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Discussion paper

The long term equilibrium interest rate and risk premiums under uncertainty

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Abstract

Both the equilibrium interest rate and the equity premium, as well as risk premiums of risky investments are all important quantities in cost-benefit analyses. In the light of the current (2008 -) financial crisis, it is of interest to study models that connect the the financial sector with the real economy. The effects of climate change has, on the other hand, been the subject of extensive discussions, for example in connection with the Stern report. The paper addresses both these issues, first based on standard assumptions. In particular we investigate what is needed to have long-term interests lower than short term rates. Our model allows us to tell what happens to risk premiums in turbulent times, consistent with observations. Next we extend the pure exchange model to a production economy. As a result we obtain an equilibrium term structure of interest rates, as well as a model for the equity premium. We end by a discussion of risk adjustments of the discount factor. For projects aimed at insuring future consumption, the interest rate is smaller than the risk free rate. Mitigation can have the characteristics of such a project.

KEYWORDS: dynamic equilibrium, the Lucas model, term structure, CIR, pure exchange, production economy, equity premium puzzle, risk free rate puzzle, climate models, Stern Review.
JEL-Code: G, D.

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1 Introduction

Starting with an the exchange economy, the short term interest rate r_t is determined in equilibrium, and represents the certain rate of return an investor at time t can secure on an investment between t and $t+1$. The value of r_{t+1} is, on the other hand random as seen from time t on. Thus the risk-free interest rate $\{r_t, t = 0, 1, 2, \cdots\}$ is really an adapted stochastic process. In contrast, if the investment is uncertain, like the return on a common stock, the corresponding return R_t is a random variable.

From dynamic equilibrium theory it follows that the real short rate is given by $r_t = -\mu_\pi(t)/\pi(t)$ in a continuous-time framework, i.e., as the exponential rate of decline of the representative agent's marginal utility. Here $\pi(t)$ are the Arrow-Debreu state prices at time t, which are given by $\pi(t) = u'(c_t, t)$, where $u(\cdot, t)$ is the representative agent's utility index, and c_t is the aggregate consumption of nondurables and services in the economy at time t. Provided that u is of additive and separable form, and aggregate consumption is a geometric Brownian motion[1], a standard formula for the equilibrium real short rate is given by

$$r = \rho + \gamma\big(g - \frac{1}{2}(1+\gamma)\sigma^2\big). \tag{1}$$

Here g is the conditional expected growth rate in aggregate consumption, ρ is the subjective rate, part of the preferences of the representative agent, γ^{-1} is the elasticity of intertemporal substitution in consumption, $(1+\gamma)$ is relative prudence of the representative agent, and σ^2 is the variance rate of the consumption growth rate. [2]

The canonical model does not assume any time dependence of these parameters, but there are of course no good reasons for this, except perhaps parsimony, and we will shall have more to say about this below.

For projects with a non-random payoff, along the optimal consumption path it is correct to accept projects with a positive present value provided future consumption gains are discounted according to (1). More precisely, the discount factor at time t for consumption at time $T > t$ in the future is given by $\Lambda_{t,T} = \frac{1}{u'(c_t,t)} E_t\{u'(c_T, T)\} = e^{-r(T-t)}$ under the above stipulated conditions, where r is given by (1). This is also the equilibrium price at time t of a zero coupon government bond paying one unit of the consumption good at at time T.

[1]This means that $\frac{dc_t}{c_t} = g dt + \sigma dB_t$, where B_t is a standard Brownian motion.

[2]When the reprentative consumer has utility index $u(c,t) = \frac{c^{1-\gamma}}{1-\gamma} e^{-\rho t}$, the consumption elasticity is $-\frac{u'(c_t)}{u''(c_t)c_t} = \gamma^{-1}$ and relative prudence is defined as $-\frac{u'''(c_t)c_t}{u''(c_t)} = (1+\gamma)$.

In the classical Ramsey model, consumption is assumed deterministic, and the last term in formula (1) disappears. This formula was used by, i.a., Leif Johansen during the 1960s in connection with public projects in post-war Norway (Johansen (1967)). Typical values for the interest rate were 5 or 7 per cent, depending upon whether the estimate for the per capita growth rate was 2 or 3 per cent. The values of $\gamma = 2$ and $\rho = 0.01$ were typically used.

The magnitude of the interest rate is a central parameter in all cost-benefit analyses. In particular for projects of long durations will the value of r play an important role in determining whether a project is profitable or not. In the public sector examples are investments in infrastructure in the transportation sector, investments in energy production, and investments geared at mitigating the adverse effects from climate change and other environmental changes. In such cases one needs an educated opinion about the interest rate during the next 20 to 30 years, while in climate and environmental policy it seems to be relatively broad consensus that the most severe damages resulting from climate changes will occur after year 2100.

During the last decade we have seen several numerical analyses of optimal climate policy. The studies are based on "Integrated assessment models" connecting models of climate change to traditional applied economic models. Perhaps the best known of these models are the DICE/RICE models developed by William Nordhaus (see Nordhaus (1994), (2008)). In the last DICE-version the value of the interest used is 4.1 per cent. This gives a strong discounting of costs/benefits in the distant future: One billion dollars in 200 years is then worth 274 700 dollars today. Accordingly, the the optimal mitigations are moderate, giving an expected global warming by the year 2200 of 3.5 degrees centigrade, while the expected global warming with "business as usual" is estimated to 5.3 degrees centigrade.

The Stern report (Stern (2007)) recommends a more dramatic mitigation policy, and an important assumption behind the analysis is an interest rate r of only 1.4 per cent, which is much lower than is usually employed. An additional important assumption is that the utility index of the representative agent can only be "moderately concave", giving a low value of γ. In addition the impatience rate ρ is set to zero.

Stern's analysis rests upon a utilitarian calculus that is standard in applied economics; each person, whether alive or yet to be born, counts as equal, except that giving the same benefit to someone who is rich counts as less valuable than giving it to someone who is poor.

The standard optimal consumption and investment theory is, perhaps, not tailor-made for climate problems. In this theory the investments usually benefit the generation who makes them, in addition to future generations, whereas in climate related problems, the investments made today will only benefit generations far into the future.

With an expected per capita growth of 1.1-1.2 per cent in a 200 year perspective, as assumed in the Stern report, it has been argued that mitigation will not be a natural conclusion for any positive real interest rate, no matter how small, so long as the utility function displays decreasing marginal utility in wealth. This argument is at the best meaningful if the growth rate does not vary much. However if interpreted as some average growth rate over this time horizon, mitigation could still follow, provided the growth rate becomes low, and perhaps negative, at long horizons. If not, the transfer from the poor to the rich that would be implicit in reducing carbon emissions is the following: We, the current generation, are the poor who are to make sacrifices for the future generations, who are likely to be much wealthier than we are. They will be the beneficiaries of our sacrifices. And so, based on the utilitarian calculus, egalitarians should be opposed to curbs on carbon.

Another related example is the public revenues from the petroleum sector in Norway that has been invested abroad in the "Norwegian Petroleum Fund", now termed the "Norwegian Government Pension Fund Global". [3] This fund owns close to 1% of the world's available stocks, and by the end of 2008 had investments of 150 billion US dollars in 55 countries spread over 7366 companies. The rationale for doing so that has been put forward is (a) the future generations should get their fair share of these revenues, and (b) demographic factors of an ever aging population makes it important to have a fund based pension system rather than a pay as you go scheme, which is the current practice. There are also other issues, among them that all these revenues can not be investments in infrastructure in Norway, in fear of inflation. The time perspective is unclear, but let us say around 50 to 70 years, in other words shorter than for climate problems. Again the utilitarian calculus combined with a forecast of a per capita annual growth rate in aggregate consumption of around $1.5 - 2.0$ per cent would give the conclusion that the establishment of such a fund, whatever its name, may be hard to defend.

In addition come the incentive problems of state employees (the agents) investing in speculative markets on behalf of future generations (the principals), and the discretion vs rules problems of politicians controlling such a fund. Regarding the latter, in the end it is the politicians who determine the use of the fund. So far there have been significant deviations from the self-imposed rule of extracting only the real rate, estimated to 4%, each year.

A fund exclusively for future generations may be consistent with the 'utilitarian' point of view if at least one of the following holds: Either the utility

[3] There are several similar funds in other nations.

function must be basically linear, so that $\gamma \approx 0$, or the annual per capita consumption growth rate g must eventually become negative. Concerning the value of γ, a typical participant in the stock market is often estimated to have a relative risk aversion around 2, or perhaps larger, and accounting for those who are not participating presumably because they are even more risk averse, the representative agent ought to have risk aversion above 2. It so happens that this relative risk aversion equals the inverse of the elasticity of intertemporal substitution in consumption in the standard model, in other words it equals γ. Weil (1989) found a value of this elasticity of near 10% to be appropriate. In all, $\gamma > 2$ is a fairly safe choice. Concerning the parameter g, one may think that an estimate of a positive growth rate of this magnitude is rather conservative, at least for a time horizon of 50 to 70 years. The increased productivity is expected to outweigh the fact that there will be relatively fewer persons in the work force relative to the population at large. In view of this, the argument in (b) is largely neutralized.

