Revelation of Preferences in Patient List Data

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ABSTRACT. In this paper we will show how the patient list model in Ubøe & Lillestøl (2007) can be used to infer strength of preferences from patient list data. We prove that we can construct unique sets of preferences that replicates patient list data, and we also show how to approach cases where we only have partial information of the system. As an illustration we apply the new theory to some patient list data from the Norwegian patient list system in general practice.

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1. Introduction

In the “Norwegian patient list system in general practice” the patients can be assigned to a doctor that agrees to have the main responsibility for his or hers patients. As there are limited numbers of doctors of each type, however, it may happen that a significant number of patients are assigned to doctors of the “wrong type”, i.e., a type of doctor that they really do not want.

Ubøe & Lillestøl (2007) suggested a new statistical framework for this scenario. It turned out, however, that to apply this model to real world data, a number of rather delicate mathematical problems had to be solved. In this paper we are able to present complete solutions to these problems. As an illustration of the theory we will consider a special case using patient list data from the Norwegian patient list system in general practice. These data describe the allocation of male and female patients to male and female doctors. Assuming that the system is cost efficient (see below for a definition of cost efficiency), we can use the allocations to infer strength of preferences among patients in each group.

The Norwegian patient list system in general practice is described in some detail in Ubøe & Lillestøl (2007), and we refer to that paper for a review of the system. In this paper we want to infer strength of preferences from observed allocations. We believe that our basic approach to this problem is novel, and it is to our knowledge the only known approach to the type of problem we consider here. Hence we will not enter into a discussion of related/alternative models.

The paper is organized as follows: In Section 2 we briefly recall the construction in Ubøe & Lillestøl (2007), and show how we can obtain unique representations of preferences. In Section 3 we consider cases with partial information, i.e., cases where parts of the data are missing, and demonstrate how we can infer strength of preferences in such cases. In Section 4 we use the constructions from Section 2 and 3 to infer strength of preferences from real world data. These data were collected from an official panel survey of Norwegian living conditions (“Levekårsundersøkelsen 2003”). The responses to preference questions were very low, however, and more so for males than females. Hence the empirical part of the paper must be considered more as an illustration of the theory, and not so much as an empirical survey in its own right. In Section 5 we offer some concluding remarks.

To enhance readability of the paper, a few proofs have been placed in Appendix 1. The models we use in this paper are strongly non-linear involving a sometimes large number of parameters.

We have developed some new numerical methods that are able to handle systems with several hundred parameters. We expect that few readers are interested in numerical remarks, however, so a survey of these numerical methods has been placed in Appendix 2.

2. Identification of utilities in the patient list model

The model in Ubøe & Lillestøl (2007) can be described briefly as follows: Assume that there are S groups of patients, T types of doctors, and let P_{ts} denote the number of patients in group s that has a doctor of type t, $s = 1, \ldots, S$, $t = 1, \ldots, T$.

- **Patients:** We assume that there is a total of E_s patients belonging to group s, $s = 1, \ldots, S$. A patient belonging to group s is assumed to have a utility U_{ts} of having a doctor of type t, $s = 1, \ldots, S$. In may sometimes happen, however, that a patient prefer to wait for a vacancy of a suitable doctor rather than being assigned to a doctor of a type that the patient dislikes. We let $P_{t(s+S)}$ denote the number of patients waiting on a doctor of type t (not being assigned to any doctor), and let $U_{t(s+S)}$ denote the utility of these patients.

- **Doctors:** Every doctor working within the system is assumed to have a certain list length, i.e., a maximum number of patients that he or she can serve. We assume that there are D_t doctors of type t, and that these doctors can serve a total of L_t patients, i.e., L_t is the sum of the list lengths of all doctors of type t. Some doctors may have vacancies, and we let $U_{t(2S+1)}$ denote the disutility per vacancy incurred by a doctor of type t.

  Utilities may of course be negative, in which case we refer to these numbers as disutilities.

