A simple approach to global optimal feedback solutions: With emphasize on certain management problems in economics.

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ABSTRACT Dynamic optimization problems covers a great class of problems in management science and technology. The classical problem formulations being the variational approach as in classical mechanics, like Hamilton's principle and the optimal control theory in economics as the Pontryagin's maximum principle. In this account we start with a general problem formulation as an alternative to an approach based on solving differential equations. We focus on creating an analytical environment aimed at deriving global bounds and approximations. Alternative sufficient and necessary conditions for global optimal solutions are formulated and practical schemes for finding concrete solutions are presented. Optimization problems in a general setting is discussed and we define some ways to extend the problem and approximate solutions. In most of the work we restrict ourselves to problems in the setting of dynamic systems in continuous time. The *Principle of extension* is outlined and we also discuss the classical formulations i. e. the Hamiltonian and Dynamic programming formulations, in the present context. Practical application of the theory is presented as well as a summary and discussion of results.

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1 Introduction

Dynamic optimization problems covers a great class of problems in management science and technology. The classical problem formulations being the variational approach as in classical mechanics, like Hamilton's principle and the optimal control theory in economics as the Pontryagin's maximum principle. In this account we start with a general problem formulation as an alternative to an approach based on solving differential equations. We focus on creating an analytical environment aimed at deriving global bounds and approximations. Alternative sufficient and necessary conditions for global optimal solutions are formulated and practical schemes for finding concrete solutions are presented In Sec. 2 , optimization problems in a general setting is discussed. In this setting we present and define some ways to extend the problem and approximate solutions. Secondly we restrict ourselves to problems in the setting of dynamic systems in continuous time in Sec. 3. The *Principle of extension* is outlined in Sec. 4, which also contains a discussion on classical formulations i. e. the Hamiltonian and Dynamic programming formulations, in the present context. Sec. 5 deals with practical applications. A summary and discussion of results is presented in Sec. 6.

2 Some General Optimization Problems. Extensions and Approximations

The problem is to minimize a given functional $I(v)$ over a given set, D, of admissible processes v. We want to determine a given element $\bar{v} \in D$ in the case of a minimum

$$d = \min_{v \in D} I(v) = I(\bar{v}) \,, \tag{1}$$

where the functional $I(v) : D \to R$ is something that are measuring or ranging different processes.

This is a typical setting for a large class of optimization problems, and we want to start out in a general way in order not to clutter the problem formulations with unnecessary information. As already mentioned this formulation assumes that an optimal element $\bar{v}$ exist in the set D. However, notice that in many actual problem formulations of our time, this is often not the case. We give this a sharper formulation below. As an example consider phenomena that occurs in some fisheries named as "pulse-fishing". This has been attempted to be explained by so called "chattering solutions", used in the context of the management of renewable marine resources [1].

An alternative formulation in more general terms is as follows

$$I(v_s) \to d = \inf_{v \in D} I(v) \quad \text{when} \quad s \to \infty \quad \text{and} \quad \{v_s\} \subset D \,, \tag{2}$$

where the sequence v_s, which is supposed to be a minimizing sequence, does not necessarily have a limit point in D. By definition, $I(v_{s+1}) \leq I(v_s)$, $\forall s$. The sequence exist by definition of the exact lower bound (as long as the set of admissible processes is not empty). The optimal element (when it exists) may of course be regarded as a minimizing sequence where $v_s = \bar{v}$, $\forall s$.

When trying to solve the problems above, we basically consider two approaches. We may try to extend the set of admissible functions to choose from or we may try to extend the functionals used in the sorting process, which involves a redefinition of $I(v)$. In the following we shall be more specific on this issue.

We suppose there exist a set $M \supset D$, and that we have at our disposal an algorithm for constructing elements in M. We use a measure of distance satisfying

$$\rho(v) \stackrel{def}{=} \begin{cases} 0 & \text{when } v \in D \,, \\ > 0 & \text{when } v \in M - D \,. \end{cases} \tag{3}$$

The procedure is now to determine a sequence $\{v_s\} \subset M$ such that $\lim_{s\to\infty} \rho(v_s) = 0$. This approach give us a tool for the construction of elements in M.

Then consider the alternative, namely a modification of the qualifying process through the *sorting* functional. For this purpose we introduce a new functional, $L(v;\phi)$, defined on the set $M \supset D$. Here $\phi \in \Sigma$, just indicates that this new functional may depend on other parameters also. By assumption we now construct L such that $L(v;\phi) = I(v), \forall v \in D$. Thus these functionals are equal on the set of *admissible processes*, v, i. e. $v \in D$, and the problem amounts to determining a sequence, $\{v_s\} \subset M$, such that

$$L(v_s;\phi) \;\to\; d \stackrel{def}{=} \inf_{v\in M} I(v) \quad \text{and} \quad \rho(v_s) \to 0 \quad \text{when} \quad s \to \infty\,. \tag{4}$$

We may now use this to determine different candidates for approximative procedures. Suppose there is given the sets D and M such that $M \supset D$, and that there exist a measure of distance, on this set, which comply with Eq. (3). Then we introduce the concept of an ϵ-extension by the following definition

Definition 2.1 An ϵ-extension: *The set $D_\epsilon = D_\epsilon(\rho, D) \stackrel{def}{=} \{v \in M \supset D \,:\, \rho(v) \leq \epsilon$ and $\epsilon > 0\}$ we call an $\epsilon-$extension of the set D in the metric ρ.*

According to this definition we may now determine an ϵ-extended solution, that is a $v_\epsilon \in D_\epsilon$, $d = \inf_{v\in D_\epsilon} I(v)$, and this solution will be a lower bound for the actual solution, and as such it will be an approximation. This is because we are now minimizing with less restrictions, i.e., on a wider class of processes than the admissible processes, $v \in D$. We may now view the problem given by Eq. (4) in an approximative setting. In this respect we demand that in addition to being an ϵ-extended solution it should also be an η-optimal solution on D_ϵ in the usual meaning expressed by

Definition 2.2 An η-optimal process: *A process v is η-optimal on D_ϵ when $\eta > 0$, exist such that*

$$L(v;\phi) - \ell_\epsilon < \eta, \quad v \in D\,,\ \phi \in \Sigma \quad \text{where} \quad \ell_\epsilon \stackrel{def}{=} \inf_{v\in D_\epsilon} L(v;\phi)\,. \tag{5}$$

A trivial consequence of this is

$$L(v;\phi) - d \leq L(v;\phi) - \ell_\epsilon < \eta\,. \tag{6}$$

This implies that v is η-optimal on D, the set of admissible processes. In addition to this we may also consider the special cases, $\epsilon = 0$ and $\eta > 0$; as well as, $\epsilon > 0$ and $\eta = 0$, which corresponds to *η-optimal* on strictly admissible processes and *strictly optimal* on ϵ-extended processes.

Notice that $\ell_\epsilon = \ell_\epsilon(D, M, \rho)$ and the problem connected to the interrelation between different classes of problems, demands that we ask the question whether $\ell_\epsilon \to d$ when $\epsilon \to 0$. This is not obvious for all possible "extensions", that is for all possible pair (M, ρ).

3 Dynamic Optimization in Continuous Time

We consider the class of problems

$$I(v) = \int_{T_1}^{T_2} f(t, x(t), u(t))dt + F_2(T_2, x(T_2)) - F_1(T_1, x(T_1))\,, \tag{7}$$

where the state vector has to satisfy

$$\dot{x}(t) \;\stackrel{def}{=}\; \frac{d}{dt}x(t) = g(t, x(t), u(t))\,, \quad (T_i, x_i) \in W_i\,, \text{ where } x_i \stackrel{def}{=} x(T_i),\ \ i \in \{1,2\}\,, \tag{8}$$

$$v(t) \;\stackrel{def}{=}\; (x(t), u(t)) \in V(t) \quad \text{and} \quad t \in A \stackrel{def}{=} \langle T_1\,, T_2\rangle\,. \tag{9}$$

The set $V(t)$, is defined as the restriction on the path of solution $x(t)$ and $u(t)$, exclusive the restriction imposed by the equation of state, Eq. (8).

The functions F_1 and F_2 represent measures of quality for different possibilities with respect to endpoints $[(T_i, x(T_i))\,, \quad i \in \{1\,,2\}]$.

This problem can be extend to the case, $T_2 \to \infty$, and $I(v)$ is unbounded only due to an infinite time domain. This is to be interpreted as follows: $\{v_s^*\}$ is a solution of the problem if $\exists \tau$ & N such that:

$$J(v) - J(v_s^*) \geq 0 \quad \text{for} \quad t > \tau \geq T_1, \quad s \geq N \geq 0 \quad \text{and} \quad \forall v \in D$$

where

$$J(v) \stackrel{def}{=} \int_{T_1}^{t} f(s, x(s), u(s))ds - F_1(T_1, x(T_1))\,.$$

This optimality definition is termed "catcing up" optimality, see [2].

At this point, we also restrict ourselves to cases where the functions

$$f \;:\; A \times X \times U \;\to\; R\,, \tag{10}$$

$$F_i \;:\; W_i \;\to\; R \quad (i \in \{1,2\}) \tag{11}$$

$$g \;:\; A \times X \times U \;\to\; R^m\,, \quad \text{where} \quad X \subseteq R^m\,,\; U \subseteq R^n\,, \tag{12}$$

are all continuous and piecewise differentiable functions.

For the case of a fixed start- and end-condition the process is $v = (x(t), u(t))$, and for a fixed starting-point and completely free terminal/end point we have $v = (x(t), u(t, T_2, x_2))$, and so forth.

We now introduce the definition of admissible processes.

