	4.65 %
0.5 %	37 %	0.972	34 %	4.63 %
1.0 %	38 %	0.947	32 %	4.61 %
1.5 %	40 %	0.900	31 %	4.58 %
2.0 %	42 %	0.824	27 %	4.53 %
2.5 %	47 %	0.684	23 %	4.45 %
3.0 %	70 %	0.399	18 %	4.35 %

Table 3: Overview Index-linked Bonds

We are free to choose the parameters  $\alpha, \delta, \theta, g$ . To find the set of parameters that gives fair contracts, we assign reasonable values to  $\alpha, \theta, g$ , and let  $\delta$  be a residual. We optimize over the set of allowable parameters.

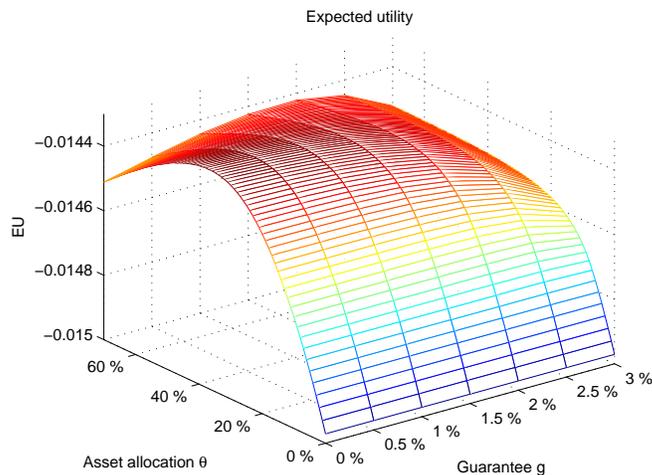


Figure 4: Optimal contract of index-linked bonds as a function of asset allocation,  $\theta$ , and guarantee,  $g$ , with  $\alpha = 0.9$

In figure 4 allocation to the equity index is shown on the x-axis, and annual guaranteed rate on the y-axis. We find that optimal  $\theta$  given  $g$  converges to the Merton solution for decreasing guarantees. This is of course due to fact that when the effect of guarantees disappear we are back to the Merton problem.<sup>3</sup> At that point the expected utility will also converge to the level of the Merton solution. A higher guarantee will increase the flat area of the  $L_T(A_T)$  (see figure 2) where the customer face no risk when t approaches T.

Further details on this problem is given in table 3. While the optimal  $\theta$  increases with the level of  $g$ , the effective exposure to the equity index,  $\Theta_0$ , decreases. Due to the non-linear time-variance of  $\theta$ , customers are reluctant to have a high  $\Theta_0$ , as the utility loss from this parameter increasing over time may be large.

### 3.5 Simple Life Insurance

The results of the simple life problem are given in figure 5. Again, we find that optimal  $\theta$  and corresponding expected utility approaches the Merton for low guarantees. For a high  $g$ , asset values in the "flat area" of  $L_T(A_T)$  (see figure 2) when t is approaching T is more likely. This is undesirable, as in this area stock market exposure approaches zero. However, the level of guarantees does not seem to matter a lot relative to the importance of asset allocation.

For a given  $\theta$  higher than 37%, however, we find that a guarantee is optimal. This is due to the  $\Theta_0$ s shown in table 4, which now are expected to be optimal close to the Merton solution because of the symmetric upside and downside of the contracts (as illustrated in figure 2). For a  $\theta$  higher than 37%, a way of reducing stock market exposure consists of introducing a

<sup>3</sup>Theoretically this only happens when  $g = -\infty$ .

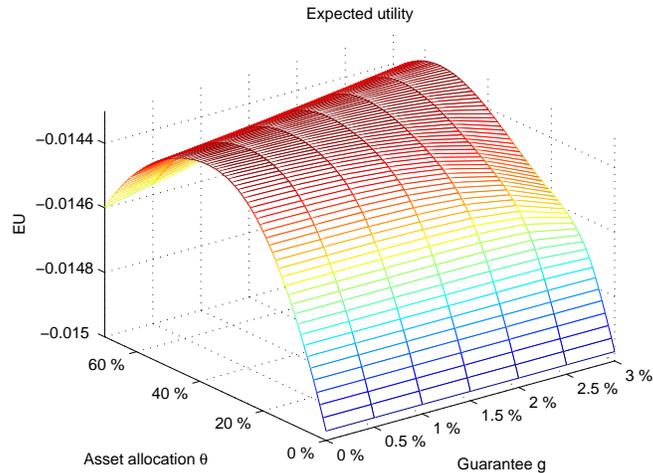


Figure 5: Optimal contract of simple life insurance as a function of asset allocation,  $\theta$ , and guarantee,  $g$ , with  $\alpha = 0.9$

Guarantee $g$	Optimal $\theta_0$	$\delta$	Effective theta $\Theta_0$	Mean return $\bar{r}_A$	Prob of default $p(d)$	Average loss given default $lgd$ of $L_0$
0.0 %	37 %	0.989	35 %	4.65 %	0.3 %	3.8 %
0.5 %	39 %	0.971	36 %	4.66 %	0.8 %	4.4 %
1.0 %	42 %	0.944	36 %	4.67 %	1.9 %	5.3 %
1.5 %	45 %	0.904	36 %	4.68 %	3.7 %	6.1 %
2.0 %	48 %	0.850	36 %	4.67 %	6.3 %	7.2 %
2.5 %	52 %	0.780	36 %	4.67 %	9.7 %	8.8 %
3.0 %	55 %	0.700	36 %	4.65 %	13.4 %	10.1 %

Table 4: Overview Simple Life

guaranteed return.