If this growth rate is time varying, on the other hand, and negative towards the end of the oil and gas producing period, the fund will still be consistent with the 'utilitarian' point of view, perhaps combined with a lower extraction rate.

In this paper we take the view that a proper use of the standard model, and its extensions, is adequate. It prescribes, in a utilitarian framework, the optimal consumption and the optimal investment policy in society. The expression (1) gives the equilibrium interest rate at which savings should take place, or the correct return to require on a project that is uncorrelated with equity. The model prescribes an optimal consumption rate in society, as well as an optimal investment policy. These models do not, however, take into account the possible negative relationship between consumption growth and climate problems. When it comes to the Norwegian Government Pension Fund Global, the economic rationale for it may be sound, so long as the real return is used for investments and consumption in Norway.

The Stern-report and other analyses of climate problems have been the subjects of considerable debate. An important issue has been the value of the interest rate in analyses with a very long time horizon. An important topic is then if one should use a time varying interest rate. Instead of focusing solely on the interest rate, one is led to study the discount factor. This leads us to incorporate market data in order to estimate the term structure of interest rates. Also, one is inevitably led to consider risky projects in the private sector. There seems to be fairly broad consensus that such information is relevant for investment projects in the public sector with durations up to 30 years. It is less obvious that there is available reliable data from the private sector that can be used to find the present value of quantities more that 50 years into the future.

One obvious omission in these analyses is that the heterogeneity between nations of the world is not taken into account. Including this would raise a host of new questions, well beyond the scope of the present presentation. [4]

In Section 2 we discuss time variations in the interest rate. In Section 3 we calibrate the uncertainty about the future, inherent in the Stern Review, to the standard pure exchange equilibrium model of Lucas (1978). Here we extend this model to allow for sudden shocks at random points in time, and demonstrate that this feature of the model may explain lower future interest rates, and higher risk premiums.

In Section 4 we bring in term structure models. It turns out that such models offer the right framework for some of the questions we are interested in. Sections 5 and 6 treat risk premiums, in particular the premium on equity, and what we can say from data in the private sector about the discount factors. Here we indicate what the model has to say about risk premium changes related to the current (2008 -) financial crisis.

In Section 7 we address the issue of risk adjustments of projects directed towards mitigation of the adverse effects from climate changes. With respect to climate problems, imagine a 'project' that does not pay off if the future state of the climate is good, and gives a positive return if the future climate is in a bad state. Such a project has the effects of smoothing consumption across time. With reference to an insurance setting, such a project is associated with a negative correlation with aggregate consumption, and will therefore result in a negative risk premium in equilibrium. The appropriate discount rate for discounting the future related to projects of this nature would be smaller than the risk free rate r_t. In other words, while a project that contributes positively to aggregate consumption is penalized with higher discounting, a project that works as an insurance of future consumption is 'rewarded' by lower discounting.

It is argued that mitigation has the properties of pooling across states/time, and has an associated return that can be negatively correlated with aggregate consumption.

[4] According to *Der Spiegel* of 18.10-2010 Angela Merkel and Nikolas Sarkozy pick up the challenge and establish a think tank of experts to analyze just these questions.

2 Time variations of the equilibrium interest rate

In may be useful to recall the basic principles behind the analysis to follow. In the standard exchange model there is a set of m agents characterized by endowment processes e^i and utility functions U_i. The agents can trade in a securities market consisting of N risky, and one risk free asset. The objective is to use this market for investments such that the utility of life time consumption is optimized. Provided a representative agent equilibrium exists, the short term interest rate r_t as well as risk premiums on the risky assets can be determined under various assumptions. For example, when (i) the aggregate endowment process $c = \sum_i e^i$ has dynamics

$$\frac{dc(t)}{c(t)} = g(t)dt + \sigma_c(t)dB(t), \tag{2}$$

where B is a Brownian motion, $g(t)$ is the conditional expected growth rate at time t, and $\sigma_c(t)^2$ is the conditional variance of the growth rate of aggregate consumption, and (ii) the utility function $U_\lambda(\cdot)$ of the representative agent, where $\lambda = (\lambda_1, \cdots, \lambda_m)$ signify the agent weights, has the form

$$U_\lambda(c) = E\Big(\int_0^T u_\lambda(c_t, t)dt\Big),$$

then the risk free interest rate is given by ($u_\lambda = u$)

$$r(t) = -\frac{\frac{\partial}{\partial t}u'(c_t,t)}{u'(c_t,t)} + \left(-\frac{u''(c_t)c_t}{u'(c_t)}\right)g(t) - \frac{1}{2}\frac{u'''(c_t)}{u'(c_t)}c(t)^2\sigma_c(t)\cdot\sigma_c(t), \tag{3}$$

and the equilibrium risk premium of any risky security having return rate $\mu_R(t)$ at time t is

$$\mu_R(t) - r_t = \left(-\frac{u''(c_t,t)c_t}{u'(c_t,t)}\right)\sigma_R(t)\sigma_c(t). \tag{4}$$

The optimal consumption c_t^i of agent i at time t is given by

$$c_t^i = u_i'^{-1}(\lambda_i^{-1}u'(c_t,t),t),$$

where $u_i'^{-1}(\cdot,t)$ inverts the marginal utility function $u_i'(\cdot,t)$. The state price π_t at time t is $\pi_t = u_\lambda'(c_t,t)$. By market clearing, the optimal consumption in society equals the aggregate endowment c in this 'fruit-tree' economy. This result tells us the optimal consumption of individual i at any time t is a non-decreasing function of aggregate consumption. This we will later refer to as *the mutuality principle*.

First we focus on the short rate r. We assume that the felicity index can be represented as $u(c_t, t) = e^{-\int_0^t \rho(s)\,ds} u(c_t)$. As a consequence

$$\rho(t) = -\frac{\frac{\partial}{\partial t} u'(c_t, t)}{u'(c_t, t)},$$

where $\rho(t)$ is interpreted as the subjective rate at time t. Furthermore we denote by

$$\gamma(t) = -\frac{u''(c_t)c_t}{u'(c_t)},$$

the inverse of the elasticity of intertemporal substitution in consumption at time t. $\gamma(t)$ is also the relative risk aversion of the representative agent.

Some economists have recently argued for a falling interest rate in the long run (e.g., Weitzman (2007) and Gollier et.al. (2008). This has motivated other researchers (e.g., Dalen et. al. (2006)) to allow the quantities in (1) to be deterministic functions of time. In the standard formula for the interest rate given in (1) all the quantities are constants ($\sigma_c = \sigma$).

In the above, if we further make the assumption that

$$u(x, t) = \frac{1}{1 - \gamma(t)} x^{1-\gamma(t)} e^{-\int_0^t \rho(s)\,ds}, \tag{5}$$

the resulting equilibrium interest rate model takes the form

$$r(t) = \rho(t) + \gamma(t)\big(g(t) - \frac{1}{2}(1 + \gamma(t))\sigma_c^2(t)\big). \tag{6}$$

Provided g and σ_c are not state dependent, the parameters in (1) are now seen to be deterministic functions of time.

Notice that $u(c_t, t)$ is not state and time separable, due to the somewhat unusual assumption (5) that the parameter γ is time dependent. We shall return to this assumption below, where we also discuss the time dependency in the other terms as well.

First observe that the discount factor $\Lambda_{t,T}$ is given by

$$\Lambda_{t,T} = \begin{cases} e^{-\int_t^T r(s)ds}, & \text{if } \gamma \text{ is a constant;} \\ e^{-\int_t^T \tilde{r}(s)ds} c_t^{(\gamma(t)-\gamma(T))}, & \text{if } \gamma = \gamma(t) \text{ is time varying.} \end{cases} \tag{7}$$

where $\tilde{r}(s) = \rho(s) + \gamma(T)(g(s) + \frac{1}{2}\sigma^2(s)(1 + \gamma(T))$. The expression for the term structure is simpler and has a more intuitive form when γ is a constant.

When this parameter varies with time, the discount factor becomes state dependent even if none of the parameters are. It is not obvious in which direction changes within and between generations would affect γ. In some studies it is found that $\gamma(t)$ is lower the higher the consumption, with 1 as a lower boundary (Dagsvik, Jia and Strøm (2006)).