Definition 3.1 Admissible Processes:

The set D, of admissible processes, v, is defined as the set of continuous and piecewise differentiable states, $x(t)$, and piecewise continuous controls $u(t)$ which satisfy Eq. (8), and eventual extra restrictions that come into play by demanding $v \in V(t)$.

This definition is a compromise. Stronger restrictions may result in situations where "a solution" only exists in the form of a minimizing sequence. Two counteracting interests meets:

1. Simplification. For example that $(x(t),\, u(t))$ are sufficiently smooth continuous functions.

2. Sufficient compleatness of the set D.

It is worth noticing that dependent on the structure regarding $V(t)$, the original optimization problem could be simplified. In this respect it could be beneficial to look at the sets D_x and D_u as the admissible sets of states and controls respectively. [1]

Let us now introduce the following notation labeling different families of optimization problems:

1. Here $\{D, I\}$ refers to the original problem.

2. Then let $\{D, L\}$ refer to the alternative family of problems.

3. Finally let $\{D_\epsilon, I\}$ refer to the ϵ-extended family of problems.

In this way we may clarify problem areas as:

[1] In general we define V_x as the projection of V on X and V^x consists of all elements in V belonging to a given x (crossection).

1. Classical calculation of variation as $\{D_x, I\}$.
2. Classical control theory like Pontryagin and Hamilton-Jacobi-Bellman type formulations as $\{D_u, I\}$.

A natural extension of our class of admissible functions is the set E, consisting of all piecewise continuous functions, $v \in V$, and in addition having a piecewise differentiable trajectory, $x(t)$.

Such functions will normally not satisfy the equation of state, Eq. (8). A natural operational choice for the metric defined in Eq. (3) is

$$\rho(v) = \int_{T_1}^{T_2} \left| \frac{d}{dt} x(t) - g(t, x(t), u(t)) \right| dt + \sum_{t \in \psi} \left| x(t^+) - x(t^-) \right| , \tag{13}$$

where ψ is the set of t-values where the state-vector, $x(t)$, is discontinuous. We shall return to a formal introduction of the set E in sec. 4.2.

4 The Principle of Extension

4.1 Preliminary Considerations

Let us now step back and reconsider the basic optimalization problem as formulated in Eq. (2). We have extended this problem to include minimalization through minimizing sequences. We now formally introduce a one-parameter representation $L(v;\phi)$ of the functional $I(v)$, where by assumption, $\phi \in \Sigma$ (to be specified later).

Definition 4.1 *Let $L(v;\phi)$ be any functional that satisfy*

$$L(v;\phi): \; M \times \Sigma \to R \,, \quad \& \quad L(v;\phi) = I(v) \,, \quad \forall v \in D \,, \quad \phi \in \Sigma \,. \tag{14}$$

Here we make an extension of the original problem by adding the basic set $M = M(\phi) \supseteq D$, and instead look at a family of minimization problems defined by $\{M, L\}$. In other words we look for a solution v, which solves the problem of finding

$$\ell(\phi) \stackrel{def}{=} \inf_{v \in M} L(v;\phi) \,, \quad -\infty < \ell(\phi) \leq d \,. \tag{15}$$

The philosophy behind this is that by proper choice of the set $M(\phi)$ (so far not specified), it may be easier to construct the elements of this set than for the original D, and we can more easily generate lower bounds for the original *infimum*, d, by properly selecting ϕ-values in a prescribed set Σ. The strategy is then to select the best of these values of lower bounds as an approximation to the solution of the original problem, thus let

$$\ell^* \stackrel{def}{=} \sup_{\phi \in \Sigma} \ell(\phi) \,, \tag{16}$$

where ℓ^* is now the best approximation obtained from this procedure. In the cases where $\ell(\phi) \to -\infty$ when $T_2 \to \infty$, there still is the possibility for further sorting concerning practical applications. The concept of e.g. "catching up optimality ", see [2] may often be applicable.

If the element, $\bar{v} = \bar{v}(\phi) \in D$, or the sequence $\{\bar{v}_s(\phi)\} \subset D$, is a solution to the extended problem $\{M, L\}$, (that is; there exist a choice of the parameter $\phi = \bar{\phi}$, that makes the members of this sequence *an admissible function*) then the following result apply:

Proposition 4.1 *Let a parameter, $\phi = \bar{\phi} \in \Sigma$, exist such that for the minimizing sequence $\bar{v}_s(\bar{\phi})$ we have $\{\bar{v}_s(\bar{\phi})\} \subset D$, $\forall s$. Then it follows that*

$$\lim_{s \to \infty} I(\bar{v}_s) = \inf_{v \in D} I(v) = \max_{\phi \in \Sigma} \ell(\phi) = \ell(\bar{\phi}) = \ell^* = d \,. \tag{17}$$

This means that the sequence, $\{\bar{v}_s(\bar{\phi})\}$, minimizes the functional I on the set D and that $\ell(\bar{\phi})$ is the exact lower bound. Furthermore the pair $(\{\bar{v}_s\},\bar{\phi})$ is the solution to the dual problem.

Proof:

Since $\bar{v}_s \in D$, by definition we have $L(\bar{v}_s,\phi_s) = I(\bar{v}_s)$. It remains to prove that $\ell(\bar{\phi}) = d$. Suppose this is not the case, then it follows that $d > \ell(\bar{\phi})$. Moreover since the sequence $\{\bar{v}_s\} \subset D$ there exist an $\epsilon > 0$ such that $L(\bar{v}_s,\bar{\phi}) = I(\bar{v}_s) \geq d \geq \ell(\bar{\phi}) + \epsilon$, and this must be true for all s. This is contrary to the definition which demands that $L(\bar{v}_s,\bar{\phi}) \to \ell(\bar{\phi})$. The case $d \leq \ell(\bar{\phi})$ is true by definition. We see that the dual problem is part of the construction (the quantities linked by the second equality sign in Eq. (17)).

Q.E.D.

This result offers us the possibility to replace the minimization problem $\{D, I\}$ with a family of problems $\{M, L\}$. These problems may now be solved in conjunction with the determination of the parameter $\phi = \bar{\phi}$, so that the solution becomes "admissible", i.e. $\bar{v}(\bar{\phi}) \in D$.

Three conditions must be fulfilled for the principle to be useful:

1. The new problem formulation must offer some structural simplifications.
2. The family of representation, $L(v;\phi)$, $\phi \in \Sigma$, must be sufficiently large so that it ensures the existence of a parameter $\bar{\phi}$ that makes $v(\bar{\phi})$ belong to the admissible processes, that is: $v(\bar{\phi}) \in D$, not only $v(\bar{\phi}) \in M(\phi)$.
3. Finally an efficient method to determine $\phi = \bar{\phi}$, must be available.

In order to continue developing these ideas we are now ready to introduce the concept of *equivalent extension*.

Definition 4.2 Equivalent-extension:

We have that $\{M, L\}$ is an equivalent extension of $\{D, I\}$ if there exist a sequence $\{v_s\} \subset D$ such that $I(v_s) \to L(v;\phi)$, $\forall\, v \in M(\phi) \supset D$.

Comment: Equivalent extension corresponds to a closure of the original problem.

4.2 Ignoring The Equation of State

As a first approach we shall ignore the dynamics (the state equations) in the control problem. The problem then reduces to finding the minimum value of $I(v)$, where v can now belong to a greater set than the set of admissible processes, on the interval $A = [T_1, T_2]$. We define a set extension that we will make frequently use of

Definition 4.3 *The set E is the set consisting of all piecewise continuous functions with a piecewise differentiable trajectory, $x(t)$, defined over the interval $A = [T_1, T_2]$, which comply with all other restrictions except the equation of state, Eq. (8).*

Thus the problem $\{E, I\}$ is an extension of $\{D, I\}$. Occasionally it may also be meaningful to drop the requirement of differentiability since the equation governing the processes (or some of them) is omitted, this may, however, require an adjustment of the measure of distance or metric presented in Eq. (13).

Then let $I(v)$ be given by Eq. (7), and for the sake of simplicity let start and end conditions be given. Then we may split the problem $\{E, I\}$ in the following family of problems: In every instant of time we seek the solution $\tilde{v}(t) = (\tilde{x}(t), \tilde{u}(t))$ that minimizes the integrand f. Let this be denoted by $w(t)$, thus

$$\min_{v \in E} f(t, v) = f(t, \tilde{v}) = w(t)\,. \tag{18}$$

Clearly we now have a lower bound on $I(v)$ on D, $w(t)$. It is also self evident that if $\tilde{v} \in D$, this is a sufficient condition for the process to be optimal, proposition 4.1. Notice that this condition for optimality is *strong* in the sense that by its nature it is *global*. We notice that the reason we may minimize pointwise in time, is because the dynamic process governing equation is omitted.

We then have that problem $\{E, I\}$ is an extension of problem $\{D, I\}$ which may be incorporated in $\{L(v;\phi), E\}$. Thus the class of minimization problems is extended to also finding a parameter $\bar{\phi}$. Every such parameter corresponds to an equivalent extension. We then have the following result:

Proposition 4.2 *If by assumption*

$$\exists \bar{\phi} \in \Sigma\,, \text{ and } \bar{v} \in D : \inf_{v \in D} I = L(\bar{v}; \bar{\phi}) = d\,, \tag{19}$$

then

$$\min_{v \in E\,,\ \phi \in \Sigma} L(v;\phi) \ \leq\ L(\bar{v};\bar{\phi}) \ \leq\ L(v;\phi)|_{v \in D,\, \phi \in \Sigma}\,. \tag{20}$$

Proof:

The first inequality is established on the basis that the minimum on the left hand side is calculated with less restrictions than the right hand side.

