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**Discussion paper**

# **The equity premium and the risk free rate in a production economy. A new perspective**

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# The equity premium and the risk free rate in a production economy. A new perspective

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## Abstract

We study a competitive equilibrium in a production economy, i.e., a system of prices at which firms' profit maximizing production decisions and individuals' preferred affordable consumption choices equate supply and demand in every market. We derive the equilibrium price of the firm and the equilibrium short term interest rate, the optimal consumption in society, as well as the risk premium on equity. Both a linear, and a nonlinear production technology are considered. For the linear one applied to the Standard and Poor's composite stock price index for the last century, a risk premium of 0.062 corresponds to a relative risk aversion of 2.27. The model provides a riskfree interest rate for the period of 0.8%. The nonlinear model, however, highlights a hedging demand for the investors related to the real economy, which would, if taken into account, make the stock market of the last century less risky than it was perceived to be.

## 1 Introduction

Rational expectations, a cornerstone of modern economics and finance, has been under attack for quite some time. Are prices too volatile relative to the information arriving in the market? Is the mean premium on equities over the riskless rate too large? Is the real interest rate too low? Is the market's risk aversion too high? Is the stochastic process representing aggregate consumption changes of nondurables and services too smooth compared to the returns in the stock market?

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Mehra and Prescott (1985) raised some of these questions in their paper, where they employed a variant of Lucas's (1978) pure exchange economy and conducted a "calibration" exercise in the spirit of Kydland and Prescott (1982). They chose the parameters of the endowment process to match the sample mean, variance and first-order autocorrelation of the annual growth rate of per capita consumption in the years 1889 -1978. They postulated that the representative agent has time- and state-separable utility. The puzzle is that they were unable to find a plausible pair of the subjective discount rate and the relative risk aversion of the representative agent to match the sample mean of the annual real rate of interest and of the equity premium over the 90-year period.

The equity premium puzzle is not an isolated observation. Hansen and Singleton (1983), Ferson (1983), Grossman, Melino, and Shiller (1987), and several others came to similar conclusions. Many theories have been suggested during the years to explain the puzzle. Constantinides (1990) introduced habit persistence in the preferences of the agents. Also Campbell and Cochrane (1999) and Haug (2001) used habit formation. These articles manage to explain high risk premiums and a low real interest rate. While this approach explained a reasonable level of a certain risk aversion parameter, the real risk aversion of the representative agent could still become arbitrarily large.

Epstein and Zin (1989) developed a framework for generalized expected utility, which allows for the separation of risk aversion from the intertemporal elasticity of substitution in consumption. That recursive utility does not solve the puzzle was demonstrated by Weil (1989), who discovered another problem, termed the riskfree rate puzzle. By using generalized expected utility, he obtained a risk premium in the same order of magnitude as the 0.35% of Mehra and Prescott, and the risk free rate he arrived at was around 20 – 25%, which is higher than what Mehra and Prescott obtained for the same value of the risk aversion. He found support for a value of around 0.1 for the elasticity of substitution in consumption.

Rietz (1988) introduced financial catastrophes in the model of Mehra and Prescott. In order to get close to a relative risk aversion of about 2, the consumption has to be reduced with 98% if a catastrophe occurs, but a fall of this magnitude has never been observed in the US. Barro (2005) develops this idea further, and tries to estimate the probability that a major crack will happen. Aase (1993a-b) introduce jumps in the continuous-time processes for consumption and dividends, and develops an equilibrium model in this incomplete setting. By assuming that the dynamics is driven by jump processes only, parts of the problems can be explained by deviations from normality of the equity index. Incomplete models have been studied

further by Weil (1992) by introducing non-diversifiable background risk in a one-period model, and by Heaton and Lucas (1996), who also introduce transaction costs in a multi-period model. While the results are moved in the right direction, there is still some way to go since the costs must be set unrealistically high in order to match the observed values of the real rate and the risk premium.

There is a rather long list of other approaches aimed to solve the puzzles, among them are borrowing constraints (Constantinides et al. (1998)), taxes (Mc Grattan and Prescott (2001)), loss aversion (Benartzi and Thaler (1995)), survivorship bias (Brown, Goetzmann and Ross (1995)), and heavy tails and parameter uncertainty (Weitzmann (2007)). While some of the proposed models may be part of the explanation, to date there does not seem to be any consensus that the puzzles have been fully resolved by any single of the proposed explanations.

Kocherlakota (1996) writes in his review paper on the equity premium puzzle:

*"The universality of the equity premium tells us that, like money, the equity premium must emerge from some primitive and elementary features of asset exchange that are probably best captured through extremely stark models. With this in mind, we cannot hope to find a resolution to the equity premium puzzle by continuing in our current mode of patching the standard models of asset exchange with transactions costs here and risk aversion there."*

Our approach starts with a linear neoclassical growth model, in a similar manner as in Cox, Ingersoll and Ross (1985a), and Duffie (1992). We then reinterpret the model as a production economy, where firms produce a single perishable consumption good, which can be used for consumption as well as for investment in production technologies. Prices are derived at which firms' profit maximizing production decisions and individuals' preferred affordable consumption choices equate supply and demand.

The firms optimal production decisions are taken as given by the consumers, who observe what the firms' shares sell for. Actual dividends paid to the shareholders are irrelevant, as the firms' investment decisions are now fixed, which is in accordance with the Miller and Modigliani (1961) result. By national accounting, in equilibrium the representative agent holds one share of the firms, and consumes the aggregate output from the firms. As a consequence of the consumers' preferences, they separate the investment decisions from the consumption choices. In the financial market the consumers behave as professional investors in the sense that they determine their optimal portfolio choices on the basis of financial market data alone. It appears that these decisions are consistent with a moderate level of risk aversion in equilibrium, when calibrated to US-stock market data of the period 1889-1978.

The riskfree interest rate is determined in the capital market as well, and combined with a reasonable level of the market's relative risk aversion, the model predicts a riskfree equilibrium interest rate of less than one percent, when calibrated to the same data.

However, this model gives the same volatility for the consumption growths as for the return on equity, which is not consistent with observations. For this reason we introduce a nonlinear technology which allows these two volatilities to differ. We obtain several new features, among them that the optimal demand for the risky asset has, in addition to the familiar term in standard finance, also a term that hedges against unfavorable changes in technology. We use a utility that can be state dependent, and show that the consumption based CAPM holds also in this model.

One interpretation is that for the investors of the last century, the stock market may have appeared more risky than it really was. Prices are endogenously determined by the agents, and are what the collective believes them to be. Since optimal investments must be perceived as rather difficult for most people, even for experts as the current (2007 -) financial crisis shows us, if agents have followed what standard financial theory prescribes, the application of this theory simply yields the results of the last century. If the agents had utilized optimally their hedging demands related to the "real" economy, they would not have demanded such a high premium to invest in equities as the one observed. At the same time this would have given a higher equilibrium short rate.

These conclusions should not come as a surprise. Most economists do not really believe that 0.80% is a reasonable value for the real interest rate, nor do they believe in an equity premium of 6%. Related to climate problems for example, the Stern (2007) report's conclusions about mitigation critically hinges on a low interest rate, in order not to discount future consumption benefits too heavily. Using a subjective impatience rate of zero percent, and a utility index that is "moderately concave", a value of 1.4% for the real rate was found appropriate. If the researchers had believed in the numbers behind the Equity Premium Puzzle, it would have been much simpler to just adopt the current "estimate" of 0.80%. Nordhaus (2008) uses the estimate 4.1% in a climate model context. In the same vein, in an interview in 2008, Ranish Mehra, one of the two authors behind the seminal 1985 -paper, suggests a reasonable premium on equity to be about one percent in the future.

McGrattan and Prescott (2003) re-examine the equity premium puzzle, taking into account some factors ignored by the Mehra and Prescott: Taxes, regulatory constraints, and diversification costs - and focus on long-term rather than short-term savings instruments. Accounting for these factors, the authors find that the difference between average equity and debt returns

during peacetime in the last century is less than one percent, with the average real equity return somewhat under five percent, and the average real debt return almost four percent.

Siegel (1992) finds that the period covered by the Mehra and Prescott study is not representative for the riskfree rate, which is typically higher in other periods.

