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Discussion paper

Long Dated Life Insurance and Pension Contracts

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Abstract

We discuss the "life cycle model" by first introducing a credit market with only biometric risk, and then market risk is introduced via risky securities. This framework enables us to find optimal pension plans and life insurance contracts where the benefits are state dependent. We compare these solutions both to the ones of standard actuarial theory, and to policies offered in practice. Two related portfolio choice puzzles are discussed in the light of recent research, one is the horizon problem, the other is related to the aggregate market data of the last century, where theory and practice diverge. Finally we present some comments on longevity risk and cohort risk.

KEYWORDS: The life cycle model, pension insurance, optimal life insurance, longevity risk, the horizon problem, equity premium puzzle

1 Introduction

Four or five decennials back life and pension insurance seemed less problematic than today, at least from the insurance companies' point of view. Prices were set by actuaries using life tables, and a "fixed calculation" interest rate. This rate was not directly linked to the spot interest rate of the market, or any other market linked quantities or indexes. The premium reserves of the individual and collective policies were invested in various assets, and when the different contracts were settled, the evolution of the premium reserve determined the final insurance compensation. If the return on the premium

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reserve had been higher than the "calculation rate", this gave rise to a bonus. For a mutual company "bonus" need not only involve a payment from the insurer to the customer, but could also involve a payment in the other direction. For a stock owned corporation the bonus could in principle only be non-negative. In most cases this did not matter all that much, since the "calculation rate" was set to the "safe side", which meant much lower than the realized return rate on the premium reserve.

In several countries the nominal interest rate was high during some parts of this period, often significantly higher than the fixed rate used in determining premiums .

In Norway, for example, this calculation rate (4%) appeared from some point in time as a legal guaranteed return rate in the contracts. For current policies this guarantee is reduced to 3%.

During the last two or three decennials this interest rate guarantee has become a major issue for many life insurance companies. What initially appeared to be a benefit with almost no value, later turned out to be rather valuable for the policy holders, and correspondingly problematic for the insurers.

In this paper we study optimal demand theory, where, among other things, we can check if such contracts have any place in the life cycle model. It turns out to be not much evidence for this. If we were to take into account also the supply side of the economy, and for example study Pareto optimal contracts, it is not likely that this would change the picture much. We know that such contracts are "smooth" unless there are frictions of some kinds.

Every downturn in the financial market has typically been accompanied by problems for the life insurance industry. In view of this, life insurance companies seem to prefer to offer "defined contribution" type policies to the more traditional "defined benefit" ones. For the former type the companies have considerably less risk than for the latter.

During the financial crisis of 2008 and onwards, casual observations seem to suggest that many *individuals* would rather prefer the defined benefit type to the other. In a particular case, the employees of a life and pension insurance company would rather prefer a collective defined benefit pension plan, but were voted down by the board. Collective pension plans organized by firms on behalf of their workers, are almost exclusively defined contribution plans these days, which appear to be the least costly of the two for the firms, and also the preferred choice to offer by the insurance companies.

The paper is organized as follows: In Section 2 we introduce consumption and saving with only a credit market available. Here we introduce some actuarial concepts related to mortality. Actuarial notation can be rather demanding at first sight, so we have tried to keep the technical details at a

minimum. In particular we study the effects from pooling. Next, in Section 3, we include mortality risk, i.e., an uncertain planning horizon, in the model of Section 2. In this setting we derive both optimal life insurance, not commonly studied, and optimal pension insurance, and investigate their properties when there is only a credit market present. In Section 4 we introduce a market for risky securities in addition to the credit market. Here we solve both the optimal portfolio choice and the pension/life insurance problem. We show that with pension insurance available, the actual consumption rate at each time is larger than without pension. The optimal portfolio choice problem is studied in Section 5, where we also point out a solution to *time horizon problem*, as well as a solution to a related empirical problem with the optimal strategy. This latter problem is also related to the celebrated *equity premium puzzle*. In Section 6 we discuss our results, and reflect on longevity risk and cohort risk in relation to the framework presented. Section 7 concludes.

2 Consumption and Saving

We start with the simplest problem in optimal demand theory, when there is no risk and no uncertainty.

Consider a person having income $e(t)$ and consumption $c(t)$ at time t . Given income, possible consumption plans must depend on the possibilities for saving and for borrowing and lending. We want to investigate the possibilities of using income during one period to generate consumption in another period.

To start, assume the consumer can borrow and lend to the same interest rate r . Given any e and c , the consumer's net saving $W(t)$ at time t is

$$W(t) = \int_0^t e^{r(t-s)}(e_s - c_s)ds. \quad (1)$$

Assuming the person wants to consume as much as possible for any e , not any consumption plan is possible. A constraint of the type $W(t) \geq a(t)$ may seem reasonable: If $a(t) < 0$ for some t , the consumer is allowed a net debt at time t . Another constraint could be $W(T) \geq B \geq 0$, where T is the planner's horizon. The consumer is then required to be solvent at time T .

The objective is to optimize the utility $U(c)$ of lifetime consumption, subject to a budget constraint. There could also be a bequest motive, but this is not the only explanation underlying life insurance.

2.1 Uncertain planning horizon

In order to formulate the most natural budget constraint of an individual, which takes into account the advantages of pooling risk, we introduce mortality. Yaari (1965), Hakansson (1969) and Fisher (1973) were of the first to include an uncertain lifetime into the theory of the consumer.

The remaining lifetime T_x of an x year old consumer at time zero is a random variable with support $(0, \tau)$ and cumulative probability distribution function $F^x(t) = P(T_x \leq t)$, $t \geq 0$. The survival function is denoted by $\bar{F}^x(t) = P(T_x > t)$. Ignoring possible selection effects, it can be shown that

$$\bar{F}^x(t) = \frac{l(x+t)}{l(x)} \quad (2)$$

for some function $l(\cdot)$ of one variable only. The decrement function $l(x)$ can be interpreted as the expected number alive in age x from a population of $l(0)$ newborns.

The force of mortality or death intensity is defined as

$$\mu_x(t) = \frac{f_x(t)}{1 - F^x(t)} = -\frac{d}{dt} \ln \bar{F}^x(t), \quad F^x(t) < 1, \quad (3)$$

where $f_x(t)$ is the probability density function of T_x . Integrating this expression yields the survival function in terms of the force of mortality

$$\bar{F}^x(t) = \frac{l(x+t)}{l(x)} = \exp \left\{ -\int_0^t \mu_x(u) du \right\}. \quad (4)$$

Suppose $y \geq 0$ a.s. is a non-negative process in L , the set of consumption processes. Later L will be a set of adapted stochastic processes y satisfying $E(\int_0^\tau y_t^2 dt) < \infty$. If T_x and y are independent, the formula

$$E\left(\int_0^{T_x} y_t dt\right) = \int_0^\tau E(y_t) \frac{l(x+t)}{l(x)} dt = \int_0^\tau E(y_t) e^{-\int_0^t \mu_x(u) du} dt \quad (5)$$

follows essentially from integration by parts, the independence assumption and Fubini's Theorem. Assuming the interest rate r is a constant, it follows that the single premium of an annuity paying one unit per unit of time is given by the actuarial formula

$$\bar{a}_x^{(r)} = \int_0^\tau e^{-rt} \frac{l_{x+t}}{l_x} dt, \quad (6)$$

and the single premium of a "temporary annuity" which terminates after time n is

$$\bar{a}_{x:\bar{n}|}^{(r)} = \int_0^n e^{-rt} \frac{l_{x+t}}{l_x} dt. \quad (7)$$

Under a typical pension plan the insured will pay a constant, or "level" premium p up to some time of retirement n , and from then on he will receive an annuity b as long as he lives. *The principle of equivalence* gives the following relationship between premium and benefit:

$$p \int_0^n e^{-rt} \frac{l_{x+t}}{l_x} dt = b \int_n^\tau e^{-rt} \frac{l_{x+t}}{l_x} dt.$$

In standard actuarial notation this is written

$$p\bar{a}_{x:\bar{n}|}^{(r)} = b(\bar{a}_x^{(r)} - \bar{a}_{x:\bar{n}|}^{(r)}). \quad (8)$$

The following formulas are sometimes useful in life insurance computations

$$\mu_x(t) = -\frac{l'(x+t)}{l(x+t)}, \quad \text{and} \quad f_x(t) = -\frac{l'(x+t)}{l(x)} = \frac{l(x+t)}{l(x)}\mu_{x+t}, \quad (9)$$

where $l'(x+t)$ is the derivative of $l(x+t)$ with respect to t . The present value of one unit payable at time of death is denoted \bar{A}_x . Using (9) and integration by parts, it can be written

$$\bar{A}_x = \int_0^\tau e^{-rt} f_x(t) dt = 1 - r\bar{a}_x^{(r)}. \quad (10)$$

This insurance contract is called *Whole life insurance*. If the premium rate p is paid until the retirement age n for a combined life insurance with z units payable upon death, and an annuity of rate b per time unit as long as the insured lives, we have the following relationship between p, b and z :

$$p\bar{a}_{x:\bar{n}|}^{(r)} = b(\bar{a}_x^{(r)} - \bar{a}_{x:\bar{n}|}^{(r)}) + z(1 - r\bar{a}_x^{(r)}). \quad (11)$$

Pension insurance and life insurance can now be integrated in the life cycle model in a natural way, as we shall see.

2.2 The effect from pooling

Continuing our discussion of consumption and saving the following quantity plays an important role:

$$E(W(T_x)e^{-rT_x}) = \text{expected discounted net savings}. \quad (12)$$

In the absence of a life and pension insurance market, one would as before consider consumption plans c such that $W(T_x) \geq B$, or

$$W(T_x)e^{-rT_x} \geq b \geq 0 \quad \text{almost surely} \quad (13)$$

e.g., debt must be resolved before the time of death. If, on the other hand, pension insurance is possible, then one can allow consumption plans where

$$E(W(T_x)e^{-rT_x}) = 0 \quad (\text{no life insurance.}) \quad (14)$$

Those individuals who live longer than average are guaranteed a pension as long as they live via the pension insurance market. The financing of this benefit comes from those who live shorter than average, which is what pooling is all about.

