A Perturbation Approach on a Class of Optimal Control Problems, Unifying the Pontryagin and Dynamic programming Approach.

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Abstract: This article consider special ways of solving time dependent (nonautomus) systems. Different types of time-dependent *Optimal Control Problems* in the setting of a Hamiltonian formulation, are considered and the emphasis is on analytic solutions by expansion techniques. Such analytic solutions serve as very powerful tools in the testing and verification procedure for numerical schemes attacking more general types of problems.

Also the Unification of the Pontryagin and Dynamic programming Approach is discussed.

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1 Introduction

In this paper we consider time dependent (non autonomus) systems. These are firstly restricted to a class of problems having the explicite time dependency described by a discount factor or *current value multiplier* of the usual exponential form used in economic applications. Secondly we also consider more general forms with explicite time dependency.

We consider problems connected to management of renewable resources, for example fish. In practical terms we often have the situation that observed data of the amount of fish is the basic reason for action. Thus the observed estimated amount of available fish determine the amount of harvest. In this context time is not an interesting parameter. Therfore one may simplify the problem by asking for solutions in phase-space instead of configuration space, that is one is happy when obtaining a pure feedback form of the solution, where the harvest or control u are completely determined by knowledge of the fish (stock) y. At which point in time this happens is really uninteresting for many situations.

The first approach will be simple and serve as a guideline for the next steps.

2 An elementary approach

2.1 General considerations

Consider a problem with the following state equation

$$\dot{y} \overset{def}{=} \frac{d}{ds}y(s) = g(s, y(s), u(s)), \; y \in \Omega \subset \mathbb{R} \text{ and } u \in U \subset \mathbb{R} \tag{1}$$

$$y(s = t) = x, \quad s \in \mathbb{R}^+, \quad g : \{A \times \Omega \times U \to \mathbb{R}\}. \tag{2}$$

We have given a utility flow function (objective function) f subject to a control, where U and Ω are closed intervals. Time is in $A = [0, T]$ where $T = \infty$ is a possibility. We define the value function, that we seek to maximize

$$V(x, t) = \max_{u \in U} \int_t^T f(s, y(s), u(s)) ds, \tag{3}$$

where $f : \{A \times \Omega \times U \to \mathbb{R}\}$. The horizon T, has a predetermined value or $T = \infty$.

In order to avoid unnecessary complications, we consider the situation where f, g and u are sufficiently regular for the Pontryagin Maximum Principle to apply[1]. A Hamiltonian for this problem with a multiplier function (Lagrangian) λ is given by

$$H(s, y, u, \lambda) = f(s, y(s), u(s)) + \lambda\, g(s, y(s), u(s)), \tag{4}$$

and the resulting Hamiltonian canonical equations are given as

$$\dot{y} = \frac{\partial H}{\partial \lambda} = g(s, y(s), u(s)), \tag{5}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial y} = -f_y(s, y(s), u(s)) - \lambda\, g_y(s, y(s), u(s)), \tag{6}$$

$$u^* = \operatorname*{argmax}_u H, \tag{7}$$

where $u = u^*$ is the optimal value of the *control* variable u. For the case of an internal maximum Eq. (7) simplifies to

$$\frac{\partial H}{\partial u} = 0 = f_u(s, y(s), u(s)) + \lambda\, g_u(s, y(s), u(s))\,. \tag{8}$$

Notice that Eq. (7) applies also when the extremum is not an internal point in the control space U.

The equations above results from a general theory in variational computation [1]. In the following we consider a problem with a constant discount rate r, and for convenience we introduce F and m by

$$F(s, y(s), u(s)) \stackrel{def}{=} e^{rs} f(s, y(s), u(s))\,, \tag{9}$$

$$m(s) \stackrel{def}{=} e^{rs}\lambda(s) \quad \Rightarrow \quad \lambda(s) = e^{-rs} m(s)\,. \tag{10}$$

It is convenient to define a new Hamiltonian density by

$$\mathcal{H} \stackrel{def}{=} F(s, y(s), u(s)) + m(s)\, g(s, y(s), u(s)) = e^{rs} H(s, y, u, \lambda)\,. \tag{11}$$

This way we obtain

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial m} = g(y, u)\,, \tag{12}$$

$$\dot{m} = r\, m - \frac{\partial \mathcal{H}}{\partial y}\,, \tag{13}$$

$$u^* = \operatorname*{argmax}_u \mathcal{H}\,, \tag{14}$$

where again $u = u^*$ is the optimal control. For the case when the extremum is an internal point in the control space U [1], as previously mentioned, Eq. (14) above can be replaced by

$$\mathcal{H}_u = F_u + m\, g_u = 0\,. \tag{15}$$

From Eq. (15) we then find

$$m(s) = M(y, u) \stackrel{def}{=} -\frac{F_u}{g_u}\,, \tag{16}$$

where we have introduced M as a new functional form of m. We shall refer to this as Problem A. We notice that $g_u = 0$ in Eq. (15) also implies that $F_u = 0$, so this case needs special attention, and Eq. (16) as an equation defining $M(y, u)$ is not obvious although the limit may still exist. We consider this case in some more detail by observing the following fact:

We factor out a function $\eta(y, u)$ which is defined in such a way that at least one of the functions $\hat{f}_u$ and $\hat{g}_u$ (see below) are different from zero for $u \in U$ and $y \in \Omega$. Thus we claim that we may write

$$F_u(y, u) + m\, g_u(y, u) = \eta(y, u)\{\hat{f}(y, u) + m\, \hat{g}(y, u)\}\,. \tag{17}$$

[1] $H_u = 0$ applies to more general cases than internal maxima. See Seierstad and Sydsæther [3] Note 3 p 86

Then defining the new functions: $h_u(y,u) \stackrel{def}{=} \eta(y,u)$, $\tilde{F}_h(y,h(y,u)) \stackrel{def}{=} \hat{f}(y,u)$ and $\tilde{g}_h(y,h(y,u)) \stackrel{def}{=} \hat{g}(y,u)$. We restrict ourselves to cases covered by the following form:

$$\mathcal{H} \stackrel{def}{=} \tilde{F}(y,h(u,y)) + m\,\tilde{g}(y,h(u,y))\,, \tag{18}$$

from which we obtain

$$\mathcal{H}_u = h_u(\tilde{F}_h + m\,\tilde{g}_h)\,, \tag{19}$$

and $\mathcal{H}_u = 0$, is satisfied by $h_u(y,u) = 0$, regardless of m. First, consider the special case, $h(y,u) = h_0(u)$, where $h_0(u)$ is a continuous function. In this case it is natural to switch to the new control variable $\tilde{u} \stackrel{def}{=} h_0(u)$. In the new setting the new Hamiltonian results in Problem A, Eq. (16). Notice that in this case the region of definition U is simply transformed to a new closed interval.

The general case may formally be rewritten by introducing a new control $\tilde{u} \stackrel{def}{=} h(y,u)$ where $\tilde{u} : \Omega \times U \to \tilde{U}$, which is a closed interval. In this way we can reformulate this problem and observe that it again reduces to Problem A, Eq. (16).

In the following we shall therefore restrict ourselves to problems having internal maxima [2] and focus on the straight forward case.

Then since F and g are not explicitly dependent on time, it is convenient to define a function $P(y,u)$, which by using Eqs. (11) and (16) may be written as

$$P(y,u) \stackrel{def}{=} \mathcal{H}(y, m = M(y,u), u) = F - \frac{F_u}{g_u} g = -\frac{g^2}{g_u}\frac{\partial}{\partial u}\left(\frac{F}{g}\right) = \frac{(\frac{F}{g})_u}{(\frac{1}{g})_u}\,. \tag{20}$$

For the case of an internal optimum we have $H_u = 0$ and in addition Hamilton's canonical equations results in the familiar equation

$$\frac{dH}{ds} = \frac{\partial H}{\partial s} = f_s(s, y(s), u(s)) + \lambda\, g_s(s, y(s), u(s))\,. \tag{21}$$

In the new variables this equation reads

$$\dot{\mathcal{H}} = \mathcal{H}_s + r\,m\,g = \mathcal{H}_s + r\,m\,\dot{y}\,, \tag{22}$$

(for details see Appendix A).