The second inequality is true because $v \in D$ and $\phi \in \Sigma$ are arbitrarily chosen and the equal sign apply when $v = \bar{v}$ and $\phi = \bar{\phi}$.

Q.E.D

These inequalities may be the basis for an approximative approach to finding a solution.

Apparently these ideas may in particular be useful in cases where the admissible processes only permit a minimizing sequence. This results in the following proposition:

Proposition 4.3 *Let the function $f(t,x,u)$ be continuous on $A \times X \times U$. The function $w(t)$ is defined and continuous according to Eq. (18). Moreover, suppose there exist a sequence $\{\bar{v}_s \in D\}$ such that*

1. *$f(t, \bar{v}_s(t)) \to w(t)$ in measure relative to A.*
2. *There exist a number Q, such that $f(t, \bar{v}_s(t)) < Q\,, \quad \forall (t,s)$.*

It then follows that

$$\lim_{s \to \infty} I(\bar{v}_s) = \inf_{v \in D} I(v) = \ell \stackrel{def}{=} \int_{T_1}^{T_2} w(t)dt\,. \tag{21}$$

Proof:

The function f is continuous and the function $\bar{v}_s(t) \in D$ is continuous almost everywhere. It follows that for any s the function $f_s = f(t, \bar{v}_s(t))$ is continuous almost everywhere and therefore f_s makes up a bounded sequence such that $w(t) \leq f(t, \bar{v}_s(t)) < Q\,, \quad \forall (t,s)$ and furthermore $f(t, \bar{v}_s(t)) \to w(t)$ on A.

From Lebesgue's theorem it follows that

$$\lim_{s \to \infty} I(\bar{v}_s) = \ell \stackrel{def}{=} \int_{T_1}^{T_2} w(t)dt\,.$$

Q.E.D.

Thus the number ℓ is a lower bound for the functional $I(v)$ over the set of admissible processes.

4.3 Equivalent Representation

We are now in the position where we have at our disposal a number of tools that make us equipped for practical use. Here we will limit ourselves to dynamic problem formulations in continuous time.

Definition 4.4 The set Σ
The set Σ is the set of all real differentiable functions: $\phi = \phi(t,x) : A \times X \to R$.

We look at the problem $\{D, I\}$ where the functional I is defined in Eq. (7), and the corresponding one-parameter family of functionals $L(v;\phi)$ (definition 4.1), which we restrict to the following form:

Definition 4.5 *Let the functional $L(v;\phi)$ be defined as*

$$L(v;\phi) \stackrel{def}{=} G_2(T_2,x_2) - G_1(T_1,x_1) - \int_{T_1}^{T_2} R(t,x(t),u(t))dt\,, \tag{22}$$

where the parameter function $\phi \in \Sigma$, and the functions in the expression above are given by:

$$G_k(t,x) \stackrel{def}{=} F_k(t,x) + \phi(t,x)\,, \qquad k \in \{0,1\}\,, \tag{23}$$

$$R(t,x,u) \stackrel{def}{=} \frac{\partial\,\phi(t,x)}{\partial t} + g(t,x,u)\cdot\frac{\partial\,\phi(t,x)}{\partial x} - f(t,x,u)\,. \tag{24}$$

In the definition above we have that $x \in X$, and $u \in U$, can both be vectors of arbitrary but given dimensions, in which case the "dot" in Eq. (24) denotes the Euclidian inner product. The functions F_k represent a quality measure related to different initial and final possibilities. We have $(T_i, x(T_i)) \in W_i$, which also implies an optimization over W_i. Fixed end conditions is represented by $G_k = 0$, $k = 1,2$.

Proposition 4.4 *The functionals $L(v;\phi)$ and $I(v)$ coincide for all admissible processes $v \in D$, and functions $\phi(t,x) \in \Sigma$.*

Proof:
The result follows trivially from the fact that for all processes $v \in D$, we may write

$$R(t,x(t),u(t)) = \frac{d}{dt}\phi(t,x) - f(t,x(t),u(t))\,.$$

Q.E.D.

Thus it is obvious that $\{D, I\}$ and $\{D, L\}$ are equivalent problems. This gives us a starting point for extension to $\{E, L\}$. We can also use this to solve an improved problem connected to the process $v \in D$ by choosing $L(v;\phi)$ in such a way that it becomes obvious how to choose an admissible process $v \in D$, and such that $L(v;\phi) = I(v) < I(v_0) = L(v_0,\phi)$.

The two functionals L and I will in general not coincide for non admissible processes ($v \notin D$). As an example consider the previously defined E (see Sec. 4.2), when $v \in E$ do not satisfy the process equation. In this case we have

$$L(v;\phi) = I(v) + \int_{T_1}^{T_2} \frac{\partial\phi(t,x)}{\partial x}\cdot h(t)dt + \Sigma_{t\in\psi}[\phi(t,x(t))]_{t^-}^{t^+}\,, \tag{25}$$

$$h(t) \stackrel{def}{=} \frac{d}{dt}x(t) - g(t,x(t),u(t))\,, \tag{26}$$

where ψ is the set of instants in time where the state vector is discontinuous, and the function $h(t)$ represent the "mismatch" in the equation of state.

The functional L has a couple of nice properties. In general it is important to bear in mind that $L(v;\phi_1) \neq L(v;\phi_2)$ for $\phi_1 \neq \phi_2$.

Proposition 4.5 *The functional L satisfy the following properties:*

1. *Invariance under special translations*

$$L(v;\phi+\eta(t)) = L(v;\phi)\,, \tag{27}$$

where $\eta(t)$ is an arbitrary differentiable function of time only.

2. *In particular we also have*

$$L(v;\phi_1) = L(v;\phi_2)\,, \quad v\in D\,, \quad \{\phi_1,\phi_2\}\subset\Sigma\,. \tag{28}$$

Proof:
We have from definition 4.5 that

$$L(v;\phi+\eta) - L(v;\phi) = \eta(T_2) - \eta(T_1) - \int_{T_1}^{T_2} \frac{d}{dt}\eta(t)dt = 0\,.$$

The last result follows from the definition given in Eq. (14).

Q.E.D.

4.4 Global Considerations

Consider the case where a minimum exist (see Eq. (1)) . We start from Eqs. (22) - (24) and make some useful definitions:

Definition 4.6

$$\mu(t) \stackrel{def}{=} \sup_{v(t)\in V} R(t,v(t))\,, \quad \forall t\in\langle T_1,\,T_2\rangle\,, \tag{29}$$

$$m \stackrel{def}{=} \min G_2(T_2,x)\,, \quad x\in W_2\,, \tag{30}$$

$$M \stackrel{def}{=} \max G_1(T_1,x)\,, \quad x\in W_1\,, \tag{31}$$

$$\ell(\phi) \stackrel{def}{=} m - M - \int_{T_1}^{T_2}\mu(s)ds\,. \tag{32}$$

Definition 4.7 Bounding function
Any $\phi=\bar\phi\in\Sigma$ that makes $\ell(\bar\phi)$ exist is called a bounding function.

When a bounding function exist, $\ell(\bar\phi)$, emerge as a lower bound for $L(v;\phi)$, i.e., $L(v;\phi)\geq \ell(\bar\phi)$ for $v\in E$ and all $\phi\in\Sigma$. Thus we have

Proposition 4.6 Lower bound *The functional $\ell(\phi)$ is a lower bound for the functional $I(v)$, or*

$$\ell(\phi)\leq I(v), \quad v\in D \quad \forall\phi\in\Sigma\,. \tag{33}$$

Proof: This result follows immediately from the definitions and the remarks above.

Q.E.D.

Proposition 4.7 Solving function I *Let a bounding function $\bar\phi(t,x)$ exist such that the associated process $\bar v=\bar v(\bar\phi)$ is defined and admissible (i.e. $\bar v\in D$). Then it follows that the pair $(\bar\phi,\bar v)$ is a solution of the dual problems*

$$I(\bar v) = \min_{v\in D} I(v) = \ell(\bar\phi) = \max_{\phi\in\Sigma}\ell(\phi)\,. \tag{34}$$

Proof: From the definition of $L(v;\phi)$, definition 4.5 and Eqs. (29) - (32) and proposition 4.6, we find $I(\bar{v}) = L(\bar{v};\bar{\phi}) = \ell(\bar{\phi})$. The last quantity is also a lower bound since $I(v) \geq \ell(\bar{\phi})$ for all $v \in D$ and by definition $\ell(\phi) \leq \ell(\bar{\phi})$ for all $\phi \in \Sigma$. Thus the proposition is proved. This result also follows from proposition 4.1.

Q.E.D.

In other words this is a global sufficiency condition and in addition the pair $(\bar{v}, \bar{\phi})$ solves the dual problem. This proposition can be given an alternative formulation as follows

Proposition 4.8 Solving function II *Let a bounding function $\bar{\phi}(t,x)$ and an admissible process $\bar{v} = (\bar{x}(t), \bar{u}(t))$ exist such that*

$$\mu(t) = R(t, \bar{x}(t), \bar{u}(t)) = \max_{(x,u)\in V(t)} R(t,x,u) \quad \text{for} \quad t \in \langle \tau, T \rangle . \tag{35}$$

Then the pair $(\bar{\phi}, \bar{v})$ solves the dual problems given by Eq. (34).

Proof: We observe that by Eq. (32), Eq. (34) implies the existence of $\ell(\phi)$. The arguments from the previous proposition, 4.8, then applies.

Q.E.D.

We also need a formulation of this proposition that can be applied to problems when the optimal element $\bar{v} \notin D$, but a $\bar{\phi} \in \Sigma$ exists.