Clearly the $(E_1, \ldots, E_S)$ patients can be allocated to the $(D_1, \ldots, D_T)$ doctors in many different ways. The basic hypothesis in Ubøe & Lillestøl (2007), however, is to assume that the system is cost efficient in the sense that states with large total utility (sum of the utility of all patients and doctors) are more probable than states with smaller total utility. If the system is cost efficient with a large number of patients in every group, it is possible to prove, see Ubøe & Lillestøl (2007), that the allocation will settle at a statistical equilibrium given by the following explicit formula:

$$
P_{ts} = \begin{cases} A_t B_s \exp[U_{ts}] & \text{if } s = 1, \dots, S \\ D_t B_{s-S} \exp[U_{ts}] & \text{if } s = S+1, \dots, 2S \\ A_t \exp[U_{ts}] & \text{if } s = 2S+1 \end{cases}
$$

$$
\sum_{s=1}^{S} P_{ts} + P_{t(2S+1)} = L_t \qquad t = 1, \dots, T \tag{1}
$$

$$
\sum_{t=1}^{T} \big(P_{ts} + P_{t(s+S)}\big) = E_s \qquad s = 1, \dots, S
$$

See Appendix 2 on how to compute the balancing factors $A_1, \dots, A_T, B_1, \dots, B_S$.

The basic problem we want to address in this paper can be formulated as follows: *Assume that the system is cost efficient and that we observe*

- *The total number of patients in each group. i.e.,* E_s, $s = 1, \dots, S$
- *The total number of doctors of each type, i.e.,* D_t, $t = 1, \dots, T$
- *The total list length of doctors of each type, i.e.,* L_t, $t = 1, \dots, T$
- *The final allocation of patients to doctors, i.e.,* P_{ts}, $s = 1, \dots, 2S+1$, *t=1,*$\dots, T$

To what extent do these observations reveal the strength of the preferences

$$
U_{ts}, \; s = 1, \dots, 2S+1, t = 1, \dots, T\,?
$$

It is easy to observe, however, that there are always an infinite number of utility matrices leading to the same final allocation. To obtain uniqueness we hence have to impose some additional restrictions. More precisely we can prove the following:

THEOREM 2.1

Assume that an observed patient list distribution $\mathbf{P}$ *can be replicated by a model that satisfies (1). Then we can find a unique utility matrix* $\mathbf{U}$ *on the form*

$$
\begin{bmatrix}
0 & 0 & 0 & \dots & 0 & v_{11} & v_{12} & \dots & v_{1S} & 0 \\
0 & u_{11} & u_{12} & \dots & u_{1(S-1)} & v_{21} & v_{22} & \dots & v_{2S} & w_1 \\
0 & u_{21} & u_{22} & \dots & u_{2(S-1)} & \vdots & \vdots & \dots & \vdots & w_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
0 & u_{(T-1)1} & u_{(T-1)2} & \dots & u_{(T-1)(S-1)} & v_{T1} & v_{T2} & \dots & v_{TS} & w_{T-1}
\end{bmatrix} \tag{2}
$$

that replicates $\mathbf{P}$.

PROOF

See Appendix 1.

□

The zeros in (2) can be interpreted as reference points and the corresponding groups as reference groups. Uniqueness is obtained when we specify how much more/less utility the other groups have in comparison to these reference groups. Clearly, reference groups can be chosen in many different ways, hence there are many alternative ways to obtain uniqueness. While the representation given by (2) has several favorable properties, results given on this form are quite hard to interpret. Hence it might be profitable to look for other representations offering more transparent interpretations.

We can obtain alternative unique representations by assuming a utility structure with sufficient identities and/or symmetries. Nevertheless, it is convenient to use (2) as a canonical form, both for algorithmic programming and for resolving theoretical issues. One important issue is that of identification, i.e., recovering the parameters of an assumed utility structure from its established canonical form. Equivalent structures are obtained by transformations of $\mathbf{U}$ that leave $\mathbf{P}$ invariant. These are:

- Add/subtract a fixed T-dimensional column vector $\mathbf{a}$ to all columns of $s = 1, \ldots, S$ and $s = 2S + 1$ (i.e. except $s = S + 1, \ldots, 2S$).
- Add/subtract a fixed $2S+1$-dimensional row vector of form ($\mathbf{b}$,$\mathbf{b}$,0) with $\mathbf{b}$ S-dimensional to all rows.
- Add/subtract a constant c to column $s = 2S+1$ and at the same time subtract/add the same constant from all columns $s = S + 1, \ldots, 2S$.

However, the easiest way to check identifiability may be to use the transform given by formula (8) in Appendix 1 and check the uniqueness of the parameter recovery.