The implication is that the individual's savings possibilities are exhausted, by allowing gambling on own life length. Clearly the above constraint in (14) is far less demanding than requiring that the discounted net savings, the random variable in (13), is larger than some non-negative number b with certainty. Integration by parts gives the following expression for the expected discounted net savings

$$E(W(T_x)e^{-rT_x}) = \int_0^{\tau} (e(t) - c(t))e^{-\int_0^t (r+\mu_{x+u})du} dt. \quad (15)$$

This expression we have interpreted as the present value of the consumer's net savings, which is seen from (15) to take place at a "spot" interest rate

$$r + \mu > r$$

where the inequality follows since the mortality rate $\mu > 0$. This is a result of the pooling effect of (life and) pension insurance. The existence of a life and pension insurance market allows the individuals to save at a higher interest rate than the spot rate r . With a pure pension insurance contract, the policyholder can consume *more* while alive, since terminal debt is resolved by pooling. This is illustrated later in an example when all the relevant uncertainty is taken into account.

Example 1. (A Pension Contract, or an Annuity). Suppose $e(t) = 0$ for $t > n$. The condition $E(W(T_x)e^{-rT_x}) = 0$ can be interpreted as the *Principle of Equivalence*:

$$\int_0^n (e(t) - c(t))P[T_x > t]e^{-rt} dt = \int_n^{\tau} c(t)e^{-rt}P[T_x > t]dt. \quad (16)$$

Here the difference $(e_t - c_t) = p_t$ is the premium (intensity) paid while working, giving rise to the pension c_t after the time of retirement n . This relationship implies that the pension is paid out to the beneficiary as long as necessary, and only then, i.e., as long as the policy holder is alive. \square

Notice the similarity between the actuarial formula in (8) and the above equation (16). Both equations are, of course, based on the same principle.

3 The optimal demand theory with only a credit market

In order to analyze the problem of optimal consumption, we need some assumptions about the preferences of the consumer. We assume the preferences are represented by a utility function $U : L \rightarrow R$ given in the additive and separable form by

$$U(c) = E \left\{ \int_0^{T_x} e^{-\rho t} u(c_t) dt + e^{-\kappa T_x} v(W_{T_x}) \right\}. \quad (17)$$

Here ρ and κ are subjective impatience rates, u is a strictly increasing and concave utility function, and v is a another utility function. The function v is connected to life insurance, and may represent a bequest motive, but as I will argue later, this is not the most natural reason for life insurance. The functions u and v are sometimes referred to as felicity indexes.

The variable $z = W(T_x)$ is the amount of life insurance. It is often assumed to be a given constant (e.g., 1) in the standard theory of life insurance, but we will allow it to be a decision variable. First we focus on pensions and annuities and set $v \equiv 0$.

The pension problem may be formulated as:

$$\max_c E \left\{ \int_0^{T_x} e^{-\rho t} u(c_t) dt \right\} \quad (18)$$

subject to (i) $E(W(T_x)e^{-rT_x}) = 0$, and (ii) $c_t \geq 0$ for all t . Ignoring the positivity constraint (ii) for the moment, we may use Kuhn-Tucker to solve this problem. The Lagrangian is

$$\mathcal{L}(c; \lambda) = \int_0^\tau u(c_t) e^{-\int_0^t (\rho + \mu_{x+s}) ds} dt + \lambda \left(\int_0^\tau (e(t) - c(t)) e^{-\int_0^t (r + \mu_{x+s}) ds} dt \right).$$

If $c^*(t)$ is optimal, there exists a Lagrange multiplier λ such that $\mathcal{L}(c; \lambda)$ is maximized at $c^*(t)$ and complementary slackness holds. Denoting the directional derivative of $\mathcal{L}(c^*; \lambda)$ in the direction c by $\nabla \mathcal{L}(c^*, \lambda; c)$, the first order condition of this unconstrained problem is

$$\nabla \mathcal{L}(c^*, \lambda; c) = 0 \quad \text{in all 'directions' } c \in L,$$

which is equivalent to

$$\int_0^\tau (u'(c_t^*) e^{-\int_0^t (\rho + \mu_{x+s}) ds} - \lambda e^{-\int_0^t (r + \mu_{x+s}) ds}) c(t) dt = 0, \quad \forall c \in L.$$

This gives the first order condition

$$u'(c_t^*) = \lambda e^{-(r-\rho)t}, \quad t \geq 0. \quad (19)$$

Notice that the force of mortality μ does not enter this expression.

Differentiating this function in t along the optimal path c^* , we deduce the following differential equation for c^*

$$\frac{dc_t^*}{dt} = (r - \rho)T(c_t^*), \quad (20)$$

where $T(x) = -\frac{u'(x)}{u''(x)}$ is the *risk tolerance function* of the consumer, the reciprocal of the *absolute risk aversion function*.

Exampel 2. (A Pension Contract for the CRRA Consumer.) Assume that the income process e_t is:

$$e_t = \begin{cases} y, & \text{if } t \leq n; \\ 0, & \text{if } t > n \end{cases} \quad (21)$$

where y is a constant, interpreted as the consumer's salary when working. The felicity index is assumed to be $u(x) = \frac{1}{1-\gamma}x^{1-\gamma}$. This index has a constant *relative risk aversion* (CRRA) of γ . We may interpret y as the agent's salary while working. The optimal consumption and pension is $c_t^* = ke^{\frac{1}{\gamma}(r-\rho)t}$, where k is an integration constant. Equality in constraint (i) determines the constant k : The optimal life time consumption ($t \in [0, n]$) and pension ($t \in [n, \tau]$) is

$$c_t^* = y \frac{\bar{a}_{x:\bar{n}|}^{(r)}}{\bar{a}_x^{(r_0)}} e^{\frac{1}{\gamma}(r-\rho)t} \quad \text{for all } t \geq 0. \quad (22)$$

Here $r_0 = r - \frac{r-\rho}{\gamma}$ and $\bar{a}_{x:\bar{n}|}^{(r)}$ and $\bar{a}_x^{(r_0)}$ are the actuarial formulas explained in (6) and (7). Although the first order conditions in (19) do not depend on mortality, the optimal consumption c_t^* does, since the Lagrange multiplier λ , or equivalently, the integration constant k , is determined from the 'average' budget constraint (i). Also, the positivity constraint (ii) is not binding at the optimum, due to the form of the felicity index u . \square

The differential equation (20) tells us that the value of the interest rate r is a crucial border value for the subjective impatience rate ρ . When $\rho > r$ the optimal consumption c_t^* is always a decreasing function of time t , but when $\rho < r$ the optimal consumption increases with time. In the first case, the 'impatient' one has already consumed so much, that he can only look forward to a decreasing consumption path. The 'patient' one can, on the other hand, look forward to a steadily increasing future consumption path. In Example

2 we see from (22) that the former has an optimal consumption path that is a decreasing exponential, while the latter has an increasing exponential consumption path. This seems to suggest that it may be difficult to compare consumption paths between different consumers. That this is not so clear-cut as this example might suggest, will follow when we introduce a securities market where the consumers are allowed to invest in risky securities as well as a risk less asset in order to maximize lifetime consumption. In Example 2 we notice that the above effects are dampened as the relative risk aversion γ increases.

3.1 Including life insurance

We can now introduce life insurance, where the goal is to determine the optimal amount of life insurance for an individual. The problem is then to solve

$$\max_{c(t), z} E \left\{ \int_0^{T_x} e^{-\rho t} u(c_t) dt + e^{-\kappa T_x} v(z) \right\}$$

subject to (i) $E(W(T_x)e^{-rT_x}) \geq E(ze^{-rT_x})$, and (ii) $c_t \geq 0$ for all t and $z \geq 0$.

The Lagrangian for the problem is (ignoring again the non-negativity constraints (ii)),

$$\begin{aligned} \mathcal{L}(c, z; \lambda) = & \int_0^{\tau} u(c_t) e^{-\int_0^t (\rho + \mu_{x+s}) ds} dt + v(z) (1 - \kappa \bar{a}_x^{(\kappa)}) \\ & - \lambda \left((1 - r \bar{a}_x^{(r)}) z - \int_0^{\tau} (e(t) - c(t)) e^{-\int_0^t (r + \mu_{x+s}) ds} dt \right). \end{aligned}$$

The first order condition (FOC) in c is the same as for pensions treated above. The FOC in the amount z of life insurance is obtained by ordinary differentiation with respect to the real variable z . This gives

$$v'(z^*) = \lambda \frac{1 - r \bar{a}_x^{(r)}}{1 - \kappa \bar{a}_x^{(\kappa)}}.$$

We can now determine both the optimal life time consumption, including pension and the optimal amount of life insurance. An example will illustrate.

Example 3: (The CRRA consumer.) Assume e_t is as in (21), the consumption felicity index is $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$, and the life insurance index is

$v(x) = \frac{1}{1-\psi}x^{1-\psi}$. The optimal life insurance amount and optimal consumption/pension are given by

$$z^* = \lambda^{-\frac{1}{\psi}} \left(\frac{1 - r\bar{a}_x^{(r)}}{1 - \kappa\bar{a}_x^{(\kappa)}} \right)^{-\frac{1}{\psi}} \quad \text{and} \quad c_t^* = \lambda^{-\frac{1}{\gamma}} e^{\frac{1}{\gamma}(r-\rho)t}. \quad (23)$$

Equality in the 'average' budget constraint (i) determines the Lagrangian multiplier λ . The equation is

$$\lambda^{-\frac{1}{\psi}} (1 - r\bar{a}_x^{(r)}) \left(\frac{1 - r\bar{a}_x^{(r)}}{1 - \kappa\bar{a}_x^{(\kappa)}} \right)^{-\frac{1}{\psi}} + \lambda^{-\frac{1}{\gamma}} \bar{a}_x^{(r_0)} = y \bar{a}_{x:\bar{n}}^{(r)}. \quad (24)$$

Notice that with life insurance included, the optimal consumption and the pension payments become smaller than without life insurance present, which is seen when comparing the expressions in (23) and (24) with (22). This just tells us the obvious: When some resources are bound to be set aside for the beneficiaries, less can be consumed while alive. The optimal amount in life insurance is an increasing function in income y , and depends on the interest rate r , the pension age n , the insured's relative risk aversion γ as well as his impatience rate ρ , the bequest relative risk aversion ψ and the corresponding impatience rate κ , the insured's age x when initializing the pension and insurance contracts, and the insured's life time distribution through the actuarial formulas in (24).