We then consider Eq. (22) for $\dot{\mathcal{H}}$ with $\frac{\partial \mathcal{H}}{\partial s} = 0$, from which we obtain

$$\dot{\mathcal{H}} = P_y \dot{y} + P_u \frac{du}{dy}\dot{y} = \mathcal{H}_s + r\,m\,\dot{y} = r\,m\,\dot{y}\,, \tag{23}$$

since $\mathcal{H}_s = 0$, thus

$$\dot{y}\{P_y + P_u \frac{du}{dy} - r\,m\} = 0\,. \tag{24}$$

When $\dot{y} \neq 0$ this gives (i.e. unless for an equilibrium or a turning point)

$$\frac{dP}{dy} = r\,m = r\,M(y,u)\,, \tag{25}$$

where

[2] A brief discussion on endpoint maxima is provided in Appendix B.

$$\frac{dP}{dy} \stackrel{def}{=} \frac{\partial P}{\partial y} + \frac{\partial P}{\partial u}\frac{du}{dy}\,. \tag{26}$$

This result can be extended to cases where the control parameter is subject to a constraint, say $u \in [\alpha, \beta]$ and where there is no internal optimum. See Appendix B for details.

At points or intervals where $\dot{y} = 0$ we may have:

1. $\dot{y} = 0$ and $\dot{m} = 0$ *(equilibrium)*: These points are equilibrium points for the system and therfore of no interest when a dynamic evolution is the interesting scenario.

2. $\dot{y} = 0$ and $\dot{m} \neq 0$ *(turning point)*: For this case in the situation we consider we have
$$\dot{m} = \dot{M} = \frac{dM}{dy}\dot{y} = (M_y + M_u\frac{du}{dy})\dot{y}\,, \tag{27}$$
thus this possibility is excluded. In our setting a turning point can exist only for a discontinuous control u.

Notice also that $\frac{\partial \mathcal{H}}{\partial u} \neq \frac{\partial P}{\partial u}$. In fact $\frac{\partial \mathcal{H}}{\partial u} = 0$, but $\frac{\partial P}{\partial u} \neq 0$ because P depends on u also through $M(y, u)$. Eq. (25) is now the only equation to be solved as a first order differential equation for $u = u(y)$. The additional equation

$$\dot{y} = g(y, u)\,, \tag{28}$$

now has the function of determining y as function of s, and initial values $s = t$ and $y(t) = x$, when the differential equation, Eq. (25), determining $u = u(y)$ is solved, then $u(t) = u(y(t))$. Also notice that $\dot{y} = 0 \;\rightarrow\; g = 0 \;\rightarrow\; \dot{\mathcal{H}} = \dot{P} = 0$.

We have in mind applications where the state y is an observable quantity that trigger the action, u. For example problems with pollution and renewable resources.

Thus, as we focus on a solution where the optimal control u is determined as a function of the state variable (stock) y it is important to bear in mined that time t becomes a more or less redundant parameter in this context. The phase plane trajectory for the optimal control $u = u^*$ is all what is needed for exercising the necessary control. However, if for some reason the time evolution is of interest this can as mentioned, be found by solving Eq. (1) or Eq. (28) with $u = u^* = u(y)$. So this determines y as well as u^* as a function of t if this is of any interest.

In this connection it is interesting to notice that working the other way around my result in unsurmountable problems. That is if we have found the solution in the time domain, and want to eliminate time in order to find the feedback form, this quickly becomes a nontrivial matter. This is can be illustrated by the example studied in Sec. **5**. This problem can easily be solved in the time domain. But starting from this solution and then eliminate time to obtain the feedback form is a nontrivial project. Therfore to obtain the feedback form of the solution the time domain should not be introduced.

2.2 Results

We can formulate the results obtained in the following:

Proposition: *The shadow price (costate equation), Eq. (13)*

$$\dot{m} = r\,m - \mathcal{H}_y\,,$$

is automatically satisfied with $m = M(y,u)$, Eq. (16), if Eq. (25) is satisfied.

Comment: Notice that from the above discussion we have excluded the case $\dot{y} = 0$ & $\dot{m} \neq 0$. The situation considered applies to a pure feedback solution: $u = u(y)$ with $\frac{d}{dt} = \dot{y}\frac{d}{dy} = g\frac{d}{dy}$.

Proof:
Consider

$$\begin{aligned}
\dot{M} &\stackrel{def}{=} M_y\,\dot{y} + M_u\,\dot{u} = M_y\,g + M_u\,g\frac{du}{dy} \\
&= P_y - \mathcal{H}_y + M_u\,g\frac{du}{dy} \\
&= \frac{d\,P}{dy} - \mathcal{H}_y - P_u\frac{du}{dy} + M_u\,g\frac{du}{dy} \\
&= rM - \mathcal{H}_y + \{M_u\,g - \mathcal{H}_u - M_u g\}\,\frac{du}{dy} \\
&= rM - \mathcal{H}_y
\end{aligned} \tag{29}$$

where we have used the definition of P, Eq. (20), Eq. (25) and Eq. (26) in addition to

$$\mathcal{H}_u \stackrel{def}{=} F_u + M\,g_u = 0\,, \tag{30}$$

$$P_u \stackrel{def}{=} F_u + M\,g_u + M_u g\,, \tag{31}$$

$$P_u = \mathcal{H}_u + M_u g = M_u g\,. \tag{32}$$

The result in Eq. (29) shows that we obtain Eq. (13) by using Eq. (25) and the definitions and relation listed above. Q.E.D.

2.3 Conclusions

We conclude that for the straight forward case as well as in the special cases outlined in the discussion following Eq. (19) and Appendix B, we have that

$$\frac{dP}{dy} = P_y + P_u\frac{du}{dy} = r\,M(y,u) = -r\frac{F_u}{g_u}\,, \tag{33}$$

is the only equation required to be solved, where P_y, P_u and $M(y,u) = -\frac{F_u}{g_u}$ are known functions. The solution of Eq. (33) will, however, determine u as a function of y and not as a function of s. This is known as a specific feedback form and is the ordinary phase plane solution mathematically speaking.

We also notice that Eq. (33) in the case of a small parameter r, offers itself to a perturbation expansion. This problem is considered in detail in the next section.

We also remark, that based on Eq. (22), we may formulate a perturbation scheme for a more general time dependency than we considered here. We will return to this problem later.

3 Different approaches

3.1 Discount rate as a parameter of smallness

There are basically two pronounced time scales in the problem we are discussing. First we have the "clock" associated with the state equation Eq. (5). Approximating the time dependency by an exponential behavior $\exp(-\gamma t)$, the characteristic rate of change with time from this equation becomes $\gamma \sim \frac{|\bar{g}|}{|\bar{y}|}$, where $\bar{g}$ and $\bar{y}$ are typical values for these functions. This should be compared with r, the discount rate. This is here considered to be much smaller, i.e. $r << \gamma$. This assumption makes a straightforward perturbation expansion of Eq. (25) tractable.

We notice here that for problems where the only time dependency occurs through the discounting factor, $\dot{y}$ is a common factor and a change in time scale seemingly only affects the discount rate r, however one must remember that Eq. (25) also contains g from Eq. (5) explicitely, and thus is affected by a change of time scale in Eq. (5). Let the discount rate r, be replaced by a characteristic time τ_0 ($r = 1/\tau_0$) and introduce a non dimensional time τ by a scale factor t_0 such that, $t = t_0\tau$, and let τ be a number of order unity, then $rt \to \frac{t_0}{\tau_0}\tau$, and the discount rate r would be replaced by a non dimensional discount rate t_0/τ_0, which is a small parameter provided $t_0 << \tau_0$. As en example consider the case of fish: One would expect the typical time change to occur over one year, whereas a discount rate of 10% would correspond to 10 years and the corresponding ratio t_0/τ_0 would be 0.1.