3. Inference under partial information

Assume that we know the number of patients on the patient lists and the number of vacancies, but do not know how many patients that are waiting for a vacancy. Is it then possible to infer the strength of preferences of the patients on the patient lists? The answer is yes, and this can be demonstrated as follows:

Assume that $P_{ts}^{(0)}$ $s = 1, \ldots, S, t = 1, \ldots, T$ is given, and let for $s = S+1, \ldots, 2S+1,\ t = 1, \ldots, T$ $P_{ts}^{(1)}$ and $P_{ts}^{(2)}$ be arbitrary numbers.

Define the following aggregated quantities

$$L_t^{(0)} = \sum_{s=1}^{S} P_{ts}^{(0)},\ E_s^{(0)} = \sum_{t=1}^{T} P_{ts}^{(0)},\ L_t^{(i)} = \sum_{s=S+1}^{2S+1} P_{ts}^{(i)},\ E_s^{(i)} = \sum_{t=1}^{T} P_{ts}^{(i)} \qquad i = 1, 2$$

THEOREM 3.1

For $i = 1, 2$ put $L_t = L_t^{(0)} + L_t^{(i)}$, $E_s = E_s^{(0)} + E_s^{(i)}$, and find a unique matrix $\mathbf{U}^{(i)}$ of the form (2) such that the system given by (1) replicates the numbers

$$P_{ts} = \begin{cases} P_{ts}^{(0)} & \text{if } s = 1, \ldots, S, t = 1, \ldots, T \\ P_{ts}^{(i)} & \text{if } s = S+1, \ldots, 2S+1,\ t = 1, \ldots, T \end{cases} \tag{3}$$

If $K = \frac{P_{1(2S+1)}^{(2)}}{P_{1(2S+1)}^{(1)}}$, then the two utility matrices $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ are connected through the formula

$$\mathbf{U}_{ts}^{(2)} = \begin{cases} \mathbf{U}_{ts}^{(1)} & \text{if } s = 1, \ldots, S,\ t = 1, \ldots, T \\ \mathbf{U}_{ts}^{(1)} + \ln[P_{ts}^{(2)}/P_{ts}^{(1)}] + \ln K & \text{if } s = S+1, \ldots, 2S,\ t = 1, \ldots, T \\ \mathbf{U}_{ts}^{(1)} + \ln[P_{ts}^{(2)}/P_{ts}^{(1)}] - \ln K & \text{if } s = 2S+1,\ t = 1, \ldots, T \end{cases} \tag{4}$$

PROOF

See Appendix 1.

□

As we can see from Theorem 3.1, the utilities $\mathbf{U}_{ts}$, $s = 1, \ldots, S, t = 1, \ldots, T$ do not depend on the values of P_{ts} for $s = S+1, \ldots, 2S+1$, $t = 1, \ldots, T$. Hence we have the following corollary:

COROLLARY 3.2

Assume that P_{ts} $s = 1, \ldots, S, t = 1, \ldots, T$ are known, while data on P_{ts} for $s = S+1, \ldots, 2S+1$, $t = 1, \ldots, T$ are missing. If we choose $P_{ts} > 0$ for $s = S+1, \ldots, 2S+1,\ t = 1, \ldots, T$ arbitrarily, we can still infer the correct values on $\mathbf{U}_{ts}$, $s = 1, \ldots, S, t = 1, \ldots, T$.

In the next section we will apply this theory to some real world data. The data we were able to obtain did not contain any information on the number of patients waiting for vacancies. Nevertheless we can appeal to Corollary 3.2 and infer preferences of the various patient groups that are registered with a doctor. Moreover, we see from the bottom line in formula (4) that we can also obtain strength of preferences for vacancies in cases where information on the number of patients waiting for vacancies are missing. Clearly, however, it is impossible to infer strength of preferences for groups of patients waiting for vacancies unless we have data for these groups.

4. Application to patient list data

Suppose that patients and doctors are grouped by gender and the issue is whether the patients want a doctor of the same gender or not. In this case $T = 2$, with groups denoted M (male) and F (female), and $S = 4$ with groups denoted mm, mf, fm and ff, where the first letter is the gender of the patient and the second letter is the preferred gender of doctor. The utility matrix is then

$$\mathbf{U} = \begin{bmatrix} U_{11} & \cdots & U_{14} & U_{15} & \cdots & U_{18} & U_{19} \\ U_{21} & \cdots & U_{24} & U_{25} & \cdots & U_{28} & U_{29} \end{bmatrix} \tag{5}$$

with the row order is M, F and the column order is mm, mf, fm, ff, mm-w, mf-w, fm-w, ff-w, vacancy, where w indicates a waiting list state. See Table 1-6 below for a more reader friendly format. Consider the following assumptions

(i) all utilities for correct patient/doctor matching are equal and (without loss of generality) taken to be zero
(ii) all disutilities of being on a waiting list are the same
(iii) the disutilities of vacancy are the same for both gender of doctors.