Comparative statics in the parameters are not straightforward, and numerical technics are necessary. As an example, when $\psi = \gamma$, it can be seen that the optimal amount of life insurance $z^*(\kappa)$ as a function of the bequest impatience rate κ is increasing for $\kappa \leq \kappa_0$ for some $\kappa_0 > 0$, and decreasing in κ for $\kappa > \kappa_0$. For reasonable values of κ this means that more impatience with respect to life insurance means a higher amount z^* of life insurance. \square

The above results deviate rather much from the standard actuarial formulas, which is to be expected since the two approaches are indeed different: The actuarial theory is primarily based on the principle of equivalence and risk neutrality. This is problematic, since risk neutral insurance customers would simply not demand any form of insurance. Therefore we assume that the individuals are risk averse, unlike what is done in actuarial theory, and use expected utility as our optimization criterion.

Going back to the actuarial relationship (11), the three quantities p , b and z representing the premium, the pension benefit and the insured amount respectively could be any non-negative numbers satisfying this relationship. In the above example, all these quantities are in addition derived so that

expected utility is optimized. The optimal contracts still maintain the actuarial logic represented by the principle of equivalence, which in our case corresponds to the budget constraint (i) on the 'average'. The present analogue to the relationship (11) is:

$$\int_0^n (y - c_t^*) \frac{l_{x+t}}{l_x} e^{-rt} dt = \int_n^\tau c_t^* \frac{l_{x+t}}{l_x} e^{-rt} dt + z^*(1 - r\bar{a}_x^{(r)}), \quad (25)$$

where the constant premium p corresponds to the time varying $p_t = (y - c_t^*)$ for $0 \leq t \leq n$, the constant pension benefit b corresponds to the optimal c_t^* for $n \leq t \leq \tau$, and the number z corresponds to z^* found in (23), where also the optimal pension c_t^* is given.

So far the insured amount is still a deterministic quantity, albeit endogenously derived. The reason for the non-randomness in z^* in the present situation is that only biometric risk is considered.

When uncertainty in the financial market is also taken into account, we shall demonstrate that the optimal insured amount becomes state dependent, and the same is true for c_t^* . Both real and nominal amounts are then of interest when comparing the results with insurance theory and practice.

Including risky securities in a financial market is our next topic.

4 A Financial Market including Risky Assets

We consider a consumer/insurance customer who has access to a securities market, as well as pension and life insurance as considered in the above. The securities market can be described by a price vector $X' = (X^{(0)}, \dots, X^{(N)})$, where (prime means transpose)

$$dX_t^{(n)} = \mu_n X_t^{(n)} dt + X_t^{(n)} \sigma^{(n)} dB_t, \quad X_0^{(n)} > 0, \quad t \in [0, T], \quad (26)$$

The vector $\sigma^{(n)}$ is the n -th row of a matrix σ consisting of constants in $R^{N \times N}$ with linearly independent rows, and μ_n is a constant. Here N is also the dimension of the Brownian motion B .

Underlying there is a probability space (Ω, \mathcal{F}, P) and an increasing information filtration \mathcal{F}_t generated by the d -dimensional Brownian motion. Each price process $X_t^{(n)}$ is a geometric Brownian motion, and we suppose that $\sigma^{(0)} = 0$, so that $r = \mu_0$ is the risk free interest rate. T is the finite horizon of the economy, so that $\tau < T$. The state price deflator π is given by

$$\pi_t = \xi_t e^{-rt}, \quad (27)$$

where the 'density' process ξ has the representation

$$\xi_t = \exp(-\eta' \cdot B_t - \frac{t}{2} \eta' \cdot \eta). \quad (28)$$

Here η is the market-price-of-risk for the discounted price process $X_t e^{-rt}$, defined by

$$\sigma \eta = \nu. \quad (29)$$

ν is the vector with n -th component $(\mu_n - r)$, the excess rate of return on security n , $n = 1, 2, \dots, N$. From Ito's lemma it follows from (28) that

$$d\xi_t = -\xi \eta' \cdot dB_t, \quad (30)$$

i.e., the density ξ_t is a martingale.

The agent is represented by an endowment process e (income) and a utility function $U : L_+ \times L_+ \rightarrow R$, where

$$L = \{c : c_t \text{ is } \mathcal{F}_t\text{-adapted, and } E(\int_0^T c_t^2 dt) < \infty\}.$$

L_+ , the positive cone of L , is the set of consumption rate processes.

The specific form of the function U is as before, namely the time additive one given in (17). The remaining life time T_x of the agent is assumed independent of the risky securities X . The information filtration \mathcal{F}_t is enlarged to account for events like $T_x > t$.

4.1 The Consumption/Portfolio Choice/Pension Problem

The consumer's problem is, for each initial wealth level w , to solve

$$\sup_{(c, \varphi)} U(c) \quad (31)$$

subject to an intertemporal budget constraint

$$dW_t = (W_t(\varphi_t' \cdot \nu + r) - c_t)dt + W_t \varphi_t' \cdot \sigma dB_t, \quad W_0 = w. \quad (32)$$

Here $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \dots, \varphi_t^{(N)})$ are the fractions of total wealth held in the risky securities. The first order condition for the problem (31) is given by the Bellman equation:

$$\sup_{(c, \varphi)} \{ \mathcal{D}^{(c, \varphi)} J(w, t) - \mu_x(t) J(w, t) + u(c, t) \} = 0, \quad (33)$$

with boundary condition

$$EJ(w, T_x) = 0, \quad w > 0. \quad (34)$$

The function $J(w, t)$ is the indirect utility function of the consumer at time t when the wealth $W_t = w$, and represents future expected utility at time t in state w , provided the optimal portfolio choice strategy is being followed from this time on. The differential operator $\mathcal{D}^{(c, \varphi)}$ is given by

$$\begin{aligned} \mathcal{D}^{(c, \varphi)} J(w, t) &= J_w(w, t)(w\varphi \cdot \nu + rw - c) + J_t(w, t) \\ &+ \frac{w^2}{2} \varphi' \cdot (\sigma \cdot \sigma') \cdot \varphi J_{ww}(w, t). \end{aligned} \quad (35)$$

The problem as it now stands is a non-standard dynamic programming problem, a so called non-autonomous problem. Instead of solving this problem directly, we solve an equivalent one. As is well known (e.g., Cox and Huang (1989) or Pliska (1987)), since the market is complete, the dynamic program (31) - (35) has the same solution as a simpler, yet more general problem, which we now explain.

4.2 An Alternative Problem Formulation

Find

$$\sup_{c \in L} U(c), \quad (36)$$

subject to

$$E \left\{ \int_0^{T_x} \pi_t c_t dt \right\} \leq E \left\{ \int_0^{T_x} \pi_t e_t dt \right\} := w \quad (37)$$

Here e is the endowment process of the individual, and is assumed that e_t is \mathcal{F}_t measurable for all t .

As before, the pension insurance element secures the consumer a consumption stream as long as needed, but only if it is needed. This makes it possible to compound risk-free payments at a higher rate of interest than r .

The optimal wealth process W_t associated with a solution c^* to the problem (36)-(37) can be implemented by some adapted and allowed trading strategy φ^* , since the marketed subspace M is equal to L (complete markets). Without mortality this is a well-known result in financial economics. We claim that by introducing the new random variable T_x this result still holds: In principal mortality corresponds to a new state of the economy, which should normally correspond to its own component in the state price, but the insurer can diversify this type of risk away by pooling over the agents, all in age x , so that the corresponding addition to the Arrow-Debreu state

price is only the term $\exp\{-\int_0^t \mu_x(u)du\}$, a non-stochastic quantity. Accordingly, adding the pension insurance contract in an otherwise complete model has no implications for the state price π other than multiplication by this deterministic function, and thus the model is still 'essentially' complete.

4.3 The Optimal Consumption/Pension

The constrained optimization problem (36)-(37) can be solved by Kuhn-Tucker and a variational argument. The Lagrangian of the problem is

$$\mathcal{L}(c; \lambda) = E\left\{ \int_0^{T_x} (u(c_t, t) - \lambda(\pi_t(c_t - e_t))) dt \right\}, \quad (38)$$

We assume that the optimal solution c^* to the problem (36)-(37) satisfies $c_t^* > 0$ for a.a. $t \in [0, T_x)$, a.s. Then there exists a Lagrange multiplier, λ , such that c^* maximizes $\mathcal{L}(c; \lambda)$ and complementary slackness holds.

Denoting the directional derivative of $\mathcal{L}(c^*; \lambda)$ in the "direction" $c \in L$ by $\nabla \mathcal{L}(c^*, \lambda; c)$, the first order condition of this unconstrained problem becomes

$$\nabla \mathcal{L}(c^*, \lambda; c) = 0 \quad \text{for all } c \in L \quad (39)$$

This is equivalent to

$$E\left\{ \int_0^\tau \left((u'(c_t^*)e^{-\rho t} - \lambda\pi_t)c(t) \right) P(T_x > t) dt \right\} = 0, \quad \text{for all } c \in L, \quad (40)$$

where the survival probability $P(T_x > t) = \frac{l(x+t)}{l(x)}$. In order for (40) to hold true for all processes $c \in L$, the first order condition is

$$u'(c_t^*) = \lambda e^{-\rho t} \pi_t = \lambda e^{-(r-\rho)t} \xi_t \quad \text{a.s.,} \quad t \geq 0 \quad (41)$$

in which case the optimal consumption process is

$$c_t^* = u'^{-1}\left(\lambda e^{-(r-\rho)t} \xi_t\right) \quad \text{a.s.,} \quad t \geq 0, \quad (42)$$

where the function $u'^{-1}(\cdot)$ inverts the function $u(\cdot)$. Comparing the first order condition to the one in (19) where only biometric risk is included, we notice that the difference is the state price density ξ_t in (41). Still mortality does not enter this latter condition.