A declining equity premium has been observed in the 1990s, and Lettau et. al. (2008) attributes this to lower macroeconomic volatility and high asset prices in this period. It will be interesting to see how the risk premiums develop in the future, after the current financial crisis.

In addition to the two puzzles discussed above, our approach can also shed some light on another but related problem in standard investment theory, namely that it prescribes a much larger fraction in equity compared to bonds than is observed.

The paper is organized as follows: A neoclassical growth model is introduced in Section 2, and reinterpreted as a production economy. Equilibrium in this latter economy is defined, and established in Section 3. Section 4 calibrates the equilibrium to the historical data, and makes the connection to the standard exchange economy. Section 5 introduces the nonlinear production technology, and a discussion appears in Section 6. Section 7 concludes.

## 2 The first model

### 2.1 A neoclassical growth model

We start by considering the following variant of the neoclassical growth model. An economy is developing over time in which  $K = K_t$  denotes the capital stock,  $c = c_t$  consumption and  $Z = Z_t$  net national product at time  $t$ , and where  $Z = f(K)$  denotes the production function. For each  $t$  we have the national accounting identity

$$\frac{dK_t}{dt} = f(K_t) - c_t$$

which means that production,  $f(K_t)$ , is divided between consumption,  $c_t$ , and investment,  $dK_t/dt$ . The problem is to find the optimal investment, or equivalently, the optimal consumption, that solves

$$\sup_{c \in \mathcal{C}} U(c) \tag{1}$$

where  $\mathcal{C}$  is the choice set, and  $U$  the central planner's utility function.

Uncertainty is introduced via a probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ , where  $\Omega$  is the set of states,  $\mathcal{F}$  is the set of events on which the probability measure  $P$  is defined, and  $\mathcal{F}_t$  is the set of possible events that may occur by time  $t$ , often referred to as the "information available" at time  $t$ . On this probability space is defined a standard Brownian motion  $B$ , that is assumed to generate the information filtration  $\mathcal{F}_t$ . The dynamics of capital stock process  $K$  is assumed to follow a process of the form

$$dK_t = (\mu(K_t, Y_t) - c_t)dt + \sigma(K_t, Y_t)dB_t; \quad K_0 > 0,$$

where  $Y$  is a vector of state variables, satisfying its own dynamic equation. In a Solow variant (no uncertainty)  $Y$  would be labor, and  $\mu$  could be a Cobb-Douglas type function. Cox, Ingersoll and Ross (1985b) specified  $Y$  to be a mean reverting diffusion process, a square root process, to capture cycles in the equilibrium interest rate. In this case  $\mu(K_t, Y_t) = \mu_K K_t Y_t$  and  $\sigma(K_t, Y_t) = \sigma_K K_t Y_t$ . As we are not primarily concerned with these issues in the following, we choose a linear production technology, and set  $Y \equiv 1$ . Our model for the capital stock is

$$dK_t = (\mu_K K_t - c_t)dt + \sigma_K K_t dB_t; \quad K_0 > 0, \quad (2)$$

where  $\mu_K$  and  $\sigma_K$  are strictly positive scalars.<sup>1</sup>

The objective is to maximize utility subject to the dynamic constraints (2) when the utility function  $U$  is time additive expected utility. The felicity index is separable with a constant coefficient of relative risk aversion  $\gamma > 0$ ,  $\gamma \neq 1$ , and a subjective rate  $\rho \geq 0$ , i.e.,  $u(c, t) = \frac{1}{1-\gamma} c^{1-\gamma} e^{-\rho t}$ . With an infinite time horizon, the objective (1) can be written

$$\sup_{c \in \mathcal{C}} E \left[ \int_0^\infty u(c_t, t) dt \right]. \quad (3)$$

The first order conditions for this problem is given by the Bellman equation, which takes the form ( $x = K_t$ )

$$\sup_{c \in R_+} \left( \mathcal{D}^c J(x) - \rho J(x) + \frac{c^{1-\gamma}}{1-\gamma} \right) = 0 \quad (4)$$

for all  $x > 0$  where  $J(\cdot)$  is the indirect utility function and

$$\mathcal{D}^c J(x) = J_x(x)(\mu_K x - c) + \frac{1}{2} J_{xx}(x) \sigma_K^2 x^2.$$

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<sup>1</sup>The model (2) could, perhaps, be considered as an extension of Domar's growth model to include uncertainty.

The solution is given by

$$c(t) = \theta K(t) \quad \text{for all } t, \quad (5)$$

where the constant  $\theta$  is

$$\theta = \frac{1-\gamma}{\gamma} \left( \frac{\gamma}{2} \sigma_K^2 + \frac{\rho}{1-\gamma} - \mu_K \right). \quad (6)$$

The detailed derivations are carried out in Appendix 1. For  $\theta > 0$  the necessary transversality condition

$$\lim_{T \rightarrow \infty} E\{e^{-\rho T} |J(K_T^c)|\} = 0 \quad (7)$$

is satisfied for all initial values of the capital stock  $K_0 > 0$  and for all admissible  $c \in \mathcal{C}$ . For the parameter ranges of interest, it can readily be verified that  $\theta \in (0, 1)$ . Accordingly the optimal consumption is a certain fraction of the capital stock. Notice that

$$\text{var}(c_t) = \theta^2 \text{var}(K_t^c) < \text{var}(K_t^c) \quad (8)$$

for all  $t$ , so the variance of the consumption at each time  $t$  is smaller than the variance of the capital stock at  $t$ . The implication of (5) is that the capital stock  $K^c(t)$  is lognormally distributed along the optimal consumption path, with dynamics

$$dK^c(t) = K^c(t)(\mu_K - \theta)dt + K^c(t)\sigma_K dB(t), \quad (9)$$

The conditional expected investment rate  $E_t("dK^c(t)/dt") = K^c(t)(\mu_K - \theta)$  for all  $t$ , where  $E_t$  signifies conditional expectation given the information set  $\mathcal{F}_t$  at time  $t$ .

## 2.2 The production/exchange economy

We now reinterpret the description in Section 2.1 as a single firm that depletes its capital stock  $K_t$  at rate  $\delta_t \in \mathcal{Y}$ , where  $\mathcal{Y}$  is the production set, and that maximizes its share price  $S_t$ . The economy is populated with one agent having preferences specified by (1) and (3), and endowment one share of the firm.

Thus  $\delta$  is the optimal real output of the firm controlling the capital stock production process and maximizing its share price, provided  $\delta_t = c_t$  for all  $t$ , where  $c_t$  is given in (5).

The consumer ignores what the firm is trying to do and merely observes that the firm's common share sells for  $S_t$  and each share pays the dividend

process  $\delta$  that the firm determines. The consumer is free to purchase any number of these shares, or to short-sell them, and can also borrow or lend at a short-rate process  $r$ . The price process of the riskfree asset is  $\beta_t$ , satisfying  $d\beta_t = r_t\beta_t dt$ . These are the only two securities available. The riskfree asset is supposed to be in zero net supply.

Let  $W_t$  be the consumer's wealth at time  $t$ , and  $n_t = (n_t^S, n_t^\beta)$  the number of stocks held in the risky asset and the riskfree asset, respectively, at time  $t$ . The agent's optimal consumption and investment strategy  $(c_t, n_t)$  satisfies

$$\sup_{(c,n) \in \mathcal{A}} E \left( \int_0^\infty \frac{1}{1-\gamma} c_t^{1-\gamma} e^{-\rho t} dt \right)$$

where the set  $\mathcal{A}$  signifies the set of permissible consumption processes  $c$  and trading strategies  $n$  that finances  $c$ . We use the following notation for the valuation functional:  $\Pi(c)$  is the value of the consumption stream  $c \in \mathcal{C}$ , where  $\Pi(\cdot)$  is defined by

$$\Pi(c) = \frac{1}{\pi_0} E \left\{ \int_0^\infty \pi_t c_t dt \right\}.$$

The state prices  $\pi$  *strictly supports* the allocation  $(c, \delta)$  provided

$$U(\tilde{c}) > U(c) \Rightarrow \Pi(\tilde{c}) > \Pi(c) \tag{10}$$

for all  $\tilde{c} \in \mathcal{C}$ , and

$$\Pi(\delta) \geq \Pi(\tilde{\delta}) \tag{11}$$

for all  $\tilde{\delta} \in \mathcal{Y}$ . Here the consumption choice set  $\mathcal{C}$  is equal to the production set  $\mathcal{Y}$ .