Proposition 4.9 Solving function III *Let a bounding function $\bar{\phi}(x,t)$ exist and a sequence of admissible processes $\{v_s\} = \{x_s(t), u_s(t)\} \subset D$, such that*

$$\int_\tau^T \{R(t, v_s(t)) - \mu(t)\} dt \quad \to 0 , \tag{36}$$

then it follows that

$$I(v_s) \quad \to \quad \inf_{v\in D} I(v) = \ell(\tilde{\phi}) = \max_{\phi\in\Sigma} \ell(\phi) . \tag{37}$$

Proof: We have a bounding function, $\bar{\phi}$, which implies by definition that

$$\ell(\bar{\phi}) = \min_{x\in W} G_2 - \max_{x\in W_0} G_1 - \int_\tau^T \mu(t) dt$$

exist. Furthermore also by definition

$$I(v_s) = L(v_s;\phi) = L(v_s;\bar{\phi}) = G_2 - G_1 - \int_\tau^T R(t, v_s) dt .$$

Then let the end-choices be the same as for $\ell(\bar{\phi})$, and consider the difference

$$I(v_s) - \ell(\bar{\phi}) = -\int_\tau^T \{R(t, v_s) - \mu(t)\} dt \to 0 .$$

This implies that

$$I(v_s) \to \inf_{v\in D} I(v) = \ell(\bar{\phi})$$

and therefore $\{v_s\}$ is a minimizing sequence. Since the maximum exist in the dual problem the last equality is trivial.

Q.E.D.

Notice that this last proposition may be extended to cover the cases where we do not have a bounding function in Σ, but only a sequence $\{\phi_s\} \subset \Sigma$, and the sequences $\{R(t, v_s)\}$ and $\{\mu_s(t)\}$. If $R_s \to \mu_s$ weakly, that is in the integrated form, then a similar proposition exist with the change that the last max operator in Eq. (37) is replaced by sup.

4.5 Classical Control Theory

The Classical Control Theory can easily be incorporated as special cases of this theory. Classical Control Theory has two principle formulations.

- The variational approach that results in a Hamiltonian formulation through Hamilton's equations.
- The principle of Dynamic Programming resulting in the Hamilton-Jacobi-Bellman equation.

4.5.1 Variational, Hamiltonian formulation

Consider the family of problems given by Eq. (22), restricted to the case where we have no end contributions. In this case the problem reduces to finding the maximum of R. For this purpose we look for a local maximum by examining

$$R_x = 0 = \phi_{tx} + H_x + \phi_{xx} H_{\phi_x} \,,$$

where we have introduced

$$H(t, x, u, \phi_x) \stackrel{def}{=} \phi_x \cdot g(t, x, u) - f(t, x, u) \,.$$

For an admissible state $x = \bar{x}(t)$, we define the function $\lambda(t) \stackrel{def}{=} \phi_x(t, \bar{x}(t))$, obtaining

$$\dot{\lambda} = -H_x \,.$$

Admissibility implies that the state equation

$$\dot{x} = g(t, x, u) \quad \text{or} \quad \dot{x} = H_\lambda \,,$$

is satisfied. In addition we also have for an internal maximum that

$$R_u = 0 \;\Leftrightarrow\; H_u = 0 \,,$$

or in a more general setting, the optimal control u^*, is given by

$$u^* = \underset{u}{\operatorname{argmax}} \; H \,.$$

These equation implies, with some regularity conditions, that we also have

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \,,$$

Thus we have recovered the usual conditions for an optimal solution formulated within the framework of a Hamiltonian setting. Incorporating end conditions resulting in transversality conditions at the endpoints require a little more work, but can easily be done.

We conclude that our formulation incorporate the usual formulation obtainable through a variational approach.

4.5.2 Dynamic Programming Principle

For the purpose of incorporating the Dynamic Programming Principle in this theory, consider a special formulation given by Eq. (22). Furthermore consider the case $G_2 \equiv 0$, and a family of initial value problems where the choice $T_1, x(T_1)$, belongs to the set of admissible inputs to the control $U(t, x) = V^x(t) \subset R^n$ (see footnote 1), which is supposed to be well defined and not empty for every $(t, x) \in A \times R^n$.

The problem of determining an optimal feedback control $u(t, x)$, is equivalent to finding the solution to the family of optimal control problems with initial condition $(T_1, x_1) \in U(t, x)$.

Let $\tilde{\phi} \in \Sigma$ be a bounding function according to definition 4.7.

Furthermore, let the control corresponding to this $\phi = \tilde{\phi}$, be $\tilde{u}(t,x)$. We now introduce a new function P by

$$P(t,x) \stackrel{def}{=} \max_{u \in U(t,x)} R(t,x,u) = R(t,x,\tilde{u}(t,x)) \,. \tag{38}$$

Alternatively we also say that the process $\tilde{v} = (\tilde{x}(t),\, \tilde{u}(t))$, corresponds to a $\tilde{\phi}(t,x)$, when a $\phi = \tilde{\phi}$, is associated with $\tilde{v}(t,x)$.

Then by assumption let a bounding function $\phi \in \Sigma$ exist such that

$$P(t,x) = c(t)\,, \quad \forall x \in X,\ \ t \in (T_1, T_2)\,, \qquad G_2(x) \stackrel{def}{=} F_2(x) + \phi(T_2,x) = C\,, \quad \forall x \in X\,. \tag{39}$$

It then follows from proposition 4.4, that the corresponding process must be optimal, and proposition 4.5, permits us to use a translation in ϕ to adjust for the function $c(t)$ and the constant C to be zero. Furthermore by assumption $F_1 \equiv 0$ (Eq.(23)) and $\phi(T_2,x) = -F_2(x)$. Since the corresponding process is optimal $d = \ell(\phi) = -m = -\phi(t,x)$, this opens for the following choice

$$\phi(t,x) = -d(t,x)\,, \quad \text{(the valuefunction)}\,.$$

Then returning to Eq. (38) we have that $P = 0$ gives

$$\max_{u \in U(t,x)} \{\phi_t + \phi_x \cdot g(t,x,u) - f(t,x,u)\} = 0\,, \tag{40}$$

or

$$\phi_t = -\max_{u \in U(t,x)} \{\phi_x \cdot g(t,x,u) - f(t,x,u)\}\ , \tag{41}$$

which we recognize as the Hamilton-Jacobi-Bellman equation. Introducing the Hamiltonian

$$H^*(t,x,\phi_x(t,x)) \stackrel{def}{=} \max_{u \in U(t,x)} H(t,x,u,\phi_x)\,. \tag{42}$$

Eq. (41) may also be written as

$$\phi_t = -H^*(t,x,\phi_x(t,x))\,. \tag{43}$$

4.6 Bounds on accuracy

An important task in this work is to determine bounds on accuracy. This is of prime importance because this is the only way to determine the quality of a given approximate solution.

In this connection we shall now focus on $\epsilon-$ optimal solutions.

Let $\bar{\phi}$ be a *bounding function*. That is $\ell(\bar{\phi})$ exist and is a lower bound on $I(v)$ and $L(v;\phi)$.

Then

$$\ell(\bar{\phi}) \le L(v;\phi)\,, \quad \forall\, v \in E \quad \text{and} \quad \forall\, \phi \in \Sigma\,. \tag{44}$$

Especially we have

$$\ell(\bar{\phi}) \le d = \inf_{v \in D} I(v)\,. \tag{45}$$

Then consider the difference

$$I(v) - d = L(v;\phi) - d \le L(v;\phi) - \ell(\bar{\phi}) \quad \forall v \in D \ \&\ \forall \phi \in \Sigma\,,$$

where

$$
\begin{aligned}
L(v;\phi) - \ell(\bar{\phi}) &\stackrel{def}{=} G_2(T,x(T)) - G_1(\tau,x(\tau)) - \int_\tau^T R\,dt - \left\{ m - M - \int_\tau^T \mu(t)\,dt \right\} \\
&= G_2(T,x(T)) - \min_{x\in W} G_2(T,x) + \max_{x\in W} G_1(\tau,x) - G_1(\tau,x(\tau)) \\
&\quad + \int_\tau^T \left\{ \max_{(x,u)\in V} R(t,x,u) - R(t,x(t),u(t)) \right\} dt\,,
\end{aligned}
$$

where W correspondes to W_2. For the case of simplicity let us consider those cases where we have a fixed starting state and final state, x_τ and x_T in addition to a given time horizon, T.

We then obtain

$$
\begin{aligned}
I(v) - d &= L(v;\phi) - d \le \Delta(\bar{\phi}) \stackrel{def}{=} L(v;\phi) - \ell(\bar{\phi}) \\
&= \int_\tau^T (\max_V R - R)dt \qquad (46) \\
&\le \int_\tau^T (\max_V R - \min_V R)dt \stackrel{def}{=} \bar{\Delta}(\bar{\phi})\,. \qquad (47)
\end{aligned}
$$

Remember that V does not contain the restriction imposed by the equation of state.

This is a drastic simplification and might result in a bound that is "too weak ". The associated ($\bar{v}$ associated with $\bar{\phi}$) is $\bar{\Delta}$-optimal, where $\bar{\Delta}$ is defined above.

4.7 Summary

The traditional approach for solving optimal dynamic control problems is the Hamiltonian formulation and the Hamilton- Jacobi - Bellman equation.

We have presented a more general approach incorporating these formulations as special cases. This was discussed in Sec. 4.5. In Sec. 4.4 the essential tools for working with this new formulation was presented, for example the inequalities (20) and (33).

Hunting for the optimal solution by solving differential equations that orignited from either standard control theory via Hamiltonian formulation or from the dynamic programming principle through the solution of the Hamilton-Jacobi-Bellman equation, can be very cumbersome.