To facilitate discussion consider the more general case

$$\mathbf{U} = \begin{bmatrix} 0 & a_2 & 0 & a_4 & b_{11} & b_{12} & b_{13} & b_{14} & c_1 \\ a_1 & 0 & a_3 & 0 & b_{21} & b_{22} & b_{23} & b_{24} & c_2 \end{bmatrix} \tag{6}$$

where $b_{ts} = b_t,\ t = 1, 2$ corresponds to equal waiting list disutilities within each gender of patients. Then assumption (ii) corresponds to $b_1 = b_2 = b$ and assumption (iii) corresponds to $c_1 = c_2 = c$.

For a given observed $\mathbf{P}$-matrix suppose we have computed the unique canonical $\mathbf{U}$-matrix. The question now is whether and how we can recover each element of the assumed matrix structure.

We will briefly explore this identification issue. Transforming $\mathbf{U}$ by formula (8) in Appendix 1 to its canonical form gives

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -a_2 - a_1 & a_3 - a_1 & -a_4 - a_1 \end{bmatrix.$$
$$\begin{matrix} b_{11} + c_1 & b_{12} + c_1 - a_2 & b_{13} + c_1 & b_{14} + c_1 - a_4 & 0 \\ b_{21} + c_1 & b_{22} + c_1 - a_2 & b_{23} + c_1 & b_{24} + c_1 - a_4 & c_2 - c_1 - a_1 \end{bmatrix} \tag{7}$$

We see that we do not have identifiability, unless we add restrictions. Assume first $c_1 \neq c_2$. To identify the a's individually, we may fix one of them, say take $a_4 = -1$, which is just a matter of choice of scale. Now the identified a's may be used in the identification of b's and c's if needed. Since b_{ts} and c_1 occur as a sum they are not individually identified, unless we add an assumption that relates them, say each b_{ts} is a multiple or fraction of c_1. With the assumption of equal b's, either within each gender or for both gender, i.e. assumption (ii), we have over-identification, and we may lump the identified b_{ts}'s together by averaging. This may not give exact replication, but a good fit in the statistical sense. The element $c_2 - c_1 - a_1$ in the south-east corner of the matrix now automatically identifies c_2. With assumption (iii), we see that this element is reduced to $-a_1$, and thereby does not contribute to the identification of the common c, but in fact identifies a_1 directly. This means that for exact identification of the others, we have to leave out the scaling assumption on a_4. On the other hand we may look at this as over-identification providing added information on a_1, and thereby also on a_2 and a_3. Numerically this case is degenerate. The limits $\lim_{c_1 \to c_2} \mathbf{U}_{ts}$ $s = 1, \ldots, S$, $t = 1, \ldots, T$ exist, however, and coincide with the values reported in Table 2 below.

The Norwegian patient list system was introduced in year 2001 and is monitored by the authorities. Data on availability of doctors are made readily available to the public, and some aggregated data on list composition and vacancies are also available for research purposes. Reliable data on doctor preference are not readily available. However, some questions on the combination (gender of respondent, gender of assigned doctor, preferred gender of doctor) were included the official panel survey of Norwegian living conditions ("Levekårsundersøkelsen 2003"). Unfortunately the responses to the preference question were very low, and more so for males than females. This also affects the distribution of doctors among gender in the data base. We have therefore scaled the data to get the marginal frequencies in accordance with the approximately known distribution of doctors at the time, namely 70% males and 30% females. The result is then given in Table 1 per 1 000 respondents. A survey made by the Norwegian Ministry of Health and Care Services (2004) reports a total of 2 026 doctors with vacancies, the average number of vacancies being 223. With the reported 4 563 751 patients served, this gives 99 vacancies per 1 000 patients. For illustrative purposes we round this in Table 1 to 100 patients per 1 000 served. We have no information on how this is distributed among the gender of doctors, and will look into how this affects the solution. If they are distributed evenly among the genders, the number will be as given in the parentheses. Officially there are no waiting lists, and data on this are hard to get, and not really needed for our illustrative purpose. Note that the number of patients who want a doctor of the same gender is higher for males than for females.