If the per capita growth rate in GNP is positive, meaning that future generations become wealthier, then $(\gamma(t) - \gamma(T)) > 0$ is consistent with this observation, in which case the discount function tends to be larger when $\gamma(t)$ is time varying, meaning that future benefits are not as heavily discounted as when γ is a constant, provided the last term in (6) is not too big, and assuming $c_t > 1$. In this case we also see that the equilibrium interest rate $r(t)$ decreases when $\gamma(t)$ decreases.

In the case where the growth rate $g(t)$ and the volatility $\sigma(t)$ are random processes, the subjective rate $\rho(t)$ is a deterministic function of time, and γ is a constant, by use of the stochastic exponential the discount factor is given by the conditional expectation

$$\Lambda_{t,T} = E_t\big(e^{-\int_t^T r(s)ds}\big), \tag{8}$$

where the interest rate $r(s,\omega) = \rho(s) + \gamma(g(s,\omega) + \frac{1}{2}\sigma^2(s,\omega)(1+\gamma))$ in equilibrium, and where ω signifies state dependence. Notice that the market price of risk is zero in this situation.

Returning to the interest rate given in (6), the effects from an increase in $\gamma^{-1}(t)$, the elasticity of intertemporal substitution in consumption, is as expected. As the substitution effect increases, the representative agent wants to save. Since this is impossible, the interest rate must fall to restore equilibrium. Because of this effect the value of $\gamma = 1$ becomes a border case. For values of $\gamma > 1$ the wealth effect will dominate, while for $\gamma < 1$ the substitution effect dominates. It would of course be of interest to separate the elasticity of intertemporal substitution from being just the reciprocal of the relative risk aversion, and there are other models of preferences that accommodate this, like recursive utility, Epstein-Zin utitliy, habit formation etc., but we shall not discuss these any further here.

There is no fundamental issue with allowing γ to be time dependent, aside from the practical problem encountered with (7). In a recent paper (Aase (2009)), it is shown that allowing this time dependence can be used to explain that the the optimal investment strategy of an individual is to reduce the exposure to equity as the investor becomes older. This effect follows if the individual's risk aversion is an increasing function of time. This behavior is consistent with empirical evidence, but does not follow from the canonical investment model. This may ultimately lead to the topic of a wealth dependent γ, but there is no definitive argument for or against decreasing relative risk aversion in wealth.

Regarding the parameter ρ, allowing the impatience rate to be time dependent introduces time inconsistency since discounting is no longer exponential, which may be viewed as reflecting an irrationality (see e.g., Johnsen and Donaldson (1985)). Consider a person who prefers to receive two apples in one year plus one day rather than one apple in one year, but prefers to receive one apple today rather than two apples tomorrow (Thaler (1981)). This person is decreasingly impatient. If his preferences between "today" and "tomorrow" remain the same for one year, and he resets the clock to zero whenever he makes a decision, then in one year from now he will prefer to receive one apple on that day rather than two apples one day later. Thus, his preferences between the two options will have changed over time.

Based on the above, we will assume that the subjective rate ρ is a constant through time. The same assumption is made for the parameter γ.

Two effects are of particular importance when discussing the equilibrium interest rate. One is the wealth effect, the other the precautionary effect. If the future growth rate increases, then the future interest rate increases, ceteris paribus. This is the wealth effect and leads to a higher discounting of future consumption benefits. The yield curve $R(t, T)$, often called the *term structure of interest rates*, is defined by

$$\Lambda_{t,T} = e^{-R(t,T)(T-t)},$$

or

$$R(t, T) = -\frac{\ln \Lambda_{t,T}}{T-t}, \qquad (9)$$

so that $R(t, T)$ is the average, annual interest in the time interval $[t, T]$, a quantity that can be observed in the market at time t provided there is a rich enough supply of zero coupon bonds with different maturities. Notice that in the rather simple situation of equation (7), this yield would be decreasing in T if $g(\cdot)$ is a decreasing function, ceteris paribus.

The precautionary effect is positive since our representative agent is prudent. What this means is the following: If future uncertainty increases, so that $\sigma(t)$ is increasing in t, the future interest rate will go down, all else equal. If this is the case, the yield curve is a decreasing function of T. In the canonical model given in (1) these two effects exactly cancel, giving a flat yield curve.

Constant yield curves are usually not observed in real markets. Most of the time the yield curve is increasing, compensating long term investors. This compensation can, perhaps, be partly interpreted as a liquidity premium. From 1900 to date there has only been seven periods with falling, or "inverted" yield curves, and in six of these cases this period was followed by a recession. The interest rates are then in nominal terms, while r is otherwise the real rate in this paper.

This behavior is an indication that the information about the yield curve is of importance for the market of risky investments, and hence for the real economy as well.

3 Adverse shocks to the economy

In this section we limit the discussion to the possibilities of obtaining an equilibrium interest rate of around 1.4 per cent in the standard model (1), or perhaps (6), as was suggested in the Stern report.

The yearly standard deviation σ of consumption growths (nondurables and services) over the last 100 years or so has been estimated to around 3.57%, consistent with the estimate of Mehra and Prescott (1985), and later updates. Unlike Stern, however, we insist on using standard values for the parameters regarding the preferences, that is, we assume that $\rho \approx .01$ and $\gamma \approx 2$ in the paper. [5]

Two key parameters in the standard exchange model of Lucas (1987) are then the future growth rate, and the future standard deviation of the growth rate of aggregate consumption of nondurables and services. One may argue that climate problems are particularly difficult to quantify, because of damages caused by non-linear meteorological phenomena that are only partially understood. Figure 6-5c of the Review shows the increasing damages of climate change on a "business as usual" policy. By year 2200, the losses in GNP have an expected value of 13.8% of what GNP would be otherwise, with a .05 percentile of about 3% and a .95 percentile of about 34%. If we let c_t represent the value at a future time t of aggregate consumption[6] with no greenhouse gas emissions, and c_t^e the corresponding quantity with greenhouse gas emissions and no mitigation, the above statement means in quantitative terms that if we define

$$X_t := \Big(\frac{c_t - c_t^e)}{c_t}\Big) 100\%,$$

then $P(X_t < 3) = .05$, $P(X_t > 34) = .05$ and $E(X_t) = 13.8$. The base rate of growth of the economy, before calculating the climate change effects, was taken to be 1.3% per year.

[5]In Sections 4 we use logarithmic utility, but then there are no restriction on the subjective rate ρ.

[6]For simplicity we assume that GNP and aggregate consumption of nondurables and services are proportional in this argument.

If we apply the modeling framework given above, which is based on normally distributed random shocks, we obtain that this kind of uncertainty about the future translates into $g = .0130$, $g^e = .0119$, and $\sigma^e = .0366$. The latter is to be compared to $\sigma = .0357$. In other words, the benefits from mitigating greenhouse gas emissions can be represented as an increase in the annual growth rate from today to 2200 from 1.19% to 1.3%, with a corresponding decline in annual standard deviation from 3.66% to 3.57%. We have to compare this benefit with the cost of stabilization.

The effect on the interest rate given in (1) from this drop in standard deviation is not enough to obtain the desired low value of r. The interest rate with no emissions would be 3.22% with the above parameter values, whereas the corresponding interest rate with emissions and no mitigation is 3%.

We think it is reason to reconsider a constant yearly standard deviation of the growth rate in consumption for the next 200 years of the order of 3.7%. This estimate is the historic one, based on data from about the last 100 years. In the above calculations we have used that a yearly standard deviation of σ leads to a standard deviation of $\sqrt{t}\sigma$ at time t years from now, so the future uncertainty automatically increases with a constant σ. Making the volatility parameter σ in addition an increasing function of time, seems as 'gilding the lily'. However, if the future uncertainty is not revealed gradually, but more at later stages, a time varying volatility could be appropriate.

This is likely to be the case here, since the time horizon is twice as long as the one the historic estimate is based on, combined with our present lack of a proper understanding the consequences of aggregated human activity, as well as other natural phenomena.

What size of uncertainty is required in the standard model to account for a low enough interest rate? A numerical example will illustrate. If by year 2200 the losses in consumption have an expected value of 52% of what consumption would be otherwise, with a .05 percentile of about 2% and a .95 percentile of about 99%, this is consistent with a growth rate $g^e = .008$ and a standard deviation $\sigma^e = .0655$. This would eventually bring the yearly interest rate down to 1.3%. However, this implies much more future uncertainty than considered in the Review.