Differentiation (41) in t along the optimal path c_t^* , by the use of Ito's lemma and diffusion invariance the following stochastic differential equation for c_t^* is obtained

$$dc_t^* = ((r - \rho)T(c_t^*) + \frac{1}{2}T^3(c_t^*) \frac{u'''(c_t^*)}{u'(c_t^*)} \eta' \cdot \eta) dt + T(c_t^*) \eta' \cdot dB_t \quad (43)$$

where $T(\cdot)$ is the risk tolerance function defined earlier.

Comparing with the corresponding differential equation (20) for c_t^* with only biometric risk present, it is seen that including market risk means that the dynamic behavior of the optimal consumption is not so crucially dependent upon whether $r < \rho$ or not. This follows since, first, there is an additional term in the drift, and, second, there is a diffusion term present under market risk. The definition of what impatience means will also change with market risk present, as we shall see.

Notice that when the market-price-of-risk $\eta = 0$, the two equations coincide. We consider an example:

Example 4. (The CRRA-consumer.) In this case the optimal consumption takes the form

$$c_t^* = (\lambda e^{-(r-\rho)t} \xi_t)^{-\frac{1}{\gamma}} \quad \text{a.s., } t \geq 0. \quad (44)$$

The budget constraint determines the Lagrange multiplier λ , where mortality comes in. Suppose we consider an endowment process e_t giving rise to a pension as in (21). Using Fubini's theorem this constraint can be written

$$\int_0^n \left(y e^{-rt} \frac{l_{x+t}}{l_x} - \lambda^{-\frac{1}{\gamma}} e^{-\frac{\rho t}{\gamma}} E(\pi_t^{(1-\frac{1}{\gamma})}) \frac{l_{x+t}}{l_x} \right) dt + \int_n^\tau (-1) \lambda^{-\frac{1}{\gamma}} e^{-\frac{\rho t}{\gamma}} E(\pi_t^{(1-\frac{1}{\gamma})}) \frac{l_{x+t}}{l_x} dt = 0. \quad (45)$$

By the properties of the state prices π_t and (27) - (30), it follows that

$$E(\pi_t^{(1-\frac{1}{\gamma})}) = e^{-[(1-\frac{1}{\gamma})(r+\frac{1}{2} \frac{1}{\gamma} \eta' \cdot \eta)]t}.$$

Accordingly, the budget constraint can be written

$$y \int_0^n e^{-rt} \frac{l_{x+t}}{l_x} dt = \lambda^{-\frac{1}{\gamma}} \int_0^\tau e^{-[\frac{\rho}{\gamma} + (1-\frac{1}{\gamma})(r+\frac{1}{2} \frac{1}{\gamma} \eta' \cdot \eta)]t} \frac{l_{x+t}}{l_x} dt.$$

Defining the quantity

$$r_1 = \frac{\rho}{\gamma} + (1 - \frac{1}{\gamma})(r + \frac{1}{2} \frac{1}{\gamma} \eta' \cdot \eta),$$

the Lagrangian multiplier is determined by

$$\lambda^{-\frac{1}{\gamma}} = y \frac{\bar{a}_{x:\bar{n}}^{(r)}}{\bar{a}_x^{(r_1)}}.$$

From this, the optimal consumption ($t \in [0, n]$) and the optimal pension ($t \in [n, \tau]$) are both given by the expression

$$c_t^* = y \frac{\bar{a}_{x:\bar{n}|}^{(r)}}{\bar{a}_x^{(r_1)}} e^{\frac{1}{\gamma}(r-\rho)t} \xi_t^{-\frac{1}{\gamma}} \quad \text{for all } t \geq 0. \quad (46)$$

which can be compared to (22) which gives the corresponding process with only mortality risk present. Notice that this latter formula follows from (46) by setting $\eta = 0$, in which case $\xi_t = 1$ for all t (a.s.) and $r_1 = r_0$.

The expected value of the optimal consumption is given by

$$E(c_t^*) = y \frac{\bar{a}_{x:\bar{n}|}^{(r)}}{\bar{a}_x^{(r_1)}} \exp \left\{ \frac{1}{\gamma} \left(r + \frac{1}{2} \eta' \cdot \eta \left(1 + \frac{1}{\gamma} \right) - \rho \right) t \right\}, \quad (47)$$

which is seen to grow with time t already when $r > \rho - \frac{1}{2} \eta' \cdot \eta \left(1 + \frac{1}{\gamma} \right)$. When the opposite inequality holds, this expectation decreases with time. In terms of expectations, the crucial border value for the impatience rate ρ is no longer r but $\left(r + \frac{1}{2} \eta' \cdot \eta \left(1 + \frac{1}{\gamma} \right) \right)$ when a stock market is present.

As an alternative derivation of c_t^* , the stochastic differential equation (43) for the optimal consumption process is

$$dc_t^* = c_t^* \left(\frac{r - \rho}{\gamma} + \frac{1}{2} \frac{1}{\gamma^2} (\gamma + 1) \eta' \cdot \eta \right) dt + c_t^* \frac{1}{\gamma} \eta' \cdot dB_t, \quad (48)$$

from which it follows that c_t^* is a geometric Brownian motion. Notice that here it the risk tolerance function $T(c) = \frac{c}{\gamma}$. The "solution" to this stochastic differential equation is

$$c_t^* = c_0 e^{\frac{1}{\gamma} \left[(r - \rho + \frac{1}{2} \eta' \cdot \eta) t + \eta' \cdot B_t \right]}, \quad t \geq 0.$$

The initial value c_0 is finally determined by the budget constraint, and (46) again results. The dynamics of c_t^* will be used later in solving the optimal portfolio choice problem. \square

When stock market uncertainty is present, since $\gamma > 0$, the solution in this example tells us that when state prices π_t are low, optimal consumer is high, and vice versa. State prices reflect what the representative consumer is willing to pay for an extra unit of consumption; in particular is π_t high in times of crises and low in good times.

In real terms the result for pensions is as for optimal consumption in society at large: In times of crises the pensions are lower than in good times. This only explains the obvious, namely that society can only pay the pensioners that the economy can manage at each time. Insurance companies,

for example, pay the pensions from funds, which in bad times are lower than in good times. The government is similarly affected. Since pensions are, presumably, paid out to the whole generation of people above a certain age, it is in principle not possible to insure the entire society against crises and bad times. A single individual can of course find a strategy to hedge against low income in certain periods, and so can an insurance company by proper use of risk management, but this types of hedging will not work for the entire population, by *the mutuality principle*: In equilibrium everyone holds a non-decreasing function of aggregate consumption. If aggregate consumption in society is down, everyone is in principle worse off.

4.4 Pensions in nominal terms

Pensions (and insurance payments) are usually not made in real, but in nominal terms. There exist index-linked contracts, but these are still more the exception than the rule. In nominal terms the optimal consumption is $c_t^* \pi_t$.

For the preferences of Example 4, the nominal consumption/pension is given by the

$$c_t^* \pi_t = (\lambda e^{\rho t})^{-\frac{1}{\gamma}} \pi_t^{(1-\frac{1}{\gamma})}$$

Here $\gamma = 1$ is seen to be a border value of the relative risk aversion in the sense that for $\gamma > 1$ both optimal consumption and pensions in nominal terms are countercyclical. This can give rise to an *illusion* of being insured against times of crises.

People with $\gamma < 1$ experience no such illusion, since nominal amounts behaves as real amounts with respect to cycles in the economy. In the situation when $0 < \gamma < 1$ the agent is sometimes called *risk tolerant*.

This phenomenon is connected to another interpretation of γ . The quantity $\alpha = 1/\gamma$ is the *elasticity of intertemporal substitution*. If $\alpha < 1$ will an increase in income of 1% lead to a higher increase in consumption today than tomorrow. If $\alpha > 1$ the substitution effect will dominate, and consumption tomorrow increases the most. If $\alpha = \gamma = 1$ the income and the substitution effect will cancel out, and the consumption at the two time points will increase equally much.

Most people seem to have relative risk aversion larger than one, so $\gamma > 1$ when γ has this interpretation, and a value larger than 2 is found in most experimental situations. This could, perhaps, explain the impression that some people have¹, namely that in good times, everyone else "seem better

¹in particular many state employees

off'. First, this person's nominal consumption is low, second a larger part of the increased income will be consumed today, than the part invested for consumption later on. This is probably a reasonable description of how many people act, although the model is, admittedly, very simplistic.

The risk tolerant individual, of which there are fewer in society, are not subject to this distorted perception: In good times both his real and nominal consumption are high, and since $\alpha > 1$ the substitution effect will dominate, and a larger part of income increases is invested rather than consumed right away. One would, perhaps, think that the investment of income for later consumption is consistent with risk averse behavior, and thus be stronger when $\gamma > 1$, but this is not so. It should be mentioned that a reasonable value for α has been found to be close to 0.1 by some researchers.

A better description of this latter issue may, perhaps, be obtained if the elasticity of intertemporal substitution could be separated from the individual's risk aversion. There are several representations of preferences that accommodate this, like recursive utility, habit formation, Kreps-Porteus utility, Epstein-Zin utility, etc. We choose the simplicity and elegance of the separable and additive framework for the present presentation, except for one small deviation later on.