Also  $(c, \delta)$  is *budget constrained* by  $\pi$  if

$$\Pi(c) \leq \Pi(n^S \delta + n^\beta r). \tag{12}$$

Here (10) and (12) are the optimality conditions for the agent, given the state prices  $\pi$ . Condition (11) is market value maximization by the firm, given  $\pi$ . Because of strict monotonicity of the utility function, the budget constraint (12) holds with equality.

### 3 Equilibrium

Consider the economy  $\mathcal{E} = [(S, \beta), \pi, \delta, r, (c, n)]$ . A triple  $(c, \delta, \pi)$  is an *equilibrium* for  $\mathcal{E}$  provided  $(c, \delta)$  is a feasible allocation that is budget constrained and strictly supported by  $\pi$ .

In a representative agent economy this means that the aggregate consumption  $c_t = \delta_t$  for all  $t \geq 0$ , and that the optimal strategy for the agent is to hold one share of the firm and no shares of the riskfree security for each  $t \geq 0$ .

In order to find an equilibrium for this economy, we start with the state price, which is given by the marginal utility at the optimal output, or  $\pi_t = u'(\delta_t, t)$ , where  $\delta_t = \theta K_t^{(\delta)}$  for any  $t$ . The state price  $\pi_t = e^{-\rho t}(\theta K_t^{(\delta)})^{-\gamma}$ , a geometric Brownian motion process, satisfies the dynamics

$$d\pi_t = -\pi_t(\gamma(\mu_K - \theta) + \rho - \frac{1}{2}\gamma(\gamma + 1)\sigma_K^2)dt - \gamma\pi_t\sigma_K dB_t. \quad (13)$$

This representation is instrumental in finding the equilibrium short term interest rate, as we do next.

### 3.1 The interest rate

Our candidate for the equilibrium riskfree rate is  $r_t = -\frac{\mu_\pi(t)}{\pi_t}$ , where  $\mu_\pi(t)$  is the drift term in (13). It follows that

$$r_t = \rho + \gamma\mu_K - \frac{1}{2}\gamma(1 + \gamma)\sigma_K^2 - \gamma\theta \quad \text{for all } t, \quad (14)$$

i.e., the equilibrium interest rate is a constant. Notice the similarities between this expression for the interest rate, and the standard one that follows in a pure exchange economy

$$r_t^{ex} = \rho + \gamma\mu_c - \frac{1}{2}\gamma(1 + \gamma)\sigma_c^2, \quad (15)$$

where the parameter  $\mu_c$  is the conditional expected growth rate in aggregate consumption and  $\sigma_c$  is the corresponding volatility parameter. We return to a comparison with the standard exchange economy in Section 4.2.

A closer examination of the expression (14) reveals that it can be written

$$r = \mu_K - \gamma\sigma_K^2, \quad (16)$$

a simple formula, indeed. Notice that we obtain the precautionary effect without using the prudence property of the CRRA utility. Also notice that the subjective rate  $\rho$  falls out.

If the conditional expected growth rate of the capital stock increases, the equilibrium interest rate will increase, which is the income effect. Faced with better prospects for the future, our consumer would like to consume more now, and hence borrow. Since this is impossible, the interest rate must increase to make the agent just indifferent to status quo.

### 3.2 The price of the firm's stock

We now turn to the candidate for the price process for the firm's shares. Given a dividend stream  $\delta_t$  from the firm and state prices  $\pi_t$ , the price  $S$  at time  $t$  equals

$$S_t = \frac{1}{\pi_t} E_t \left( \int_t^\infty \pi_s \delta_s ds \right). \quad (17)$$

By carrying out this computation, first we obtain by Fubini's theorem that

$$S_t = \theta K_t^{(\delta)} \int_t^\infty e^{-\rho(s-t)} E_t \left\{ \exp \left( (1-\gamma)(\mu_K - \theta - \frac{1}{2}\sigma_K^2)(s-t) + (1-\gamma)\sigma_K(B_s - B_t) \right) \right\} ds.$$

Next, by the moment generating function of the normal distribution we get

$$S_t = \theta K_t^{(\delta)} \int_t^\infty e^{[(1-\gamma)(\mu_K - \theta) - \frac{1}{2}\gamma(1-\gamma)\sigma_K^2 - \rho](s-t)} ds = \frac{\theta}{\alpha} K_t^{(\delta)},$$

where

$$\alpha = -[(1-\gamma)(\mu_K - \theta) - \frac{1}{2}\gamma(1-\gamma)\sigma_K^2 - \rho].$$

Finally it can be verified that  $\alpha = \theta$ , so the spot price is  $S_t = K_t^{(\delta)}$  for all  $t$ . As we have shown that  $K_t^{(\delta)}$  is lognormal when  $\delta = c$ , and  $c$  is given by (5), it follows that our candidate price process  $S_t$  is a geometric Brownian motion process, where the conditional expected return on the capital gains are  $(\mu_S - \theta) = (\mu_K - \theta)$ , and the associated volatility  $\sigma_S = \sigma_K$ . This means, for example, that the securities market model is dynamically complete.

Recall when there are dividends, we adjust the price process for dividends and obtain the gains process  $G_t$ , sometimes called the adjusted price process, defined by

$$G_t = S_t + \int_0^t \delta_s ds \quad (18)$$

Using the above results the gains process is

$$dG_t = (\mu_K - \theta)S_t dt + \delta_t dt + \sigma_S S_t dB_t,$$

or, since  $\delta_t = \theta S_t$  we obtain

$$dG_t = \mu_S S_t dt + \sigma_S S_t dB_t. \quad (19)$$

The cumulative-return process  $R_t$  for this security is defined by  $dG_t = S_t dR_t$ , so that

$$dR_t = \mu_S dt + \sigma_S dB_t. \quad (20)$$

The process  $R_t$  takes into account both the capital gains and the dividends over the small time interval  $(t, t + dt]$ . This expression shows that  $R$  is a Brownian motion with drift. Because of this relation, we sometimes write  $\mu_R$  instead of  $\mu_S$ , and similarly  $\sigma_R$  instead of  $\sigma_S$ .

### 3.3 The optimal consumption and portfolio problem

Having a candidate for the price process of the firm's stock, we can now reformulate the consumer's optimal consumption and portfolio choice problem. The problem is to solve

$$\sup_{(c, \varphi)} E \left( \int_0^\infty \frac{1}{1-\gamma} c_t^{1-\gamma} e^{-\rho t} dt \right)$$

subject to the dynamic budget constraint

$$dW_t = \left( W_t(\varphi_t(\mu_S - r_t) + r_t) - c_t \right) dt + W_t \varphi_t \sigma_S dB_t, \quad W_0 = S_0, \quad (21)$$

where  $\varphi_t = \frac{n_t^S G_t}{W_t}$  is the fraction of wealth held in the risky asset at time  $t$ .

In formulating the budget constraint (21) we have made use of the dynamics of the price process  $G_t$  that adjusts for dividends. This problem is now well suited for dynamic programming, and the Bellman equation is

$$\sup_{c, \varphi} \left\{ \mathcal{D}^{c, \varphi} J(w) - \rho J(w) + \frac{c^{(1-\gamma)}}{1-\gamma} \right\} = 0, \quad w > 0,$$

where ( $w = W_t$ )

$$\mathcal{D}^{c, \varphi} J(w) = J_w(w) (\varphi(\mu_S - r)w + rw - c) + \frac{1}{2} w^2 \varphi^2 \sigma^2 J_{ww}(w).$$

The first order condition in  $\varphi$  is

$$J_w(w) (\mu_S - r)w + w^2 \varphi \sigma^2 J_{ww}(w) = 0 \quad \text{for all } w > 0,$$

which gives in terms of the dynamics of  $G$  that

$$\varphi_t = \left( - \frac{J_w(W_t)}{J_{ww}(W_t) W_t} \right) \frac{\mu_S - r}{\sigma_S^2}, \quad (22)$$

Here  $\varphi$  is proportional to the relative risk tolerance of the agent's indirect utility, increases with the risk premium  $(\mu_S - r)$ , and decreases as the volatility parameter  $\sigma_S$  increases, ceteris paribus.