Increasing the uncertainty only through an increased standard deviation is a bit vague. Instead we suggest to include possible jumps in aggregate consumption at random points in time, in addition to the normally distributed infinitesimal shocks of the standard model. Assuming that the frequency of these jumps are λ per year and the jump sizes Z_n in per cent of aggregate consumption are independent with the same probability distribution, we have the following expression for the equilibrium interest rate (Aase (1993a-b, 2007)

$$r = \rho + \gamma\big(g - \frac{1}{2}(1+\gamma)\sigma^2\big) - \lambda\big(\gamma E(Z) + E(1+Z)^{-\gamma} - 1\big). \tag{10}$$

The last term is the influence on the interest rate from the random shocks to the growth in GNP. For the present purposes, we use a Taylor series approximation to obtain

$$r = \rho + \gamma g - \frac{1}{2}\gamma(1+\gamma)\big(\sigma^2 + \lambda E(Z^2)\big) + \frac{1}{6}\gamma(\gamma+1)(\gamma+2)\lambda E(Z^3) + \cdots \tag{11}$$

In this expression, the term $\sigma^2 + \lambda E(Z^2)$ is the new variance rate of the growth rate, and the last term measures the additional, higher order effects on the equilibrium interest rate from introducing non-Gaussian "disasters". As a numerical illustration, assume there is between 2 and 3 disasters every 100 years, each one bringing down the aggregate output by 34%. This would bring the interest rate down to $r = .017$ or 1.7% interest, which is comparable to the Stern report. Here we have a total standard deviation rate of 6.5%, but now we at least know where the additional uncertainty comes from. Barro (2006) and also Rietz (1988) have considered jumps from an applied point of view. The above parameter estimates are consistent with the numbers presented in Barro (2006), based on data for about the last 100 years.

In isolation higher uncertainty brings down the equilibrium risk-free interest rate $r(t)$, but as we shall see in Section 5, it increases the risk premium on investments having an uncertain outcome.

4 Term Structure Models

Related to a time changing interest rate, there is a theory in financial economics, often referred to as the Theory of Fixed Income, that partly deals with these issues. It has been observed that there may be risk premiums associated with government bonds. Even if we disregard any risk associated with the final payments, meaning there is only a very low probability that a sovereign nation will fail on its debt issues, still the interest rate is a stochastic process, meaning that the price of a zero coupon bond will also vary randomly, consistent with the formula (8). It would not be surprising if this uncertainty leads to a risk premium for holding bonds, but this does not follow from the above theory, which is consistent only with the 'pure expectation hypothesis'.

We now turn to models that allow for such risk premiums. More importantly, as noted by several researchers, the interest rate r in the distant future involve considerable amounts of uncertainty. For example is it an important point in Weizman (1998) that this uncertainty is transfered to the discount factor at a future point in time. It is the discount factors at different future time points which are relevant in a cost-benefit analysis, and since this factor is a non-linear function of the interest rate, we can not account for the uncertainty by just replacing the uncertain interest rate by the expected rate. In term structure models, this problem is partly resolved in an elegant way, but only for future time points where there exist government bonds, which means about 30 years into the future, at the maximum.

The models can be both of the equilibrium type as we have seen above, or of the so called "no arbitrage" type. The latter will be of less interest to us, since there is no clear connection between the growth of the economy and the interest rate.

In general the price of a zero coupon bond can be expressed as

$$\Lambda_{t,T} = E_t^Q\big(e^{-\int_t^T r(s)ds}\big). \tag{12}$$

Formally this looks like the expression in (8) but with one notable exception: The conditional expectation is taken with respect to a risk adjusted probability measure Q instead of the given P. The connection between the two probability measures goes through the market price of risk process η_t. If this quantity is identically equal to zero, then $P = Q$. [7]

In addition to the spot interest rate $r(t)$, this theory is also concerned with future interest rates that can be locked in today, called forward rates, or long interest rates. They are defined by the identity

$$R(t,T) = \frac{1}{T-t}\int_t^T f(t,u)du, \tag{13}$$

where $f(t,u)$ is interpreted as the rate of return at the future date $u \leq T$, that can be locked in at the present date t, on risk free investments. Provided there are zero coupon bonds maturing at all dates between t and T, these forward rates are observable in the market at each time t. The instantaneous forward rate can also be expressed by the price $\Lambda_{t,T}$ of a zero coupon bond as follows

$$f(t,T) = -\frac{\partial \ln(\Lambda_{t,T})}{\partial T}, \qquad f(t,t) = r_t.$$

Next we turn to an equilibrium model of term structure of interest rates.

[7] Here $dQ/dP = \xi_T$, i.e., $Q(B) = \int_B \xi_T(\omega)dP(\omega)$ for any event B, where the density process $\xi_t = E_t(\xi_T)$, and $\xi_t = \exp\{-\int_0^t \eta_s dB_s - \frac{1}{2}\int_0^t \eta_s \cdot \eta_s ds\}$. Under Q the discounted bond prices are martingales. The state price $\pi_t = \xi_t e^{-\int_0^t r(s)ds}$.

4.1 An equilibrium term structure model with production

The Cox, Ingersoll and Ross (1985b) term structure theory is an important contribution to asset pricing theory for a variety of reasons. The term structure of interest rates, or prices of default-free zero coupon bonds, is an important piece of information in financial markets. The term structure embodies the expectations of the market about future events and decisions worth billions of dollars made every day based on it.

Prior to the CIR, there were many hypotheses about the term structure. But they were only that: Hypotheses. Phenomena like liquidity preference, preference habitat or expectations hypotheses were proposed to explain how the term structure behaves. The CIR model is the first theory, a consistent set of results, that describes the term structure. Like Vasicek (1977), CIR also get closed form solutions, which makes the theory testable.

Before we present this model, recall the following situation describing how much a nation should save. Consider an economy developing over time where $K = K_t$ denotes the capital stock, $c = c_t$ consumption and $Z = Z_t$ net national product at time t, where $Z = f(K)$ denotes the production function. For each t we have the national accounting identity

$$\dot{K}_t = f(K_t) - c_t$$

which means that production, $f(K_t)$, is divided between consumption, c_t, and investment, $\dot{K}_t := dK_t/dt$. The problem is to find the optimal investment, or equivalently, the optimal consumption, that solves

$$\sup_c E\Big[\int_0^T u(c_t, t)\, dt\Big]. \tag{14}$$

In their theory for the term structure of interest rates, Cox-Ingersoll-Ross (1985) assume that the rate of growth of capital stock in a given, linear production technology is determined by a "random shock", or state variable process Y satisfying

$$dY_t = (b_Y - \kappa Y_t)dt + \sigma_Y \sqrt{Y_t} dB; \qquad Y_0 > 0, \qquad t \geq 0, \tag{15}$$

where b_Y, κ and σ_Y are strictly positive scalars with $2b_Y > \sigma_Y^2$, and where B is a standard Brownian motion. The capital stock process K is then assumed to be of the form

$$dK_t = (\mu_K Y_t K_t - c_t)dt + \sigma_K K_t \sqrt{Y_t} dB_t; \qquad K_0 > 0, \tag{16}$$

where μ_K and σ_K are strictly positive scalars with $\mu_K > \sigma_K^2$. We may think of Y as a "shock" process that affects the productivity of capital.

The objective is to solve (14) subject to the dynamic constraints (15) and (16), when $u(x,t) = \log(x)e^{-\rho t}$, i.e., the objective is as in sections 1 - 3, only with $\gamma = 1$ instead of a general γ.

The first order conditions for this problem is given by the Bellman equation, and the solution for the optimal consumption is found to be

$$c_t = \frac{\rho K_t}{1 - e^{-\rho(T-t)}}, \qquad t \in [0,T]. \tag{17}$$

Assuming that the aggregate consumption process in society follows (17), we can now utilize the results of the standard Lucas type fruit tree exchange economy in order to find prices; here the state price deflator $\pi_t = e^{-\rho t}/c_t$. With the aid of Itô's Formula, we get

$$d\pi_t = (\sigma_K^2 - \mu_K)Y_t\pi_t dt - \pi_t\sigma_K\sqrt{Y_t}dB_t. \tag{18}$$

The short rate process r is given as

$$r_t = -\frac{\mu_\pi(t)}{\pi_t} = (\mu_K - \sigma_K^2)Y_t.$$

Since $dr_t = (\mu_K - \sigma_K^2)dY_t$, we find that the (endogenous) dynamics for the short rate is

$$dr_t = \kappa(r^* - r_t)dt + \sigma_r\sqrt{r_t}dB_t, \tag{19}$$

where $r^* = b_Y(\mu_K - \sigma_K^2)/\kappa$ and $\sigma_r = \sigma_Y\sqrt{\mu_K - \sigma_K^2}$. As for the Vasicek model, we obtain here a mean reverting interest rate process, reverting towards r^*. Here the spot rate $r_t > 0$ for all t, and its expectation satisfies

$$E(r_t) = r^* + (r_0 - r^*)e^{-\kappa t}.$$

Thus, $E(r_t) \to r^*$ exponentially with t, and r^* is the long-run mean of the short rate process. Likewise we can also show that the conditional expected value of r_s given the information available at time t, where $s > t$, is given by

$$E_t(r_s) = r^* + (r_t - r^*)e^{-\kappa(s-t)} \qquad \text{almost surely.}$$

The market price of risk process $\eta_t = \frac{\sigma_K}{\sqrt{\mu_K - \sigma_K^2}}\sqrt{r_t}$, which is only equal to zero if there is no volatility in the capital stock process K_t.