Patient group	mm	mf	fm	ff	mm-w	mf-w	fm-w	ff-w	vac
M-doctor	455	12	69	164	–	–	–	–	(70)
F-doctor	19	14	2	265	–	–	–	–	(30)
Totalling	474	26	71	429	–	–	–	–	100

Table 1: Observed counts in each group per 1000 patients served

From the data in Table 1 we get the uniquely defined disutilities of Table 2, where the data in parenthesis affect the computed disutility in the parenthesis only. Identification according to the assumed structure (7) gives Table 3. Note that our model provides perfect fit to data, and that traditional statistical estimation and sampling error analysis do not apply. Hence it makes no sense to report standard errors in these cases.

Patient group	mm	mf	fm	ff	mm-w	mf-w	fm-w	ff-w	vac
M-doctor	0	0	0	0	–	–	–	–	0
F-doctor	0	3.33	−0.37	3.66	–	–	–	–	(2.33)

Table 2: Canonical utilities using the representation in (2)

Patient group	mm	mf	fm	ff	mm-w	mf-w	fm-w	ff-w	vac
M-doctor	0	−0.67	0	−1	–	–	–	–	(c_1)
F-doctor	−2.66	0	−3.03	0	–	–	–	–	$(c_1 - 0.33)$

Table 3: Alternative utilities using the condition $a_4 = -1$ in (7)

We see that this reveals a structure where the felt nuisance of a mismatched male patient who wants a female doctor is less than the corresponding mismatch for female patients wanting a female doctor. Furthermore we see that, for both male and female patients, the felt nuisance of getting a female doctor when wanting a male in considerably higher, and highest for female patients.

Moreover we see that $c_1 > c_2$ for the given data, i.e., the disutility for a vacant entry appears larger for male doctors than for female doctors. We may study how the solution depends on the assumed vacancy counts $(x, 100 - x)$. It turns out that $c_2 - c_1$ is a decreasing function of x and is zero for the added datum $x = 63$. Thus when the costs of vacancy are equal for both gender of doctors, we expect less than proportionate vacancy at male doctors. This is so because more males patients prefer a doctor of the same gender than female patients, and despite the expected harder pressure on doctors of the scarce gender, which may come as a surprise.

For the data given in Table 1 the identification of utilities under the condition (iii) $c_1 = c_2 = c$ is given in Table 4.

Patient group	mm	mf	fm	ff	mm − w	mf − w	fm − w	ff − w	vac
M-doctor	0	-1.00	0	-1.32	–	–	–	–	(c)
F-doctor	-2.33	0	-2.70	0	–	–	–	–	(c)

Table 4: Identified utilities using the conditions $c_1 = c_2 = c$ and the vacancies $(70, 30)$

If we compare the numbers in Table 3 and 4, we see that the two representations are not very different, and that the remarks below Table 3 also apply to the numbers reported in Table 4.

For illustrative purposes we now add the artificial data for persons on waiting lists as given in Table 5. This gives the complete set of canonical utilities given in Table 6.

Patient group	mm	mf	fm	ff	mm-w	mf-w	fm-w	ff-w	vac
M-doctor	455	12	69	164	8	4	9	1	70
F-doctor	19	14	2	265	2	6	4	6	30
Totalling	474	26	71	429	10	10	13	7	100

Table 5: Observed counts with added artificial waiting list data

Patient group	mm	mf	fm	ff	mm-w	mf-w	fm-w	ff-w	vac
M-doctor	0	0	0	0	-7.66	-4.71	-5.65	-8.71	0
F-doctor	0	3.33	-0.37	3.66	-8.19	-3.46	-5.61	-6.08	2.33

Table 6: Canonical utilities using representation (2) together with data from Table 5

Assuming (ii) and (iii) with all $b_{ts} = b = c$, we get $b = -3.42$ by the proposed averaging. The nuisance of being on a waiting list without being assigned to a doctor is therefore somewhat higher than being assigned to a doctor of wrong gender, as we would expect it to be. If we instead assume all $b_{ts} = b = 2c$, we get the somewhat stronger felt nuisance $b = -4.56$.

5. Concluding remarks

Ubøe and Lillestøl (2007) proposed a new type of statistical model to study allocation of groups of patients to different types of doctors. The problem of non-uniqueness of preferences was mentioned briefly in the concluding remarks of that paper. Only later we realized the seriousness of this problem, i.e., that special methods had to be developed to classify and interpret the results. In this paper we have made the model operational in the sense that it can now be used to infer strength of preferences from observed patient list data, and the problem of non-uniqueness has been solved completely by Theorem 2.1.