An example using the theory given here is presented in Sec. 5.

3.2 Time dependent discount rate

We consider a more general class of problems where the discount rate may be time dependent. Thus we consider a generalized discount rate as in the following problem:

$$V(x,t) = \max_{u \in U} \int_t^T e^{-r(s)} f(s,y,u)ds\,, \tag{34}$$

subject to

$$\dot{y} = g(s,y,u)\,, \quad y(t) = x\,, \quad r(s) = \epsilon \int_0^s \rho(\tau)d\tau\,, \tag{35}$$

where $\epsilon\,\rho(s)$ is the instantaneous rate and $r(s)$ is the accumulated rate. If the instantaneous rate is a constant r_0, then the accumulated rate is $r_0 s$.

The corresponding Hamiltonian may be written as

$$H = e^{-r(s)} f(s,y,u) + \lambda\, g(s,y,u)\,, \tag{36}$$

where λ is a Lagrangian multiplier. We replace λ by

$$m(s) \stackrel{def}{=} e^{r(s)}\lambda\,. \tag{37}$$

The current value Hamiltonian is now

$$\mathcal{H} \stackrel{def}{=} e^{r(s)} H = f(s,y,u) + m(s)\, g(s,y,u) = \mathcal{H}(s,y,m,u)\,. \tag{38}$$

Then we restrict ourselves to internal extremum, and

$$H_u = 0 \quad \Rightarrow \quad \mathcal{H}_u = 0\,. \tag{39}$$

From Eq. (6)

$$\dot{\lambda} = -H_y\,. \tag{40}$$

It then follows that

$$\begin{aligned} \dot{m} &= \dot{r}\,m + e^{r(s)}\dot{\lambda} \\ &= \epsilon\,\rho(s)\,m - \mathcal{H}_y\,. \end{aligned} \tag{41}$$

Furthermore

$$\dot{\mathcal{H}} = \frac{d\mathcal{H}}{ds} = \epsilon\,\rho(s)\,m\,\dot{y} + \mathcal{H}_s\,. \tag{42}$$

So far this approach is to general. We shall restrict ourselves to certain classes of problems where all explicite time dependency occurs through $\epsilon\,\rho(s)$ only, i.e.

$$\mathcal{H}_s = 0\,, \quad \Rightarrow \quad \dot{\mathcal{H}} = \frac{d\mathcal{H}}{ds} = \epsilon\,\rho(s)\,m\,\dot{y} = \epsilon\,\rho(s)\,m\,g\,. \tag{43}$$

Further more we consider a $\epsilon\,\rho(s)$ which is constant to leading order. Then to leading order s is a redundant parameter which can be eliminated. In order to reflect this fact explicitly we introduced M by Eq. (16) and the corresponding new Hamiltonian P by Eq. (20) and we obtain the basic equation

$$\dot{P}(y,u) = \epsilon\,\rho(s)M(y,u)\,\dot{y}\,, \tag{44}$$

or

$$P_y(y,u)\dot{y} + P_u(y,u)(u_y\dot{y} + u_s) = \epsilon\,\rho(s)M(y,u)\,\dot{y}\,, \tag{45}$$

where we now consider u to be given as $u = u(s,y)$ and $\dot{u} = u_y\dot{y} + u_s$. Thus in the case where $u_s = 0$, i.e. when u depend only on the state y, Eq. (45) simplifies to

$$P_y(y,u) + P_u(y,u)\,u_y = \epsilon\,\rho(s)\,M(y,u)\;, \tag{46}$$

since $\dot{y}$ is a common factor.[3] Then if $\epsilon\,\rho(s) << 1\,,\ \forall s$, we have a problem formulation suitable for a perturbation expansion. We notice that whereas $\mathcal{H}_u = 0$ we have $P_u = M_u\,g \neq 0$, when the system is not in an equilibrium. We also notice that Eq. (46), for the case $\rho =$constant, is exactly Eq. (33).

Then consider the case with $r(s)$ given as

$$r(s) = s\epsilon + \alpha_2(s)\epsilon^2 + \alpha_3(s)\epsilon^3 + \cdots \tag{47}$$

then

$$\epsilon\,\rho(s) = \dot{r} = \epsilon + \dot{\alpha}_2(s)\epsilon^2 + \dot{\alpha}_3(s)\epsilon^3 + \cdots\,. \tag{48}$$

For this expansion to be uniformly valid in s, it is necessary that $\alpha_k(s)$ are bounded for all k and $s \in \{t_0, T\}$, where T may be infinite if this is the range for s in the actual problem.

[3] We notice that also in the case $\frac{\partial u}{\partial t} \neq 0$, $\dot{y}$ is a common factor in Eq. (48), since we have, $P_u = H_u + M_u g = M_u g = M_u \dot{y}$.

We then have the following expansion scheme

$$P = P_0 + \epsilon P_1 + \mathcal{O}(\epsilon^2)\,, \tag{49}$$
$$u = u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2)\,, \tag{50}$$

$$\begin{aligned}
\{P'\}_0 &= 0\,, \quad \rightarrow \quad P_0 = C_0 = \text{const.} \quad \rightarrow \quad u_0 = u_0(y_0; C_0)\,, && (51)\\
\{P'\}_1 &= \{P_y(y,u)\dot{y} + P_u(y,u)[u_y\dot{y} + u_s]\}_1 \\
&= M_0\,, && (52)\\
\{P'\}_2 &= \{P_y(y,u)\dot{y} + P_u(y,u)[u_y\dot{y} + u_s]\}_2 \\
&= (M_1 + \dot{\alpha}_2(s)M_0)\dot{y}\,, && (53)\\
&\cdots\,,
\end{aligned}$$

and $M_0 = M(y_0, u_0)$, $M_1 = M_u(y_0, u_0)\,u_1$. Although this procedure could in principle work it becomes quickly rather complicated. We shall investigate a different approach in the next section.

3.3 Alternative approach

An alternative approach turns out to be useful. In this approach we first integrate Eq. (44)

$$\dot{P} = \epsilon\,\rho\,M\,\dot{y}\,,$$

and obtain

$$P = C + \epsilon \int_0^t \rho(s)M(y(s),u(s))\dot{y}ds = C + \epsilon \int_x^y \rho(\tau(y'))M(u(y'),\tau(y'))dy'\,, \tag{54}$$

where the last form of the integral requires that one can find a relationship between y and t. For this purpose we use Eq. (5), $\dot{y} = g(y,u)$ to define a new quantity $\tau(y)$ as

$$\tau(y) \stackrel{def}{=} \int_x^y \frac{dy'}{g(y',u(y'))}\,, \tag{55}$$

generally speaking this quantity may not be single valued, however, this is connected to points where $g(y,u) = 0$ i. e. equilibrium or turning points.

The main focus here is the path towards the equilibrium and not the equilibrium itself. Assuming that we do not integrate through or past any turning points, multiple values are not an issue.

However, this approach does not lead anywhere in general, the trick here is to make an expansion and solve the problem using one of the two integral representation. Either way one has to solve the problem order by order. This is outlined in sections **3.3.1** and **3.3.2**.