4.5 The connection to actuarial theory and insurance practice

In standard actuarial theory the nominal pension is nonrandom, at least is this the case in most textbooks on this subject. Referring to the above theory, this is only consistent with $\gamma = 1$ corresponding to logarithmic utility, the case when the substitution effect and the income effect cancel each other. In addition this theory commonly uses the principle of equivalence to price insurance contracts, where the state price density $\xi_t \equiv 1$. This implies that the agent is really risk neutral, so $\gamma = 0$ should follow. There seems to be an inconsistency inherent in this theory.

In insurance practice, which actuaries are engaged in, we can distinguish between two main types of contracts; (a) defined benefits, and (b) defined contributions. With regard to the first, before possible profit sharing the nominal value is usually constant, although as we have noticed, sometimes is the real value also constant. The latter case is not consistent with any finite value of γ . Attached to this contract is usually a return rate guarantee. Many life insurance companies are having difficulties with this guarantee in times when the stock market is down. Lately, in times of crises, this tends to go together with a low interest rate (like in the financial crisis of (2008,

-)) due to government interference. In such cases life insurance companies suffer twofold, and must rely on built-up reserves before, possibly, equity is being used.

Defined contribution contracts are actively marketed by the insurance companies. For such contracts the insurance customers take all the financial risk, only the mortality risk remains with the companies. Also such contracts have no rate of return guarantees, and function much like unit linked pension contracts. Thus the nominal, as well as the real pensions are state dependent, in accordance with the general theory outlined above. In neither case does a guaranteed return enter the optimal pension contract. A guarantee affects the insurance company's optimal portfolio choice plan. Typically, due to the nature of the guarantees and regulatory constraints, the companies are led to sell when the market goes down, and buy when the market rises, which is just the opposite of what is known to be optimal, at least under certain conditions, to be demonstrated in the next section.

Guarantees may seem attractive to customers, and insurers may decide to offer such contracts in order to be competitive. There are different reasons why such guarantees originated in the insurance business. In Norway for example, it became part of the legal terms of the contracts, more or less by an oversight, in times where the market interest rate was considerably higher than the 4% that was used in the premium calculations, and which the standard actuarial tables were based on.

In times of crises, defined benefit pension contracts seem most attractive to the customers, at least as long as they ignore the possibility that their insurance company may go bankrupt. In the crisis referred to above, many life insurance companies failed, and individuals all over the world lost parts of, or even their entire pensions. In times of rising stock prices, the defined contribution contracts seem more attractive for many individuals. What alternative the individuals find best may thus depend upon where in the business cycle an individual decides to retire.

In the life cycle model optimal consumption and pension insurance are intertwined and analyzed in one stroke, reflected in our analysis. In real life consumers are likely to separate the two. An optimal pension may then be regarded as an insurance against a bad state in the economy when the consumer becomes retired. Regarded this way a pension is considered as a minimum subsistence level when alternative forms of savings fail. With this in mind, defined benefits can be a rational contract, even if it does not follow from our simple analysis.

Insurance companies should, on the other side, be especially well equipped to take on market risk, since they normally have a long term perspective. This should enable them to obtain the risk premiums in the market, which

are after all averages.

4.6 Pensions versus ordinary consumption

In this section we demonstrate that with pension insurance allowed, the actual consumption at each time t in the life of the consumer is at least as large as the corresponding consumption when the possibility of "gambling" on own life length is not allowed, provided the value of life time consumption w is fixed. This demonstrates a very concrete effect of pooling.

To this end, consider the random, remaining life time T_x of an x -year old as we have worked with all along, and for comparison, the deterministic life length T , where $T = E(T_x) = \bar{e}_x$ is the expected remaining life time of an x -year old pension insurance customer.

We consider the situation with a CRRA-customer with general coefficient of relative risk aversion γ as in Example 4, and denote the value of life time consumption by w , i.e.,

$$\frac{1}{\pi_0} E\left(\int_0^{T_x} \pi_t c_t^* dt\right) = w.$$

Using (44) this can be written $\lambda^{-\frac{1}{\gamma}} \bar{a}_x^{(r_1)} = w$, or

$$\lambda^{\frac{1}{\gamma}} = \frac{\bar{a}_x^{(r_1)}}{w}, \quad (49)$$

where we have set $\pi_0 = 1$ without loss of generality. The corresponding value of life time consumption w for the deterministic time horizon T is determined by

$$\frac{1}{\pi_0} E\left(\int_0^T \pi_t c_t dt\right) = w,$$

where it is assumed that these two values are the same for the deterministic and the stochastic life times. In other words, in the two situations the budget constraints are the same. Again the optimal consumption/pension c_t is given in (44), however, the Lagrange multipliers determining the optimal consumption/pension are different in the two cases. In order to distinguish, we denote the optimal consumptions by c_t^* and c_t , respectively. The multiplier for the deterministic situation is determined by

$$\lambda_{(T)}^{-\frac{1}{\gamma}} \int_0^T e^{-r_1 t} dt = w,$$

using Fubini's theorem, which in actuarial notation is equivalent to

$$\lambda_{(T)}^{\frac{1}{\gamma}} = \frac{\bar{a}_{\overline{T}|}^{(r_1)}}{w}. \quad (50)$$

The function $\bar{a}_{\overline{t}|}^{(r_1)} = \int_0^t e^{-r_1 t} dt = \frac{1}{r_1}(1 - e^{-r_1 t})$ is convex in t , which means that $\bar{a}_x^{(r_1)} = E\left(\int_0^{T_x} \pi_t c_t^* dt\right) = E(\bar{a}_{\overline{T_x}|}^{(r_1)}) < \bar{a}_{\overline{T}|}^{(r_1)}$ by Jensen's inequality, since $T = E(T_x)$. By (49) and (50) this means that $\lambda^{\frac{1}{\gamma}} < \lambda_{(T)}^{\frac{1}{\gamma}}$, and using (44), since the state price density ξ_t is the same in both cases, it follows that for all states $\omega \in \Omega$ of the world is

$$c_t^* > c_t \quad \text{for all } w \text{ and for each } t \geq 0. \quad (51)$$

With pension insurance available the individual obtains a higher consumption rate at each time t that he/she is alive. This demonstrates the benefits from pooling when it comes to pensions.

4.7 Including Life Insurance

We are now in position to analyze life insurance in the problem formulation of this section. We assume that the felicity index u and the utility function v are as in Example 3 of Section 3.1: The problem can then be formulated as follows:

$$\max_{z, c \geq 0} E \left\{ \int_0^{T_x} e^{-\rho t} \frac{1}{1-\gamma} c_t^{1-\gamma} dt + e^{-\kappa T_x} \frac{1}{1-\psi} z^{1-\psi} \right\}$$

subject to

$$E \left\{ e^{-r T_x} W(T_x) \right\} \geq E \left\{ \pi_{T_x} z \right\},$$

where z is the amount of life insurance, here a random decision variable. The Lagrangian of the problem is:

$$\begin{aligned} \mathcal{L}(c, z; \lambda) = E \left\{ \int_0^T e^{-\rho t} \frac{1}{1-\gamma} c_t^{1-\gamma} \frac{l_{x+t}}{l_x} dt + e^{-\kappa T_x} \frac{1}{1-\psi} z^{1-\psi} \right. \\ \left. - \lambda \left[\pi_{T_x} z - \int_0^T (e_t - c_t) \frac{l_{x+t}}{l_x} dt \right] \right\}. \end{aligned}$$

The first order condition in c is:

$$\nabla_c \mathcal{L}(c^*, z^*; \lambda; c) = 0, \quad \forall c \in L_+$$

which is equivalent to

$$E\left\{\int_0^\tau ((c_t^*)^{-\gamma} e^{-\rho t} - \lambda \pi_t) c_t \frac{l_{x+t}}{l_x} dt\right\} = 0, \quad \forall c \in L_+$$

and this leads to the optimal consumption/pension

$$c_t^* = (\lambda e^{\rho t} \pi_t)^{-\frac{1}{\gamma}} \quad \text{a.s. } t \geq 0$$

as we have seen before in (44). The first order condition in the amount of life insurance z is:

$$\nabla_z \mathcal{L}(c^*, z^*; \lambda; z) = 0, \quad \forall z \in L_+$$

which is equivalent to

$$E\left\{((z^*)^{-\psi} e^{-\kappa T_x} - \lambda \pi_{T_x}) z\right\} = 0, \quad \forall z \in L_+ \quad (52)$$

Notice that both z^* and z are $\mathcal{F} \vee \sigma(T_x)$ - measurable. For (52) to hold true, it must be the case that

$$z^* = (\lambda e^{\kappa T_x} \pi(T_x))^{-\frac{1}{\psi}} \quad \text{a.s.}, \quad (53)$$

showing that the optimal amount of life insurance z^* is a state dependent \mathcal{F}_{T_x} - measurable quantity. If the state is relatively good at the time of death, the state price π_{T_x} is then low and $(\pi_{T_x})^{-\frac{1}{\psi}}$ is relatively high (when $\psi > 0$). Thus this life insurance contract covaries positively with the business cycle. In practice this could be implemented by linking the payment z^* to an equity index.

One can of course wonder how desirable this positive correlation with the economy is. With pensions we found it quite natural, but life insurance is something else. This product possess many of the characteristics of an ordinary, (non-life) insurance contact. In some cases it may seem reasonable that a life insurance contract is countercyclical to the economy, thereby providing real insurance in time of need. For this to be the result, however, the function v must be convex, corresponding to *risk proclivity* which here means that $\psi < 0$, but risk loving people do not buy insurance.

The expected value of z^* is found by conditioning: It is given by the formula

$$E(z^*) = \lambda^{-\frac{1}{\psi}} \int_0^\tau \exp\left\{\frac{1}{\psi}\left(r + \frac{1}{2}\eta' \cdot \eta\left(1 + \frac{1}{\psi}\right) - \kappa\right)t\right\} \frac{l_{x+t}}{l_x} dt. \quad (54)$$

For a given value of budget constraint (λ), this expectation is seen to be larger if $r + \frac{1}{2}\eta' \cdot \eta(1 + \frac{1}{\psi}) > \kappa$ than if the opposite inequality holds. As for pensions, in terms of expectation has the impatience cutt-off-point increased from r to $(r + \frac{1}{2}\eta' \cdot \eta(1 + \frac{1}{\psi}))$. In other words, not only the market interest rate r , but also the market-price-of-risk and the relative risk aversion of the function v determines what it means to be impatient, when a stock market is present.