As an illustration of this theory we have applied the model to patient list data from the Norwegian patient list system in general practice. It is quite clear, however, that this type of model can be used to infer preferences from much more refined systems than the one we have studied in the empirical part of this paper. Here we only made use of two types of doctors and 4 groups of patients. Our model allows arbitrary many types of doctors and arbitrary many groups of patients. The numerical methods developed in Appendix 2 are very powerful, and a system with, e.g., 10 types of doctors and 10 different patient groups can be computed without problems.

The revealed preferences from the Norwegian patient list data turned out to be very reasonable, and mostly in accordance with prior beliefs. Despite the weakness of such data, they may give some backing for the health authorities, e.g., when asking questions like: What changes are likely to happen when the fraction of female doctors are on the rise? This may be answered by using the model in the forward manner, as described in Ubøe and Lillestøl (2007), and in more detail in Lillestøl et.al. (2007). Revealed disutilities are then used as input, representing the current preference status. It would clearly be of interest to have periodic updates on patient allocations and preferences to investigate the stability of disutilities.

6. Appendix 1: Proofs

Consider the following matrix transformation

$$\tilde{U}_{ts} = \begin{cases} U_{ts} - U_{t1} - U_{1s} + U_{11} & \text{if } s = 1, \ldots, S \\ U_{ts} - U_{1(s-S)} + U_{1(2S+1)} & \text{if } s = S+1, \ldots, 2S \\ U_{ts} - U_{t1} - U_{1(2S+1)} + U_{11} & \text{if } s = 2S+1 \end{cases} \tag{8}$$

LEMMA 6.1

Let $\mathbf{U} = \{U_{ts}\}_{s,t=1}^{M,N}$ *be given, let* $\tilde{\mathbf{U}}$ *be defined by (8) and let* $\mathbf{P}$ *and* $\tilde{\mathbf{P}}$ *denote the corresponding distributions of patients in (1) when we use* $\mathbf{U}$ *and* $\tilde{\mathbf{U}}$*, respectively. Then* $\mathbf{P} = \tilde{\mathbf{P}}$*.*

PROOF

Let $A_1, \ldots, A_T, B_1, \ldots, B_S$ denote the balancing factors solving (1) when we use $\mathbf{U}$, and define

$$\begin{aligned} \tilde{A}_t &= A_t \exp[U_{t1} + U_{1(2S+1)} - U_{11}] & t = 1, \ldots, T \\ \tilde{B}_s &= B_s \exp[U_{1s} - U_{1(2S+1)}] & s = 1, \ldots, S \end{aligned} \tag{9}$$

If $s = 1, \ldots, S$, we get

$$\tilde{A}_t \tilde{B}_s \exp[\tilde{U}_{ts}] = A_t B_s \exp[U_{ts}] \tag{10}$$

If $s = S+1, \ldots, 2S$, we get

$$\begin{aligned} D_t \tilde{B}_{s-S} \exp[\tilde{U}_{ts}] &= D_t B_{s-S} \exp[U_{1(s-S)} - U_{1(2S+1)}] \exp[U_{ts} - U_{1(s-S)} + U_{1(2S+1)}] \\ &= D_t B_{s-S} \exp[U_{ts}] \end{aligned} \tag{11}$$

If $s = 2S+1$, we get

$$\begin{aligned} \tilde{A}_t \exp[\tilde{U}_{ts}] &= A_t \exp[U_{t1} + U_{1(2S+1)} - U_{11}] \exp[U_{ts} - U_{t1} - U_{t(2S+1)} + U_{11}] \\ &= A_t \exp[U_{ts}] \end{aligned} \tag{12}$$

which proves the lemma.

□

PROPOSITION 6.2

Let $\mathbf{U^{(1)}}$ and $\mathbf{U^{(2)}}$ denote two utility matrices, and assume that $\mathbf{P}^{(1)} = \mathbf{P}^{(2)}$ in (1). Using the transformation in (8) we have $\tilde{\mathbf{U}}^{(1)} = \tilde{\mathbf{U}}^{(2)}$.