Next we find the first order condition for optimization in the consumption variable  $c$ . From the Bellman equation it is seen to be

$$-J_w(w) + c^{-\gamma} = 0,$$

which implies that

$$c = (J_w(w))^{-\frac{1}{\gamma}}, \quad \text{or} \quad c_t = (J_w(W_t))^{-\frac{1}{\gamma}}$$

in terms of the random wealth process  $W_t$ . Notice how the consumption choice problem is separated from the investment problem. In Appendix 2 it is shown that the solution is

$$c_t = \eta W_t \tag{23}$$

where the constant  $\eta$  is

$$\eta = \left[ \frac{1-\gamma}{\gamma} \left( \frac{\rho}{1-\gamma} - r - \frac{1}{2} \frac{1}{\gamma} \frac{(\mu_S - r)^2}{\sigma_S^2} \right) \right]. \tag{24}$$

The agent optimally consumes a constant proportion of current wealth.

Returning to the optimal investment policy, it is seen to be

$$\varphi_t = \frac{1}{\gamma} \frac{\mu_S - r}{\sigma_S^2}, \tag{25}$$

i.e., the relative risk tolerance of the indirect utility function is the same as the relative risk tolerance of the felicity index. The optimal investment ratio is of the same form as the classical solution in the no-dividend case, known to follow when the price process is lognormal (e.g., Mossin (1968), Samuleson (1969), Merton (1971)<sup>2</sup>). The difference is that in (25) the parameter  $\mu_S$  is the return rate of capital gains plus dividends, while only the capital gains appears in the standard formulation.

Since we have only one consumer in our model, he is interpreted as the representative agent in the context of equilibrium. For the above to be an equilibrium, it must be the case that the price  $S_t$  of the firm and the interest rate  $r_t$  are both set at each time  $t$  such that the agent's fraction of wealth in the risky asset is always equal to 1, or  $\varphi_t = 1$  for all (or, a.a.)  $t$  (a.s.). From (25) it follows that in equilibrium it must be the case that

$$\mu_S - r = \gamma \sigma_S^2. \tag{26}$$

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<sup>2</sup>Notice that these references do not deal with equilibrium; the prices of the risky assets are given exogenously.

The above investment strategy is only feasible if the dividends  $\delta$  from the firm equals the optimal consumption  $c$  derived in (23). This is indeed the case: By comparing  $c$  to the optimal consumption in (5), derived in the centralized economy of Section 2.1, we can show that equating these two expressions is equivalent to the equilibrium relation (26). In other words

$$\delta_t = \theta K_t = \eta W_t = c_t \text{ for all } t \Leftrightarrow \mu_S - r = \gamma \sigma_S^2. \quad (27)$$

Thus taking the output from the single firm  $\delta_t$  to be equal to the optimal consumption  $c_t$  in the centralized economy of Section 2.1, we have shown that this is also equal to the optimal consumption of the representative agent, denoted  $c_t$  as well, in the decentralized economy, provided that (26) holds.

Returning to (26) and recalling that our candidate for the riskfree rate is

$$r = \gamma(\mu_K - \theta) + \rho - \frac{1}{2}\gamma(1 + \gamma)\sigma_K^2,$$

it follows that

$$\mu_S = \gamma(\mu_K + \sigma_S^2 - \theta) + \rho - \frac{1}{2}\gamma(1 + \gamma)\sigma_K^2$$

Inserting for  $\theta$  from (6) we obtain that  $\mu_S = \mu_K$ , which is consistent with our earlier conjecture for the stock price.

Notice that the wealth of the representative agent can always be found by a prospective point of view as

$$W_t = \frac{1}{\pi_t} E_t \left( \int_t^\infty \pi_s \delta_s ds \right),$$

which by (17) means that  $W_t = S_t$  for all  $t$ .

What remains to be verified for an equilibrium to be satisfied is profit maximization at the state prices  $\pi_t$ . To this end, recall that the securities market is dynamically complete. This means that the dynamic optimization problem in Section 2.1 is equivalent to the following "static" problem

$$\sup_{\tilde{\delta}} U(\tilde{\delta}) \quad \text{subject to} \quad \Pi(\tilde{\delta}) \leq w,$$

where  $w = S_0 \cdot 1 = K_0$ , and  $\Pi(\tilde{\delta}) = \frac{1}{\pi_0} E \{ \int_0^\infty \tilde{\delta}_t \pi_t dt \}$ . Since we have shown that the solution  $\delta$  to this problem satisfies  $\Pi(\delta) = S_0$ , the problem can be written

$$\sup_{\tilde{\delta}} U(\tilde{\delta}) \quad \text{subject to} \quad \Pi(\tilde{\delta}) \leq \Pi(\delta),$$

or,

$$U(\tilde{\delta}) \leq U(\delta) \Leftrightarrow \Pi(\tilde{\delta}) \leq \Pi(\delta) \quad \text{for any } \tilde{\delta} \in \mathcal{Y},$$

which shows that the requirement (11) holds, i.e., the optimal output  $\delta$  from the firm maximizes profits at prices  $\pi$ .

	Expectation	Standard deviation
Consumption growth	1,83%	3,57%
Return S&P500	6,98%	16,67%
Government bonds	0,80%	5,67%
Equity premium	6,18%	16,54%

Table 1: Key numbers for the time period 1889-1978

## 4 Comparisons and calibrations

In this section we first we relate our results to the corresponding results of the pure exchange economy, that is most commonly employed in the present setting. Then we perform a numerical calibration of our model to the data used by Mehra and Prescott (1985). We interpret our firm as the US production economy, and the risk premium of the risky asset as the equity premium.

### 4.1 A numerical calibration exercise

We refer to Table 1 for the key estimates for the period 1889–1978, data used by Mehra and Prescott (1985). Using these results for the volatility of equity and the equity premium, we have an estimate of 0.165 for the parameter  $\sigma_S$ . For an observed, average equity premium over the period of 0.062, our relation (26) then provides an estimate of 2.27 for the relative risk aversion parameter  $\gamma$ . This numerical value is well within the acceptable region.

The return on equity was estimated to 0.0698 over the same period, which is then an estimate for the parameter  $\mu_R$ . Recalling that  $\mu_R = \mu_K$ , the expression (14), or equivalently (16), gives an estimate of 0.0080, or 0.80% for the equilibrium interest rate  $r$ . The latter number is exactly the one estimated by Mehra and Prescott (1985) (to the fourth decimal place) for the time period 1889 to 1978. We conclude that in this linear model there seems to be no equity premium puzzle, and there is no riskfree rate puzzle. The model has, however, one weakness, to which we now turn.

### 4.2 The connection to the CCAPM

One result of our analysis is that the optimal consumption  $c_t = \theta K_t$ , which means that the optimal consumption has the dynamics

$$dc_t = (\mu_K - \theta)c_t dt + \sigma_K c_t dB_t, \quad (28)$$

or the growth rate in consumption can be expressed as follows

$$\frac{dc_t}{c_t} = (\mu_K - \theta)dt + \sigma_K dB_t. \quad (29)$$

We define the growth rate of the per capita real consumption by  $C$ , or  $dc_t = c_t dC_t$ , so that

$$dC_t = \mu_C dt + \sigma_C dB_t,$$

where  $\mu_C = \mu_c$ ,  $\sigma_C = \sigma_c$ . Recall the corresponding expression for the cumulative-return process  $R_t$  of the firm in (20). Using this, the consumption based CAPM has the following form

$$\mu_R - r = \gamma \sigma_R \sigma_C \quad (30)$$

in the pure exchange economy - the canonical model, where  $\sigma_R = \sigma_S$ . From the equation for the consumption growth in (29), we see that the risk premium in (30) can be written

$$\mu_R - r = \gamma \sigma_S \sigma_K. \quad (31)$$

Note that this is consistent with our result (26), since  $\sigma_S = \sigma_K$  because  $S_t = K_t^{(\delta)}$ , diffusion invariance, and the equality  $\mu_R = \mu_S$ .