Using the budget constraint with equality, we find an equation for the Lagrange multiplier λ ;

$$E\left\{\pi_{T_x} z^* - \int_0^{\tau} (e_t - c_t^*) \pi_t \frac{l_{x+t}}{l_x} dt\right\} = 0.$$

With an income of y up to the time n of retirement, and an optimal pension c_t^* thereafter as in (21), we obtain the equation

$$\lambda^{-\frac{1}{\psi}}(1 - r_2 \bar{a}_x^{(r_2)}) + \lambda^{-\frac{1}{\gamma}} \bar{a}_x^{(r_1)} = y \bar{a}_{x:\bar{n}|}^{(r)},$$

where

$$r_1 = \frac{\rho}{\gamma} + r(1 - \frac{1}{\gamma}) + \frac{1}{2}\eta' \cdot \eta(1 - \frac{1}{\gamma})\frac{1}{\gamma},$$

as in Section 4.3, and

$$r_2 = \frac{\kappa}{\psi} + r(1 - \frac{1}{\psi}) + \frac{1}{2}\eta' \cdot \eta(1 - \frac{1}{\psi})\frac{1}{\psi}.$$

In the special situation where $\kappa = \rho$ and $\psi = \gamma$ so that $u = v$, it follows that $r_1 = r_2$ and

$$\lambda^{-\frac{1}{\gamma}} = \frac{y \bar{a}_{x:\bar{n}|}^{(r)}}{(1 + (1 - r_1) \bar{a}_x^{(r_1)})}.$$

It is at this point that pooling takes place in the contract. In this situation the optimal consumption/pension is given by

$$c_t^* = \frac{y \bar{a}_{x:\bar{n}|}^{(r)}}{(1 + (1 - r_1) \bar{a}_x^{(r_1)})} e^{((r-\rho)/\gamma)t} \xi_t^{-\frac{1}{\gamma}}, \quad (55)$$

and the optimal amount of life insurance at time T_x of death of the insured is

$$z^* = \frac{y \bar{a}_{x:\bar{n}|}^{(r)}}{(1 + (1 - r_1) \bar{a}_x^{(r_1)})} e^{((r-\rho)/\gamma)T_x} \xi_{T_x}^{-\frac{1}{\gamma}}. \quad (56)$$

One could, perhaps, say that these contracts represent an "innovation" in life insurance theory.

If large parts of the population buys life insurance products, a positive correlation with the business cycle seems like a natural property, and is really the only one that is economically sustainable. Unlike pension insurance, however, life insurance is a product that not everybody seems to demand. We can single out two different family situations where life insurance is of particular interest. The first concerns a relatively young family with small children. Then one of the parents, usually the wife, can not work full time, which means that the other is the main provider. If this person dies, in for example an accident, this is of course dramatic for this family. Life insurance then plays the role of substitution for part of the loss of a life time income. As can be seen from (56), is the insured amount proportional to the present value at time zero of life time income $y\bar{a}_{x:\bar{n}|}^{(r)}$. If death comes early, T_x is relatively small so the factor $e^{((r-\rho)/\gamma)T_x}$ is close to one.

The other situation is the traditional one attached to the bequest motive, usually meaning that an older person wants to transfer money to his or her heirs. The social need for this insurance seems less obvious than in the first situation described. Here the factor $e^{((r-\rho)/\gamma)T_x}$ may be large for the patient life insurance customer, implying a large insured sum to the beneficiaries. Despite of the all the good reasons for a life insurance contract for the young family, its seems far less widespread than life insurance with the bequest motive, which is ironic.

In climate problems the bequest idea could be interesting in the following sense. By paying a premium (e.g., by reducing consumption) today, one may "roll over" a more sustainable society to future generations by "inter-personal transfers". This is discussed further elsewhere (e.g., Aase 2011b).

One objection to the optimal solutions (53) and (56) is that the amount payable has not been subject to "enough pooling" over the individuals. The pooling element is present, since it is used in the budget constraints, but the amount payable is here crucially dependent on the actual time of death T_x of the insured, which is unusual in life insurance theory.

One alternative approach is to integrate out mortality in the first order condition (52). Notice that this is strictly speaking not the correct solution to the optimization problem, but must instead be considered as a suboptimal pooling approximation. This results in the following approximative first order condition:

$$E_{z,z^*} \left\{ \left((z^*)^{-\gamma} (1 - \rho \bar{a}_x^{(\rho)}) - \lambda \int_0^\tau \pi_t f_x(t) dt \right) z \right\} = 0, \quad \forall z,$$

assuming again that $\kappa = \rho$ and $\psi = \gamma$. The solution to this problem also a

random variable, and given by

$$\bar{z}^* = \left(\frac{\lambda \int_0^\tau \xi_t e^{-rt} f_x(t) dt}{1 - \rho \bar{a}_x^{(\rho)}} \right)^{-\frac{1}{\gamma}} \quad \text{a.s.} \quad (57)$$

However, this contract is seen to depend on the state of the economy from time 0 when the insured is in age x , to the end of the insured's horizon τ . At time of death $T_x (< \tau)$ this quantity is not known, which is a consequence of our approximative procedure. Ignoring this information problem for the moment, by employing the budget constraint, the Lagrange multiplier λ is found as

$$\lambda^{-\frac{1}{\gamma}} = \frac{y \bar{a}_{x:\bar{n}|}^{(r)}}{\bar{a}_x^{(r1)} + \frac{E[(\int_0^\tau \xi_t e^{-rt} f_x(t) dt)^{(1-\frac{1}{\gamma})}]}{(1-\rho \bar{a}_x^{(\rho)})^{-\frac{1}{\gamma}}}} \quad (58)$$

Inserting λ from (58) into (57), the suboptimal insured amount results.

When stock market uncertainty goes to zero, i.e., when $\xi_t \rightarrow 1$ a.s., \bar{z}^* converges to the corresponding contract of Section 3.1 when only biometric risk is present.

We can derive an insured amount z^{**} that is consistent with the information available at time of death of the insured as the following conditional expectation

$$z^{**} := E\{\bar{z}^* | \mathcal{F}_{T_x}\}.$$

This is a random variable at the time when the life insurance contract is initialized, and an observable quantity at the time of death of the insured, and thus solves the information problem mentioned above, but is otherwise, of course, somewhat ad hoc.

Note that such a contract would benefit the young family in the case of early death of the provider, since those who die early are subsidized by those who live long when the insured sum is subject to enough averaging.

The advantage with this contract is that it takes into account pooling over life contingencies at two stages of the analysis. Furthermore it is consistent with the standard analysis when there is "no market risk in the limit".

5 The optimal portfolio choice problem

We have barely touched upon the portfolio choice problem in Section 4.1, but could there proceed without really having to solve it. This is due to the fact that in the model that we discuss, we may separate the the consumer's portfolio choice problem from his or her optimal consumption choice. In the present section we do solve the investment problem explicitly. For this we

need the agent's net wealth W_t at time t . For the general CRRA-consumer it is given by

$$W_t = \frac{1}{\pi_t} E_t \left\{ \int_t^{T_x} \pi_s c_s^* ds \right\} = \frac{1}{\pi_t} E_t \left\{ \int_t^{T_x} \pi_s^{(1-\frac{1}{\gamma})} \lambda^{-\frac{1}{\gamma}} e^{-\frac{\rho}{\gamma} s} ds \right\},$$

where we have used (44). Here E_t means conditional expectation given the information filtration $\mathcal{F}_t \vee (T_x > t)$, i.e., given the financial information available at time t and the fact that the individual is alive then. Recalling that at time t the agent is in age $x+t$, we get, using Fubini's theorem

$$W_t = \frac{1}{\pi_t} \lambda^{-\frac{1}{\gamma}} \int_t^{\tau} E_t(\pi_s^{(1-\frac{1}{\gamma})}) e^{-\frac{\rho}{\gamma} s} \frac{l_{x+s}}{l_{x+t}} ds.$$

The conditional expectation appearing in the integrand is computed as follows:

$$\begin{aligned} E_t(\pi_s^{(1-\frac{1}{\gamma})}) &= E_t\left(\pi_t^{(1-\frac{1}{\gamma})} e^{(1-\frac{1}{\gamma})(-r-\frac{1}{2}\eta'\cdot\eta)(s-t)+(1-\frac{1}{\gamma})\eta'\cdot(B_s-B_t)}\right) = \\ &\pi_t^{(1-\frac{1}{\gamma})} e^{-[(1-\frac{1}{\gamma})r+\frac{1}{2}\frac{1}{\gamma}(1-\frac{1}{\gamma})\eta'\cdot\eta](s-t)}, \end{aligned}$$

where we have used the lognormal representation for the state price π and the moment generating function of the normal distribution. This gives for the wealth process

$$W_t = \pi_t^{-\frac{1}{\gamma}} \lambda^{-\frac{1}{\gamma}} e^{-\frac{\rho}{\gamma} t} \bar{a}_{x+t}^{(r_1)} = c_t^* \bar{a}_{x+t}^{(r_1)}, \quad (59)$$

where r_1 is as given in Section 4.3. This shows that the wealth at any time t in the life of the consumer, who is then in age $(x+t)$, is equal to the actuarial value of receiving the optimal consumption c_t^* per time unit for the rest of his or her life, discounted at the rate r_1 . For logarithmic utility, $r_1 = \rho$ the subjective interest rate; when $\gamma \neq 1$ this discount rate depends on the volatility of the state prices, or the market price of risk η , ρ , r as well as of γ . In fact, r_1 can be interpreted as the a risk adjusted return rate.