PROOF

It follows from Lemma 2.1 that $\tilde{\mathbf{P}}^{(1)} = \tilde{\mathbf{P}}^{(2)}$. Let $s = 2S+1, t = 1$, and observe from (8) that $\tilde{U}^{(1)}_{(2S+1)1} = \tilde{U}^{(2)}_{(2S+1)1} = 0$ by definition. Since

$$\tilde{A}^{(1)}_1 \exp[\tilde{U}^{(1)}_{(2S+1)1}] = \tilde{A}^{(1)}_2 \exp[\tilde{U}^{(2)}_{(2S+1)1}] \tag{13}$$

it follows that $\tilde{A}^{(1)}_1 = \tilde{A}^{(2)}_1$. Now put $t = 1$ and $s = 1, \ldots, S$, and observe from (8) that $\tilde{U}^{(1)}_{s1} = \tilde{U}^{(2)}_{s1} = 0$ by definition. Hence from (1) we get

$$\tilde{A}^{(1)}_1 \tilde{B}^{(1)}_s \exp[\tilde{U}^{(1)}_{s1}] = \tilde{A}^{(2)}_1 \tilde{B}^{(2)}_s \exp[\tilde{U}^{(2)}_{s1}] \tag{14}$$

It then follows from (14) that $\tilde{B}^{(1)}_s = \tilde{B}^{(2)}_s$ for all $s = 1, \ldots, S$. We then put $s = 1$ and $t = 1, \ldots, T$, and observe from (8) that $\tilde{U}^{(1)}_{1t} = \tilde{U}^{(2)}_{1t} = 0$ by definition. From (1) again we get

$$\tilde{A}^{(1)}_t \tilde{B}^{(1)}_1 \exp[\tilde{U}^{(1)}_{st}] = \tilde{A}^{(2)}_t \tilde{B}^{(2)}_1 \exp[\tilde{U}^{(2)}_{st}] \tag{15}$$

Since $\tilde{B}^{(1)}_1 = \tilde{B}^{(2)}_1$, it follows that $\tilde{A}^{(1)}_t = \tilde{A}^{(2)}_t$ for all $t = 1, \ldots, T$. We have hence proved that all the balancing factors must be equal, and then it follows from (1) that all the utilities must be equal as well.

□

Proof of Theorem 2.1

By assumption we can find a matrix $\mathbf{U}$ that replicates $\mathbf{P}$. According to Lemma 6.1 $\tilde{\mathbf{U}}$ also replicates $\mathbf{P}$. Note that by construction $\tilde{\mathbf{U}}$ is on the special format given by (2). Hence there exist a matrix on the form (2) that replicates $\mathbf{P}$. Conversely if a matrix is of the form given by (2), it does not change when we apply the transformation given by (8). Uniqueness then follows from Proposition 6.2.

□

Proof of Theorem 3.1

Define a new utility matrix $\tilde{\mathbf{U}}$

$$\tilde{\mathbf{U}}_{ts} = \begin{cases} \mathbf{U}^{(1)}_{ts} & \text{if } s = 1, \dots, S,\, t = 1, \dots, T \\ \mathbf{U}^{(1)}_{ts} + \ln[P^{(2)}_{ts}/P^{(1)}_{ts}] + \ln K & \text{if } s = S+1, \dots, 2S,\, t = 1, \dots, T \\ \mathbf{U}^{(1)}_{ts} + \ln[P^{(2)}_{ts}/P^{(1)}_{ts}] - \ln K & \text{if } s = 2S+1,\, t = 1, \dots, T \end{cases} \tag{16}$$

and let $A^{(1)}_t$, $t = 1, \dots, T$ and $B^{(1)}_s$, $s = 1, \dots, S$ denote the balancing factors solving (1) using the replicating utilities $\mathbf{U}^{(1)}$. Now put $A^{(2)}_t = A^{(1)}_t \cdot K$ and $B^{(2)}_s = B^{(1)}_s/K$. If $s = 1, \dots, S$, $t = 1, \dots, T$, we get

$$A^{(2)}_t B^{(2)}_s \exp[\tilde{\mathbf{U}}_{ts}] = A^{(1)}_t \cdot K \cdot B^{(1)}_s/K \exp[\mathbf{U}^{(1)}_{ts}] = A^{(1)}_t B^{(1)}_s \exp[\mathbf{U}^{(1)}_{ts}] = P^{(0)}_{ts}$$