Very few problems can be solved exactly. One has to resort to approximate solutions for most realistic problems, either by finding approximate numeric solutions or approximate analytic solutions. In this context the question of accuracy is an important issue.

Our approach makes it much easier to evaluate accuracy, especially when dealing with closed form approximate solutions. This new approach offers a straight forward technique, its main virtue is simplicity in terms of how to implement it.

Regarding the accuracy some general remarks may be appropriate. One should bear in mind that all *models* are inaccurate when it comes to describing the "real world". Thus it may not be meaningful to spend too much effort in order to improve accuracy, or finding accurate solutions to a specific problem unless one is sure that features has not been left out from the model that might introduce greater errors.

In particular we want to point out that our approach can be implemented in a systematic way also in conjunction with perturbation/expansion techniques used to find approximate solutions through differential equations formulations. This will be demonstrated in a following section.

In conclusion it is our belief that for a large class of problems sufficiently accurate solutions can easily be obtained by our direct approach.

5 Applications

5.1 A special class of problems

In management of renewable resources the following class of problems are of interest,

$$\max_u \int_{T_1}^{T_2} e^{-\delta t}\Pi(x,u)dt\,, \tag{48}$$

where $x \in X$ and $u \in U$ with $X = [0,k]$, $U = [0,\infty\rangle$, $\Pi \stackrel{def}{=} \gamma u - \Gamma u^2$ where $\gamma = \gamma(x)$ and $\Gamma = \Gamma(x)$ are given non negative functions of the argument x and u may be interpreted as the harvest rate. The maximization is to be performed subject to the constraint (state equation),

$$\dot{x} \stackrel{def}{=} \frac{dx}{dt} = g(x) - u\,. \tag{49}$$

From Eq. (24) we find

$$R = \phi_t + \phi_x(g-u) + e^{-\delta t}\Pi \tag{50}$$

Since ϕ is arbitrary (bounding) in this formulation we can make the following choice

$$\phi = W(x)e^{-\delta t} - \frac{1-e^{-\delta t}}{\delta}K\,. \tag{51}$$

Here ϕ can be interpreted as a value function. By this choice we have

$$\begin{aligned} e^{\delta t}R &= -K - \delta W + W'(g-u) + \gamma u - \Gamma u^2 \\ &= -K - \delta W + W'g + \frac{1}{4\Gamma}(\gamma - W')^2 - \Gamma\left\{u - \frac{\gamma - W'}{2\Gamma}\right\}^2\,. \end{aligned} \tag{52}$$

This form is quadratic in u, which means that the optimal value of u is obtained by eliminating the last term, and thereby maximizing R. This way u is determined as

$$u = \frac{\gamma - W'}{2\Gamma}\,. \tag{53}$$

Since u in our context represent a harvest function it follows that $u \geq 0$. Thus we have

$$u = \max\left\{0, \frac{\gamma - W'}{2\Gamma}\right\}\,. \tag{54}$$

Introducing $\tilde{P}$ for this particular choice of R, Eq. (52), we may write

$$e^{\delta t}\tilde{P}(x,t) \stackrel{def}{=} \begin{cases} -K - \delta W + W'g + \frac{1}{4\Gamma}(\gamma - W')^2\,, & \gamma \geq W' \\ -K - \delta W + W'g\,, & \gamma < W' \end{cases}\,, \tag{55}$$

where $W' \in C^1$ and W and K, are in principle free. However, we now want to focus on a particular choice of these parameters which comply with our previous perturbation expansion approach [4] Eq. (114) and Eq. (116), we obtain to leading and first order in the expansion

$$-K_0 + e^{\delta t}P(x,u_0) = 0 \tag{56}$$

$$-W_0 - K_1 + e^{\delta t}P_u(x,u_0)u_1 = 0 \tag{57}$$

where for this case

$$e^{\delta t}P(x,u) \stackrel{def}{=} \Pi + \Pi_u\,(g-u)\,. \tag{58}$$

Thus it follows that

$$e^{\delta t}P(x,u) = \Gamma u^2 - 2\Gamma g u + \gamma g\,. \tag{59}$$

We now pick as the first approximation the zeroth order expansion solution which imply, when $\gamma \geq W'$, that

$$u = u_0 = \frac{\gamma - W_0'}{2\Gamma}\,, \quad W = W_0\,, \quad K = K_0\,. \tag{60}$$

From Eq. (56) we then obtain

$$-K_0 + \Gamma u_0^2 - 2\Gamma u_0 g + \gamma g = -K_0 + \Gamma u_0^2 + W_0' g = 0\,,$$

or

$$K_0 = \Gamma u_0^2 + W_0' g\,. \tag{61}$$

Making use of this result in Eq. (55) we obtain $\tilde{P} \to \tilde{P}_1$, where

$$\tilde{P}_1 \stackrel{def}{=} -\delta W_0 e^{-\delta t}\,. \tag{62}$$

Notice that $\tilde{P}_1 = \mathcal{O}(\delta^2)$, relative to a "typical" value determined by Eq. (48) when sitting in "a fixed equilibrium point". This value turns out to be of $\mathcal{O}(\frac{1}{\delta})$. We conclude that the zeroth order perturbation solution makes $\tilde{P}$ a first order quantity in the parameter of smallness, δ, whereas the integrated form, the value function is of zeroth order and first order in the relative sense.

Continuing we consider the first order perturbation as the approximate solution with

$$\begin{aligned} u &= u_0 + \delta\, u_1\,, \\ W &= W_0 + \delta\, W_1\,, \\ K &= K_0 + \delta\, K_1\,. \end{aligned}$$

From Eq. (57) we obtain

$$-W_0 - K_1 + [2\Gamma u_0 - 2\Gamma g]u_1 = -W_0 - K_1 + 2\Gamma u_0 u_1 + g W_1' = 0 \tag{63}$$

where we have used that according to Eq. (53) we have

$$u_1 = -\frac{W_1'}{2\Gamma}\,.$$

We then obtain

$$K_1 = -W_0 + 2\Gamma u_0 u_1 + g W_1' = -W_0 + (g - u_0) W_1'\,. \tag{64}$$

Further more we obtain

$$e^{\delta\, t}\tilde{P}(x,u) = -K_0 - \delta\, K_1 - \delta\, W_0 - \delta^2\, W_1 + W_0' g + \delta\, W_1' g + \Gamma(u_0 + \delta\, u_1)^2 = \delta^2 \left[\frac{1}{4\Gamma}(W_1')^2 - W_1\right]\,.$$

Thus invoking the first order perturbation expansion as our approximate solution makes $\tilde{P} \to \tilde{P}_2$. The zeroth and first order contributions to $\tilde{P}$ vanish and the result is

$$\tilde{P} \to \tilde{P}_2 \stackrel{def}{=} \delta^2 e^{-\delta\, t}\left[\frac{1}{4\Gamma}(W_1')^2 - W_1\right]\,. \tag{65}$$

Also notice that

$$\tilde{P}_1 = -\delta\, W_0 e^{-\delta\, t} < 0\,. \tag{66}$$

An observation we make from this result is that the value function need only be determined to one order lower than the control. This is important, since it relaxes the necessary amount of computations to be performed when determining the optimal control.

5.2 An Example

Consider the following example relevant to fisheries, with a utility function

$$\hat{\Pi} = e^{-\delta s}\left\{(1-\frac{z_0}{z})h - \hat{\Gamma}h^2\right\}, \tag{67}$$

and a state equation

$$\dot{z} = \tilde{R}z(1-\frac{z}{k}) - h, \tag{68}$$

where s is time, z the stock, z_0 refers to a break even point in the utility function, h is harvest, k is carrying capacity and $\hat{R}$ represent the reproduction rate or intrinsic growth rate in the stock. $\hat{\Gamma}$ is related to a nonlinear cost of operation or downward sloping demand. We now make the following substitutions/definitions:

$$u \stackrel{def}{=} \frac{1}{\hat{R}k}h, \quad x \stackrel{def}{=} \frac{z}{k}, \quad x_0 \stackrel{def}{=} \frac{z_0}{k}, \quad t \stackrel{def}{=} \hat{R}s, \quad r \stackrel{def}{=} \frac{\delta}{\hat{R}}, \quad \Gamma \stackrel{def}{=} \hat{R}k\hat{\Gamma}. \tag{69}$$

By these substitutions the problem simplifies as follows

$$\Pi \stackrel{def}{=} \frac{1}{\hat{R}k}\hat{\Pi} = e^{-rt}\{(1-\frac{x_0}{x})u - \Gamma u^2\} \tag{70}$$

$$\dot{x} = x(1-x) - u. \tag{71}$$

We notice that the problem in this setting has three parameters r, x_0 and Γ.