The linear relationship  $c_t = \theta S_t$  between consumption and equity has as a consequence that (8) holds, or  $\text{var}(c_t) = \theta^2 \text{var}(S_t) < \text{var}(S_t)$ , since  $\theta \in (0, 1)$ . Thus very different levels of variances of equity and consumption are allowed. However, as we have demonstrated, the linear relationship leads to the same percentage-wise changes in consumption and equity, so the values for parameters  $\sigma_R$  and  $\sigma_C$  are the same. If the estimates in Table 1 are correct, this is not consistent with the model.

Returning to the equilibrium interest rate, when aggregate consumption is taken as exogenously given, and, moreover, is lognormal as in (28), the equilibrium interest rate for our CRRA consumer is known to have the form

$$r_t = \rho + \gamma \mu_c - \frac{1}{2} \gamma (1 + \gamma) \sigma_c^2 \quad (32)$$

in the canonical model, as was remarked in (15). Since the growth rate in aggregate consumption is  $\mu_c = (\mu_K - \theta)$  and the volatility of the growth rate of consumption is  $\sigma_c = \sigma_K$ , it follows that (32) can be written

$$r_t = \rho + \gamma (\mu_K - \theta) - \frac{1}{2} \gamma (1 + \gamma) \sigma_K^2,$$

which is seen to be the the same as our expression (14) for the equilibrium short term interest rate in the production economy. Accordingly our results are consistent with those of the standard pure exchange economy.

## 5 The general set up

The model we present here is in the same spirit as the one of sections 2 and 3, and will have the advantage that it overcomes the weakness of the linear model, since it allows  $\sigma_C \neq \sigma_R$ .

There exists one production good, which is also the consumption good. This good may be consumed or invested in two technologies. One is riskfree, the other consists of the capital stock  $K$  satisfying the dynamics

$$dK_t = K_t \mu_K(K_t, Y_t) dt + K_t \sigma_K(K_t, Y_t) dB_t, \quad (33)$$

where  $Y$  is a state variable satisfying its own dynamics

$$dY_t = Y_t \mu_Y(Y_t) dt + Y_t \sigma_Y(Y_t) dB_t. \quad (34)$$

The terms  $\mu_K$  and  $\mu_Y$  are allowed to be nonlinear, in fact the former is required to be so. If we interpret  $Y$  as labor, it is clear that the utility function  $u$  must depend upon leisure, so that  $u = u(c_t, Y_t)$  at time  $t$ , where the utility function is decreasing in its second variable. Because of ease of exposition, at the present we leave out the state variable  $Y$  in the list of arguments of  $u$ , but will return to it in Section 5.5. At first the agent is not allowed to use the riskfree technology. The problem to be solved is the following

$$\max_{c \in \mathcal{C}} E \left\{ \int_0^\infty u(c_t) e^{-\rho t} dt \right\} \quad (35)$$

subject to the wealth dynamics

$$dW_t = (W_t \mu(K_t) - c_t) dt + W_t \sigma_K(K_t) dB_t.$$

It is here assumed that the agent invests everything in the production technology. The Bellman equation for this problem is

$$\sup_c \left\{ \mathcal{D}^c J(w, k) - \rho J(w, k) + u(c) \right\},$$

where

$$\begin{aligned} \mathcal{D}^c J(w, k) = & J_w(w, k) (\mu_K(k) w - c) + J_k(w, k) k \mu_K(k) + \\ & \frac{1}{2} J_{ww}(w, k) w^2 \sigma_K(k) \sigma_K(k) + \frac{1}{2} J_{kk}(w, k) k^2 \sigma_K(k) \sigma_K(k) + \\ & J_{wk}(w, k) w \sigma_K(k) \sigma_K(k) k. \end{aligned} \quad (36)$$

Assuming an interior solution, the first order condition in the consumption variable  $c$  is,

$$-J_w(w, k) + u_c(c) = 0. \quad (37)$$

Further, assuming that the marginal utility  $u_c$  is invertible, and that the indirect utility function  $J$  is well defined and sufficiently smooth, the optimal consumption is given by

$$c^*(t) = u_c^{-1}(J_w(W_t, K_t)). \quad (38)$$

## 5.1 The equilibrium real interest rate

As in CIR (1985a), we may first introduce riskless borrowing and lending, and second a securities market. Considering the first, in equilibrium the representative agent is just indifferent to holding the riskfree asset, so the short term equilibrium interest rate  $r$  is determined from the constraint that the agent invests everything in the risky technology.

The equilibrium interest  $r$  may either be less or greater than  $\mu_K$ , the expected return on optimally invested wealth. Although investment in the production process exposes an individual to uncertainty about the output received, it may also allow him to hedge against the risk of less favorable changes in technology. An individual investing only in locally riskless lending would be unprotected against this latter risk. This is, for example, the case with the individual in the first part of the paper, when the riskless rate is

$$r = \mu_K - \gamma\sigma_K^2,$$

which does not take into account the covariance between wealth and the capital stock. In general, either effect may dominate.

As noted in the first part, the spot rate can be determined from the state price deflator  $\pi$  as follows

$$r_t = -\mu_\pi(t)/\pi_t, \quad (39)$$

where the state price deflator is  $\pi_t = u_c(c^*(t))e^{-\rho t} = J_w(W_t, K_t)e^{-\rho t}$ . In terms of the dynamics for the quantity  $J_w(W_t, K_t)$ , by Ito's formula we then get the following dynamics of  $\pi$

$$d\pi_t = \mu_\pi(t)dt + e^{-\rho t}(J_{ww}(W_t, K_t)W_t\sigma_K(K_t) + J_{wk}(W_t, K_t)K_t\sigma_K(K_t))dB_t \quad (40)$$

where the drift term  $\mu_\pi$  is the following

$$\begin{aligned} \mu_\pi(t) = & -\rho\pi_t + e^{-\rho t}(J_{ww}(W_t, K_t)(W_t\mu_K(K_t) - c^*(t)) \\ & + J_{wk}(W_t, K_t)(K_t\mu_K(K_t) + \frac{1}{2}J_{www}(W_t, K_t)W_t^2\sigma_K(K_t)\sigma_K(K_t) \\ & + J_{wwk}(W_t, K_t)W_tK_t\sigma_K(K_t)\sigma_K(K_t) \\ & + \frac{1}{2}J_{wkk}(W_t, K_t)K_t^2\sigma_K(K_t)\sigma_K(K_t)). \end{aligned} \quad (41)$$

From this it follows that the equilibrium short rate is

$$\begin{aligned}
r_t = & \rho + \left( \frac{-J_{ww}W_t}{J_w} \right) \left( \mu_K(K_t) - \frac{u_c^{-1}(J_w)}{W_t} \right) + \left( \frac{-J_{wk}K_t}{J_w} \right) \mu_K(K_t) \\
& + \left\{ \frac{1}{2} \left( \frac{-J_{www}W_t^2}{J_w} \right) + \frac{1}{2} \left( \frac{-J_{wkk}K_t^2}{J_w} \right) \right. \\
& \left. + \left( \frac{-J_{wwk}W_tK_t}{J_w} \right) \right\} \sigma_K(K_t) \sigma_K(K_t), \quad \text{for all } t \geq 0.
\end{aligned} \tag{42}$$

This may be compared to equation (14) for the corresponding linear technology, which is

$$r_t = \rho + \gamma(\mu_K - \theta) - \frac{1}{2}\gamma(1 + \gamma)\sigma_K^2.$$

In the above the term  $\frac{u_c^{-1}(J_w)}{W_t} = \frac{c_t^*}{W_t} = \theta$  in the linear model, the next term has no counterpart in this model, the fourth term on the right hand side of (42) corresponds to last term above, while the last terms have no counterparts.

## 5.2 The price of the firm's stock

Next we introduce a securities market. The setting and notation are the same as in Section 3.3. The equilibrium price process of the firm is denoted by  $S_t$  and is given by equation (17) with the state price  $\pi$  satisfying the dynamic equation (40), and the dividends  $\delta(t) = c^*(t)$ , the latter given in (38). The gains process  $G_t$ , the price process adjusted for dividends, has the representation

$$dG_t = \mu_G(W_t, K_t)dt + \sigma_G(W_t, K_t)dB_t,$$

where the wealth  $W_t$  depends on the optimal dividends given in (38). Defining the cumulative-return process  $R$  of this security by  $dG_t = S_t dR_t$ , we may write

$$dR_t = \mu_R(W_t, K_t)dt + \sigma_R(W_t, K_t)dB_t,$$

where  $\mu_R(W_t, K_t) = \frac{1}{S_t}\mu_G(W_t, K_t)$  and  $\sigma_R(W_t, K_t) = \frac{1}{S_t}\sigma_G(W_t, K_t)$ , assuming  $S_t > 0$  a.s. for all  $t$ .