Defaults in the simple life problem will always take place in the last period. Of course, both the probability and severity of defaults increase with  $g$ . The default probabilities, particularly for high  $g$ 's, seem high, compared to the market focus on the solidity of the life insurance companies.

### 3.6 Annual Guarantees

In figure 6 we plot the expected utility for the annual guarantee case. We find that optimal  $\theta$  is increasing in guarantees as for the simple life alternative. But optimal  $\theta$  is higher, e.g. for  $g = 2\%$  optimal  $\theta = 58\%$ , versus 48% for simple life. Possible explanations for this include compensation for early defaults (when all assets are invested risk-free) and/or the fact that the bonus reserve serve as a smoothing mechanism. For low guarantees we still approach the Merton solution.

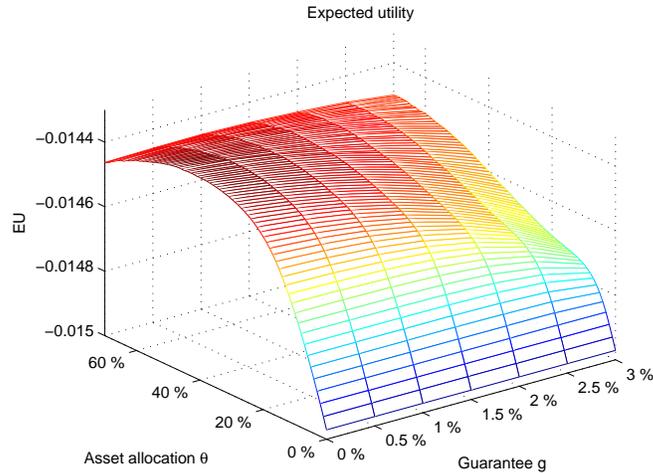


Figure 6: Optimal contract of annual guarantees as a function of asset allocation,  $\theta$ , and guarantee,  $g$ , with  $\alpha = 0.9$  and  $b = 0.2$

Guarantee $g$	Optimal $\theta$	$b$	$\delta$	Mean return $\bar{r}_A$	Prob of default $p(d)$	Average loss given default $lgd$ of $L_0$
0.0 %	43 %	20 %	0.851	4.56 %	6 %	2.7 %
0.5 %	46 %	20 %	0.793	4.53 %	9 %	2.9 %
1.0 %	48 %	20 %	0.731	4.49 %	13 %	3.4 %
1.5 %	55 %	20 %	0.640	4.48 %	21 %	4.0 %
2.0 %	58 %	20 %	0.565	4.45 %	26 %	4.6 %
2.5 %	63 %	20 %	0.484	4.43 %	32 %	5.3 %
3.0 %	68 %	20 %	0.408	4.41 %	37 %	5.9 %

Table 5: Overview Annual Guarantees

The penalty for larger guarantees is significantly larger than in the simple life case. With annual guarantees a higher guarantee means a higher probability of default at a time  $\tau < T$ , when all assets again are invested unfavorably in the risk-free asset. This must be weighted against the requirement of compensate low  $\delta$ s with high  $\theta$ s to get an optimal exposure to the stock market. In the high guarantee scenarios this trade-off turns out to be difficult to make, causing a utility loss even for an optimal  $\theta$ .

Default probabilities given in table 5 are now accumulated over  $t = 0, \dots, T$ . Probabilities are higher than for simple life due to the possibility of bankruptcy before expiry. However, the average loss in the case of default is now lower, as losses are not allowed to accumulate. Again, comparing to the real world the probabilities of default seem high. However, one should take into account that  $\theta$ s are much higher than the 15 – 20% commonly seen in continental European life insurers.

### 3.7 Comparison

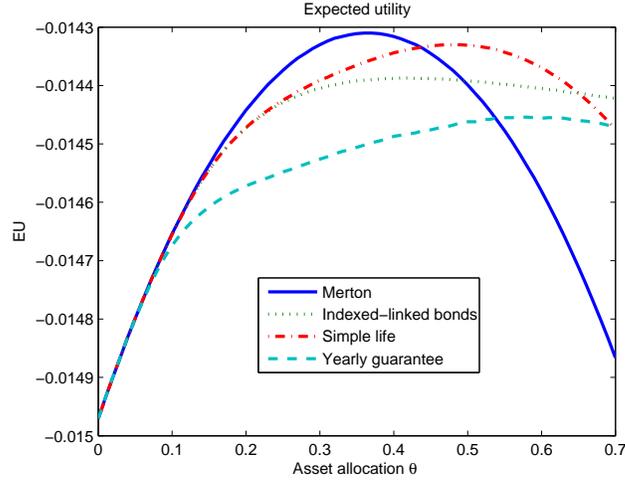


Figure 7: Optimal asset allocation under expected utility, given  $\alpha = 0.9$ ,  $g = 0.02$ , and  $b = 0.2$ .