Using the dynamics for c_t^* given in (48), by Ito's lemma we obtain the following dynamic representation for the wealth W_t :

$$dW_t = \mu_W(t)dt + \frac{1}{\gamma} W_t \eta' \cdot dB_t,$$

for some drift term $\mu_W(t)$. Comparing this to the intertemporal budget constraint (32) of Section 4.1, we may apply diffusion invariance to determine the the optimal fractions $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \dots, \varphi_t^{(N)})$ of total wealth held in

the risky securities at each time t . By equating the two diffusion terms, we obtain that

$$\frac{1}{\gamma}\eta' = \varphi_t \cdot \sigma.$$

and recalling that $\sigma\eta = \nu$, it follows from this that the optimal investment fractions are

$$\varphi = \frac{1}{\gamma}(\sigma\sigma')^{-1}\nu, \tag{60}$$

where ν , with components $\nu_n = \mu_n - r$, $n = 1, 2, \dots, N$, is the vector of risk premiums for the N risky securities. These ratios are all seen to be constants, meaning that they do not depend upon the age ($x + t$) of the investor, the state of the economy π , or on the investor's death intensity μ_{x+t} .

This result is the same as the one found by Mossin (1968), Samuelson (1969) and Merton (1971) without pension insurance present. A random time horizon simply does not alter this result.

The formula (60) basically tells us that when prices of stocks increase, it is optimal to sell, and when prices fall it is optimal to buy. From an insurance perspective companies are often led to do the opposite, as we have mentioned before, which is of course unsatisfactory.

One immediate objection to this result is that the optimal strategy does not depend upon the investor's horizon, or put differently, is independent of the investor's age ($x + t$) at the time of investment. This is, however, against empirical evidence, and also against the typical recommendations of portfolio managers. The typical advice is that as the horizon gets shorter, the investor should gradually go out of equities, and thus take on less financial risk.

One of the reasons for the advice that younger people should hold a higher fraction in equities is the tendency for stocks to outperform bonds or bills over the long run, despite the higher stock market volatility. This should not be mistaken as a "time diversification" advice, which is a different but related issue, typically arising after each down-turn in the stock market (e.g., Delong (2008), Bodie (2009)). For example, following the 2008/09 market crash it is evident that many people around the world have lost their pensions, partly or entirely. For many old people it seems obvious that they have too short remaining lifetimes to regain what has been lost.

Paul A. Samuelson has explained, in many articles over the years, what is wrong with time diversification. In Samuelson (1989) for example, he demonstrates that under the standard assumptions of the financial market, the optimal portfolio strategy based on maximizing expected utility of consumption over the investor's lifetime, beats various buy-and-hold strategies by clear margins. The standard assumptions are: 1) asset returns are i.i.d., 2) agents have additively separable constant relative risk aversion (CRRA)

utility, 3) agents have no non-tradeable assets, and 4) markets are frictionless and complete. If portfolio choice is going to depend on age and/or on wealth, then one or more of these standard assumptions must be relaxed. Aase (2009) has discussed this problem by a slight reformulation of assumption 2), which we discuss next.

5.1 The horizon problem

In this part we examine the effect of horizon and wealth on portfolio choice. We assume that the felicity index $u(x, t)$ satisfies the following

Assumption 1

$$u(x, t) = \begin{cases} \frac{1}{1-\gamma(t)} x^{(1-\gamma(t))} e^{-\rho t}, & \text{if } \gamma(t) \neq 1; \\ \ln(x) e^{-\rho t}, & \text{if } \gamma(t) = 1. \end{cases} \quad (61)$$

where $\gamma : [0, \tau) \rightarrow R_+$ is a continuous and strictly positive function of time.

Notice that in this case $u(x, t)$ is not time and state separable, but this is the only relaxation of the standard assumptions 1) - 4) that is done. Using this assumption, Aase (2009) shows that under Assumption 1 the optimal fractions in the risky assets are

$$\varphi(t) = \frac{1}{\gamma(\tilde{t}_t)} (\sigma \sigma')^{-1} \nu, \quad (62)$$

where \tilde{t}_t is an \mathcal{F}_t -measurable random time satisfying $\tilde{t}_t \in (t, \tau)$. It is determined at each time t by the equation

$$\gamma(\tilde{t}_t) = \frac{\int_t^\tau g(s, t) ds}{\int_t^\tau g(s, t) \frac{1}{\gamma(s)} ds} := \frac{W(t)}{Y(t)}. \quad (63)$$

Here $W(t)$ is the agent's optimal wealth at time t , given by equation

$$W_t = \int_t^\tau (\lambda e^{\rho s})^{-\frac{1}{\gamma(s)}} \pi_t^{-\frac{1}{\gamma(s)}} \exp \left\{ - \left(r + \frac{1}{2} \frac{1}{\gamma(s)} \eta' \cdot \eta \right) \left(1 - \frac{1}{\gamma(s)} \right) (s - t) \right\} \frac{l(x + s)}{l(x + t)} ds. \quad (64)$$

Notice that when the function $\gamma(t) \equiv \gamma$, then the wealth in this equation becomes the same as the wealth in (59), as the case should be. Clearly the quantity $Y(t)$ can be computed from the expression for W_t in (64) and the function $\gamma(t)$.

	Expectation	Standard deviation
Consumption growth	1.83%	3.57%
Return S&P500	6.98%	16.67%
Government bonds	0.80%	5.67%
Equity premium	6.18%	16.54%

Table 1: Key numbers for the time period 1889-1978

The consequences of this result are several, and the above reference gives the details. Here we only point out that if the risk aversion function $\gamma(t)$ is increasing with time, then this result implies that individuals should invest more in the risky asset when they have a longer horizon, i.e., when they are young, and gradually move into bonds as they grow older. This is then in agreement with both advice from investment professionals, and with empirical studies of actual behavior.

It seems natural, with this assumption, that the investor should pick some average time in the remaining horizon when deciding on today's portfolio choice.

5.2 A second portfolio choice puzzle

In connection with the optimal portfolio choice result (60), there is also another empirical puzzle. Mehra and Prescott (1985) studied consumption and market data in the US-economy for the period of 1889-1978. The data are summarized in Table 1. Newer data are of course somewhat different, but the main conclusions remain.

Using the data of Table 1, for a relative risk aversion of around two, the optimal fraction in equity is 132% based on the standard, first term in (65) (when $el_W(c_t^*) = 1$). In contrast, depending upon estimates, the typical household holds between 6% to 20% in equity. Conditional on participating in the stock market, this number increases to about 40% in financial assets.

Based on an equilibrium model with production and capital stock (K), and a non-linear production function, Aase (2011) shows, among other things, that the equilibrium demand for the risky asset is given by

$$\varphi_t = \left(- \frac{u_c(c_t^*)}{u_{cc}(c_t^*)c_t^*} \right) \frac{1}{el_W(c_t^*)} \frac{\mu_R - r}{\sigma_R \sigma_R} - \frac{el_K(c_t^*)}{el_W(c_t^*)} \frac{\sigma_R \sigma_K}{\sigma_R \sigma_R}, \quad (65)$$

where σ_K is the volatility of capital, σ_R is the volatility of the return on equity, μ_R is the expected rate of return on equity, and $el_W(c_t^*)$ and $el_K(c_t^*)$ are partial consumption elasticities with respect to wealth and capital stock, respectively.

The first term is seen to be the solution corresponding to (60), reformulated to the present situation, in the case when $el_W(c_t^*) = 1$.

Implied by results in Aase (2011), the observed risk premium $(\mu_R - r) = 6.18$ and the observed value for the short rate $r = 0.0080$ of the last century follow from the production model for a value of the relative risk aversion of $\gamma = 2.27$, provided the investors only use the first term in (65), with $el_W = 1$.

The last term in (65) reflects the investor's demand for the risky asset to hedge against unfavorable changes in technology. If the investors take into account also information conveyed by the real economy, they find that the stock market may have appeared more risky than it really was. The consumption based capital asset pricing model still holds in the production economy, which implies that the risk premium should have been closer to 1% than to the observed 6% of the last century for a reasonable value of the relative risk aversion γ .

If this explanation holds, from the first term alone, φ is down to 20% in equities, for $\gamma = 2.27$, $\mu_R - r = 0.013$ when $el_W = 1$. The last term in (65) further adjusts this number in the right direction.

Conditional on these results, also the second portfolio choice puzzle can be explained. Of course, the above mentioned paper aims at solving two other celebrated puzzles; namely the *equity premium puzzle*, and also the *risk free rate puzzle*. If we accept the results of that paper, all these puzzles are more or less explained.

6 Discussion and extensions

The life cycle model is analyzed in two steps; first with only a credit market and mortality risk, then with a securities market added. The analysis provides an optimal demand theory from the point of view of the consumers, who are also life and pension insurance customers. We have derived several conclusions from the analysis, some with more predictive power than others, which we now summarize.

The first result was related to the optimal consumption path in the situation with only a credit market. When there is life time uncertainty, the optimal consumption paths are shown to be crucially dependent on the impatience rate. The impatient consumer ($\rho > r$) must always look forward to an ever decreasing optimal consumption, since $dc_t^*/dt = (r - \rho)T(c_t^*)$. The patient agent ($\rho < r$), on the other hand, can look forward to an ever increasing optimal consumption.

While this gives an interesting and intuitive interpretation of the impatience rate, or the subjective interest rate ρ , it is not likely to give reliable

predictions. With a securities market included this property is diluted, by both a new addition to the drift term and a diffusion term. In particular the latter will dictate consumption paths to deviate from the simple, deterministic description given above. This opens up for interpersonal comparisons of consumption behavior at the same time in agents' life cycles. Impatience is more naturally discussed in terms of expectations when a stock market is present, in which case both the market-price-of-risk and the relative risk aversion must be taken into account when characterizing this property.