Finally we let the agent trade freely in the capital market consisting of the firm's shares and the riskfree asset.

## 5.3 The optimal consumption and portfolio problem

The consumer/investor is initially endowed with one share of the firm, and solves the problem

$$\sup_{c, \varphi} E \left\{ \int_0^\infty e^{-\rho t} u(c_t) dt \right\},$$

subject to the dynamic wealth constraint

$$dW_t = (W_t[\varphi_t(\mu_R(W_t, K_t) - r_t) + r_t] - c_t)dt + W_t\varphi_t\sigma_R(W_t, K_t)dB_t,$$

where  $W_0 = S_0$ . Here the wealth  $W_t$  depends on the optimal consumption  $c$ , while the capital stock  $K_t$  does not. The associated Bellman equation is

$$\sup_{c, \varphi} \left\{ \mathcal{D}^{c, \varphi} J(w, k) - \rho J(w, k) + u(c) \right\} = 0, \quad w > 0,$$

where

$$\begin{aligned} \mathcal{D}^{c, \varphi} J(w, k) &= J_w(w, k)(\varphi(\mu_R(w, k) - r_t)w + r_t w - c) + J_k(w, k)k\mu_K(k) \\ &\quad + \frac{1}{2}J_{ww}(w, k)w^2\varphi^2\sigma_R(w, k)\sigma_R(w, k) + \frac{1}{2}J_{kk}(w, k)k^2\sigma_K(k, k)\sigma_K(k) \\ &\quad + J_{wk}(w, k)wk\varphi\sigma_R(w, k)\sigma_K(k). \end{aligned}$$

The first order condition in  $\varphi$  is

$$\begin{aligned} J_{ww}(w, k)w^2\sigma_R(w, k)\sigma_R(w, k)\varphi + J_w(w, k)(\mu_R(w, k) - r_t)w \\ + J_{wk}(w, k)wk\sigma_R(w, k)\sigma_K(k) = 0. \end{aligned}$$

This gives for the optimal demand of the risky asset

$$\begin{aligned} W_t\varphi_t &= \left( -\frac{J_w(W_t, K_t)}{J_{ww}(W_t, K_t)} \right) \left( \frac{\mu_R(W_t, K_t) - r_t}{\sigma_R(W_t, K_t)\sigma_R(W_t, K_t)} \right) \\ &\quad + \left( -\frac{J_{kw}(W_t, K_t)K_t}{J_{ww}(W_t, K_t)} \right) \left( \frac{\sigma_R(W_t, K_t)\sigma_K(K_t)}{\sigma_R(W_t, K_t)\sigma_R(W_t, K_t)} \right). \quad (43) \end{aligned}$$

The demand function is seen to have two components: The first one is the usual demand function for a risky asset, similar to the one encountered by a single-period mean-variance maximizer. This is what an investor can relate to when he only has access to the financial market. For the linear model is the only term that appears in the demand function, as can be seen from (22). In this respect the time continuous model with the linear production technology has much in common with the widely taught, one-period mean-variance model.

The last term reflects the investor's demand for the risky asset to hedge against unfavorable shifts in the investment opportunity set, here represented by less favorable changes in technology. This term is the hedging demand, available when the investor also uses information about the production part of the economy. For the linear model of the first part, this hedging component is not present, and as a consequence, if the production technology is nonlinear in reality, the stock market may have appeared more risky than it really was.

## 5.4 The risk premium

The representative agent is initially endowed with one share of the firm, in which case the market clearing condition is  $\varphi_t = 1$  a.s. for all  $t$ . From the expression (43) we get the equilibrium risk premium

$$\begin{aligned} \mu_R(W_t, K_t) - r_t = & \left( - \frac{J_{ww}(W_t, K_t)W_t}{J_w(W_t, K_t)} \right) \sigma_R(W_t, K_t) \sigma_R(W_t, K_t) \\ & + \left( - \frac{J_{wk}(W_t, K_t)K_t}{J_w(W_t, K_t)} \right) \sigma_R(W_t, K_t) \sigma_K(K_t). \end{aligned} \quad (44)$$

Comparing with the linear model of the first part, we see from (26) that the second term on the right-hand side in the above expression is missing. For investors who only focus on the stock market, this leads to a risk premium of about 6% for the data of the last century. The second term on the right hand side appears in our framework because of the nonlinear production function.

Considering the expression in (44), could it be, for example, that the first term on the right hand side is approximately equal to the relative risk aversion  $\gamma$ , times the variance rate of the return, and that the last term is small compared to the first term, such that  $\mu_R - r \approx \gamma \sigma_R^2$ ? If this were the case, our present model would give the same nice fit to the data of the last century as the model of the first part of the paper. That this is not so, will now be explained.

To this end, we seek an interpretation of the terms of the risk premium in (44). First we find the dynamics of the quantity  $e^{-\rho t} u_c(c_t^*)$ , and compare this to the dynamics of the state price deflator  $\pi_t$  given in (40). By diffusion invariance and the envelope theorem, it follows that

$$u_{cc}(c_t^*)c_W^* = J_{ww}(W_t, K_t) \quad \text{and} \quad u_{cc}(c_t^*)c_K^* = J_{wk}(W_t, K_t)$$

where  $c_W^*$  is the partial derivative of  $c^*$  with respect to wealth, and  $c_K^*$  is the partial derivative of  $c^*$  with respect to the state variable  $K$ . Using this, the risk premium can be represented in the following convenient form

$$\begin{aligned} \mu_R(W_t, K_t) - r_t = & \left( - \frac{u_{cc}(c_t^*)c_t^*}{u_c(c_t^*)} \right) \left( el_W(c_t^*) \sigma_R(W_t, K_t) \sigma_R(W_t, K_t) \right. \\ & \left. + el_K(c_t^*) \sigma_R(W_t, K_t) \sigma_K(K_t) \right), \end{aligned} \quad (45)$$

where  $el_W(c_t^*) = \frac{c_W^* W_t}{c_t^*}$ , and  $el_K(c_t^*) = \frac{c_K^* K_t}{c_t^*}$  are the partial consumption elasticities with respect to wealth and capital stock, respectively.

Similarly, the equilibrium demand for the risky asset is given by

$$\varphi_t = \left( -\frac{u_c(c_t^*)}{u_{cc}(c_t^*)c_t^*} \right) \frac{1}{el_W(c_t^*)} \frac{\mu_R - r}{\sigma_R \sigma_R} - \frac{el_K(c_t^*)}{el_W(c_t^*)} \frac{\sigma_R \sigma_K}{\sigma_R \sigma_R}. \quad (46)$$

The first term is seen to be the classical one in standard finance in the case when  $el_W(c_t^*) = 1$ , that is known to be the only term in the pure demand theory (Mossin (1968), Samuelson (1969), Merton (1971)). The last term is the hedging demand. The result means that the agent is supposed to actively employ macro data related to both the capital market and the production sector of the "real" economy to find the optimal investment.

The fraction  $\frac{el_K(c_t^*)}{el_W(c_t^*)}$  is the marginal substitution ratio between  $K$  and  $W$ , multiplied by the ratio  $\frac{K}{W}$ . It is a measure of the elasticity of substitution in consumption between capital and wealth (but is not exactly this quantity as it is usually defined).

Using the above elasticities, the short term interest rate in (42) can be written

$$r_t = \rho + \left( -\frac{u_{cc}(c_t^*)c_t^*}{u_c(c_t^*)} \right) \left\{ (el_W(c_t^*) + el_K(c_t^*))\mu_K(K_t) - el_W(c_t^*) \frac{u_c^{-1}(J_W)}{W_t} \right\} + \dots \quad (47)$$

where we have omitted the higher order terms. In the situation where the elasticities add to one and take values between zero and one, while  $\theta$  remains close to the quantity  $\frac{u_c^{-1}(J_W)}{W_t}$ , the short term can be seen to be larger than the one produced by the linear model. The omitted terms may further strengthen this effect.