The term structure of interest rates can be solved, and the results are the following, for any time $\tau < T$:

$$\Lambda_{t,\tau} = E_t^Q\big(e^{-\int_t^\tau r_s ds}\big) = A(t,\tau)e^{-B(t,\tau)r_t},$$

where

$$A(t,\tau) = \Big(\frac{2\gamma_1 e^{\gamma_2(\tau-t)/2}}{\gamma_2(e^{\gamma_1(\tau-t)}-1)+2\gamma_1}\Big)^{\frac{2\kappa r^*}{\sigma_r^2}}$$

and

$$B(t,\tau) = \Big(\frac{2\gamma_1 e^{\gamma_2(\tau-t)/2}-1}{\gamma_2(e^{\gamma_1(\tau-t)}-1)+2\gamma_1}\Big),$$

and the constants γ_1 and γ_2 are given by

$$\gamma_1 = \sqrt{(\kappa+\sigma_Y\sigma_K)^2+2\sigma_r^2},$$

and

$$\gamma_2 = \gamma_1 + \kappa + \sigma_Y\sigma_K.$$

The long run average interest rate is given by

$$R(\infty) := \lim_{\tau\to\infty} R(t,\tau) = \frac{2r^*}{1+\frac{\sigma_Y\sigma_K}{\kappa}+\sqrt{(1+\frac{\sigma_Y\sigma_K}{\kappa})^2+2\big(\frac{\sigma_r}{\kappa}\big)^2}}. \tag{20}$$

Also, the slope of the yield curve can be determined:

- If $r_t \leq \frac{\kappa r^*}{\gamma_1+\kappa+\sigma_Y\sigma_K}$, then $\frac{\partial R}{\partial \tau} > 0$, we have an upward sloping yield curve;
- If $\frac{\kappa r^*}{\gamma_1+\kappa+\sigma_Y\sigma_K} < r_t \leq \frac{\kappa r^*}{\kappa+\sigma_Y\sigma_K}$, then we get a humped yield curve;
- If $\frac{\kappa r^*}{\kappa+\sigma_Y\sigma_K} < r_t$, then $\frac{\partial R}{\partial \tau} < 0$, the yield curve is downward sloping.

4.2 The connection to the growth rate of the economy

We want to establish a connection between the conditional growth rate of the economy and long term interest rates, and compare to the results of sections 1-3 of the paper. To our knowledge, these results are new in the literature.

There may be a good reason why seasonality, or business cycles, are not found of interest in climate problems. It is not the cycles per se that interest us. Rather it is the ergodic nature of the economy - it allows for a long term perspective, and the fact that consumption is endogenous.

Recall that in the sections 1-3 the growth rate was denoted by g, and its relation to the spot interest rate r was given by equations (1), (6), (10) or (11), depending upon circumstances.

To start, we derive the dynamics of the optimal, aggregate consumption given in (17). By Itô's Formula it follows that

$$dc_t = \mu_c(t)dt + \sigma_c(t)dB_t \tag{21}$$

where the drift term is

$$\mu_c(t) = \frac{\rho}{1 - e^{-\rho(T-t)}}\Big((\mu_K Y_t - \frac{\rho e^{-\rho(T-t)}}{1 - e^{-\rho(T-t)}}) K_t - c_t \Big),$$

and the diffusion term is

$$\sigma_c(t) = \frac{\rho}{1 - e^{-\rho(T-t)}} K_t \sigma_K \sqrt{Y_t}.$$

The conditional expected growth rate $\mu_c(t)/c_t$ of the aggregate consumption process at time t is by definition

$$\frac{\mu_c(t)}{c_t} := \frac{d}{du} E_t\Big(\frac{c_u}{c_t}\Big)|_{u=t} = \Big(\mu_K Y_t - \frac{\rho e^{-\rho(T-t)}}{1 - e^{-\rho(T-t)}} - 1\Big),$$

where we have used (17) in the above expression for $\mu_c(t)$. From the dynamics of the state variable Y given in (15) it follows that for $s > t$

$$E_t(Y_s) = \frac{b_Y}{\kappa} + (Y_t - \frac{b_Y}{\kappa})e^{-\kappa(s-t)} \qquad \text{and} \qquad E(Y_t) = \frac{b_Y}{\kappa} + (Y_0 - \frac{b_Y}{\kappa})e^{-\kappa t}.$$

Let us use the notation $g(t) := \mu_c(t)/c_t$, and $\bar{g}_t := E(g(t))$. We conclude that

$$\bar{g}_t := \mu_K\Big(\frac{b_Y}{\kappa} + (Y_0 - \frac{b_Y}{\kappa})e^{-\kappa t}\Big) - \frac{\rho e^{-\rho(T-t)}}{1 - e^{-\rho(T-t)}} - 1. \tag{22}$$

Equation (22) gives a connection between the mean growth rate of the aggregate consumption process and the other parameters of the present production economy.

Related to climate problems discussed in sections 1-3, it is of interest to look at very long term relationships, which we here can do by first letting the time horizon T grow, then increasing t from its origin. T plays the role of the "end of the world" in this model, as can be seen from the equation (17) for the optimal consumption process: As t approaches T, the consumption rate goes to infinity. As $T \to \infty$ we see that

$$\bar{g}_t = \mu_K\Big(\frac{b_Y}{\kappa} + (Y_0 - \frac{b_Y}{\kappa})e^{-\kappa t}\Big) - 1. \tag{23}$$

Notice the transient nature of the subjective interest rate ρ is this relationship; at long horizons the effect from ρ disappears. In the Stern Review it has been argued that $\rho = 0$ in climate related problems since it is difficult to give any good ethical arguments for treating the generations differently just based on the fact that they are borne at different points in time. Provided one accepts this argument, and at the same time individuals have a relatively high ρ in their evaluations of own consumption during their life time, this could be used as an argument for a lower long term interest rate than short term rate.

Recall that r^* is the long-run mean of the short rate process, where $r^* = b_Y(\mu_K - \sigma_K^2)/\kappa$. By letting t grow in (23), we get a direct connection between the long-run mean growth rate $\bar{g}$ of aggregate consumption and the long-run short rate r^* as follows

$$r^* = (\bar{g}+1)\big(1 - \frac{\sigma_K^2}{\mu_K}\big). \tag{24}$$

Note that t can either be interpreted as the present, where t is large relative to the origin 0, which we could take to be around, say, 1900 for the data that we have in mind, or the present could be 0 and t some distant future point in time, less than 200 years from now.

This relationship is of a different nature than the connection between r and g in (1), as is (22), but is of interest in the present setting: Since $\mu_K > \sigma_K^2$, the factor $(1 - \frac{\sigma_K^2}{\mu_K}) < 1$, and the long-run mean short rate may be small. For example if the uncertainty in the capital stock is large relative to the expected growth of the capital stock, this will happen. Note that in such a situation σ_r will be small, so the the spot rate uncertainty is then low, and the short term interest rate is close to constant. In other words, this production economy allows the equilibrium long-run short rate to be relatively low, even if the long-run mean growth rate in aggregate consumption is relatively high.

Recall that we now have a production economy, which also involves investments. The growth rate in consumption is $g(t)$, and we must in addition take into account the investments in order to obtain the overall growth rate in GNP.

To this end, consider the conditional expected growth rate of the *net national product* of the economy. Denoting this quantity by $h(t)$, it is given by

$$h(t) := \frac{d}{du} E_t\Big(\frac{1}{K_t}\int_t^u (dK_s + c_s ds)\Big)|_{u=t}. \tag{25}$$

Note that this definition takes into account both consumption growth and growth in investment. From the dynamics of the capital stock in (16) it follows that

$$h(t) = \mu_K Y_t.$$

From the dynamics of the shock process Y it follows that for $s > t$

$$E_t(h(s)) = \mu_K\Big(\frac{b_Y}{\kappa} + (Y_t - \frac{b_Y}{\kappa})e^{-\kappa(s-t)}\Big),$$

and from this it we deduce that

$$E_t(h(s)) \to \mu_K \frac{b_Y}{\kappa} := \bar{h} \qquad \text{as } s \to \infty,$$

where $\bar{h}$ is the long-run mean growth rate of the net national product. Accordingly

$$r^* = \bar{h} - \frac{b_Y \sigma_K^2}{\kappa}. \tag{26}$$

This relationship gives an equilibrium connection between the long-run short rate r^* and and the long-run mean growth rate $\bar{h}$ of the net national product. For example, since $b_Y > 0$ and $\kappa > 0$, it is clear that $r^* < \bar{h}$. The Stern Review indicates the number $\bar{h} = 0.012$, or a 1.2% growth rate during the next 200 years with no mitigation. This allows for a long-run interest rate smaller that 1.2%, which is of the same order of magnitude as the 1.4% proposed by Stern.