The results for a fixed  $g = 2\%$  are given in figure 7. Highest utility is as previously noted offered by the Merton solution for a  $\theta = 37\%$ . The simple life contract is close to the Merton solution, but with a higher optimal  $\theta$ . Index-linked bond has a lower optimum due to the previously identified problems with  $\Theta_t$  changing significantly over time. However, for large  $\theta$ s this contract preforms best, as the guarantee makes sure there is no downside risk.

The contract with yearly guarantees does not perform as well as the other contract due to the problems of balancing the need for a high  $\theta$  against the problem of having too large likelihood of default. However, for a high  $\theta$ , it touches the simple life contract, possibly because the bonus reserve serves as a smoothing mechanism.

For low  $\theta$ s all contracts are performing at the same level, due to the fact that guarantees are never effective. Of the same reason, the level of  $g$  will have limited impact in this area, as long as  $g$  is significantly lower than  $r$ .

Welfare calculations are done in the form of expected utility. To better compare the different cases we define the certainty equivalent (CEQ):

$$u(CEQ) = E[u(L_T)]. \quad (36)$$

We can interpret  $CEQ$  as the amount of wealth to be received at the horizon with certainty that would give the customer the same expected utility as he receives under the other strategies. For the power utility function CEQ is given by

$$CEQ = [E[u(L_T)](1 - \gamma)]^{\frac{1}{1-\gamma}}. \quad (37)$$

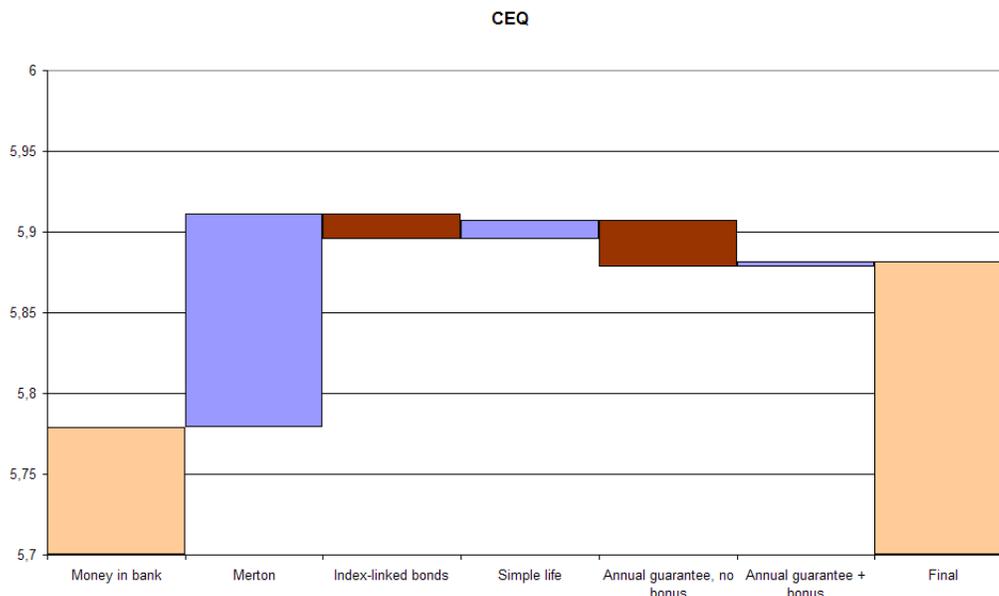


Figure 8: Development of certainty equivalents under expected utility, as contracts become more sophisticated, given  $\alpha = 0.9$ ,  $g = 0.02$ , and  $b = 0.2$ .

Figure 8 shows the certainty equivalent of different types of contracts. It is similar to figure 7 but only compares results for optimal  $\theta$ s for each contract type. We include a second type of annual guarantee contract, without a bonus reserve. The impact of the bonus reserve seems to be limited, but positive as utility increases with  $b$ . The overall picture, even though annual guarantees clearly weaken the contract, is that effects are rather small ( $< 1\%$  of the initial amount).

## 4 Results with Cumulative Prospect Theory (CPT)

### 4.1 Cumulative Prospect Theory (CPT)

In section 3 we found that life and pension insurance products are not optimal within the standard expected utility framework. Why do customers still buy these products? One reason may be that the expected utility function does not describe the preferences of customers. Expected utility maximization is a description of how rational households should choose, its descriptive accuracy has come under attack as experimental psychologists have demonstrated that households systematically deviate from the choice predictions that it implies. Several alternative behavioral models of human choice have been proposed. The most fully developed and thoroughly investigated so far is Tversky and Kahneman (1992)'s Cumulative Prospect Theory (CPT). It is a descriptive theory, based on experimental evidence, of how people eval-

uate risk. CPT combines the concepts of loss aversion (LA) and a nonlinear rank-dependent weighting of probability assessments.