We now elaborate on this example according to the procedure used in Sec. 5.1. From Eq. (48) we have $\gamma(x) \stackrel{def}{=} 1 - \frac{x}{x_0}$ and Γ is a constant. According to Eq. 58 we find

$$\begin{aligned} P(x,u) &= (1-\frac{x_0}{x})u - \Gamma u^2 + \{(1-\frac{x_0}{x}) - 2\Gamma u\}\{x(1-x) - u\} \\ &= \Gamma u^2 - 2\Gamma u x(1-x) + (x-x_0)(1-x) \end{aligned} \tag{72}$$

From [4] Eqs. (121) and (122) we find that K_0 and the proper equilibrium point $y^{**} \in \{\tilde{x}\}$ ($\tilde{x}$ the set of possible equilibrium points), is determined as the value that maximizes H (the detailed account of this procedure is included here for the convenience of easy reading):

$$e^{\delta t}H \stackrel{def}{=} \mathcal{H} = \Pi(x, u_0(x)) + M(x, u_0(x) \cdot \dot{x}, \tag{73}$$

at the point where $\dot{x} = 0$, or

$$y^{**} \stackrel{def}{=} \underset{G(x,u_0(x))=0}{\operatorname{argmax}} \Pi(x, g(x)). \tag{74}$$

In this particular case we have

$$\Pi(x, g(x)) = (1-\frac{x_0}{x})\, g(x) - \Gamma g^2(x), \tag{75}$$

where $\dot{x} = 0 \Rightarrow u_0 = g(x) = x(1-x)$. This way $K_0 \stackrel{def}{=} -F(y^{**}, u_0(y^{**}))$ is determined and we find for the case $\Gamma = 0.1$, $x_0 = 0.2$ that $y^{**} = 0.6050$, $u_{00} = u_0(y^{**}) = 0.23897$, $P(y^{**}, u_0(y^{**})) = F(y^{**}, u_0(y^{**})) = 0.15426$ or $K_0 = 0.15426$. By solving $P(x, u_0(x) = K_0$ for the given value of K_0 we find the feedback solution drawn in the following plot presented in Fig. 1.

The curved line above the horizontal axis represents natural growth function for the stock, and at the equilibrium point for the system $u = u_{00} = u_0(y^{**}) = 0.23897$. There are two branches for the feedback solutions. The proper one starts with a positive slope and moves towards the equilibrium point from either side. This curve is plotted. The other branch is unstable i.e. moves away from the equilibrium point on either side. This curve is not plotted. We conclude that we have the zeroth order solution and can also find the

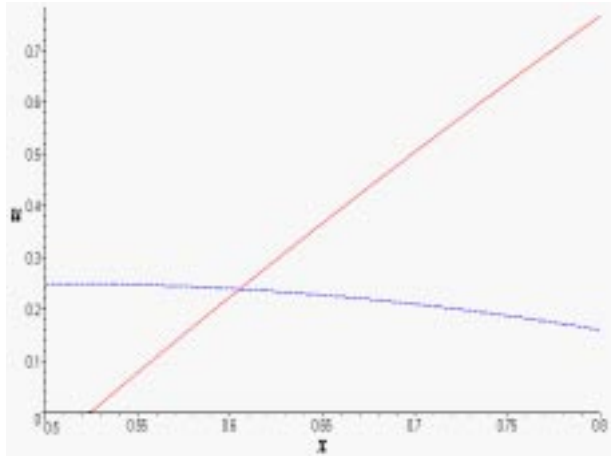

Figure 1: Plot for u versus x for the cases $r = 0$ and $dx/dt = 0$ or $u = x(x-1)$, as the curved line just above the x-axis. $\Gamma = 0.1$.

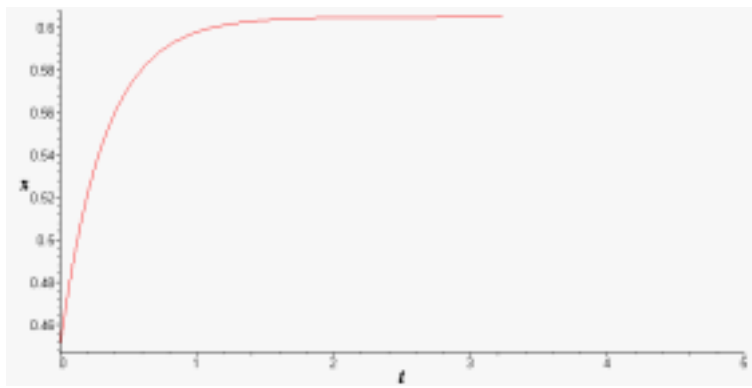

Figure 2: Plot for stock, x versus time, t for the case $r = 0$, $\Gamma = 0.1$.

evolution of the stock with respect to time by integrating Eq. (71), we obtain the following result plotted in Fig. 2.

Turning to u_1 we have

$$u_1 = \frac{\int_a^x M(\tilde{x}, u_0(\tilde{x}))d\tilde{x} - K_1}{P_u(x, u_0(x))}\,, \tag{76}$$

where a is any suitable arbitrary chosen constant. Notice here that $P_u = H_u + H_\lambda M_u = G\,M_u = 0$, at the zeroth order equilibrium point x^{**} given by Eq. (74) (known quantity). Then regularity of u_1 at this point require

$$\int_a^{x^{**}} M(\tilde{x}, u_0(\tilde{x}))d\tilde{x} - K_1 = 0$$

or

$$K_1 = \int_a^{x^{**}} M(\tilde{x}, u_0(\tilde{x}))d\tilde{x}\,. \tag{77}$$

Finally we find

$$u_1 = \frac{\int_{x^{**}}^x M(\tilde{x}, u_0(\tilde{x}))d\tilde{x}}{P_u(x, u_0(x))}\,. \tag{78}$$

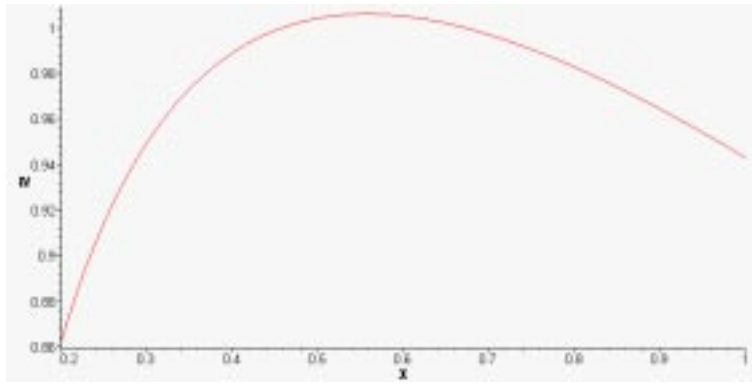

Figure 3: Plot for the first order component, u_1, of solution u versus x for arbitrary but "small" r, $\Gamma = 0.1$. Thus to get the proper value this solution must be multiplied by r.

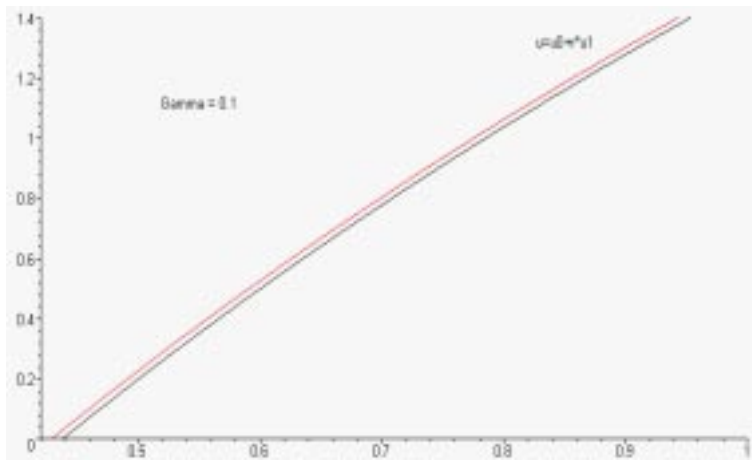

Figure 4: The red curve is a plot for $u = u_0 + r\,u_1$ versus x for the case $r = 0.3$. The black curve is the numerical solution of the exact problem. We see that the approximate solution to first order compares very well with the "exact" solution.

This determines u_1, and this way the procedure continues. Notice that this expression for u_1 is singular at the equilibrium point, since $P_u = 0$ there, as pointed out. In order to determine u_1 at such points one may use L'Hôpitals rule. For this particular case we can solve for this solution numerically and obtain the results shown in Fig. 3. In Fig. 4 the solution obtained by perturbation/expansion correct to first order is plotted (red curve) and we observe that it fits very well with the exact solution (black curve).

We shall later argue that by proper adjustment of the zeroth order solution, it may not be necessary to even go to the trouble of finding the above mentioned first order solution.

5.3 A numerical approach

These kind of problems can be solved numerically without any approximations in the governing equations. A way of solving this problem is to locate the equilibrium points, make the choice that maximizes the Hamiltonian; which at these points is equal to $\Pi(x_E, u_E)$, where x_E, u_E is the equilibrium point, see Eq. (75). Then we observe that the only solutions making contact with the equilibrium point are the separatrices emerging from or approaching this point. Notice that one may not start the solution at the equilibrium point, since this point may not be part of the trajectory leading to it i. e. the separatrix. Numerically one may solve for the separatrices for the exact problem, by starting the solution just outside the equilibrium point in a point that is consistent with the calculated

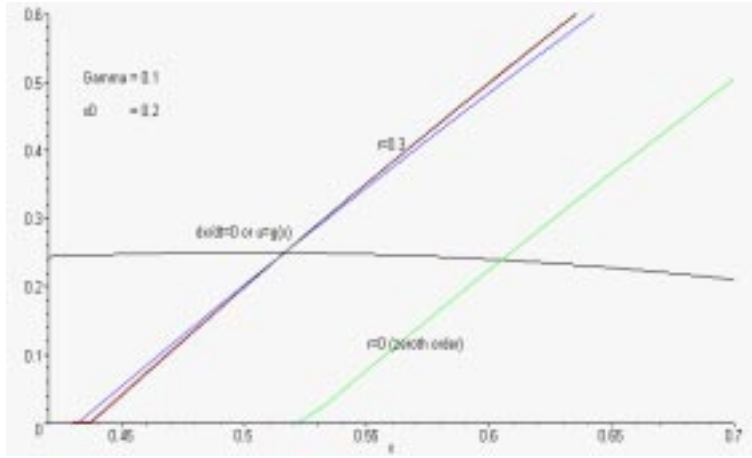

Figure 5: Plot for u versus x for two cases $r = 0$ (*green*) and $r = 0.3$ (*black*) and $dx/dt = 0$ or $u = x(x-1)$ (*black*). Notice that the *blue* curve, corresponding to $r = 0$ (*green*), is created by moving the $r = 0$ - curve (green) parallel to itself so that the corresponding equilibrium points coincides in the equilibrium point for the $r = 0.3$ curve (*black*). The red curve is created by a simpler procedure, just lifting the zeroth-order (*green*) curve until it passes through the equilibrium point for the $r = 0.3$ curve (*black*). Notice that for this case $\Gamma = 0.1$.

value of the derivative at the equilibrium point. In the following figure this has been done.