3.3.1 First approach

In the first part of Eq. (54) we substitute

$$P = P_0 + \epsilon P_1 + \mathcal{O}(\epsilon^2), \quad \epsilon \rho(s) = \epsilon\eta_1(s) + \epsilon^2\eta_2(s) + \mathcal{O}(\epsilon^2),, \tag{56}$$

and so onwards. Thus we obtain

$$\begin{aligned} P_0 + \epsilon P_1 + \epsilon^2 P_2 + \mathcal{O}(\epsilon^3) &= C_0 + \epsilon C_1 + \epsilon^2 C_2 \\ &+ \epsilon \int_0^t (\eta_1(s) + \epsilon\eta_2(s)) \\ &\times (M_0 + \epsilon M_1 + \cdots)([\dot{y}]_0 + \epsilon[\dot{y}]_1) ds + \mathcal{O}(\epsilon^3) . \end{aligned}$$

The order by order solutions are:

First order:

$$P(y, u_0(y)) = C_0 . \tag{57}$$

This relation determines $u_0 = u_0(y; C_0)$ in terms of a constant of integration C_0. We shall return to the determination of C_0. For the case of argument we consider C_0 as known in the following discussion.

Second order:

$$P_1 = P_u(y, u_0)\, u_1(y) = C_1 + \int_0^t \eta_1(s)\, M_0(y, u_0)\, g(y, u_0) ds , \tag{58}$$

where we needed to know $y = y(s)$ in order to perform the last integration. This information was obtained from path Γ_0 :

$$[\dot{y}]_0 = g(y, u_0(y; C_0)) . \tag{59}$$

Notice that the righthand side of Eq. (59) is now completely known, so as to make the integration of this equation possible. If a solution can not be obtained analytically, one can always resort to a numerically determined solution. Thus the problem associated with the non autonomous systems of this kind, is that we need to integrate only along the autonomous path (the non discounted path) Γ_0. And in the higher order approximations we can relax the integration to a known path. Thus in principle Eq. (58) now determines $u_1 = u_1(y; C_1)$. And again C_1 is a constant of integration that must be determined.

Higher order :
The process above can now be continued to determine the higher order solution, and can in principle be continued. However, one must expect as usual for such procedures an increase in complexity as the process is carried on. Therefore in practical terms one may find great difficulties in performing such an expansion scheme beyond the first order solution. In this respect we want to point out that nonlinearities are kept even to zeroth order in this approach. This fact will relax the need for carrying this procedure very far in order to obtain useful approximations.

3.3.2 Second approach

In this approach we use the second part of Eq. (54) and handle Eq. (55) differently i. e.

$$\tau(y) = \int_x^y \frac{dy'}{g(y', u(y'))} ,$$

and rewrite Eq. (54) as

$$P = C_0 + \epsilon\, C_1 + \epsilon^2\, C_2 + \mathcal{O}(\epsilon^3) + \int_x^y \rho(\tau(y'))M(u(y'), \tau(y'))dy' .$$

Please notice that $P = P(u, y)$ is completely known as a function, however the argument u is unknown. We now solve this problem order by order using the following expansion scheme:

$$\begin{aligned} u &= u_0 + \epsilon\, u_1 + \epsilon^2\, u_2 + \mathcal{O}(\epsilon^3) , \\ \tau &= \tau_0 + \epsilon\, \tau_1 + \epsilon^2\, \tau_2 + \mathcal{O}(\epsilon^3) , \end{aligned}$$

where

$$\tau_0 = \int_x^y \frac{dz}{g(u_0, z)} , \qquad (60)$$

$$\tau_1 = -\int_x^y \frac{u_1\, g_u(u_0, z)}{g^2(u_0, z)} dz , \qquad (61)$$

where $u_0\, , u_1\, , \ldots$ are determined from order by order solutions of the equation

$$\begin{aligned} P(u, y) &= P(u_0, y) + \epsilon\, u_1\, P_u(u_0, y) + \epsilon^2 \{ \frac{1}{2} P_{uu}(u_0, y) u_1^2 + P_u(u_0, y)\, u_2 + \mathcal{O}(\epsilon^3) \} \\ &= C_0 + \epsilon \{ C_1 + \int_x^y \rho(\tau_0) M(u_0, y) dy \} \\ &\quad + \epsilon^2 \{ C_2 + \int_x^y [\rho'(\tau_0)\tau_1 M(u_0, y) + \rho(\tau_0) M_u(u_0, y)\, u_1] dy \} \} + \mathcal{O}(\epsilon^3) . \end{aligned}$$

From this equation we now obtain

$$\begin{aligned} u_0 &= u_0(y, C_0) \\ u_1 &= \{ C_1 + \int_x^y \rho(\tau_0) M(u_0, y) dy \} \frac{1}{P_u(u_0, y)} \\ u_2 &= \{ C_2 + \int_x^y [\rho'(\tau_0)\tau_1 M(u_0, y) + \rho(\tau_0) M_u(u_0, y)\, u_1] dy \\ &\quad - \frac{1}{2} P_{uu}(u_0, y) u_1^2 \} \frac{1}{P_u(u_0, y)} . \end{aligned} \qquad (62)$$

This process can be continued and the problem is solved order by order.

However, one problem remain, that is, the constants C_0, C_1, C_2, ... are still unknown.

C_0: To zeroth order we require the solution to be the same as in the corresponding problem with zero discount rate. Now we have

$$P = P_0 = P(y, u_0) = C_0 = H(y, m, u)\,, \tag{63}$$

and the maximum principle tell us that $H(y, m, u)$ should be maximized. In addition we consider a solution that asymptotically approaches equilibrium ($g(y, u) = 0$) and since $H(y, m, u) = f(y, u) + m\,g(y, u)$, we must seek the maximum of $f(y, u)$ with the constraint $g(y, u) = 0$ or

$$C_0 \stackrel{def}{=} \max_{g(y,u)=0} f(y, u)$$

Let this occur for $y^{**} \in \Omega$ and $u^{**} \in U$.

This way C_0 is uniquely determined.

C_1: For the next step we notice that $P_u = M_u g = 0$, for $y = y^{**}$ thus Eq. (62) leave us with no other choice than

$$\{C_1 + \int_x^{y^{**}} \rho(\tau_0) M(u_0, y) dy\} = 0$$

and since $\int_x^y \rho(\tau_0) M(u_0, y) dy = \int_x^{y^{**}} \rho(\tau_0) M(u_0, y) dy + \int_{y^{**}}^y \rho(\tau_0) M(u_0, y) dy = -C_1 + \int_{y^{**}}^y \rho(\tau_0) M(u_0, y) dy$ we find from Eq. (62)

$$u_1 = \frac{1}{P_u(u_0, y)} \{C_1 + \int_x^y \rho(\tau_0) M(u_0, y) dy\} = \frac{1}{P_u(u_0, y)} \int_{y^{**}}^y \rho M(y', u(y')) dy'\,.$$

By this choice u_1 is completely determined since y^{**} now is defined by the equilibrium solution found for C_0

$$\max_{g(y,u)=0} f(y, u) \quad \rightarrow (y^{**}, u_0^{**})$$

C_2: Again $P_u = 0$ for $y = y^{**}$, in the denominator in Eq. (63) require that the parenthesis

$$\{C_2 + \int_x^y [\rho'(\tau_0)\tau_1 M(u_0, y) + \rho(\tau_0) M_u(u_0, y)\, u_1] dy - \frac{1}{2} P_{uu}(u_0, y) u_1^2\} = 0$$

and this determines C_2 and finally u_2 by splitting the integral as above.

$$u_2 = \frac{1}{P_u} \left\{ \int_{y^{**}}^y [\rho'(\tau_0)\tau_1 M(u_0, y) + \rho(\tau_0) M_u(u_0, y)\, u_1] dy - \left[\frac{1}{2} P_{uu} u_1^2\right]_{y^{**}}^y \right\}, \tag{64}$$

where $[Q(y)]_{y_1}^{y_2} \stackrel{def}{=} Q(y_2) - Q(y_1)$, and as before u_2 at the equilibrium point y^{**} is determined by the limit when approaching this point.

Now this process continues in the same way and the constants C_n can in principle be determined to arbitrary order n.