CPT has two principal components. First, the individuals are not taking absolute levels of wealth into account, but rather, gains and losses measured relative to a reference point. There is a value function defined over gains, similar to the utility function in expected utility. Over losses there is a loss aversion function that transforms the specific finding that individuals are much more sensitive to losses than to gains of the same magnitude. Here  $\lambda > 1$  describes how much more sensitive an individual is to a loss relative to a gain. The LA function allows individuals to be risk averse over gains but risk seeking over losses, and for losses to matter more than gains. This is described by an S-shaped utility function. The sensitivity to increasing gains or losses is measured by  $\phi$ . Finally, there is a weighting function used to transform probability distributions into a function where individuals put more emphasis on extreme outcomes.

Cumulative prospect theory treats gains and losses separately. We define surplus wealth as current wealth relative to a reference point,  $\Gamma$ . The initial amount invested is frequently referred to as a good reference point. We define  $\Gamma = L_0 e^{\rho T}$  and let  $\rho$  be a parameter we can freely choose.

Assume a gamble is composed of  $m + n + 1$  outcomes,  $L_{T,-m} < \dots < \Gamma < \dots < L_{T,n}$ , which occur with probabilities  $p_{-m}, \dots, p_n$ , respectively. The corresponding gamble can be denoted by the pair  $(L, p)$ , where  $L = (L_{T,-m}, \dots, L_{T,n})$  and  $p = (p_{-m}, \dots, p_n)$ . We define

$$V^+(L; p) = w(p_n)u(L_{T,n}) + \sum_{k=1}^n \left[ w\left(\sum_{j=0}^k p_{n-j}\right) - w\left(\sum_{j=0}^{k-1} p_{n-j}\right) \right] u(L_{T,n-k}), \quad (38)$$

and

$$V^-(L; p) = w(p_{-m})u(L_{T,-m}) + \sum_{k=1}^m \left[ w\left(\sum_{j=0}^k p_{-(m-j)}\right) - w\left(\sum_{j=0}^{k-1} p_{-(m-j)}\right) \right] u(L_{T,-(m-k)}). \quad (39)$$

The preference value of the gamble  $(L, p)$  is given by

$$V(L; p) = V^+(L; p) + V^-(L; p) \quad (40)$$

where  $V^+(L; p)$  measures contribution of gains, and  $V^-(L; p)$  the contribution of losses. The function  $w(p)$  is a probability weighting function assumed to be increasing from  $w(0) = 0$  until  $w(1) = 1$ . Prelec (1998) offers a single parameter version of the weighting function:

$$w(p) = e^{-(-\ln p)^\varphi} \quad (41)$$

where  $\varphi$  is a "free" parameter. Prelec (1998)'s weighting function is almost identical to Tversky and Kahneman's weighting function. The key difference is that Prelec's specification

is based on behavioral axioms rather than the convenience of the functional form. We note that with  $\varphi = 1$ ,  $w(p)$  degenerates to  $w(p) = p$ . Hence, we are back to the expected utility framework with a non-standard utility function. We will later use this as a special case, see section 4.8.

Finally, the utility function is defined as follows:

$$u(L_T) = \begin{cases} u_G(L_T) = (L_T - \Gamma)^\phi & L_T \geq \Gamma \\ \lambda u_L(L_T) = -\lambda(\Gamma - L_T)^\phi & L_T < \Gamma \end{cases} \quad (42)$$

## 4.2 Parameters

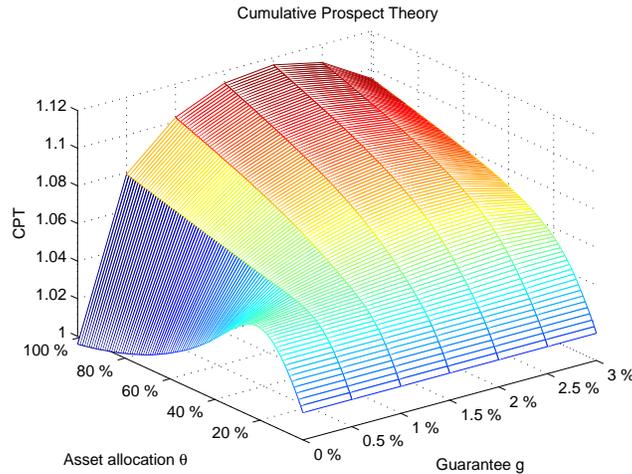


Figure 9: Optimal contract of index-linked bonds under CPT as a function of asset allocation  $\theta$  and guarantee,  $g$ , with  $\alpha = 0.9$

$$\begin{aligned} & \text{Basecase} \\ L_0 &= 4.75 & r &= 0.04 \\ \mu &= 0.065 & \sigma &= 0.15 \\ T &= 5 & b &= 0.2 \\ \rho &= 0 & \alpha &= 0.90 \\ \lambda &= 2.25 & \phi &= 0.5 \\ & & \varphi &= 0.75 \end{aligned}$$