The comparison with the analysis in sections 2 and 3 is not entirely fair, since we here have a representative agent with relative risk aversion $\gamma = 1$. In Section 3 we claimed that the interest rate is 3.22% with no emissions, using $\gamma = 2$. The corresponding number with $\gamma = 1$ is 2.17%, using the other parameter values in Section 3. So, after adjusting for the discrepancies in risk aversion, the present theory can still explain smaller long-run equilibrium interest rates for any given growth rate of GNP.

We can also find an equilibrium connection between $R(t, T)$ and the *average* mean growth rate of GNP over the period (t, T) defined by

$$\frac{1}{T-t} \int_t^T E_t\big(h(s)\big)\, ds.$$

It is seen that this quantity equals

$$\mu_K \frac{b_Y}{\kappa} + \big(Y_t - \frac{b_Y}{\kappa}\big) \frac{1 - e^{-\kappa(T-t)}}{(T-t)\kappa}.$$

When the horizon T grows, it follows that

$$\frac{1}{T-t} \int_t^T E_t\big(h(s)\big)\, ds \to \mu_K \frac{b_Y}{\kappa} := \bar{h}_\infty, \qquad \text{as } (T-t) \to \infty.$$

From the above we note that $\bar{h}_\infty = \bar{h}$, and from the expression for the long run average interest rate $R(\infty)$ in (20), we find that

$$R(\infty) = \frac{2\big(\bar{h}_\infty - \frac{\sigma_K^2 b_Y}{\kappa}\big)}{1 + \frac{\sigma_Y \sigma_K}{\kappa} + \frac{\gamma_1}{\kappa}}. \tag{27}$$

Again, there is nothing pathological in having a low average interest rate, that is, averaged from the present to the distant future, even if the long-run, averaged mean growth rate of the net national product is relatively high, provided the uncertainty in the capital stock is large enough.

From the parameter restriction $2b_Y > \sigma_Y^2$ the following inequalities hold:

$$R(\infty) < \frac{2\bar{h}_\infty - \frac{\sigma_K^2 \sigma_Y^2}{\kappa}}{1 + \frac{\sigma_Y \sigma_K}{\kappa} + \frac{\gamma_1}{\kappa}} \qquad \text{and} \qquad r^* < \bar{h} - \frac{1}{2}\frac{\sigma_Y^2 \sigma_K^2}{\kappa}. \tag{28}$$

Although formally different from the model of the equilibrium interest rate of Section 1, the two models have many similar features, as is to be expected. For example, precautionary savings was seen to have a decreasing effect on the interest rate, so that when uncertainty of the growth rate in aggregate consumption increases, r decreases.

A similar pattern is seen to take place in the present model, now related to an increase in the uncertainty of the capital stock and in the shock process Y. From (28) we notice that when the product $\sigma_K^2 \sigma_Y^2$ increases, then the interest decreases. Also, when κ increases, i.e., when the force by which mean reversion takes place is intensified, both the interest rates in (28) increase. Note that the representative agent of the present section is also prudent, so precautionary savings ought to take place in some form or the other.

The present model for the term structure can be embedded in a stock-market equilibrium with decentralized production decisions. In this equilibrium c_t of equation (17) is the optimal real output rate process of a firm controlling the capital stock production process and maximizing its share price $S(t)$. This stock price can be shown to be equal to K_t, and from the expressions for $\mu_c(t)$ and $\sigma_c(t)$ in the dynamic equation (21) for consumption, we notice that the consumption process with this reinterpretation depends on investments in equity.

For projects that are correlated with equity, or more precisely with aggregate consumption, the proper discount factor should be adjusted for risk. This is the topic to which we now turn.

5 Risk premiums

When projects have uncertain returns, the situation is no longer so simple. There is, for example, no "term structure" for such projects, since this depends on risk-free payments in the future. In Section 2 we suggested that the risk premium of any risky asset is given by equation (4). Below we shall return to this relationship.

We start by discussing a model for the equilibrium asset prices. If Q is a risk adjusted probability measure, then the price today, S_t, of a risky security is given by the formula

$$S_t = E_t^Q\Big[\exp\big(-\int_t^T r_u du\big)S_T + \int_t^T \exp\big(-\int_t^s r_u du\big)dD_s\Big]. \tag{29}$$

Here D is the dividend process of the security, and r the short rate. The connection between the state prices π_t and the risk adjusted probability Q goes through the discounted density process as follows: $\pi_t = \xi_t e^{-\int_0^t r_u du}$ where $\xi_t = E_t(\xi_T)$ and $\frac{dQ}{dP} = \xi_T$.

Provided the dividends can be represented as a rate, i.e., $D_t = \int_0^t \delta_s ds$, the present value can be expressed in term of the state prices under the given probability measure P as follows:

$$S_t = \frac{1}{\pi_t} E_t\Big[\pi_T S_T + \int_t^T \pi_s dD_s\Big], \tag{30}$$

with dD_s replaced by $(dD_s - \sigma_D(s)\eta_s ds)$, where η is the market-price-of-risk, if the dividends are modeled by an Itô-process with a non-zero diffusion σ_D.

Under certain circumstances it is possible to find risk adjusted returns in each period as functions of primitives at time t, but the conditions are of course strict. If the project has a long time horizon, it is more demanding to find the relevant required returns over the entire future period. Provided the state prices are known today, and their future evolution has a known probability distribution, and if all joint distributions dictated by (29) are known, then it should be possible to find the correct discount factors in all future time periods as seen from today, but one cannot expect simple formulas.

The most popular approach is given by the Capital Asset Pricing Model (CAPM) developed in the mid 1960's by Mossin, Lindtner and Sharpe. This is a one-period model which gives the difference between the expected return $E(R_j)$ of a risky security j and the risk-less rate r in terms of the expected return on the market portfolio $E(R_m)$ as follows

$$ER_j - r = \beta_{j,m}(ER_m - r) \qquad j = 1, 2, \cdots, N, \tag{31}$$

for each of N risky securities, where

$$\beta_{j,m} = \mathrm{cov}(R_j, R_m)/\mathrm{var}(R_m).$$

Thus the risk premium of asset j is proportional to the risk premium of the market portfolio. Since the constant of proportionality $\beta_{j,m}$ can also be interpreted as a regression coefficient, the model invites for time series estimates of this parameter for each risky security.

However, this very methodology presumes that the model is valid in a multiperiod setting. If the conditions are strong for the one period CAPM to hold, they are considerably stronger for this to be true in several periods. The problem is that while in the one period model final wealth equals consumption, in many periods one must also consider investments. Suppose there is a state variable $Y = Y_t$, a vector of random variables observable at each time t. Assuming that Y is a Markov process, an approximate "multi-beta" model can be derived in a discrete time framework, that in some sense generalizes the single period CAPM. However, only when the state variable $Y(1), Y(2), \cdots, Y(T)$ is a sequence of independent random variables will the multiperiod problem be transformed into a sequence of disconnected single period problems, and a conditional CAPM will hold period by period, in which the rate of return on the market portfolio is the pivotal variable.

In a continuous-time framework an exact multi-beta asset pricing relation has been developed by Merton (1973), the Intertemporal Capital Asset Pricing Model (ICAPM), in which case a stochastic investment opportunity set is allowed. Because the state variables are generic, the model is robust in the sense that it can capture investors' desires to hedge against other economic events in addition to shifts in the investment opportunity set. To either test or apply the model requires, of course, that the relevant state variables of the environment can be identified.

One approach to solving this identification problem is the purely empirical one, generally associated with the application of the Ross' (1976) Arbitrage Pricing Theory. From the derived specification of asset-return dynamics, factor analysis can be applied directly to the historical time series of returns to calculate the "implied" state variables of the environment. An alternative approach is to specify the state variables from a priori theoretical reasoning. A particularly imaginative and fundamental contribution to this theoretical approach is Breeden's (1979) Consumption-Based Capital Asset Pricing Model (CCAPM), to which we return below.

When there are no state variables, Merton's (1973) version of the ICAPM can be considered a continuous-time version of the ordinary CAPM of equation (31): The risk premiums are given by

$$\mu_i - r = \beta_{i,m}(\mu_m - r) \tag{32}$$

where

$$\beta_{i,m} = \frac{\operatorname{cov}(\frac{dS_i}{S_i}, dr_m)}{\operatorname{cov}(dr_m, dr_m)}.$$

The assumptions are, among others, that the security prices S_i are all log-normally distributed *equilibrium* prices, where μ_i is the drift rate of the cumulative-return process $R_i(t)$ defined by $dR_i(t) = dS_i(t)/S_i(t)$, dr_m is the instantaneous return on the value weighted market portfolio, and μ_m is the expected instantaneous return on the market portfolio.