If $s = S+1, \dots, 2S$, $t = 1, \dots, T$, we get

$$D_t B^{(2)}_{s-S} \exp[\tilde{\mathbf{U}}_{ts}] = D_t B^{(1)}_{s-S}/K \exp[\mathbf{U}^{(1)}_{ts} + \ln[P^{(2)}_{ts}/P^{(1)}_{ts}] + \ln K] = P^{(1)}_{ts} \cdot \frac{P^{(2)}_{ts}}{P^{(1)}_{ts}} = P^{(2)}_{ts}$$

If $s = 2S+1$, $t = 1, \dots, T$, we get

$$A^{(2)}_t \exp[\tilde{\mathbf{U}}_{ts}] = A^{(1)}_t \cdot K \exp[\mathbf{U}^{(1)}_{ts} + \ln[P^{(2)}_{ts}/P^{(1)}_{ts}] - \ln K] = P^{(1)}_{ts} \cdot \frac{P^{(2)}_{ts}}{P^{(1)}_{st}} = P^{(2)}_{ts}$$

The marginal constraints are automatically satisfied when the model replicates each entry in the matrix. Note that

$$\tilde{\mathbf{U}}_{1(2S+1)} = \mathbf{U}^{(1)}_{1s} + \ln[P^{(2)}_{1(2S+1)}/P^{(1)}_{1(2S+1)}] - \ln K = \mathbf{U}^{(1)}_{ts} = 0$$

and that if $s = 1, \dots, S$, then $\tilde{\mathbf{U}}_{ts} = \mathbf{U}_{ts}$. This proves that $\tilde{U}$ is of the form (2). Hence if we put $\mathbf{U}^{(2)} = \tilde{\mathbf{U}}$, this matrix is the unique matrix on the form (2) that replicates the system in (3) when $i = 2$.

□

7. Appendix 2: Numerical methods

In this appendix we describe the main algorithms we used to compute the models in this paper. The problems we solve are strongly non-linear, and uses a sometimes large set of parameters. Taking this into account, the algorithms below are surprisingly simple. They are easily implemented on a standard computer, and no special software is needed.

I How to find a numerical solution to (1) when utilities **U** and marginal constraints L and **E** are given:

We need to find numerical values for the $S + T$ balancing factors $A_1, \dots, A_T, B_1, \dots, B_S$. This is done as follows: Initially we put all the balancing factors equal to 1. Then for $t = 1, \dots, T$ we update A_t using

$$A_t = \frac{L_t}{\left(\sum_{s=1}^{S} B_s \exp[U_{ts}]\right) + P_{t(2S+1)}} \tag{17}$$

Once these are updated, then for $s = 1, \dots, S$ we update B_s using

$$B_s = \frac{E_s}{\sum_{t=1}^{T} A_t \exp[U_{ts}] + D_t \exp[U_{t(s+S)}]} \tag{18}$$

We then repeat the updates in (17) and (18) until the system settles. The algorithm is a variant of the Bregman balancing algorithm, see Bregman (1967). Like the standard Bregman algorithm this algorithm is surpricingly efficient, and solves large systems in a very short time.

II How to infer utilities when allocation data **P** is given:

From the allocation data we can quickly compute the marginal constraints L and **E**. To solve the problem we must construct numerical values for the u, v and w's in (2). Note that if S and T are fairly large, this system has a large number of parameters. Even in the small case covered in this paper, i.e., $T = 2$, $S = 4$, we are left with 12 unknown parameters, and standard replication software can hardly cover that case. A tailor made algorithm solves these problems very quickly, however. The construction can be described as follows:

Initially we put all the parameters in (2) equal to zero. Then we fix all parameters except u_{11}, and find a value for u_{11} such that P_{22} from (1) is equal to $P_{22}^{(\text{observed})}$. Note that we only make a match in one particular entry, the other entries may of course be very different. We update u_{11} to the value above. Then we fix all parameters except u_{12}, and find u_{12} such that P_{23} from (1) is equal to $P_{23}^{(\mathrm{observed})}$. We continue like that until all the non-zero entries in (2) has been updated. Note that this construction only involves one variable at the time, and due to the extreme speed of the Bregman type algorithm above, these updates can be made very quickly. Once all parameters have been updated, we repeat the process until the system settles at a replicating state. In the case reported in this paper, we obtain perfect replication within a few minutes. We have tested this algorithm on much larger systems, however, and cases with more than one hundred parameters can be solved within reasonable time (i.e. a few days) on a standard computer. A supercomputer using parallel processing would probably be able to handle extremely large systems of this kind.

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