Estimates of the parameters of CPT can be found in several studies. A challenge for CPT is to move the empirical estimates from experimental data to real world choice scenarios. Tversky and Kahneman (1992) estimated  $\phi = 0.88$ ,  $\lambda = 2.25$ ,  $\varphi_{\text{gain}} = 0.75$ , and  $\varphi_{\text{loss}} = 0.69$ , but they used the parameter  $\varphi$  for a slightly different weighting function than we use. Camerer and Ho (1994) estimates  $\phi = 0.32$  and  $\varphi = 0.56$ . Wu and Gonzalez (1996) also estimate the

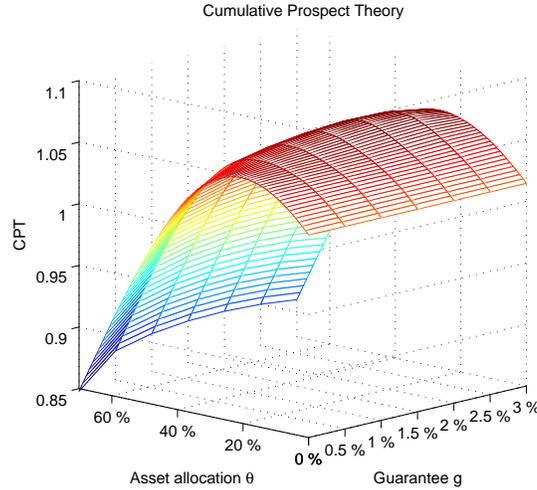


Figure 10: Optimal contract of simple life insurance under CPT as a function of asset allocation  $\theta$  and guarantee,  $g$ , with  $\alpha = 0.9$

Prelec’s weighting function yielding  $\phi = 0.48$  and  $\varphi = 0.72$ . Based on all these different studies we assign the following figures to our free parameters:  $\phi = 0.5$ ,  $\varphi = 0.75$ , and  $\lambda = 2.25$ . With  $\rho$  equal 0 the reference point is equal initial invested amount,  $\Gamma = L_0$ .

### 4.3 The Merton Problem

Even though there is done some work with optimal portfolio choice and loss aversion,<sup>4</sup> not much is done with portfolio choice and CPT. Davies and Satchell (2005) investigate closed-form solutions with strict assumptions on the asset process. In our simulation world we can easily solve the standard portfolio choice problem numerically. As shown in figure 12, the optimal allocation to the equity index is about 18% for our basecase. In the same figure we also plot the value for a 2% guarantee of the index-linked bond case and the simple life case. The value is higher for the index-linked bond contract than for Mertons problem. Thus, the customer is better off with a fair priced index-linked bond than just a portfolio consisting of a combination of bonds and an equity index.

### 4.4 Index-linked Bonds

We plot the value for the index-linked bonds case in figure 9. On the x-axis the allocation to the equity index is plotted, and the annual guaranteed rate is on the y-axis. We find that optimal  $\theta$  increases for increasing guarantee. Also opposite to the standard utility problem we find that having a guarantee is better than not having.

<sup>4</sup>See Gomes (2005) and Berkelaar, Kouwenberg, and Post (2003).

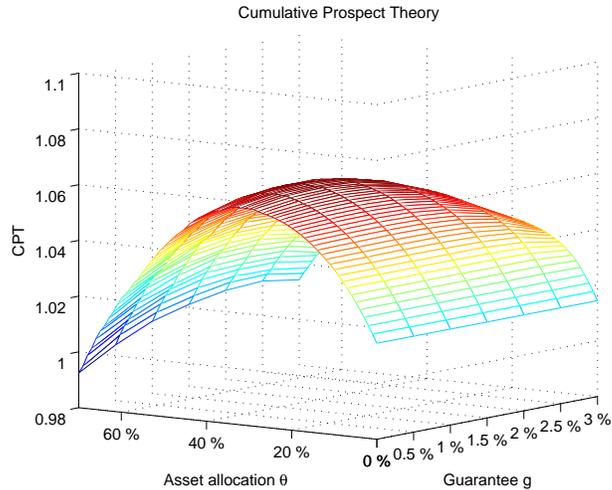


Figure 11: Optimal contract with annual guarantees under CPT as a function of asset allocation  $\theta$  and guarantee,  $g$ , with  $\alpha = 0.9$  and  $b = 0.2$

Guarantee $g$	Optimal $\theta_0$	$\delta$	Effective theta $\Theta_0$	Mean return $\bar{r}_A$
0.0 %	0.2	1.000	20 %	4.37 %
0.5 %	0.7	0.841	48 %	4.85 %
1.0 %	0.7	0.792	44 %	4.78 %
1.5 %	0.7	0.727	38 %	4.70 %
2.0 %	0.7	0.644	33 %	4.61 %
2.5 %	0.7	0.537	26 %	4.49 %
3.0 %	0.7	0.399	18 %	4.35 %

Table 6: Overview Index-linked Bonds

For guarantees equal or below zero we obtain an inner optimal guarantee. For higher guarantees we find that the customer seeks as large  $\theta$  as possible. The reason for this change is due to the level of the reference point which is equal to a guarantee of zero per cent.