## 5.5 The consumption based capital asset pricing model

Returning to the risk premium in (45), we want to explore in what sense it is different from the risk premium obtained in the linear production model. For example, if  $el_W(c_t^*) = el_K(c_t^*) \approx \frac{1}{2}$ , these two risk premiums would yield approximately the same numerical results, provided  $\sigma_R = \sigma_K$ . Recall that we now operate with a nonlinear production technology, so, in particular it is no longer true that the optimal consumption is proportional to wealth, or that the price of the firm's stock is equal to the capital stock. Thus the volatilities  $\sigma_R$  and  $\sigma_K$  are not necessarily equal. It turns out that also in the production based model of this section, the risk premium can ultimately be expressed as

$$\mu_R(W_t, K_t) - r_t = \left( -\frac{u_{cc}(c_t^*)c_t^*}{u_c(c_t^*)} \right) \sigma_C(t) \sigma_R(t), \quad (48)$$

i.e., the CCAPM holds true also here. The simplest way to demonstrate this is to find the dynamics of  $c_t^*$  using the representation in (38), which is  $c^*(t) = u_c^{-1}(J_w(W_t, K_t))$ . By Itô's lemma we get

$$dc_t^* = \mu_{c^*}(t)dt + \left( \frac{J_{ww}(W_t, K_t)}{u_{cc}(c_t^*)} \right) \sigma_W dB_t + \left( \frac{J_{wk}(W_t, K_t)}{u_{cc}(c_t^*)} \right) K_t \sigma_K dB_t.$$

From this we see that the volatility  $\sigma_C(t)$  of the consumption growths is

$$\sigma_C(t) = \left( \frac{J_{ww}(W_t, K_t)}{u_{cc}(c_t^*)c_t^*} \right) \sigma_W + \left( \frac{J_{wk}(W_t, K_t)K_t}{u_{cc}(c_t^*)c_t^*} \right) \sigma_K.$$

Accordingly is

$$\begin{aligned} \left( - \frac{u_{cc}(c_t^*)c_t^*}{u_c(c_t^*)} \right) \sigma_C(t) \sigma_R(t) &= \left( - \frac{J_{ww}(W_t, K_t)W_t}{J_w(W_t, K_t)} \right) \sigma_R \sigma_R \\ &+ \left( - \frac{J_{wk}(W_t, K_t)K_t}{J_w(W_t, K_t)} \right) \sigma_R \sigma_K = \mu_R(t) - r_t, \end{aligned} \quad (49)$$

where we have used the first order conditions in (37), and the expression for the risk premium in (44) accounts for the last equality.

There is one more point to be made here: We also need to investigate the case with labor  $Y$  in the production function as well, and leisure must then appear in the utility function, as mentioned before, and salary could be output in the one agent world. Otherwise the analysis would not be complete, since we have left out an important factor in the economy. Carrying out the analysis, the risk premium will include a term representing the investor's demand to hedge against unfavorable shifts in the supply of labor. This term takes the form

$$\left( - \frac{u_{cc}(c_t^*, Y_t)c_t^*}{u_c(c_t^*, Y_t)} \right) (e l_Y(c_t^*) \sigma_Y(t) \sigma_R(t)).$$

The risk premium can be expressed as

$$\begin{aligned} \mu_R(t) - r_t &= \left( - \frac{u_{cc}(c_t^*, Y_t)c_t^*}{u_c(c_t^*, Y_t)} \right) \sigma_C(t) \sigma_R(t) \\ &+ \left( \frac{u_{cc}(c_t^*, Y_t)}{u_c(c_t^*, Y_t)} \right) \left( \frac{\partial}{\partial y} u_c^{-1}(J_w(W_t, K_t, Y_t), Y_t) \right) Y_t \sigma_Y(t) \sigma_R(t), \end{aligned} \quad (50)$$

where the function  $u_c^{-1}(\cdot, y)$  inverts  $u_c(\cdot, y)$ , meaning that  $u_c^{-1}(u_c(x, y), y) = x$  for all  $(x, y)$ . The partial derivative with respect to labor in the last term appears because the utility index depends on leisure (labor). One may of

course wonder about the sign of this last term in the risk premium, but using the first order condition (38) in consumption, the partial derivative term becomes

$$\frac{\partial}{\partial y} u_c^{-1}(u_c(c_t^*, Y_t), Y_t) = 0 \quad \text{for all values of } c_t^* \text{ and } Y_t \text{ a.s.,}$$

which means that in equilibrium the risk premium is

$$\mu_R(t) - r_t = \left( - \frac{u_{cc}(c_t^*, Y_t) c_t^*}{u_c(c_t^*, Y_t)} \right) \sigma_C(t) \sigma_R(t),$$

that is, the CCAPM holds in this particular form. Thus this model is fairly robust.

With a relative risk aversion of  $\gamma = 2.27$  and a subjective rate of  $\rho = 0.01$  for the representative consumer, the standard model demands a risk premium of 1.35% and a short term interest rate of around 4.7% for the consumption and equity moment estimates in Table 1.

## 6 Discussion of the results

From the linear production model, which is consistent with the widely taught standard financial theories, the consumer/investor is being told to separate his consumption choice problem from his investment problem. In particular this means that he will look at the financial market in isolation. With a relative risk aversion close to two ( $\gamma = 2.27$ ), a risk premium ( $\mu_R - r$ ) of around six percent will emerge from a observed market volatility of equity of 16.67%. In its turn this leads to an equilibrium short rate of 0.80%, both numbers consistent with the observations of Mehra and Prescott (1985). However, as we have also noticed, this model gives the same volatility for the consumption growths  $C$  as for the return  $R$  on equity, which is not consistent with observations.

Investors have, of course, no feeling for the consumption based CAMP in making their investment decisions, since they do not really use aggregate consumption as one of their inputs to this problem. When we introduce a nonlinear production technology, which allows the resulting model to have different volatilities for  $C$  and  $R$ , we observe several new features, the most important being: The optimal demand for the risky asset has, in addition to the familiar term in standard finance, also terms that will hedge against unfavorable changes in the state variables of the economy.

In contrast to the CCAPM, this provides the investor with tools to actively use the real economy when investing in the capital markets.

One interpretation is that for the investors of the last century, the stock market may have appeared more risky than it really was. By this I mean that for the given volatility of equity (16.67%), the investors have demanded a too high risk premium. Since prices are not determined only by exogenous factors, as, for example, physical processes in the natural sciences, but by the agents in the economy, they are of course, in a sense, always "right"; prices are the results of the acts of people. Since optimal investments must be perceived as rather difficult for most people, even experts as the current (2007 -) financial crisis shows us, agents may have largely done as they have been taught, or advised by, for example, standard financial theory. If the agents had utilized optimally their hedging demands related to the real economy, they would not have required such high premiums to invest in equities as the ones observed. At the same time this would have yielded a higher equilibrium short rate.

In addition to the two puzzles discussed above, our approach can also shed some light on another but related problem with the standard theory. Recall the following investment puzzle: Using the data of Table 1, for a relative risk aversion of around two, the optimal fraction in equity is 132% based on the standard, first term in (46) (when  $el_W(c_t^*) = 1$ ). In contrast, depending upon estimates, the typical household holds between 6% to 20% in equity. Conditional on participating in the stock market, this number increases to about 40% in financial assets.

Implied by our above discussion, the risk premium ( $\mu_R - r$ ) of the last century should be closer to 1% than to 6%, in which case from the first term alone  $\varphi$  is down to 20% in equities (for  $\gamma = 2.27$ ,  $\mu_R - r = 0.013$  and  $el_W = 1$ ). The last term in (46) further adjusts this number in the right direction.

## 7 Conclusions

When presenting our simple, linear production and exchange economy as a possible explanation of the two well known empirical puzzles discussed in the introduction, this should of course not be taken literally. However, in all the the major aspects our production model is at about the same level of complexity as the standard pure exchange model, so we should expect about the same level of explanatory power from either of these two approaches.