For the one-period CAPM, it is evident from Mossin (1966) that prices are assumed to be in equilibrium in his exposition. Sufficient conditions for equilibrium to exist for this model was only given as late as in 1990 (see Nielsen (1990)).

5.1 The Consumption-Based Capital Asset Pricing Model (CCAPM)

We are particularly interested in risk premiums in times of crises, like the current financial turmoil of 2008 and onwards, and would also like to add some comments to risk premiums for the long run, related to our discussion of climate change in sections 1-3.

In a continuous-time framework, assume processes of interest are Itô-processes, so that, for example, the price of a risky security is a strictly positive process of the form

$$dS_t = \mu_S(t)dt + \sigma_S(t)dB_t.$$

We define the cumulative-return process R by $dR_t = dS_t/S_t$, having expected return rate $\mu_R(t)$ and volatility $\sigma_R(t)$. Starting with a pure no-arbitrage restriction on expected returns, provided there exists a state price deflator π meaning that deflated prices $S^\pi(t) = S_t\pi_t$ are martingales, the martingale property immediately gives the relationship for the risk premium

$$\mu_R(t) - r_t = -\frac{1}{\pi_t}\sigma_R(t) \cdot \sigma_\pi(t). \tag{33}$$

In an equilibrium of the type described in the first part of the paper, the state price deflator $\pi_t = u'(c_t, t)$. Supposing the aggregate consumption satisfies the exogenous dynamics

$$dc_t = c_t\mu_c(t)dt + c_t\sigma_c(t)dB_t,$$

an application of Itô's formula gives that

$$\sigma_\pi(t) = u''(c_t, t)c_t\sigma_c(t),$$

Using the no-arbitrage restriction (33), we immediately obtain the CCAPM:

$$\mu_R(t) - r_t = \gamma(t)\sigma_R(t) \cdot \sigma_c(t). \tag{34}$$

where

$$\gamma(t) = -\frac{u''(c_t, t)c_t}{u'(c_t, t)},$$

i.e., $\gamma(t)$ is the intertemporal coefficient of relative risk aversion of the representative consumer. We indicated this relationship already in equation (4) of Section 2. When this consumer has the additive and separable utility function

$$U(c) = E\Big\{ \int_0^T e^{-\int_0^t \rho(s)ds} u(c_t)dt \Big\},$$

the equilibrium representation for the interest rate is from (3)

$$r_t = \rho(t) + \gamma(t)\mu_c(t) - \frac{1}{2}\frac{u'''(c_t)c_t^2}{u'(c_t)}\sigma_c(t) \cdot \sigma_c(t). \tag{35}$$

We notice from (34) that higher consumption uncertainty measured by σ_c leads to higher risk premiums and a smaller interest rate, all else equal.

These two relationships have in principle been tested on the data of the previous century by Mehra and Prescott (1985), also followed by many others, and the empirical performance is mixed. Many possible explanations have been put forth. There may be significant indivisibilities and transactions costs in consumption, and especially consumer durables[8]. Since the aggregate consumption process c_t involves a considerable amount of smoothing, both across goods and across agents, it is unlikely that it is possible to estimate the covariance rate term $\sigma_R \cdot \sigma_c$ appearing in (34) with the required degree of accuracy. The likelihood of measurement errors is enhanced by the specification requirement that the observation interval between measured, successive changes must be of short duration. Real-world asset returns are quite volatile and exhibit little high-frequency correlation with lagged variables of any sort. Hence lagged, or smoothed estimates of "true" aggregate consumption can severely influence estimates of covariance between contemporaneous changes in consumption and asset returns. Despite many efforts, a resolution of the empirical-performance issue for the CCAPM has remained an open question to date (e.g., Kocherlakota (1996)).

Implied by the Mehra and Prescott (1985) study, there is a risk free rate of 0.80% annually. This estimate has been found to be too low by other researchers, like Siegel (1992). Seen in light of the Stern Review's 1.4% however, this low estimate should be regarded as promising, and one may wonder why it has not simply been adopted in the Review. One reason could be that Mehra and Prescott also found a premium on equity to be

[8]The fruit-tree economy of Lucas (1978) assumes the consumption good to be nondurable and services, also including the nondurable parts of durables.

6% during the period. Since all investments with a time horizon of about 200 years are necessarily risky, this could mean no good news for mitigation today. However, this depends on the nature of the project, as we shall discuss below. Another reason may be that researchers simply do not believe in a risk free rate of 0.80% for the last century. This we also return to below.

The above observations lead directly to the Equity Premium Puzzle: In order for (34) to explain such a high risk premium on equity, the covariance between equity and aggregate consumption increases must be relatively large. For reasonable values of the risk aversion γ, the model does not appear to fit the data.

In the same vein, since the estimate of the variance of consumption increases is so low, the last term in (35) becomes too small in absolute value to explain the low risk-free rate at an annual growth rate in aggregate consumption of about 1.8%. In order to fit this equation, the subjective rate ρ becomes negative, which is nonsense. This is related to the Risk-Free Rate Puzzle (Weil (1989)). Solved together with the standard estimates, one obtains $\hat{\gamma} = 10.2$ and $\hat{\rho} = -.10$ as a "method of moment estimates" for the parameters γ and ρ.

In an interview in 2008, Ranish Mehra, one of the two authors of the seminal 1985-article, suggests a reasonable risk premium on equity to be about one percent in the future. In this case, both puzzles would disappear.

McGrattan and Prescott (2003) re-examine the equity premium puzzle, taking into account some factors ignored by the Mehra and Prescott: Taxes, regulatory constraints, and diversification costs - and focus on long-term rather than short-term savings instruments. Accounting for these factors, the authors find that the difference between average equity and debt returns during peacetime in the last century is less than one per cent, with the average real equity return somewhat under five per cent, and the average real debt return almost four per cent.

In a recent paper Aase (2011) employs a production model instead of the standard pure exchange model to study these issues: If investors follow the standard financial theory paradigm, the simplest linear production model studied in that paper, similar in spirit to the model in (16), implies that they will approach the financial market more or less in isolation from the rest of the economy. With a relative risk aversion close to two, a risk premium of around six per cent will automatically emerge from such an approach with a market volatility of 17 per cent. In its turn this leads to an equilibrium short rate of 0.80 per cent. However, the linear model implies that the volatility of the consumption growths is the same as the volatility of return on equity. With a non-linear production model, on the other hand, possibly including capital and labor as state variables, the investors are invited to hedge against unfavorable changes in the state variables as well. When these effects are taken into account, one interpretation is that the stock market of the last century may have been perceived to be more risky than it really was. This is so since also in the more general production model the CCAPM also holds. Notice that this approach takes the market volatility as given.

Let us return to the question of how the risk premium behaves in times of "turbulence".

5.2 Will risk premiums increase when consumption uncertainty increases?

To answer this question, let us assume there exists an asset, having return process R^*, with the property that its variance rate $\sigma^*(t) = k(t)\sigma_c(t)$ for some strictly positive, real valued process k. This asset has accordingly a perfect instantaneous correlation with aggregate consumption. The CCAPM gives for this asset

$$\mu^*(t) - r_t = \gamma(t)\sigma^*(t) \cdot \sigma_c(t) = \gamma(t)k(t)\sigma_c(t) \cdot \sigma_c(t)$$

Now define the "beta" of any risky security with this particular asset as follows

$$\beta_{R,R^*}(t) = \frac{\sigma_R(t) \cdot \sigma^*(t)}{\sigma^*(t) \cdot \sigma^*(t)}.$$

The CCAPM then takes the form

$$\mu_R(t) - r_t = \beta_{R,R^*}(t)(\mu^*(t) - r_t), \tag{36}$$

for any risky security with return rate process $R(t)$. Now recall that β measures risk *within* a market, and $\beta_{R^*,R^*}(t) = 1$ for all t. In turbulent times, some firms' betas may increase, and others may decrease, but by and large the betas do not systematically change on average, simply by construction. However, the risk premium $(\mu^*(t) - r_t)$ is seen to increase when the variance rate of aggregate consumption increases. Thus, when the uncertainty in aggregate consumption increases, most risk premiums will increase.

This could be consistent with many different scenarios, one being that prices will fall, which is what happened in 2008. Similarly the model predicts that the equilibrium interest rate will fall, as is evident from the expression in (35) for a prudent representative consumer ($u''' > 0$). One strength with this model is that it links the financial economy with aggregate consumption in a simple and direct manner.