#### 4.5 Simple Life Insurance

In figure 10 we plot the value for the simple life case. The optimal guarantee is about  $g = 2.5\%$ , with a corresponding optimal  $\theta$  of 28%. For  $g = 0\%$ , the optimal  $\theta$  is about 20%. For increasing  $g$ ,  $\theta$  increases with a maximum  $\theta = 29\%$  for  $g = 2.0\%$ . We observe in general that  $\theta$  is lower than the corresponding scenarios with expected utility. This is due to the aversion for bankruptcies. Table 7 shows that bankruptcy risk is now rather low.

Similarly to the index-linked bond, a guarantee of zero per cent is clearly unfavorable, as we then touch the reference point of the utility function.

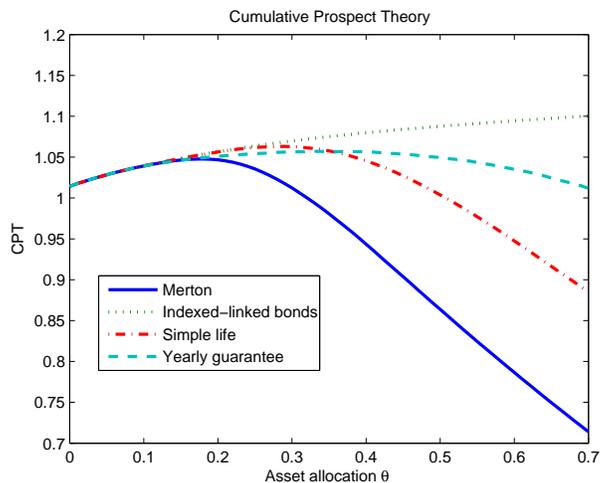


Figure 12: Optimal asset allocation under cumulative prospect theory, given  $\alpha = 0.9$ ,  $g = 0.02$ , and  $b = 0.2$ .

Guarantee $g$	Optimal $\theta_0$	$\delta$	Effective theta $\Theta_0$	Mean return $\bar{r}_A$	Prob of default $p(d)$	Average loss given default $lgd$ of $L_0$
0.0 %	0.2	1.000	20 %	4.37 %	0.0 %	0.0 %
0.5 %	0.25	0.996	25 %	4.46 %	0.0 %	1.9 %
1.0 %	0.28	0.984	26 %	4.50 %	0.1 %	2.9 %
1.5 %	0.29	0.963	26 %	4.49 %	0.4 %	3.4 %
2.0 %	0.29	0.928	24 %	4.45 %	0.9 %	3.6 %
2.5 %	0.28	0.872	21 %	4.39 %	1.4 %	3.6 %
3.0 %	0.25	0.783	16 %	4.29 %	1.6 %	3.4 %

Table 7: Overview Simple Life

#### 4.6 Annual Guarantees

Figure 11 shows the value for the annual guarantee case. The optimal  $\theta$  is about 32% independent of the guarantee. Again the guarantee has an inner optimum, at about 1%. The penalty for being close to the breakpoint is now less severe, as hitting the guarantee one year can be off-set by higher returns in other years.

On the other hand, there seems to be a bankruptcy penalty on high guarantees not unlike the effect we found with standard expected utility (see section 3). However, table 8 shows that the cumulative probability of bankruptcy is high. Compared to the simple life case, bankruptcies are not as dramatic, as the magnitude is lower, and as part of the loss can be off-set by investing risk-free, to bring the total return above the reference point (0% return).

Guarantee g	Optimal $\theta_0$	$b$	$\delta$	Mean return $\bar{r}_A$	Prob of default $p(d)$	Average loss given default $lgd$ of $L_0$
0.0 %	0.33	0.2	0.915	4.49 %	1 %	1.7 %
0.5 %	0.33	0.2	0.882	4.45 %	2 %	1.9 %
1.0 %	0.34	0.2	0.829	4.42 %	3 %	2.1 %
1.5 %	0.35	0.2	0.761	4.38 %	5 %	2.3 %
2.0 %	0.33	0.2	0.701	4.31 %	5 %	2.3 %
2.5 %	0.36	0.2	0.576	4.27 %	10 %	2.7 %
3.0 %	0.36	0.2	0.452	4.20 %	13 %	2.7 %

Table 8: Overview Annual Guarantees

## 4.7 Comparison

In figure 12 we found that guarantees are not effective for low  $\theta$ s. For higher  $\theta$ s the probability of large bankruptcies is the dominant feature of the contracts. This means that the index-linked bonds (with no bankruptcies) performs the best, while the limited losses of yearly guarantees also do fairly well. In the optimal  $\theta$ , however, simple life outperforms the yearly guarantees, as losses are moderate and "unnecessary bankruptcies" should be avoided.