Summary:

We now have

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \mathcal{O}(\epsilon^3) \tag{65}$$

where

Zeroth order: We find u_0 from the implicit relation $P(y, u_0) = C_0$ with C_0 given in Eq. (66) as

$$C_0 \stackrel{def}{=} \max_{g(y,u)=0} f(y, u)\,. \tag{66}$$

Notice that $P(y, u)$ is a known function of its arguments.

First order:

$$u_1 = \frac{1}{P_u(u_0, y)} \int_{y^{**}}^{y} \rho M(y', u(y'))dy' \tag{67}$$

Second order:

$$u_2 = \frac{1}{P_u(u_0, y)} \left\{ \int_{y^{**}}^{y} [\rho'(\tau_0)\tau_1 M(u_0, y) + \rho(\tau_0) M_u(u_0, y)\, u_1] dy - \left[\frac{1}{2} P_{uu} u_1^2\right]_{y^{**}}^{y} \right\}, \tag{68}$$

Conclusion: Above we have presented explicit algorithms for determining u to $\mathcal{O}(\epsilon^3)$. This procedure can in principle be carried on to any order. However, for any practical purposes this is hardly interesting since the basic information is contained in these first terms. We also want to point out that even the zeroth order term contain the genuine nonlinearity of the problem. This is very basic and powerful for this procedure that it is able to catch the nonlinear behavior from the start.

In the case of zero discount rate (zeroth order), the equilibrium point may be a saddle point, in which case one has to choose the ingoing separatrices. This means that for $y < y^{**}$ we have $g((y, u_0(y)) > 0$ and vice versa.

3.4 A general formulation by transformation

Consider the generic problem with an equation of state

$$\dot{y} = g(s, y(s), u(s))\,,\ y \in \Omega \subset \mathbb{R} \text{ and } u \in U \subset \mathbb{R} \tag{69}$$

$$y(s = t) = x\,, \quad s \in \mathbb{R}^+\,. \tag{70}$$

Given a utility function (objective function) f, that we seek to maximize subject to the control $u \in U$, where $U \subset \mathbb{R}$, is a given control space and $s \in A \subset \mathbb{R}$. We define a value function as

$$V(x, t) = \max_{u \in U} \int_t^T f(s, y(s), u(s))ds\,. \tag{71}$$

where $f : \{A \times \Omega \times U \rightarrow \mathbb{R}\}$ and $g : \{A \times \Omega \times U \rightarrow \mathbb{R}\}$. The horizon T has a predetermined value.

Then consider a class of problems where $g(s, y(s), u(s))$, is sufficiently regular and smooth so that we may define a new function $\tilde{u}(s)$ by $\tilde{u}(s) = g(s, y(s), u(s))$. By assumption this relation may be inverted so that $u(s) = v(s, y(s), \tilde{u}(s))$, in principle is known when $\tilde{u}$ is known.

By this change of control variable from an $u(s)$ to a $\tilde{u}(s)$ representation, we arrive at the following problem in the new setting:

$$V(x,t) = \max_{u \in U} \int_t^T F(s, y(s), \tilde{u}(s))ds\,, \quad \dot{y} = u\,. \tag{72}$$

The corresponding Hamiltonian is

$$H = F + M\,G = F + M\tilde{u}\,, \tag{73}$$

and $H_{\tilde{u}} = F_{\tilde{u}} + MG_{\tilde{u}} = F_{\tilde{u}} + M = 0$, and the condition for an internal optimum, gives $M = -F_{\tilde{u}}$. Thus the corresponding P, which is H in the new representation [4], is given by

$$P = F - \tilde{u}F_{\tilde{u}}\,, \tag{74}$$

where $\dot{H} = H_s$ transforms into

$$\dot{P} = F_s\,. \tag{75}$$

Now make the following transformation

$$\tilde{F} = F + A(s,y) + B(s,y)\tilde{u}\,, \tag{76}$$

then

$$\begin{aligned}
\tilde{P} &= \tilde{F} - \tilde{F}_{\tilde{u}}\tilde{u} = F + A + B\tilde{u} - (F_{\tilde{u}} + B)\tilde{u} = F + A - F_{\tilde{u}}\tilde{u}\,, \\
\dot{\tilde{P}} &= \frac{D}{Dt}(F - F_{\tilde{u}}\tilde{u}) + \dot{A} \\
&= F_s + A_s + B_s\tilde{u} - B_s\tilde{u} + A_y\dot{y} \\
&= \frac{\partial}{\partial s}(F + A + B\tilde{u}) - B_s\tilde{u} + A_y\tilde{u} \\
&= \tilde{F}_s\,,
\end{aligned}$$

or

$$\dot{\tilde{P}} = \tilde{F}_s\,, \tag{77}$$

provided $A_y = B_s$. Thus the condition $A_y(s,y) = B_s(s,y)$ makes Eq. (75) invariant with respect to the transformation given by Eq.(76). The aim here is to find a transformation where the right-hand side of Eq. (77) is zero or a small quantity making this equation suitable for an expansion as a perturbation problem as given in the preceding sections. See sections **3.3.1** and **3.3.2**.

We consider an example

[4] This corresponds to a Legendre-transformation, see [2].

Example 3.1 Let

$$F = -u^2 + (1 + \epsilon\, g(y)\, t)u + h(y) \tag{78}$$

then

$$\tilde{F} = F + A(t, y) + B(t, y)\, u\,, \tag{79}$$

where

$$B(t, x) = -(1 + \epsilon g(y)\, t)\ ,\ \text{with } A_y = B_t = -\epsilon g(y)\,,$$

thus

$$A = -\epsilon G(y)\,, \quad \text{where} \quad G'(y) = g(y)\,, \tag{80}$$

and

$$\tilde{F} = -u^2 + h(y) - \epsilon G(y)\,, \quad \dot{\tilde{P}} = \tilde{F}_s = 0 \ \Rightarrow\ \tilde{P} = C \ \text{(constant)}, \tag{81}$$

where $\tilde{P} = \tilde{F} - u\tilde{F}_u = u^2 + h(y) - \epsilon\, G(y)$. From this we obtain

$$u = U(y; C)\,. \tag{82}$$

Furthermore

$$H = F - uF_u = \tilde{F} - u\tilde{F}_u - A = C - A = C + \epsilon G(y)\,. \tag{83}$$

On the other hand we also have $H = F + \lambda \dot{y}$. Then considering an equilibrium point ($\dot{y} = 0$), we obtain

$$C = F|_{u=0} - \epsilon G(y) \stackrel{def}{=} \tilde{C}(y). \tag{84}$$

Now by choice, the constant C is determined by the maximum value of $\tilde{C}(y)$. This should be compared to the original F, (Eq. (78)), and the corresponding H given by

$$H = -u^2 + (1 + \epsilon\, g(y)\, t)u + h(y) + \lambda u\,. \tag{85}$$

Thus

$$H_u = -2u + (1 + \epsilon g(y)t) + \lambda = 0\,, \quad \Rightarrow \quad \lambda = 2u - (1 + \epsilon g(y)\, t)\,, \tag{86}$$

and the corresponding $P = F - uF_u$ is given by

$$P = -u^2 + (1 + \epsilon\, g(y)\, t)u + 2u^2 - (1 + \epsilon\, g(y)\, t)u = u^2 + h(y)\,, \tag{87}$$

from which we obtain

$$\frac{d}{dt}[u^2 + h(y)] = F_t = \epsilon\, g(y)\, u = \epsilon\, g(y)\, \dot{y} = \frac{d}{dt}[\epsilon\, G(y)]\,, \tag{88}$$

or

$$u^2 + h(y) - \epsilon\, G(y) = \text{constant}\,. \tag{89}$$

Thus for this case this equation is easily integrated, giving the same result as above. Note that frequently this rout to solution is easier than solving the equivalent system

$$\dot{x} = H_\lambda\,,$$
$$\dot{\lambda} = -H_y\,,$$
$$u = \underset{u}{\text{argmax}}\; H\,.$$