The consumers' investment problems can be separated from the optimal consumption choices, and as a consequence the consumers' behavior in the financial market can be explained from financial market data alone. Accordingly, at a reasonable level of the representative agent's risk aversion, the model matches both the historical risk premium and the interest rate.

Summarizing, the linear production economy can explain: the large mean premium on equities over the riskless rate, the low real rate, a smooth time series of aggregate consumption of nondurables and services, and volatile prices.

This model can not explain: a lower variability in the consumption than in the stock market dividends, and a lower variability in the growth rate of per capita real consumption of nondurables and services, than in the real return in the stock market.

With respect to the first issue, this is ruled out by national accounting: In equilibrium it must be the case that  $c$  equals  $\delta$ . The second issue is ruled out from the model's logic: In equilibrium the optimal consumption  $c_t = \theta K_t$ , and the equilibrium market price of the firm  $S_t = K_t$ . As a consequence  $c_t = \theta S_t$ . While this allows for a low standard deviation of the aggregate consumption  $c$  compared to the the standard deviation of the stock index  $S$ , the percentage-wise changes in these two quantities are the same in equilibrium.

In order to address the second issue, we introduce a nonlinear production function. The resulting model is not solved in full detail as the linear model, in fact, we have not even specified the form of production function. However, the new model explains several interesting features. The perhaps most important one is that the investor will demand hedging possibilities related to the real economy. This makes the stock market less risky than in the linear production model, resulting in a lower risk premium and a higher equilibrium short rate. This model also opens up for a better explanation of optimal investment behavior for the representative household.

Whether utility is state dependent or not, the CCAPM is still true. When we compare the two models, one interpretation is that for investors of the last century, the stock market has appeared more risky than it really was, at the given level of the volatility of equity.

## Appendix 1

### Solution of the Bellman equation in the centralized economy

The conjectured solution of the Bellman equation (4) of Section 2.1 is ( $x = K_t$ )

$$J(x) = A \frac{1}{1-\gamma} x^{1-\gamma}, \quad J_x(x) = Ax^{-\gamma}, \quad J_{xx}(x) = -\gamma Ax^{-\gamma-1},$$

where  $A$  is some constant. Maximization in the Bellman equation gives

$$-J_x(x) + c^{-\gamma} = 0,$$

which implies that

$$c = (J_x(x))^{-\frac{1}{\gamma}},$$

or, in terms of the underlying random process, here the capital stock  $K$ , the optimal consumption takes the form

$$c_t = A^{-\frac{1}{\gamma}} K_t \quad \text{for all } t.$$

Inserting this conjecture into the Bellman equation reveals that our guess is successful in that the equation separates:

$$x^{-\gamma+1} \left( A(\mu_K - A^{-\frac{1}{\gamma}}) - \frac{\gamma}{2} A \sigma_K^2 - \frac{\rho A}{1-\gamma} + \frac{1}{1-\gamma} (A^{-\frac{1}{\gamma}})^{1-\gamma} \right) = 0$$

for all  $x > 0$ . Accordingly the constant  $A$  must satisfy the equation

$$\frac{\gamma}{1-\gamma} A^{\frac{\gamma-1}{\gamma}} + A \left( \mu_K - \frac{\gamma}{2} \sigma_K^2 - \frac{\rho}{1-\gamma} \right) = 0.$$

One solution is  $A = 0$ , which gives infinite consumption, and is thus not feasible. Dividing through by  $A$  we get

$$A = \left[ \frac{1-\gamma}{\gamma} \left( \frac{\gamma}{2} \sigma_K^2 + \frac{\rho}{1-\gamma} - \mu_K \right) \right]^{-\gamma}.$$

It follows that the optimal consumption is given by (5) as  $c_t = A^{-\frac{1}{\gamma}} K_t = \theta K_t$ , where  $\theta$  is given in (6).

An application of The Verification Theorem reveals that our conjectured solution solves the problem.

As for the transversality condition, we have to verify that

$$\lim_{T \rightarrow \infty} E \{ e^{-\rho T} |J(K_T^c)| \} = 0$$

As a consequence of what we just have shown, the capital stock  $K_t^{(c)}$  satisfies the following dynamics along the optimal consumption path:

$$dK_t^{(c)} = K_t^{(c)} (\mu_K - \theta) dt + K_t^{(c)} \sigma_K dB_t$$

which means that

$$(K_T^{(c)})^{(1-\gamma)} = K_0^{(1-\gamma)} \exp \left\{ (1-\gamma) \left( \mu_K - \theta - \frac{1}{2} \sigma_K^2 \right) T + (1-\gamma) \sigma_K B_T \right\}.$$

Using the moment generating function of the normal probability distribution, we obtain that

$$E((K_T^{(c)})^{(1-\gamma)}) = K_0^{(1-\gamma)} e^{[(1-\gamma)(\mu_K - \theta) - \frac{1}{2}\gamma(1-\gamma)\sigma_K^2]T}.$$

From this it follows that the transversality condition is satisfied provided

$$(1-\gamma)(\mu_K - \theta) - \frac{1}{2}\gamma(1-\gamma)\sigma_K^2 - \rho < 0.$$

Denoting by

$$\alpha = -[(1-\gamma)(\mu_K - \theta) - \frac{1}{2}\gamma(1-\gamma)\sigma_K^2 - \rho],$$

it can be checked that  $\alpha = \theta$ . In other words, the transversality condition is satisfied if  $-\alpha < 0$ , which is equivalent to  $\theta > 0$ , as claimed in Section 2.1.

## Appendix 2

### Solution of the Bellman equation in the decentralized economy

The conjectured solution of the Bellman equation (4) of Section 2.1 is ( $x = W_t$ )

$$J(x) = B \frac{1}{1-\gamma} x^{1-\gamma}, \quad J_x(x) = Bx^{-\gamma}, \quad J_{xx}(x) = -\gamma Bx^{-\gamma-1},$$

where  $B$  is some constant. Using (22) this conjecture immediately leads to

$$\varphi = \frac{1}{\gamma} \frac{\mu_S - r}{\sigma_S^2},$$

which is (25). Next we find the first order condition for optimization in the variable  $c$ . From the Bellman equation it is seen to be

$$-J_w(w) + c^{-\gamma} = 0,$$

which implies that

$$c = (J_w(w))^{-\frac{1}{\gamma}}.$$

By our conjecture this means that in terms of the underlying stochastic process, here the agent's wealth, the optimal consumption takes the form

$$c_t = B^{-\frac{1}{\gamma}} W_t \quad \text{for all } t.$$

Inserting our candidate optimal portfolio rule and optimal consumption into the Bellman equation, we get the following

$$w^{-\gamma+1} \left[ B \left( \frac{1}{\gamma} \frac{(\mu_S - r)^2}{\sigma_S^2} + r - B^{-\frac{1}{\gamma}} \right) - \frac{1}{2} B \frac{1}{\gamma} \frac{(\mu_S - r)^2}{\sigma_S^2} - \frac{\rho B}{1 - \gamma} + \frac{1}{1 - \gamma} (B^{-\frac{1}{\gamma}})^{(1-\gamma)} \right] = 0.$$

Notice that this equation separates, indicating that our conjecture is promising. Since the constant  $B > 0$ , it is determined as

$$B = \left[ \frac{1 - \gamma}{\gamma} \left( \frac{\rho}{1 - \gamma} - r - \frac{1}{2} \frac{1}{\gamma} \frac{(\mu_S - r)^2}{\sigma_S^2} \right) \right]^{-\gamma}.$$

From this the optimal consumption is

$$c_t = (J_w(W_t))^{-\frac{1}{\gamma}} = (B W_t^{-\gamma})^{-\frac{1}{\gamma}} = \left[ \frac{1 - \gamma}{\gamma} \left( \frac{\rho}{1 - \gamma} - r - \frac{1}{2} \frac{1}{\gamma} \frac{(\mu_S - r)^2}{\sigma_S^2} \right) \right] W_t,$$

which is the solution (23) - (24) given in Section 3.3. Again we use The Verification Theorem to confirm that our conjectured solution solves the problem.

Finally, the transversality condition must be checked, and it holds provided  $\theta > 0$ , where the equilibrium restriction  $\mu_S - r = \gamma \sigma_S^2$  has been utilized.

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