To better compare the different cases we also for CPT define the certainty equivalent (CEQ) in a similar way as section in 3.7:

$$V(CEQ) = V(L; p). \quad (43)$$

For the loss aversion function CEQ is given by

$$CEQ = \begin{cases} \Gamma + [V(L; p)]^{\frac{1}{\phi}} & L_T \geq \Gamma \\ \Gamma - \left[ \frac{-V(L; p)}{\lambda} \right]^{\frac{1}{\phi}} & L_T < \Gamma \end{cases} \quad (44)$$

The figures show that all realizations of value are positive, thus  $L_T \geq \Gamma$ . Figure 13 shows the CEQ for the different contracts. Opposite to the situation with standard expected utility, the index-linked bonds contract gives highest value. Hence, for the customer under CPT, the effect of combining no bankruptcies (losses) with the opportunity of taking high risk if  $E(L_T)$  is high, is highly appreciated.

## 4.8 The Non-standard Expected Utility Case

CPT is based on experimental evidence and clearly outside of the expected utility framework. However, we can rationalize the guarantees within expected utility, but with a non-standard utility function. By using the special case  $\varphi = 1$  in equation (41), and the loss-aversion utility function from equation (42), we are still within the expected utility framework. The results are similar to the result from section 4.6. We still get an internal optimum, but the optimal annual guarantee is now slightly below 0%. The full results with annual guarantees are shown

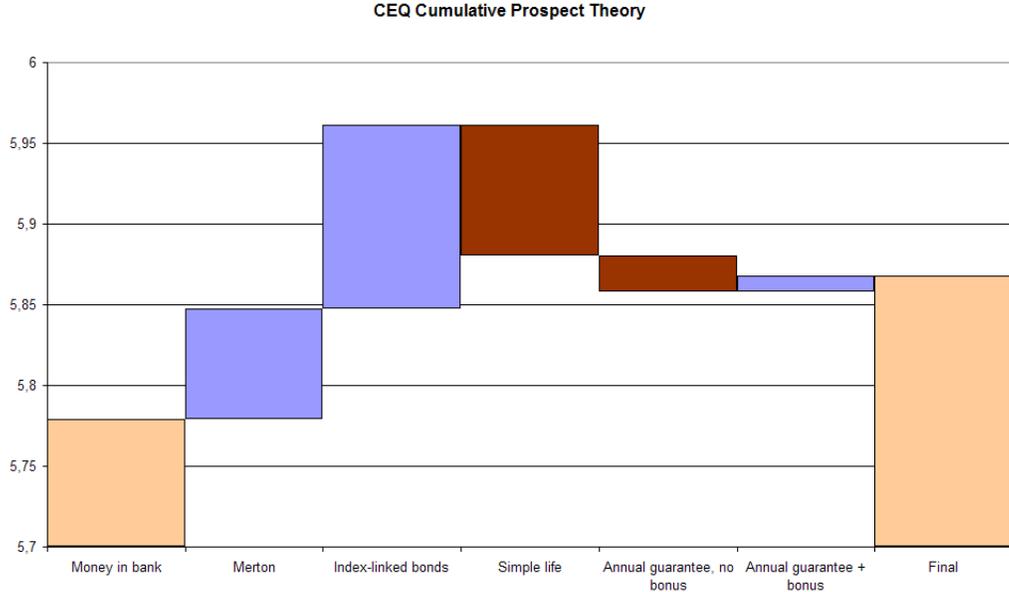


Figure 13: Development of certainty equivalents under cumulative prospect theory, as contracts become more sophisticated, given  $\alpha = 0.9$ ,  $g = 0.02$ , and  $b = 0.2$ .

in figure 14. Similarly, the other types of contracts give results in the same fashion as what is shown in sections 4.3-4.5, but with slightly different optima.

## 5 Conclusion

We have presented a framework for optimizing pension insurance design by combining pricing principles with utility theory. Not surprisingly is the Merton solution optimal with standard expected utility. The design of the contracts is a less important factor than stock market participation in itself.

With CPT index-linked bonds outperform the other alternatives. Contracts including both insurance against losses and stock market participation tend to give high expected utility. The contract design is now more important than the decision whether to participate in the stock market or not. In both cases annual guarantee contracts are outperformed by simpler products.

All in all we can not explain the large demand for structured products in the world of standard expected utility. A possible explanation may be that the customers have CPT preferences. However, potentially important features of the contracts, such as transaction costs, taxes, and actuarial elements are left for further research. Furthermore, more sophisticated models may include other sources of revenues, such as labor income or revenues from alternative pension system(s).

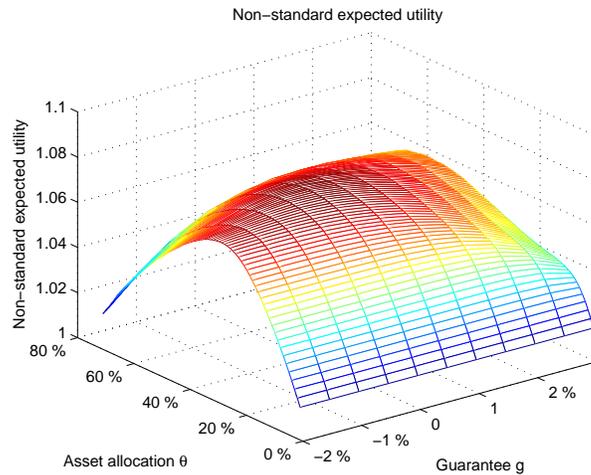


Figure 14: Optimal contract with annual guarantees with loss-aversion (within expected utility framework) as a function of asset allocation  $\theta$  and guarantee,  $g$ , with  $\alpha = 0.9$  and  $b = 0.2$