3.5 More General Time Dependency

Starting from the basic equations Eq. (12) to Eq. (14) we have three equations for determining three unknowns u, y, m. Eq. (22) is therefore not independent, and we argue that it is convenient to use Eq. (22) instead of the costate equation Eq. (13). Thus we start from (see Eq. (42)

$$\dot{\mathcal{H}} = \mathcal{H}_s + \epsilon\rho(s)\, m\, g = \mathcal{H}_s + \epsilon\rho(s)\, m\dot{y}\,, \tag{90}$$

where the constant discount rate r is now replaced by $\dot{r}(s) \stackrel{def}{=} \epsilon\rho(s)$. In addition we have

$$\dot{y} \stackrel{def}{=} \frac{d}{ds}y(s) \quad = \quad g(y(s), u(s), s)\,, \tag{91}$$
$$\mathcal{H}_u \quad = \quad F_u(y(s), u(s), s) + m(s)\, g_u(y(s), u(s), s) = 0\,. \tag{92}$$

We notice, however, that the last equation above, which is satisfied for any internal stationary point, can be used for elimination of m from this system, i.e.

$$m(s) = M(y(s), u(s), s) \stackrel{def}{=} -\frac{F_u}{g_u}\,. \tag{93}$$

In this process we introduce a new function P which has the same values as the Hamiltonian $\mathcal{H}$, but a different functional dependency of its arguments.

$$\begin{aligned} P(y(s), u(s), s) \quad &\stackrel{def}{=} \quad \mathcal{H}(y(s), u(s), M(y(s), u(s), s), s) \\ &= \quad F - \frac{F_u}{g_u}\, g\,, \end{aligned} \tag{94}$$

and Eq. (90) becomes

$$\dot{y}P_y + \dot{u}P_u + P_s = \mathcal{H}_s + \epsilon\rho(s)\, M\, g\,, \tag{95}$$

or

$$\dot{y}P_y + \dot{u}P_u = \mathcal{H}_s - P_s + \epsilon\rho(s)\, M\, g\,, \tag{96}$$

and in addition we have

$$\dot{y} = g\,, \tag{97}$$

where $Q = Q(y(s), u(s), s)$ applies to g, P_u, P_s, P_y, M. Notice that we have $\mathcal{H}_u = 0$, but $P_u \neq 0$.

Thus in this approach we are left with only two equations.

One may carry this one step further and consider the feedback type solution i. e.

$$u = u(y, s), \qquad \dot{u} = \dot{y}\, u_y + u_s\,, \tag{98}$$

that is u is determined by y, the state variable, but may also be a function of time explicitly. We may then write Eq. (95) as

$$\begin{aligned}
\frac{D}{Dy}P &= \epsilon\rho(s)M + \frac{1}{g}\left(\mathcal{H}_s - P_s\right) - \frac{1}{g}u_s P_u = \epsilon\rho(s)M - M_s - \frac{1}{g}u_s P_u,\\
&= \epsilon\rho(s)M - M_s - M_u\, u_s,\\
&= \epsilon\rho(s)M - \frac{DM}{Ds} = -e^{r(s)}\frac{D}{Ds}\left[e^{-r(s)}M\right],
\end{aligned}$$

or

$$\frac{D}{Dy}[e^{-r(s)}P] + \frac{D}{Ds}[e^{-r(s)}M] = 0 \tag{99}$$

where now

$$\frac{D}{Dy} \stackrel{def}{=} \frac{\partial}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial}{\partial u}\,,$$

and

$$\frac{D}{Ds} \stackrel{def}{=} \frac{\partial}{\partial s} + \frac{\partial u}{\partial s}\frac{\partial}{\partial u}\,.$$

Although Eq. (99) is compact and elegant it has so far not proved to useful for practical purposes. We therfore list the more useful form

$$\frac{D}{Dy}P = \epsilon\rho(s)M - M_s - \frac{1}{g}u_s P_u,\ , \tag{100}$$

where the basic assumption is that we have $M_s = \mathcal{O}(\epsilon)$ and $u_s = \mathcal{O}(\epsilon)$. The first of these assumptions is at our control when setting up the problem. The second assumption has to be verified when the solution is found.

In summary we notice that if the explicite time dependency reflected by the terms M_s and u_s is weak, then these terms as well as the term containing $\epsilon\rho(s)$, may be treated like a perturbation on the system for a small $\epsilon\rho(s)$. Then the system is brought into a form suitable for a perturbation expansion in the small parameter ϵ.

We recover Eq. (25) by assuming no explicite time dependency in M in the last term of Eq. (99), thus $M_s = u_s = 0$. The condition $M_s = 0$ means $\frac{\partial}{\partial s}\left(\frac{F_u}{g_u}\right) = 0$.

Notice that several types of systems having explicite time dependency are included even when this condition is satisfied.

Also notice that this formulation cover all the previously considered special cases. The details of finding explicite approximate solution by perturbation techniques for this more general class of systems are similar to what is already presented, we are therefore not repeating them here. However, one now must let the constants of integration (previously called $C_0,\ C_1,\ \ldots$) **be dependent on time** t**.**

3.6 The Hamilton-Jacobi-Bellman (HJB) equation

The preceeding discussion can also be viewed in the context of the HJB-equation. The problem formulated, Eqs. (1) - (3), can also be cast in the following form

$$V_t(x,t) + \max_{u \in U} H(t,x,u,\lambda) = 0\,. \tag{101}$$

where $\lambda = V_x$ and $H(t,x,u,\lambda) = f(t,x,u) + \lambda\, g(t,x,u)$, or

$$V_t(x,t) + \max_{u \in U} \left[f(t,x,u) + V_x(x,t) \cdot g(t,x,u)\right] = 0\,. \tag{102}$$

For an internal optimum we have

$$f_u + V_x \cdot g_u = 0. \tag{103}$$

Since f and g are given this determines the value of V_x as a given function t, x, u.

We now restrict ourselves to the special case where the explicite time dependency can be accounted for by a constant discount rate. In this case we have

$$H = e^{-\delta t} F(x,u) + \lambda G(x,u)\,. \tag{104}$$

We then define the current value Hamiltonian by

$$\mathcal{H} \stackrel{def}{=} e^{\delta t} H = F(x,u) + \Lambda G(x,u) \tag{105}$$

where

$$F(x,u) \stackrel{def}{=} e^{\delta t} f(.), \quad G(x,u) \stackrel{def}{=} g(.) \quad \text{and} \quad \Lambda \stackrel{def}{=} \lambda\, e^{\delta t} = e^{\delta t} V_x\,. \tag{106}$$

We consider the following class of solutions (where now the only explicite time dependency is due to a constant discount rate - the well known current value approach):

$$V(x,t) = W(x;\delta)\, e^{-\delta t} + \frac{1 - e^{-\delta t}}{\delta} K(\delta) \tag{107}$$

or

$$e^{\delta t}\, V(x,t) = W(x;\delta) + \frac{e^{\delta t} - 1}{\delta} K(\delta) \tag{108}$$

where $K(\delta) = K_0 + \delta K_1 \ldots$ is constant with respect to t, x, u and $W(x;\delta) = W_0(x) + \delta W_1(x) + \ldots$. Thus we impose the restriction of the existence of a regular expansion of K and W in δ. Also notice that by this choice of $V(x,t)$ we obtain in the limit $\delta \to 0$

$$\lim_{\delta \to 0} V(x,t) = W_0(x) + K_0\, t\,, \tag{109}$$

where $W_0 \stackrel{def}{=} W(x;0)$ and $K_0 \stackrel{def}{=} K(0)$. Notice that the problem we obtain in the limit of zero discount rate is the proper solution of the actual zero discounting problem. In this limit $V_x = W_0'$.