The optimal pensions in these two situations differ only by a random factor with a stock market included. This factor is reciprocal to the state price density, a fact which was found to have several interesting implications. In particular the optimal pensions are found to be positively correlated with the economy in the sense that when stock prices are high, the pensions are also high, and vice versa. This is a quite natural property, in particular for the aggregate economy, since such a consumption pattern is consistent with what the economy can deliver.

The discussion of nominal pensions revealed a weakness with the additive and separable framework of preference representations that this paper is build on. The reason is that the parameter γ has two different, and sometimes conflicting interpretations.

We have a strikingly simple demonstration of the advantages of pooling with regard to pensions. It is shown that, with the same economic resources, the optimal yearly pension is strictly larger with pooling, than without. This shows the mutuality idea is still fruitful, a fact that is worth a reminder, in particular since we live in a time of individualism, seemingly picturing a world in which we are solely responsible for our own successes and failures.

Optimal life insurance, where the insured amount is endogenously determined, is analyzed, and its properties are found reasonable. Like pension insurance, also the insured amounts in life insurance are co-cyclical with the economy.

It should be pointed out that we know little about the specification of the function v , when it serves a bequest motive, as compared to u . Life insurance is after all only a financial tool for controlling inter-personal transfers, which necessitates references to the theory of transfers (like e.g., Bernheim, Schleifer and Summers (1985)). We show that if the insured amount is to be countercyclical to the economy, and thus be a bona fide insurance against tough times for the beneficiaries, this requires *risk proclivity* of the bequest function v . This effectively rules out this possibility. A countercyclical insured amount does not appear directly irrational in a finance setting, but risk proclivity does.

We compared our results to both actuarial theory and insurance practice.

With regard to pensions it was found that defined contribution plans are most in line with the optimal contracts found in this paper. These are also the type of contracts that insurers seem most comfortable with at the present time. The argument commonly used by corporate officers is that for these contracts the financial risk is held by the customers. However, insurers should be aware of the fact that if they are unwilling to take risk, risk premiums in the business will be low, and consequently, so will profits. If insurance companies avoid financial risk altogether, they can only expect a return rate equal to the risk-less rate on government bonds, which may not impress the owners. This holds if financial theory works in practice the way it is supposed to. Since insurance companies normally have a long term perspective, they should be especially well suited to take risk, and be able to earn the equilibrium risk premiums in the long run.

Much has been written about the recent financial crises of 2008. One major criticism of the financial industry that has been put forth is that the banks were eager to collect fees for their services, by inventing all kinds of products that were difficult to understand for ordinary customers. Such fees can not be directly considered as a compensation for risks, and were the basis for bonuses to the CEO's and other leaders in the industry. As long as prices went up, this worked, but as soon as confidence in the system started to fail, the collapse came partly as a consequence of failed risk management, and, too low equity, among other factors.

With defined contribution products, the insurers' equity can be kept low, and the return on equity can only be made high provided the insurers are clever in collection fees from the customers. If the products are largely standardized, competition should bring down these fees, and also the profit margins for the insurers. For this reason insurers are sometimes ingenious in tailor making products to customers, where terms are opaque and difficult to compare.

In some countries there are state guarantees issued for individual pensions. As with banks, where the government has a stake because it insures deposits, the reason is to preserve the stability of the financial system, which is important to preserving the stability of the economy. If such an insurance company gets into a situation of distress, the government has to come in to honor its commitments to the insurance customers, which can be done by conservatorship. Because of the importance of thrust between the population and the life insurance industry, it is common that life insurance companies in distress are taken over by other companies in the industry. If instead, as happened in the 2008-09 crisis in the US, the government chooses to provide funds to the financial firms with no strings attached, this may distort both risk management in the future, as well as pricing of the products, due to

moral hazard.

Financial firms trading in derivatives may access unbounded liability exposures and are granted limited liability. Under such circumstances an all equity firm holds a call option, whereby it receives a free option to put losses back to the taxpayers (e.g., Eberlein and Madan (2010)). In such a situation increasing volatility increases the value of both assets and the liabilities, thereby creating perverse incentives.

It is essential that the financial industry and the population at large learn from this, so that future crises become less severe. In order for the relevant requirement on equity and reserves to be appropriate, both incentives must be aligned with societal goals, and governments must get in place a proper regulatory regime that works.

Finally the paper discusses optimal portfolio choice strategies. This culminates with the formula (60), characterizing the optimal plan. As with all simple formulas, there are pros and cons. The advantage is the simple logic this formula conveys, the drawback is that it is framed in a very simple model of a complete, frictionless financial market, which is, perhaps often taken too literally. One particular assumption about this market is that the investment opportunity set is constant. When this is not the case, as in the real world, other state variables must be taken into account. We refer to an article where this was done indirectly through a production economy, instead of using only a pure exchange economy that is most common when analyzing such questions. When the state variables are capital and labor, the message is that provided the information about these quantities is being utilized, the picture changes and the above formula does not capture all the key elements of the risks. The investors have in reality hedging possibilities related to the "real" economy, and when these are properly evaluated, the stock market may not appear quite so risky as it was perceived to be during the last century. These insights are then used to explain a portfolio choice puzzle, as well as the equity premium puzzle.

Another weakness with the theory of optimal portfolio choice is related to the "horizon problem". Here we make the only deviation from the additive and separable preference representation: We relax the separability of state and time in the felicity index $u(x, t)$. This can be used to explain observed behavior, namely that as investors grow older, they invest a larger proportion of their wealth in government bonds.

6.1 Longevity and cohort risk.

We round off by discussing some issues that does not directly come as a result of the analysis, but which are related to problems commonly discussed

in connection with pensions.

In comparing longevity risk with cohort risk, it is tempting to dismiss the latter as not being of such fundamental importance as the former. By cohort risk is meant that some periods have larger numbers of retired people than other periods. This is a transient phenomenon that will eventually pass away, and not a structural one, as longevity risk. Of course, when these two risks materialize at the same time, this causes extra problems for any nation's welfare programs. This seems to be the case in many western countries when the large broods borne right after World War II become pensioners. In addition these cohorts tend to live longer than the generations before them.

In some countries the actuarial tables are modified every year, like in Canada, in other countries the same tables as were constructed in 1963 were still used in 2009, like in Norway. The theory in this paper assumes that the tables capture the real mortality risk, and pooling works so that there is no economic risk premium associated with mortality. As long as the proper measures have been taken regarding reserving for longevity risk, there should be few problems for the private insurance industry with respect to either of these two types of risk.

For government welfare programs, the situation is of course different. Many developed countries have a social security system that pays a basic pension to its citizens. This is usually independent of what the individuals have arranged in terms of pensions from the insurance industry. In Norway, for example, the country that I know best, the government pensions are determined by the principle of "pay as you go". For those only acquainted with the premium reserving of private or mutual insurance companies, this may not look like a sound principle. In the parliament (Stortinget) the politicians determine a basic amount each year, called one G upon which the pensions are based. In 2010 the size of $G = \text{NoK } 75.641$, corresponding to USD 13.000. The more registered work effort an individual has put in, and the higher the salary, the higher the pension. First let us consider the incentives: By and large this arrangement means that the daughters and sons of the beneficiaries determine the benefits. Thus the "weak" part - the pensioners - seem protected, or they get what they have deserved. Second, what about economic sustainability? Since all pensions are determined from the basic amount G , by making this amount state dependent, matters can be arranged such that the nation each year pays the pensions it can afford.

In practice, to set G lower one year than the previous year may require a great deal of political determination and courage, which means that the system represents no guarantee that the nation will not consume beyond its means. Here rules rather than discretion may be the solution.

In addition to this basic pension from the government, and possible pri-

vate pensions with the insurance industry, in many countries there are pensions also from the employers. These collective pensions are usually arranged between the employers and private insurers. The pensions depend upon how long an employee has been with the company, and what the salary has been, and the premium reserve moves with the worker as he or she changes jobs.

The two types of risk, longevity risk and cohort risk, are problematic for governments' welfare schemes. One solution has been pointed out in a recent report². By increasing the pensionable age by a few years, the projected increase in the state's pension expenses may be mitigated. In particular this report claims that by increasing the pension age by two years, this increases the state's income of about four per cent of GDP. For an average working period of 40 years, an increase of two years means that the total work effort is society has increased by five per cent. In other words, society can become five per cent richer if people work two more years.

This suggestion has of course its weaknesses, since for once it "assumes away" unemployment, which is not negligible in many western countries. It is therefore also likely to be controversial. That it is politically difficult, we know from protests and demonstrations in 2010 in countries like Greece, Ireland, France, Portugal, Spain, etc. However, it is no secret that some countries seem to have more "slack" than others. As an illustration, in Table 2 is shown the employment frequency for people between 60 and 64 years for a number of European countries and the USA. It starts at about 7% in Austria, goes via 40% in the USA and ends with 58% in Sweden and Norway.

Freq.	5	12	12	21	22	31	35	40	43	58	58
Nation	Au	Fr	It	Sp	Ger	Gree	Den	US	UK	Swe	Nor

Table 2: Employment frequency in per cent, 60-64 years. Source: Eurostat.

The official and the real pension age also vary across the European countries, highest in Iceland with 67 and 66 years, and lowest in France with 60 and 59 years, respectively. The problems with longevity and cohort risk are thus seen to have both macro, public, and political economic perspectives.

7 Summary

In the paper we analyzed the "life cycle model" in a modern setting, by first introducing a credit market with biometric risk only, and then market risk via risky securities. This approach enabled us to investigate pension

²<http://www.dn.no/forsiden/borsMarked/article2029034.ece>

and life insurance step by step. We found optimal pension plans and life insurance contracts where the benefits were state dependent. We compared these solutions both to the ones of standard actuarial theory, and to policies offered in practice.

We discussed two related portfolio choice puzzles in the light of recent research, one is the horizon problem, the other is related to the aggregate market data of the last century where theory and practice diverge, and suggested resolutions to these problems.

Finally we presented some comments on longevity risk and cohort risk, and found that these problems are, perhaps, best analyzed in the perspective of macro, public, and political economics.

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