## References

- Berkelaar, A., R. Kouwenberg, and T. Post, 2003, “Optimal Portfolio Choice Under Loss Aversion,” *Working Paper*.
- Black, F., and M. Scholes, 1973, “The Pricing of Options and Corporate Liabilities,” *Journal of Political Economy*, 81, 637–59.
- Borch, K., 1962, “Equilibrium in a Reinsurance Market,” *Econometrica*, 30, 424–44.
- Brennan, M. J., 1993, “Aspects of Insurance, Intermediation and Finance,” *Geneva Papers on Risk and Insurance Theory*, 18, 7–30.
- Briys, E., and F. de Varenne, 1994, “Life Insurance in a Contingent Claim Framework: Pricing and Regulatory Implications,” *The Geneva Papers on Risk and Insurance Theory*, 19, 53–72.
- Briys, E., and F. de Varenne, 2001, *Insurance: From Underwriting to Derivatives : Asset Liability Management in Insurance Companies*, John Wiley & Sons.
- Camerer, C. F., and T.-H. Ho, 1994, “Violations of the Betweenness Axiom and Nonlinearity in Probability,” *Journal of Risk and Uncertainty*, 8, 167–96.
- Campbell, J. Y., 2006, “Household Finance,” *Presidential Address to the American Finance Association*.
- Consiglio, A., D. Saunders, and S. A. Zenios, 2006, “Asset and Liability Management for Insurance Products with Minimum Guarantees: The UK Case,” *Journal of Banking and Finance*, p. forthcoming.
- Davies, G., and S. E. Satchell, 2005, “Continuous Cumulative Prospect Theory and Individual Asset Allocation,” *Cambridge Working Papers in Economics*.

- Gomes, F. J., 2005, “Portfolio Choice and Trading Volume with Loss-Averse Investors,” *Journal of Business*, 78 (2).
- Grosen, A., and P. Jørgensen, 2000, “Fair Valuation of Life Insurance Liabilities: The Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies,” *Insurance: Mathematics and Economics*, 26, 37–57.
- Hansen, M., and K. R. Miltersen, 2002, “Minimum Rate of Return Guarantees: The Danish Case,” *Scandinavian Actuarial Journal*, 4, 280–318.
- Hansen, M. S., and M. Hansen, 2003, “Portfolio Choice And Fair Pricing in Life Insurance Companies,” *Working Paper*.
- Harrison, J. M., and D. M. Kreps, 1979, “Martingales and Arbitrage in Multiperiod Securities Markets,” *Journal of Economic Theory*, 20, 381–408.
- Jensen, B. A., and C. Sørensen, 2001, “Paying for minimum interest rate guarantees: Who should compensate who?,” *European Financial Management*, 7, 183–211.
- Merton, R. C., 1971, “Optimum Consumption and Portfolio Rules in a Continuous-Time Model,” *Journal of Economic Theory*, 3, 373–413.
- Miltersen, K. R., and S.-A. Persson, 2003, “Guaranteed Investment Contracts: Distributed and Undistributed Excess Return,” *Scandinavian Actuarial Journal*, 4, 257–79.
- Prelec, D., 1998, “The Probability Weighting Function,” *Econometrica*, 60, 497–528.
- Tversky, A., and D. Kahneman, 1992, “Advances in Prospect Theory: Cumulative Representation of Uncertainty,” *Journal of Risk and Uncertainty*, 5, 297–323.
- von Neumann, J., and O. Morgenstern, 1944, “Theory of Games and Economic Behavior,” 1953 edition, Princeton, NJ: Princeton University Press.
- Wu, G., and R. Gonzalez, 1996, “Curvature of the Probability Weighting Function,” *Management Science*, 42, 1676–90.

## Appendix A: Numerical Methods

Both the pricing of the contracts and the computation of the expected utility is based on standard Monte Carlo techniques. For every alternative we calculate a multi-dimensional grid of different contracts. For all alternatives except annual guarantees, there exists a closed-form solution to find fair contracts. To be able to find fair  $\delta$  for the latter case, we simulate  $m = 100000$  paths of the value of equity under the risk-neutral measure,  $Q$ , (see e.g. equation (17)). Since the value of equity is monotonic decreasing in  $\delta$ , we can utilize Newton’s method to find a fair  $\delta$  for each contract.

For the set of fair contracts we use the same  $m$  runs, but under the real measure  $P$ , to calculate expected utility for both the power utility and CPT. Optimal expected utility for power utility is given by

$$U^* = \max_{\alpha, \theta, g, b} \frac{1}{m} \sum_{i=1}^m u(L_{i,T} + B_{i,T}) \quad (45)$$

and in the case of CPT,

$$U^* = \max_{\alpha, \theta, g, b} V(L_{i,T} + B_{i,T}). \quad (46)$$