One observation we make is that if we take the zeroth order solution and move it parallel, so that the point corresponding to the zeroth order equilibrium point now coincide with the exact equilibrium point for the given $r = 0.3$ curve, then this adjusted zeroth order solution makes a very good approximation to the exact solution in the interval of interest i.e. for $0.44 < x < 1.0$.

In Fig. 6 a different situation is plotted where the constant Γ is replaced by a function, $\Gamma = \frac{0.1}{x^2}$.

We conclude from these figures that one policy is to move the zeroth order curve in such a way that the two equilibrium points coincide. Alternatively one may simply *lift* the zeroth-order curve by adding an amount Δu to u, so that the adjusted curve passes through the "exact" equilibrium point. The point to be made is: Determining the equilibrium point numerically is a straight forward approach for any given problem of interest. Finding the zeroth order solution is likewise straight forward, and can also be done by solving algebraic equations, i.e $P(x, u_0(x)) =$ constant. We then argue that for a large class of problems this may be a sufficiently accurate approximation, and the only representation for the approximate solution that is needed. Thus we now have a very simple procedure for finding a closed form approximate solution that is very likely to be sufficiently accurate for most cases of interest. This solution is obtained as follows:

1. **Find the zeroth order solution.**
2. **Move this solution vertically until it passes through the exact equilibrium point**.

Both these steps are simple to perform also in practical terms. We now turn back to a more specific discussion regarding error control.

5.4 Analytic approximation

The usefulness of the approximation theory presented in section 4.6 needs to be demonstrated through application. We apply some of our results to the following class of problems.

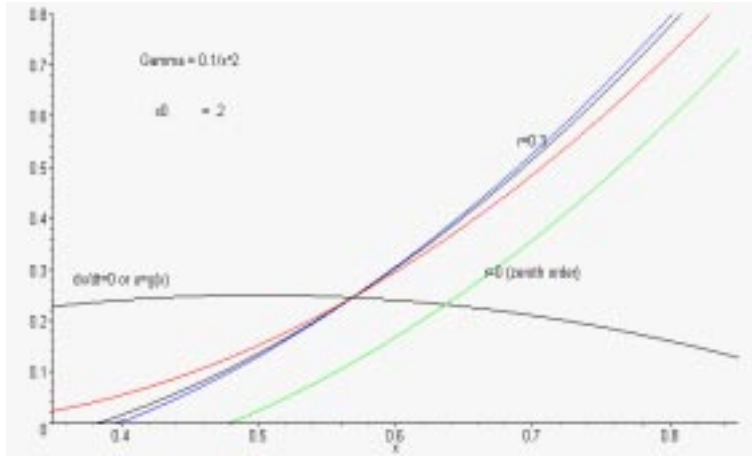

Figure 6: Plot for u versus x for two cases $r = 0$ (green) and $r = 0.3$ (black) and $dx/dt = 0$ or $u = x(x-1)$ (black). Notice that the blue curve, corresponding to $r = 0$ (green), is created by moving the $r = 0$ curve parallel to itself so that the corresponding equilibrium points coincides in the equilibrium point for the $r = 0.3$ curve (black). Likewise the red curve is produced by simply lifting the green curve vertically until it passes through the equilibrium point for the $r = 0.3$ case. Notice that $\Gamma = \frac{0.1}{x^2}$ for this case.

Consider an example relevant to renewable resource management, with a quadratic utility function as previously introduced by Eqs. (48) and (49)

$$\begin{aligned} I(v) &= \max_{u \geq 0} \int_0^\infty e^{-\delta t}(\gamma u - \Gamma u^2)dt\,, \\ \dot{x} &= g(x) - u \qquad x \in [0,1]\,, \end{aligned}$$

where $\gamma = \gamma(x) \geq 0$ and $\Gamma = \Gamma(x) \geq 0$ and the stock x is measured relative to carrying capacity. We shall adapt the following strategy:

1. Solve the zeroth-order expansion problem i.e. the associated problem with zero discounting. Normally this will be associate with the separatrices approaching or passing through such a point.

2. The next part is to determine the *improved* solution which is obtained by taking the zeroth order solution and move it parallel to itself so that it passes through the corresponding exact equilibrium point (this procedure is not unique). The *exact* equilibrium point can easily be determined. This procedure is highlighted in the Figures 5, 6 and 7. We shall show that there exist a way of doing this which yields an explicit expression for the feedback solution which is $\mathcal{O}(\delta^2)$ optimal in the global sense.

Thus let

$$\Pi(x,u) \stackrel{def}{=} \gamma(x)\,u - \Gamma(x)\,u^2\,, \tag{79}$$

then we introduce the following definitions:

$$S(x) \stackrel{def}{=} \Pi(x, g(x)) = \gamma(x)\,g(x) - \Gamma(x)\,g^2(x)\,. \tag{80}$$

$$\mu(x) \stackrel{def}{=} \Pi_u(x, g(x)) = \gamma(x) - 2\Gamma(x)\,g(x)\,. \tag{81}$$

Equilibrium is now determined by solving $S'(x) - \delta\mu = 0$. From H =const we find

$$\Gamma\dot{x}^2 + S = H = S^*\,, \tag{82}$$

with S^* a constant defined below and we have that the last equality sign above applies only to the case $\delta = 0$. For details see [11]. In the case of zero discount rate, $\delta = 0$, we introduce

$$u_0 \stackrel{def}{=} u_0(x) = g(x) + \nu\sqrt{\frac{S^* - S(x)}{\Gamma}}\,, \quad \tilde{u}_0 \stackrel{def}{=} \max\{0, u_0(x)\}\,, \tag{83}$$

where $\nu = \operatorname{sgn}(x - x^*)$, $x^* \stackrel{def}{=} \operatorname{argmax} S$, i.e. equilibrium, $x \in [0,1]$ and $S^* \stackrel{def}{=} S(x^*)$, $u_0(x^*) = g(x^*)$.

In the case of several equilibrium points one select that one which corresponds to the largest S.

$$x^{**} \stackrel{def}{=} \operatorname{argmax} \tilde{S}\,, \quad \text{determined by} \quad \tilde{S}' = 0 \quad \text{where} \quad \tilde{S}' \stackrel{def}{=} S' - \delta\,\mu\,. \tag{84}$$

Then consider a parallel movement of the zero discount rate solution so that it passes through the equilibrium point for the problem with nonzero discount rate.

Normally with a discount rate, $\delta > 0$, we have a saddle point at equilibrium, x^{**}.

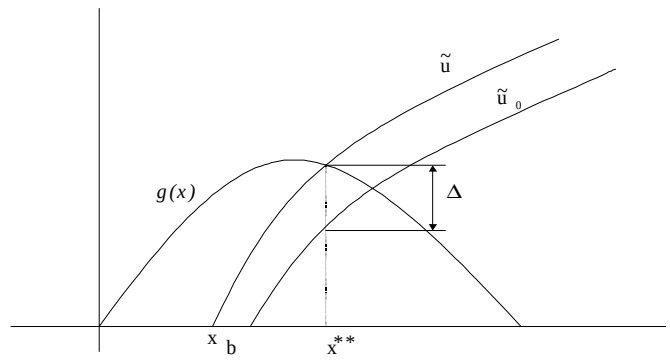

Figure 7: Plot for $\tilde{u}$ and u_0 ($\delta = 0$) versus x. The curve $\tilde{u}$ is produced by simply moving the u_0-curve vertically until it passes through the equilibrium point for the $\delta \neq 0$ case. Here x_b is the barrier/moratorium for the approximate policy $\tilde{u}$. At $x = x^{**}$ we have $\tilde{u}(x^{**}) = u_0(x^{**}) + \Delta = g(x^{**})$.