For arbitrary δ we obtain from Eq. (101)

$$-\delta\, W + K + \max_u \mathcal{H}(x,u,\Lambda) = 0 \tag{110}$$

with $\Lambda = W'$. An internal optimum now means

$$\mathcal{H}_u = 0 = F_u + \Lambda G_u \quad \Rightarrow \quad M(x,u) \stackrel{def}{=} \Lambda = -\frac{F_u}{G_u} = e^{\delta t} V_x(x,t)\,, \tag{111}$$

which is Eq. (16). Further more

$$\begin{aligned} \delta W &= K + P(x,u)\,, & (112)\\ W' &= M(x,u)\,, & (113) \end{aligned}$$

where $P(x,u)$ given by Eq. (20), has the same value as the optimal Hamiltonian, even though it functionally is different. It is easily seen that Eqs. (112) and (113) represent a first integral of Eq. (25).

Now we proceed by solving this problem by a regular perturbation expansion order by order.

Zeroth order:

$$\begin{aligned} 0 &= K_0 + P(x,u_0)\,, \text{ this is an algebraic Eq. determining } u_0(x), & (114)\\ W_0' &= M(x,u_0)\,, \quad W_0' \text{ is now a known function through } u_0(x). & (115) \end{aligned}$$

First order:

$$\begin{aligned} W_0 &= K_1 + P_u(x,u_0)\,u_1, \quad \text{this determines } \; u_1(x), & (116)\\ W_1' &= M_u(x,u_0)\,u_1, \quad W_1' \text{ is now a known function.} & (117) \end{aligned}$$

in principle this procedure can now be continued. We shall stop and look at some details. Notice that u_0 and u_1 also depend on the parameters K_0 and K_1. Also notice that u_0 and W_0 corresponds to the solution of the problem with zero discount rate.

$$0 = K_0 + P(x,u_0(x))\,, \tag{118}$$

or

$$K_0 = -P(.) = -\mathcal{H}_0^*\,, \tag{119}$$

where a * refers to the optimal solution, $u^* = u_0(x)$, in this approximation $(\delta = 0)$. In the zero discount limit we have

$$H = \mathcal{H} = F(x,u_0(x)) + M(x,u_0(x) \cdot \dot{x}\,. \tag{120}$$

Suppose $y = x$ is an equilibrium point then $\dot{x} = 0$ and

$$K_0 = -F(x,u_0(x)) \quad \text{and} \quad G(x,u_0(x)) = 0\,, \tag{121}$$

where the latter condition determines the set of equilibrium points, $\{\tilde{x}\}$. Then K_0 and the proper equilibrium point $y^{**} \in \{\tilde{x}\}$, is determined as the value that maximizes H, or

$$y^{**} \stackrel{def}{=} \operatorname*{argmax}_{G(x,u_0(x))=0} F(x,u_0(x))\,. \tag{122}$$

This way $K_0 \overset{def}{=} -F(y^{**}, u_0(y^{**})$ is determined. Turning to u_1 we have

$$u_1 = \frac{\int_a^x M(\tilde{y}, u_0(\tilde{y}))d\tilde{y} - K_1}{P_u(x, u_0(x))}\,, \tag{123}$$

where a is any suitable arbitrary chosen constant. Notice here that $P_u = H_u + H_\lambda M_u = G\, M_u = 0$, at the zeroth order equilibrium point y^{**} given by Eq. (122) (known quantity). Then regularity of u_1 at this point require

$$\int_a^{y^{**}} M(\tilde{y}, u_0(\tilde{y}))d\tilde{y} - K_1 = 0$$

or

$$K_1 = \int_a^{y^{**}} M(\tilde{y}, u_0(\tilde{y}))d\tilde{y}\,. \tag{124}$$

Finally we find

$$u_1 = \frac{\int_{y^{**}}^x M(\tilde{y}, u_0(\tilde{y}))d\tilde{y}}{P_u}\,. \tag{125}$$

This determines u_1, and this way the procedure continues.

Looking back at Sec. **3.3.2** we observe that we now have reproduced the same results, for the case of a constant discount rate, from the HJB-equation.

4 A game problem

Consider a game problem with n players described by n Hamiltonians that each player tries to optimize subject to the same state condition

$$\dot{y} = g(y, u, t)\,, \quad x(0) = x \quad u = u_1 + u_2 + \ldots + u_n\,, \quad \text{and} \quad t \geq 0\,. \tag{126}$$

Each player has its own utility function, and player number k has the utility function f_k. We then have a system of n Hamiltonians

$$H_k = f_k + \lambda_k g\,, \qquad k = 1,\, 2,\, \ldots, n\,, \tag{127}$$

with

$$\dot{\lambda}_k = -\frac{\partial H_k}{\partial x} - \sum_{j \neq k} \frac{\partial H_k}{\partial u_j}\frac{\partial u_j}{\partial x}\,, \tag{128}$$

$$\dot{y} = \frac{\partial H_k}{\partial \lambda_k} = g\,, \tag{129}$$

$$\frac{\partial H_k}{\partial u_k} = 0\,, \quad \text{necessary condition.} \tag{130}$$

Eq. (128)is selected by choice, this can be done because λ_k is arbitrary. The implication of this choice is Eq. (130), as a necessary condition for a stationary value of H_k. We then obtain

$$\begin{aligned}
\dot{H}_k &= \frac{\partial}{\partial t}H_k + \frac{\partial H_k}{\partial u_k}\dot{u}_k + \frac{\partial H_k}{\partial y}\dot{y} + \frac{\partial H_k}{\partial \lambda_k}\dot{\lambda}_k + \sum_{j\neq k}\frac{\partial H_k}{\partial u_j}\dot{u}_j \\
&= \frac{\partial}{\partial t}H_k + \frac{\partial H_k}{\partial x}\frac{\partial H_k}{\partial \lambda_k} + \frac{\partial H_k}{\partial \lambda_k}\left\{-\frac{\partial H_k}{\partial x} - \sum_{j\neq k}\frac{\partial H_k}{\partial u_j}\frac{\partial u_j}{\partial y}\right\} + \sum_{j\neq k}\frac{\partial H_k}{\partial u_j}\dot{u}_j \\
&= \frac{\partial}{\partial t}H_k - \frac{\partial H_k}{\partial \lambda_k}\left\{\sum_{j\neq k}\frac{\partial H_k}{\partial u_j}\frac{\partial u_j}{\partial y}\right\} + \sum_{j\neq k}\frac{\partial H_k}{\partial u_j}\dot{u}_j \\
&= \frac{\partial}{\partial t}H_k + \sum_{j\neq k}\frac{\partial H_k}{\partial u_j}\frac{\partial}{\partial t}u_j\,,
\end{aligned}$$

where we have used that

$$\begin{aligned}
\dot{u}_j &= \frac{\partial}{\partial t}u_j + \frac{\partial u_j}{\partial y}\dot{y} = \frac{\partial}{\partial t}u_j + \frac{\partial u_j}{\partial y}\frac{\partial H_k}{\partial \lambda_k}\,, \\
\frac{\partial H_k}{\partial u_k} &= 0, \qquad \text{optimum condition for player no. } k\,, \\
\frac{\partial H_k}{\partial \lambda_k} &= g = \dot{y}\,, \qquad \text{state equation}\,.
\end{aligned}$$

Thus the final result may be written as

$$\dot{H}_k = \frac{\partial}{\partial t}H_k + \sum_{j\neq k}\frac{\partial H_k}{\partial u_j}\frac{\partial}{\partial t}u_j\,. \tag{131}$$

Defining discounted Hamiltonians $\mathcal{H}_k \stackrel{def}{=} e^{rt}H_k$, and following the same procedure as in Appendix A we obtain

$$\dot{\mathcal{H}}_k = \frac{\partial}{\partial t}\mathcal{H}_k + rm_k g + \sum_{j\neq k}\frac{\partial \mathcal{H}_k}{\partial u_j}\frac{\partial}{\partial t}u_j\,, \tag{132}$$

where $m_k \stackrel{def}{=} e^{rt}\lambda_k$.