6 Adverse shocks to the economy and the risk premium

As with the interest rate, there is a corresponding CCAPM when there are sudden shocks in the exogenous variables. When the same underlying shocks generate jumps in both equilibrium asset prices and aggregate consumption, the risk premium analogue of equation (33) is (Aase 1993a-b, 2007)

$$\mu_R(t) - r_t = -\frac{1}{\pi_t}\Big(\sigma_R(t)\cdot\sigma_\pi(t) + \int_Z \gamma_\pi(t,z)\gamma_R(t,z)\lambda dF(z)\Big). \qquad (37)$$

The last term stems from the jumps, with "internal" shocks having values in a set Z with cumulative distribution function $F(z)$, and frequency λ, and where $\gamma_\pi(t,z)$ is the jump size in the state price deflator, $\gamma_R(t,z)$ the corresponding shock in the equilibrium return of a risky asset, when the shocks in the aggregate consumption are $\gamma_c(t,z)$. The analogue of the CCAPM in (34) is

$$\begin{aligned}\mu_R(t) - r_t &= \gamma(t)\sigma_R(t)\cdot\sigma_c(t)\\ &- \int_Z \Big(\frac{u'(c_{t-}(1+\gamma_c(t,z)),t) - u'(c_{t-},t)}{u'(c_t,t}\Big)\gamma_R(t,z)\lambda dF(z).\end{aligned} \qquad (38)$$

In order to se how the sudden shocks may influence risk premiums, let us expand in a Taylor series. The result is

$$\begin{aligned}\mu_R(t) - r_t &= \gamma(t)\Big(\sigma_R(t)\cdot\sigma_c(t) + \int_Z \gamma_c(t,z)\gamma_R(t,z)\lambda dF(z)\Big)\\ &- \frac{1}{2}\Big(\frac{u'''(c_t,t)c_t^2}{u'(c_t,t)}\Big)\int_Z \gamma_c(t,z)^2\gamma_R(t,z)\lambda dF(z) + \cdots\end{aligned} \qquad (39)$$

The shocks in the price S_t of a risky asset are negative when the quantity $\gamma_R(t,z)$ is negative. Thus, in times of turmoil the quantities $\gamma_c(t,z)$ and $\gamma_R(t,z)$ both negative at the same time. The second term on the right-hand side of (39) will be positive in this case, and, since for a prudent representative agent the third derivative of the utility index is positive, the third term on the right-hand side is also positive. This means that the risk premium will increase in times of economic crises, consistent with the arguments of the previous section.

Related to the Equity Premium Puzzle, the second term on the right-hand side will have to be calibrated to the data, but the third term will yield a positive contribution to the risk premium provided only disasters are modeled by the jump term, as we have just seen. Thus the introduction of jumps in the stochastic models may help explain higher risk premiums.

Regarding the Risk Free Rate Puzzle, consider equation (11) for the equilibrium interest rate. The third term on the right-hand side is the variance rate of the aggregate consumption increases, and will have to be calibrated to the data. However, considering only the possibility of negative shocks to aggregate consumption, the fourth term is accordingly negative, helping to get the subjective rate ρ in the right direction in this regard.

With the US-data of Mehra and Prescott (1985) in mind, we have been able to calibrate the parameters γ and ρ to a pure jump model of the kind just described to the following sizes: $\hat{\gamma} = 5.1$ and $\hat{\rho} = -.05$, which is, of course, better that the continuous model can manage alone, but still not satisfactory. The results are in general better for the risk aversion parameter than for the subjective interest rate. As suggested above, however, Aase (2011) has a different explanation for these two puzzles.

7 Discounting and mitigation

We have argued that, with a relative risk aversion around two and a subjective rate of $\rho = 0.01$ for the representative consumer, the standard model demands an equity premium of about one per cent and a short term interest rate of around four per cent for the consumption and equity moment estimates used by Mehra and Prescott (1985). This value for the risk free interest rate is consistent with Nordhaus (2008), and also with McGrattan and Prescott (2003).

A declining equity premium has been observed in the 1990s, and Lettau et. al. (2008) attributes this to lower macroeconomic volatility and high asset prices in this period. It will be interesting to see how the risk premiums develop in the future.

With respect to climate problems, imagine a project whose return is negatively correlated with aggregate consumption. It follows from the CCAPM that such a project is associated with a negative risk premium in equilibrium. The resulting rate for such a project would be lower than the risk free rate r_t. In other words, while a project that contributes positively to aggregate consumption is penalized with a higher discounting, a project that works as an insurance of future consumption is 'rewarded' by a lower discounting. In an insurance market context, insurance benefits the one who owns insurance in the bad state, at the expense of those 'issuing' insurance. Such an interpretation is, however, not fruitful when it comes to insuring society as a whole.

Normally it is difficult to get private entrepreneurs to invest in projects with a negative expected rate of return. Individuals, however, buy insurance products with precisely this property. 'Projects' with counter-cyclical payoff structure in the aggregate, as national defense, are more naturally relegated to the public sector. According to the mutuality principle, everyone will optimally hold increasing functions of the aggregate consumption. Abatements will typically have the effect of smoothing aggregate consumption, so that, in cases where the climate is in a bad state, consumption is not as low as it would have been without abatement. Risk averse individuals would normally prefer such pooling over states/time.

Abatement administered by the government on behalf of the population may possess this insurance property. On the other hand, 'abatement projects' are not 'owned' by some individuals at the 'expense' of the 'issuers' in the bad state. Thus such projects may function as insuring the society as a whole. Such insurance of aggregate consumption is however not transfer of risk (as in insurance markets) and not between individuals, but reduction in aggregate variability across states. Our theory then tells us that mitigation can be associated with a discount rate that is lower than the risk free rate.

As an illustration, consider a situation with a growth rate of 1.3% and a risk free rate of 3.2% as was suggested in Section 3. Considering the extreme case where the effect from mitigation has a volatility of the order of equity, this leads to a discount rate of around 2%. From the model of Section 4, on the other hand, we could explain a risk free rate as low as 1.2%, which would then imply a rather low discount rate for such projects.

We do not claim that references to all projects of mitigation are negatively correlated with aggregate consumption. In the production economy of Aase (2011), for example, a project can have negative correlation with some state variables, like capital, labor or wealth, without being negatively correlated with aggregate consumption, in which case the proper interest rate may be close to the risk-less one.

The problem still remains that mitigation hurts the generations that carry it out, and rewards future, yet unborn generations. The above smoothing argument is normally valid for one individual throughout his or her lifetime, and thus it is perfectly valid under infinitely lived generations and - if deemed realistic - multigenerational models that retain their features. At this point it seems appropriate to bring in an analogy to life insurance. The insurance customer pays a premium today in exchange for a payment upon death. In a model of life cycles, we associate this observation with a bequest motive. Life insurance can be viewed as a financial tool for controlling inter-personal transfers (e.g., Bernheim, Shleifer and Summers (1985)). This is obviously one direction that could be pursued further regarding climate projects. Here we only make the point that markets for life insurance do exist.

8 Summary

We have studied models that connect the the financial sector to the real economy, in order to (a) investigate the effects of sudden negative shocks to the economy, and (b) study the economic reactions to the possible effects of climate change. One problem of interest is under what conditions are long-term interests lower than short term rates. The traditional model is by and large silent about this, and ad hoc arguments are commonplace. We introduce jump dynamics in the aggregate consumption process, which allows us to obtain lower long term interest rates subject to negative shocks, when, at the same time, the uncertainty in consumption growth only increases moderately in the future. Alternatively we use a production economy that provides an entire term structure of interest rates to answer such questions. One result is that in the long run will the equilibrium interest rate not depend on the subjective rate, and this model can be consistent with a fairly low long term interest rate.

We also study what impact negative shocks in aggregate consumption has on risk premiums. Here we consider two different continuous-time models; one with a continuous state dynamics, and one which allows for sudden shocks at random time points. Both models predict, perhaps not surprisingly, increased risk premiums in turbulent times.

Based on a production model, we conclude that the best estimate of the short rate is about four per cent, with an equity premium of around one per cent, based on historical estimates for volatilities and mean rates of equity and aggregate consumption. If these volatilities increase in the future, this leads to higher risk premiums, and also lower interest rates.

Our theory tells us that projects correlated with aggregate consumption must include adjustments for this uncertainty in discounting. In particular, for specific projects aimed at insuring future consumption, the risk premium is typically negative, which means that the discount factors for such projects are higher than discount factors for uncorrelated projects. It is argued that such projects constitute the proper social insurance, which may then become feasible from a cost/benefit perspective.

With reference to global warming, we have questioned whether the utilitarian calculus is the right ethical framework. The ethics of the common Norwegian farmer may be an appropriate guide: To leave the farm to the next generation in a condition at least as good as it was received. This leads to a rights-based notion of ethics rather than a utilitarian. With some common sense in place, and, perhaps, if supplemented with the bequest motive, we believe the utilitarian platform is still appropriate.

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