The parallel shift is made so that $u^{**} \stackrel{def}{=} u(x^{**}) = g(x^{**})$:

$$\tilde{u}(x) \stackrel{def}{=} \max(0, u_0(x) + \Delta)\,, \tag{85}$$

and according to Eq. (83)

$$\Delta \stackrel{def}{=} g(x^{**}) - u_0(x^{**}) = \sqrt{(S^* - S^{**})/\Gamma^{**}}\,, \tag{86}$$

where $S^{**} \stackrel{def}{=} S(x^{**})$ and $\Gamma^{**} \stackrel{def}{=} S(x^{**})$. We then compute R (see Eq. (24)), in the functional $L(v;\phi)$, which is given by

$$\begin{aligned} R &= \{\phi_t + \phi_x(g - u)\} + \gamma u - \Gamma u^2 \\ &= -\delta\, W + W'\, g + \Gamma \hat{u}^2 - \Gamma(u - \hat{u})^2\,, \end{aligned}$$

where we have substituted for the bounding function ϕ by $\phi = e^{-\delta t} W(x)$ and introduced the new parameter

$$\hat{u} \stackrel{def}{=} \frac{1}{2\Gamma}(\gamma - W')\,. \tag{87}$$

Regarding W this is a free function that we may utilize to make a choice so that our corresponding ϕ becomes a bounding function associated with our $\tilde{u}$ i. e. $\hat{u} = \tilde{u}$. This implies that we can choose

$$W \stackrel{def}{=} \int_{x^{**}}^{x} (\gamma(x') - 2\Gamma(x')\tilde{u}(x'))dx' + \frac{S^{**}}{\delta}\,, \tag{88}$$

where the constant term S^{**}/δ is a choice of convenience as will become clear later (see the proof following proposition 5.1). We conclude from this result that R is maximized by $u = \tilde{u}$. By a simple rewriting we obtain

$$P(t,x) \stackrel{def}{=} \max_u R(t,x,u) = (-\delta\, W + W' g + \Gamma\tilde{u}^2)e^{-\delta t} = \left\{-\delta\, W + S + \Gamma(g-\tilde{u})^2\right\} e^{-\delta t}\,, \tag{89}$$

where S is defined in Eq. (80). Eq. (89) may again be rewritten in the following form

$$e^{\delta t} P \stackrel{def}{=} \mathcal{P} = S - \delta\, W + \Gamma\dot{x}^2 = S - \delta\, W + \Gamma(g-\tilde{u})^2\,. \tag{90}$$

Then let us consider the interval $[x_b,\, x^{**}] \subset [0,1]^2$, which for most cases is the region of interest when the model is made dimensionless so that $x \in [0,1]$ means x less or equal to carrying capacity. Let

$$D \stackrel{def}{=} \Gamma\dot{x}^2 = \Gamma(\dot{x}_0 - \Delta)^2 = D_0 - 2\Gamma\Delta\dot{x}_0 + \Gamma\Delta^2 \quad \Rightarrow \quad D' = -\frac{\dot{x}}{\dot{x}_0}S' - \Gamma'\Delta\,\dot{x}\,,$$

where a prime means derivative with respect to x and we have used $S^* = S + D_0$, $D_0 \stackrel{def}{=} \Gamma\dot{x}_0^2$, see Eq. (82). We assume that $\mathcal{P}(x)$ (Eq. (90), is a concave function and look for a maximum by discussing the derivative $\mathcal{P}'(x) = 0$. We obtain

$$\mathcal{P}'(x) = \tilde{S}' - \frac{\dot{x}}{\dot{x}_0}\left\{S' + 2\delta\,\Gamma\,\dot{x}_0 + \Gamma'\,\Delta\,\dot{x}_0\right\}\,, \quad \tilde{S}' \stackrel{def}{=} S' - \delta\,\mu\,, \tag{91}$$

where we have $\Gamma\,\dot{x}_0 = \sqrt{\Gamma\,(S^* - S)}$ and $\Gamma\,\dot{x} = \sqrt{\Gamma\,(S^* - S)} - \Gamma\,\Delta$. We formulate the basic result at this point in the following proposition

Proposition 5.1 Concave $\mathcal{P}$

Suppose

$$\mathcal{P}(x) = S(x) - \delta\, W(x) + \Gamma(x)\left\{g(x) - \tilde{u}^2(x)\right\} \tag{92}$$

is concave on an $x-$inervall, $\tilde{X}$, where $x^{**} \in \tilde{X}$. Then it follows that

$$\max_{\tilde{X}} \mathcal{P}(x) = \mathcal{P}(x^{**}) = 0\,.$$

Proof:

Fore $x = x^{**}$ we have $\dot{x} = 0$ in Eq. (91) and from Eq. (84) we have $\tilde{S}' = 0$. Thus it follows that $\mathcal{P}'(x^{**}) = 0$ and $\max \mathcal{P} = \mathcal{P}(x^{**}) = S^{**} - \delta\, W(x^{**}) + \Gamma\dot{x}^2 = 0$, where the definition of W, Eq. (88) has been employed.

Q.E.D.

Proposition 5.2 An $\mathcal{O}(\delta^2)$ bound on relative error

Suppose $0 \le \Delta \le u_0$ and $\mathcal{P} \le 0$ for x in an intervall $\tilde{X} \ni x^{**}$.

Then it follows that the policy $\tilde{u}$ at most has a relative error of order δ^2.

[2] See Fig. 7 and the associated text

Proof:
From proposition 5.1 or proposition 5.2 - (2a), it follows that

$$\eta \stackrel{def}{=} \max \mathcal{P} - \mathcal{P} = \delta W - S - D = -\mathcal{P} \geq 0. \tag{93}$$

We continue by evaluating η.

$$\begin{aligned} \delta W - S &= S^{**} - S + \delta \int_{x^{**}}^{x} \{\gamma(x') - 2\Gamma(x')\,\tilde{u}(x')\}\,dx' \\ &= S^{**} - S + \delta \int_{x^{**}}^{x} \Pi_u(x', \tilde{u}(x'))dx' \end{aligned}$$

$$D = \Gamma \dot{x}^2 = \Gamma(\dot{x}_0 - \Delta)^2 = D_0 - 2\Gamma\Delta\dot{x}_0 + \Gamma\Delta^2 = (S^* - S) - 2\Gamma\Delta\dot{x} - \Gamma\,\Delta^2\,.$$

$$\begin{aligned} \eta &= \delta\,W - S - \Gamma\dot{x}^2 = \delta\,W - S - D \\ &= (\Gamma - \Gamma^{**})\Delta^2 + 2\Gamma\,\Delta\,\dot{x} + \delta \int_{x^{**}}^{x} \Pi_u(x', \tilde{u}(x'))\,dx' \\ &\leq (\Gamma - \Gamma^{**})\Delta^2 + 2\Gamma\,\Delta\,\dot{x}\,, \end{aligned}$$

where we have made use of Eq. (86). The last inequality follows from the assumption that $\Pi_u \geq 0$ [3], so that the integral gives a negative contribution for $x < x^{**}$. It follows that

$$\hat{\epsilon} \stackrel{def}{=} \int_0^\infty e^{-\delta t}\eta dt \leq 2\bar{\Gamma}\,\Delta + \frac{\bar{\Gamma} - \Gamma^{**}}{\delta}\Delta^2\,, \tag{94}$$

or

$$\epsilon \stackrel{def}{=} \frac{\hat{\epsilon}}{W^{**}} = \frac{2\bar{\Gamma}\delta\,\Delta + (\bar{\Gamma} - \Gamma^{**})\Delta^2}{S^{**}} = \mathcal{O}(\delta^2)\,. \tag{95}$$

We have that $\bar{\Gamma} \stackrel{def}{=} \max \Gamma(x)$ where $x \in A$. Thus $\bar{\Gamma} - \Gamma^{**} \geq 0$.
Notice that Eq. (95) is not merely an asymptotic error bound, but emerge as a fixed upper bound.

Q.E.D.

6 Summary and Conlusions

The first part formulates and reviews optimization problems in a general setting. The *Principle of Extension* plays a central role in this connection. Opening up the problem formulation to a wider context and introducing free functions that can be utilized for simplification purposes is important in this context. Restricting ourselves to *Dynamic Optimization in Continuous Time* this is given special attention in Sec. 3. The *Principle of Extension* is given further attention in Sec. 4, where one especially focus on the drastic approximation of leaving out the equation of state, as a first approach. Equivalent representation is an other important tool that is discussed. A special derivation of *The Variational Hamiltonian Formulation* as well as *The Dynamic Programming Principle* from our more general setting is provided. The problem of determining the accuracy of an approximation is also given special attention. The general results obtained are discussed in the setting of practical problem formulations in Sec. 5.

[3] There are sound economic reasons to limit on selves to this restriction.

References

[1] C. W. Clark: Mathematical Bioeconomics: The Optimal Management of Renewable Resources (New York, John Wiley & Sons, 1990.

[2] Atle Seierstad and Knut Sydsæter: Optimal Control Theory with Economic Applications Second reprint, North-Holland (1999).

[3] Kamien and Schwartz: Dynamic Optimization. (second ed.) North-Holland 1991.

[4] L. K. Sandal and G. Berge: A Perturbation Approach on a Class of Optimal Control Problems, Unifying the Pontryagin and Dynamic Programming Approach. Discussion paper 15/2000. June 2000.

[5] Leif K. Sandal and Stein I. Steinshamn: A Stochastic Feedback Model for Optimal Management of Renewable Resources. Natural Resource Modeling, 1997, 10: 31-52 ,

[6] Leif K. Sandal and Stein I. Steinshamn: A feedback model for optimal management of renewable natural capital stocks. Canadian Journal of Fisheries and Aquatic Sciences, 1997, 54: 2475-2482,

[7] Leif K. Sandal and Stein I. Steinshamn: Dynamic corrective taxes with flow and stock externalities: A feedback approach. Natural Resource Modeling, 1998, 11:217-239.

[8] R.Q. Grafton , Leif K. Sandal and Stein I. Steinshamn: How to improve the management of renewable resources: The case of Canada's northern cod fishery. American Journal of Agricultural Economics, 2000, 82: 570 - 580.

[9] Leif K. Sandal and Stein I. Steinshamn: The Cost of Underutilisation of Labour in Fisheries. Annals of Operations Research, forthcoming.

[10] Leif K. Sandal and Stein I. Steinshamn: A simplified feedback approach to optimal resource management. Natural Resource Modeling, forthcoming.

[11] Leif K. Sandal and Stein I. Steinshamn: Optimal Steady States and the Effect of Discounting Marine Resource Economics, 1997, 12.

[12] Leif K. Sandal and Stein I. Steinshamn: A Stochastic Feedback Model for Optimal Management of Renewable Resources. Natural Resource Modeling 1997, 10.