5 Example: Feedback type solution

In Sec. 2, a theory was presented, where a procedure for finding feedback type solutions were discussed. As a test of this theory we consider the following example

$$F(y,u) = a\,y - \frac{1}{2}y^2 - c\,u^2\,, \qquad \dot{y} = g(y,u) = u - b\,y\,, \tag{133}$$

$$M = -\frac{F_u}{g_u} = 2cu \tag{134}$$

$$P = F + M\,g = a\,y - \frac{1}{2}y^2 - c\,u^2 + M\,(u - b\,y) = a\,y - (\frac{1}{2} + b^2 c)y^2 + c(u - by)^2 \tag{135}$$

$$\begin{aligned} P_y &= a - y - 2bcu\,, \\ P_u &= 2c(u - b\,y)\,, \end{aligned}$$

and

$$P_u \frac{du}{dy} + P_y = -r\frac{F_u}{g_u} \quad \Rightarrow \quad (u - b\,y)\frac{du}{dy} - (r+b)u - \frac{y}{2c} + \frac{a}{2c} = 0\,. \tag{136}$$

First consider the case $r = 0$, then we have

$$P(y,u_0) = a\,y - (\frac{1}{2} + b^2 c)y^2 + c(u_0 - by)^2 = C_0\,, \tag{137}$$

and

$$u_0 = b\,y \pm \frac{1}{\sqrt{c}}\sqrt{C_0 + (\frac{1}{2} + b^2 c)y^2 - a\,y}\,. \tag{138}$$

Using Eq. (66) we find

$$C_0 = \max[ay - (\frac{1}{2} + b^2 c)y^2]\,, \quad \text{notice} \quad g = u - by = 0\,, \tag{139}$$

this occur when

$$y = y^{**} = \frac{a}{1 + 2b^2 c} \tag{140}$$

and

$$C_0 = \frac{1}{2}ay^{**} = \frac{1}{2}\frac{a^2}{1 + 2b^2 c}\,. \tag{141}$$

We observe that this choice of C_0 makes the expression under the square root in Eq. (138) a complete square[5], and we obtain

$$u_0 = by + \frac{1}{\sqrt{c}}\sqrt{\frac{1}{2} + b^2 c}\;(y - y^{**}) \tag{142}$$

We continue to find u_1 by using Eq. (67) and Eq.(142)

[5] If the value of the constant C_0 is chosen differently, this will complicate the solution for u_1 with a logarithmic term.

$$u_1 = \frac{1}{P_u(u_0, y)} \int_{y^{**}}^{y} \rho M(y', u(y'))dy' = \frac{\rho_0}{u_0 - b\,y} \int_{y^{**}}^{y} u_0(z)dz\,, \tag{143}$$

and since $u_0 - by = \frac{1}{\sqrt{c}}\sqrt{\frac{1}{2} + b^2 c}\;(y - y^{**})$ we finally obtain

$$u_1 = \frac{b\sqrt{c}}{\sqrt{\frac{1}{2} + b^2 c}} \frac{\rho_0}{2}(y + y^{**}) + \frac{\rho_0}{2}(y - y^{**})\,, \tag{144}$$

where in our case $r = \epsilon\rho_0$ and ρ_0 may be chosen equal to one but with dimension $\frac{1}{t}$ where t is time. With a non dimensional time in the problem this factor can simply be put equal to one. Furthermore this problem was chosen simple enough so that a closed form exact solution can be found.

This solution can be written in implicit compact form as

$$G(y, u) \stackrel{def}{=} \frac{1}{R} e^{\frac{-2r}{\sqrt{D}} \arctan\phi} = C = \text{const.}\,, \tag{145}$$

where

$$\phi \stackrel{def}{=} -\frac{r + 2b}{\sqrt{D}} + \frac{2(u - b\,q)}{\sqrt{D}(y - q)}\,, \tag{146}$$

$$D \stackrel{def}{=} -r^2 - 4b^2 - 4r\,b - 4\tilde{c} = -(r + 2b)^2 - 4\tilde{c}\,, \tag{147}$$

$$q \stackrel{def}{=} \frac{d}{b^2 + \tilde{c} + r\,b}\,, \tag{148}$$

$$\begin{aligned} R &\stackrel{def}{=} -(u - b\,q)^2 + (2b + r)(y - q)(u - b\,q) + \tilde{c}(y - q)^2 \\ &= f(y - q)^2 - [u - b\,x - \frac{r}{2}(y - q)]^2\,, \end{aligned} \tag{149}$$

$$f \stackrel{def}{=} (b + \frac{r}{2})^2 + \tilde{c} = -\frac{1}{4}D\,. \tag{150}$$

Notice that

$$\sqrt{D} = \pm 2i\sqrt{f}\,.$$

We observe that even the exact solution may not be to useful for practical purposes. However, we can now recover the same result as that previously obtained for u_0 and u_1 by a Taylor expansion of the exact solution, Eq. (145), using the discount rate r as an expansion parameter. This calculation (which has been performed) is for general initial conditions long and tedious and show the power of perturbation techniques for finding approximate solutions. Other examples using this technique dealing with economic problems can be found in the literature. See [4] -[9].

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6 APPENDICES

A Some Details

The derivation of Eq. (21)

$$\dot{\mathcal{H}} = \mathcal{H}_s + r\,m\,g = \mathcal{H}_s + r\,m\dot{y}\,. \tag{151}$$

We have

$$\mathcal{H} = e^{rs}H = F + mg = e^{rs}f + e^{rs}\lambda\,g = e^{rs}f + m\,g\,, \tag{152}$$

$$\begin{aligned}
\dot{\mathcal{H}} &= r\,\mathcal{H} + e^{rs}\frac{dH}{ds} = r\,\mathcal{H} + e^{rs}\frac{\partial H}{\partial s} \\
&= r\,\mathcal{H} + e^{rs}\frac{\partial f}{\partial s} + m\frac{\partial g}{\partial s} \\
&= r\,e^{rs}f + r\,m\,g + \frac{\partial}{\partial s}\left(e^{rs}f\right) - r\,e^{rs}f + m\,\frac{\partial g}{\partial s} \\
&= \frac{\partial \mathcal{H}}{\partial s} + r\,m\,g = \frac{\partial \mathcal{H}}{\partial s} + r\,m\,\dot{y}\,,
\end{aligned}$$

where $\mathcal{H}_s = F_s + mg_s$, thus m is to be considered as constant under the $\frac{\partial}{\partial s}$ operation here. In this new formulation we have $\mathcal{H} = \mathcal{H}(s, y, u, m))$, where m is replacing λ as variable.

B Constrained Control

Let $u \in [\alpha, \beta]$ and the condition $\mathcal{H}_u \geq 0$. Then for a binding constraint at $u = \beta$ we have

$$\mathcal{H} = F(y,\beta) + M(y,\beta)g(y,\beta)\,, \tag{153}$$

and

$$\frac{d\mathcal{H}}{dy} = F_y + M\,g_y + M_y g\,.$$

From Eq. (13) we find

$$\begin{aligned}
\dot{m} &= r\,m - \frac{\partial \mathcal{H}}{\partial y}\,,\\
\dot{y}\,\frac{\partial M}{\partial y} &= rM - F_y - M\,g_y\,,\\
g\,\frac{\partial M}{\partial y} &= rM - F_y - M\,g_y\,, \qquad \text{or}\\
\frac{d\mathcal{H}}{dy} &= F_y + M\,g_y + g\,\frac{\partial M}{\partial y} = rM\,,
\end{aligned}$$

and we see that we again recover Eq. (25).